A New and Deterministic Scheme for Characterizing The Organization of Prime Numbers

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The fundamental theorem of arithmetic states that any composite natural integer can be expressed in one and only one way as a product of prime numbers. This sets the understanding of the organization of prime numbers at the core of number theory. In this work we present a simple, self-consistent and deterministic scheme allowing to investigate further the intrinsic organization of prime numbers. Using this scheme, we establish an algorithm that yields the complete list of prime numbers below any preassigned limit x. Counting the latter yields $\pi(x)$, the number of prime numbers below x. Based on preliminary tests on computing clusters available, a considerable gain in computational speed and algorithmic simplicity towards producing complete lists of large prime numbers is observed. At the core of the new scheme lays its ability to provide, in a deterministic way, complete lists of consecutive and composite odd numbers below any preassigned limit x. The complete list of prime numbers below x is deduced from the latter. The two key ingredients of the scheme are a set of eleven generic tables, coupled with a three-criteria test applied on the differences between pairs of the consecutive composite odd numbers initially obtained. Since it leads to counting all the elements of a complete list of prime numbers up to x, our deterministic scheme provides a new approach to the long standing problem of "how many prime numbers are there below any preassigned limit x". The said scheme therefore potentially contributes towards studies aimed at unveiling the organization of prime numbers. We illustrate the latter in a follow-up paper, Paper II [3], where we propose a new perspective on the Riemann hypothesis.

Keywords: Number Theory, Composite Odd Numbers, Prime Numbers

I. INTRODUCTION

Some numbers have the special property that they cannot be expressed as the product of two or more smaller numbers, e.g. $2, 3, 5, 7, etc.$ Such numbers are called prime numbers, and they play an important role, both in pure mathematics and its applications. Prime numbers occur in a very irregular way within the sequences of natural numbers. In particular, the distribution of prime numbers exhibits a local irregularity but a global regularity. One of the best existing results illustrating that global regularity is the prime number theorem giving the number of prime number, $\pi(x)$, not exceeding an upper limit x. In epoch-marking works published in the late nineteenth century, it was shown that $\pi(x)$ behaves asymptotically as $\frac{x}{\ln x}$ [2, 7]. Prime numbers derive most of their peculiar importance from the fundamental theorem of arithmetic stating that any composite natural integer can be expressed in one and only one way as a product of prime number [1]. Given the importance of primes in the construction of all natural numbers, a problem that presents itself at the very roots of mathematics is therefore the organization of the primes among the integers.Formulated in other terms, a fundamental tool for number theory would be a scheme allowing to obtain, at a relatively low computational cost - when compared to existing approaches, the complete list of primes below any upper limit n. In the present work (Paper I), we propose a simple and self-consistent deterministic scheme allowing us to investigate further the intrinsic organization of prime numbers. From the said scheme, we have established an algorithm that systematically yields the complete list of prime numbers below any preassigned limit x at a relatively low computational cost. The reader might wish to test our scheme online for values of $x \leq 10^9$, on this website `http://univgandhi-guinee.com/algo/public/home/nombres-premiers`[1]. In Paper II ([3]), based on our deterministic scheme, we offer a new perspective on Riemann hypothesis.

[1] The datafiles also provided on this database come with minimal copyright requirements specified onsite. Feel free to contact the authors for more.

II. A NEW AND DETERMINISTIC SCHEME YIELDING THE COMPLETE LIST OF PRIMES UP TO A PREASSIGNED LIMIT x, WITHOUT ANY PRIMALITY TEST

Towards clarifying our approach, let us first make clear our terminology with a few definitions.

Definition 1. $\mathbb{N}^* = \mathbb{N} - \{0\}$

Definition 2. $\forall\ p \in \mathbb{N}$, p is a composite odd number if and only if

(1) p is an odd number;

(2) p is not a prime number.

Definition 3. For any n *and* p *elements of* $\mathbb{N}$, n and p are consecutive composite odd numbers if and only if there exists no other composite odd number k such that $n < k < p$.

Our deterministic scheme is built upon the set of composite odd numbers. From these, and as explained below, we deduce non-composite odd numbers i.e prime numbers.

Except 2 and 5, any prime number (PN) has to end with 1 *or* 3 *or* 7 *or* 9. The trivial proof relies on realizing that all other cases are excluded by definition of prime numbers. From the result above we deduce that except 2 and 5, any prime number can be written as

Theorem II.1. *(i)* $PN_1 \equiv 10n + 1,\ \forall n \in \mathbb{N}$, *for all primes ending with* 1*;*

(ii) $PN_3 \equiv 10n + 3,\ \forall n \in \mathbb{N}$, *for all primes ending with* 3*;*

(iii) $PN_7 \equiv 10n + 7,\ \forall n \in \mathbb{N}$, *for all primes ending with* 7*;*

(iv) $PN_9 \equiv 10n + 9,\ \forall n \in \mathbb{N}$, *for all primes ending with* 9*;*

Including trivial prime numbers ($\{2,\ 3,\ 5,\ 7\}$), the results above states that all prime numbers are also odd numbers. More precisely, all prime numbers are non-composite odd numbers (nocoon in short). For convenience, we will ignore $\{2, 3, 5, 7\}$, the trivial prime numbers. We are dominantly interested in the prime numbers greater or equal to 9. Let us therefore introduce the set Ω_{nocoon} of all non-composite odd numbers (i.e prime numbers) greater or equal to 9.

Definition 4. $\Omega_{nocoon} = \{p = 10n + k$ with $n \in \mathbb{N}^*$, p non-composite odd number, and $k \in \{1, 3, 7, 9\}\}$.

Within $\Omega = \{p \in \mathbb{N}, p \geq 9$ and p is a odd number$\}$, the set that is complementary to Ω_{nocoon} is $\Omega_{coon} = \{p \geq 9$ is a composite odd number$\}$, the set comprising all composite odd numbers greater or equal to 9. Hence

Corollary II.2. $\Omega = \Omega_{coon} \bigcup \Omega_{nocoon}$ *with* $\Omega_{coon} \bigcap \Omega_{nocoon} = \emptyset$.

Corollary II.3. *All odd numbers are either*

i *Composite or;*

ii *Prime (non-composite).*

From corollary II.2, we read that any odd number that is not composite is therefore a prime number (i.e a non-composite odd number). By construction, the organization of prime numbers (non-composite odd numbers) is therefore directly correlated to the organization of composite odd numbers. The set Ω_{coon} of composite odd number is at the core of the present work: we deduce all complete list of prime numbers below any preassigned limit s from complete list of composite odd numbers below x.

Lemma 1. *Given* $\mathcal{C}$*, the complete list of all consecutive composite odd numbers below any integer* x*, any odd number not listed in* $\mathcal{C}$ *is a non-composite odd number i.e. a prime number.*

Our deterministic scheme consists therefore in

(1) Constructing the complete and ordered list of all composite odd numbers below any preassigned limit x. The tables in corollary II.9 are the backbone of this task;

(2) Based on the test(s) laid down in Theorem II.10 applied to the list obtained in (1) above, one deduces the complete and ordered list of all non-composite odd numbers (i.e prime numbers) smaller than x.

With these preliminaries set, and as a way to illustrate the establishment of our deterministic scheme, let us briefly and momentarily focus on all non-trivial prime numbers (≥ 10) ending with 7 and below any preassigned x, i.e. all prime numbers that can be written as PN_7 and that are smaller than any preassigned limit x. The same reasoning will be applied to all prime numbers that can be written as PN_1, PN_3, PN_9 since we established earlier that except 2 and 5, primes have to end with 1 *or* 3 *or* 7 *or* 9. For practical convenience, in all the subsequent sections, $2, 3, 5, 7$ will be regarded as trivial prime numbers. The census will therefore be done for all prime numbers larger than 10 and smaller than any preassigned limit x.

We wanted to start by the general problem of listing all the odd numbers ending with 7.

Theorem II.4. *Any composite odd number ending with 7*

1. *is of the form* $10n + 7$, *for* $n \in \mathbb{N}$*;*
2. *can only be written either as*

   (i) $10n + 7 = (10n_1 + 7)(10n_2 + 1)$*;*

   (ii) or as or as $10n + 7 = (10n_3 + 3)(10n_4 + 9)$,

   where $n, \; n_1, \; n_3, \; n_4$ *are all elements of* $\mathbb{N}$ *and* $n_2 \in \mathbb{N}^*$.

Proof. It is trivial to prove that any odd number (ON) ends with 1 *or* 3 *or* 5 *or* 7 *or* 9, since the excluded cases (0 *or* 2 *or* 4 *or* 6 *or* 8) lead to even numbers. Hence

(i) For any odd number ON_1 ending with 1, $ON_1 = 10n_1 + 1, \text{ for } n_1 \in \mathbb{N}$;

(ii) For any odd number ON_3 ending with 3, $ON_3 = 10n_3 + 3, \text{ for } n_3 \in \mathbb{N}$;

(iii) For any odd number ON_5 ending with 5, $ON_5 = 10n_5 + 5, \text{ for } n_5 \in \mathbb{N}$;

(iv) For any odd number ON_7 ending with 7, $ON_7 = 10n_7 + 7, \text{ for } n_7 \in \mathbb{N}$;

(v) For any odd number ON_9 ending with 9, $ON_9 = 10n_9 + 9, \text{ for } n_9 \in \mathbb{N}$.

Clearly, any odd number ending with 7 can only be written as $10n + 7$ where $n \in \mathbb{N}$.

Equally trivial is the proof that any composite odd number can only be written as a product of odd numbers (composite or not). If not one ends up with the absurd situation of having the existence of odd numbers also equal to some even number. A straightforward reasoning by the absurd also proves that any finite product of composite odd numbers yields a composite odd number.

Lemma 2. $\forall \; p \in \Omega_{coon}, \; \exists \; s_1, \; s_2, \; ...., \; s_n$ *each in* Ω_{coon} *such that* $p = s_1 * s_2 * ... * s_n$. *By consequence, one can always write* $p = k * l * m$ *where* l *and* m *are each one a composite odd number and* $k \in \mathbb{N}$

The latter result is a consequence of the fundamental theorem of arithmetic. Let us therefore list all possible products $ON_i * ON_j$ for $i, \; j \in \{1, 3, 5, 7, 9\}$, of composite odd numbers, since these underlie the decomposition of any composite odd number. There are $C_5^2 = \frac{5!}{(5-2)!2!}$ such products i.e. fifteen of these products that we list below.

Lemma 3.

$$\begin{aligned}
P_1 &= (10n_1+1)(10n_1+1) \Leftrightarrow P_1 = 10(10n_1^2+2n_1)+1,\ n_1 \in \mathbb{N}^* \\
P_2 &= (10n_1+1)(10n_2+3) \Leftrightarrow P_2 = 10(10n_1n_2+3n_1+n_2)+3,\ n_2 \in \mathbb{N} \\
P_3 &= (10n_1+1)(10n_3+5) \Leftrightarrow P_3 = 10(10n_1n_3+5n_1+n_3)+5,\ n_3 \in \mathbb{N} \\
P_4 &= (10n_1+1)(10n_4+7) \Leftrightarrow P_4 = 10(10n_1n_4+7n_1+n4)+7,\ n_4 \in \mathbb{N} \\
P_5 &= (10n_1+1)(10n_5+9) \Leftrightarrow P_5 = 10(10n_1n_5+9n_1+n3)+9,\ n_5 \in \mathbb{N} \\
P_6 &= (10n_2+3)(10n_2+3) \Leftrightarrow P_6 = 10(10n_2^2+6n_2)+9,\ n_2 \in \mathbb{N} \\
P_7 &= (10n_2+3)(10n_3+5) \Leftrightarrow P_7 = 10(10n_2n_3+5n_2+3n_3+1)+5,\ n_1 \in \mathbb{N} \\
P_8 &= (10n_2+3)(10n_4+7) \Leftrightarrow P_8 = 10(10n_2n_4+7n_2+3n_4+2)+1,\ n_4 \in \mathbb{N} \\
P_9 &= (10n_2+3)(10n_5+9) \Leftrightarrow P_9 = 10(10n_2n_5+9n_2+3n_5+2)+7,\ n_5 \in \mathbb{N} \\
P_{10} &= (10n_3+5)(10n_3+5) \Leftrightarrow P_{10} = 10(10n_3^2+10n_3+2)+5,\ n_3 \in \mathbb{N} \\
P_{11} &= (10n_3+5)(10n_4+7) \Leftrightarrow P_{11} = 10(10n_3n_4+7n_3+5n_3+3)+5,\ n_4 \in \mathbb{N} \\
P_{12} &= (10n_3+5)(10n_5+9) \Leftrightarrow P_{12} = 10(10n_3n_5+9n_3+5n_5+4)+5,\ n_5 \in \mathbb{N} \\
P_{13} &= (10n_4+7)(10n_4+7) \Leftrightarrow P_{13} = 10(10n_4^2+14n_4+4)+9,\ n_4 \in \mathbb{N} \\
P_{14} &= (10n_4+7)(10n_5+9) \Leftrightarrow P_{14} = 10(10n_4n_5+9n_4+7n_5+6)+3,\ n_5 \in \mathbb{N} \\
P_{15} &= (10n_5+9)(10n_5+9) \Leftrightarrow P_{15} = 10(10n_5^2+18n_5+80)+1,\ n_5 \in \mathbb{N}
\end{aligned}$$

The fifteen (15) expressions in Lemma 3 define all the possible forms for any composite odd number. We also know that any composite odd number always ends with 1 *or* 3 *or* 5 *or* 7 *or* 9. We then deduce from these that all composite odd numbers of the $10n+7$ i.e. ending with 7 are all and only of the type P_4 or P_9. In other words, Any composite odd number ending with 7 can only be written either as $10n+7 = (10k_1+1)(10k_2+7)$ or as $10n+7 = (10k_3+3)(10k_4+9)$, where $k_2,\ k_3,\ k_4$ are all elements of $\mathbb{N}$ and $k_1 \in \mathbb{N}^*$ □

Similarly, the set of 15 equations in Lemma 3 yields the following results.

Equations P_1 and P_{15} are special cases of the generic forms for any composite odd number ending with 1. Since we are only interested in the last digit (1), P_1 and P_{15} can therefore be extended into $\tilde{P}_1$ and $\tilde{P}_{15}$ defined by $\tilde{P}_1 = (10k_1+1)(10k_2+1)$ and $\tilde{P}_{15} = (10k_3+9)(10k_4+9)$ where $k_1,\ k_2,\ k_3,$ *and* k_4 are all elements of $\mathbb{N}$ and $k_1,\ k_2 \in \mathbb{N}^*$. Coupled with P_8 we therefore get that

Theorem II.5. *All composite odd numbers of the form* $10n+1$ *i.e. ending with* 1 *are all and only of the type* $\tilde{P}_1$ *or* P_8 *or* $\tilde{P}_{15}$*. In other words, any composite odd number ending with 1 (i.e. of the form* $10n+1$*) can only be written either as* $10n+1 = (10k_1+3)(10k_2+7)$ *or as* $10n+1 = (10k_3+9)(10k_4+9)$*, or as* $10n+1 = (10k_5+1)(10k_6+1)$*, where* $k_1,\ k_2,\ k_3,\ k_4$ *are all elements of* $\mathbb{N}$ *and* $k_5,\ k_6 \in \mathbb{N}^*$*. Generalization of* P_1 *and* P_{15} *may lead to counting more then once some composite odd number ending with 9. This does not fundamentally alter or affect the fact that we will get final lists made of all consecutive composite odd numbers ending with* 1*. The same applies to all subsequent situations where duplicates are included in the final lists. We are interested in the final complete list with or without duplicate. Only the fact to it is complete matters.*

Theorem II.6. *Any composite odd number ending with 3 (i.e. of the form* $10n+3$*) can only be written either as* $10n+3 = (10k_1+3)(10k_2+1)$ *or as* $10n+3 = (10k_3+7)(10k_4+9)$*, where* $k_1,\ k_3,\ k_4$ *are all elements of* $\mathbb{N}$ *and* $k_2 \in \mathbb{N}^*$*.*

Theorem II.7. *Combining* P_3*,* P_7*,* P_{10}*,* P_{11}*, and* P_{15} *we obtain the trivial result that any composite odd number ending with 5 (i.e. of the form* $10n+5$*) can only be written as* $10n+5 = 5(2k+1)$ *where* $k \in \mathbb{N}$*.*

Theorem II.8. *Equations* P_6 *and* P_{13} *are special cases of the generic forms for any composite odd number ending with 9. Since we are only interested in the last digit (9),* P_6 *and* P_{13} *can therefore be extended into* $\tilde{P}_6$ *and* $\tilde{P}_{13}$ *defined by* $\tilde{P}_6 = (10k_3+3)(10k_4+3)$ *and* $\tilde{P}_{13} = (10k_5+7)(10k_6+7)$ *where* $k_3,\ k_4,\ k_5,\ k_6$ *are all elements of* $\mathbb{N}$*. Coupled with* P_5 *we therefore get that any composite odd number ending with 9 i.e. of the form* $10n+9$ *can only be written either as* $10n+9 = (10k_1+1)(10k_2+9)$ *or as* $10n+9 = (10k_3+3)(10k_4+3)$*, or as* $10n+9 = (10k_5+7)(10k_6+7)$*, where* $k_2,\ k_3,\ k_4,\ k_5,$ *and* k_6 *are all elements of* $\mathbb{N}$ *and* $k_1 \in \mathbb{N}^*$*.*

The results in theorems II.4, II.5, II.6, II.7 and II.8 define all the possible forms for any composite odd number greater or equal to 10. They can all be summarized in the form of eleven (11) tables given below. Three (03), two (02), one (01), two (02) and three (03) tables are needed for all composite odd numbers ending with 1 *or* 3 *or* 5*or* 7 *or* 9, respectively. As implicit in corollary II.9, all complete lists of consecutive prime numbers larger or equal to 9 (always ending with 1 *or* 3 *or* 7 *or* 9) and below any pre-assigned limit x will be deduced from trivial and systematic operations on elements of these tables, coupled with a test that we establish in Theorem II.10. The reason why some elements above and in the tables are taken in $\mathbb{N}^*$ is because we are looking for composite odd numbers and the first one of these is 9.

Corollary II.9. *Based on the fifteen equations in Lemma 3 and since* 5 *is the only prime number ending with* 5*, ten* $(10 = 11 - 1)$ *of the eleven* (11) *tables given in Appendix* **A** *will be effectively used in our deterministic scheme to deduce the complete list of consecutive prime numbers larger than* 10 *and smaller than any pre-assigned limit* x*. The ten tables III, IV, V, VI, VII, VIII, IX, X, XI and XII given in Appendix* **A** *constitute the numerical backbone of our deterministic scheme. These eleven Tables yield the complete list of all consecutive composite odd numbers greater or equal to* 9 *and smaller than any pre-assigned limit* x*. In each table, for each computation* $p = (10k + m)(10l + q)$, $k, m, l, q \in \mathbb{N}$*, the constrain is that each product* p *has to be smaller than* x*.*

How then do we deduce, from the previous, the complete list of all consecutive prime numbers below any pre-assigned limit x? The following theorem yields an answer.

Theorem II.10 (Main Theorem)**.** $\forall$ N_1 *and* N_2 *two consecutive composite odd numbers each larger than* 9*,*

(1) $N_2 - N_1 = 2$ *or* $N_2 - N_1 = 4$ *or* $N_2 - N_1 = 6$*,*

(2) If $N_2 - N_1 = 2$ *then there is no prime number between* N_1 *and* N_2*,*

(3) If $N_2 - N_1 = 4$ *then there is one prime numbers between* N_1 *and* N_2*,*

(4) If $N_2 - N_1 = 6$ *then there are two prime numbers between* N_1 *and* N_2*.*

Proof. (1) We prove item (1) in Theorem II.10. The result is immediate for any composite odd number from $1, 3, 5, 7$. Let us introduce the ordered set $\Omega(x)$ of all composite odd numbers smaller or equal to than $x \in \mathbb{N}$. By construction, 9 is the first element of $\Omega(x)$. $\Omega(x) = \bigcup_{k=1,\ k\in\mathbb{N}}^{m} \Omega_k$ and $\Omega_k = \{p \in \mathbb{N}, p \text{ is a composite odd number and } 3(2k+1) \leq p \leq 3(2k+3),\ k \in \mathbb{N}^* \} \subseteq \mathbb{N}$ $\Omega(x)$ can be visualized as the line of integers (consecutive composite odd numbers) shown in Figure 1.

$\forall$ $k \in \mathbb{N}^*$, $\Omega_k \subseteq \mathbb{N}$ is the interval whose extremities are consecutive and composite odd numbers that are each a multiple of 3. It is trivial to prove that pairs of consecutive odd numbers that are multiple of 3 are separated by a distance of 6. Of all the elements of $\Omega(x)$, those making up Ω_k are between pairs of consecutive composite odd numbers that are separated by the smallest distance of all (equal to 6). By construction, any pair of consecutive composite odd numbers within any union of at least two Ω_k is separated by a distance strictly larger than 6. As illustrated in Figure 1, intervals of length 6, from any Ω_k can be understood as a "unit interval", $\mathcal{U}$: Any interval larger than Ω_k has a length that is a multiple of the length of $\mathcal{U}$.

(a) We are interested in the length of all intervals between pairs of consecutive composite odd numbers within $\Omega(x)$;

(b) Ω_k constitutes the shortest interval of integers between pairs of consecutive composite odd numbers within $\Omega(x)$, so it matters;

(c) As illustrated in Figure 1, whatever happens with odd numbers within any Ω_k will be mirrored within larger intervals of $\Omega(x)$;

(d) As illustrated in Figure 1 and expressed in Lemma 4 below, any number from $\Omega(x)$ i.e. any composite odd number will appear within one " unit interval", Ω_k.

Let us therefore investigate what happens with numbers within any "unit interval" Ω_k.

Lemma 4. $\forall p_1$ *and* p_2 $\in \mathbb{N}$*, two consecutive and composite odd numbers multiple of* 3*, there always exist two and only two odd numbers between* p_1 *and* p_2*.*

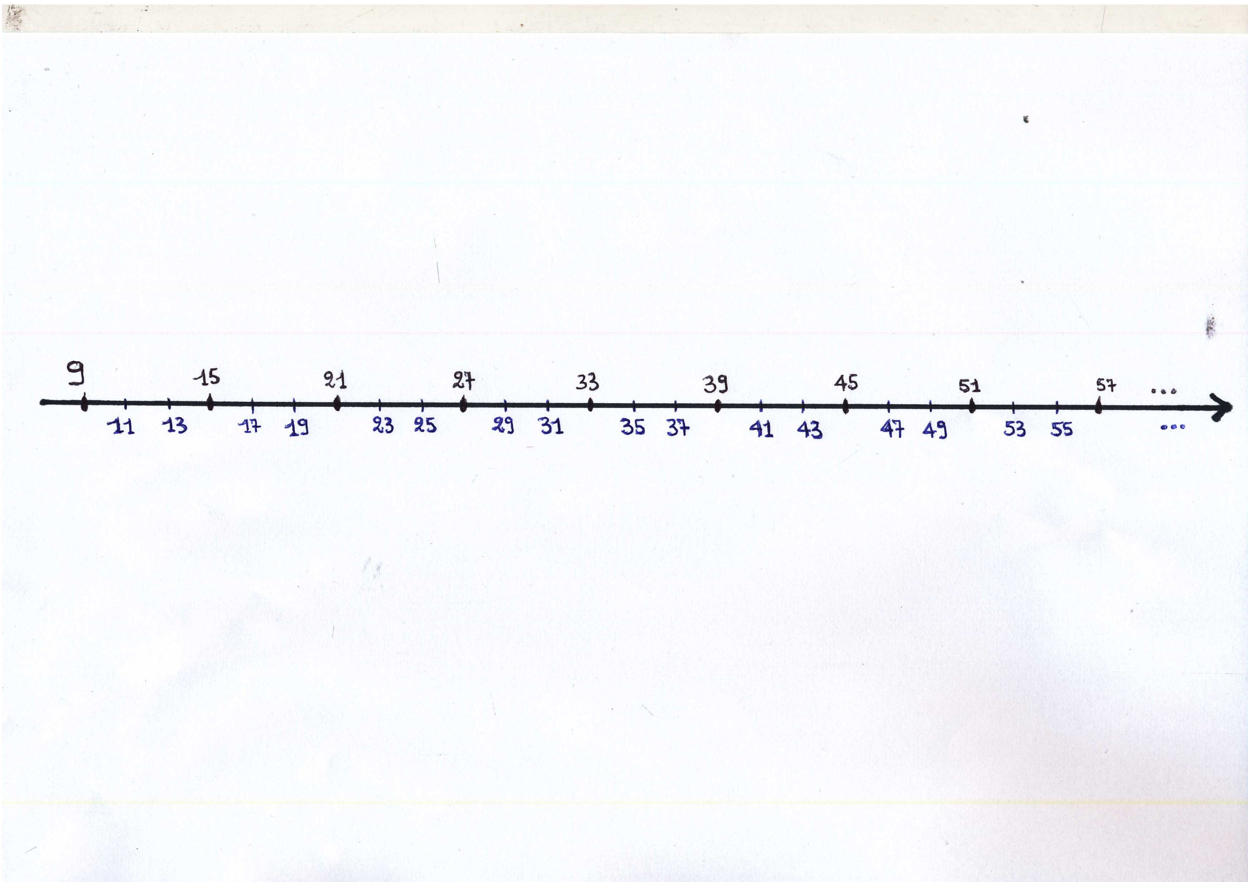

FIG. 1: Partially illustrated here is the ordered set of all composite odd numbers multiple of 3 and smaller or equal to some $x \in \mathbb{N}$ (upper ticks in black). By construction, 9 is the first (smallest) composite odd number. $\Omega_{coon}(x)$ is the set of all consecutive composite odd numbers smaller or equal to x. $\Omega_{coon}(x) = \bigcup_{k=1,\ k\in\mathbb{N}}^{m} \Omega_k$ and $\Omega_k = \{p \in \mathbb{N}, p \text{ is a composite odd number and } 3(2k+1) \leq p \leq 3(2k+3),\ k \in \mathbb{N}^* \} \subseteq \mathbb{N}$ is ordered. Any pair of consecutive composite odd numbers within any union of at least two Ω_k is separated by a distance strictly larger than 6. As illustrated here, any Ω_k is an interval of consecutive composite odd numbers. Each Ω_k has a length equal to 6 and can be understood as a "unit interval" of consecutive composite odd numbers. Whatever happens with consecutive composite odd numbers within any Ω_k will be mirrored within larger intervals of $\Omega_{coon}(x)$. Hence the importance of Ω_k in quantifying the maximal distance between any pair of consecutive composite odd numbers, since any of these composite odd number will fall within one Ω_k.: illustrated here as *some* of the integers in blue.

We prove it. Let p_1 and p_2 be any two two consecutive and composite odd numbers multiple of 3 :

$$\begin{aligned} p_1 &= 3(2n+1),\ n \in \mathbb{N} \\ &= 6n+3 \end{aligned} \tag{1}$$

and

$$\begin{aligned} p_2 &= 3(2n+3),\ n \in \mathbb{N} \\ &= 6n+9. \end{aligned} \tag{2}$$

We have

$$\begin{aligned} p_1 + 2 &= 3(2n+1)+2,\ n \in \mathbb{N} \\ &= 6n+5. \\ &= p_{11} \end{aligned} \tag{3}$$

and

$$p_1 + 4 = 3(2n+1) + 4,\ n \in \mathbb{N} \\ = 6n + 7 \\ = p_{12}. \tag{4}$$

p_{11} and p_{12} both between p_1 and p_2 are uniquely defined and are both odd numbers. There are three possibilities for p_{11} and p_{12}:

(i) p_{11} and p_{12} are both composite odd numbers;

(ii) one of the two is a prime and the other is a composite number.

(iii) p_{11} and p_{12} are both prime numbers;

Lemma 5. *The three options above encompass all possibilities for any pair of composite odd numbers appearing in the ordered set $\Omega(x)$ and as illustrated in Figure 1. Let us sum it up.*

(a) If $p_{11} = 6n+5$ is a composite odd number (and $p_{12} = 6n+7$ is a composite odd number) then $p_1 = 6n+3$ and p_{11} are consecutive composite odd numbers and their difference is equal to 2;

(b) If $p_{12} = 6n+7$ is a composite odd number and $p_{11} = 6n+5$ is a prime number then $p_1 = 6n+3$ and $p_{12} = 6n+7$ are consecutive composite odd numbers and their difference equal to 4;

(c) $p_{11} = 6n+5$ and $p_{12} = 6n+7$ are prime numbers with $p_1 = 6n+3$ and $p_2 = 6n+9$ consecutive and composite odd number. The difference $p_2 - p_1 = 6$.

This all happens within every "unit interval" Ω_k. Given that $\Omega(x) = \bigcup_{k=1\ k\in\mathbb{N}}^{m} \Omega_k$. Lemma 5 encapsulates the description of the way $\Omega(x)$ is systematically populated. Having listed, in Lemma 5, all possible occurrences of pairs of consecutive composite odd numbers within $\Omega(x)$, the constrain emerging is that for any pair (p_1, p_2) of consecutive composite odd numbers, their difference is always smaller or equal to 6. In other words, we have established that

$$\forall\ p_1 \textit{ and } p_2 \text{ two consecutive composite odd numbers with } p_1 < p_2, \text{ one has } p_2 - p_1 \leq 6.$$

On the other hand p_1 and p_2 being consecutive, $p_2 - p_1 > 0$. This proves the item (1) in Theorem II.10.

(2) Here we prove item (2) in Theorem II.10. We look at all of the only six possibilities. $\forall\ N_1 \geq 9$ *and* $N_2 \geq 9$ two consecutive composite odd numbers, such that $N_2 - N_1 = 2$,

(i) if N_1 ends with 1 therefore N_2 only ends with 3 and there is no prime number between N_1 and N_2 since the only integer between both will be an even number ending with 2.

(ii) if N_1 ends with 3 therefore N_2 only ends with 5 and there is no prime number between N_1 and N_2 since the only integer between both will be an even number ending with 4.

(iii) if N_1 ends with 5 therefore N_2 only ends with 7 and there is no prime number between N_1 and N_2 since the only integer between both will be an even number ending with 6.

(iv) if N_1 ends with 7 therefore N_2 only ends with 9 and there is no prime number between N_1 and N_2 since the only integer between both will be an even number ending with 8.

(v) if N_1 ends with 9 therefore N_2 only ends with 1 and there is no prime number between N_1 and N_2 since the only integer between both will be an even number ending with 0.

Hence, $\forall\ N_1 \geq 9$ *and* $N_2 \geq 9$ two consecutive composite odd numbers, If $N_2 - N_1 = 2$ then there is no prime number between N_1 and N_2. This also proves that:

Lemma 6. *$\forall\ N_1 \geq 9$ and $N_2 \geq 9$ two consecutive composite odd numbers, If $N_2 - N_1 = 2$ then there is one and only one integer between N_1 and N_2 and that integer is even.*

(3) Here we prove item (3) in Theorem II.10. Let $N_1 \geq 9$ *and* $N_2 \geq 9$ be any two consecutive composite odd numbers such that $N_2 - N_1 = 4$.

(i)

$$\begin{aligned} N_2 &= N_1 + 4 \\ &= (N_1 + 2) + 2 \\ &= \tilde{N}_1 + 2 \text{ with } \tilde{N}_1 = N_1 + 2. \end{aligned} \tag{5}$$

$N_2 = \tilde{N}_1 + 2$ and $\tilde{N}_1 = N_1 + 2$ yields

$$\tilde{N}_1 = \frac{N_1 + N_2}{2}.$$

So,

$$\begin{aligned} \tilde{N}_1 &= \frac{N_1 + N_2}{2} \\ &= 2 + N_1 \\ &= 2 + (2k_1 + 1) \text{ with } k_1 \in \mathbb{N} \text{ since } N_1 \text{ is an odd number} \\ &= 2\tilde{k}_1 + 1. \end{aligned} \tag{6}$$

In other words, $\tilde{N}_1$ is also an odd integer. $\tilde{N}_1$ and N_2 are therefore two consecutive odd numbers of difference 2. The same goes for N_1 and $\tilde{N}_1$. From Lemma 6, we deduce that $\exists\,!\ p_1,\ p_2 \in\ \mathbb{N}$ all even integers and such that $N_1 \leq p_2 \leq \tilde{N}_1$ and $\tilde{N}_1 \leq p_1 \leq N_2$. We also have $N_1 \leq \tilde{N}_1 \leq N_2$. In other words there exist three and only three uniquely defined integers $p_1, \tilde{N}_1$ *and* p_2 in the interval $[N_1,\ N_2]$.

Since $N_1 (\neq\ \tilde{N}_1)$ and $N_2 (\neq\ \tilde{N}_1)$ are consecutive composite odd numbers and there are only three options for $N_2 - N_1$ via item (1) in Theorem II.10. We therefore deduce that $\tilde{N}_1$ is a non-composite odd number, otherwise N_1 and N_2 would not be consecutive composite odd numbers as given in Definition 3. We conclude then that $\tilde{N}_1$ is a prime number. This also proves that:

Lemma 7. *$\forall\ N_1 \geq 9$ and $N_2 \geq 9$ two consecutive composite odd numbers. If $N_2 - N_1 = 4$ then there exists one and only one prime number p_1 between N_1 and N_2. That prime number is given by $p_1 = \frac{N_1+N_2}{2}$.*

(4) Here we prove item (4) in Theorem II.10. Let $N_1 \geq 9$ *and* $N_2 \geq 9$ be any two consecutive composite odd numbers such that $N_2 - N_1 = 6 \Leftrightarrow N_2 - N_1 = 4 + 2 \Leftrightarrow N_2 = N_1 + 4 + 2$ Here we take advantage of the previous results.

(a)

$$\begin{aligned} N_2 &= (N_1 + 4) + 2, \\ &= \tilde{N}_1 + 2 \text{ with } \tilde{N}_1 = N_1 + 4, \\ N_2 &= \tilde{N}_2 \text{ with } \tilde{N}_2 = \tilde{N}_1 + 2. \end{aligned} \tag{7}$$

It is trivial to prove that $\tilde{N}_1$ is an odd integer, as is N_1. Therefore Lemma 7 states that there exists one and only one prime number p_1 between N_1 and $\tilde{N}_1$, given by $p_1 = \frac{N_1+\tilde{N}_1}{2} \ \Leftrightarrow\ p_1 = \frac{N_1+N_2}{2} - 1$.

(b) Similarly,

$$N_2 - N_1 = 4 + 2 \ \Leftrightarrow\ N_1 - N_2 = -4 - 2.$$

In other words,

$$\begin{aligned} N_1 &= N_2 - 4 - 2 \\ &= \bar{N}_2 - 2 \text{ with } \bar{N}_2 = N_2 - 4, \\ &= \bar{N}_1 \text{ with } \bar{N}_1 = \bar{N}_2 - 2. \end{aligned} \tag{8}$$

It is trivial to prove that $\bar{N}_2$ is an odd integer, as is N_2. Therefore Lemma 7 states that there exists one and only one prime number p_2 between $\bar{N}_2$ and N_2, given by $p_2 = \frac{\bar{N}_2+N_2}{2} \ \Leftrightarrow\ p_2 = \frac{N_1+N_2}{2} + 1$.

This also proves that:

Lemma 8. *$\forall$ $N_1 \geq 9$ and $N_2 \geq 9$ two consecutive composite odd numbers. If $N_2 - N_1 = 6$ then there exists two and only two prime numbers p_1 and p_2 between N_1 and N_2. These prime numbers given by $p_1 = \frac{N_1+N_2}{2} - 1$ and $p_1 = \frac{N_1+N_2}{2} + 1$.*

□

Corollary II.9 and Theorem II.10 thus constitute the two foundational components of our deterministic scheme for establishing the complete list of prime numbers up to any preassigned limit x. As evidenced in the results above, no single known primality test is performed. Our scheme operates much like a "compositeness" test. The dramatic gain in computational speed and algorithmic simplicity in obtaining complete lists of small or large prime numbers up to any preassigned limit x opens up unexpected avenues of fertile research on the intrinsic organization of prime numbers and associated. New conceptual developments in the field of number theory are naturally to follow. In Paper II [3], a follow-up paper to the present one, we illustrate the immense potential attached to our new deterministic scheme by using its key components to offer a new perspective on Riemann Hypothesis.

III. BRIEF NUMERICAL ILLUSTRATIONS AND CONSIDERATIONS

A forthcoming publication will delve further into a few relevant numerical applications of our deterministic scheme. Tables I and II provide a few outputs and statistics for similar runs performed on a high performance computing infrastructure, for values of $x \geq 10^9$. All ten tables in corollary II.9 were used for Table II while only Tables V, VI and VII were used for Table I. The latter highlights how the scheme can be fine-tuned to investigate desired sub-sets of the ensemble of prime numbers, such as all prime numbers ending with either $1, 3, 7,$ *or* 9 and all located below any preassigned limit x. This approach may prove useful when, for example, targeting large prime numbers.

By construction, Tables in corollary II.9 yield duplicated values. This does not fundamentally affect the robustness of the final result as these duplicates are easily removed. The final lists typically encompass all the possibilities. The final list is then ordered. We are left with adding - by hand - the very few trivial ones, which by construction of the tables, are not listed since the smallest values from each one of the table is set by the product of the two integer appearing in each factor in the the product $(10k + l)\,(10m + n)$, i.e. the product mn.
Shown in Table I are a few statistics from preliminary runs on facilities hosted by the South African Centre for High Performance Computing[2] in Cape Town, South Africa. The results shown are for composite odd numbers ending with 1. We now know that within our approach, sets made of consecutive composite odd numbers are the gateways into prime numbers. Not fully disclosed here is that computational options exists to restrain the search for both composite and therefore primes within any preassigned interval $[x_1, x_2]$ with $x_1, x_2 \in \mathbb{N}$. This will figure in a forthcoming paper. When compared with Table II, the preliminary execution times given in Table I suggests forthcoming breakthroughs towards computing - much faster and much simpler - prime numbers, small and large, via a deterministic way. Table II gives a few statistics from preliminary runs on the South African Centre for High Performance Computing infrastructure. In Figures 2 and 4, we show evolutions of the values of $\pi(x)$ and $\rho(x) = \frac{\pi(x)}{x}$ for bin $[10^5 k, 10^5(k+1)]$, with $k = 1, ..., 1000$. On Figures 3 and 5, one notes the same trend as the well-known function $\tilde{\rho}(x) \equiv \frac{1}{log(x)}$, approximating $\rho(x)$ and that can be deduced from the prime number theorem [2, 7]. A forthcoming publication will delve further into a few relevant numerical applications of our deterministic scheme. The reader is advised to write us at the email provided to gain access to some of the complete list mentioned in the tables below. Complete lists of prime numbers up to 10^9 are freely available on this website `http://univgandhi-guinee.com/algo/public/home/nombres-premiers`[3].

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[2] The specifications and world ranking of the South African Centre for High Performance Computing are available at `https://www.top500.org/system/178793` or `www.chpc.ac.za` .
[3] Data provided come with minimal copyright requirements. Contact the authors for more.

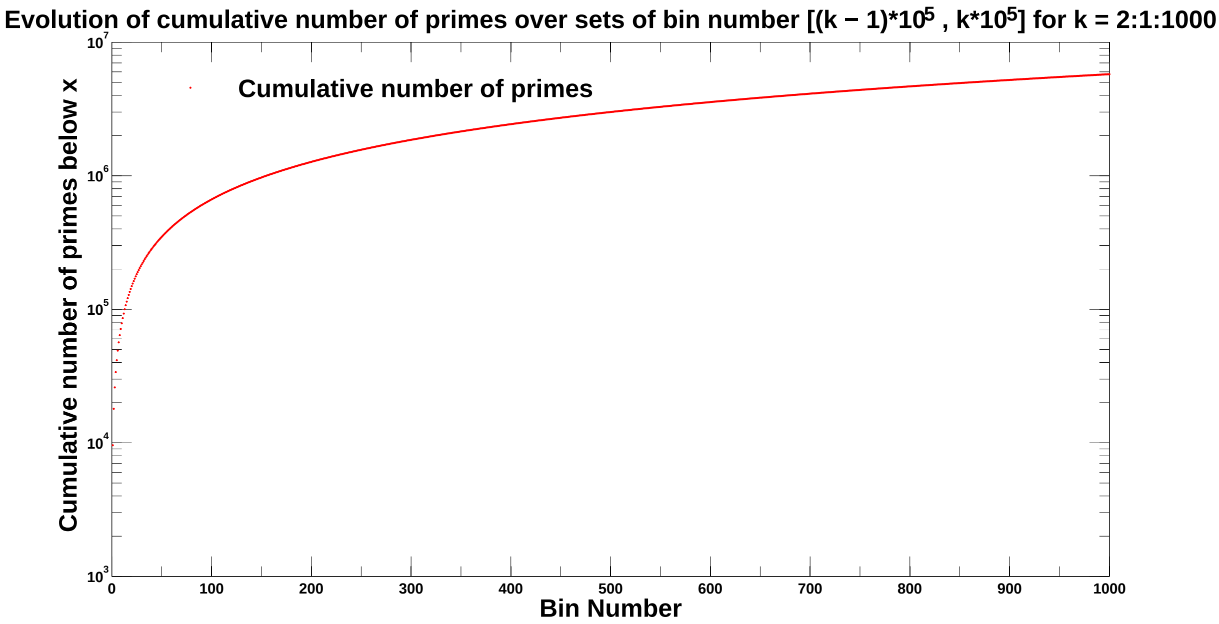

FIG. 2: Shown here is the evolution of the cumulative number of primes over sets of bin $[(k-1)*10^5, k*10^5]$ for $k \in \mathbb{N}$, $2 \leq k \leq 1000$.

TABLE I: A few statistics from preliminary runs on a high performance computing infrastructure. Shown below are results for composite odd numbers ending with 1. When compared with Table II, the preliminary execution times given in this table suggests potential breakthroughs towards computing - much faster and mush simpler - prime numbers, small and large.

Composite odd number ending with 1 and below $x =$	Execution time on a high performance computing infrastructure
10^9	01 hour 10 minutes
$5 * 10^9$	43 minutes
$10 * 10^9$	02 hours

TABLE II: A few statistics from preliminary runs on a high performance computing infrastructure: Shown below are results for prime numbers below a preassigned limit x. These are all deduced from associated list of composite odd numbers below x.

Prime numbers below $x =$	Number of prime numbers below x	Execution time on a high performance computing infrastructure
$0.1 * 10^9$	5761455	5 minutes
$0.25 * 10^9$	13679318	14 minutes
$0.5 * 10^9$	26355867	31 minutes
10^9	50847534	01 hour 53 minutes

IV. CONCLUSION AND FUTURE WORK

We have established a deterministic scheme that yields the complete list of prime numbers below any preassigned limit x. We have observed a considerable gain in computational speed and algorithmic simplicity towards obtaining complete lists of prime numbers both small or large. Our deterministic scheme shines a new and unique light on the importance of the set of composite odd numbers. From these composite odd numbers, non-composite odd number i.e prime numbers are deduced. At the core of the deterministic scheme is a set of eleven (11) generic tables fully characterized and leading to complete lists of ordered composite odd numbers below x. The said tables are coupled with a three-criteria systematic test applied on differences between pairs of the consecutive composite odd numbers obtained. Counting all the elements of the obtained list of prime numbers therefore provides a new and fertile approach in tackling the long standing problem of "how many prime numbers are there below any preassigned limit x. Our

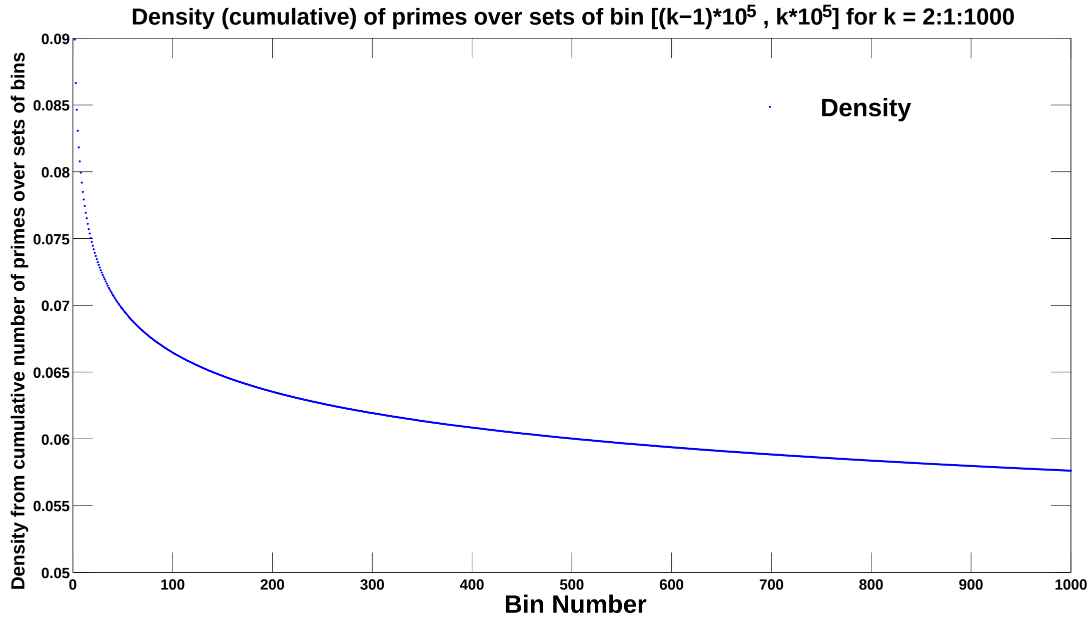

FIG. 3: Shown here is the density from the cumulative number of primes over sets of bin $[(k-1)*10^5, k*10^5]$ for $k \in \mathbb{N}$, $2 \leq k \leq 1000$. Visible here and in Figure 5 is the well-known function $\tilde{\rho}(x) \propto \frac{1}{log(x)}$, approximating $\rho(x)$ that can be deduced from the prime number theorem [2, 7]

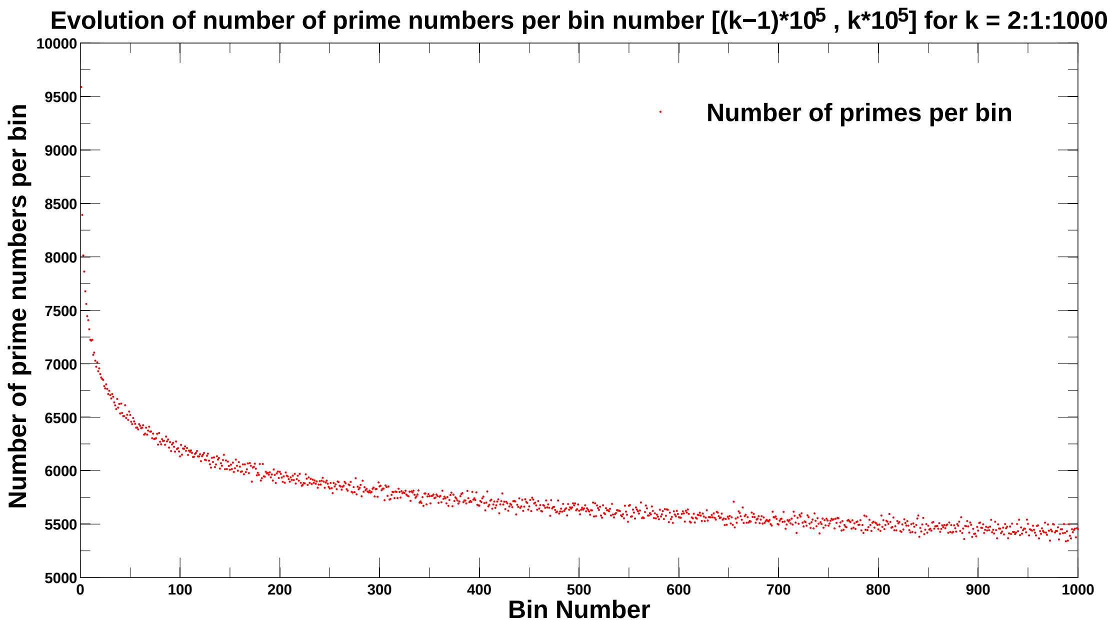

FIG. 4: Shown here is the evolution of the number of primes per bin number $[(k-1)*10^5, k*10^5]$ for $k \in \mathbb{N}$, $2 \leq k \leq 1000$. Evident here is the well known increased scarcity of prime numbers as the preassigned limit x increases.

scheme presented here suggests a solution to that problem in a simple, coherent and deterministic way. In a follow-up paper [3], we propose - based on our scheme - a new perspective on Riemann hypothesis. Thus highlighting the immense potential, for number theory and related fields, embedded in the new approach presented in this work.

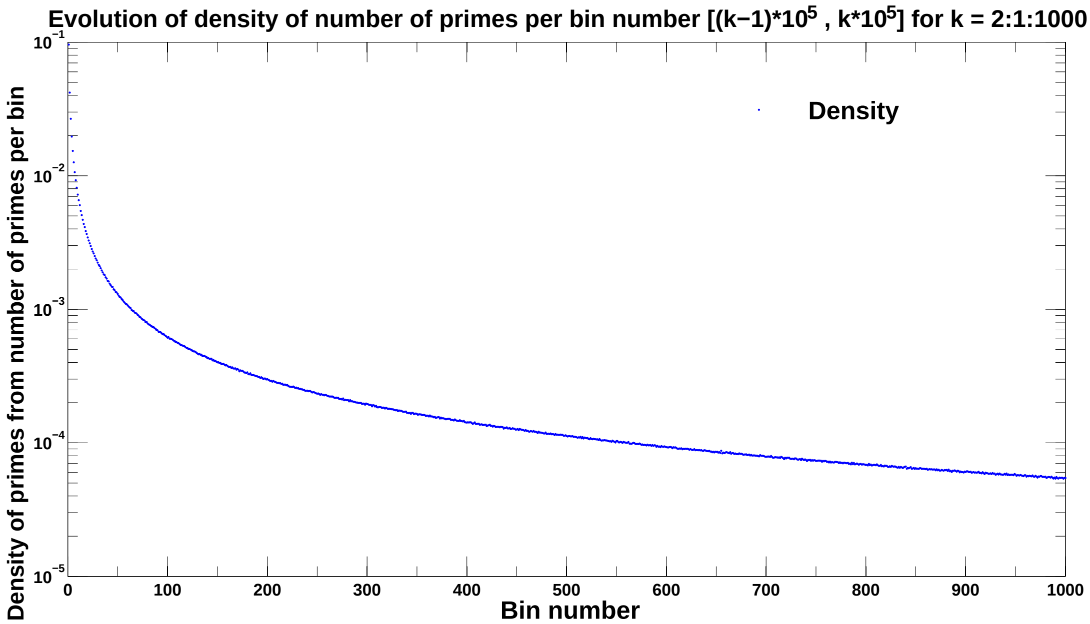

FIG. 5: Shown here is the evolution of the number of primes per bin number $[(k-1)*10^5, k*10^5]$ for $k \in \mathbb{N}$, $2 \leq k \leq 1000$.

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[1] *C. F. Gauss*, Disquisitiones Arithmeticae, 2nd ed., Yale University Press, 1965.
[2] *J. Hadamard*, Article in a journal, Bull. Soc. Math. France **24** (1986), 119.
[3] *G. Hawing* and *P. M. Okouma*, Riemann Hypothesis Revisited, (2016), Submitted.
[4] *A. E. Ingham*, The Distribution of Prime Numbers, 2nd ed., Cambridge University Press, 1990.
[5] *J. Korevaar*, Article in a journal, Math. Intell. **4** (1982), 108.
[6] *D. J. Newman*, Article in a journal, Am. Math. Month. **87** (1980), 693.
[7] *C. de la la Vallée-Poussin*, Article in a journal, Ann. Soc. Sci. Bruxelles **202** (1896), 183.

Appendix A: The Eleven Generic Tables for Our Deterministic Scheme

In each table and for each computation of $p = (10k + m)\,(10l + q)$, $k, m, l, q \in \mathbb{N}$, the constrain is that each product p has to be smaller than x, the preassigned limit.

TABLE III: Simplifying representation in table for $\{(10k_1+1)(10k_2+7),\ k_2\ \in \mathbb{N}\ and\ k_1 \in \mathbb{N}^*\}$ leading to composite odd numbers ending with 7

k_1	k_2	$(10k_1+1)$	$(10k_2+7)$
1	0	11	7
2	1	21	17
..	...	...	...
∞	∞	∞	∞

TABLE IV: Simplifying representation in table for $\{(10k_3+3)(10k_4+9),\ k_3,\ k_4\ \in \mathbb{N}\}$ leading to composite odd numbers ending with 7

k_3, k_4	$(10k_3+3)$	$(10k_4+9)$
0	3	9
1	13	19
2	23	29
...	...	...
∞	∞	∞

TABLE V: Simplifying representation in table for $\{(10k_1+3)(10k_2+7),\ k_1,\ k_2\ \in \mathbb{N}\}$ leading to composite odd numbers ending with 1

k_1, k_2	$(10k_1+3)$	$(10k_2+7)$
0	3	7
1	13	17
2	23	27
...	...	...
∞	∞	∞

TABLE VI: Simplifying representation in table for $\{(10k_3+9)\,(10k_4+9)\,,\ k_3,\ k_4\ \in \mathbb{N}\}$ leading to composite odd numbers ending with 1

k_3, k_4	$(10k_3+9)$	$(10k_4+9)$
0	9	9
1	19	19
2	29	29
...	...	...
∞	∞	∞

TABLE VII: Simplifying representation in table for $\{(10k_5+1)\,(10k_6+1)\,,\ k_5,\ k_6\ \in \mathbb{N}^*\}$ leading to composite odd numbers ending with 1

k_5, k_6	$(10k_5+1)$	$(10k_6+1)$
1	11	11
2	21	21
...	...	...
∞	∞	∞

TABLE VIII: Simplifying representation in table for $\{(10k_1+3)\,(10k_2+1)\,,\ k_1,\ k_2\ \in \mathbb{N}\ and\ k_2 \in \mathbb{N}^*\}$ leading to composite odd numbers ending with 3

k_1	k_2	$(10k_1+3)$	$(10k_2+1)$
0	1	3	11
1	2	13	21
.. .	...	...	...
∞	∞	∞	∞

TABLE IX: Simplifying representation in table for $\{(10k_3+7)\,(10k_4+9)\,,\ k_3,\ k_4\ \in \mathbb{N}\}$ leading to composite odd numbers ending with 3

k_3, k_4	$(10k_3+7)$	$(10k_4+9)$
0	7	9
1	17	19
...	...	...
∞	∞	∞

TABLE X: Simplifying representation in table for $\{(10k_1+1)\,(10k_2+9)\,,\ k_1,\ k_2\ \in \mathbb{N}\ and\ k_1 \in \mathbb{N}^*\}$ leading to composite odd numbers ending with 9

k_1	k_2	$(10k_1+1)$	$(10k_2+9)$
1	0	11	9
2	1	21	19
...	...	...	...
∞	∞	∞	∞

TABLE XI: Simplifying representation in table for $\{(10k_3+3)\,(10k_4+3)\,,\ k_3,\ k_4\ \in \mathbb{N}\}$ leading to composite odd numbers ending with 9

k_3, k_4	$(10k_3+3)$	$(10k_4+3)$
0	3	3
1	13	13
...	...	...
∞	∞	∞

TABLE XII: Simplifying representation in table for $\{(10k_5+7)(10k_6+7),\ k_5,\ k_6\ \in \mathbb{N}\}$ leading to composite odd numbers ending with 9

k_5, k_6	$(10k_5+7)$	$(10k_6+7)$
0	7	7
1	17	17
...	...	...
∞	∞	∞

TABLE XIII: Simplifying representation in table for $\{5(2k+1),\ k\ \in \mathbb{N}\}$ leading to composite odd numbers ending with 5

k	$5(2k+1)$
0	5
1	15
2	25
...	...
∞	∞