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# Null Controllability for Wave Equations with Memory

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## Abstract

We study the memory-type null controllability property for wave equations involving memory terms. The goal is not only to drive the displacement and the velocity (of the considered wave) to rest at some time-instant but also to require the memory term to vanish at the same time, ensuring that the whole process reaches the equilibrium. This memory-type null controllability problem can be reduced to the classical null controllability property for a coupled PDE-ODE system. The latter can be viewed as a degenerate system of wave equations, in which the velocity of propagation for the ODE component vanishes. This fact requires the support of the control to move to ensure the memory-type null controllability to hold, under the so-called Moving Geometric Control Condition. The control result is proved by duality by means of an observability inequality which employs measurements done on a moving observation open subset of the domain where the waves propagate.

## Résumé

Nous étudions le contrôle vers l'équilibre de l'équations des ondes avec mémoire. Ceci exige non seulement de contrôler le déplacement et la vitesse de l'onde considérée, mais aussi d'assurer que le terme de mémoire atteigne l'état d'équilibre à un instant donné. Ce problème de contrôle à zéro de type mémoire peut être réduit à un problème de contrôle à zéro pour un système couplé PDE-ODE. Ce dernier peut être considéré comme un système dégénéré d'équations d'ondes, pour lequel la vitesse de propagation de la composante ODE est nulle. Pour aboutir au résultat, le support du contrôle doit se déplacer le long du domaine où les ondes se propagent, assurant ainsi, non seulement l'absorption des rayons classiques de l'équation des ondes, mais aussi l'absorption des rayons qui ne se propagent pas et qui sont associés à l'ODE. On appelle cette condition : Condition de contrôle géométrique en mouvement. Le résultat de contrôle est prouvé moyennant une inégalité d'observabilité.

*Keywords:* Wave equations with memory, memory-type null controllability, moving control, moving geometric control condition, coupled PDE-ODE system

*2010 MSC:* 93B05, 74D05, 35L05, 93B07

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## 1. Introduction

This paper is devoted to analyzing the controllability properties of the following model for wave propagation involving a memory term:

$$\begin{cases} y_{tt} - \Delta y + \alpha y_t + \beta y + \int_0^t M(t, s)y(s)ds = \chi_{O(t)}u & \text{in } (0, +\infty) \times \Omega, \\ y = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ y(0) = y_0, y_t(0) = y_1 & \text{in } \Omega. \end{cases} \quad (1)$$

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Here,  $\Omega \subset \mathbb{R}^d$  ( $d \in \mathbb{N}$ ) is a given bounded domain with a  $C^\infty$ -smooth<sup>1</sup> boundary  $\partial\Omega$ ,  $\alpha, \beta \in C^\infty(\overline{\Omega})$ ,  $M \in C([0, T] \times [0, T])$  and  $T > 0$  is a given control time. System (1) is a controlled wave equation with a memory term entering as a lower order term perturbation, and the control being applied on an open subset  $O(t)$  of the domain  $\Omega$  where the waves propagate. The support  $O(t)$  of the control  $u(\cdot)$  at time  $t$  may move in time. This is reflected in the structure of the control in the right hand side of the equation where  $\chi_{O(t)} = \chi_{O(t)}(x)$  stands for the characteristic function of the set  $O(t)$ . The state of the system is given by  $(y, y_t)$  and the initial state by  $(y_0, y_1)$ . The control  $u \in L^2(O)$  is an applied force localised in  $O(t)$ , where  $O \equiv \{(t, x) \mid t \in (0, T), x \in O(t)\}$ . We shall also employ the notations  $Q = (0, T) \times \Omega$  and  $\Sigma = (0, T) \times \partial\Omega$ .

Roughly speaking, the main contributions of this paper can be described as follows:

- To show that the system (1) cannot be fully controlled if the support of the control does not move;
- To prove that the system (1) can be controlled, if the control moves in a suitable manner that we shall make precise.

Evolution models involving memory terms are ubiquitous. Natural and social phenomena are often affected not only by its current state but also by its history. Some classical examples are viscoelasticity, non-Fickian diffusion and thermal processes with memory. In this setting, in view of the locality of partial differential operators, relevant models need to include non-local memory terms, leading to partial differential equations with memory. We refer readers to [3, 20, 34] and the rich references therein for more details. In particular, some studies for (1) can be found in [4, 5, 6].

In the literature, the controllability problems for evolution equations with memory terms have been studied extensively (See [2, 15, 17, 21, 22, 25, 26, 28, 32, 33, 35] and the references therein). However, in most of the existing works the problem has been addressed analyzing whether the state can be driven to zero at time  $t = T$ , without paying attention to the memory term. But this is insufficient to guarantee that the dynamics can reach the equilibrium. Obviously, for an evolution equation without memory terms, once its solution is driven to rest at time  $T$  by a control, then it vanishes for all  $t \geq T$  in the absence of control thereafter. This is not the case for evolution equations with memory terms.

To illustrate the above fact, let us consider the following simple controlled system:

$$\begin{cases} \frac{d\eta}{dt} + \int_0^t \eta(s)ds = v & \text{in } [0, +\infty), \\ \eta(0) = 1. \end{cases} \quad (2)$$

Assume that  $v \in L^2(0, T)$  is a control such that  $\eta(T) = 0$ . If we do not pay attention to the accumulated memory, i.e. if  $\int_0^T \eta(s)ds \neq 0$ , then the solution  $\eta(\cdot)$  will not stay at the rest after time  $T$  as  $t$  evolves. In other words, to ensure that the system reaches the equilibrium  $\eta(t) = 0$  for  $t \geq T$ , it would be also necessary that the memory term reaches the null value, that is,  $\int_0^T \eta(s)ds = 0$ .

The above example indicates that the correct notion of controllability for the system (1) at time  $t = T$  should require not only that

$$y(T) \equiv y_t(T) \equiv 0, \quad (3)$$

as considered in the existing literature, which is actually a partial controllability result, but also that

$$\int_0^T M(T, s)y(s)ds = 0. \quad (4)$$

This paper is devoted to a study of the above property (for the system (1)) that we refer to as memory-type null controllability (Precise definition will be given later).

As in our previous work addressed to the heat equation ([9]) we shall view the wave model involving the memory term as the coupling of a wave-like PDE with an ODE. This will allow us to show, first, that the memory-type controllability property cannot hold if the support  $O(t)$  of the control  $u(\cdot)$  is time-independent, unless where  $O = Q$ . We shall then introduce a sharp sufficient condition for memory-type controllability, the so-called Moving Geometric Control Condition (MGCC, for short). Inspired by the classical Geometric Control Condition (GCC, for short) introduced in

<sup>1</sup> Here and henceforth,  $C^\infty$ -regularity is assumed to simplify the presentation although most of the results in this paper hold for less regular data.

[1] for the control of the wave equation, the MGCC takes into account that the ODE component of the system involves characteristic rays which do not propagate in space and time. Accordingly, the support of the control set  $O(t)$ , moving in time, has to ensure not only that it observes all rays of Geometric Optics for the wave equation, but also that it covers the whole domain  $\Omega$  on its motion.

In the recent work [23] it has been shown that the classical GCC suffices for the control of the wave equation (without memory terms), even when the support of the control moves. The main result of our present paper shows that, under the stronger MGCC condition, the memory term can also be controlled. For this to hold some technical assumptions on the memory kernel will be required.

The memory wave equation (1) is well posed in a suitable functional setting that we describe below.

Set  $V = H^2(\Omega) \cap H_0^1(\Omega)$ , and denote by  $V'$  the dual space of  $V$  with respect to the pivot space  $L^2(\Omega)$ . It is easy to see that  $H^{-1}(\Omega) \subset V' \subset H^{-2}(\Omega)$  topologically and algebraically.

Define an unbounded linear operator  $A$  on  $L^2(\Omega)$  as follows:

$$\begin{cases} D(A) = V, \\ A\varphi = -\Delta\varphi, \quad \forall \varphi \in D(A). \end{cases} \quad (5)$$

Our system (1) is well-posed, as stated in the following result:

**Proposition 1.1.** *For any  $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$  and  $u \in L^2(O)$ , the system (1) admits a unique solution  $y \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ . Moreover,*

$$\|y\|_{C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))} \leq C(\|(y_0, y_1)\|_{H_0^1(\Omega) \times L^2(\Omega)} + \|u\|_{L^2(O)}). \quad (6)$$

We refer to the Appendix at the end of the paper for a proof of Proposition 1.1.

We are now ready to define the property of memory-type controllability.

**Definition 1.1.** *System (1) is said to be memory-type null controllable at time  $T$  if for any  $(y_0, y_1) \in V \times H_0^1(\Omega)$ , there is a control  $u \in L^2(O)$  such that the corresponding solution  $y$  satisfies that*

$$y(T) = 0, \quad y_t(T) = 0 \quad \text{and} \quad \int_0^T M(T, s)y(s)ds = 0 \quad \text{in } \Omega. \quad (7)$$

**Remark 1.1.** *The concept of memory-type null controllability for evolution equations with memory terms was introduced in [9] for controlled ODEs and parabolic equations with memory terms.*

When  $M \equiv 0$  the model under consideration reduces to the classical wave equation. But this paper is devoted to studying, mainly, the effect of the presence of a non-trivial memory term at the level of controllability.

At this point it is important to note that the memory-type null controllability is not sufficient to ensure the system (1) to stay at rest for  $t \geq T$ . This actually depends on the structure of the memory kernel. For instance, if  $M(\cdot, \cdot) \equiv 1$ , then, (7) ensures that  $y(t) = y_t(t) = 0$  for  $t \geq T$ , provided that  $u(t) = 0$  for  $t \geq T$ . However, this is not the case for general kernels  $M(\cdot, \cdot)$ . A detailed analysis will be given later.

Before ending this section, we remark that, the main motivation for considering systems in the form of (1) is to study the heat equations with memory and the linear viscoelastic systems. Let us give below a brief explanation.

Since the classical heat equation admits an infinite speed of propagation for a finite thermal pulse, it is not really physical. To give a more precise model for the heat transfer process, people modified Fourier's law and introduced heat equations with memory ([18]):

$$\begin{cases} w_t - \int_{-\infty}^t a(t-s)\Delta w(s)ds = 0 & \text{in } (0, +\infty) \times \Omega, \\ w = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ w = \eta & \text{in } (-\infty, 0) \times \Omega, \\ w(0) = w_0 & \text{in } \Omega. \end{cases} \quad (8)$$

Here  $a(\cdot)$  is a suitable function, called the heat-flux relaxation function; while  $(\eta, w_0)$  is a given history of the temperature, called an initial history. Such kind of equations were studied extensively in the literature (see [12, 14, 31, 36] and the references therein).

A typical case is that  $\eta \equiv 0$  and  $a \in C^\infty([0, +\infty))$  with  $a(0) = 1$ . In this case, the equation (8) with a control reads

$$\begin{cases} w_t - \int_0^t a(t-s)\Delta w(s)ds = \chi_{O(t)}u & \text{in } (0, +\infty) \times \Omega, \\ w = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ w(0) = w_0 & \text{in } \Omega. \end{cases} \quad (9)$$

Set

$$v(t, x) = \int_0^t a(t-s)w(s, x)ds. \quad (10)$$

Then,

$$v_t(t, x) = w(t, x) + \int_0^t a_t(t-s)w(s, x)ds. \quad (11)$$

By the classical result of the theory of integral equations (e.g. [16]), one can find a  $\gamma \in C^\infty([0, +\infty))$  such that,

$$\begin{aligned} w(t, \cdot) &= v_t(t, \cdot) + \int_0^t \gamma(t-s)v_t(s, \cdot)ds \\ &= v_t(t, \cdot) + \gamma(0)v(t, \cdot) + \int_0^t \gamma_t(t-s)v(s, \cdot)ds \quad \text{in } \Omega. \end{aligned} \quad (12)$$

It follows from (10) and (12) that

$$\begin{aligned} w_t - \int_0^t a(t-s)\Delta w(s)ds \\ = v_{tt} - \Delta v + \gamma(0)v_t + \gamma_t(0)v + \int_0^t \gamma_{tt}(t-s)v(s)ds = \chi_{O(t)}u. \end{aligned}$$

Hence, we reduce (9) to an equation in the form of (1).

On the other hand, for any  $b \in C^\infty([0, +\infty))$ , let us consider the following controlled linear viscosity system:

$$\begin{cases} w_{tt} - \Delta w - \int_0^t b(t-s)\Delta w(s)ds = \chi_{O(t)}u & \text{in } (0, +\infty) \times \Omega, \\ w = 0 & \text{on } (0, +\infty) \times \partial\Omega, \\ w(0) = w_0, \quad w_t(0) = w_1 & \text{in } \Omega. \end{cases} \quad (13)$$

Set

$$\Upsilon(t, x) = w(t, x) + \int_0^t b(t-s)w(s, x)ds. \quad (14)$$

Similar to the above, we may find a  $\rho \in C^\infty([0, +\infty))$  such that,

$$w(t, \cdot) = \Upsilon(t, \cdot) + \int_0^t \rho(t-s)\Upsilon(s, \cdot)ds \quad \text{in } \Omega. \quad (15)$$

From (14) and (15), we get that that

$$\begin{aligned} w_{tt} - \Delta w - \int_0^t b(t-s)\Delta w(s)ds \\ = \Upsilon_{tt} - \Delta \Upsilon + \rho(0)\Upsilon_t + \rho_t(0)\Upsilon + \int_0^t \rho_{tt}(t-s)\Upsilon(s)ds = \chi_{O(t)}u. \end{aligned}$$

65 Hence, (13) is transformed to an equation in the form of (1).

At least for some special cases, if the solution to (1) stay at rest for  $t \geq T$  by a control  $u$ , then so does the solution to (9)/(13) by the same control. Indeed, we have the following elementary result (See the Appendix for its proof):

**Proposition 1.2.** i) Let  $\lambda \in \mathbb{R}$  and  $a(s) = e^{-\lambda s}$  for  $s \geq 0$ . If  $v$  defined by (10) satisfies that  $v(t) \equiv 0$  for all  $t \geq T$ , then so does the solution  $w$  to (9).

70 ii) Let  $\lambda \neq 1$  and  $b(s) = -e^{-\lambda s}$  for  $s \geq 0$ . If  $\Upsilon$  defined by (14) satisfies that  $\Upsilon(t) \equiv 0$  for all  $t \geq T$ , then, the solution  $w$  to (13) satisfies that  $w(t) = w_t(t) \equiv 0$  for all  $t \geq T$ .

The rest of this paper is organized as follows. Section 2 is addressed to an analysis of the memory kernels. The main result of this paper, i.e., Theorem 3.1 will be presented in Section 3. In Section 4, we show that the memory-type null controllability of (1) can be obtained by the null controllability of a coupled system of a wave equation and an ODE with a memory term. Section 5 is devoted to the proof of Theorem 3.1. At last, in Section 6, we present some further comments and open problems.

## 2. Analysis of the memory kernels

We first give an example of memory system to show that, even for linear scalar ODEs, the final condition (7) does not suffice for the system to reach the equilibrium.

Let us first consider the following controlled ODE:

$$\begin{cases} \frac{d\eta}{dt} + \int_0^t M(t, s)\eta(s)ds = v & \text{in } [0, +\infty), \\ \eta(0) = 1. \end{cases} \quad (16)$$

Assume that there is a control  $v \in L^2(0, +\infty)$  with  $v(\cdot) = 0$  on  $(T, +\infty)$ , such that the corresponding solution  $\eta(\cdot)$  to the system (16) satisfies that

$$\eta(t) = 0, \quad \forall t \geq T. \quad (17)$$

Then, from (16), we have that

$$\int_0^T M(T, s)\eta(s)ds = \int_0^t M(t, s)\eta(s)ds = 0, \quad \forall t \geq T. \quad (18)$$

Now we show that for some kernels  $M(\cdot, \cdot)$ , (18) implies that  $\eta(\cdot) = 0$  on  $(0, T)$ . This shows that the memory-type null controllability cannot hold for this kind of kernels.

**Example 2.1.** Let  $M(t, s) = (s + 1)^t$ . Then, from (17), we get that

$$\frac{d\eta(t)}{dt} = 0 \quad \text{for all } t \geq T.$$

Using (17) again, we find that

$$\int_0^t M(t, s)\eta(s)ds = \int_0^T M(t, s)\eta(s)ds + \int_T^t M(t, s)\eta(s)ds = \int_0^T M(t, s)\eta(s)ds, \quad \forall t \geq T. \quad (19)$$

According to (18) and (19), and noting that  $M(t, s) = (s + 1)^t$ , we see that

$$\int_0^T (s + 1)^t \eta(s)ds = 0, \quad \forall t \geq T. \quad (20)$$

Let us take the derivative of the left hand side of (20) with respect to  $t$  and let  $t = T, T + 1, \dots, T + k, \dots$ . Then, from (20), it holds that

$$\int_0^T (s + 1)^k [(s + 1)^T \ln(s + 1)\eta(s)]ds = \int_1^{T+1} s^k [s^T (\ln s)\eta(s - 1)]ds = 0, \quad \forall k \in \{0\} \cup \mathbb{N}. \quad (21)$$

This, together with Weierstrass approximation theorem, implies that

$$s^T (\ln s)\eta(s - 1) = 0, \quad \forall s \in (1, T + 1).$$

Hence,

$$\eta(\cdot) = 0 \quad \text{in } (0, T).$$

Since  $\eta(\cdot)$  is continuous, we see that  $\eta(0) = 0$ .

The above example shows that the condition of memory-null controllability (7) does not guarantee the solutions to remain in the equilibrium. But it suffices for a large class of memory kernels, including special cases such as  $M(t, s) = e^{\alpha(t-s)}$  with  $\alpha \in \mathbb{R}$  and  $M(t, s) = f(s)$ . More generally, (7) suffices to guarantee solutions to remain in the equilibrium for  $t \geq T$  if the kernel  $M(t, s)$  satisfies

$$M(t_1, t_3) = \widetilde{M}(t_1, t_2)M(t_2, t_3), \quad (22)$$

for all  $t_1, t_2$  and  $t_3$  with  $0 \leq t_3 \leq t_2 \leq t_1 < \infty$ , and some function  $\widetilde{M}(\cdot, \cdot) \in C([0, \infty) \times [0, \infty))$ . Indeed, if (22) holds, then for any  $\sigma > T$ ,

$$\begin{aligned} \int_0^\sigma M(\sigma, s)y(s)ds &= \widetilde{M}(\sigma, T) \int_0^T M(T, s)y(s)ds + \int_T^\sigma M(\sigma, s)y(s)ds \\ &= \int_T^\sigma M(\sigma, s)y(s)ds. \end{aligned}$$

Therefore, if (7) and (22) hold, then the solution to (1) with the control  $u = 0$  on  $[T, +\infty)$  satisfies

$$\begin{cases} y_{tt} - \Delta y + \int_T^t M(t, s)y(s)ds = 0 & \text{in } (T, +\infty) \times \Omega, \\ y = 0 & \text{on } (T, +\infty) \times \partial\Omega, \\ y(T) = 0, y_t(T) = 0 & \text{in } \Omega. \end{cases} \quad (23)$$

It is clear that  $y = 0$  is the unique solution to (23), which shows that the solution to (1) vanishes for  $t > T$ .

### 3. MGCC and the main result

85 The lower order terms  $\alpha y_t$  and  $\beta y$  would not affect the controllability property of the system (1). Hence, in what follows, for simplicity of notations, we assume that  $\alpha = \beta = 0$ .

We shall address the memory-type control problem through the dual notion of observability. For this purpose, we first introduce the following equation:

$$\begin{cases} p_{tt} - \Delta p + \int_t^T M(s, t)p(s)ds + M(T, t)q_0 = 0 & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(T) = p_0, p_t(T) = p_1 & \text{in } \Omega, \end{cases} \quad (24)$$

where  $(p_0, p_1) \in L^2(\Omega) \times H^{-1}(\Omega)$  and  $q_0 \in L^2(\Omega)$ . Similar to the proof of Proposition 1.1, one can show that there is a unique solution  $p \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ .

90 Our first result establishes the equivalence between the memory-type null controllability and the observability of this system.

**Proposition 3.1.** *System (1) is memory-type null controllable if and only if there is a constant  $C > 0$  such that*

$$\begin{aligned} |p(0)|_{H^{-1}(\Omega)}^2 + |p_t(0)|_{V'}^2 &\leq C|p|_{L^2(Q)}^2, \\ \forall (p_0, p_1, q_0) &\in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega), \end{aligned} \quad (25)$$

where  $p(\cdot)$  is the solution to the equation (24).

Although Proposition 3.1 is a Corollary of [9, Proposition 2.1], we shall give a proof in an Appendix at the end of this paper for the sake of completeness.

95 Observe that (25) is the usual observability inequality that is assured in the context of wave equations if the GCC is satisfied (see [38] for a discussion of other methods to achieve these observability inequalities for the classical wave equations). But note that in the classical literature of wave equations without memory, the adjoint system does not involve either the memory term or the non-homogeneous one containing  $q_0$ . Of course, it is natural that the adjoint system involves a memory term. But the addition of the non-homogenous term is required to ensure that the

memory term is under control. This is a very important issue that requires a complete revision of the methods to get observability inequalities and, eventually, we need to impose the new condition MGCC.

In order to understand better the possibility that a system of the form (24) satisfies the observability inequality and how this needs of a moving control, as in our previous papers [8, 9], we reduce this complex equation to a coupled system of simpler equations (see e.g. [11] for the use of these ideas in the context of well-posedness).

To present the idea, let us first consider the model case  $M(\cdot, \cdot) \equiv 1$ .

Let  $z = \int_0^t y(s)ds$ . Then, the system (1) can be transformed into the following one:

$$\begin{cases} y_{tt} - \Delta y + z = \chi_{O(t)}u & \text{in } Q, \\ z_t = y & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(0) = y_0, y_t(0) = y_1, z(0) = 0 & \text{in } \Omega. \end{cases} \quad (26)$$

Similarly, the adjoint system (24) can be reduced to the following system:

$$\begin{cases} p_{tt} - \Delta p + q = 0 & \text{in } Q, \\ q_t = -p & \text{in } Q, \\ p = q = 0 & \text{on } \Sigma, \\ p(T) = p_0, p_t(T) = p_1, q(T) = q_0 & \text{in } \Omega. \end{cases} \quad (27)$$

From the second equation in (27), we have that

$$q_{tt} = -p_t \text{ in } Q.$$

Hence, the system (24) can be regarded as two coupled wave equations in which one of them degenerates, having null velocity of propagation. Enlightened by the Geometric Optics interpretation of the property of observability for the waves we could say that there are vertical rays in the  $(x, t)$  which do not propagate at all in the space variable  $x$ . Thus, inspired by the necessity of the GCC for the control of waves ([1]), and in view of the presence of these vertical rays, if we want to establish an observability estimate for the solution to (24) from a cylindrical subset  $(0, T) \times \omega \subset (0, T) \times \Omega$ , the only possibility is that  $\omega = \Omega$ . This means that we have to act with the control on the whole domain  $\Omega$  to control the system (1).

But, of course, with applications in view, one is interested in controlling the system with a minimal amount of control and, in particular, minimizing its support. This motivates the use of moving control supports  $O(t)$ .

This strategy was employed successfully in the study of the null controllability of viscoelasticity equations with viscous Kelvin-Voigt and frictional damping terms in [8, 27].

Taking into account that the system under consideration combines not only vertical rays that require the control/observation support to move, but also wave components that propagate with unit velocity, following the classical laws of Geometric Optics, inspired by [8, 23] we introduce the following:

**Definition 3.1.** We say that an open set  $U \subset Q$  satisfies the Moving Geometric Control Condition (MGCC for short), if

- 1) All rays of geometric optics of the wave equation enter into  $U$  before the time  $T$ ;
- 2) For all  $x_0 \in \Omega$ , the vertical line  $\{(s, x_0) \mid s \in \mathbb{R}\}$  enters into  $U$  before the time  $T$  and

$$L_U \triangleq \inf_{x \in \Omega} \sup_{(t_1, t_2) \times \{x\} \subset U} (t_2 - t_1) > 0. \quad (28)$$

**Remark 3.1.** The above Condition 2 needs that vertical rays, which do not propagate in space, also reach the control set and stay in it for some time. In practice this means that the cross section  $U(t)$  of  $U$  has to move as time  $t$  evolves covering the whole domain  $\Omega$ .

**Remark 3.2.** Controllability with moving controls was previously studied with different purposes (See [7, 8, 9, 24, 37] and the references therein). For example, in [7], the author used moving controls to obtain the exact controllability for the one dimensional wave equations with pointwise controls; in [24, 37], the authors used moving controls to get the rapid exact controllability of wave equations; in [8], the authors take advantage of moving controls to establish the null controllability of viscoelasticity equations with viscous Kelvin-Voigt and frictional dampings; particularly, in [9], the authors employ the moving control to get the memory-type null controllability for heat equations with memory.



The main result of the paper, stated as follows, ensures the memory-type null controllability of the system (1) under the MGCC.

**Theorem 3.1.** *Suppose that  $O$  fulfills the MGCC and that the memory kernel  $M$  satisfies*

$$M(\cdot, \cdot) \in C^3([0, T] \times [0, T]) \quad \text{and} \quad M(t, 0)M(T, t) \neq 0, \quad \forall t \in [0, T]. \quad (29)$$

*Then the system (1) is memory-type null controllable.*

**Remark 3.3.** *Both the regularity condition on  $M(\cdot, \cdot)$  and the assumption that  $M(t, 0)M(T, t)$  does not vanish for any  $t \in [0, T]$  are, very likely, unnecessary. However, we use them in the proof. For instance, in (59) below, we need the third order derivative of  $M$ . Furthermore, in the definition of the adjoint system (33) and in view of the structure of the auxiliary kernels  $M_1$  and  $M_2$ , we need to assume that  $M(t, 0)M(T, t) \neq 0$  for any  $t \in [0, T]$ .*

#### 4. Reduction of the memory-type null controllability problem to the null controllability problem of a coupled system

In this section, we reduce the memory-type null controllability problem of the system (1) to the null controllability problem of a suitable coupled system. For convenience, we first introduce some subsets of  $O$  as follows.

For any  $\varepsilon > 0$  and  $A \subset \mathbb{R}^{1+d}$ , write  $O_\varepsilon(A) = \{z \in \mathbb{R}^{1+d} \mid \text{dist}(z, A) < \varepsilon\}$ . Put

$$O_\varepsilon \triangleq O \setminus \overline{O_\varepsilon(\partial O \setminus \Sigma)}. \quad (30)$$

Since  $O$  fulfills the MGCC, there exists an  $\varepsilon_0 > 0$  such that  $O_{\frac{3}{2}\varepsilon_0}$  (and hence  $O_{\varepsilon_0}$ ) still fulfills the MGCC.

Let  $\rho \in C^\infty(\overline{Q})$  satisfy that

$$\begin{cases} 0 \leq \rho \leq 1, \\ \rho = 1 \text{ in } O_{\varepsilon_0}, \\ \rho = 0 \text{ in } O \setminus O_{\frac{\varepsilon_0}{2}}. \end{cases} \quad (31)$$

Clearly,  $\text{supp } \rho \subset \overline{O}$ .

Instead of (1), we consider the following controlled system:

$$\begin{cases} y_{tt} - \Delta y + M(t, 0)z = \rho u & \text{in } Q, \\ z_t = M_1(t, t)y + \int_0^t M_{1,t}(t, s)y(s)ds & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(0) = y_0, \quad y_t(0) = y_1, \quad z(0) = z_0 & \text{in } \Omega, \end{cases} \quad (32)$$

where

$$M_1(t, s) = \frac{M(t, s)}{M(t, 0)}.$$

Although there is still a memory term in the system (32), it appears in the ODE part, which is easier to handle, as we shall see below.

**Definition 4.1.** *The system (32) is called null controllable if for any  $(y_0, y_1, z_0) \in V \times H_0^1(\Omega) \times V$ , there is a control  $u \in L^2(O)$  such that the corresponding solution  $(y, z)$  satisfies  $y(T) = 0$ ,  $y_t(T) = 0$  and  $z(T) = 0$  in  $\Omega$ .*

**Remark 4.1.** *Clearly, if  $z(0) = 0$ , then the solution  $y$  to (32) solves (1). Hence, the null controllability of (32) implies the memory-type null controllability of (1). On the other hand, the null controllability of (1) implies a partial null controllability of (32) (with  $z_0 = 0$ ).*

To study the null controllability of the system (32), let us introduce the adjoint system:

$$\begin{cases} p_{tt} - \Delta p + M(T, t)q = 0 & \text{in } Q, \\ q_t = -M_2(t, t)p + \int_t^T M_{2,t}(s, t)p(s)ds & \text{in } Q, \\ p = q = 0 & \text{on } \Sigma, \\ p(T) = p_0, \quad p_t(T) = p_1, \quad q(T) = q_0 & \text{in } \Omega, \end{cases} \quad (33)$$

where  $p_0 \in L^2(\Omega)$ ,  $p_1 \in H^{-1}(\Omega)$ ,  $q_0 \in L^2(\Omega)$  and

$$M_2(s, t) = \frac{M(s, t)}{M(T, t)}.$$

The memory term in (33) is also in the ODE part. But, as we shall see later, it only leads to a term which can be got rid of by a classical compactness-uniqueness argument.

**Definition 4.2.** *The equation (33) is said to be initially observable on  $O$  if,*

$$\begin{aligned} |p(0)|_{H^{-1}(\Omega)}^2 + |p_t(0)|_{V'}^2 + |q(0)|_{V'}^2 &\leq C|p|_{L^2(O)}^2, \\ \forall (p_0, p_1, q_0) &\in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega). \end{aligned} \quad (34)$$

By means of the standard duality argument, we can obtain the following result.

**Proposition 4.1.** *The system (32) is null controllable if and only if the equation (33) is initially observable on  $O$ .*

155 The left hand side of the inequality (34) contains terms involving norms in negative Sobolev spaces, which makes the analysis harder. Therefore, we first consider the controllability and observability problems for (32) and (33), respectively, in the following alternative functional setting.

**Definition 4.3.** i) *The system (32) with initial data in  $L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$  is said to be null controllable if for any  $(y_0, y_1, z_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ , there is a control  $u \in L^2(0, T; V')$  such that the corresponding solution  $(y, z)$  satisfies*

$$y(T) = 0, \quad y_t(T) = 0 \quad \text{and} \quad z(T) = 0, \quad \text{in } \Omega. \quad (35)$$

ii) *The equation (33) with final data in  $V \times H_0^1(\Omega) \times V$  is called initially observable on  $O$  with the weight  $\rho$  if there is exists a constant  $C > 0$  such that*

$$\begin{aligned} |p(0)|_{H_0^1(\Omega)}^2 + |p_t(0)|_{L^2(\Omega)}^2 + |q(0)|_{L^2(\Omega)}^2 &\leq C|\rho p|_{H^2(O)}^2, \\ \forall (p_0, p_1, q_0) &\in V \times H_0^1(\Omega) \times V. \end{aligned} \quad (36)$$

160 **Remark 4.2.** *In Definition 4.3, we put the attributives “with initial data in  $L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ ” and “with final data in  $V \times H_0^1(\Omega) \times V$ ” to emphasize that we are considering a functional setting different from those in Definitions 4.1 and 4.2. Once the null controllability problem is solved for the system (32) with  $(y_0, y_1, z_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$  and  $u \in L^2(0, T; V')$ , we can obtain the null controllability of the system (32) in the sense of Definition 4.1.*

We have the following result.

**Proposition 4.2.** *The following statements are equivalent:*

- 165 i) *The equation (33) with final data in  $V \times H_0^1(\Omega) \times V$  is initially observable on  $O$  with the weight  $\rho$ ;*  
ii) *The system (32) with initial data in  $L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$  is null controllable;*  
iii) *Solutions to (33) satisfy*

$$|p(0)|_{H_0^1(\Omega)}^2 + |p_t(0)|_{L^2(\Omega)}^2 + |q(0)|_{L^2(\Omega)}^2 \leq C|\Delta(\rho p)|_{L^2(O)}^2, \quad (37)$$

for all  $(p_0, p_1, q_0) \in V \times H_0^1(\Omega) \times V$ .

We refer to Subsection 5.1 for a proof of Proposition 4.2.

By Proposition 4.2, to get the null controllability of (32) with initial data in  $L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ , we only need to establish the inequality (36), which is true according to the following theorem.

170 **Theorem 4.1.** *Suppose that  $O$  fulfills the MGCC and that the memory kernel  $M$  satisfies the condition (29). Then the system (33) with final data in  $V \times H_0^1(\Omega) \times V$  is initially observable on  $O$  with the weight  $\rho$ . Moreover, when  $M(\cdot, \cdot)$  is a nonzero constant, one cannot replace the term  $|\rho p|_{H^2(O)}$  (in the right hand side of (36)) by  $|\rho p|_{H^s(O)}$  for any  $s < 2$ .*

We refer to Subsection 5.2 for a proof of Theorem 4.1.

By Proposition 4.2 and Theorem 4.1, we can obtain the following null controllability result for the system (32).

175 **Corollary 4.1.** *Suppose that  $O$  fulfills the MGCC and that the memory kernel  $M$  satisfies the condition (29). Then the system (32) with initial data in  $L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$  is null controllable.*

As shown in [13], if the initial datum is more regular, then we can choose more regular control functions.

**Corollary 4.2.** *Suppose that  $O$  fulfills the MGCC and that the memory kernel  $M$  satisfies the condition (29). Then the system (32) is null controllable.*

180 **Remark 4.3.** *We can obtain the memory-type null controllability for the system (1) as an immediate corollary of Corollary 4.2, and via which, Theorem 3.1 follows.*

## 5. Proof of Theorem 3.1

This section is addressed to the proof of Theorem 3.1. To complete this task, as we have shown in Section 4, we only need to prove Corollary 4.2. We first prove Proposition 4.2 and Theorem 4.1.

### 185 5.1. Proof of Proposition 4.2

*Proof of Proposition 4.2:* Let us first derive an equality (equality (41) below), which will be used in later. By multiplying the first equation of (32) by  $p$  and by integrating by parts, one has that

$$\begin{aligned} & \langle p_0, y_t(T) \rangle_{V, V'} - \langle p_1, y(T) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \langle p(0), y_1 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ & + \langle p_t(0), y_0 \rangle_{L^2(\Omega)} + \int_0^T [\langle p, M(t, 0)z \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \langle M(T, t)q, y \rangle_{H_0^1(\Omega), H^{-1}(\Omega)}] dt \\ & = \langle \rho p, u \rangle_{L^2(0, T; V), L^2(0, T; V')}. \end{aligned} \quad (38)$$

It follows from the second equations in (32) and (33) that

$$\begin{aligned} & \int_0^T \langle p, M(t, 0)z \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt \\ & = \int_0^T \langle M(t, 0)p(t), z_0 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt + \int_0^T \left\langle p(t), \int_0^t M(t, s)y(s)ds \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt \\ & = M(T, 0)\langle q(0) - q_0, z_0 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt + \int_0^T \left\langle p(t), \int_0^t M(t, s)y(s)ds \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt \end{aligned} \quad (39)$$

and

$$\begin{aligned} & \int_0^T \langle M(T, t)q, y \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt \\ & = \int_0^T \langle q_0, M(T, t)y(t) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt + \int_0^T \left\langle \int_t^T M(s, t)p(s)ds, y(t) \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt \\ & = M(T, 0)\langle q_0, z(T) - z_0 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} dt + \int_0^T \left\langle p(s), \int_0^s M(s, t)y(t)dt \right\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} ds. \end{aligned} \quad (40)$$

According to (38)–(40), we have that

$$\begin{aligned} & \langle p_0, y_t(T) \rangle_{V, V'} - \langle p_1, y(T) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \langle p(0), y_1 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ & + \langle p_t(0), y_0 \rangle_{L^2(\Omega)} - M(T, 0)\langle q_0, z(T) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} + M(T, 0)\langle q(0), z_0 \rangle_{L^2(\Omega)} \\ & = \langle \rho p, u \rangle_{L^2(0, T; V), L^2(0, T; V')}. \end{aligned} \quad (41)$$

**i)⇒ii).** Denote by  $\mathcal{Y}$  the Hilbert space which is the completion of

$$\left\{ (p_0, p_1, q_0) \in V \times H_0^1(\Omega) \times V \mid \int_0^T |(\partial_{tt} + \Delta)(\rho p)|^2 dx dt < \infty \right\} \quad (42)$$

with respect to the norm

$$\|(p_0, p_1, q_0)\|_{\mathcal{Y}} \triangleq \left( \int_0^T |(\partial_{tt} + \Delta)(\rho p)|^2 dx dt \right)^{\frac{1}{2}},$$

where  $p$  solves (33) with the final datum  $(p_0, p_1, q_0)$ .

We claim that  $\mathcal{Y} \subset H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ . Indeed, if  $(p, q)$  is a solution to (33), then it also solves the following equation:

$$\begin{cases} \tilde{p}_t - \Delta \tilde{p} + M(T, t)\tilde{q} = 0 & \text{in } Q, \\ \tilde{q}_t = -M_2(t, t)\tilde{p} + \int_t^T M_{2,t}(s, t)\tilde{p}(s)ds & \text{in } Q, \\ \tilde{p} = \tilde{q} = 0 & \text{on } \Sigma, \\ \tilde{p}(0) = p(0), \tilde{p}_t(0) = p_t(0), \tilde{q}(0) = q(0) & \text{in } \Omega. \end{cases} \quad (43)$$

From (36), we know that if  $(p_0, p_1, q_0) \in \mathcal{Y}$ , then

$$(p(0), p_t(0), q(0)) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \subset L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega).$$

This, together with the well-posedness of (43), implies that

$$(\tilde{p}, \tilde{q}) \in [C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))] \times C^1([0, T]; L^2(\Omega)). \quad (44)$$

From (43), we know that

$$\begin{cases} \tilde{p}_t - \Delta \tilde{p} + M(T, t)\tilde{q} = 0 & \text{in } Q, \\ \tilde{p} = 0 & \text{on } \Sigma, \\ \tilde{p}(0) = p(0), \tilde{p}_t(0) = p_t(0) & \text{in } \Omega. \end{cases} \quad (45)$$

Since  $(p(0), p_t(0)) \in H_0^1(\Omega) \times L^2(\Omega)$  and  $\tilde{q} \in C^1([0, T]; L^2(\Omega))$ , we have that

$$\tilde{p} \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)). \quad (46)$$

This, together with (44), implies that  $(p_0, p_1, q_0) = (\tilde{p}(T), \tilde{p}_t(T), \tilde{q}(T)) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ .

Fix any  $(y_0, y_1, z_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ , and define a functional  $J : \mathcal{Y} \rightarrow \mathbb{R}$  by

$$\begin{aligned} J(p_0, p_1, q_0) &= \frac{1}{2} \int_O |(\partial_t + \Delta)(\rho p)|^2 dxdt + \langle p(0), y_1 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &\quad - (p_t(0), y_0)_{L^2(\Omega)} - M(T, 0)(q(0), z_0)_{L^2(\Omega)}, \end{aligned} \quad (47)$$

where  $(p, q)$  solves (33) with the final datum  $(p_0, p_1, q_0) \in \mathcal{Y}$ . Clearly,  $J(\cdot, \cdot, \cdot)$  is continuous and strictly convex. From (36), we have that

$$\begin{aligned} &J(p_0, p_1, q_0) \\ &\geq \frac{1}{2} \int_O |(\partial_t + \Delta)(\rho p)|^2 dxdt - |p(0)|_{H_0^1(\Omega)} |y_1|_{H^{-1}(\Omega)} - |p_t(0)|_{L^2(\Omega)} |y_0|_{L^2(\Omega)} - |M(T, 0)| |q(0)|_{L^2(\Omega)} |z_0|_{L^2(\Omega)} \\ &\geq C_1 |\rho p|_{H^2(O)}^2 - (|p(0)|_{H_0^1(\Omega)} |y_1|_{H^{-1}(\Omega)} + |p_t(0)|_{L^2(\Omega)} |y_0|_{L^2(\Omega)} + |M(T, 0)| |q(0)|_{L^2(\Omega)} |z_0|_{L^2(\Omega)}) \\ &\geq C_1 |\rho p|_{H^2(O)}^2 - C_2 |p|_{H^2(O)} (|y_1|_{H^{-1}(\Omega)} + |y_0|_{L^2(\Omega)} + |z_0|_{L^2(\Omega)}), \end{aligned} \quad (48)$$

where  $C_1$  and  $C_2$  are independent of  $(p, q)$ .

By (48),  $J(\cdot, \cdot, \cdot)$  is coercive. Thus,  $J(\cdot, \cdot, \cdot)$  admits a unique minimizer  $(\hat{p}_0, \hat{p}_1, \hat{q}_0)$  in  $\mathcal{Y}$ . Denote by  $(\hat{p}, \hat{q})$  the solution to (33) with the final datum  $(\hat{p}_0, \hat{p}_1, \hat{q}_0)$ . Then, for any  $(p_0, p_1, q_0) \in V \times H_0^1(\Omega) \times V$  and  $\delta \in \mathbb{R}$ ,

$$\begin{aligned} 0 &\leq J(\hat{p}_0 + \delta p_0, \hat{p}_1 + \delta p_1, \hat{q}_0 + \delta q_0) - J(\hat{p}_0, \hat{p}_1, \hat{q}_0) \\ &= \frac{1}{2} \int_O |(\partial_t + \Delta)[\rho(\hat{p} + \delta p)]|^2 dxdt + \langle \hat{p}(0) + \delta p(0), y_1 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &\quad - (\hat{p}_t(0) + \delta p_t(0), y_0)_{L^2(\Omega)} - M(T, 0)(\hat{q}(0) + \delta q(0), z_0)_{L^2(\Omega)} \\ &\quad - \frac{1}{2} \int_O |(\partial_t + \Delta)(\rho \hat{p})|^2 dxdt + \langle \hat{p}(0), y_1 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - (\hat{p}_t(0), y_0)_{L^2(\Omega)} - M(T, 0)(\hat{q}(0), z_0)_{L^2(\Omega)} \\ &= \delta \int_O (\partial_t + \Delta)(\rho \hat{p})(\partial_t + \Delta)(\rho p) dxdt + \frac{\delta^2}{2} \int_O |(\partial_t + \Delta)(\rho p)|^2 dxdt \\ &\quad + \delta \langle p(0), y_1 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - \delta (p_t(0), y_0)_{L^2(\Omega)} - \delta M(T, 0)(q(0), z_0)_{L^2(\Omega)}. \end{aligned} \quad (49)$$

Thus,

$$\begin{aligned} 0 &= \lim_{\delta \rightarrow 0} \frac{J(\hat{p}_0 + \delta p_0, \hat{p}_1 + \delta p_1, \hat{q}_0 + \delta q_0) - J(\hat{p}_0, \hat{p}_1, \hat{q}_0)}{\delta} \\ &= \int_O (\partial_{tt} + \Delta)(\rho \hat{p})(\partial_{tt} + \Delta)(\rho p) dx dt + \langle p(0), y_1 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &\quad - (p_t(0), y_0)_{L^2(\Omega)} - M(T, 0)(q(0), z_0)_{L^2(\Omega)}. \end{aligned} \quad (50)$$

We claim that

$$(\partial_{tt} + \Delta)^2(\rho \hat{p}) \in L^2(0, T; V'). \quad (51)$$

To see this, write

$$\check{p} \triangleq (\partial_{tt} + \Delta)(\rho \hat{p}), \quad \check{q} \triangleq (\partial_{tt} + \Delta)(\rho \hat{q}). \quad (52)$$

From the definition of  $\mathcal{Y}$ , we see that

$$(\partial_{tt} + \Delta)(\rho \hat{p}) \in L^2(O). \quad (53)$$

Since  $(\hat{p}_0, \hat{p}_1, \hat{q}_0) \in \mathcal{Y} \subset H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ , similar to the proof of (44) and (46), we have

$$(\hat{p}, \hat{q}) \in (C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))) \times C^1([0, T]; L^2(\Omega)). \quad (54)$$

This implies that

$$2\rho_t \hat{p}_t + \rho_{tt} \hat{p} \in C([0, T]; L^2(\Omega)) \quad \text{and} \quad 2\nabla \rho \cdot \nabla p + \hat{p} \Delta \rho \in C([0, T]; L^2(\Omega)).$$

Since  $(\hat{p}, \hat{q})$  is the solution to (33), it is easy to see that  $(\rho \hat{p}, \rho \hat{q})$  satisfies

$$\begin{cases} (\rho \hat{p})_{tt} - \Delta(\rho \hat{p}) + M(T, t)\rho \hat{q} = 2\rho_t \hat{p}_t + \rho_{tt} \hat{p} - 2\nabla \rho \cdot \nabla \hat{p} - \hat{p} \Delta \rho & \text{in } Q, \\ (\rho \hat{q})_t = -M_2(t, t)\rho \hat{p} + \rho \int_t^T M_{2,t}(s, t)p(s) ds + \rho_t \hat{q} & \text{in } Q, \\ \rho \hat{p} = \rho \hat{q} = 0 & \text{on } \Sigma, \\ (\rho \hat{p})(T) = 0, (\rho \hat{p})_t(T) = 0, (\rho \hat{q})(T) = 0 & \text{in } \Omega. \end{cases} \quad (55)$$

According to (52) and (55), we get that  $(\check{p}, \check{q})$  solves

$$\begin{cases} \check{p}_{tt} - \Delta \check{p} + M(T, t)\check{q} \\ = -2M_t(T, t)(\rho \hat{q})_t - M_{tt}(T, t)\rho \hat{q} + (\partial_{tt} + \Delta)(2\rho_t \hat{p}_t + \rho_{tt} \hat{p} - 2\nabla \rho \cdot \nabla \hat{p} - \hat{p} \Delta \rho) & \text{in } Q, \\ \check{q}_t = (\partial_{tt} + \Delta) \left[ -M_2(t, t)\rho \hat{p} + \rho \int_t^T M_{2,t}(s, t)\hat{p}(s) ds + \rho_t \hat{q} \right] & \text{in } Q, \\ \check{p} = \check{q} = 0 & \text{on } \Sigma, \\ \check{p}(T) = 0, \check{p}_t(T) = 0, \check{q}(T) = 0 & \text{in } \Omega. \end{cases} \quad (56)$$

From (33) and (54), we see that

$$\begin{aligned} &(\partial_{tt} + \Delta)(\rho_t \hat{p}_t) \\ &= \rho_{tt} \hat{p}_t + 2\rho_{tt} \hat{p}_t + \rho_t \hat{p}_{tt} + \Delta \rho_t \hat{p}_t + 2\nabla \rho_t \nabla \hat{p}_t + \rho_t \Delta \hat{p}_t \\ &= \rho_{tt} \hat{p}_t + 2\rho_{tt}(\Delta \hat{p} - M(T, t)\hat{q}) + \rho_t(\Delta \hat{p}_t - M(T, t)\hat{q}_t - M_t(T, t)\hat{q}) + \Delta \rho_t \hat{p}_t \\ &\quad + 2\nabla \rho_t \nabla \hat{p}_t + \rho_t \Delta \hat{p}_t \in C([0, T]; V'). \end{aligned} \quad (57)$$

Similarly, we can obtain that

$$(\partial_{tt} + \Delta)(\rho_{tt} \hat{p} - 2\nabla \rho \cdot \nabla \hat{p} - \hat{p} \Delta \rho) \in C([0, T]; V') \quad (58)$$

and

$$(\partial_{tt} + \Delta) \left[ -M_2(t, t)\rho \hat{p} + \rho \int_t^T M_{2,t}(s, t)\hat{p}(s) ds + \rho_t \hat{q} \right] \in C([0, T]; V'). \quad (59)$$

It follows from (56) and (59) that

$$\check{q} \in C^1([0, T]; V'). \quad (60)$$

By means of (53), we find that

$$\Delta(\partial_t + \Delta)(\rho\hat{p}) \in L^2(0, T; V'). \quad (61)$$

Combining (56), (57), (58), (60) and (61), we conclude that

$$\partial_t(\partial_t + \Delta)(\rho\hat{p}) \in L^2(0, T; V'). \quad (62)$$

From (61) and (62), we obtain (51).

Put

$$u = (\partial_t + \Delta)^2(\rho\hat{p}). \quad (63)$$

By (51) and (63), and noting the equation (33), we see that

$$\int_O (\partial_t + \Delta)(\rho\hat{p})(\partial_t + \Delta)(\rho p) dx dt = \langle \rho p, u \rangle_{L^2(0, T; V), L^2(0, T; V')}. \quad (64)$$

From (50), (64) and (41), we conclude that for all  $(p_0, p_1, q_0) \in V \times H_0^1(\Omega) \times V$ ,

$$\langle p_0, y_t(T) \rangle_{V, V'} - \langle p_1, y(T) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} - M(T, 0) \langle q_0, z(T) \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} = 0, \quad (65)$$

which implies that

$$y(T) = 0 \text{ in } H^{-1}(\Omega), \quad y_t(T) = 0 \text{ in } V' \quad \text{and} \quad z(T) = 0 \text{ in } H^{-1}(\Omega).$$

**ii)⇒iii).** Since the system (32) is null controllable, for any given  $(y_0, y_1, z_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ , there is a control  $u \in L^2(0, T; V')$  driving the corresponding solution to the rest. From the proof of (41), we have that

$$-\langle p(0), y_1 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} + \langle p_t(0), y_0 \rangle_{L^2(\Omega)} + M(T, 0) \langle q(0), z_0 \rangle_{L^2(\Omega)} = \langle \rho p, u \rangle_{L^2(0, T; V), L^2(0, T; V')}. \quad (66)$$

Define a bounded linear operator  $\mathcal{L} : \mathcal{Y} \rightarrow H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$  as follows:

$$\mathcal{L}(p_0, p_1, q_0) = (p(0), p_t(0), q(0)),$$

where  $(p(0), p_t(0), q(0))$  is the value at time  $t = 0$  of the solution to the equation (33) with the final datum  $(p_0, p_1, q_0)$ .

We now use a contradiction argument to prove that solutions to the equation (33), which satisfy (37), are unique. If this was false, then, one could find a sequence  $\{(p_0^k, p_1^k, q_0^k)\}_{k=1}^\infty \subset \mathcal{Y}$  with  $(p_0^k, p_1^k, q_0^k) \neq (0, 0, 0)$  for all  $k \in \mathbb{N}$ , such that the corresponding solutions  $(p^k, q^k)$  to (33) (with  $(p_0, p_1, q_0)$  replaced by  $(p_0^k, p_1^k, q_0^k)$ ) satisfy that

$$\int_O |\Delta(\rho p^k)|^2 dx dt \leq \frac{1}{k^2} (|p^k(0)|_{H_0^1(\Omega)}^2 + |p_t^k(0)|_{L^2(\Omega)}^2 + |q^k(0)|_{L^2(\Omega)}^2). \quad (67)$$

Write

$$\lambda_k = \frac{\sqrt{k}}{\sqrt{|p^k(0)|_{H_0^1(\Omega)}^2 + |p_t^k(0)|_{L^2(\Omega)}^2 + |q^k(0)|_{L^2(\Omega)}^2}}$$

and

$$\tilde{p}_0^k = \lambda_k p_0^k, \quad \tilde{p}_1^k = \lambda_k p_1^k, \quad \tilde{q}_0^k = \lambda_k q_0^k,$$

and denote by  $(\tilde{p}^k, \tilde{q}^k)$  the corresponding solution to (33) (with  $(p_0, p_1, q_0)$  replaced by  $(\tilde{p}_0^k, \tilde{p}_1^k, \tilde{q}_0^k)$ ). Then, it follows from (67) that, for each  $k \in \mathbb{N}$ ,

$$\int_O |\Delta(\rho \tilde{p}^k)|^2 dx dt \leq \frac{1}{k} \quad (68)$$

and

$$|\mathcal{L}(\tilde{p}_0^k, \tilde{p}_1^k, \tilde{q}_0^k)|_{H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)} = \sqrt{k}. \quad (69)$$

In view of (66), we have that

$$\begin{aligned} & -\langle \tilde{p}^k(0), y_1 \rangle_{H_0^1(\Omega), H^{-1}(\Omega)} + \langle \tilde{p}_t^k(0), y_0 \rangle_{L^2(\Omega)} + M(T, 0) \langle \tilde{q}^k(0), z_0 \rangle_{L^2(\Omega)} \\ & = \langle \rho \tilde{p}^k, u \rangle_{L^2(0, T; V), L^2(0, T; V')}. \end{aligned} \quad (70)$$

By (68) and (70), we have that

$$\mathcal{L}(\tilde{p}_0^k, \tilde{p}_1^k, \tilde{q}_0^k) \text{ tends to 0 weakly in } H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \text{ as } k \rightarrow +\infty$$

Hence, by the Principle of Uniform Boundedness, the sequence  $\{\mathcal{L}(\tilde{p}_0^k, \tilde{p}_1^k, \tilde{q}_0^k)\}_{k=1}^\infty$  is uniformly bounded in  $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ , which contradicts (69).

**iii)⇒i).** This is obvious. Hence we complete the proof of Proposition 4.2.  $\square$

*Proof:* Under the MGCC, we have that (see [23] for the proof)

$$|p|_{H^1(Q)}^2 \leq C(|p_t|_{L^2(O_{\varepsilon_0})}^2 + |q|_{L^2(Q)}^2). \quad (71)$$

For any  $t \in (0, T)$  and  $x \in O_{\varepsilon_0}(t)$ , it follows from (33) that

$$|q(s, x)|^2 \leq C(|q(t, x)|^2 + \int_0^T |p(x, \sigma)|^2 d\sigma), \quad \forall s \in (0, T). \quad (72)$$

Since  $O_{\varepsilon_0}$  fulfills the MGCC, by integrating (72) on  $O_{\varepsilon_0}$ , we get that (recall (28) for the definition of  $L_{O_{\varepsilon_0}}$ )

$$\begin{aligned} L_{O_{\varepsilon_0}} \int_{\Omega} |q(s, x)|^2 dx &\leq \int_{O_{\varepsilon_0}} |q(s, x)|^2 dx dt \\ &\leq C \left( \int_{O_{\varepsilon_0}} |q(t, x)|^2 dx dt + \int_{O_{\varepsilon_0}} \int_0^T |p(x, \sigma)|^2 d\sigma dx dt \right), \quad \forall s \in (0, T). \end{aligned} \quad (73)$$

This implies that

$$L_{O_{\varepsilon_0}} \int_0^T \int_{\Omega} |q(s, x)|^2 dx ds \leq C \left( \int_{O_{\varepsilon_0}} |q(t, x)|^2 dx dt + \int_0^T \int_{\Omega} |p(t, x)|^2 dx dt \right),$$

that is,

$$|q|_{L^2(Q)}^2 \leq C(|q|_{L^2(O_{\varepsilon_0})}^2 + |p|_{L^2(Q)}^2). \quad (74)$$

From (71) and (74), we find that

$$|p|_{H^1(Q)}^2 + |q|_{L^2(Q)}^2 \leq C(|p_t|_{L^2(O_{\varepsilon_0})}^2 + |q|_{L^2(O_{\varepsilon_0})}^2 + |p|_{L^2(Q)}^2). \quad (75)$$

Now we are going to get rid of the last term in the right hand side of (75) by a compactness-uniqueness argument, that is, we will prove the following inequality:

$$|p|_{H^1(Q)}^2 + |q|_{L^2(Q)}^2 \leq C(|p_t|_{L^2(O_{\varepsilon_0})}^2 + |q|_{L^2(O_{\varepsilon_0})}^2). \quad (76)$$

If (76) was false, then there would be a sequence  $\{p^k, q^k\}_{k=1}^{\infty} \subset H^1(Q) \times L^2(Q)$  solving (33) such that for all  $k \in \mathbb{N}$ ,

$$|(p^k, q^k)|_{H^1(Q) \times L^2(Q)} = 1 \quad (77)$$

and

$$|p_t^k|_{L^2(O_{\varepsilon_0})}^2 + |q^k|_{L^2(O_{\varepsilon_0})}^2 \leq \frac{1}{k}. \quad (78)$$

From (77), we know that there is a subsequence  $\{p^{k_j}, q^{k_j}\}_{j=1}^{\infty}$  of  $\{p^k, q^k\}_{k=1}^{\infty}$  such that

$$(p^{k_j}, q^{k_j}) \text{ converges weakly to some } (p^*, q^*) \text{ in } H^1(Q) \times L^2(Q). \quad (79)$$

It is clear that  $(p^*, q^*)$  is a weak solution to (33). By (79), we get that

$$p^{k_j} \text{ converges strongly to } p^* \text{ in } L^2(Q). \quad (80)$$

This, together with (74) and (78), implies that

$$q^{k_j} \text{ converges strongly to } q^* \text{ in } L^2(Q). \quad (81)$$

By (79), we have that

$$|p_t^*|_{L^2(O_{\varepsilon_0})}^2 + |q^*|_{L^2(O_{\varepsilon_0})}^2 \leq \liminf_{j \rightarrow \infty} (|p_t^{k_j}|_{L^2(O_{\varepsilon_0})}^2 + |q^{k_j}|_{L^2(O_{\varepsilon_0})}^2) = 0. \quad (82)$$

Hence

$$p_t^* = q^* = 0 \text{ in } O_{\varepsilon_0} \quad (83)$$

and

$$|p^*|_{H^1(Q)}^2 + |q^*|_{L^2(Q)}^2 \leq C|p^*|_{L^2(Q)}^2. \quad (84)$$

From (75), (77) and (78), we see that

$$1 \leq \frac{C}{k} + C|p^k|_{L^2(Q)}^2, \quad \forall k \in \mathbb{N}. \quad (85)$$

According to (80) and (85), we get that

$$|p^*|_{L^2(Q)} > 0. \quad (86)$$

Thus,  $(p^*, q^*)$  is not zero.

Let us introduce a linear subspace of  $H^1(Q) \times L^2(Q)$  as follows:

$$\mathcal{E} \triangleq \left\{ (p, q) \in H^1(Q) \times L^2(Q) \mid (p, q) \text{ satisfies the first two equations in (33),} \right. \\ \left. p|_{\Sigma} = 0, \text{ and } p_t = q = 0 \text{ in } O_{\varepsilon_0} \right\}. \quad (87)$$

Clearly,  $(p^*, q^*)$  given in (79) belongs to  $\mathcal{E}$ . Consequently,  $\mathcal{E} \neq \{0\}$ . Now we are going to prove that  $\mathcal{E} = \{0\}$ , which is a contradiction.

We claim that

$$\mathcal{E} \subset H^4(Q) \times H^3(Q). \quad (88)$$

Indeed, since  $p_t = q = 0$  in  $O_{\varepsilon_0}$ , it follows from (33) that

$$-\Delta p = 0 \quad \text{in } O_{\varepsilon_0},$$

which implies that

$$p \in H^{l+1}(O_{\frac{3}{2}\varepsilon_0}), \quad \forall l \in \mathbb{N}. \quad (89)$$

Since  $O_{\frac{3}{2}\varepsilon_0}$  satisfies the MGCC, similar to the proof of (74), we obtain that

$$|q|_{H^1(Q)}^2 \leq C(|q|_{H^1(O_{\varepsilon_0})}^2 + |p|_{H^1(Q)}^2) \leq C|p|_{H^1(Q)}^2. \quad (90)$$

By the classical result on the propagation of singularities for the wave equation (see [10, Section 4.1] for example), we have that

$$p \in H^2(Q). \quad (91)$$

By the energy estimate for the ODE part of (33) again, we have that

$$|q|_{H^2(Q)}^2 \leq C(|q|_{H^2(O_{\varepsilon_0})}^2 + |p|_{H^2(Q)}^2) \leq C|p|_{H^2(Q)}^2. \quad (92)$$

This, together with the classical result for the propagation of singularities for the wave equation, implies that

$$p \in H^3(Q). \quad (93)$$

Repeating the similar argument once more, we conclude (88).

Next, we prove that  $\mathcal{E}$  is a finite dimensional space. Let  $\{p^i, q^i\}_{i=1}^{\infty} \subset \mathcal{E}$  satisfying

$$|p^i|_{H^1(Q)}^2 + |q^i|_{L^2(Q)}^2 = 1 \text{ for all } i \in \mathbb{N}.$$

Then, there is a subsequence  $\{p^{i_j}, q^{i_j}\}_{j=1}^{\infty} \subset \mathcal{E}$  such that

$$(p^{i_j}, q^{i_j}) \text{ converges weakly to some } (\hat{p}, \hat{q}) \text{ in } H^1(Q) \times L^2(Q) \text{ as } j \rightarrow +\infty.$$

Therefore,

$$p^{i_j} \text{ converges strongly to } \hat{p} \text{ in } L^2(Q) \text{ as } j \rightarrow +\infty. \quad (94)$$



From (75), we have that

$$|p|_{H^1(Q)}^2 + |q|_{L^2(Q)}^2 \leq C|p|_{L^2(Q)}^2, \quad \forall (p, q) \in \mathcal{E}.$$

This, together with (94), implies that

$$(p^{ij}, q^{ij}) \text{ converges strongly to } (\hat{p}, \hat{q}) \text{ in } H^1(Q) \times L^2(Q) \text{ as } j \rightarrow +\infty.$$

200 Hence,  $\dim \mathcal{E} < \infty$ .

For any  $(p, q) \in \mathcal{E}$ , by (88), noting  $O_{\varepsilon_0}$  fulfills the MGCC and  $q = 0$  in  $O_{\varepsilon_0}$ , we see that  $q = 0$  on  $\Sigma$ , and

$$\begin{cases} (\Delta p)_{tt} - \Delta(\Delta p) + M(T, t)\Delta q = 0 & \text{in } Q, \\ (\Delta q)_t = -M_2(t, t)(\Delta p) + \int_t^T M_{2,t}(s, t)(\Delta p)(s) ds & \text{in } Q, \\ \Delta p = \Delta q = 0 & \text{on } \Sigma. \end{cases} \quad (95)$$

Thus,  $(\Delta p, \Delta q)$  is also a solution to (33). Further, since

$$(p_t, q) = 0 \text{ in } O_{\varepsilon_0},$$

we have that

$$((\Delta p)_t, \Delta q) = 0 \text{ in } O_{\varepsilon_0}.$$

Hence  $(\Delta p, \Delta q) \in \mathcal{E}$ .

Since  $\mathcal{E}$  is a finite dimensional space, the operator  $\Delta$  has an eigenvalue  $\lambda \in \mathbb{C}$  and an eigenvector  $(\tilde{p}, \tilde{q}) \in \mathcal{E} \setminus \{0\}$ . We claim that  $\lambda \neq 0$ . Indeed, if  $\lambda = 0$ , then for any  $t \in (0, T)$ ,

$$\begin{cases} -\Delta \tilde{p}(t) = 0 & \text{in } \Omega, \\ \tilde{p}(t) = 0 & \text{on } \partial\Omega. \end{cases}$$

This concludes that

$$\tilde{p}(t) = 0 \text{ in } \Omega \text{ for all } t \in (0, T).$$

Then, from (33), we find that  $\tilde{q} = 0$  in  $Q$ . Hence  $(\tilde{p}, \tilde{q}) = 0$ , which is a contradiction.

Noting that this eigenfunction  $(\tilde{p}, \tilde{q})$  solves (33), we get that

$$\begin{cases} \tilde{p}_{tt} - \lambda \tilde{p} + M(T, t)\tilde{q} = 0 & \text{in } Q, \\ \tilde{q}_t = -M_2(t, t)\tilde{p} + \int_t^T M_{2,t}(s, t)\tilde{p}(s) ds & \text{in } Q, \\ \tilde{p} = \tilde{q} = 0 & \text{on } \Sigma. \end{cases} \quad (96)$$

Since

$$\tilde{p}_t = \tilde{q} = 0 \text{ in } O_{\varepsilon_0},$$

we see from (96) that

$$\tilde{p} = M(T, t) \frac{\tilde{q}}{\lambda} = 0 \text{ in } O_{\varepsilon_0}.$$

For a fixed  $t_0 \in (0, T)$  and  $x_0 \in O_{\varepsilon_0}(t_0)$ , it follows from (96) that  $(\tilde{p}(\cdot, x_0), \tilde{q}(\cdot, x_0))$  is the solution to

$$\begin{cases} \tilde{p}_{tt}(t, x_0) - \lambda \tilde{p}(t, x_0) + M(T, t)\tilde{q}(t, x_0) = 0 & \text{in } (0, T), \\ \tilde{q}_t(t, x_0) = -M_2(t, t)\tilde{p}(t, x_0) + \int_t^T M_{2,t}(s, t)\tilde{p}(s, x_0) ds & \text{in } (0, T), \\ \tilde{p}(t_0, x_0) = 0, \quad \tilde{p}_t(t_0, x_0) = 0, \quad \tilde{q}(t_0, x_0) = 0. \end{cases} \quad (97)$$

Clearly,

$$\tilde{p}(t, x_0) = \tilde{q}(t, x_0) = 0 \text{ for any } t \in (0, T).$$

Since MGCC holds, by the above argument, we can show that for any  $x \in \Omega$ ,

$$\tilde{p}(t, x) = \tilde{q}(t, x) = 0 \text{ for any } t \in (0, T),$$

that is,

$$\tilde{p} = \tilde{q} = 0 \text{ in } Q,$$

which implies that  $\mathcal{E} = \{0\}$ . This leads to a contradiction that  $(p^*, q^*)$  is not zero. Therefore, we obtain (76).

Now, we are going to get rid of the observation on  $q$ , i.e., the term  $|q|_{L^2(O_{e_0})}$  in the right hand side of (76). Since

$$q = \frac{1}{M(T, t)}(-p_{tt} + \Delta p), \quad (98)$$

from (76), we obtain that

$$|p|_{H^1(Q)}^2 + |q|_{L^2(Q)}^2 \leq C|p|_{H^2(O_{e_0})}^2. \quad (99)$$

This, together with the energy estimate of (33), implies that

$$|p(0)|_{H_0^1(\Omega)}^2 + |p_t(0)|_{L^2(\Omega)}^2 + |q(0)|_{L^2(\Omega)}^2 \leq C|p|_{H^2(O_{e_0})}^2 \leq C|\rho p|_{H^2(O)}^2. \quad (100)$$

Finally, we prove that (100) is sharp, i.e., we show that

$$|p(0)|_{H_0^s(\Omega)}^2 + |p_t(0)|_{L^2(\Omega)}^2 + |q(0)|_{L^2(\Omega)}^2 \leq C|p|_{H^s(O)}^2 \quad (101)$$

does not hold for any  $s < 2$ . Without loss of generality, let us assume that  $M(\cdot, \cdot) = 1$ . We achieve this goal by a contradiction argument.

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Denote by  $\{\lambda_j\}_{j=1}^\infty$  (with  $0 < \lambda_1 < \lambda_2 \leq \dots$ ) the eigenvalues of  $A$  (defined by (5)) and  $\{\varphi_j\}_{j=1}^\infty$  with  $|\varphi_j|_{L^2(\Omega)} = 1$  ( $j \in \mathbb{N}$ ) the corresponding eigenvectors. Put

$$a_j = -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\lambda_j^3}{27}}, \quad b_j = -\frac{1}{2} - \sqrt{\frac{1}{4} + \frac{\lambda_j^3}{27}} \quad \text{and} \quad \mu_j = \sqrt[3]{a_j} + \sqrt[3]{b_j}. \quad (102)$$

Then,  $\mu_j \in \mathbb{R}$  satisfies that

$$\mu_j^3 + \lambda_j \mu_j + 1 = 0 \quad (103)$$

and

$$|\mu_j| = \left| \sqrt[3]{a_j} + \sqrt[3]{b_j} \right| = \left| \frac{a_j + b_j}{\sqrt[3]{a_j^2 - \sqrt[3]{a_j b_j} + \sqrt[3]{b_j^2}}} \right| \leq \left| \frac{1}{\sqrt[3]{a_j^2}} \right|.$$

Since  $\lambda_j \rightarrow +\infty$  as  $j \rightarrow +\infty$ , we know that there is a constant  $j_0 > 0$  such that for all  $j \geq j_0$ ,

$$|\mu_j| \leq \left| \frac{1}{\sqrt[3]{a_j^2}} \right| < \frac{6}{\lambda_j}. \quad (104)$$

Put

$$p^j = e^{\mu_j(T-t)} \varphi_j \quad \text{and} \quad q^j = \frac{1}{\mu_j} e^{\mu_j(T-t)} \varphi_j.$$

Then,

$$\begin{aligned} & p_{tt}^j - \Delta p^j + q^j \\ &= \mu_j^2 e^{\mu_j t} \varphi_j + \lambda_j e^{\mu_j(T-t)} \varphi_j + \frac{1}{\mu_j} e^{\mu_j(T-t)} \varphi_j \\ &= (\mu_j^3 + \lambda_j \mu_j + 1) \frac{1}{\mu_j} e^{\mu_j(T-t)} \varphi_j = 0. \end{aligned}$$

Further,

$$p^j = e^{\mu_j(T-t)} \varphi_j = q^j = \frac{1}{\mu_j} e^{\mu_j(T-t)} \varphi_j = 0 \quad \text{on } \Sigma.$$

Thus,  $(p^j, q^j)$  is a solution to (33). For any  $j \geq j_0$ ,

$$|p^j(0)|_{H^1(\Omega)}^2 + |p_t^j(0)|_{L^2(\Omega)}^2 + |q^j(0)|_{L^2(\Omega)}^2 \geq \int_{\Omega} \left| \frac{1}{\mu_j} \varphi_j \right|^2 dx dt = \frac{1}{\mu_j^2} \geq \frac{\lambda_j^2}{36}. \quad (105)$$

On the other hand, for any  $j \in \mathbb{N}$ ,

$$|p^j|_{H^s(O)}^2 \leq |p^j|_{H^s(Q)}^2 \leq |e^{\mu_j t} \varphi_j|_{H^s(Q)}^2 \leq C\lambda_j^s. \quad (106)$$

From (99), (105) and (106), we get that

$$\lambda_j^2 \leq C(s)\lambda_j^s, \quad \forall j \geq j_0, \quad (107)$$

which is impossible.  $\square$

### 5.3. Proof of Theorem 3.1

*Proof of Theorem 3.1:* We only need to prove Corollary 4.2, which, by Proposition 4.1, is equivalent to the following inequality:

$$\begin{aligned} |p(0)|_{H^{-1}(\Omega)}^2 + |p_t(0)|_{V'}^2 + |q(0)|_{V'}^2 &\leq C \int_O |p|^2 dxdt, \\ \forall (p_0, p_1, q_0) &\in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega). \end{aligned} \quad (108)$$

For a given  $(p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ , put (Recall (5) for  $A$ )

$$(\tilde{p}_0, \tilde{p}_1, \tilde{q}_0) = (A^{-1}p_0, A^{-1}p_1, A^{-1}q_0) \in V \times H_0^1(\Omega) \times V.$$

Denote by  $(\tilde{p}, \tilde{q})$  and  $(p, q)$  the solutions to (33) with the final data  $(\tilde{p}_0, \tilde{p}_1, \tilde{q}_0)$  and  $(p_0, p_1, q_0)$ , respectively. From (33), we have that

$$\begin{cases} (A^{-1}p)_{tt} - \Delta(A^{-1}p) + M(T, t)A^{-1}q = 0 & \text{in } Q, \\ (A^{-1}q)_t = -M_2(t, t)A^{-1}p + \int_t^T M_{2,t}(s, t)A^{-1}p(s)ds & \text{in } Q, \\ A^{-1}p = A^{-1}q = 0 & \text{on } \Sigma, \\ (A^{-1}p)(T) = A^{-1}p_0, (A^{-1}p)_t(T) = A^{-1}p_1, (A^{-1}q)(T) = A^{-1}q_0 & \text{in } \Omega. \end{cases} \quad (109)$$

This concludes that

$$(\tilde{p}, \tilde{q}) = (A^{-1}p, A^{-1}q) \text{ in } Q.$$

By Theorem 4.1 and Proposition 4.2, we see that

$$|A^{-1}p(0)|_{H_0^1(\Omega)}^2 + |A^{-1}p_t(0)|_{L^2(\Omega)}^2 + |A^{-1}q(0)|_{L^2(\Omega)}^2 \leq C|\Delta(\rho A^{-1}p)|_{L^2(O)}^2, \quad (110)$$

which implies that

$$|p(0)|_{H^{-1}(\Omega)}^2 + |p_t(0)|_{V'}^2 + |q(0)|_{V'}^2 \leq C|p|_{L^2(O)}^2. \quad (111)$$

$\square$

## 6. Further comments and open problems

- Our strategy of proving Theorem 3.1 is to reduce the memory-type null controllability of (1) to the null controllability of the coupled system (32). Nevertheless, in order to obtain the memory-type null controllability of the system (1), one only needs the following observability estimate:

$$|p(0)|_{H_0^1(\Omega)}^2 + |p_t(0)|_{L^2(\Omega)}^2 \leq C|p|_{H^s(O)}^2. \quad (112)$$

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Theorem 4.1 concludes that (36) is sharp. However, the reason for this is that we put the term  $|q(0)|_{L^2(\Omega)}^2$  on the left hand side of (36). Indeed, to prove that (36) is sharp, we construct a sequence of solutions  $(p^j, q^j) = (e^{\mu_j t} \varphi_j, \frac{1}{\mu_j} e^{\mu_j t} \varphi_j)$  of (33), which shows that the right hand side of (36) cannot be replaced by some  $|p|_{H^s(O)}$  for  $s < 2$ . Unfortunately, this argument fails to show that the right hand side of (112) cannot be replaced by some  $|p|_{H^s(O)}$  for  $s < 2$ . Whether the right hand side of (112) can be replaced by some  $|p|_{H^s(O)}$  for  $s < 2$  is an interesting open problem.

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- We have studied the memory-type null controllability of the wave equation with a memory term  $\int_0^t M(t, s)y(s)ds$ . It is more natural and interesting to study the same problem but for the system below:

$$\begin{cases} y_{tt} - \Delta y - \int_0^t M(t, s)\Delta y(s)ds = u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, y_t(0) = y_1 & \text{in } \Omega, \end{cases} \quad (113)$$

where  $(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$ , and  $u \in L^2(0, T; V')$  with  $\text{supp } u \subset \bar{O}$ .

Following the method used in this paper, we can introduce a coupled system:

$$\begin{cases} y_{tt} - \Delta y - M(t, 0)\Delta z = u & \text{in } Q, \\ z_t = M_1(t, t)y + \int_0^t M_{1,t}(t, s)y(s)ds & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(0) = y_0, y_t(0) = y_1, z(0) = z_0 & \text{in } \Omega. \end{cases} \quad (114)$$

However, we do not know how to establish the null controllability of (114) except if  $O = Q$ . Indeed, the adjoint system of (114) reads

$$\begin{cases} p_{tt} - \Delta p - M(T, t)\Delta q = 0 & \text{in } Q, \\ q_t = -M_2(t, t)p + \int_t^T M_{2,t}(s, t)p(s)ds & \text{in } Q, \\ p = q = 0 & \text{on } \Sigma, \\ p(T) = p_0, p_t(T) = p_1, q(T) = q_0 & \text{in } \Omega. \end{cases} \quad (115)$$

Here  $p_0 \in V$ ,  $p_1 \in H_0^1(\Omega)$  and  $q_0 \in V$ . If we follow the proof of Theorem 4.1, we get that

$$|p|_{H^1(Q)}^2 \leq C(|p_t|_{L^2(O)}^2 + |\Delta q|_{L^2(O)}^2)$$

and

$$|\Delta q|_{L^2(O)}^2 \leq C(|\Delta q|_{L^2(O)}^2 + |\Delta p|_{L^2(O)}^2),$$

which lead to

$$|p|_{H^1(Q)}^2 + |q|_{L^2(O)}^2 \leq C(|p_t|_{L^2(O)}^2 + |q|_{L^2(O)}^2 + |\Delta p|_{L^2(O)}^2). \quad (116)$$

We do not know how to get rid of the last term in the right hand side of (116) since it is not compact with respect to the terms in the left hand of (116).

Nevertheless, at least for some special form of the kernel  $M(\cdot, \cdot)$ , similar to what we have done for the systems (9) and (13), by setting

$$w(t, x) = y(t, x) + \int_0^t M(t, s)y(s, x)ds,$$

which leads to

$$y(t, x) = w(t, x) + \int_0^t \tilde{M}(t, s)w(s, x)ds$$

for some kernel  $\tilde{M}(\cdot, \cdot)$ , we may reduce the system (113) to the form of (1).

- Our argument in Subsection 5.2 works well for time dependent memory kernels. However, it seems that it cannot be applied to wave equations with a space dependent memory kernel. For example, let us consider the following system:

$$\begin{cases} y_{tt} - \Delta y + \int_0^t M(t, s, x)y(s)ds = \chi_O u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, y_t(0) = y_1 & \text{in } \Omega. \end{cases} \quad (117)$$

Following the method used in this paper, we can introduce a coupled system:

$$\begin{cases} y_{tt} - \Delta y + M(t, 0, x)z = \chi_{OU} & \text{in } Q, \\ z_t = M_1(t, t, x)y + \int_0^t M_{1,t}(t, s, x)y(s)ds & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ y(0) = y_0, y_t(0) = y_1, z(0) = z_0 & \text{in } \Omega, \end{cases} \quad (118)$$

and its adjoint system:

$$\begin{cases} p_{tt} - \Delta p + M(T, t, x)q = 0 & \text{in } Q, \\ q_t = -M_2(t, t, x)p + \int_t^T M_{2,t}(s, t, x)p(s)ds & \text{in } Q, \\ p = q = 0 & \text{on } \Sigma, \\ p(T) = p_0, p_t(T) = p_1, q(T) = q_0 & \text{in } \Omega. \end{cases} \quad (119)$$

Here  $M_1(t, s, x) = \frac{M(t, s, x)}{M(t, 0, x)}$ ,  $M_2(t, s, x) = \frac{M(t, s, x)}{M(T, t, x)}$ ,  $p_0 \in V$ ,  $p_1 \in H_0^1(\Omega)$  and  $q_0 \in V$ . Similar to the proof of (75), we can obtain that

$$|p|_{H^1(Q)}^2 + |q|_{L^2(Q)}^2 \leq C(|p_1|_{L^2(\Omega)}^2 + |q_0|_{L^2(\Omega)}^2 + |p|_{L^2(Q)}^2). \quad (120)$$

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We do not know how to get rid of the last term in the right hand side of (120). Indeed, it seems that the compactness-uniqueness argument does not work since we do not know how to establish the desired unique continuation property for (119).

- We only consider the memory-type null controllability for the linear wave equation with a linear memory term. The same problems could be studied for wave equations with some nonlinear lower order terms or a nonlinear memory term. Nevertheless, the method of proof used in this paper, which allows dealing with linear equations with special memory kernels, does not apply in the nonlinear context. For example, let us consider the memory-type null controllability of the following semi-linear equation:

$$\begin{cases} y_{tt} - \Delta y + f(y) + \int_0^t M(t, s)y(s)ds = \chi_{OU} & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0) = y_0, y_t(0) = y_1 & \text{in } \Omega, \end{cases} \quad (121)$$

where  $f$  is a suitable nonlinear function.

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Usually, the controllability of semilinear systems is achieved by combining a controllability for the linearized system of the nonlinear one and a fixed point method. To do this, we should first consider a linear equation involving a  $(t, x)$ -dependent potential. However, the approach developed to derive the observability estimate for (33) does not apply in this case.

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- We need the assumption (29) to prove the main result of this paper. We believe that the system (1) is still memory-type null controllable without (29). However, as we explain in Remark 3.3, it is really needed for our proof. How to establish the memory-type null controllability of the system (1) for continuous  $M(\cdot, \cdot)$  is an interesting problem.

## 7. Appendix: Some Technical Proofs

In this appendix, we present the proofs of Propositions 1.1, 1.2 and 3.1.

### 7.1. Proof of Proposition 1.1

The proof is almost standard. We give it here for the sake of completeness. Denote by  $\mathcal{Z}$  the space  $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  with the following norm:

$$|f|_{\mathcal{Z}} \triangleq (|e^{-\alpha t} f|_{C([0, T]; H_0^1(\Omega))}^2 + |e^{-\alpha t} f_t|_{C([0, T]; L^2(\Omega))}^2)^{\frac{1}{2}},$$

235 where  $\alpha$  is a positive real number whose value will be given below.

Clearly,

$$e^{-\alpha T} \|f\|_{C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))} \leq \|f\|_{\mathcal{Z}} \leq \|f\|_{C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))}.$$

Therefore,  $\mathcal{Z}$  is a Banach space with the norm  $\|\cdot\|_{\mathcal{Z}}$  and  $\mathcal{Z}$  equals  $C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  algebraically and topologically.

Define a map  $\mathcal{F}$  on  $\mathcal{Z}$  as

$$\hat{y} = \mathcal{F}(\tilde{y}),$$

where  $\tilde{y} \in \mathcal{Z}$ , and  $\hat{y}$  is the corresponding solution to (1) with  $\int_0^t M(t, s)y(s)ds$  being replaced by  $\int_0^t M(t, s)\tilde{y}(s)ds$ .

From the well-posedness result for wave equations with nonhomogeneous terms, we have that

$$\begin{aligned} \|\hat{y}\|_{\mathcal{Z}} &\leq \|\hat{y}\|_{C([0,T];H_0^1(\Omega)) \cap C^1([0,T];L^2(\Omega))} \\ &\leq C\left(\|(y_0, y_1)\|_{H_0^1(\Omega) \times L^2(\Omega)} + \|u\|_{L^2(\Omega)} + \left\| \int_0^t M(\cdot, s)\tilde{y}(s)ds \right\|_{L^2(\Omega)}\right) \\ &\leq C\left(\|(y_0, y_1)\|_{H_0^1(\Omega) \times L^2(\Omega)} + \|u\|_{L^2(\Omega)} + \|\tilde{y}\|_{L^2(\Omega)}\right). \end{aligned} \quad (122)$$

Hence,  $\mathcal{F}(\mathcal{Z}) \subset \mathcal{Z}$ .

Next, for any  $\tilde{y}, \tilde{\tilde{y}} \in \mathcal{Z}$ ,

$$\begin{aligned} &\left| \mathcal{F}(\tilde{y})(t) - \mathcal{F}(\tilde{\tilde{y}})(t) \right|_{H_0^1(\Omega)} + \left| \mathcal{F}(\tilde{y})_t(t) - \mathcal{F}(\tilde{\tilde{y}})_t(t) \right|_{L^2(\Omega)} \\ &\leq C \int_0^t \int_{\Omega} \left| \int_0^t M(t, s)[\tilde{y}(s, x) - \tilde{\tilde{y}}(s, x)]ds \right|^2 dx dt. \end{aligned}$$

Thus,

$$\begin{aligned} &e^{-2\alpha t} \left| \mathcal{F}(\tilde{y})(t) - \mathcal{F}(\tilde{\tilde{y}})(t) \right|_{H_0^1(\Omega)}^2 + e^{-2\alpha t} \left| \mathcal{F}(\tilde{y})_t(t) - \mathcal{F}(\tilde{\tilde{y}})_t(t) \right|_{L^2(\Omega)}^2 \\ &\leq C \int_0^t \int_{\Omega} \left| \int_0^t e^{-\alpha t} M(t, s)[\tilde{y}(s, x) - \tilde{\tilde{y}}(s, x)]ds \right|^2 dx dt \\ &\leq C |M|_{C([0,T] \times [0,T])} \int_0^t \int_{\Omega} \int_0^t e^{-2\alpha t} |\tilde{y}(s, x) - \tilde{\tilde{y}}(s, x)|^2 ds dx dt \\ &\leq C |M|_{C([0,T] \times [0,T])} \int_0^t \int_{\Omega} \int_0^t e^{-2\alpha(t-s)} e^{-2\alpha s} |\tilde{y}(s, x) - \tilde{\tilde{y}}(s, x)|^2 ds dx dt \\ &\leq CT |M|_{C([0,T] \times [0,T])} \int_0^t e^{-2\alpha(t-s)} ds \\ &\quad \times \sup_{s \in [0,t]} \left( e^{-2\alpha s} \|\tilde{y}(s) - \tilde{\tilde{y}}(s)\|_{H_0^1(\Omega)} + e^{-2\alpha s} \|\tilde{y}_t(s) - \tilde{\tilde{y}}_t(s)\|_{L^2(\Omega)} \right) \\ &\leq CT |M|_{C([0,T] \times [0,T])} \frac{1 - e^{-2\alpha T}}{2\alpha} \\ &\quad \times \sup_{s \in [0,T]} \left( e^{-2\alpha s} \|\tilde{y}(s) - \tilde{\tilde{y}}(s)\|_{H_0^1(\Omega)} + e^{-2\alpha s} \|\tilde{y}_t(s) - \tilde{\tilde{y}}_t(s)\|_{L^2(\Omega)} \right). \end{aligned}$$

This implies that

$$\|\mathcal{F}(\tilde{y}) - \mathcal{F}(\tilde{\tilde{y}})\|_{\mathcal{Z}} \leq \left( CT |M|_{C([0,T] \times [0,T])} \frac{1 - e^{-2\alpha T}}{2\alpha} \right)^{\frac{1}{2}} \|\tilde{y} - \tilde{\tilde{y}}\|_{\mathcal{Z}}. \quad (123)$$

Let us take  $\alpha = CT |M|_{C([0,T] \times [0,T])}$ . Then (123) implies that

$$\|\mathcal{F}(\tilde{y}) - \mathcal{F}(\tilde{\tilde{y}})\|_{\mathcal{Z}} \leq \frac{1}{2} \|\tilde{y} - \tilde{\tilde{y}}\|_{\mathcal{Z}},$$

240 which concludes that  $\mathcal{F}$  is a contractive mapping. Hence, there is a unique fixed point of  $\mathcal{F}$ , which is the solution to (1).

Let  $y$  be the solution to (1). We have that

$$\begin{aligned} &\|y(t)\|_{H_0^1(\Omega)}^2 + \|y_t(t)\|_{L^2(\Omega)}^2 \\ &\leq C\left(\|(y_0, y_1)\|_{H_0^1(\Omega) \times L^2(\Omega)} + \|u\|_{L^2(\Omega)} + \int_0^t \int_{\Omega} \left| \int_0^t M(t, s)\tilde{y}(s)ds \right|^2 dx dt\right) \\ &\leq C\left(\|(y_0, y_1)\|_{H_0^1(\Omega) \times L^2(\Omega)} + \|u\|_{L^2(\Omega)} + |M|_{C([0,T] \times [0,T])} \int_0^t \int_{\Omega} |\tilde{y}(s)|^2 ds dx\right). \end{aligned}$$

This, together with Gronwall's inequality, implies that

$$|y(t)|_{H_0^1(\Omega)}^2 + |y_t(t)|_{L^2(\Omega)}^2 \leq C(|(y_0, y_1)|_{H_0^1(\Omega) \times L^2(\Omega)} + |u|_{L^2(\Omega)}).$$

Thus, we get (6). □

### 7.2. Proof of Proposition 1.2

We first consider the assertion i). From (10), it follows that

$$v(t, \cdot) = \int_0^t e^{-\lambda(t-s)} w(s, \cdot) ds = 0 \quad \text{for all } t \geq T. \quad (124)$$

Thus,

$$v_t(t, \cdot) = w(t, \cdot) - \lambda \int_0^t e^{-\lambda(t-s)} w(s, \cdot) ds = 0 \quad \text{for all } t \geq T. \quad (125)$$

By (124) and (125), we conclude that

$$w(t, \cdot) = 0 \quad \text{for all } t \geq T.$$

Next we prove the assertion ii). From (14), we get that

$$\Upsilon(t, \cdot) = w(t, \cdot) - \int_0^t e^{-\lambda(t-s)} w(s, \cdot) ds = 0 \quad \text{for all } t \geq T. \quad (126)$$

Hence,

$$\Upsilon_t(t, \cdot) = w_t(t, \cdot) - w(t, \cdot) + \lambda \int_0^t e^{-\lambda(t-s)} w(s, \cdot) ds = 0 \quad \text{for all } t \geq T. \quad (127)$$

It follows from (127) and (126) that

$$w_t(t, \cdot) + (\lambda - 1)w(t, \cdot) = 0 \quad \text{for all } t \geq T. \quad (128)$$

Therefore,

$$w(t, \cdot) = e^{-(\lambda-1)(t-T)} w(T, \cdot). \quad (129)$$

By (14) again, we obtain that

$$\Delta \Upsilon(t, \cdot) = \Delta w(t, \cdot) - \int_0^t e^{-\lambda(t-s)} \Delta w(s, \cdot) ds = 0 \quad \text{for all } t \geq T. \quad (130)$$

This, together with (13), implies that

$$w_{tt}(t, \cdot) = (\lambda - 1)^2 e^{-(\lambda-1)(t-T)} w(T, \cdot) = 0 \quad \text{for all } t \geq T.$$

Thus, it follows from  $\lambda \neq 1$  and (129) that  $w(T, \cdot) = 0$ . □

### 245 7.3. Proof of Proposition 3.1

The “if” part. Fix a  $(y_0, y_1) \in V \times H_0^1(\Omega)$ . Let

$$\mathcal{U} = \{ \chi_{Op}(\cdot) \mid p(\cdot) \text{ solves (23) for some } (p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega) \}.$$

Then,  $\mathcal{U}$  is a linear subspace of  $L^2(O)$ . Let us define a linear functional  $\mathcal{L}$  on  $\mathcal{U}$  as follows:

$$\mathcal{L}(\chi_{Op}) = -\langle p(0), y_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle p_t(0), y_0 \rangle_{V', V}, \quad \forall \chi_{Op} \in \mathcal{U}.$$

From (25) we know that  $\mathcal{L}$  is a bounded linear functional on the normed linear space  $\mathcal{U}$  (with the norm inherited from  $L^2(O)$ ). By the Hahn-Banach Theorem,  $\mathcal{L}$  can be extended to a bounded linear functional on  $L^2(O)$ . Then, by the Riesz Representation Theorem, there is a  $u(\cdot) \in L^2(O)$  such that

$$\int_0^T p(t, x) u(t, x) dx dt = \mathcal{L}(\chi_{Op}) = -\langle p(0), y_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle p_t(0), y_0 \rangle_{V', V}. \quad (131)$$

This  $u(\cdot)$  is the desired control. Indeed, for any  $(p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ , by multiplying (1) by  $p(\cdot)$  and interating by parts, we obtain that

$$\begin{aligned} & (p_0, y_t(T))_{L^2(\Omega)} - \langle p(0), y_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \langle p_1, y(T) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle p_t(0), y_0 \rangle_{V', V} \\ &= \left( q_0, \int_0^T M(T, t)y(t)dt \right)_{L^2(\Omega)} + \int_O p(t, x)u(t, x)dxdt. \end{aligned} \quad (132)$$

According to (131) and (132), we get that for any  $(p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ ,

$$(p_0, y_t(T))_{L^2(\Omega)} - \langle p_1, y(T) \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} - \left( q_0, \int_0^T M(T, t)y(t)dt \right)_{L^2(\Omega)} = 0.$$

We deduce that  $y(T) = 0$ ,  $y_t(T) = 0$  and  $\int_0^T M(T, t)y(t)dt = 0$ .

The ‘‘only if’’ part. We use the contradiction argument. Assume that (25) was untrue. Then, there is a sequence  $\{(p_0^k, p_1^k, q_0^k)\}_{k=1}^\infty \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$  such that the corresponding solutions  $p^k(\cdot)$  to (24) (with  $(p_0, p_1, q_0)$  replaced by  $(p_0^k, p_1^k, q_0^k)$ ) satisfy

$$0 \leq \int_O |p^k(t, x)|^2 dxdt < \frac{1}{k^2} |(p_0^k, p_1^k, q_0^k)|_{L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)}^2, \quad \forall k \in \mathbb{N}. \quad (133)$$

Put

$$(\tilde{p}_0^k, \tilde{p}_1^k, \tilde{q}_0^k) = \frac{\sqrt{k}}{|(p_0^k, p_1^k, q_0^k)|_{L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)}} (p_0^k, p_1^k, q_0^k).$$

Denote by  $\tilde{p}^k(\cdot)$  the corresponding solution to (24) (with  $(p_0, p_1, q_0)$  replaced by  $(\tilde{p}_0^k, \tilde{p}_1^k, \tilde{q}_0^k)$ ). Let us define a bounded linear operator  $\tilde{\mathcal{L}} : L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega) \rightarrow H^{-1}(\Omega) \times V'$  as

$$\tilde{\mathcal{L}}(p_0, p_1, q_0) = (p(0), p_t(0)), \quad \forall (p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega).$$

According to (133), for each  $k \in \mathbb{N}$ , it holds that

$$\int_O |p(t, x)|^2 dxdt < \frac{1}{k}, \quad |\tilde{\mathcal{L}}(\tilde{p}_0^k, \tilde{p}_1^k, \tilde{q}_0^k)|_{Y'} = \sqrt{k}. \quad (134)$$

Noting that (1) is memory-type null controllable, for any  $(y_0, y_1) \in V \times H_0^1(\Omega)$ , there is a control  $u(\cdot) \in L^2(O)$  such that (7) holds. For any  $(p_0, p_1, q_0) \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Omega)$ , from (131), we have that

$$\begin{aligned} \int_O p(t, x)u(t, x)dxdt &= -\langle p(0), y_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \langle p_t(0), y_0 \rangle_{V', V} \\ &= \langle \tilde{\mathcal{L}}(p_0, p_1, q_0), (y_1, y_0) \rangle_{H^{-1}(\Omega) \times V', H_0^1(\Omega) \times V}. \end{aligned}$$

Thus,

$$\int_O \tilde{p}^k(t, x)u(t, x)dxdt = \langle \tilde{\mathcal{L}}(\tilde{p}_0^k, \tilde{p}_1^k, \tilde{q}_0^k), (y_1, y_0) \rangle_{H^{-1}(\Omega) \times V', H_0^1(\Omega) \times V}. \quad (135)$$

By (135) and the first inequality in (134), we see that  $\mathcal{L}(\tilde{p}_0^k, \tilde{p}_1^k, \tilde{q}_0^k)$  tends to 0 weakly in  $H^{-1}(\Omega) \times V'$ . Hence, by the Principle of Uniform Boundedness, the sequence  $\{\mathcal{L}(\tilde{p}_0^k, \tilde{p}_1^k, \tilde{q}_0^k)\}_{k=1}^\infty$  is uniformly bounded in  $H^{-1}(\Omega) \times V'$ . It contradicts the second equality in (134). This completes the proof of Proposition 3.1.  $\square$

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