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Flux through a Markov chain

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Abstract

In this paper we study the flux through a finite Markov chain of a quantity, that we will call mass, which moves through the states of the chain according to the Markov transition probabilities. Mass is supplied by an external source and accumulates in the absorbing states of the chain. We believe that studying how this conserved quantity evolves through the transient (non-absorbing) states of the chain could be useful for the modelization of open systems whose dynamics has a Markov property.

1 Introduction

Markov chains, [1], [2], [3], [4], are probably the most popular models for stochastic dynamics. They have a huge range of applications, going from physics and chemistry, [5], [6], [7], to social science, [8], [9], including also informatics, [10], [11] or engineering, [13], [14], and biology [15], [16], [17]. In this paper, we are interested in the problem of a flux through a Markov chain.

The time evolution of a probability distribution in a conventional Markov chain may be seen as a redistribution of the probability among the states of the chain at any time step. From the point of view of dynamical systems this is a closed system. Likewise, it is natural to consider the case of an open system where we allow for a continue inflow as well as an associated outflow of a quantity that has the property of moving through the states of the chain according to the Markov transition probabilities. To describe the properties of this flow through the states of the chain is the main subject of our analysis. We believe that this could be interesting for the modelization of open physical or biological systems, where energy, mass or population is constantly injected from the outside, and is then dissipated or eliminated.

We consider a Markov Chain with a finite number of states $\{1, 2, \ldots, N\}$ in discrete time t. We note $\mathbf{P}$ be the corresponding $N \times N$ stochastic matrix of transition probabilities between states.

We will focus on a positive quantity μ, that we will call "mass". This quantity has the property of being transported in time from a state i to any state j according to the transition probability $\mathbf{P}_{ij}$, and to be additive. More explicitly, let μ_t^i be the mass present in state i at time t; when time passes from t to $t+1$ a fraction $\mu_t^i \mathbf{P}_{ij}$ of μ_t^i is transferred to state j. The total mass present in state j at time $t+1$ is the sum of the masses arriving from any state. This conservation law is encoded in the following Markov recurrence:

$$\mu_{t+1} = \mu_t \mathbf{P} \tag{1}$$

where μ is the row vector with positive entries representing the mass distribution, which is not necessarily normalized ($\sum_{i=1}^N \mu_t^i$ can be different from one).

We also suppose that mass is injected in the system from an external source at every instant of time t with (positive) distribution $\sigma_t = (\sigma_t^1, \ldots, \sigma_t^N)$ on the states of the chain.

We consider the case where some of the states of the chain are absorbing: once the system gets in these states, the probability of going back to any other state is zero. We will name any such state a sink. This guarantees that the incoming mass accumulates in the end in the absorbing states.

As a consequence, contrary to the case of usual absorbing Markov chains, we are concerned with a double open system since there is a continuous access as well as a departure of mass from the system itself.

A framework where similar issues are considered is the one of dynamics of populations modelled by branching processes, [19], [20]. In this context, other individuals can eventually come from the outside (branching processes with migration), but since they give birth at each time step to a random number

of individuals there is no conservation of "mass" in this case. This kind of phenomena have of course a Markov property, since the random future evolution depends on the past only through the state of the process in the present. The eventual formalisation in terms of Markov chains can be done either considering that the states of the Markov chain are defined by the number of individuals itself, or, in the case of multi-type branching processes, by the type of individuals (in this case the *mean matrix* plays the role of the Markov transition matrix).

Another context where these ideas could be applied is the modelization of traffic flow in transport networks. The incoming vehicles represent the source of mass, the outgoing roads play the role of sink states, the Markov transition probabilities apply to the road junctions or to the road destinations. This example is presented in section 6.

Let us now define more precisely how the injection and transport of mass in the chain take place, in order to compute the mass distribution $\mu_t = (\mu_t^1, \ldots, \mu_t^N)$ as a function of the source σ and of the Markov transition probabilities.

When time is incremented from t to $t+1$ two operations take place, in a definite order: first, the total mass in each state is redistributed according to the transition probability matrix $\mathbf{P}$; second, the external source injects the mass σ_t^i in each state i of the chain, and this mass sums up with the quantity μ_t^i already present in state i (if we had chosen to perform these operations in the inverse order, formulas would change a little, but the results would remain the same).

At time $t = 0$, let us note $\mu_{t=0} = \mu_0$ the row vector giving the mass initially present in the chain. When time passes from $t = 0$ to $t = 1$ the total mass is redistributed and then the mass $\sigma_{t=0} = \sigma_0$ injected in the chain, so that we have at $t = 1$:

$$\mu_{t=1} = \sigma_0 + \mu_0 \, \mathbf{P} \tag{2}$$

Similarly, when time passes from $t-1$ to t we have the relation

$$\mu_t = \sigma_{t-1} + \mu_{t-1} \, \mathbf{P} \tag{3}$$

which gives

$$\mu_t = \sigma_{t-1} + \sigma_{t-2} \, \mathbf{P} + \cdots + \sigma_0 \, \mathbf{P}^{t-1} + \mu_0 \, \mathbf{P}^t = \sum_{k=0}^{t-1} \sigma_{t-1-k} \mathbf{P}^k + \mu_0 \, \mathbf{P}^t \tag{4}$$

Without loss of generality, we will identify the absorbing states with the last states of the chain: if n is the number of absorbing states, we label them $\{N-n+1, N-n+2, \ldots, N\}$.

We also notice that the generalization of this approach to the case of infinite countable number of states is possible ([18]) with only minor modifications.

In the following, we will study the behavior of the mass distribution μ_t. We determine analytical expressions for it, in the cases where the source term is constant or periodic in time. We work out all calculations explicitly for a 3 states system, and finally discuss the case of random sources.

2 Constant source

We will study the special case where the same amount of mass is injected in the chain at every time step, namely

$$\sigma_t = \sigma_0 = (\sigma_0^1, \ldots, \sigma_0^N) \quad \forall\, t \tag{5}$$

From now on, we will suppose that the chain is initially empty, $\mu_0 = 0$. This assumption has no influence on the asymptotic distribution of mass, since when $t \gg 1$ the additional term $\overline{\mu}_0 \mathbf{Q}^t$ that would appear below in equation (10) following the term $\mu_0 \mathbf{P}^t$ in equation (4) tends to zero.

Equation (4) gives in this case

$$\mu_t = \sigma_0 \left(\mathbf{1} + \mathbf{P} + \mathbf{P}^2 + \cdots + \mathbf{P}^{t-1}\right) \tag{6}$$

One could be tempted to use the formulas of the geometric series to compute the recurrence as:

$$\mu_t = \sigma_0 \, (\mathbf{1} - \mathbf{P}^t)(\mathbf{1} - \mathbf{P})^{-1} \tag{7}$$

but this is incorrect because, due to the Perron-Frobenius maximal eigenvalue $\lambda_0 = 1$ of $\mathbf{P}$, the matrix $(\mathbf{1} - \mathbf{P})$ is not invertible. Of course the reason for this is the accumulation of mass in the sink states, since by definition for these states j we have $\mathbf{P}_{jj} = 1$ (and $\mathbf{P}_{ji} = 0 \;\; \forall i \neq j$).

Therefore, as in the case of classical absorbing chains, we shall separate the sinks from the other states in order to analyse the flow of mass in the system. Let us then define the matrix $\mathbf{Q}$ as the $(N-n) \times (N-n)$ submatrix of $\mathbf{P}$ restricted to the non-absorbing states $\{1, 2, \ldots, N-n\}$:

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{8}$$

Here $\mathbf{R}$ is the $(N-n) \times n$ submatrix of the fractions of mass transfers from the non-absorbing states to the sinks and $\mathbf{1}$ is the $n \times n$ identity matrix. A simple algebraic computation gives:

$$\mathbf{P}^j = \begin{pmatrix} \mathbf{Q}^j & (\mathbf{Q}^{j-1} + \mathbf{Q}^{j-2} + \cdots + 1)\mathbf{R} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{9}$$

And finally:

$$\mathbf{1} + \mathbf{P} + \cdots + \mathbf{P}^{t-1} = \begin{pmatrix} \mathbf{1} + \mathbf{Q} + \cdots + \mathbf{Q}^{t-1} & \mathbf{R} + (\mathbf{Q} + \mathbf{1})\mathbf{R} + \cdots + (\mathbf{Q}^{t-2} + \cdots + \mathbf{1})\mathbf{R} \\ \mathbf{0} & t\mathbf{1} \end{pmatrix} \tag{10}$$

We conclude from (10) that the time evolution of the mass distribution restricted to the non-absorbing states, denoted $\overline{\mu}$, is given by:

$$\overline{\mu}_t = \overline{\sigma}_0 (\mathbf{1} + \mathbf{Q} + \mathbf{Q}^2 + \cdots + \mathbf{Q}^{t-1}) \tag{11}$$

where $\overline{\sigma}_0 = (\sigma_0^1, \ldots, \sigma_0^{N-n})$.

If $\mathbf{R} \neq 0$, the matrix $\mathbf{Q}$ is a substochastic matrix with all eigenvalues with modulus strictly less than 1 insuring that the geometric series converges. We get:

$$\overline{\mu}_t = \overline{\sigma}_0 (\mathbf{1} - \mathbf{Q}^t)(\mathbf{1} - \mathbf{Q})^{-1} \tag{12}$$

so that

$$\lim_{t \to \infty} \overline{\mu}_t = \overline{\sigma}_0 (\mathbf{1} - \mathbf{Q})^{-1} \tag{13}$$

As a direct consequence of (13) we have:

Fixed Point Theorem: There is an invariant distribution $\overline{\mu}$ of the non-absorbing part of the system that reads:

$$\overline{\mu} = \overline{\sigma}_0(\mathbf{1} - \mathbf{Q})^{-1} \tag{14}$$

We can also remark that the fixed point (14) equation can be written as

$$\overline{\sigma}_0 + \overline{\mu}\mathbf{Q} = \overline{\mu} \tag{15}$$

which clearly shows that $\overline{\mu}$ is a fixed point of an affine transformation contrary to the usual iteration of Markov chains which is instead a linear transformation.

Mass balance: Equation (15) has a clear physical meaning. Each component of

$$\overline{\sigma}_0 = \overline{\mu}(\mathbf{1} - \mathbf{Q}) \tag{16}$$

corresponds to the mass balance of a non-absorbing state. The left hand side represents the incoming of the external mass at each time step. The right hand side gives the internal balance of the state with each of the others as well as with the sinks.

If the matrix $\mathbf{Q}$ is diagonalisable, the computation of the invariant distribution (14) may be done using the (non-ortoghonal) basis of eigenvectors $\overline{\varphi}_i$ of $\mathbf{Q}$. By computing first the components α_i of $\overline{\sigma}_0$ on this basis:

$$\overline{\sigma}_0 = \sum_{i=1}^{N-n} \alpha_i \overline{\varphi}_i \tag{17}$$

and then using the linearity of $\mathbf{Q}$ we get:

$$\overline{\mu} = \sum_{i=1}^{N-n} \alpha_i \, \frac{1}{1-\lambda_i} \, \overline{\varphi}_i \tag{18}$$

where the λ_i are the eigenvalues of $\mathbf{Q}$. Notice that the fixed point relation (15) also allows the computation of the stationary distribution by solving a system of $N-n$ linear equations. In the next section we shall use both formulas to compute the stationary distribution in a simple example.

The mass distribution $\overline{\mu}_t$ at time t can also easily be expressed as a function of the eigenvalues and eigenvectors of $\mathbf{Q}$. Equations (12) and (14) give

$$\overline{\mu}_t = \overline{\mu}\,(\mathbf{1} - \mathbf{Q}^{\mathbf{t}}) \tag{19}$$

so that

$$\overline{\mu}_t = \overline{\mu} - \sum_{i=1}^{N-n} \alpha_i \, \frac{\lambda_i^t}{1-\lambda_i} \, \overline{\varphi}_i \tag{20}$$

which shows how $\overline{\mu}_t$ tends exponentially to $\overline{\mu}$ as time grows.

Notice that the eigenvectors $\overline{\varphi}$ of $\mathbf{Q}$ can be computed independently from the complete eigenvectors φ of $\mathbf{P}$. In effect, thanks to the particular form of $\mathbf{P}$ the equation $\varphi\mathbf{P} = \lambda\varphi$ for the eigenvalues and eigenvectors of (8) becomes:

$$(\varphi^1, \ldots, \varphi^{N-n})\mathbf{Q} = \lambda(\varphi^1, \ldots, \varphi^{N-n}) \tag{21}$$

$$(\varphi^1, \ldots, \varphi^{N-n})\mathbf{R} + (\varphi^{N-n+1}, \ldots, \varphi^N) = \lambda(\varphi^{N-n+1}, \ldots, \varphi^N) \tag{22}$$

The system (21) of $N-n$ equations determines the eigenvectors of $\mathbf{Q}$, $\overline{\varphi} = (\varphi^1, \ldots, \varphi^{N-n})$. Once these are solved, the n remaining equations (22) allow to compute the other components $(\varphi^{N-n+1}, \ldots, \varphi^N)$ of the eigenvectors of $\mathbf{P}$.

The n eigenvectors relative to the absorbing states are such that their first $N-n$ components are equal to zero: $\varphi = (0, 0, \ldots, 0, \varphi^{N-n+1}, \ldots, \varphi^N)$. Equations (22) then imply that the corresponding eigenvalues are equal to one, as it should be.

In the case where $\mathbf{Q}$ is not diagonalisable, i.e., it does not possess $N-n$ linearly independent eigenvectors, one can use a Jordan base to compute the invariant mass distribution through equation (12). Let $\lambda_1, \ldots, \lambda_m$ ($m < N-n$ in this case) be the eigenvalues of $\mathbf{Q}$, and $k_1, \ldots, k_m$ their respective multiplicities. Let $\{\overline{\varphi}_{i,1}, \ldots, \overline{\varphi}_{i,k_i}\}$ be the Jordan vectors generating the subspace relative to the eigenvalue λ_i, that satisfy:

$$\overline{\varphi}_{i,j}\mathbf{Q} = \lambda_i \overline{\varphi}_{i,j} \quad \text{if} \quad j = 1\ , \qquad \overline{\varphi}_{i,j}\mathbf{Q} = \lambda_i \overline{\varphi}_{i,j} + \overline{\varphi}_{i,j-1} \quad \text{if} \quad j = 2, \ldots, k_i \tag{23}$$

(notice that with this definition, the Jordan blocks have ones below the diagonal). Let

$$\overline{\sigma}_0 = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \alpha_{i,j} \overline{\varphi}_{i,j} \qquad \text{and} \qquad \overline{\mu}_t = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \beta_{i,j}^t \overline{\varphi}_{i,j} \tag{24}$$

be the decompositions of the source and the mass distribution at time t on this basis. Starting from equation (12), that we may rewrite as

$$\overline{\mu}_t - \overline{\mu}_t \mathbf{Q} = \overline{\sigma}_0 - \overline{\sigma}_0 \mathbf{Q}^{\mathbf{t}} \tag{25}$$

the coefficients $\beta_{i,j}^t$ can be expressed as functions of the $\alpha_{i,j}$ in the following way, for all $i = 1, \ldots, m$:

$$(1-\lambda_i) \sum_{j=1}^{k_i} \beta_{i,j}^t \overline{\varphi}_{i,j} - \sum_{j=2}^{k_i} \beta_{i,j}^t \overline{\varphi}_{i,j-1} = \tag{26}$$

$$= \sum_{j=1}^{k_i} \alpha_{i,j} \overline{\varphi}_{i,j} - \sum_{l=0}^{t} \binom{t}{l} \lambda_i^{t-l} \sum_{j=l+1}^{k_i} \alpha_{i,j} \overline{\varphi}_{i,j-l}$$

where by definition any sum $\sum_{j=n_1}^{n_2}$ is zero if $n_1 > n_2$. Expression (26) allows then to compute the $\beta_{i,j}^t$ by identifying the coefficients of the independent base vectors $\overline{\varphi}_{i,j}$.

We will now explicitly give the expression of the asymptotic mass distribution in the Jordan base. Let us first remark that the double sum on l, j on the right-hand-side of equation (26) goes to zero exponentially as time goes to infinity, since its terms vanish for $l \geq k_i$, and thus the index l is bounded as t grows. After a straightforward calculation, we get for the coefficients $\beta_{i,j}^\infty$ of the asymptotic distribution $\overline{\mu}$:

$$\beta_{i,j}^\infty = \sum_{l=j}^{k_i} \frac{1}{(1-\lambda_i)^{l-j+1}}\ \alpha_{i,l} \qquad i = 1, \ldots, m \tag{27}$$

which coincides with expression (18) in the diagonalisable case $k_i = 1 \; \forall \; i$.

There is an interesting difference between the quasi stationary distribution, [18] [25], associated to the linear Markov case corresponding to the recurrence

$$\overline{\mu}_t = \overline{\mu}_{t-1}\mathbf{Q} \tag{28}$$

and the asymptotic distribution $\overline{\mu}$ of our affine model. In the linear case the limit of the distribution (28) is zero and the conditioned distribution (to the trajectories staying in the internal system) converges to the direction of the eigenvector $\overline{\varphi}_1$. The logharitm of λ_1 plays the role of a pressure in the Gibbs formalism of statistical mechanics, [25].

Here $\overline{\mu}$ depends of σ_0 and has contributions depending on the λ_i from all the eigenvectors of $\mathbf{Q}$. In particular, as a fixed point of an affine transformation, $\overline{\mu}$ is not zero. In our case, the role of the largest eigenvalue λ_1 is to give the dominant exponential rate at which $\overline{\mu}_t$ approaches $\overline{\mu}$ at large times, the gap $\overline{\mu}_t - \overline{\mu}$ being approximately in the direction of the corresponding eigenvector $\overline{\varphi}_1$.

3 Periodic source

Until now, we have considered the case of a source that is constant in time: $\sigma_t = \sigma_0$. We will now treat the case of a source that is periodic in time. In the case of period $T = 2$ we will have:

$$\begin{aligned} \sigma_t = \sigma_0 = (\sigma_0^1, \ldots, \sigma_0^N) \qquad t = 0, 2, 4, \ldots \\ \sigma_t = \sigma_1 = (\sigma_1^1, \ldots, \sigma_1^N) \qquad t = 1, 3, 5, \ldots \end{aligned} \tag{29}$$

Expression (4) is still valid and gives in this case (always supposing that the chain is initially empty, $\mu_0 = 0$):

$$\mu_t = \sigma_0 \, \mathbf{P}^{t-1} + \sigma_1 \mathbf{P}^{t-2} + \sigma_0 \mathbf{P}^{t-3} + \sigma_1 \mathbf{P}^{t-4} + \cdots + \sigma_{t-1} \tag{30}$$

where $\sigma_{t-1} = \sigma_0$ if t is odd and $\sigma_{t-1} = \sigma_1$ if t is even. The distribution of mass for the non-absorbing states $\overline{\mu}_t$ at time t is then given by

$$\overline{\mu}_{t=1} = \overline{\sigma}_0 \tag{31}$$

$$\overline{\mu}_t = (\overline{\sigma}_0 \mathbf{Q} + \overline{\sigma}_1)(1 - \mathbf{Q}^t)(1 - \mathbf{Q}^2)^{-1} \qquad t = 2, 4, 6, \ldots \tag{32}$$

$$\overline{\mu}_t = \overline{\sigma}_0 (1 - \mathbf{Q}^{t+1})(1 - \mathbf{Q}^2)^{-1} + \overline{\sigma}_1 \mathbf{Q}(1 - \mathbf{Q}^{t-1})(1 - \mathbf{Q}^2)^{-1} \qquad t = 3, 5, 7, \ldots \tag{33}$$

So, as t grows, the mass distribution oscillates with the same period as the source between the two asymptotic values:

$$\overline{\mu}_{even} = (\overline{\sigma}_0 \mathbf{Q} + \overline{\sigma}_1)\,(1 - \mathbf{Q}^2)^{-1} \qquad \text{and} \qquad \overline{\mu}_{odd} = (\overline{\sigma}_0 + \overline{\sigma}_1 \mathbf{Q})\,(1 - \mathbf{Q}^2)^{-1} \tag{34}$$

The generalization to a source of any other period is straightforward. For a source of period T, the asymptotic mass distribution oscillates with the same period as the source between the T asymptotic values:

$$\begin{aligned} \overline{\mu}_{(T)} &= (\overline{\sigma}_0 \mathbf{Q}^{T-1} + \overline{\sigma}_1 \mathbf{Q}^{T-2} + \cdots + \overline{\sigma}_{T-2}\mathbf{Q} + \overline{\sigma}_{T-1})\,(1 - \mathbf{Q}^T)^{-1} \\ \overline{\mu}_{(T+1)} &= (\overline{\sigma}_0 + \overline{\sigma}_1 \mathbf{Q}^{T-1} + \cdots + \overline{\sigma}_{T-2}\mathbf{Q}^2 + \overline{\sigma}_{T-1}\mathbf{Q})\,(1 - \mathbf{Q}^T)^{-1} \\ \cdots \\ \overline{\mu}_{(2T-1)} &= (\overline{\sigma}_0 \mathbf{Q}^{T-2} + \overline{\sigma}_1 \mathbf{Q}^{T-3} + \cdots + \overline{\sigma}_{T-2} + \overline{\sigma}_{T-1}\mathbf{Q}^{T-1})\,(1 - \mathbf{Q}^T)^{-1} \end{aligned} \tag{35}$$

The asymptotic periodic orbit (35) is therefore made of the fixed point of the system corresponding to $\mathbf{Q}^T$ driven by a constant source, and of its iterates, as prescribed by equation (14).

4 The 3-states model

We will now discuss the simplest but non trivial case where all the quantities of interest are easily and analytically computable. We consider the case of a three states Markov chain ($N = 3$), with one absorbing state ($n = 1$). Moreover, we suppose that only one of the two transient states is directly connected to the sink. The transition matrix reads then:

$$\mathbf{P} = \begin{pmatrix} (1-p) & p & 0 \\ q & (1-q-r) & r \\ 0 & 0 & 1 \end{pmatrix} \tag{36}$$

where $0 < p, q < 1$ and $r < (1-q)$ so that the matrix $\mathbf{P}$ is stochastic.

The matrix $\mathbf{Q}$ which is the restriction of $\mathbf{P}$ to the transient states is:

$$\mathbf{Q} = \begin{pmatrix} (1-p) & p \\ q & (1-q-r) \end{pmatrix} \tag{37}$$

whose eigenvalues λ_1, λ_2 are:

$$\lambda_1 = \frac{1}{2}\left[2 - p - q - r + \sqrt{p^2 + 2p(q-r) + (q+r)^2}\right] \tag{38}$$

$$\lambda_2 = \frac{1}{2}\left[2 - p - q - r - \sqrt{p^2 + 2p(q-r) + (q+r)^2}\right] \tag{39}$$

The two eigenvalues coincide in the case $q = 0$, $p = r$. Here we will rather discuss the non-degenerate case $\lambda_1 \neq \lambda_2$. The eigenvectors $\overline{\varphi}_1$, $\overline{\varphi}_2$ (that we take to be normalized) are then:

$$\overline{\varphi}_1 = \frac{1}{\sqrt{\left(\frac{q}{\lambda_1+p-1}\right)^2 + 1}} \left(\frac{q}{\lambda_1 + p - 1}, 1\right) \tag{40}$$

$$\overline{\varphi}_2 = \frac{1}{\sqrt{\left(\frac{q}{\lambda_2+p-1}\right)^2 + 1}} \left(\frac{q}{\lambda_2 + p - 1}, 1\right) \tag{41}$$

We want to compute the asymptotic distribution of mass $\overline{\mu}$ given by equation (14) corresponding to a general constant source $\overline{\sigma}_0 = (\gamma_a, \gamma_b)$, that we can express as a linear combination of the elementary sources $\overline{\sigma}_{0,a} = (1, 0)$ (mass is injected only in the first state of the chain) and $\overline{\sigma}_{0,b} = (0, 1)$ (mass is injected only in the second state of the chain) : $\overline{\sigma}_0 = \gamma_a \overline{\sigma}_{0,a} + \gamma_b \overline{\sigma}_{0,b}$. If $\overline{\mu}_a$, $\overline{\mu}_b$ are the asymptotic distributions corresponding to $\overline{\sigma}_{0,a}$, $\overline{\sigma}_{0,b}$ we have by linearity:

$$\overline{\mu} = \gamma_a \overline{\mu}_a + \gamma_b \overline{\mu}_b \tag{42}$$

To compute $\overline{\mu}_a$, one possibility is to use directly equation (16):

$$(1, 0) = (\overline{\mu}_a^1, \overline{\mu}_a^2) \begin{pmatrix} p & -p \\ -q & (q+r) \end{pmatrix} \tag{43}$$

which gives

$$\overline{\mu}_a = \left(\frac{q+r}{rp}, \frac{1}{r}\right) \tag{44}$$

The same result can of course be obtained by decomposing the source $\overline{\sigma}_{0,a}$ on the eigenvectors (40) and (41) of $\mathbf{Q}$. As in the case of the standard Markov chains this computation allows a geometrical interpretation of the time evolution of the mass distribution $\overline{\mu}_t$ as it will be seen below.

The relation

$$\overline{\sigma}_{0,a} = (1,0) = \alpha_a \overline{\varphi}_1 + \beta_a \overline{\varphi}_2 \tag{45}$$

gives

$$\alpha_a = \frac{\overline{\varphi}_2^2}{\overline{\varphi}_1^1\overline{\varphi}_2^2 - \overline{\varphi}_1^2\overline{\varphi}_2^1}\ , \qquad \beta_a = -\frac{\overline{\varphi}_1^2}{\overline{\varphi}_1^1\overline{\varphi}_2^2 - \overline{\varphi}_1^2\overline{\varphi}_2^1} \tag{46}$$

Therefore

$$\overline{\mu}_a = \alpha_a \frac{1}{1-\lambda_1}\overline{\varphi}_1 + \beta_a \frac{1}{1-\lambda_2}\overline{\varphi}_2 \tag{47}$$

where λ_1 and λ_2 are the eigenvalues (38) and (39) of $\mathbf{Q}$.

The same kind of computation for the source $\overline{\sigma}_{0,b} = (0,1)$ gives, for its decomposition on the eigenvectors of $\mathbf{Q}$:

$$\alpha_b = \frac{\overline{\varphi}_2^1}{\overline{\varphi}_1^2\overline{\varphi}_2^1 - \overline{\varphi}_1^1\overline{\varphi}_2^2}\ , \qquad \beta_b = -\frac{\overline{\varphi}_1^1}{\overline{\varphi}_1^2\overline{\varphi}_2^1 - \overline{\varphi}_1^1\overline{\varphi}_2^2} \tag{48}$$

and for the fixed point distribution:

$$\overline{\mu}_b = \alpha_b \frac{1}{1-\lambda_1}\overline{\varphi}_1 + \beta_b \frac{1}{1-\lambda_2}\overline{\varphi}_2 = \left(\frac{q}{rp}, \frac{1}{r}\right) \tag{49}$$

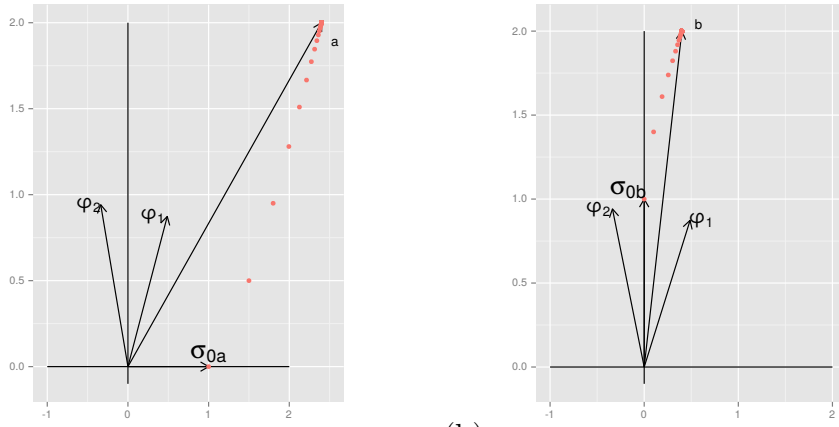
For a general input distribution $\overline{\sigma}_0 = (\gamma_a, \gamma_b)$ we have then:

$$\overline{\mu} = \left((\gamma_a+\gamma_b)\frac{q}{rp} + \gamma_a\frac{1}{p}, (\gamma_a+\gamma_b)\frac{1}{r}\right) \tag{50}$$

The mass distribution $\overline{\mu}_t$ tends exponentially in time to (50) in the way expressed by equation (20). Figure 1 gives a geometrical picture of the situation. There we see two trajectories represented in the two-dimensional phase space: one corresponding to the source $\overline{\sigma}_{0,a}$: Figure 1(a), the other to $\overline{\sigma}_{0,b}$: Figure 1(b). They are essentially driven by $\overline{\varphi}_1$ but with a smaller component on the direction $\overline{\varphi}_2$. The combination of these two dynamical components sums up to the two different asymptotes according to (47) and (49).

Any asymptotic mass distribution expressed by equation (50) can be rapidly reached by the chain, thanks to the exponential convergence properties. If we suppose that the transition probabilities (i.e., the parameters p, q, r) are fixed this can be done by choosing a suitable source (γ_a, γ_b). In a situation where, on the contrary, the source is imposed but the structure of the Markov chain can be modified, any mass distribution can be reached by varying the parameters p, q, r.

In Figure 2, we show a numerical simulation of the time evolution of the mass distribution $\overline{\mu}_t$ in a case where a unit of mass is injected in the second state of the Markov chain at every time step (i.e., the constant source term is $\overline{\sigma}_0 = (0,1)$), and the transition probabilities are modified when the system has almost attained the asymptotic regime corresponding to the initial value of the parameters $p = 1/2$,



(a) (b)

Figure 1: The trajectories of μ_t (dots) in time corresponding to the sources $\overline{\sigma}_{0,a}$ (Figure (a)) and $\overline{\sigma}_{0,b}$ (Figure (b)), represented in the two-dimensional phase space. Here $(p, q, r) = (0.5, 0.1, 0.5)$.

$q = 1/2$, $r = 1/3$ ($\overline{\mu} = (3, 3)$). To reach a situation where the second state contains 1.5 times more mass than the first, the transition probability $p = 1/2$ is changed to $p' = 3/4$ at $t = 51$. The effect of this is to decrease the asymptotic quantity of mass in the first state by a factor 2/3, as given by formula (49): at $t = 60$ the mass distribution in the chain has reached the new asymptotic value $\overline{\mu}' = (2, 3)$. Notice that the memory of the mass $\overline{\mu}$ present in the chain at the moment when p is changed is rapidly lost, since its contribution after t times steps is $\overline{\mu}\mathbf{Q}^t$, which again goes exponentially to zero as t grows.

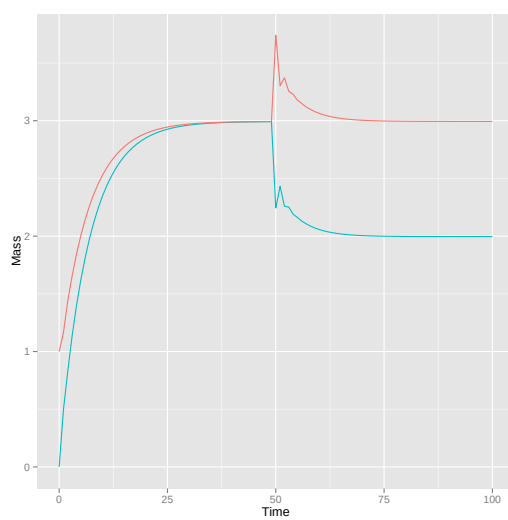


Figure 2: Time evolution of the mass distribution $\overline{\mu}_t$ in the case where the constant source is $\overline{\sigma}_0 = (0, 1)$, the chain is initially defined by the parameters $p = 1/2$, $q = 1/2$, $r = 1/3$, and is initially empty. After 50 time steps, the transition probability p is changed to $p' = 3/4$, while q, r keep the same value. Bottom/Blue line: first component of $\overline{\mu}_t$, the amount of mass in the first state of the chain. Top/Red line: second component of $\overline{\mu}_t$, the amount of mass in the second state of the chain.

Finally, in Figure 3 we illustrate the case of a time dependent source of period two, by a numerical simulation showing that the mass distribution $\overline{\mu}_t$ rapidly reaches a period-2 asymptotic state, according to equation (34).

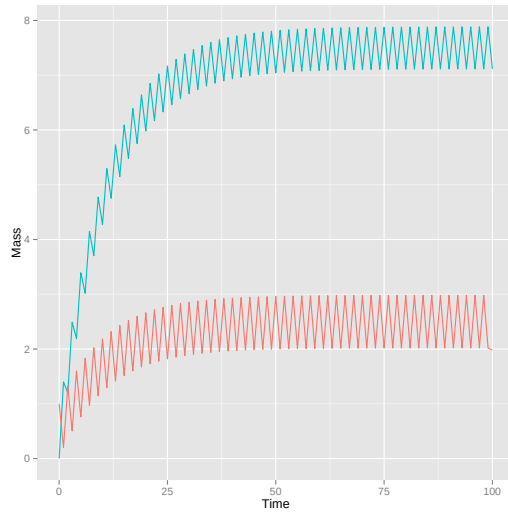


Figure 3: Time evolution of the mass distribution $\overline{\mu}_t$ for a periodic source defined by $\overline{\sigma}_0 = (0, 1)$, $\overline{\sigma}_1 = (1, 0)$. The values of the transition probabilities are $p = 0.8$, $q = 0.4$, $r = 0.4$. For times larger than $t = 25$, the system is in the period-2 asymptotic regime described by equation (34). Top/Blue line: first component of $\overline{\mu}_t$, the amount of mass in the first state of the chain. Bottom/Red line: second component of $\overline{\mu}_t$, the amount of mass in the second state of the chain.

5 Trajectories of individual particles and fluctuating source

The preceding sections are devoted to the dynamics of the mass distribution. Here we take another viewpoint, describing the trajectories of "individual particles". The individual trajectories are probabilistic realizations of the process and cylinders are the basic sets to be considered in this case, [23]. A cylinder is a set $[s_1, s_2, \dots s_m]_t^{t+m-1}$ of trajectories having in common a given path of states $[s_1, s_2, \dots s_m]$ during a fixed time window $[t, t+m-1]$. Clearly for an "individual particle" to follow a path during the interval $[t, t+m-1]$ it is necessary to be inside the system at time t. The trajectories of the particles entering the system after this time cannot belong to a cylinder based on $[t, t+m-1]$. On the other hand Markovianity implies that all the information needed to treat trajectories starting before t lies in the measure μ_t. Therefore the mass of the cylinder $[s_1, s_2, \dots s_m]_t^{t+m-1}$ is given by :

$$\mu[s_1, s_2, \dots s_m]_t^{t+m-1} = \mu_t^{s_1} \mathbf{P}_{s_1 s_2} \mathbf{P}_{s_2 s_3} \dots \mathbf{P}_{s_{m-1} s_m} \tag{51}$$

Our process is a sum of an infinite sequence of independent Markov chains, one for each starting time. Some natural questions about the statistics of the individual trajectories have a computational answer using the fundamental matrix $\mathbf{Z}$ of this process, [3], [4]. Not surprisingly, the matrix $\mathbf{Z} = (\mathbf{1} - \mathbf{Q})^{-1}$ also plays a central role in all the analysis before.

An interesting situation that may be analysed using the notion of cylinders is the one of a time dependent source whose fluctuations are defined in terms of a random walk on the set of the extremal distribution sources $\mathcal{S} = \{\overline{e}_i, 1 \leq i \leq N-n\}$ with transition probability matrix $\mathbf{S}$ that we take ergodic and with prescribed initial distribution ξ_i. By definition, the probability that the source injects mass $\overline{e}_i$ at time t and mass $\overline{e}_j$ at time $t+1$ is given by S_{ij} for any t (where $\overline{e}_i$ is the $(N-n)$-dimensional row vector with zero entries everywhere except a 1 at position i).

Recall that if the source injects mass $\overline{e}_i$ at time t we get:

$$\overline{\mu}_{t+1} = \overline{e}_i + \overline{\mu}_t \mathbf{Q} \tag{52}$$

Therefore to any realisation $(\overline{e}_{i_0}, \dots, \overline{e}_{i_t}, \dots)$ of this random walk defining the time evolution of the source will correspond a dynamical trajectory on the space of mass distributions; the trajectory will be given by the equivalent of equation (4) for the non-absorbing states:

$$\overline{\mu}_t = \overline{e}_{i_{t-1}} \mathbf{1} + \overline{e}_{i_{t-2}} \mathbf{Q} + \cdots + \overline{e}_{i_1} \mathbf{Q}^{t-2} + \overline{e}_{i_0} \mathbf{Q}^{t-1} \tag{53}$$

The interesting point is that the probability of a finite section $[\overline{\mu}]_t^{t+s}$ to be observed is correlated to the probability of the corresponding cylinder in the random walk on $\mathcal{S}$, in particular the asymptotic behavior of the system will be an itinerary visiting the fixed point distributions according the equilibrium distribution of the source process $\mathbf{S}$.

5.1 Contraction Property

Let $\overline{\mu}_{\infty,i}$ denote the asymptotic mass distribution corresponding to a constant source $\overline{e}_i$. We have, according to equation (14):

$$\overline{\mu}_{\infty,i} = \overline{e}_i (\mathbf{1} - \mathbf{Q})^{-1} \tag{54}$$

or, equivalently

$$\overline{\mu}_{\infty,i} = \overline{e}_i + \overline{\mu}_{\infty,i} \mathbf{Q} \tag{55}$$

Using (52) and (55) we have:

$$\overline{\mu}_{t+1} - \overline{\mu}_{\infty,i_t} = (\overline{\mu}_t - \overline{\mu}_{\infty,i_t})\, \mathbf{Q} \tag{56}$$

Now, see for example [26], since the spectral radius $\rho(\mathbf{Q})$ is strictly smaller than one, for each sufficiently small positive ϵ we can choose an appropriate subordinate norm such that there is $c \leq \rho(\mathbf{Q}) + \epsilon < 1$ and:

$$\|\overline{\mu}_{t+1} - \overline{\mu}_{\infty,i_t}\| \leq \|\overline{\mu}_t - \overline{\mu}_{\infty,i_t}\| \|\mathbf{Q}\| \leq c \|\overline{\mu}_t - \overline{\mu}_{\infty,i_t}\| \tag{57}$$

Equation (57) means that, if the source switches to the value $\overline{e}_{i_t}$ at time t, then the mass distribution reacts by exponentially approaching to the asymptotic value $\overline{\mu}_{\infty,i_t}$. The smaller the spectral radius of $\mathbf{Q}$, the closer $\overline{\mu}_{t+1}$ gets to the asymptotic value $\overline{\mu}_{\infty,i_t}$ corresponding to injection of mass $\overline{e}_{i_t}$ at time $t+1$. Notice that this result, [26], is also true for Jordan block matrixs.

5.2 Asymptotic Distribution Average

We can also give the time average $\overline{\mu}_{av}$ of the mass distribution, that we can compute using the ergodic properties of the transition matrix $\mathbf{S}$. $\overline{\mu}_{av}$ is obtained by summing on all the possible $\overline{\mu}_t$, weighted by the corresponding probability, and taking the limit $t \to \infty$. Recall that to each cylinder $[\overline{e}_{i_0}, \dots, \overline{e}_{i_{t-1}}]$

(having measure $\xi_{i_0} S_{i_0 i_1} S_{i_1 i_2} \dots S_{i_{t-2} i_{t-1}}$) corresponds by (53) a given $\overline{\mu}_t$. Also $\xi_{i_0} = 1$ in our model since the initial distribution of the source is set with probability 1 in one of the states $\overline{e}_{i_0}$ of the system.

Therefore,

$$\overline{\mu}_{av} = \lim_{t\to\infty} \sum_{i_1 \dots i_{t-1}} S_{i i_1} S_{i_1 i_2} \dots S_{i_{t-2} i_{t-1}} (\overline{e}_{i_{t-1}} \mathbf{1} + \overline{e}_{i_{t-2}} \mathbf{Q} + \dots + \overline{e}_i \mathbf{Q}^{t-1}) \tag{58}$$

Equation (58) says that $\overline{\mu}_{av}$ is the i-th row of the matrix $\mathbf{\Gamma}_\infty = \lim_{t\to\infty} \mathbf{\Gamma}(t)$, where we have defined:

$$\mathbf{\Gamma}(t) = \sum_{k=0}^{t-1} \mathbf{S}^k \mathbf{Q}^{t-k-1} \tag{59}$$

Proof:

$$\sum_{i_1 \dots i_{t-1}} S_{i i_1} S_{i_1 i_2} \dots S_{i_{k-1} i_k} S_{i_k i_{k+1}} \dots S_{i_{t-2} i_{t-1}} \overline{e}_{i_k} \mathbf{Q}^{t-k-1} = \sum_{i_1 \dots i_k} S_{i i_1} S_{i_1 i_2} \dots S_{i_{k-1} i_k} \overline{e}_{i_k} \mathbf{Q}^{t-k-1} \tag{60}$$

since

$$\sum_{i_{k+1} \dots i_{t-1}} S_{i_k i_{k+1}} \dots S_{i_{t-2} i_{t-1}} = 1 \tag{61}$$

From the definition of $\mathbf{\Gamma}(t)$, equation (59), it follows that :

$$\mathbf{\Gamma}(t+1) = \mathbf{\Gamma}(t)\mathbf{Q} + \mathbf{S}^t \tag{62}$$

which implies

$$\mathbf{\Gamma}_\infty = \mathbf{S}_\infty (\mathbf{1} - \mathbf{Q})^{-1} \tag{63}$$

where $\mathbf{S}_\infty = \lim_{t\to\infty} \mathbf{S}^t$ is the matrix whose identical rows are the stationary distribution $s = (s^1, \dots, s^{N-n})$ of the Markov chain described by $\mathbf{S}$, which give the long-run fraction of time of presence of the system in the different states. Notice that $\mathbf{\Gamma}_\infty$ inherited from $\mathbf{S}_\infty$ the property of having all the rows identical and here equal to $\overline{\mu}_{av}$. As expected,

$$\overline{\mu}_{av} = \sum_{i=1}^{N-n} s^i \, \overline{\mu}_{\infty,i} \tag{64}$$

Equation (64) says that $\overline{\mu}_{av}$ is simply the barycentre of the $\overline{\mu}_{\infty,i}$'s weighted by the fraction of time during which the source is equal to $\overline{e}_i$. This also implies that $\overline{\mu}_{av}$ coincides with the asymptotic mass distribution corresponding to a constant average source $\overline{\sigma} = (s^1, \dots, s^{N-n})$.

Let us remark that in the case where the source varies periodically in time instead of randomly, the time average of the mass distribution is still given by an expression equivalent to (64). Indeed, for a source of period T, the time average of the asymptotic mass distribution is, using the notations of equation (35):

$$\overline{\mu}_{av} = \frac{1}{T} \left(\overline{\mu}_{(T)} + \overline{\mu}_{(T+1)} + \dots + \overline{\mu}_{(2T-1)} \right) \tag{65}$$

Equation (35) gives then

$$\overline{\mu}_{av} = \frac{1}{T} (\overline{\sigma}_0 + \overline{\sigma}_1 + \dots + \overline{\sigma}_{T-1}) \, (\mathbf{1} + \mathbf{Q} + \dots + \mathbf{Q}^{T-1}) \, (1 - \mathbf{Q}^T)^{-1} = \frac{1}{T} \sum_{i=0}^{T-1} \overline{\sigma}_i \, (1 - \mathbf{Q})^{-1} \tag{66}$$

which is again the barycentre of the asymptotic mass distributions corresponding to the T constant sources $\overline{\sigma}_i$, weighted by the fraction of time during which the source is equal to each $\overline{\sigma}_i$, that is, $1/T$ for all $i = 1, \ldots, T-1$.

Once the average is known it is tempting to search for bounds of the fluctuations around the mean. However, at each time step the source adds a quantity $\overline{e}_{i_t}$ which is always of order one and independent of the eigendirections of $\mathbf{Q}$; therefore, as it will be clear in section 6, the fluctuation size in the long run is of the order of the inverse of the asymptotic masses. Despite of that we shall see that the average dynamics may be an useful indicator in many cases.

Let us illustrate these ideas through the 3-state model. Here:

$$\mathbf{S} = \begin{pmatrix} (1-a) & a \\ b & (1-b) \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} (1-p) & p \\ q & (1-q-r) \end{pmatrix} \tag{67}$$

Therefore:

$$\mathbf{S}_\infty = \frac{1}{(a+b)} \begin{pmatrix} b & a \\ b & a \end{pmatrix} \tag{68}$$

and

$$\text{rows of } \mathbf{S}_\infty (\mathbf{1} - \mathbf{Q})^{-1} = \left(\frac{q}{rp} + \frac{b}{(a+b)\,p}, \frac{1}{r} \right) = \lim_{t \to \infty} \frac{1}{t} \sum_{t'=1}^{t} \overline{\mu}_{t'} \tag{69}$$

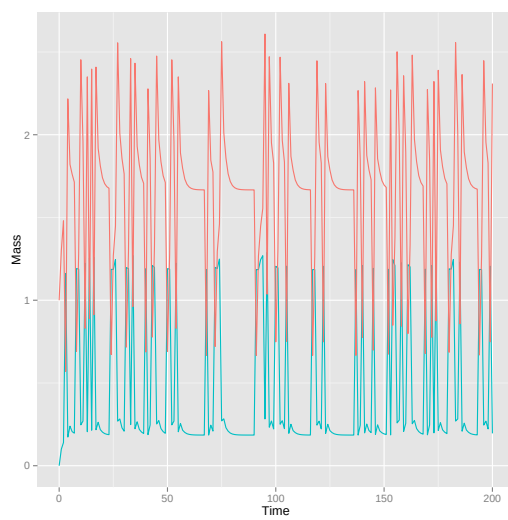


Figure 4: System with a fluctuating source driven by a Markov chain. Bottom/Blue line: first component of $\overline{\mu}_t$, the amount of mass in the first state of the chain. Top/Red line: second component of $\overline{\mu}_t$, the amount of mass in the second state of the chain.

In Figure 4 we show an example of this situation. The parameters of the system are in this case $(a, b) = (0.6, 0.2)$, $(p, q, r) = (0.1, 0.3, 0.6)$.

We observe a random oscillation between the two asymptotic solutions. In this case the convergence rates are sufficiently fast. Here the maximal eigenvalue of $\mathbf{Q}$ is $\lambda_1 = 0.9359$, the second eigenvalue of $\mathbf{S}$ is $\xi_2 = 0.2$, (see section 5.1), and the diagonal elements sufficiently high to insure visible residence times close to each equilibrium of the original $\mathbf{Q}$ system with constant sources $\overline{\sigma}_0$ and $\overline{\sigma}_1$. Instead, following section 3, the trajectory will be close to the periodic asymptotes if $\mathbf{S}$ is close to a periodic chain.

6 Application: A Model of Road Network Dynamics

A natural context of application of our model is transport networks where the traffic flow is modelled by a network representing roads and junctions where cars can move. Recent access to direct recorded imaging data of the path followed by each single vehicle within the flow or to the collective behavior when they arrive to a road junction attracts new interest to this type of models. Moreover there is now a large number of dedicated softwares that can be used to simulate with more or less detail this traffic situations. One of the most popular of such mobility simulators, SUMO, [27], [28] is a friendly open source microscopic road traffic simulation package allowing comparison with real or virtual data.

Clearly Markov chains were employed in the past to build simple or sophisticated models representing vehicular mobility in urban environment, see for example [29] for a documented presentation. However up to our knowledge, all this models, being standard Markov chains, are closed systems.

Here we modify a known road network Markov model, [30], [31], in order to introduce external sources and sinks of vehicles. It is out of the scope of this work to develop in detail the comparison of our results with real data. Instead by comparing our simulations with the ones made in [30], [31], we hope that it is possible to confirm the relevance of our paradigm.

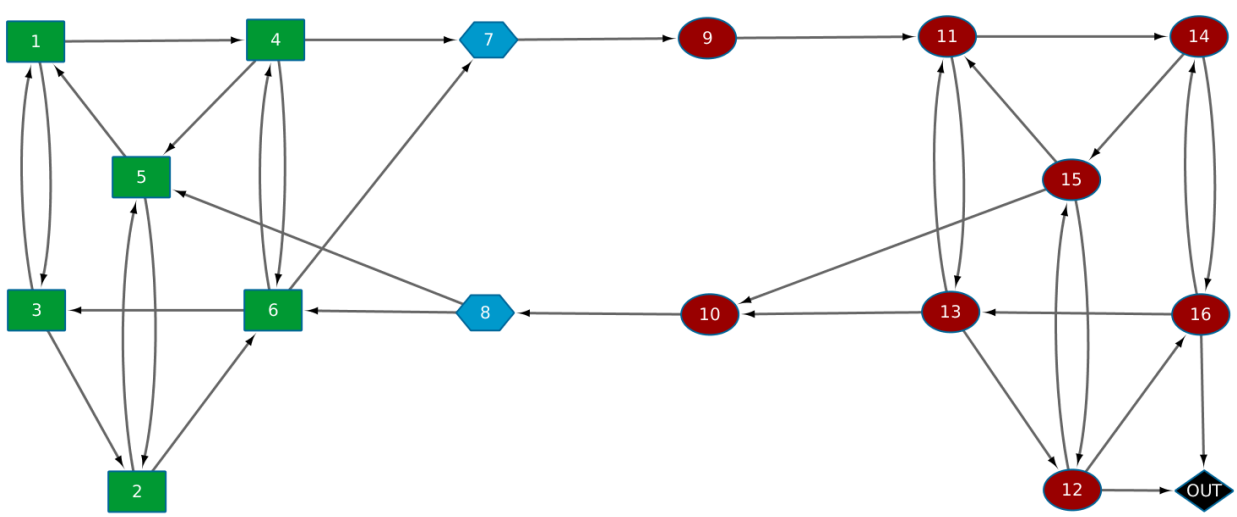


Figure 5: Dual graph of a road network , [30]: nodes are roads and arrows are junctions. From left to right: West side (green), nodes $1-6$, Bridges (blue), nodes $7-8$ and East side (red), nodes $9-16$. In black an additional absorbing external node.

It is natural to introduce in this context a directed graph $(\mathbf{G}, V, E)$ where nodes V correspond to junctions and directed edges to connecting roads. This representation is sometimes called primal

in contrast with a dual representation where the streets correspond to nodes and junctions to edges. Following [30], [31], we shall use the second formulation, see [32] for a discussion of this choice.

By adding to the topological information encoded in the dual graph the turning probabilities at each junction (available for instance from webcam counts), in [30] the authors define a transition stochastic matrix and therefore a Markov chain in the obvious way. They present a case study with seven nodes in the primal graph (A, B, C, D, E, F, G) and sixteen in the dual graph, in alphabetic order : $(AB, AC, BA, BC, CA, CB, CD, DC, DE, ED, EF, EG, FE, FG, GE, GF)$, hereby denoted 1 to 16, Figure 5. They study two different versions of the (16×16) transition matrix: $\mathbf{P}_0$ and $\mathbf{P}_x$. The first one, $\mathbf{P}_0$, has all the diagonal elements equal to zero and the non diagonal terms simply correspond to the empirical turning probabilities. The second matrix, $\mathbf{P}_x$, takes in account the different travel times to cover each single road (a node of the graph). In this case the diagonal elements are no more zero and the non diagonal terms are renormalized accordingly. Notice that there is a slight difference between the classical De Bruijn graphs, [33], and the dual graph in that some of the arrows of the 2-De Bruijn are missing in the dual graph because the corresponding turnaround is not possible in the actual configuration of the real road connections. For instance, there are no direct connections between the bridges CD, (node 7) and DC, (node 8) in the dual graph of Figure 5.

Starting from $\mathbf{P}_0$ and $\mathbf{P}_x$, [30], we define the corresponding sub-stochastic matrixs $\mathbf{Q}_0$ and $\mathbf{Q}_x$ in order to introduce the sinks. They are defined by adding first an absorbing node 17 (the sink) to the $\mathbf{P}$s and then deciding that the exit of vehicles from the network to the sink uses only nodes 12 and 16 and it affects 40% of the vehicles of each of these two roads.

Here we focus on the more realistic case of the matrix $\mathbf{Q}_x$ where the different travel times to cover each road is taken in account (the case $\mathbf{Q}_0$ can be treated in the same way). The matrix $\mathbf{Q}_x$ reads:

$$\mathbf{Q}_x = \begin{pmatrix}
0.168 & 0 & 0.083 & 0.749 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.62 & 0 & 0 & 0.038 & 0.304 & 0.038 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.041 & 0.361 & 0.598 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.35 & 0.45 & 0.1 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.35 & 0.05 & 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.717 & 0.08 & 0 & 0.203 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.049 & 0 & 0.951 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.469 & 0.469 & 0 & 0.062 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.935 & 0 & 0.065 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.099 & 0 & 0.09 & 0.811 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0 & 0 & 0.1 & 0.3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.05 & 0.05 & 0.1 & 0.8 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.65 & 0.15 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0.5 & 0.1 & 0 & 0 & 0.3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.45 & 0.1 & 0 & 0.05
\end{pmatrix} \tag{70}$$

We notice that these matrices show a clear block-like structure with a first block (West side) consisting of nodes 1 to 6, a second block (East side) corresponding to nodes 9 to 16 and two bridge nodes 7 and 8 connecting the West and East sides, as it appears from the structure of the network shown in Figure 5.

We then introduce the source terms in the model. We shall consider two different cases corresponding to different traffic conditions. In the first case the vehicles enter the network at West side through junction A, therefore through nodes 1 and 2 in the dual graph, and leave the network by the far East side using nodes 12 and 16. In this case the source transfer matrix $\mathbf{S}_2$ has dimension 2 and acts on the two first states of the system. Here we use as Markov source the matrix producing the plot shown in Figure 4:

$$\mathbf{S}_2 = \begin{pmatrix} 0.4 & 0.6 \\ 0.2 & 0.8 \end{pmatrix} \tag{71}$$

In the second case the vehicles enter the network at any junction and therefore at any node of the dual graph except at the bridges (nodes 7 and 8). The corresponding Markov transfer matrix $\mathbf{S}_{14}$ has

dimension 14 and acts on the West and East nodes. We chose a simple example of a reversible ergodic chain with uniform stationary distribution so that in the long run any node gathers the same amount of vehicles from the outside:

$$\mathbf{S}_{14} = \begin{pmatrix} 0.1 & 0.1 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.2 \\ 0.1 & 0.1 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0.1 & 0.1 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.1 & 0.1 & 0 & 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0 & 0 & 0.1 & 0.1 & 0.2 & 0.2 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2 & 0.2 & 0.1 & 0 & 0.1 & 0.2 & 0 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.2 & 0.1 & 0.1 & 0 & 0.2 & 0.2 & 0.2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0.2 & 0 & 0.1 & 0.1 & 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0 & 0.2 & 0.1 & 0 & 0.1 & 0.2 & 0.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0.2 & 0.1 & 0.1 & 0 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0.1 & 0.1 & 0.2 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0 & 0.1 & 0.1 & 0.2 & 0.2 \\ 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0.1 & 0.1 \\ 0.2 & 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.2 & 0.2 & 0.1 & 0.1 \end{pmatrix} \tag{72}$$

The results of the simulations of this system are partially reproduced in Figures (6), (7) and (8). Figure (6) shows the time plots of the nodes of West side for runs of Markov injection of vehicles entering the network according to the matrix $\mathbf{S}_2$ (Figure 6-a) and to the matrix $\mathbf{S}_{14}$ (Figure 6-b). Figure (7) shows the corresponding flows for the East side. It is clear that the amount of vehicles present in the West side is much larger than on the East side in both situations, in fact any West node copes with more vehicles that any East node. Moreover concerning their mass contents the nodes are ordered in the same manner in both the incoming situations of $\mathbf{S}_2$ or $\mathbf{S}_{14}$. This suggests that in this case the internal structure of the network and in particular the exit locations are the main factors governing the traffic flow as compared with the distribution of the influx nodes. In fact the nodes are ordered in the same way here as the components of the Perron-Frobenius eigenvector of $\mathbf{Q}_x$, [1] that is independent of $\mathbf{S}$. This is not always the case for we know from the previous example, (69), how the asymptotic behavior may depend on the injection system. On the contrary we can see that the fluctuations levels around the mean are intimately related with vicinity of the arrival nodes as can be checked by comparing the (a) side of the this figures with the (b) side. We notice that the dynamics on the network in our case is quite different of the one of the closed counterpart described in [30] where the traffic is equally distributed in both sides of the network and among the nodes (up to the bridges) and there is no fluctuations at all.

In section 5.2 we prove, (64), that the averaged dynamics of a network driven by an injection matrix $\mathbf{S}$ corresponds to the one of a constant source equal to the stationary distribution of $\mathbf{S}$. Let us define $\overline{\mu}^{av}(t)$ the t iterated of the mass distribution associated to this constant source. It is natural to define a fluctuation function $\delta_i(t)$ for each node i and each Markov run of the system by:

$$\delta_i(t) = |\overline{\mu}_i(t) - \overline{\mu}_i^{av}(t)| \tag{73}$$

In Figure 8(a) we show a time plot of the Bridge nodes dynamics in the case of vehicles entering by all the roads of the network, and in Figure 8(b) the corresponding average evolution. Notice that, since the Bridge nodes are a cut of the graph, the difference of the two asymptotic values in Figure 8(b) represents the net averaged outcome of vehicles from West to East per unit time. As anticipated at the end of section 5.2, in Figure 8(a) we see large fluctuations (of order one) at any time. In Figure 9 this fluctuations are characterized by the values of the fluctuation function $\delta_i(t)$ as defined in (73). Due to the

[1] $\Psi_1 = (0.0654, 0.2017, 0.1866, 0.0887, 0.1311, 0.093, 0.0176, 0.0065, 0.0170, 0.0064, 0.0328, 0.0213, 0.0460, 0.0354, 0.0367, 0.0125)$

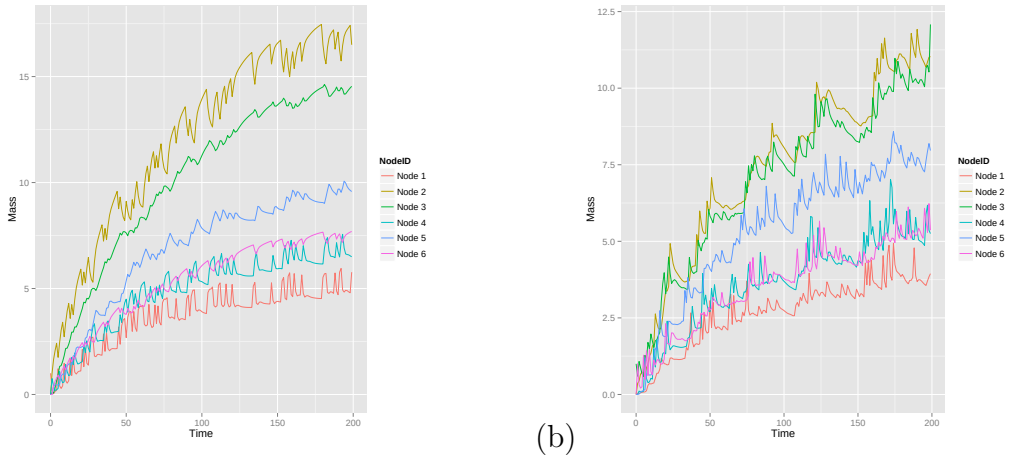


Figure 6: West side network. (a) vehicles enter by nodes 1 and 2, (matrix $\mathbf{S}_2$). From top to bottom (at $t = 200$): nodes 2, 3, 5, 6, 4, 1.(b) vehicles enter by all nodes, (matrix $\mathbf{S}_{14}$). From top to bottom (at $t = 200$): nodes 3, 2, 5, 6, 4, 1.

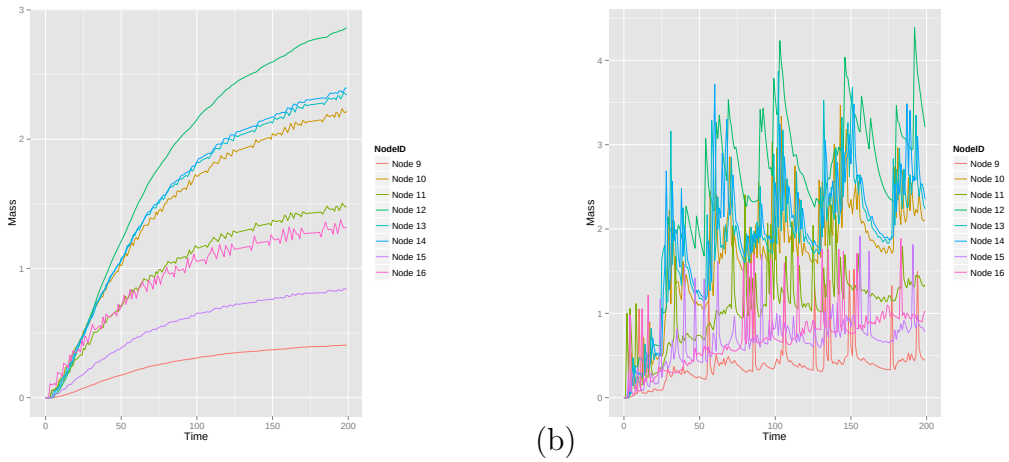


Figure 7: East side network. (a) vehicles enter by nodes 1 and 2, (matrix $\mathbf{S}_2$). (b) vehicles enter by all nodes, (matrix $\mathbf{S}_{14}$). In both cases, vehicles leave the network by nodes 12 and 16 and from top to bottom (at $t = 200$): nodes 13, 15, 14, 11, 12, 9, 16, 10.

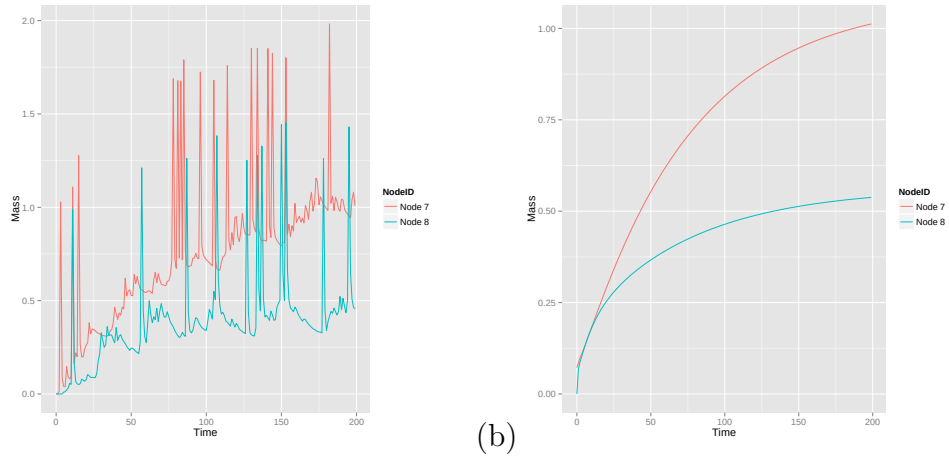


Figure 8: Bridges : vehicles enter by all nodes, (matrix $\mathbf{S}_{14}$). (a) time plot of the Markov run, (b) time plot of the average evolution. In both cases, from top to bottom (at $t = 200$): node 7 (Bridge West to East) and node 8 (Bridge East to West).

vicinity of the incoming nodes there is a net difference between the case of the source $\mathbf{S}_2$, Figure 9(a) and the case of the source $\mathbf{S}_{14}$, Figure 9(b). In the first situation we observe small fluctuations softened by the distance to the source, quite the opposite of the second situation where the δ_is are many times close to one. Even though large fluctuations cannot be neglected in the long run, it is clear that the average is an useful indication of the rates of the traffic in the network. To theoretically quantify such fluctuations in a manner that would take into account the specificity of each node gives rise to interesting questions, leading to the possibility of handling the issue of traffic control, that are out of the scope of the present work and that we leave for a forthcoming study. Nevertheless we hope that we have convinced the reader of the interest of the model for studying road traffic situations.

It is clear from the preceding analysis that our model is suitable in other cases where transport of some conserved quantity is present. A pertinent well documented case is the one of electric networks, [24], and [34] and [35] for more recent developments. In this context the presence of electronic variable resistors may lead to a stochastic source where the consideration of a Markov injection model may be of interest. In this situation, the reversibility of the chain, which is a characteristic attribute of this case, may open to interesting simplifications, and therefore to more precise conclusions. The characteristics of our model would be particularly adapted to a situation where the transient dynamics is relevant.

7 Conclusions

In this paper we study the flux through a finite Markov chain of a quantity, called mass, that has the property of moving through the states of the chain according to the Markov transition probabilities. Mass is supplied by a source and the chain possesses one or more absorbing states where mass accumulates. In the vast range of systems whose dynamics can be modelled by Markov chains, this approach could be useful to treat the case of open systems where the quantity of interest is provided by an external source, evolves inside the system and is eliminated in the end.

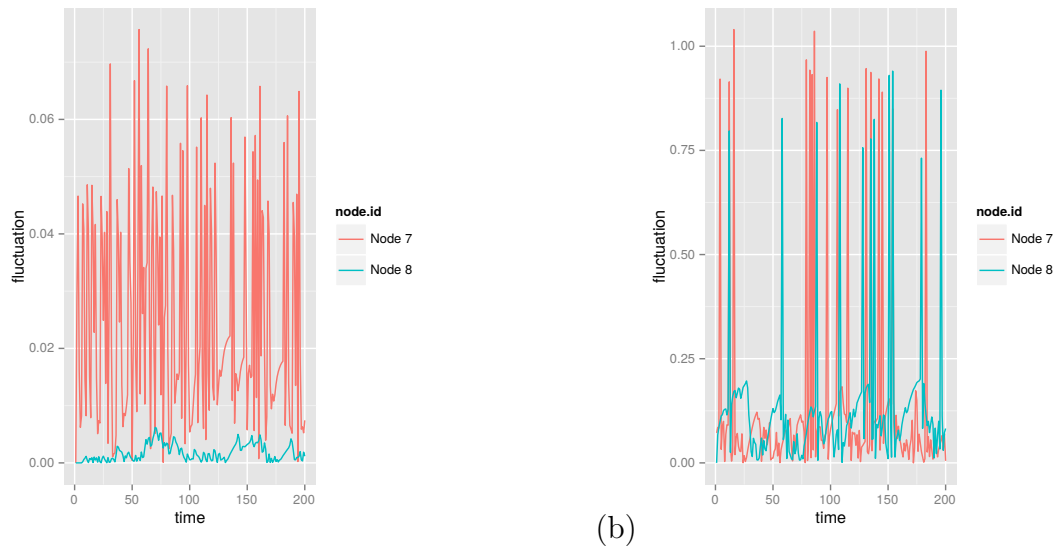


Figure 9: Fluctuations $\delta_i(t)$, $i = 7, 8$ with respect to the average defined in (73) in the Bridge nodes : (a) vehicles enter by nodes 1 and 2, (matrix $\mathbf{S}_2$), (b) vehicles enter by all nodes, (matrix $\mathbf{S}_{14}$).

In the case where the source is constant in time, we give the expression $\overline{\mu}_t$ of the distribution of mass in the system at any time, as a function of the source and of the eigenvalues and eigenvectors of $\mathbf{Q}$, the transition matrix restricted to the non-absorbing states. We show that $\overline{\mu}_t$ converges to an asymptotic distribution $\overline{\mu}$ with an exponential rate that is dominated by the largest eigenvalue of $\mathbf{Q}$.

In the case where the source is periodic in time, the distribution of mass at any time t can also be computed analytically. The asymptotic distribution is found to have the same periodicity as the source and, as in the case of constant source, the gap between $\overline{\mu}$ and $\overline{\mu}_t$ decreases exponentially with a rate depending on the largest eigenvalue of $\mathbf{Q}$.

Nevertheless, in contrast with a conventional absorbing Markov chain the asymptotic distribution $\overline{\mu}$ depends of the source flux $\overline{\sigma}$ and has contributions from all the eigendirections of $\mathbf{Q}$. In particular, as a fixed point of an affine transformation, $\overline{\mu}$ is not zero.

All quantities of interest have been explicitly computed and these ideas illustrated in the case of a 3-states Markov chain.

A concrete example of application of this framework to traffic flow in a road network is presented and partially worked out in section 6.

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