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# CHARACTERIZATION OF PROJECTIVE SPACES AND $\mathbb{P}^r$ -BUNDLES AS AMPLE DIVISORS

JIE LIU

ABSTRACT. Let  $X$  be a projective manifold of dimension  $n$ . Suppose that  $T_X$  contains an ample subsheaf. We show that  $X$  is isomorphic to  $\mathbb{P}^n$ . As an application, we derive the classification of projective manifolds containing a  $\mathbb{P}^r$ -bundle as an ample divisor by the recent work of D. Litt.

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## 1. INTRODUCTION

Projective spaces are the simplest algebraic varieties. They can be characterized in many ways. A very famous one is given by the Hartshorne's conjecture, which was proved by S. Mori.

**Theorem A.**[Mor79] Let  $X$  be a projective manifold. Then  $X$  is a projective space if and only if  $T_X$  is ample.

This result has been generalized, over the field of complex number, by several authors [Wah83, CP98, AW01].

**Theorem B.**[AW01] Let  $X$  be a projective manifold of dimension  $n$ . If  $T_X$  contains an ample locally free subsheaf  $\mathcal{E}$  of rank  $r$ , then  $X \cong \mathbb{P}^n$  and  $\mathcal{E} \cong \mathcal{O}(1)^{\oplus r}$  or  $\mathcal{E} \cong T_{\mathbb{P}^n}$ .

This theorem was successively proved for  $r = 1$  by J. Wahl [Wah83] and latter for  $r \geq n - 2$  by F. Campana and T. Peternell [CP98]. The proof was

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finally completed by M. Andreatta and J.A. Wiśniewski [AW01]. The main aim of the present article is to prove the following generalization.

**1.1. Theorem.** *Let  $X$  be a projective manifold of dimension  $n$ . Suppose that  $T_X$  contains an ample subsheaf  $\mathcal{F}$  of positive rank  $r$ , then  $(X, \mathcal{F})$  is isomorphic to  $(\mathbb{P}^n, T_{\mathbb{P}^n})$  or  $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r})$ .*

Note that this theorem was proved in [AKP08] under the additional assumption  $\rho(X) = 1$ . One interesting and important case is when  $\mathcal{F}$  comes from the image of an ample vector bundle  $E$  over  $X$ . To be more precise, we have the following corollary, which was originally conjectured by D. Litt in [Lit16].

**1.2. Corollary.** *Let  $X$  be a projective manifold of dimension  $n$ ,  $E$  an ample vector bundle on  $X$ . If there exists a non-zero map  $E \rightarrow T_X$ , then  $X \cong \mathbb{P}^n$ .*

As an application, we derive the classification of projective manifolds containing a  $\mathbb{P}^r$ -bundle as an ample divisor. This problem has already been investigated by several authors [Som76, Bäd84, FSS87, BS95, BI09]. Recently, D. Litt proved that it can be reduced to Corollary 1.2 [Lit16]. More precisely, we have the following result.

**1.3. Theorem.** *Let  $X$  be a projective manifold of dimension  $n \geq 3$ , let  $A$  be an ample divisor on  $X$ . Assume that  $A$  is a  $\mathbb{P}^r$ -bundle,  $p: A \rightarrow B$ , over a manifold  $B$  of dimension  $b$ . Then one of the following holds:*

- 1)  $(X, A) = (\mathbb{P}(E), H)$  for some ample vector bundle  $E$  over  $B$  such that  $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$ .  $p$  is equal to the restriction to  $A$  of the induced projection  $\mathbb{P}(E) \rightarrow B$ .
- 2)  $(X, A) = (\mathbb{P}(E), H)$  for some ample vector bundle  $E$  over  $\mathbb{P}^1$  such that  $H \in |\mathcal{O}_{\mathbb{P}(E)}(1)|$ .  $H = \mathbb{P}^1 \times \mathbb{P}^{n-2}$  and  $p$  is the projection to the second factor.
- 3)  $(X, A) = (Q^3, H)$ , where  $Q^3$  is a smooth quadric threefold and  $H$  is a smooth quadric surface with  $H \in |\mathcal{O}_{Q^3}(1)|$ .  $p$  is the projection to one of the factors  $H \cong \mathbb{P}^1 \times \mathbb{P}^1$ .
- 4)  $(X, A) = (\mathbb{P}^3, H)$ .  $H$  is a smooth quadric surface and  $H \in |\mathcal{O}_{\mathbb{P}^3}(2)|$ , and  $p$  is again a projection to one of the factors of  $H \cong \mathbb{P}^1 \times \mathbb{P}^1$ .

**Convention.** We always work over the field  $\mathbb{C}$  of complex numbers. Varieties are always assumed to be integral separated schemes of finite type over  $\mathbb{C}$ . If  $D$  is a Weil divisor on a projective normal variety  $X$ , we denote by  $\mathcal{O}_X(D)$  the reflexive sheaf associated to  $D$ . Given a coherent sheaf  $\mathcal{F}$  on a variety  $X$  of generic rank  $r$ , then we denote by  $\mathcal{F}^\vee$  the sheaf  $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$  and by  $\det(\mathcal{F})$  the sheaf  $(\wedge^r \mathcal{F})^{\vee\vee}$ . We denote by  $\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)$  the fiber of  $\mathcal{F}$  at  $x \in X$ . If  $\mathcal{F}$  is a coherent sheaf on a variety  $X$ , we denote by  $\mathbb{P}(\mathcal{F})$  the Grothendieck projectivization  $\text{Proj}(\bigoplus_{m \geq 0} \text{Sym}^m \mathcal{F})$ . If  $f: X \rightarrow Y$  is a morphism between projective normal varieties, we denote by  $\Omega_{X/Y}^1$  the relative differential sheaf. Moreover, if  $Y$  is smooth, we denote by  $K_{X/Y}$

the relative canonical divisor  $K_X - f^*K_Y$  and by  $\omega_{X/Y}$  the reflexive sheaf  $\omega_X \otimes f^*\omega_Y^\vee$ .

## 2. AMPLE SHEAVES AND RATIONAL CURVES

Let  $X$  be a projective manifold. In this section, we gather some results about the behavior of an ample subsheaf  $\mathcal{F} \subset T_X$  with respect to a family of minimal rational curves on  $X$ .

**2.A. Ample sheaves.** Recall that an invertible sheaf  $\mathcal{L}$  on a quasi-projective variety  $X$  is said to be *ample* if for every coherent sheaf  $\mathcal{G}$  on  $X$ , there is an integer  $n_0 > 0$  such that for every  $n \geq n_0$ , the sheaf  $\mathcal{G} \otimes \mathcal{L}^n$  is generated by its global sections [Har77, II, Section 7]. In general, a coherent sheaf  $\mathcal{F}$  on a quasi-projective variety  $X$  is said to be *ample* if the invertible sheaf  $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$  is ample on  $\mathbb{P}(\mathcal{F})$  [Kub70].

Well-known properties of ampleness of locally free sheaves still hold in this general setting.

- (1) A sheaf  $\mathcal{F}$  on a quasi-projective variety  $X$  is ample if and only if, for any coherent sheaf  $\mathcal{G}$  on  $X$ ,  $\mathcal{G} \otimes \text{Sym}^m \mathcal{F}$  is globally generated for  $m \gg 1$  [Kub70, Theorem 1].
- (2) If  $i: Y \rightarrow X$  is an immersion, and  $\mathcal{F}$  is an ample sheaf on  $X$ , then  $i^*\mathcal{F}$  is an ample sheaf on  $Y$  [Kub70, Proposition 6].
- (3) If  $\pi: Y \rightarrow X$  is a finite morphism with  $X$  and  $Y$  quasi-projective varieties, and  $\mathcal{F}$  is a coherent sheaf on  $X$ , then  $\mathcal{F}$  is ample if and only if  $\pi^*\mathcal{F}$  is ample. Note that  $\mathbb{P}(\pi^*\mathcal{F}) = \mathbb{P}(\mathcal{F}) \times_X Y$  and  $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$  pulls back, by a finite morphism, to  $\mathcal{O}_{\mathbb{P}(\pi^*\mathcal{F})}(1)$ .
- (4) Any quotient of an ample sheaf is ample [Kub70, Proposition 1]. In particular, the image of an ample sheaf under a non-zero map is also ample.
- (5) If  $\mathcal{F}$  is a locally free ample sheaf of rank  $r$ , then the  $s^{\text{th}}$  exterior power  $\wedge^s \mathcal{F}$  is ample for any  $1 \leq s \leq r$  [Har66, Corollary 5.3].
- (6) If  $\mathcal{L}$  is an ample invertible sheaf on a quasi-projective variety  $X$ , then  $\mathcal{L}^m$  is very ample for some  $m > 0$ , i.e. there is an immersion  $i: X \rightarrow \mathbb{P}^n$  for some  $n$  such that  $\mathcal{L}^m = i^*\mathcal{O}_{\mathbb{P}^n}(1)$  [Har77, II, Theorem 7.6].

**2.B. Minimal rational curves.** Let  $X$  be a normal projective variety and  $\mathcal{V}$  an irreducible component of  $\text{RatCurves}^n(X)$ .  $\mathcal{V}$  is said to be a *covering family of rational curves* on  $X$  if the corresponding universal family dominates  $X$ . A covering family  $\mathcal{V}$  of rational curves on  $X$  is called *minimal* if its general members have minimal anti-canonical degree. If  $X$  is a uniruled projective manifold, then  $X$  carries a minimal covering family of rational curves. We fix such a family  $\mathcal{V}$ , and let  $[\ell] \in \mathcal{V}$  be a general point. Then the tangent bundle  $T_X$  can be decomposed on  $\ell$  as  $\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \oplus \mathcal{O}_{\mathbb{P}^1}^{\oplus(n-d-1)}$ , where  $d+2 = \det(T_X) \cdot \ell \geq 2$  is the *anticanonical degree* of  $\mathcal{V}$ .

Let  $\bar{\mathcal{V}}$  be the normalization of the closure of  $\mathcal{V}$  in  $\text{Chow}(X)$ . We define the following equivalence relation on  $X$ . Two points  $x, y \in X$  are  $\bar{\mathcal{V}}$ -equivalent if they can be connected by a chain of 1-cycle from  $\bar{\mathcal{V}}$ . By [Cam81], there exists a proper surjective morphism  $\varphi_0: X_0 \rightarrow T_0$  from an open subset of  $X$  onto a normal variety  $T_0$  whose fibers are  $\bar{\mathcal{V}}$ -equivalence classes. We call this map the  $\mathcal{V}$ -rationally connected quotient of  $X$ .

The first step towards Theorem 1.1 is the following result which was essentially proved in [Ara06].

**2.1. Theorem.**[ADK08, Proposition 2.7] *Let  $X$  be a projective uniruled manifold,  $\mathcal{V}$  a minimal covering family of rational curves on  $X$ . If  $T_X$  contains a subsheaf  $\mathcal{F}$  of rank  $r$  such that  $\mathcal{F}|_\ell$  is an ample vector bundle for a general member  $[\ell] \in \mathcal{V}$ , then there exists a dense open subset  $X_0$  of  $X$  and a  $\mathbb{P}^{d+1}$ -bundle  $\varphi_0: X_0 \rightarrow T_0$  such that any curve on  $X$  parametrized by  $\mathcal{V}$  and meeting  $X_0$  is a line on a fiber of  $\varphi_0$ . In particular,  $\varphi_0$  is the  $\mathcal{V}$ -rationally connected quotient of  $X$ .*

Recall that the singular locus  $\text{Sing}(\mathcal{S})$  of a coherent sheaf  $\mathcal{S}$  over  $X$  is the set of all points of  $X$  where  $\mathcal{S}$  is not locally free.

**2.2. Remark.** The hypothesis that  $\mathcal{F}$  is locally free over a general member of  $\mathcal{V}$  is automatically satisfied. Since  $\mathcal{F}$  is torsion free and  $X$  is smooth,  $\mathcal{F}$  is locally free in codimension one. By [Kol96, II, Proposition 3.7], a general member of  $\mathcal{V}$  is disjoint from  $\text{Sing}(\mathcal{F})$ , hence  $\mathcal{F}$  is locally free over a general member of  $\mathcal{V}$ .

As an immediate application of Theorem 2.1, we derive a weak version of [AKP08, Theorem 4.2].

**2.3. Corollary.** *Let  $X$  be a projective uniruled manifold with  $\rho(X) = 1$ ,  $\mathcal{V}$  a minimal covering family of rational curves on  $X$ . If  $T_X$  contains a subsheaf  $\mathcal{F}$  of rank  $r$  such that  $\mathcal{F}|_\ell$  is ample for a general member  $[\ell] \in \mathcal{V}$ , then  $X \cong \mathbb{P}^n$ .*

**2.4. Remark.** In [AKP08, Theorem 4.2], they proved that the torsion-free sheaf  $\mathcal{F}$  is actually locally free. Hence  $\mathcal{F}$  is isomorphic to  $T_{\mathbb{P}^n}$  or  $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$  by Theorem B. In particular, to prove Theorem 1.1, it is sufficient to show that  $X$  is a projective space if  $T_X$  contains an ample subsheaf.

If  $X$  is a projective manifold and  $T_X$  contains an ample subsheaf, then  $X$  is uniruled and carries a minimal covering family of rational curves [Miy87, Corollary 8.6]. In particular, we obtain the following result.

**2.5. Corollary.**[AKP08, Corollary 4.3] *Let  $X$  be a projective manifold with  $\rho(X) = 1$ . Assume that  $T_X$  contains an ample subsheaf, then  $X \cong \mathbb{P}^n$ .*

### 3. FOLIATIONS AND PFAFF FIELDS

Let  $\mathcal{S}$  be a subsheaf of  $T_X$  on a quasi-projective manifold  $X$ . We denote by  $\mathcal{S}^{reg}$  the largest open subset of  $X$  such that  $\mathcal{S}$  is a subbundle of  $T_X$  over  $\mathcal{S}^{reg}$ . Note that in general  $\text{Sing}(\mathcal{S})$  is a proper subset of  $X \setminus \mathcal{S}^{reg}$ .

**3.1. Definition.** *Let  $X$  be a quasi-projective manifold and let  $\mathcal{S} \subsetneq T_X$  be a coherent subsheaf of positive rank.  $\mathcal{S}$  is called a foliation if it satisfies the following conditions:*

- 1)  $\mathcal{S}$  is saturated in  $T_X$ , i.e.  $T_X/\mathcal{S}$  is torsion free.
- 2) The sheaf  $\mathcal{S}$  is closed under the Lie bracket.

*In addition,  $\mathcal{S}$  is called an algebraically integrable foliation if the following holds.*

- 3) *For a general point  $x \in X$ , there exists a projective subvariety  $F_x$  passing through  $x$  such that*

$$\mathcal{S}|_{F_x \cap \mathcal{S}^{reg}} = T_{F_x}|_{F_x \cap \mathcal{S}^{reg}} \subset T_X|_{F_x \cap \mathcal{S}^{reg}}.$$

*We call  $F_x$  the  $\mathcal{S}$ -leaf through  $x$ .*

**3.2. Remark.** Let  $X$  be a projective manifold and  $\mathcal{S}$  a saturated subsheaf of  $T_X$ . To show that  $\mathcal{S}$  is an algebraically integrable foliation, it is sufficient to show that it is an algebraically integrable foliation over a Zariski open subset of  $X$ .

**3.3. Example.** Let  $X \rightarrow Y$  be a fibration with  $X$  and  $Y$  projective manifolds. Then  $T_{X/Y} \subset T_X$  defines an algebraically integrable foliation on  $X$  such that the general leaves are the fibers.

**3.4. Example.**[AD13, 4.1] Let  $\mathcal{F}$  be a subsheaf  $\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus r}$  of  $T_{\mathbb{P}^n}$  on  $\mathbb{P}^n$ . Then  $\mathcal{F}$  is an algebraically integrable foliation and it is defined by a linear projection  $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r}$ . The set of points of indeterminacy  $S$  of this rational map is a  $r-1$ -dimensional linear subspace. Let  $x \notin S$  be a point. Then the leaf passing through  $x$  is the  $r$ -dimensional linear subspace  $L$  of  $\mathbb{P}^n$  containing both  $x$  and  $S$ .

**3.5. Definition.** *Let  $X$  be a projective variety, and  $r$  a positive integer. A Pfaff field of rank  $r$  on  $X$  is a nonzero map  $\partial: \Omega_X^r \rightarrow \mathcal{L}$ , where  $\mathcal{L}$  is an invertible sheaf on  $X$ .*

**3.6. Lemma.**[ADK08, Proposition 4.5] *Let  $X$  be a projective variety and  $n: \tilde{X} \rightarrow X$  its normalization. Let  $\mathcal{L}$  be an invertible sheaf on  $X$ ,  $r$  positive integer, and  $\partial: \Omega_X^r \rightarrow \mathcal{L}$  a Pfaff field. Then  $\partial$  can be extended uniquely to a Pfaff field  $\tilde{\partial}: \Omega_{\tilde{X}}^r \rightarrow n^*\mathcal{L}$ .*

Let  $X$  be a projective manifold and  $\mathcal{S} \subset T_X$  a subsheaf with positive rank  $r$ . We denote by  $K_{\mathcal{S}}$  the canonical class  $-c_1(\det(\mathcal{S}))$  of  $\mathcal{S}$ . Then there is a natural associated Pfaff field of rank  $r$ :

$$\Omega_X^r = \wedge^r(\Omega_X^1) = \wedge^r(T_X^\vee) = (\wedge^r T_X)^\vee \rightarrow \mathcal{O}_X(K_{\mathcal{S}}).$$

**3.7. Lemma.** [AD13, lemma 3.2] *Let  $X$  be a projective manifold, and  $\mathcal{S}$  an algebraically integrable foliation on  $X$ . Then there is a unique irreducible projective subvariety  $W$  of  $\text{Chow}(X)$  whose general point parametrizes a general leaf of  $\mathcal{S}$ .*

**3.8. Remark.** Let  $X$  be a projective manifold, and  $\mathcal{S}$  an algebraically integrable foliation of rank  $r$  on  $X$ . Let  $W$  be the subvariety of  $\text{Chow}(X)$  provided in Lemma 3.7. Let  $Z \subset W$  be a general closed subvariety of  $W$  and let  $U \subset Z \times X$  be the universal cycle over  $Z$ . Assume that  $\tilde{Z}$  and  $\tilde{U}$  are the normalizations of  $Z$  and  $U$  respectively. We claim that the Pfaff field  $\Omega_X^r \rightarrow \mathcal{O}_X(K_{\mathcal{S}})$  can be extended to a Pfaff field  $\Omega_{\tilde{U}/\tilde{Z}}^r \rightarrow n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$ .

$$\begin{array}{ccccc} \tilde{U} & \xrightarrow{n} & U & \subset & Z \times X & \xrightarrow{p} & X \\ \tilde{q} \downarrow & & \downarrow q & & \downarrow q & & \\ \tilde{Z} & \longrightarrow & Z & \xrightarrow{=} & Z & & \end{array}$$

Let  $V$  be the universal cycle over  $W$  with  $v: V \rightarrow X$ . By the proof of [AD13, Lemma 3.2], the Pfaff field  $\Omega_X^r \rightarrow \mathcal{O}_X(K_{\mathcal{S}})$  extends to be a Pfaff field  $\Omega_V^r \rightarrow v^*\mathcal{O}_X(K_{\mathcal{S}})$ . It induces a Pfaff field  $\Omega_U^r \rightarrow p^*\mathcal{O}_X(K_{\mathcal{S}})$ . Note that  $U$  is irreducible since  $Z$  is a general subvariety. By Lemma 3.6, it can be uniquely extended to a Pfaff field  $\Omega_{\tilde{U}}^r \rightarrow n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$ .

Let  $\mathcal{K}$  be the kernel of the morphism  $\Omega_{\tilde{U}}^r \rightarrow \Omega_{\tilde{U}/\tilde{Z}}^r$ . Let  $F$  be a general fiber of  $\tilde{q}$  such that its image under  $p \circ n$  is a  $\mathcal{S}$ -leaf and the morphism  $p \circ n$  restricted on  $F$  is finite and birational. Let  $x \in F$  be a point such that  $F$  is smooth at  $x$  and  $p \circ n$  is an isomorphism at a neighborhood of  $x$ . Then the composite map  $\Omega_{\tilde{U}}^r|_F \rightarrow \Omega_{\tilde{U}/\tilde{Z}}^r|_F \rightarrow \Omega_F^r$  implies that the composite map

$$\mathcal{K} \rightarrow \Omega_{\tilde{U}}^r \rightarrow n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$$

vanishes in a neighborhood of  $x$ , hence it vanishes generically over  $\tilde{U}$ . Since the sheaf  $n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$  is torsion-free, it vanishes identically and finally yields a Pfaff field  $\Omega_{\tilde{U}/\tilde{Z}}^r \rightarrow n^*p^*\mathcal{O}_X(K_{\mathcal{S}})$ .

Let  $X$  be a projective manifold, and  $\mathcal{S} \subset T_X$  a subsheaf. We define its saturation  $\overline{\mathcal{S}}$  as the kernel of the natural surjection  $T_X \rightarrow (T_X/\mathcal{S}) / (\text{torsion})$ . Then  $\overline{\mathcal{S}}$  is obviously saturated.

**3.9. Theorem.** *Let  $X$  be a projective manifold. Assume that  $T_X$  contains an ample subsheaf  $\mathcal{F}$  of rank  $r < \dim(X)$ . Then its saturation  $\overline{\mathcal{F}}$  defines an algebraically integrable foliation on  $X$ , and the  $\overline{\mathcal{F}}$ -leaf passing through a general point is isomorphic to  $\mathbb{P}^r$ .*

*Proof.* Let  $\varphi_0: X_0 \rightarrow T_0$  be as the morphism provided in Theorem 2.1. Since  $\mathcal{F}$  is locally free in codimension one, we may assume that no fiber of  $\varphi_0$  is completely contained in  $\text{Sing}(\mathcal{F})$ .

The first step is to show that  $\mathcal{F}|_{X_0} \subset T_{X_0/T_0}$ . Since  $\varphi_0: X_0 \rightarrow T_0$  is smooth, we get a short exact sequence of vector bundles,

$$0 \rightarrow T_{X_0/T_0} \rightarrow T_X|_{X_0} \rightarrow \varphi_0^* T_{T_0} \rightarrow 0.$$

The composite map  $\mathcal{F}|_{X_0} \rightarrow T_X|_{X_0} \rightarrow \varphi_0^* T_{T_0}$  vanishes on a Zariski open subset of every fiber. Since  $\varphi_0^* T_{T_0}$  is torsion-free, it vanishes identically, and it follows  $\mathcal{F}|_{X_0} \subset T_{X_0/T_0}$ .

Next we show that, after shrinking  $X_0$  and  $T_0$  if necessary,  $\mathcal{F}$  is actually locally free over  $X_0$ . By generic flatness theorem [Gro65, Théorème 6.9.1], after shrinking  $T_0$ , we can suppose that  $(T_X/\mathcal{F})|_{X_0}$  is flat over  $T_0$ . Let  $F \cong \mathbb{P}^{d+1}$  be an arbitrary fiber of  $\varphi_0$ . The following short exact sequence of sheaves

$$0 \rightarrow \mathcal{F}|_{X_0} \rightarrow T_X|_{X_0} \rightarrow (T_X/\mathcal{F})|_{X_0} \rightarrow 0$$

induces a long exact sequence of sheaves

$$\mathcal{T}or((T_X/\mathcal{F})|_{X_0}, \mathcal{O}_F) \rightarrow \mathcal{F}|_F \rightarrow T_X|_F \rightarrow (T_X/\mathcal{F})|_F \rightarrow 0.$$

Since  $(T_X/\mathcal{F})|_{X_0}$  is flat over  $T_0$ , it follows that  $\mathcal{F}|_F$  is a subsheaf of  $T_X|_F$ , in particular,  $\mathcal{F}|_F$  is torsion-free. Without loss of generality, we may assume that the restrictions of  $\mathcal{F}$  on all fibers of  $\varphi_0$  are torsion-free. By Remark 2.4, the restrictions of  $\mathcal{F}$  on all fibers of  $\varphi_0$  are locally free, it yields particularly that the dimension of the fibers of  $\mathcal{F}$  is constant on every fiber of  $\varphi_0$  due to  $\mathcal{F}(x) = (\mathcal{F}|_F)(x)$ . Note that no fiber of  $\varphi_0$  is contained in  $\text{Sing}(\mathcal{F})$ , we conclude that the dimension of the fibers  $\mathcal{F}(x)$  of  $\mathcal{F}$  is constant over  $X_0$ . Hence  $\mathcal{F}$  is locally free over  $X_0$ .

Now we claim that  $\overline{\mathcal{F}}$  actually defines an algebraically integrable foliation on  $X_0$ . Let  $F \cong \mathbb{P}^{d+1}$  be an arbitrary fiber of  $\varphi_0$ . We know that  $(F, \mathcal{F}|_F)$  is isomorphic to  $(\mathbb{P}^{d+1}, T_{\mathbb{P}^{d+1}})$  or  $(\mathbb{P}^{d+1}, \mathcal{O}_{\mathbb{P}^{d+1}}(1)^{\oplus r})$  (c.f. Theorem B), therefore  $\mathcal{F}$  defines an algebraically integrable foliation over  $X_0$  (c.f. Example 3.4). Note that we have  $\mathcal{F}|_{X_0} = \overline{\mathcal{F}}|_{X_0}$  since  $\mathcal{F}|_{X_0}$  is saturated in  $T_{X_0}$ . Hence  $\overline{\mathcal{F}}$  also defines an algebraically integrable foliation over  $X$  (c.f. Remark 3.2).  $\square$

### 3.10. Remarks.

- (1) If  $\mathcal{F}$  is of rank  $n = \dim(X)$  and  $\mathcal{F} \subsetneq T_X$ , then  $(X, \mathcal{F}) = (\mathbb{P}^n, \mathcal{O}(1)^{\oplus n})$  and  $\mathcal{F}$  also defines an algebraically integrable foliation on  $X$ .
- (2) Since  $\mathcal{F}$  is locally free on  $X_0$ , it follows that  $\mathcal{O}_X(-K_{\mathcal{F}})|_{X_0}$  is isomorphic to  $\wedge^r(\mathcal{F}|_{X_0})$  and the invertible sheaf  $\mathcal{O}_X(-K_{\mathcal{F}})$  is ample over  $X_0$ . Moreover, as  $\mathcal{F}$  is locally free in codimension one, there exists an open subset  $X' \subset X$  containing  $X_0$  such that  $\text{codim}(X \setminus X') \geq 2$  and  $\mathcal{O}_X(-K_{\mathcal{F}})$  is ample on  $X'$ .

## 4. PROOF OF MAIN THEOREM

The aim of this section is to prove Theorem 1.1. Let  $X$  be a normal projective variety, and  $X \rightarrow C$  a surjective morphism with connected fibers onto a smooth curve. Let  $\Delta$  be an effective Weil divisor on  $X$  such that

$(X, \Delta)$  is log-canonical over the generic point of  $C$ . In [AD13, Theorem 5.1], they proved that  $-(K_{X/C} + \Delta)$  cannot be ample. In the next theorem, we give a variant of this result which is the key ingredient in our proof of Theorem 1.1.

**4.1. Theorem.** *Let  $X$  be a normal projective variety, and  $f: X \rightarrow C$  a surjective morphism with connected fibers onto a smooth curve. Let  $\Delta$  be a Weil divisor on  $X$  such that  $K_X + \Delta$  is Cartier and  $\Delta^{hor}$  is reduced. Assume that there exists an open subset  $C_0$  such that the pair  $(X, \Delta^{hor})$  is snc over  $X_0 = f^{-1}(C_0)$ . If  $X' \subset X$  is an open subset such that no fiber of  $f$  is completely contained in  $X \setminus X'$  and  $X_0 \subset X'$ , then the invertible sheaf  $\mathcal{O}_X(-K_{X/C} - \Delta)$  is not ample over  $X'$ .*

*Proof.* To prove the theorem, we assume to the contrary that the invertible sheaf  $\mathcal{O}_X(-K_{X/C} - \Delta)$  is ample over  $X'$ . Let  $A$  be an ample divisor supported on  $C_0$ . Then for some  $m \gg 1$ , the sheaf  $\mathcal{O}_X(-m(K_{X/C} + \Delta) - f^*A)$  is very ample over  $X'$  [Har77, II, Exercise 7.5]. It follows that there exists a prime divisor  $D'$  on  $X'$  such that the pair  $(X', \Delta^{hor}|_{X'} + D')$  is snc over  $X_0$  and

$$D' \sim (-m(K_{X/C} + \Delta) - f^*A)|_{X'}.$$

It implies that there exists a rational function  $h \in K(X') = K(X)$  such that the restriction of the Cartier divisor  $D = \text{div}(h) - m(K_{X/C} + \Delta) - f^*A$  on  $X'$  is  $D'$  and  $D^{hor}$  is the closure of  $D'$  in  $X$ . Note that we can write  $D = D_+ - D_-$  for some effective divisors  $D_+$  and  $D_-$  with no common components. Then we have  $\text{Supp}(D_-) \subset X \setminus X'$ , in particular, no fiber of  $f$  is supported on  $D_-$ . By [Kol13, Theorem 4.15], there exists a log-resolution  $\mu: \tilde{X} \rightarrow X$  such that:

- (1) Then induced morphism  $\tilde{f} = f \circ \mu: \tilde{X} \rightarrow C$  is prepared (c.f. [Cam04, Section 4.3]).
- (2) The birational morphism  $\mu$  is an isomorphism over  $X_0$ .
- (3)  $\mu_*^{-1}\Delta^{hor} + \mu_*^{-1}D^{hor}$  is a snc divisor.

Let  $E$  be the exceptional divisor of  $\mu$ . Note that we have  $\tilde{f}_*(E) \neq C$ . Moreover, we also have

$$K_{\tilde{X}} + \mu_*^{-1}\Delta + \frac{1}{m}\mu_*^{-1}D_+ = \mu^*(K_X + \Delta + \frac{1}{m}D) + \frac{1}{m}\mu_*^{-1}D_- + E_+ - E_-.$$

where  $E_+$  and  $E_-$  are effective  $\mu$ -exceptional divisors with no common components.

Set  $\tilde{D} = m\mu_*^{-1}\Delta + \mu_*^{-1}D_+ + mE_-$ . Then  $\tilde{D}^{hor} = m\mu_*^{-1}\Delta^{hor} + \mu_*^{-1}D^{hor}$  is a snc effective divisor with coefficients  $\leq m$ . Since  $D$  is linearly equivalent to  $-m(K_{X/C} + \Delta) - f^*A$ , we can write

$$K_{\tilde{X}/C} + \frac{1}{m}\tilde{D} \sim_{\mathbb{Q}} -\frac{1}{m}\tilde{f}^*A + \frac{1}{m}\mu_*^{-1}D_- + E_+.$$

After multiplying by some  $l$  divisible enough, we may assume that  $lmE_+$  and  $lmE_-$  are of integer coefficients. By replacing  $\tilde{D}$  by  $l\tilde{D}$ , the weak positivity theorem [Cam04, Theorem 4.13] implies that the following direct image sheaf

$$\begin{aligned} \tilde{f}_*(\omega_{\tilde{X}/C}^{lm} \otimes \mathcal{O}_{\tilde{X}}(\tilde{D})) &\simeq \tilde{f}_*(\mathcal{O}_{\tilde{X}}(-lf^*A + lmE_+ + l\mu_*^{-1}D_-)) \\ &\simeq \mathcal{O}_C(-lA) \otimes \tilde{f}_*\mathcal{O}_{\tilde{X}}(lmE_+ + l\mu_*^{-1}D_-) \end{aligned}$$

is weakly positive.

Observe that  $\tilde{f}_*(\mathcal{O}_{\tilde{X}}(lmE_+ + l\mu_*^{-1}D_-)) = \mathcal{O}_C$ . Indeed,  $E_+$  is a  $\mu$ -exceptional divisor, it follows  $\mu_*(\mathcal{O}_{\tilde{X}}(lmE_+ + l\mu_*^{-1}D_-)) = \mathcal{O}_X(lD_-)$ . Note that we have  $f_*(\mathcal{O}_X(lD_-)) = \mathcal{O}_C(P)$  for some effective divisor  $P$  on  $C$  such that  $\text{Supp}(P) \subset f(\text{Supp}(D_-))$ . Let  $V$  be an open subset of  $C$  and let  $\lambda \in H^0(V, \mathcal{O}_C(P))$ , that is,  $\lambda$  is a rational function on  $C$  such that  $\text{div}(\lambda) + P \geq 0$  over  $V$ . It follows that  $\text{div}(\lambda \circ f) + lD_- \geq 0$  over  $f^{-1}(V)$ . Since there is no fiber of  $f$  completely supported on  $D_-$ , the rational function  $\lambda \circ f$  is regular over  $f^{-1}(V)$ . Consequently, the rational function  $\lambda$  is regular over  $V$ . It implies that the natural inclusion  $\mathcal{O}_C \rightarrow \mathcal{O}_C(P)$  is surjective, which yields  $\tilde{f}_*(\mathcal{O}_{\tilde{X}}(lmE_+ + l\mu_*^{-1}D_-)) = \mathcal{O}_C$ . However, this shows that  $\mathcal{O}_C(-lA)$  is weakly positive, a contradiction. Hence  $\mathcal{O}_X(-K_{X/C} - \Delta)$  is not ample over  $X'$ .  $\square$

**4.2. Lemma.** *Let  $X$  be a normal projective variety, and  $f: X \rightarrow C$  a surjective morphism with reduced and connected fibers onto a smooth curve  $C$ . Let  $D$  be a Cartier divisor on  $X$ . If there exists a nonzero morphism  $\Omega_{X/C}^r \rightarrow \mathcal{O}_X(D)$ , where  $r$  is the relative dimension of  $f$ , then there exists an effective Weil divisor  $\Delta$  on  $X$  such that  $K_{X/C} + \Delta = D$ .*

*Proof.* Since all the fibers of  $f$  are reduced, the sheaf  $\Omega_{X/C}^r$  is locally free in codimension one. Hence the reflexive hull of  $\Omega_{X/C}^r$  is  $\omega_{X/C} \simeq \mathcal{O}_X(K_{X/C})$ . Note that  $\mathcal{O}_X(D)$  is reflexive, the nonzero morphism  $\Omega_{X/C}^r \rightarrow \mathcal{O}_X(D)$  induces a nonzero morphism  $\omega_{X/C} \rightarrow \mathcal{O}_X(D)$ . This shows that there exists an effective divisor  $\Delta$  on  $X$  such that  $K_{X/C} + \Delta = D$ .  $\square$

As an application of Theorem 4.1, we derive a special property about foliations defined by an ample subsheaf of  $T_X$ . A similar result was established for Fano foliations with mild singularities in the work of C. Araujo and S. Druel [AD13, Proposition 5.3] and we follow the same strategy.

**4.3. Proposition.** *Let  $X$  be a projective manifold. If  $\mathcal{F} \subset T_X$  is an ample subsheaf of rank  $r < n = \dim(X)$ , then there is a common point in the general leaves of  $\overline{\mathcal{F}}$ .*

*Proof.* Since  $\mathcal{F}$  is torsion-free and  $X$  is smooth,  $\mathcal{F}$  is locally free over an open subset  $X' \subset X$  such that  $\text{codim}(X \setminus X') \geq 2$ , in particular,  $\mathcal{O}_X(-K_{\mathcal{F}})$  is ample over  $X'$ . By Theorem 2.1, there exists an open subset  $X_0 \subset X$  and a  $\mathbb{P}^{d+1}$ -bundle  $\varphi_0: X_0 \rightarrow T_0$ . Moreover, from the proof of Theorem 3.9,

the saturation  $\overline{\mathcal{F}}$  defines an algebraically integrable foliation on  $X$  and we may assume that  $\mathcal{F}$  is locally free over  $X_0$ . In particular, we have  $X_0 \subset X'$ . In view of Lemma 3.7, we will denote by  $W$  the subvariety of  $\text{Chow}(X)$  parametrizing the general leaves of  $\overline{\mathcal{F}}$  and  $V$  the normalization of the universal cycle over  $W$ . Let  $p: V \rightarrow X$  and  $\pi: V \rightarrow W$  be the natural projections. Note that there exists an open subset  $W_0$  of  $W$  such that  $p(\pi^{-1}(W_0)) \subset X_0$ .

To prove our proposition, we assume to the contrary that there is no common point in the general leaves of  $\overline{\mathcal{F}}$ .

First we show that there exists a smooth curve  $C$  with a finite morphism  $n: C \rightarrow n(C) \subset W$  such that:

- (1) Let  $U$  be the normalization of the fiber product  $V \times_W C$  with projection  $\pi: U \rightarrow C$ . Then the induced morphism  $\tilde{p}: U \rightarrow X$  is finite onto its image.
- (2) There exists an open subset  $C_0$  of  $C$  such that the image of  $U_0$  under  $p$  is contained in  $X_0$ . In particular,  $U_0 = \pi^{-1}(C_0)$  is a  $\mathbb{P}^r$ -bundle over  $C_0$ .
- (3) For any point  $c \in C$ , the image of the fiber  $\pi^{-1}(c)$  under  $\tilde{p}$  is not contained in  $X \setminus X'$ .
- (4) All the fibers of  $\pi$  are reduced.

We denote  $X \setminus X'$  by  $Y$ , then  $\text{codim}(Y) \geq 2$ . We consider the subset

$$Z = \{w \in W \mid \pi^{-1}(w) \subset p^{-1}(Y)\}.$$

Since  $\pi$  is equidimensional, it is a surjective universally open morphism [Gro66, Théorème 14.4.4]. Therefore the subset  $Z$  is closed. Note that the general fiber of  $\pi$  is disjoint from  $p^{-1}(Y)$ , so  $\text{codim}(Z) \geq 1$ . Moreover, by the definition of  $Z$ , we have  $p(\pi^{-1}(Z)) \subset Y$  and  $\text{codim} Y \geq 2$ , hence we can choose some very ample divisors  $H_i$  ( $1 \leq i \leq n$ ) on  $X$  such that the curve  $B$  defined by complete intersection  $\tilde{p}^*H_1 \cap \cdots \cap \tilde{p}^*H_n$  satisfies the following conditions:

- (a) There is no common point in the general fibers of  $\pi$  over  $\pi(B)$ .
- (b)  $\pi(B) \cap W_0 \neq \emptyset$ .
- (c)  $\pi(B) \subset W \setminus Z$ .

Let  $B' \rightarrow B$  be the normalization, and  $V_{B'}$  the normalization of the fiber product  $V \times_B B'$ . The induced morphism  $V_{B'} \rightarrow V$  is denoted by  $\mu$ . Then it is easy to check that  $B'$  satisfies (1), (2) and (3). By [BLR95, Theorem 2.1], there exists a finite morphism  $C \rightarrow B'$  such that all the fibers of  $U \rightarrow C$  are reduced, where  $U$  is the normalization of  $U \times_{B'} C$ . Then we see at once that  $C$  is the desired curve.

The next step is to get a contradiction by applying Theorem 4.1. From Remark 3.8, we see that the Pfaff field  $\Omega_X^r \rightarrow \mathcal{O}_X(K_{\overline{\mathcal{F}}})$  extends to a Pfaff field  $\Omega_{V_{B'}/B'}^r \rightarrow \mu^*p^*\mathcal{O}_X(K_{\overline{\mathcal{F}}})$ , and it induces a Pfaff field  $\Omega_{U/C}^r \rightarrow \tilde{p}^*\mathcal{O}_X(K_{\overline{\mathcal{F}}})$ . The natural inclusion  $\mathcal{F} \hookrightarrow \overline{\mathcal{F}}$  induces a morphism  $\mathcal{O}_X(K_{\overline{\mathcal{F}}}) \rightarrow \mathcal{O}_X(K_{\mathcal{F}})$ . It implies that we have a Pfaff field  $\Omega_{U/C}^r \rightarrow \tilde{p}^*\mathcal{O}_X(K_{\mathcal{F}})$ . By Lemma 4.2, there exists an effective Weil divisor  $\Delta$  on  $U$  such that  $K_{U/C} + \Delta = \tilde{p}^*K_{\mathcal{F}}$ .

Let  $\Delta^{hor}$  be the  $\pi$ -horizontal part of  $\Delta$ . After shrinking  $C_0$ , we may assume that  $\Delta|_{U_0} = \Delta^{hor}|_{U_0}$ . According to the proof of Theorem 3.9, for any fiber  $F \cong \mathbb{P}^r$  over  $C_0$ , we have  $(\tilde{p}^*K_{\mathcal{F}})|_F - K_F = 0$  or  $H$  where  $H \in |\mathcal{O}_{\mathbb{P}^r}(1)|$ . This shows that either  $\Delta^{hor}$  is zero or  $\Delta^{hor}$  is a prime divisor such that  $\Delta|_{U_0} = \Delta^{hor}|_{U_0} \in |\mathcal{O}_{U_0}(1)|$ . In particular, the pair  $(U, \Delta^{hor})$  is snc over  $U_0$  and  $\Delta^{hor}$  is reduced. Note that  $\tilde{p}: U \rightarrow \tilde{p}(U)$  is a finite morphism, so the invertible sheaf  $\tilde{p}^*\mathcal{O}_X(-K_{\mathcal{F}})$  is ample over  $U' = U \cap \tilde{p}^{-1}(X')$ , i.e. the sheaf  $\mathcal{O}_U(-K_{U/C} - \Delta)$  is ample over  $U'$ , which contradicts to Theorem 4.1.  $\square$

Now we are in the position to prove our main theorem.

*Proof of Theorem 1.1.* According to Theorem 2.1, there exists an open subset  $X_0 \subset X$  and a normal variety  $T_0$  such that  $X_0 \rightarrow T_0$  is a  $\mathbb{P}^{d+1}$ -bundle and  $d+1 \geq r$ . Without loss of generality, we may assume  $r < \dim(X)$ . By Theorem 3.9 followed by Proposition 4.3,  $\overline{\mathcal{F}}$  defines an algebraically integrable foliation over  $X$  such that there is a common point in the general leaves of  $\overline{\mathcal{F}}$ . However, this cannot happen if  $\dim(T_0) \geq 1$ . Thus  $\dim T_0 = 0$  and  $X \cong \mathbb{P}^n$ .  $\square$

## 5. $\mathbb{P}^r$ -BUNDLES AS AMPLE DIVISORS

As an application of Theorem 1.1, we classify projective manifolds  $X$  containing a  $\mathbb{P}^r$ -bundle as an ample divisor. This is originally conjectured by Beltrametti and Sommese [BS95, Conjecture 5.5.1]. In the sequel, we follow the same notation and assumptions as in Theorem 1.3.

The case  $r \geq 2$  follows from Sommese's extension theorem [Som76] (see also [BS95, Theorem 5.5.2]). For  $r = 1$  and  $b = 1$ , it is due to Bădescu [Băd84, Theorem D] (see also [BS95, Theorem 5.5.3]). For  $r = 1$  and  $b = 2$ , it was done by the work of several authors [BI09, Theorem 7.4]. Recently, D. Litt proved the following result by which we can deduce Theorem 1.3 from Corollary 1.2.

**5.1. Proposition.**[Lit16] *Let  $X$  be a projective manifold of dimension  $\geq 3$ , and let  $A$  be an ample divisor. Assume that  $p: A \rightarrow B$  is a  $\mathbb{P}^1$ -bundle, then either  $p$  extends to a morphism  $\hat{p}: X \rightarrow B$ , or there exists an ample vector bundle  $E$  on  $B$  and a non-zero map  $E \rightarrow T_B$ .*

For the reader's convenience, we outline the argument of D. Litt that reduces Theorem 1.3 to Corollary 1.2 [Lit16].

*Proof of Theorem 1.3.* Since the case  $r \geq 2$  is already known, we can assume that  $r = 1$ , i.e.  $p: A \rightarrow B$  is a  $\mathbb{P}^1$ -bundle.

If  $p$  extends to a morphism  $\hat{p}: X \rightarrow B$ , then the result follows from [BI09, Theorem 5.5] and we are in the case 1) of our theorem.

If  $p$  does not extend to a morphism  $X \rightarrow B$ , by Proposition 5.1, there exists an ample vector bundle  $E$  over  $B$  with a non-zero map  $E \rightarrow T_B$ . Own to Corollary 1.2, we have  $B \cong \mathbb{P}^b$ . As the case  $b \leq 2$  is also known, we may

assume that  $b \geq 3$ . In this case, by [FSS87, Theorem 2.1], we conclude that  $X$  is a  $\mathbb{P}^{n-1}$ -bundle over  $\mathbb{P}^1$  and we are in the case 2) of our theorem.  $\square$

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