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# Effective computation of the asymptotic expansion for a small inclusion in elastic media

Makrem ARFAOUI, \*    Mohamed Rafik BEN HASSINE, †    Maher MOAKHER, ‡  
Yves RENARD, §    Gregory VIAL. ¶

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## Abstract

This paper presents a new method to compute the influence of a small inclusion of arbitrary shape in elastic media. Conventional homogenization techniques, while effective in many applications, fail to provide accurate predictions because of the averaged values used. A model based on the technique of asymptotic expansions is then proposed. At each order we provide corrective terms modeling the influence of the inclusion using techniques of scaling and multi-scale asymptotic expansions. Expansions are constructed for the Poisson and linear elasticity two-dimensional problems.

**Keywords:** Transmission problem, asymptotic expansion, Multi-scale analysis, Infinite domains, linear elasticity.

## Introduction

The interface problem investigated in this paper originates from a mechanical model for small inclusions coated with elastic bodies. Such arrangements are widely used for instance in tire's design in order to provide the desired structure stiffnesses. In this work, we consider two bi-dimensional approximations of the transmission problem between an elastic matrix and an elastic small inclusion of different stiffness. The challenge is to know the exact influence of the presence of each inclusion on the rest of the structure.

Classical analyses are based on homogenization techniques, see [22], that use average values and thus cannot predict the exact influence of each inclusion on the elastic domain. So, we try in this paper to build a more specific approach based on asymptotic expansion with respect to

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\*Université de Tunis El Manar, Laboratoire de Mécanique Appliquée et Ingénierie, École Nationale d'Ingénieurs de Tunis, 1002 Tunis, Tunisia. email: makremarfaoui@yahoo.fr

†INSA-Lyon ENI-Tunis, CNRS, INSA-Lyon, ICJ UMR5208, LaMCoS UMR5259, F-69621, Villeurbanne, France. ENI-Tunis, Laboratoire de Modélisation Mathématique et Numérique dans les Sciences de l'Ingénieur, Tunisie. email: mohamed-rafik.ben-hassine@insa-lyon.fr

‡LAMSIN-ENIT, University of Tunis El Manar, B.P. 37, 1002 Tunis-Belvédère, Tunisia. email: maher.moakher@enit.rnu.tn

§Université de Lyon, CNRS, INSA-Lyon, ICJ UMR5208, LaMCoS UMR5259, F-69621, Villeurbanne, France. email: yves.renard@insa-lyon.fr

¶Ecole Centrale de Lyon, CNRS, INSA-Lyon, ICJ UMR5208, LaMCoS UMR5259, F-69621, Villeurbanne, France. email: gregory.vial@ec-lyon.fr

the small parameter of the inclusion  $\varepsilon$  using techniques issued from [28], [9], [26], [3], [16] and [15].

Some works have treated the case of perturbations whether for inclusions, for transmission problems essentially for conductivity [26] and Helmholtz equation [3], [16] or even for perforated domains and cavities. The case of perforated domains has been studied in [24], [25], [17], [20], [9], [10] using techniques relying on the notion of profile, a normalized solution of the Laplace equation in the exterior domain obtained by blow-up of the perturbation. In fact, the influence of a boundary perforation of a smooth domain is studied in [9] and an asymptotic expansion of the shape function in terms of a size parameter is derived. The case of inclusions was treated for the first time by [12] giving a method to find the stresses set up in an elastic solid when the inclusion undergoes a change of form (for an ellipsoidal region whose elastic constants differ from those of the remaining material). The most important result is that the field inside the inclusion can be determined without having to find the field outside it. The resulting elastic field is found with the help of a sequence of imaginary cutting, straining and welding operators and using a biharmonic potential. The determination of the field outside the inclusion was briefly treated in this paper but studied deeply in [13] where the elastic field outside an ellipsoidal inclusion is expressed entirely in terms of the harmonic potential of a solid ellipsoid by replacing the biharmonic potential introduced in [12] by the harmonic potential of a certain surface distribution. Note that Eshelby's inclusion solution is one of the most significant contributions to micro-mechanics in the twentieth century and is based on the assumption that an inclusion is embedded in an unbounded ambient space. Li, Sauer and Wang deepened in [21] the precise characterization of the elastic fields due to inclusions in finite elastic media. It is important to say that, before the work done in [21], Kroner [18], [19] and Mazilu [23] tried to study the inclusion problem in a finite domain searching a Green's function of Navier's equation in a finite domain. The attempt ended in failure because of the mathematical difficulty involved in obtaining a closed form solution of Green's function in a finite domain.

On another side, Ammari and Kang described in [1] techniques for the reconstruction of small inclusions using only boundary measurements. These techniques rely on asymptotic expansions of the boundary perturbations due to the presence of the inclusions. The approach used to derive the asymptotic expansions is based on layer potential techniques allowing to deal with rough boundaries. By the mean of Muskhelishvili's complex variable method and the Schwartz-Christoffel conformal mapping technique and motivated by the high stresses that occur at the junction between the fibers and the matrix in multi-fiber composite sheets which can cause failure, [8] presented a solution for determining the stresses in an infinite elastic plate containing a rigid rectangular inclusion subject to a uniform stress field. Using also the complex variable function method, the case of an arbitrary shape inclusion embedded in a half space is studied in [27]. Solutions are obtained by the mean of techniques of analytic extension, analytic continuation and conformal mapping. In [4] an asymptotic expansion of the difference between the displacement at the boundary with and without inclusion is presented (both the body and the inclusions are allowed to be anisotropic). To derive the asymptotic expansion, the authors follow the approach introduced by Capdeboscq and Vogelius in [6] (see also [7]) for the conductivity problem. Note that the formula obtained generalizes those already available in the case of homogeneous isotropic bodies with thin inhomogeneities of small diameters.

The outline of this paper is as follows: The problem of interest is formulated for two approximations in Sections (2) and (7) respectively: The Poisson problem and the linear elasticity one. Error estimates are established for the two cases in Sections (3) and (8). Finally, we are led to deal with functions defined on infinite domains in the two cases. We reconsider proof of existence and uniqueness using the generalized Hardy's inequalities and adequate weighted Sobolev spaces,

see [2].

## 1 The two transmission problems

We consider two different approximations of an elastic transmission problem: a scalar antiplane approximation and a linear plane strain approximation. Let us first define the domain of interest. Let  $\Omega_f^\varepsilon \subset \mathbb{R}^2$  be a bounded connected domain of characteristic dimension  $\varepsilon$  representing the inclusion (also called fiber) and being small enough compared to the characteristic size of the matrix represented by  $\Omega_m^\varepsilon \subset \mathbb{R}^2$  of radius  $R$  with smooth boundary  $\Gamma$ . The interface  $\Gamma^\varepsilon$  is also considered to be smooth. Fig. 1 makes explicit the geometrical setting.

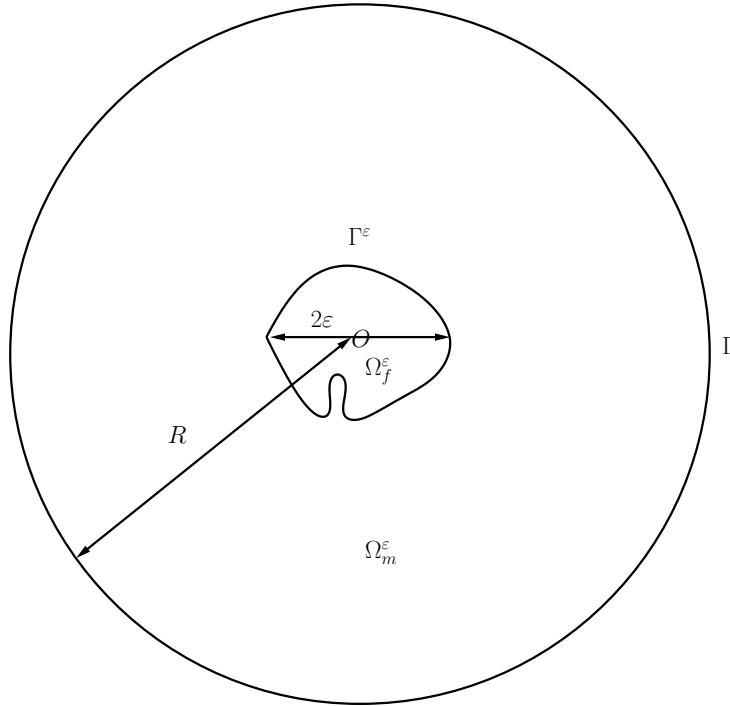


Figure 1: Initial domain  $\Omega^\varepsilon = \Omega_f^\varepsilon \cup \Gamma^\varepsilon \cup \Omega_m^\varepsilon$ .

The antiplane problem reduces the mechanical problem to a Poisson one and is characterized by a displacement reduced to the normal component to the plane  $(O, \mathbf{x}_1, \mathbf{x}_2)$  under the form  $u_3 = u(x_1, x_2)$ . Transmission conditions consisting on the continuity conditions are set on the interface  $\Gamma^\varepsilon$  and a non homogeneous Dirichlet condition is set on the fixed boundary  $\Gamma$ . The strong formulation of the antiplane transmission problem reads:

$$\alpha_f \Delta u_f^\varepsilon = 0 \quad \text{in } \Omega_f^\varepsilon, \quad (1.1a)$$

$$\alpha_m \Delta u_m^\varepsilon = 0 \quad \text{in } \Omega_m^\varepsilon, \quad (1.1b)$$

$$u_f^\varepsilon = u_m^\varepsilon \quad \text{on } \Gamma^\varepsilon, \quad (1.1c)$$

$$\alpha_f \partial_{\mathbf{n}} u_f^\varepsilon = \alpha_m \partial_{\mathbf{n}} u_m^\varepsilon \quad \text{on } \Gamma^\varepsilon, \quad (1.1d)$$

$$u_m^\varepsilon = F \quad \text{on } \Gamma. \quad (1.1e)$$

Where  $\alpha_f$  and  $\alpha_m$  are the shear modulus of the two materials,  $u_f^\varepsilon$  is the solution of the problem in  $\Omega_f^\varepsilon$ ,  $u_m^\varepsilon$  is the solution of the problem in the matrix area  $\Omega_m^\varepsilon$ ,  $F$  a prescribed displacement on

$\Gamma$  and  $\partial_{\mathbf{n}}$  denotes the normal derivative (outer for  $\Omega_f^\varepsilon$ , inner for  $\Omega_m^\varepsilon$ ). Considering  $F \in H^{\frac{1}{2}}(\Gamma)$  and thanks to Lax Milgram theorem the existence and uniqueness of the solution of Problem (1.1) in a weak form is guaranteed.

The second transmission problem treated is a plane strain linear elasticity problem with two different fourth-order stiffness constant tensors  $\mathbf{E}_f$  and  $\mathbf{E}_m$  in the inclusion and in the matrix, respectively. The constitutive law linking stress to strain is given by  $\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon}$  (where  $\boldsymbol{\varepsilon}$  is the strain tensor and  $\boldsymbol{\sigma}$  is the stress one). Transmission conditions consisting on the continuity conditions are still set on the interface  $\Gamma^\varepsilon$  and a non homogeneous Dirichlet one on the fixed boundary  $\Gamma$ . The strong formulation of the linear elasticity problem reads:

$$\operatorname{div} \boldsymbol{\sigma}_f(\mathbf{u}_f^\varepsilon) = \mathbf{0} \quad \text{in } \Omega_f^\varepsilon, \quad (1.2a)$$

$$\operatorname{div} \boldsymbol{\sigma}_m(\mathbf{u}_m^\varepsilon) = \mathbf{0} \quad \text{in } \Omega_m^\varepsilon, \quad (1.2b)$$

$$\mathbf{u}_f^\varepsilon = \mathbf{u}_m^\varepsilon \quad \text{on } \Gamma^\varepsilon, \quad (1.2c)$$

$$\boldsymbol{\sigma}_f(\mathbf{u}_f^\varepsilon) \cdot \mathbf{n}_f = -\boldsymbol{\sigma}_m(\mathbf{u}_m^\varepsilon) \cdot \mathbf{n}_m \quad \text{on } \Gamma^\varepsilon, \quad (1.2d)$$

$$\mathbf{u}_m^\varepsilon = \mathbf{F} \quad \text{on } \Gamma. \quad (1.2e)$$

Where  $\mathbf{u}_f^\varepsilon : \Omega_f^\varepsilon \rightarrow \mathbb{R}^2$  is the solution of the problem in the field of inclusion  $\Omega_f^\varepsilon$ ,  $\mathbf{u}_m^\varepsilon : \Omega_m^\varepsilon \rightarrow \mathbb{R}^2$  is the solution of the problem on the matrix area  $\Omega_m^\varepsilon$  and  $\mathbf{F}$  is a prescribed displacement. The vectors  $\mathbf{n}_f$  and  $\mathbf{n}_m$  denote the outgoing normals to  $\Omega_f^\varepsilon$  and  $\Omega_m^\varepsilon$ , respectively. The existence and uniqueness of the solution of Problem (1.2) in a weak form is also guaranteed thanks to Lax Milgram theorem for  $\mathbf{F} \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^2)$ .

## 2 Asymptotic expansion for the antiplane problem

In this section, we build the asymptotic expansion with respect to the influence of the inclusion  $\Omega_f^\varepsilon$  on the displacement field of the matrix  $\Omega_m^\varepsilon$  in the case of the antiplane Problem (1.1).

### 2.1 Comparison with the solution without inclusion

Introducing  $u^{(0)}$  the solution of Problem (1.1) when  $\varepsilon = 0$ , defined on the domain without inclusion  $\Omega^0$ , we introduce the difference  $d_{0,k}^\varepsilon = u_k^\varepsilon - u^{(0)}$  ( $k \in \{f, m\}$ ). Then  $d_{0,k}^\varepsilon$  is solution to:

$$\alpha_f \Delta d_{0,f}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_f^\varepsilon, \quad (2.1a)$$

$$\alpha_m \Delta d_{0,m}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_m^\varepsilon, \quad (2.1b)$$

$$d_{0,f}^\varepsilon(\mathbf{x}) = d_{0,m}^\varepsilon(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (2.1c)$$

$$\alpha_f \partial_{\mathbf{n}} d_{0,f}^\varepsilon(\mathbf{x}) = \alpha_m \partial_{\mathbf{n}} d_{0,m}^\varepsilon(\mathbf{x}) + (\alpha_m - \alpha_f) \partial_{\mathbf{n}} u^{(0)}(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (2.1d)$$

$$d_{0,m}^\varepsilon(\mathbf{x}) = 0 \quad \text{on } \Gamma. \quad (2.1e)$$

We note the emergence of the term  $(\alpha_m - \alpha_f) \partial_{\mathbf{n}} u^{(0)}$  in the transmission equation (2.1d).

### 2.2 Scaling and approximation in infinite domain

The original problem is posed on domains depending on  $\varepsilon$ . Then, the first step is to formulate the problem on fixed domains. To this aim, following references [28], [9], we introduce the scaling  $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$ . This scaling will allow us to fix the field of inclusion and the position of the interface. Denoting  $\Omega^{m,\varepsilon} = \frac{\Omega_m^\varepsilon}{\varepsilon}$ ,  $D_{0,k}^\varepsilon(\mathbf{y}) = d_{0,k}^\varepsilon(\mathbf{x})$  ( $k \in \{f, m\}$ ) is the solution to the following scaled

problem:

$$\alpha_f \Delta D_{0,f}^\varepsilon(\mathbf{y}) = 0 \quad \text{in } \Omega^f, \quad (2.2a)$$

$$\alpha_m \Delta D_{0,m}^\varepsilon(\mathbf{y}) = 0 \quad \text{in } \Omega^{m,\varepsilon}, \quad (2.2b)$$

$$D_{0,f}^\varepsilon(\mathbf{y}) = D_{0,m}^\varepsilon(\mathbf{y}) \quad \text{on } \Gamma^1, \quad (2.2c)$$

$$\alpha_f \partial_{\mathbf{n}} D_{0,f}^\varepsilon(\mathbf{y}) = \alpha_m \partial_{\mathbf{n}} D_{0,m}^\varepsilon(\mathbf{y}) + \varepsilon(\alpha_m - \alpha_f) \partial_{\mathbf{n}} u^{(0)}(\varepsilon \mathbf{y}) \quad \text{on } \Gamma^1, \quad (2.2d)$$

$$D_{0,m}^\varepsilon(\mathbf{y}) = 0 \quad \text{on } \frac{\Gamma}{\varepsilon}. \quad (2.2e)$$

Now, as  $\varepsilon$  is considered to be small compared to the dimensions of the geometry and to eliminate completely the dependence on  $\varepsilon$ , we approximate Problem (2.2) by the following problem posed on the dilated geometry  $\Omega^\infty = \lim \Omega^{m,\varepsilon}$  when  $\varepsilon \rightarrow 0$  as seen in Fig. 2:

$$\alpha_f \Delta D_{0,f}^\varepsilon(\mathbf{y}) = 0 \quad \text{in } \Omega^1, \quad (2.3a)$$

$$\alpha_m \Delta D_{0,m}^\varepsilon(\mathbf{y}) = 0 \quad \text{in } \Omega^\infty, \quad (2.3b)$$

$$D_{0,f}^\varepsilon(\mathbf{y}) = D_{0,m}^\varepsilon(\mathbf{y}) \quad \text{on } \Gamma^1, \quad (2.3c)$$

$$\alpha_f \partial_{\mathbf{n}} D_{0,f}^\varepsilon(\mathbf{y}) = \alpha_m \partial_{\mathbf{n}} D_{0,m}^\varepsilon(\mathbf{y}) + \varepsilon(\alpha_m - \alpha_f) \partial_{\mathbf{n}} u^{(0)}(\varepsilon \mathbf{y}) \quad \text{on } \Gamma^1, \quad (2.3d)$$

$$D_{0,m}^\varepsilon(\mathbf{y}) = 0 \quad \text{when } |\mathbf{y}| \rightarrow \infty. \quad (2.3e)$$

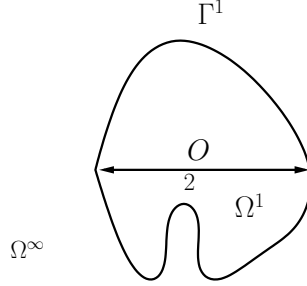


Figure 2: Dilated geometry  $\Omega^\infty$ .

The next step is to expand  $\partial_{\mathbf{n}} u^{(0)}(\varepsilon \mathbf{y})$ .

### 2.3 Asymptotic expansion: first-order term

Let us first introduce a problem which will allow us to construct the different orders of the asymptotic expansion of the inclusion contribution. It is also posed on the dilated geometry  $\Omega^\infty$ . For each order  $p$  we consider the function  $V^{(p)}$  solution to:

$$\alpha_f \Delta V_f^{(p)}(\mathbf{y}) = 0 \quad \text{in } \Omega^1, \quad (2.4a)$$

$$\alpha_m \Delta V_m^{(p)}(\mathbf{y}) = 0 \quad \text{in } \Omega^\infty, \quad (2.4b)$$

$$V_f^{(p)}(\mathbf{y}) = V_m^{(p)}(\mathbf{y}) \quad \text{on } \Gamma^1, \quad (2.4c)$$

$$\alpha_f \partial_{\mathbf{n}} V_f^{(p)}(\mathbf{y}) = \alpha_m \partial_{\mathbf{n}} V_m^{(p)}(\mathbf{y}) + \varphi^{(p)}(\mathbf{y}) \quad \text{on } \Gamma^1, \quad (2.4d)$$

$$V_m^{(p)}(\mathbf{y}) \rightarrow 0 \quad \text{when } |\mathbf{y}| \rightarrow \infty. \quad (2.4e)$$

The  $V^{(p)}$  functions are introduced to offset terms appearing on the boundary of  $\Gamma^\varepsilon$  where the data  $\varphi^{(p)}(\mathbf{y})$  will be adapted to each order  $p$ . In fact, the smaller these terms, the smaller the

residual and the expansion is driven to a higher order. The existence and uniqueness of  $V^{(p)}$  functions will be established later in Section (5). From the regularity assumptions on the domain we can write a Taylor expansion of  $\nabla u^{(0)}(\varepsilon \mathbf{y})$  for  $\mathbf{y} \in \Gamma^1$ :

$$\nabla u^{(0)}(\varepsilon \mathbf{y}) = \sum_{i=0}^K \varepsilon^i \boldsymbol{\omega}_0^{(i)}(\mathbf{y}) + O(\varepsilon^{K+1}). \quad (2.5)$$

Then, we construct the “first-order” approximation by compensating the first term in the expansion of  $(\alpha_m - \alpha_f) \partial_{\mathbf{n}} u^{(0)}(\varepsilon \mathbf{y})$ . With  $V^{(0)}$  solving Problem (2.4) for  $\varphi^{(0)}(\mathbf{y}) = (\alpha_m - \alpha_f) \boldsymbol{\omega}_0^{(0)}(\mathbf{y}) \cdot \mathbf{n} = (\alpha_m - \alpha_f) \boldsymbol{\omega}_0^{(0)}(\mathbf{y})$  (with  $\boldsymbol{\omega}_0^{(0)}(\mathbf{y}) = \nabla u^{(0)}(\mathbf{0})$ ) we update the residual:

$$d_{1,k}^\varepsilon(\mathbf{x}) = u_k^\varepsilon(\mathbf{x}) - u^{(0)}(\mathbf{x}) - \varepsilon V_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right); k \in \{f, m\}. \quad (2.6)$$

The update of the difference allows us to obtain the first-order expansion:

$$u_{\varepsilon,k}^{[1]}(\mathbf{x}) = u^{(0)}(\mathbf{x}) + \varepsilon V_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right); k \in \{f, m\}. \quad (2.7)$$

Then  $d_{1,k}^\varepsilon$  ( $k \in \{f, m\}$ ) is solution to:

$$\alpha_f \Delta d_{1,f}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_f^\varepsilon, \quad (2.8a)$$

$$\alpha_m \Delta d_{1,m}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_m^\varepsilon, \quad (2.8b)$$

$$d_{1,f}^\varepsilon(\mathbf{x}) = d_{1,m}^\varepsilon(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (2.8c)$$

$$\alpha_f \partial_{\mathbf{n}} d_{1,f}^\varepsilon(\mathbf{x}) = \alpha_m \partial_{\mathbf{n}} d_{1,m}^\varepsilon(\mathbf{x}) + (\alpha_m - \alpha_f) (\partial_{\mathbf{n}} u^{(0)}(\mathbf{x}) - \boldsymbol{\omega}_0^{(0)}(\mathbf{y})) \quad \text{on } \Gamma^\varepsilon, \quad (2.8d)$$

$$d_{1,m}^\varepsilon(\mathbf{x}) = -\varepsilon V_m^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) \quad \text{on } \Gamma. \quad (2.8e)$$

## 2.4 Building the second order

We built a “first-order” expansion of the solution with  $u^{(0)}$  and  $V^{(0)}$  independent of  $\varepsilon$ . The next step is about building the second order. At first sight, we must do the same thing we did for the term that appeared on the interface  $\Gamma^\varepsilon$  with the term that appears this time on  $\Gamma$ . In other words, we have two terms to compensate:  $(\alpha_m - \alpha_f) (\partial_{\mathbf{n}} u^{(0)} - \boldsymbol{\omega}_0^{(0)})$  which has already been compensated in the first order and that has yet to be, and the new term  $(-\varepsilon V_m^{(0)}(\frac{\mathbf{x}}{\varepsilon}))$  on  $\Gamma$ . But by analyzing things a little deeper, we can see that the term on  $\Gamma^\varepsilon$  is of order  $\varepsilon$  while that the one on  $\Gamma$  is of order  $\varepsilon^2$  (expansion of  $V$  at infinity see Section (4)). Thus, there is no need at this stage to offset the term on  $\Gamma$ . To conclude, at this stage we merely compensate the term on the interface  $\Gamma^\varepsilon$  as done in the previous section. The term on  $\Gamma$  will wait for the next order. Now with  $V^{(1)}$  solution of Problem (2.4) for  $\varphi^{(1)}(\mathbf{y}) = (\alpha_m - \alpha_f) \boldsymbol{\omega}_0^{(1)}(\mathbf{y}) \cdot \mathbf{n} = (\alpha_m - \alpha_f) \boldsymbol{\omega}_0^{(1)}(\mathbf{y})$  we write the second-order term:

$$d_{2,k}^\varepsilon(\mathbf{x}) = d_{1,k}^\varepsilon(\mathbf{x}) - \varepsilon^2 V_k^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right); k \in \{f, m\}. \quad (2.9)$$

The update of the difference allows us to obtain the second-order approximation:

$$u_{\varepsilon,k}^{[2]}(\mathbf{x}) = u^{(0)}(\mathbf{x}) + \varepsilon V_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 V_k^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right); k \in \{f, m\}. \quad (2.10)$$

Then  $d_{2,k}^\varepsilon$  ( $(k \in \{f, m\})$ ) solves the following problem:

$$\alpha_f \Delta d_{2,f}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_f^\varepsilon, \quad (2.11a)$$

$$\alpha_m \Delta d_{2,m}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_m^\varepsilon, \quad (2.11b)$$

$$d_{2,f}^\varepsilon(\mathbf{x}) = d_{2,m}^\varepsilon(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (2.11c)$$

$$\alpha_f \partial_{\mathbf{n}} d_{2,f}^\varepsilon(\mathbf{x}) = \alpha_m \partial_{\mathbf{n}} d_{2,m}^\varepsilon(\mathbf{x})$$

$$+(\alpha_m - \alpha_f)(\partial_{\mathbf{n}} u^{(0)}(\varepsilon \mathbf{y}) - \omega_0^{(0)}(\mathbf{y}) - \varepsilon \omega_0^{(1)}(\mathbf{y})) \quad \text{on } \Gamma^\varepsilon, \quad (2.11d)$$

$$d_{2,m}^\varepsilon(\mathbf{x}) = -\varepsilon V_m^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^2 V_m^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) \quad \text{on } \Gamma. \quad (2.11e)$$

## 2.5 Third-order expansion

It's time to compensate the two terms that are now of the same order ( $\varepsilon^2$ ). Functions introduced by Problem (2.4) are used to correct the term on  $\Gamma^\varepsilon$  and we will introduce equivalent functions to do the same work but in this case on the boundary  $\Gamma$ . To this aim, we consider the following problem:

$$\alpha_m \Delta_{\mathbf{x}} v^{(q)}(\mathbf{x}) = 0 \quad \text{in } \Omega^0, \quad (2.12a)$$

$$v^{(q)}(\mathbf{x}) = \psi^{(q)}(\mathbf{x}) \quad \text{on } \Gamma. \quad (2.12b)$$

The first thing to do is to expand  $V_m^{(0)}$  at infinity as follows ( $\theta$  is the angular polar coordinate):

$$V_m^{(0)}(\mathbf{y}) = \sum_{i=1}^K \frac{a_0^{(i)}(\theta)}{|\mathbf{y}|^i} + O\left(\frac{1}{|\mathbf{y}|^{K+1}}\right) = \sum_{i=1}^K \varepsilon^k \frac{a_0^{(i)}(\theta)}{|\mathbf{x}|^i} + O(\varepsilon^{K+1}); \quad \mathbf{x} \in \Gamma. \quad (2.13)$$

This expansion at infinity is justified in Section (4).

Following the same approach, with  $v^{(2)}$  solution of Problem (2.12) for  $\psi^{(2)}(\mathbf{x}) = -\frac{a_0^{(1)}(\theta)}{|\mathbf{x}|}$ , we expand as done before  $\nabla v^{(2)}(\varepsilon \mathbf{y})$  as following:

$$\nabla v^{(2)}(\varepsilon \mathbf{y}) = \sum_{i=0}^K \varepsilon^{(i)} \omega_2^{(i)}(\mathbf{y}) + O(\varepsilon^{K+1}). \quad (2.14)$$

With  $V^{(2)}$  solution of Problem (2.4) for  $\varphi^{(2)}(\mathbf{y}) = (\alpha_m - \alpha_f)(\omega_0^{(2)}(\mathbf{y}) + \omega_2^{(0)}(\mathbf{y})) \cdot \mathbf{n} = (\alpha_m - \alpha_f)(\omega_0^{(2)}(\mathbf{y}) + \omega_2^{(0)}(\mathbf{y}))$  we reduce the residual to order 3:

$$d_{3,k}^\varepsilon(\mathbf{x}) = d_{2,k}^\varepsilon(\mathbf{x}) - \varepsilon^2 v^{(2)}(\mathbf{x}) - \varepsilon^3 V_k^{(2)}\left(\frac{\mathbf{x}}{\varepsilon}\right); \quad k \in \{f, m\}. \quad (2.15)$$

The update of the difference allows us to obtain the third-order expansion:

$$u_{\varepsilon,k}^{[3]}(\mathbf{x}) = u^{(0)}(\mathbf{x}) + \varepsilon V_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 V_k^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 v^{(2)}(\mathbf{x}) + \varepsilon^3 V_k^{(2)}\left(\frac{\mathbf{x}}{\varepsilon}\right); \quad k \in \{f, m\}. \quad (2.16)$$



Then  $d_{3,k}^\varepsilon$  ( $k \in \{f, m\}$ ) solves the following problem:

$$\alpha_f \Delta d_{3,f}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_f^\varepsilon, \quad (2.17a)$$

$$\alpha_m \Delta d_{3,m}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_m^\varepsilon, \quad (2.17b)$$

$$d_{3,f}^\varepsilon(\mathbf{x}) = d_{3,m}^\varepsilon(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (2.17c)$$

$$\begin{aligned} & \alpha_f \partial_{\mathbf{n}} d_{3,f}^\varepsilon(\mathbf{x}) = \alpha_m \partial_{\mathbf{n}} d_{3,m}^\varepsilon(\mathbf{x}) \\ & + (\alpha_m - \alpha_f) (\partial_{\mathbf{n}} u^{(0)}(\varepsilon \mathbf{y}) - \omega_0^{(0)}(\mathbf{y}) - \varepsilon \omega_0^{(1)}(\mathbf{y}) - \varepsilon^2 \omega_0^{(2)}(\mathbf{y})) \\ & \quad + \varepsilon^2 (\alpha_m - \alpha_f) (\partial_{\mathbf{n}} v^{(2)}(\varepsilon \mathbf{y}) - \omega_2^{(0)}(\mathbf{y})) \quad \text{on } \Gamma^\varepsilon, \end{aligned} \quad (2.17d)$$

$$d_{3,m}^\varepsilon(\mathbf{x}) = \varepsilon \left( \varepsilon \frac{a_0^{(1)}(\theta)}{|\mathbf{x}|} - V_m^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^2 V_m^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^3 V_m^{(2)}\left(\frac{\mathbf{x}}{\varepsilon}\right) \right) \quad \text{on } \Gamma. \quad (2.17e)$$

## 2.6 Fourth-order expansion

Why going to fourth order? From problems solved by  $V^{(0)}$ ,  $V^{(1)}$  and  $V^{(2)}$  we are able to conclude about the general form of functions  $\varphi^{(n)}$  involved in problems solved by  $V^{(n)}$ . In fact, step by step we arrive at the fact that  $V^{(n)}$  solves Problem (2.4) for  $\varphi^{(n)}(\mathbf{y}) = (\alpha_m - \alpha_f) \sum_{p+q=n} \omega_q^{(p)}(\mathbf{y}) \cdot \mathbf{n}$ , ( $q \neq 1$ ) but we need the fourth order to conclude about the general expression of functions  $\psi^{(n)}$  involved in problems solved by  $v^{(n)}$ .

Now we expand  $V_m^{(1)}$  (see Section (4)):

$$V_m^{(1)}(\mathbf{y}) = \sum_{i=1}^K \frac{a_1^{(i)}(\theta)}{|\mathbf{y}|^i} + O\left(\frac{1}{|\mathbf{y}|^{K+1}}\right) = \sum_{i=1}^K \varepsilon^k \frac{a_1^{(i)}(\theta)}{|\mathbf{x}|^i} + O(\varepsilon^{K+1}); \quad \mathbf{x} \in \Gamma.$$

Following once again the same approach, with  $v^{(3)}$  solution of Problem (2.12) for  $\psi^{(3)}(\mathbf{x}) = -\frac{a_1^{(1)}(\theta)}{|\mathbf{x}|} - \frac{a_0^{(2)}(\theta)}{|\mathbf{x}|^2}$ , as done before we expand  $\nabla v^{(3)}(\varepsilon \mathbf{y})$  as follows:

$$\nabla v^{(3)}(\varepsilon \mathbf{y}) = \sum_{i=0}^K \varepsilon^{(i)} \omega_3^{(i)}(\mathbf{y}) + O(\varepsilon^{K+1}). \quad (2.18)$$

With  $V^{(3)}$  solution of Problem (2.4) for  $\varphi^{(3)}(\mathbf{y}) = (\alpha_m - \alpha_f) (\omega_0^{(3)}(\mathbf{y}) + \omega_2^{(1)}(\mathbf{y}) + \omega_3^{(0)}(\mathbf{y})) \cdot \mathbf{n} = (\alpha_m - \alpha_f) (\omega_0^{(3)}(\mathbf{y}) + \omega_2^{(1)}(\mathbf{y}) + \omega_3^{(0)}(\mathbf{y}))$  we reduce the residual to order 4:

$$d_{4,k}^\varepsilon(\mathbf{x}) = d_{3,k}^\varepsilon(\mathbf{x}) - \varepsilon^3 v^{(3)}(\mathbf{x}) - \varepsilon^4 V_k^{(3)}\left(\frac{\mathbf{x}}{\varepsilon}\right); \quad k \in \{f, m\}. \quad (2.19)$$

The update of the difference allows us to obtain the fourth-order approximation ( $k \in \{f, m\}$ ):

$$u_{\varepsilon,k}^{[4]}(\mathbf{x}) = u^{(0)}(\mathbf{x}) + \varepsilon V_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 V_k^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 v^{(2)}(\mathbf{x}) + \varepsilon^3 V_k^{(2)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^3 v^{(3)}(\mathbf{x}) + \varepsilon^4 V_k^{(3)}\left(\frac{\mathbf{x}}{\varepsilon}\right). \quad (2.20)$$

Then  $d_{4,k}^\varepsilon$  ( $k \in \{f, m\}$ ) solves the following problem:

$$\alpha_f \Delta d_{4,f}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_f^\varepsilon, \quad (2.21a)$$

$$\alpha_m \Delta d_{4,m}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_m^\varepsilon, \quad (2.21b)$$

$$d_{4,f}^\varepsilon(\mathbf{x}) = d_{4,m}^\varepsilon(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (2.21c)$$

$$\begin{aligned} & \alpha_f \partial_{\mathbf{n}} d_{4,f}^\varepsilon(\mathbf{x}) = \alpha_m \partial_{\mathbf{n}} d_{4,m}^\varepsilon(\mathbf{x}) \\ & + (\alpha_m - \alpha_f) (\partial_{\mathbf{n}} u^{(0)}(\varepsilon \mathbf{y}) - \omega_0^{(0)}(\mathbf{y}) - \varepsilon \omega_0^{(1)}(\mathbf{y}) - \varepsilon^2 \omega_0^{(2)} - \varepsilon^3 \omega_0^{(3)}(\mathbf{y})) \\ & + \varepsilon^2 (\alpha_m - \alpha_f) (\partial_{\mathbf{n}} v^{(2)}(\varepsilon \mathbf{y}) - \omega_2^{(0)}(\mathbf{y}) - \varepsilon \omega_2^{(1)}(\mathbf{y})) \\ & + \varepsilon^3 (\alpha_m - \alpha_f) (\partial_{\mathbf{n}} v^{(3)}(\varepsilon \mathbf{y}) - \omega_3^{(0)}(\mathbf{y})) \quad \text{on } \Gamma^\varepsilon, \end{aligned} \quad (2.21d)$$

$$\begin{aligned} d_{4,m}^\varepsilon(\mathbf{x}) &= \varepsilon (\varepsilon^2 \frac{a_0^{(2)}(\theta)}{|\mathbf{x}|^2} + \varepsilon \frac{a_0^{(1)}(\theta)}{|\mathbf{x}|} - V_m^{(0)}(\frac{\mathbf{x}}{\varepsilon})) \\ & + \varepsilon^2 (\varepsilon \frac{a_1^{(1)}(\theta)}{|\mathbf{x}|} - V_m^{(1)}(\frac{\mathbf{x}}{\varepsilon})) - \varepsilon^3 V_m^{(2)}(\frac{\mathbf{x}}{\varepsilon}) \quad \text{on } \Gamma. \end{aligned} \quad (2.21e)$$

## 2.7 The general case

From the previous section we are able to deduce the general forms of  $\varphi^{(n)}$  and  $\psi^{(n)}$  so we can update our residual to the order  $n$ ,  $n \geq 3$ , as follows:

$$d_{n,k}^\varepsilon(\mathbf{x}) = d_{n-1,k}^\varepsilon(\mathbf{x}) - \varepsilon^{n-1} v^{(n-1)}(\mathbf{x}) - \varepsilon^n V_k^{(n-1)}(\frac{\mathbf{x}}{\varepsilon}); \quad k \in \{f, m\}. \quad (2.22)$$

$v^{(n-1)}$  solves Problem (2.12) for  $\psi^{(n-1)}(\mathbf{x}) = -\sum_{p+q=n-2} \frac{a_q^{(p)}(\theta)}{|\mathbf{x}|^p}$ , ( $p \geq 1$ ) then as usual we expand  $\nabla v^{(n-1)}(\varepsilon \mathbf{y})$ :

$$\nabla v^{(n-1)}(\varepsilon \mathbf{y}) = \sum_{i=0}^{K-n+1} \varepsilon^i \omega_{n-1}^{(i)}(\mathbf{y}) + O(\varepsilon^{K+2-n}). \quad (2.23)$$

Using the last expansion we take for  $V^{(n-1)}$  the solution of Problem (2.4) with  $\varphi^{(n-1)} = (\alpha_m - \alpha_f) \sum_{p+q=n-1} \omega_q^{(p)}(\mathbf{y}) \cdot \mathbf{n}$ , ( $q \neq 1$ ). We are now able to build an expansion  $u_\varepsilon^{[K]}(\mathbf{x})$  to the order ' $K$ ', ( $K \geq 3$ ):

$$u_{\varepsilon,k}^{[K]}(\mathbf{x}) = u^{(0)}(\mathbf{x}) + \sum_{i=2}^{K-1} \varepsilon^i v^{(i)}(\mathbf{x}) + \sum_{i=1}^K \varepsilon^i V_k^{(i-1)}(\frac{\mathbf{x}}{\varepsilon}); \quad k \in \{f, m\}. \quad (2.24)$$

**Remark 2.1.** *Indeed, this is not the expansion to the order  $K$  in the true sense of term. It is the expansion to the order  $K$  truncated of a term. To see this more clearly, let us take the example where  $K$  would be equal to two, analyzing the expression of  $u_\varepsilon^{[2]}(\mathbf{x})$  we notice initially that the expansion has a term in  $\varepsilon^3$  ( $\varepsilon^2 V^{(1)}(\frac{\mathbf{x}}{\varepsilon})$ ). Second, we needed to bring up the residual to order 2 to obtain this expansion but if we look at the residual at the order 3 we can see that a term in  $\varepsilon^2$  appears.*

Now, replacing  $u_\varepsilon^{[K]}(\mathbf{x})$  in Problem (1.1) we obtain the following problem:

$$\alpha_f \Delta u_\varepsilon^{[K]}(\mathbf{x}) = 0 \quad \text{in } \Omega_f^\varepsilon, \quad (2.25a)$$

$$\alpha_m \Delta u_\varepsilon^{[K]}(\mathbf{x}) = 0 \quad \text{in } \Omega_m^\varepsilon, \quad (2.25b)$$

$$\llbracket u_\varepsilon^{[K]}(\mathbf{x}) \rrbracket = 0 \quad \text{on } \Gamma^\varepsilon, \quad (2.25c)$$

$$\llbracket \alpha \partial_{\mathbf{n}} u_\varepsilon^{[K]}(\mathbf{x}) \rrbracket = g_\varepsilon(\mathbf{y}) \quad \text{on } \Gamma^\varepsilon, \quad (2.25d)$$

$$u_\varepsilon^{[K]}(\mathbf{x}) = h_\varepsilon(\mathbf{x}) + F \quad \text{on } \Gamma. \quad (2.25e)$$

$$g_\varepsilon(\mathbf{y}) = (\alpha_m - \alpha_f)\partial_{\mathbf{n}}u^{(0)}(\varepsilon\mathbf{y}) + \sum_{i=2}^{K-1} \varepsilon^i(\alpha_m - \alpha_f)\partial_{\mathbf{n}}v^{(i)}(\varepsilon\mathbf{y}) + \sum_{i=1}^K \varepsilon^{i-1} \llbracket \alpha\partial_{\mathbf{n}}V^{(i-1)}(\mathbf{y}) \rrbracket_{\Gamma_1}. \quad (2.26)$$

Using the expressions of  $\partial_{\mathbf{n}}v^{(i)}(\varepsilon\mathbf{y})$  and the jump  $\llbracket \alpha\partial_{\mathbf{n}}V^{(i-1)}(\mathbf{y}) \rrbracket_{\Gamma_1}$  we obtain:

$$\begin{aligned} g_\varepsilon(\mathbf{y}) &= (\alpha_m - \alpha_f)\partial_{\mathbf{n}}u^{(0)}(\varepsilon\mathbf{y}) + \sum_{i=2}^{K-1} \varepsilon^i [(\alpha_m - \alpha_f) \sum_{l=0}^{K-i} \boldsymbol{\omega}_i^{(l)}(\mathbf{y}) \cdot \mathbf{n}\varepsilon^l + O(\varepsilon^{K-i+1})] \\ &\quad + \sum_{i=1}^K \varepsilon^{i-1} [(\alpha_f - \alpha_m) \sum_{p+q=i-1; q \neq 1} \boldsymbol{\omega}_q^{(p)}(\mathbf{y}) \cdot \mathbf{n}] + O(\varepsilon^{K+1}) = \varepsilon^K f(\mathbf{y}). \end{aligned} \quad (2.27)$$

The last step is to evaluate the  $L^2$  norm of  $\varepsilon^K f(\mathbf{y})$  on  $\Gamma^\varepsilon$ :

$$\|\varepsilon^K f(\frac{\mathbf{x}}{\varepsilon})\|_{L^2(\Gamma^\varepsilon)}^2 = \int_{\Gamma^\varepsilon} \varepsilon^{2K+1} |f(\frac{\mathbf{x}}{\varepsilon})|^2 d\mathbf{x} = \int_{\Gamma^1} \varepsilon^{2K+1} |f(\mathbf{y})|^2 \varepsilon d\mathbf{y} = C\varepsilon^{2K+1}. \quad (2.28)$$

Finally, we obtain:

$$\|g_\varepsilon(\mathbf{y})\|_{L^2(\Gamma^\varepsilon)} = \|\varepsilon^K f(\frac{\mathbf{x}}{\varepsilon})\|_{L^2(\Gamma^\varepsilon)} = O(\varepsilon^{K+\frac{1}{2}}). \quad (2.29)$$

By applying exactly the same approach we obtain:

$$\|h_\varepsilon(\mathbf{x})\|_{H^s(\Gamma)} = O(\varepsilon^K). \quad (2.30)$$

With  $d_{K,k}^\varepsilon = u_k^\varepsilon - u_{\varepsilon,k}^{[K]}(\mathbf{x})$ ;  $k \in \{f, m\}$  we have:

$$\alpha_f \Delta d_{K,f}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_f^\varepsilon, \quad (2.31a)$$

$$\alpha_m \Delta d_{K,m}^\varepsilon(\mathbf{x}) = 0 \quad \text{in } \Omega_m^\varepsilon, \quad (2.31b)$$

$$d_{K,f}^\varepsilon(\mathbf{x}) = d_{K,m}^\varepsilon(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (2.31c)$$

$$\alpha_f \partial_{\mathbf{n}} d_{K,f}^\varepsilon(\mathbf{x}) = \alpha_m \partial_{\mathbf{n}} d_{K,m}^\varepsilon(\mathbf{x}) + O_{L^2(\Gamma^\varepsilon)}(\varepsilon^{K+\frac{1}{2}}) \quad \text{on } \Gamma^\varepsilon, \quad (2.31d)$$

$$d_{K,m}^\varepsilon(\mathbf{x}) = O_{H^{\frac{1}{2}}(\Gamma)}(\varepsilon^K) \quad \text{on } \Gamma. \quad (2.31e)$$

### 3 Error estimate for the antiplane problem

In this section we establish an error estimate for the studied problem by increasing the  $H^1$  norm by powers of  $\varepsilon$  (characteristic dimension of the inclusion). First we begin by considering the following problem for a given  $g \in H^{-\frac{1}{2}}(\Gamma^\varepsilon)$  and  $h \in H^{\frac{1}{2}}(\Gamma)$ :

$$\alpha_f \Delta_x z_f(\mathbf{x}) = 0 \quad \text{in } \Omega_f^\varepsilon, \quad (3.1a)$$

$$\alpha_m \Delta_x z_m(\mathbf{x}) = 0 \quad \text{in } \Omega_m^\varepsilon, \quad (3.1b)$$

$$z_f(\mathbf{x}) = z_m(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (3.1c)$$

$$\alpha_f \partial_{\mathbf{n}} z_f(\mathbf{x}) = \alpha_m \partial_{\mathbf{n}} z_m(\mathbf{x}) + g(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (3.1d)$$

$$z_m(\mathbf{x}) = h(\mathbf{x}) \quad \text{on } \Gamma. \quad (3.1e)$$

By passing to a weak form we obtain the following equivalent problem: Find  $z \in H^1(\Omega)$  with  $z = h$  on  $\Gamma$  such that:

$$\int_{\Omega_f^\varepsilon} (\nabla z) \cdot (\nabla w) \, d\mathbf{x} + \int_{\Omega_m^\varepsilon} (\nabla z) \cdot (\nabla w) \, d\mathbf{x} + \int_{\Gamma^\varepsilon} g w \, d\sigma = 0 \text{ for all } \omega \in H^1(\Omega) \text{ such that } \omega = 0 \text{ on } \Gamma. \quad (3.2)$$

**Lemma 3.1.** *The existence and the uniqueness of the solution of Problem (3.1) in a weak form (3.2) is guaranteed thanks to Lax Milgram theorem and there is a constant  $C > 0$  independent of  $\varepsilon$  such that:*

$$\|z\|_{H^1(\Omega^\varepsilon)} \leq C(\|h\|_{H^{\frac{1}{2}}(\Gamma)} + \|g\|_{H^{-\frac{1}{2}}(\Gamma^\varepsilon)}).$$

*Proof.* The surjectivity of the trace operator guarantee the existence of a continuous lifting operator  $H \in H^1(\Omega)$  of  $h \in H^{\frac{1}{2}}(\Gamma)$  (see [11], [14]): Then  $\bar{z} = z - H$  solves the following equation for all  $\omega \in H^1(\Omega)$  such that  $\omega = 0$  on  $\Gamma$ :

$$\int_{\Omega_f^\varepsilon} (\nabla \bar{z}) \cdot (\nabla w) \, d\mathbf{x} + \int_{\Omega_m^\varepsilon} (\nabla \bar{z}) \cdot (\nabla w) \, d\mathbf{x} + \int_{\Gamma^\varepsilon} g w \, d\sigma + \int_{\Omega^0} (\nabla H) \cdot (\nabla w) \, d\mathbf{x} = 0. \quad (3.3)$$

With  $a(u, v) = \int_{\Omega_f^\varepsilon} (\nabla u) \cdot (\nabla v) \, d\mathbf{x} + \int_{\Omega_m^\varepsilon} (\nabla u) \cdot (\nabla v) \, d\mathbf{x}$  the bilinear form associated to the weak formulation, it is easy to prove that  $a(u, v)$  is coercive ( $\alpha \|\bar{z}\|_{H^1(\Omega^\varepsilon)}^2 \leq a(\bar{z}, \bar{z})$ ) (see [11], [14]). Using the coercivity of the bilinear form we obtain:

$$\alpha \|\bar{z}\|_{H^1(\Omega^\varepsilon)}^2 \leq - \int_{\Gamma^\varepsilon} g \bar{z} \, d\sigma - \int_{\Omega^\varepsilon} (\nabla H) \cdot (\nabla \bar{z}) \, d\mathbf{x}. \quad (3.4)$$

$$\alpha \|\bar{z}\|_{H^1(\Omega^\varepsilon)}^2 \leq \|g\|_{H^{-\frac{1}{2}}(\Gamma^\varepsilon)} \|\bar{z}\|_{H^{\frac{1}{2}}(\Gamma^\varepsilon)} + \|\nabla H\|_{L^2(\Omega^\varepsilon)} \|\nabla \bar{z}\|_{L^2(\Omega^\varepsilon)}. \quad (3.5)$$

Using the facts that  $\|\bar{z}\|_{H^{\frac{1}{2}}(\Gamma^\varepsilon)} \leq C \|\bar{z}\|_{H^1(\Omega^\varepsilon)}$ ,  $\|\nabla \bar{z}\|_{L^2(\Omega^\varepsilon)} \leq C \|\bar{z}\|_{H^1(\Omega^\varepsilon)}$  and  $\|\nabla H\|_{L^2(\Omega^\varepsilon)} \leq C \|H\|_{H^1(\Omega^\varepsilon)}$  we obtain:

$$\alpha \|\bar{z}\|_{H^1(\Omega^\varepsilon)}^2 \leq C(\|g\|_{H^{-\frac{1}{2}}(\Gamma^\varepsilon)} + \|H\|_{H^1(\Omega^\varepsilon)}) \|\bar{z}\|_{H^1(\Omega^\varepsilon)}. \quad (3.6)$$

The continuity of the lifting operator gives us that  $\|H\|_{H^1(\Omega^\varepsilon)} \leq C \|h\|_{H^{\frac{1}{2}}(\Gamma)}$  so we can write:

$$\|\bar{z}\|_{H^1(\Omega^\varepsilon)} \leq C(\|g\|_{H^{-\frac{1}{2}}(\Gamma^\varepsilon)} + \|h\|_{H^{\frac{1}{2}}(\Gamma)}). \quad (3.7)$$

□

We can now state the main result:

**Theorem 3.2.** *There exists a constant  $C > 0$  independent of  $\varepsilon$  such that:*

$$\|d_K^\varepsilon\|_{H^1(\Omega^\varepsilon)} \leq C \varepsilon^K ; \text{ for all } K \in \mathbb{N},$$

where  $d_K^\varepsilon$  is the residue introduced in Section (2).

*Proof.* With  $z = d_K^\varepsilon$  we have  $g \in O_{L^2(\Gamma^\varepsilon)}(\varepsilon^{K+\frac{1}{2}})$  and  $h \in O_{H^{\frac{1}{2}}(\Gamma)}(\varepsilon^K)$  (see Section (2.7)). □

We are now able to show the contribution of a small inclusion in an elastic media at any order  $K$  for the Poisson problem. We observe the appearance of two types of functions due to the correction of terms on the interfaces  $\Gamma^\varepsilon$  and  $\Gamma$ . There are  $v^{(k)}$  posed on a fixed domain independent of  $\varepsilon$   $\Omega^0$  and  $V^{(k)}$  posed on an infinite domain  $\Omega^\infty$ .

## 4 Expansion of the $V_m^{(p)}$ function at infinity

Recall that the  $V_m^{(p)}$  functions solve Problem (2.4). Let  $B(O, R)$  be the ball of center  $O$  and radius  $R > 0$  such that  $\Omega^1 \subset B(O, R)$ . Then the trace of  $V_m^{(p)}$  on the curve  $\partial B(O, R) \cap \Omega^\infty$  is smooth and can be written as the sum of its Fourier series:

$$V_m^{(p)}(R, \theta) = \alpha_0^{(p)} + \sum_{n \geq 1} \alpha_n^{(p)} \cos(n\theta) + \beta_n^{(p)} \sin(n\theta). \quad (4.1)$$

Using Poisson's Kernel, we then get that:

$$V_m^{(p)}(r, \theta) = \alpha_0^{(p)} + \sum_{n \geq 1} \frac{R^n}{r^n} (\alpha_n^{(p)} \cos(n\theta) + \beta_n^{(p)} \sin(n\theta)). \quad (4.2)$$

The behavior of the  $V_m^{(p)}$  at infinity prescribes (i.e.  $V_k^{(p)} \in V$  see Section (5))  $\alpha_0^{(p)} = 0$ .

## 5 Existence and uniqueness in unbounded domains for the Poisson problem

To ensure the existence and the uniqueness of  $V^{(p)}$  functions the first thing to do is to transform the strong formulation (Problem (2.4)) into a weak one as follows:

$$\alpha_f \int_{\Omega^1} (\nabla V_f^{(p)}) \cdot (\nabla w) \, d\mathbf{x} + \alpha_m \int_{\Omega^\infty} (\nabla V_m^{(p)}) \cdot (\nabla w) \, d\mathbf{x} = \int_{\Gamma^1} \varphi w \, d\sigma ; \text{ for all } w \in V. \quad (5.1)$$

The space  $W_{0,0}^{1,2}$  introduced in the annex contains constants and the Poincaré-type inequality is established on  $W_{0,0}^{1,2}/P^0$  (the quotient space of the adequate weighted Sobolev space with constants) in annex (A). To fix the constant, we introduce an additional condition. In fact, introducing space  $V = \{v \in W_{0,0}^{1,2}; \int_{\Gamma_{R_0}} v = 0\}$  ( $0 < R_0 < R$  an arbitrary real number) we can see that the bilinear form  $a(z, w) = \alpha_f \int_{\Omega^1} (\nabla z) \cdot (\nabla w) \, d\mathbf{x} + \alpha_m \int_{\Omega^\infty} (\nabla z) \cdot (\nabla w) \, d\mathbf{x}$  is coercive on  $V$ . In addition the bilinear form  $a(\cdot, \cdot)$  and the linear one  $l(w) = \int_{\Gamma^1} \varphi w \, d\sigma$  are obviously continuous on  $V$ . Then thanks to Lax Milgram theorem there is existence and uniqueness of  $V^{(p)}$  functions in  $V$ .

## 6 Solution for a circular inclusion using the variable-separation method

In this section, we solve first Problem (1.1) by the variable-separation method then we solve Problem (2.4) by the same method and compare the two results for the first order of the asymptotic expansion. We will use the same geometrical configuration but with a circular shape for the edges  $\Gamma^\varepsilon$  and  $\Gamma^1$  (see Fig. 3).

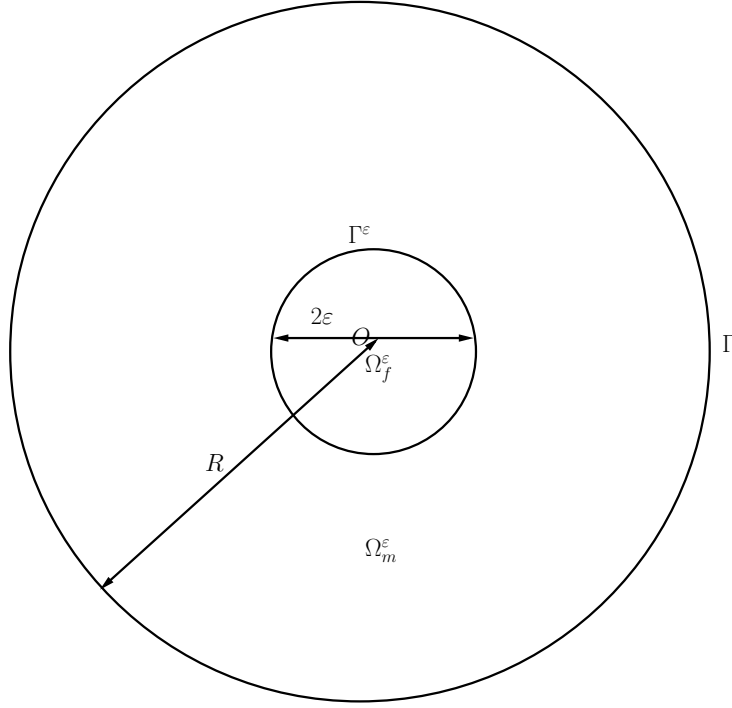


Figure 3: circular inclusion.

### 6.1 Antiplane problem solve using the variable-separation method

Considering polar coordinates  $(r, \theta)$  and the following particular form for the  $u_k^\varepsilon$ ;  $k \in \{f, m\}$ :

$$u_k^\varepsilon(r, \theta) = f_k(r)g_k(\theta). \quad (6.1)$$

we obtain from equations (1.1a) and (1.1b) the following equations:

$$r^2 \frac{f_k''(r)}{f_k(r)} + r \frac{f_k'(r)}{f_k(r)} = -\frac{g_k''(\theta)}{g(\theta)}; k \in \{f, m\}. \quad (6.2)$$

The left hand side of these equations is independent of  $\theta$  and the right one is independent of  $r$ . Then they are both equal to the same constant, called separation constant, that can be proven to be equal to  $n_k^2$  with  $n_k \in \mathbb{N}$ . This leads to the system of two decoupled equations:

$$r^2 f_k''(r) + r f_k'(r) - n_k^2 f_k(r) = 0, \quad (6.3a)$$

$$g_k''(\theta) + n_k^2 g_k(\theta) = 0. \quad (6.3b)$$

Using the equation (1.1c) we can see that  $g_1(\theta) = g_2(\theta)$  and then  $n_k = n$ . We observe now that we can search solutions of the forms of  $r^\alpha$  and thus we obtain:

$$f_f(r) = C_1 r^n + c r^{-n}, \quad (6.4a)$$

$$f_m(r) = C_2 r^n + C_3 r^{-n}. \quad (6.4b)$$

The solution must be bounded at the origin then  $c = 0$ . The next step is to use the continuity of the fields and their normal derivatives throughout the interface  $\Gamma^\varepsilon$  to determinate the constants

$C_1$ ,  $C_2$  and  $C_3$ . In fact, from equations (1.1c) and (1.1d) we obtain with  $\beta = \frac{\alpha_m - \alpha_f}{\alpha_m + \alpha_f}$  and  $h = \frac{\varepsilon}{R}$  the following expressions:

$$C_1(n) = \frac{R^{-n}(1 + \beta)}{1 + \beta h^{2n}}, \quad (6.5a)$$

$$C_2(n) = \frac{R^{-n}}{1 + \beta h^{2n}}, \quad (6.5b)$$

$$C_3(n) = \frac{R^n \beta h^{2n}}{1 + \beta h^{2n}}. \quad (6.5c)$$

By developing  $C_1$ ,  $C_2$  and  $C_3$  with respect to the small parameter  $h$  and taking  $n = 1$  we can bring up the first contribution of the inclusion in the expression of matrix field:

$$u_m^\varepsilon = u_m^{(0)} + h u_m^{(1)} + h^2 u_m^{(2)}. \quad (6.6)$$

since  $u_m^{(1)}$  is zero and  $u_m^{(0)}$ ,  $u_m^{(2)}$  are independent of  $\varepsilon$  so we can conclude that the first contribution of the inclusion is of order  $\varepsilon^2$ .

## 6.2 Solving $V^{(0)}$ by the variable-separation method

Thanks to the variable-separation method and using polar coordinates  $(r_s, \theta)$  ( $r_s = \frac{r}{\varepsilon}$ ) we decompose  $V_k^{(0)}$  functions as follows:

$$V_k^{(0)}(r_s, \theta) = \sum_{n \geq 1} f_n^{(k)}(r_s) \cos(n\theta) + g_n^{(k)}(r_s) \sin(n\theta); k \in \{f, m\}. \quad (6.7)$$

Injecting (6.7) in (2.4a) and (2.4b) and solving the differential equations we obtain:

$$f_n^k = A_k r_s^n + a_k r_s^{-n}, \quad (6.8a)$$

$$g_n^k = B_k r_s^n + b_k r_s^{-n}. \quad (6.8b)$$

Using the interface conditions prescribed by (2.4c) and (2.4d) we obtain:

$$V_m^{(0)}(r_s, \theta) = \frac{\alpha_m - \alpha_f}{r_s} \left( \frac{a}{\alpha_f + \alpha_m} \cos \theta + \frac{b}{\alpha_f + \alpha_m} \sin \theta \right). \quad (6.9)$$

with  $\nabla u^{(0)}(\mathbf{0}) = (a \ b)^t$ . Thus, Solving directly Problem (1.1) by the variable-separation method we obtain that the first contribution of the inclusion is in  $\varepsilon^2$ . Now injecting (6.9) in (2.7) we obtain:

$$u_{\varepsilon, m}^{[1]}(r, \theta) = u^{(0)} + \frac{\varepsilon^2}{r} \left( \frac{(\alpha_m - \alpha_f)a}{\alpha_f + \alpha_m} \cos \theta + \frac{(\alpha_m - \alpha_f)b}{\alpha_f + \alpha_m} \sin \theta \right). \quad (6.10)$$

The first contribution is indeed in  $\varepsilon^2$  which is in accordance with the result presented in Section (6.1).

## 7 Asymptotic expansion for the plane strain linear elasticity problem

In this section, we build the asymptotic expansion with respect to the influence of the inclusion  $\Omega_f^\varepsilon$  on the displacement field of the matrix  $\Omega_m^\varepsilon$  in the case of plane strain Problem (1.2).

## 7.1 Comparison with the solution without inclusion

Introducing  $\mathbf{u}^{(0)}$  the solution of Problem (1.2) with  $\varepsilon = 0$ , defined on the domain without inclusion  $\Omega^0$ , we introduce the difference  $\mathbf{d}_{0,k}^\varepsilon = \mathbf{u}_k^\varepsilon - \mathbf{u}^{(0)}$  ( $k \in \{f, m\}$ ). Then  $\mathbf{d}_0^\varepsilon$  is solution to:

$$\operatorname{div}_x \boldsymbol{\sigma}_f(\mathbf{d}_{0,f}^\varepsilon)(\mathbf{x}) = -\operatorname{div}_x (\mathbf{E}_f : \boldsymbol{\varepsilon}(\mathbf{u}^{(0)}))(\mathbf{x}) \quad \text{in } \Omega_f^\varepsilon, \quad (7.1a)$$

$$\operatorname{div}_x \boldsymbol{\sigma}_m(\mathbf{d}_{0,m}^\varepsilon)(\mathbf{x}) = \mathbf{0} \quad \text{in } \Omega_m^\varepsilon, \quad (7.1b)$$

$$\mathbf{d}_{0,f}^\varepsilon(\mathbf{x}) = \mathbf{d}_{0,m}^\varepsilon(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (7.1c)$$

$$\boldsymbol{\sigma}_f(\mathbf{d}_{0,f}^\varepsilon(\mathbf{x})) \cdot \mathbf{n}_f = -\boldsymbol{\sigma}_m(\mathbf{d}_{0,m}^\varepsilon(\mathbf{x})) \cdot \mathbf{n}_m + [(\mathbf{E}_m - \mathbf{E}_f) : \boldsymbol{\varepsilon}(\mathbf{u}^{(0)}(\mathbf{x}))] \cdot \mathbf{n}_f \quad \text{on } \Gamma^\varepsilon, \quad (7.1d)$$

$$\mathbf{d}_{0,m}^\varepsilon(\mathbf{x}) = \mathbf{0} \quad \text{on } \Gamma. \quad (7.1e)$$

This time, we note the emergence of two terms  $[(\mathbf{E}_m - \mathbf{E}_f) : \boldsymbol{\varepsilon}(\mathbf{u}^{(0)}(\mathbf{x}))] \cdot \mathbf{n}_f$  on  $\Gamma^\varepsilon$  which is similar to the term  $(\alpha_m - \alpha_f) \partial_{\mathbf{n}} u^{(0)}(\mathbf{x})$  of the antiplane problem and a new term  $-\operatorname{div}(\mathbf{E}_f : \boldsymbol{\varepsilon}(\mathbf{u}^{(0)}))(\mathbf{x})$  in  $\Omega_f^\varepsilon$ .

## 7.2 Scaling and approximation in infinite domain

The original problem is posed on domains depending on  $\varepsilon$ . Then, as done for the Poisson problem in Section (2.2), the first step is to formulate the problem on fixed domains. Using the scaling  $\mathbf{y} = \frac{\mathbf{x}}{\varepsilon}$  and denoting  $\Omega^{m,\varepsilon} = \frac{\Omega_m^\varepsilon}{\varepsilon}$ ,  $\mathbf{D}_{0,k}(\mathbf{y}) = \mathbf{d}_{0,k}(\mathbf{x})$  ( $k \in \{f, m\}$ ) is solution to the following scaled problem:

$$\operatorname{div}_y \boldsymbol{\sigma}_f(\mathbf{D}_{0,f}^\varepsilon)(\mathbf{y}) = -\varepsilon^2 \operatorname{div}_x (\mathbf{E}_f : \boldsymbol{\varepsilon}(\mathbf{u}^{(0)}))(\varepsilon \mathbf{y}) \quad \text{in } \Omega^1, \quad (7.2a)$$

$$\operatorname{div}_y \boldsymbol{\sigma}_m(\mathbf{D}_{0,m}^\varepsilon)(\mathbf{y}) = \mathbf{0} \quad \text{in } \Omega^{m,\varepsilon}, \quad (7.2b)$$

$$\mathbf{D}_{0,f}^\varepsilon(\mathbf{y}) = \mathbf{D}_{0,m}^\varepsilon(\mathbf{y}) \quad \text{on } \Gamma^1, \quad (7.2c)$$

$$\begin{aligned} \boldsymbol{\sigma}_f(\mathbf{D}_{0,f}^\varepsilon)(\mathbf{y}) \cdot \mathbf{n}_f &= -\boldsymbol{\sigma}_m(\mathbf{D}_{0,m}^\varepsilon)(\mathbf{y}) \cdot \mathbf{n}_m \\ &+ \varepsilon [(\mathbf{E}_m - \mathbf{E}_f) : \boldsymbol{\varepsilon}_x(\mathbf{u}^{(0)}(\varepsilon \mathbf{y}))] \cdot \mathbf{n}_f \quad \text{on } \Gamma^1, \end{aligned} \quad (7.2d)$$

$$\mathbf{D}_{0,m}^\varepsilon(\mathbf{y}) = \mathbf{0} \quad \text{on } \frac{\Gamma}{\varepsilon}. \quad (7.2e)$$

Now, as  $\varepsilon$  is considered small compared to the dimensions of the geometry, we approximate Problem (7.2) by the following problem posed on the dilated geometry  $\Omega^\infty = \lim \Omega^{m,\varepsilon}$  when  $\varepsilon \rightarrow 0$ :

$$\operatorname{div}_y \boldsymbol{\sigma}_1(\mathbf{D}_{0,f}^\varepsilon)(\mathbf{y}) = -\varepsilon^2 \operatorname{div}_x (\mathbf{E}_f : \boldsymbol{\varepsilon}(\mathbf{u}^{(0)}))(\varepsilon \mathbf{y}) \quad \text{in } \Omega^1, \quad (7.3a)$$

$$\operatorname{div}_y \boldsymbol{\sigma}_m(\mathbf{D}_{0,m}^\varepsilon)(\mathbf{y}) = \mathbf{0} \quad \text{in } \Omega^\infty, \quad (7.3b)$$

$$\mathbf{D}_{0,f}^\varepsilon(\mathbf{y}) = \mathbf{D}_{0,m}^\varepsilon(\mathbf{y}) \quad \text{on } \Gamma^1, \quad (7.3c)$$

$$\begin{aligned} \boldsymbol{\sigma}_f(\mathbf{D}_{0,f}^\varepsilon)(\mathbf{y}) \cdot \mathbf{n}_f &= -\boldsymbol{\sigma}_m(\mathbf{D}_{0,m}^\varepsilon)(\mathbf{y}) \cdot \mathbf{n}_m \\ &+ \varepsilon [(\mathbf{E}_m - \mathbf{E}_f) : \boldsymbol{\varepsilon}_x(\mathbf{u}^{(0)}(\varepsilon \mathbf{y}))] \cdot \mathbf{n}_f \quad \text{on } \Gamma^1, \end{aligned} \quad (7.3d)$$

$$\mathbf{D}_{0,m}^\varepsilon(\mathbf{y}) = \mathbf{0} \quad \text{when } |\mathbf{y}| \rightarrow \infty. \quad (7.3e)$$

## 7.3 Asymptotic expansion: first-order term

Let us first introduce two problems which will allow us to construct the different orders of the asymptotic expansion of the inclusion contribution. The two of them are posed on the dilated



geometry  $\Omega^\infty$ . For each order  $p$  we consider the functions  $\mathbf{V}^{(p)}$  and  $\mathbf{Z}^{(p)}$  solutions to:

$$\operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma}_f(\mathbf{V}_f^{(p)})(\mathbf{y}) = \mathbf{0} \quad \text{in } \Omega^1, \quad (7.4a)$$

$$\operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma}_m(\mathbf{V}_m^{(p)})(\mathbf{y}) = \mathbf{0} \quad \text{in } \Omega^\infty, \quad (7.4b)$$

$$\mathbf{V}_f^{(p)}(\mathbf{y}) = \mathbf{V}_m^{(p)}(\mathbf{y}) \quad \text{on } \Gamma^1, \quad (7.4c)$$

$$\boldsymbol{\sigma}_f(\mathbf{V}_f^{(p)})(\mathbf{y}) \cdot \mathbf{n}_f = -\boldsymbol{\sigma}_m(\mathbf{V}_m^{(p)})(\mathbf{y}) \cdot \mathbf{n}_m + \boldsymbol{\varphi}^{(p)}(\mathbf{y}) \quad \text{on } \Gamma^1, \quad (7.4d)$$

$$\mathbf{V}_m^{(p)}(\mathbf{y}) \rightarrow \mathbf{0} \quad \text{when } |\mathbf{y}| \rightarrow \infty. \quad (7.4e)$$

$$\operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma}_f(\mathbf{Z}_f^{(p)})(\mathbf{y}) = \boldsymbol{\kappa}^{(p)}(\mathbf{y}) \quad \text{in } \Omega^1, \quad (7.5a)$$

$$\operatorname{div}_{\mathbf{y}} \boldsymbol{\sigma}_m(\mathbf{Z}_m^{(p)})(\mathbf{y}) = \mathbf{0} \quad \text{in } \Omega^\infty, \quad (7.5b)$$

$$\mathbf{Z}_f^{(p)}(\mathbf{y}) = \mathbf{Z}_m^{(p)}(\mathbf{y}) \quad \text{on } \Gamma^1, \quad (7.5c)$$

$$\boldsymbol{\sigma}_f(\mathbf{Z}_f^{(p)})(\mathbf{y}) \cdot \mathbf{n}_f = -\boldsymbol{\sigma}_m(\mathbf{Z}_m^{(p)})(\mathbf{y}) \cdot \mathbf{n}_m \quad \text{on } \Gamma^1, \quad (7.5d)$$

$$\mathbf{Z}_m^{(p)}(\mathbf{y}) \rightarrow \mathbf{0} \quad \text{when } |\mathbf{y}| \rightarrow \infty. \quad (7.5e)$$

$\mathbf{V}^{(p)}$  and  $\mathbf{Z}^{(p)}$  functions are introduced to offset terms appearing on the boundary  $\Gamma^\varepsilon$  and in the domain  $\Omega_f^\varepsilon$  where the data  $\boldsymbol{\varphi}^{(p)}(\mathbf{y})$  and  $\boldsymbol{\kappa}^{(p)}(\mathbf{y})$  will be adapted to each order  $p$ . The existence and uniqueness of the functions  $\mathbf{V}^{(p)}$  and  $\mathbf{Z}^{(p)}$  are established later in Section (10). From the regularity assumptions on the domain we can write Taylor expansions of  $\varepsilon_{\mathbf{x}}(\mathbf{u}^{(0)})(\varepsilon\mathbf{y})$  for  $\mathbf{y} \in \Gamma^1$  and  $-\operatorname{div}_{\mathbf{x}}(\mathbf{E}_f : \varepsilon(\mathbf{u}^{(0)}))(\varepsilon\mathbf{y})$  for  $\mathbf{y} \in \Omega^1$ :

$$\varepsilon_{\mathbf{x}}(\mathbf{u}^{(0)})(\varepsilon\mathbf{y}) = \sum_{i=0}^K \varepsilon^i \boldsymbol{\omega}_0^{(i)}(\mathbf{y}) + O(\varepsilon^{K+1}). \quad (7.6)$$

$$-\operatorname{div}_{\mathbf{x}}(\mathbf{E}_f : \varepsilon(\mathbf{u}^{(0)}))(\varepsilon\mathbf{y}) = \sum_{i=0}^K \varepsilon^i \boldsymbol{\gamma}_0^{(i)}(\mathbf{y}) + O(\varepsilon^{K+1}). \quad (7.7)$$

Then we construct the “first order” by compensating the first term in the expansion of  $[(\mathbf{E}_m - \mathbf{E}_f) : \varepsilon(\mathbf{u}^{(0)})(\varepsilon\mathbf{y})] \cdot \mathbf{n}_f$  and the first one in the development of  $-\operatorname{div}_{\mathbf{x}}(\mathbf{E}_f : \varepsilon(\mathbf{u}^{(0)}))(\varepsilon\mathbf{y})$ . With  $\mathbf{V}^{(0)}$  solving Problem (7.4) for  $\boldsymbol{\varphi}^{(0)}(\mathbf{y}) = [(\mathbf{E}_m - \mathbf{E}_f) : \boldsymbol{\omega}_0^{(0)}(\mathbf{y})] \cdot \mathbf{n}_f$  and  $\mathbf{Z}^{(0)}$  solving Problem (7.5) for  $\boldsymbol{\kappa}^{(0)}(\mathbf{y}) = \boldsymbol{\gamma}_0^{(0)}(\mathbf{y})$  we update the residual:

$$\mathbf{d}_{1,k}^\varepsilon(\mathbf{x}) = \mathbf{u}_k^\varepsilon(\mathbf{x}) - \mathbf{u}^{(0)}(\mathbf{x}) - \varepsilon \mathbf{V}_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^2 \mathbf{Z}_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right); \quad k \in \{f, m\}. \quad (7.8)$$

The update of the difference allows us to obtain the first-order expansion:

$$\mathbf{u}_{\varepsilon,k}^{[1]}(\mathbf{x}) = \mathbf{u}^{(0)}(\mathbf{x}) + \varepsilon \mathbf{V}_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 \mathbf{Z}_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right); \quad k \in \{f, m\}. \quad (7.9)$$

Then  $\mathbf{d}_{1,k}^\varepsilon$  ( $k \in \{f, m\}$ ) is solution to:

$$\operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_f(\mathbf{d}_{1,f}^\varepsilon)(\mathbf{x}) = -\operatorname{div}_{\mathbf{x}}(\mathbf{E}_f : \varepsilon(\mathbf{u}^{(0)}))(\varepsilon\mathbf{y}) - \boldsymbol{\gamma}_0^{(0)}(\mathbf{y}) \quad \text{in } \Omega_f^\varepsilon, \quad (7.10a)$$

$$\operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_m(\mathbf{d}_{1,m}^\varepsilon)(\mathbf{x}) = \mathbf{0} \quad \text{in } \Omega_m^\varepsilon, \quad (7.10b)$$

$$\mathbf{d}_{1,f}^\varepsilon(\mathbf{x}) = \mathbf{d}_{1,m}^\varepsilon(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (7.10c)$$

$$\begin{aligned} & \boldsymbol{\sigma}_f(\mathbf{d}_{1,f}^\varepsilon)(\mathbf{x}) \cdot \mathbf{n}_f = -\boldsymbol{\sigma}_m(\mathbf{d}_{1,m}^\varepsilon)(\mathbf{x}) \cdot \mathbf{n}_m \\ & + (\mathbf{E}_m - \mathbf{E}_f) : [\varepsilon_{\mathbf{x}}(\mathbf{u}^{(0)})(\varepsilon\mathbf{y})] - \boldsymbol{\omega}_0^{(0)}(\mathbf{y}) \cdot \mathbf{n}_f \quad \text{on } \Gamma^\varepsilon, \end{aligned} \quad (7.10d)$$

$$\mathbf{d}_{1,m}^\varepsilon(\mathbf{x}) = -\varepsilon \mathbf{V}_m^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^2 \mathbf{Z}_m^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) \quad \text{on } \Gamma. \quad (7.10e)$$

## 7.4 Building the second-order

We built a “first-order” expansion of the solution with  $\mathbf{u}^{(0)}$ ,  $\mathbf{V}^{(0)}$  and  $\mathbf{Z}^{(0)}$  independent of  $\varepsilon$ . The next step is about building the second order by compensating  $(\mathbf{E}_m - \mathbf{E}_f) : [\varepsilon(\mathbf{u}^{(0)}(\varepsilon\mathbf{y})) - \boldsymbol{\omega}_0^{(0)}(\mathbf{y})] \cdot \mathbf{n}_f$  on  $\Gamma^\varepsilon$  and  $-\operatorname{div}_{\mathbf{x}}(\mathbf{E}_f : \varepsilon(\mathbf{u}^{(0)}))(\varepsilon\mathbf{y}) - \gamma_0^{(0)}(\mathbf{y})$  in  $\Omega_1^\varepsilon$ . So, with  $\mathbf{V}^{(2)}$  solution of Problem (7.4) for  $\boldsymbol{\varphi}^{(1)}(\mathbf{y}) = [(\mathbf{E}_m - \mathbf{E}_f) : \boldsymbol{\omega}_0^{(1)}(\mathbf{y})] \cdot \mathbf{n}_f$  and  $\mathbf{Z}^{(1)}$  solving Problem (7.5) for  $\boldsymbol{\kappa}^{(1)}(\mathbf{y}) = \gamma_0^{(1)}(\mathbf{y})$  we update our residual as following:

$$\mathbf{d}_{2,k}^\varepsilon(\mathbf{x}) = \mathbf{d}_{1,k}^\varepsilon(\mathbf{x}) - \varepsilon^2 \mathbf{V}_k^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^3 \mathbf{Z}_k^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right). \quad (7.11)$$

The update of the difference allows us to obtain the second-order expansion:

$$\mathbf{u}_{\varepsilon,k}^{[2]}(\mathbf{x}) = \mathbf{u}^{(0)}(\mathbf{x}) + \varepsilon \mathbf{V}_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 \mathbf{Z}_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 \mathbf{V}_k^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^3 \mathbf{Z}_k^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right); k \in \{f, m\}. \quad (7.12)$$

Then  $\mathbf{d}_{2,k}^\varepsilon$  ( $k \in \{f, m\}$ ) solves the following problem:

$$\operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_f(\mathbf{d}_{2,f}^\varepsilon)(\mathbf{x}) = -\operatorname{div}_{\mathbf{x}}(\mathbf{E}_f : \varepsilon(\mathbf{u}^{(0)}))(\varepsilon\mathbf{y}) - \gamma_0^{(0)}(\mathbf{y}) - \varepsilon \gamma_0^{(1)}(\mathbf{y}) \text{ in } \Omega_f^\varepsilon, \quad (7.13a)$$

$$\operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_m(\mathbf{d}_{2,m}^\varepsilon)(\mathbf{x}) = \mathbf{0} \text{ in } \Omega_m^\varepsilon, \quad (7.13b)$$

$$\mathbf{d}_{2,f}^\varepsilon(\mathbf{x}) = \mathbf{d}_{2,m}^\varepsilon(\mathbf{x}) \text{ on } \Gamma^\varepsilon, \quad (7.13c)$$

$$\boldsymbol{\sigma}_f(\mathbf{d}_{2,f}^\varepsilon)(\mathbf{x}) \cdot \mathbf{n}_f = -\boldsymbol{\sigma}_m(\mathbf{d}_{2,m}^\varepsilon)(\mathbf{x}) \cdot \mathbf{n}_m$$

$$+(\mathbf{E}_m - \mathbf{E}_f) : [\varepsilon(\mathbf{u}^{(0)}(\varepsilon\mathbf{y})) - \boldsymbol{\omega}_0^{(0)}(\mathbf{y}) - \varepsilon \boldsymbol{\omega}_0^{(1)}(\mathbf{y})] \cdot \mathbf{n}_f \text{ on } \Gamma^\varepsilon, \quad (7.13d)$$

$$\mathbf{d}_{2,m}^\varepsilon(\mathbf{x}) = -\varepsilon \mathbf{V}_m^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^2 \mathbf{Z}_m^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^2 \mathbf{V}_m^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^3 \mathbf{Z}_m^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) \text{ on } \Gamma. \quad (7.13e)$$

## 7.5 Third-order expansion

We introduce here an equivalent problem to Problem (2.12):

$$\operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_m(\mathbf{v}^{(q)})(\mathbf{x}) = \mathbf{0} \quad \text{in } \Omega^0, \quad (7.14a)$$

$$\mathbf{v}^{(q)}(\mathbf{x}) = \boldsymbol{\psi}^{(q)}(\mathbf{x}) \quad \text{on } \Gamma. \quad (7.14b)$$

we need to expand  $\mathbf{V}_2^{(0)}$  at  $\infty$ :

$$\mathbf{V}_m^{(0)}(\mathbf{y}) = \sum_{i=1}^K \frac{\mathbf{a}_0^{(i)}(\theta)}{|\mathbf{y}|^i} + O\left(\frac{1}{|\mathbf{y}|^{K+1}}\right) = \sum_{i=1}^K \varepsilon^i \frac{\mathbf{a}_0^{(i)}(\theta)}{|\mathbf{x}|^i} + O(\varepsilon^{K+1}) \quad \mathbf{x} \in \Gamma. \quad (7.15)$$

The previous expansion is proven in Section (9) in the isotropic case.

Following the same approach, with  $\mathbf{v}^{(2)}$  solution of Problem (7.14) for  $\boldsymbol{\psi}^{(2)}(\mathbf{x}) = -\frac{\mathbf{a}_0^{(1)}(\theta)}{|\mathbf{x}|}$  we expand as done before  $\varepsilon(\mathbf{v}^{(2)})(\varepsilon\mathbf{y})$  as follows:

$$\varepsilon(\mathbf{v}^{(2)})(\varepsilon\mathbf{y}) = \sum_{i=0}^K \varepsilon^i \boldsymbol{\omega}_2^{(i)}(\mathbf{y}) + O(\varepsilon^{K+1}). \quad (7.16)$$

With  $\mathbf{V}^{(2)}$  solution of Problem (7.4) for  $\boldsymbol{\varphi}^{(2)}(\mathbf{y}) = (\mathbf{E}_m - \mathbf{E}_f) : (\boldsymbol{\omega}_0^{(2)}(\mathbf{y}) + \boldsymbol{\omega}_2^{(0)}(\mathbf{y})) \cdot \mathbf{n}_f$  and  $\mathbf{Z}^{(2)}$  solving Problem (7.5) for  $\boldsymbol{\kappa}^{(2)}(\mathbf{y}) = \gamma_0^{(2)}(\mathbf{y})$  we write the third-order expansion:

$$\mathbf{d}_{3,k}^\varepsilon(\mathbf{x}) = \mathbf{d}_{2,k}^\varepsilon(\mathbf{x}) - \varepsilon^2 \mathbf{v}^{(2)}(\mathbf{x}) - \varepsilon^3 \mathbf{V}_k^{(2)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^4 \mathbf{Z}_k^{(2)}\left(\frac{\mathbf{x}}{\varepsilon}\right); k \in \{f, m\}. \quad (7.17)$$

The update of the difference allows us to obtain the third-order expansion ( $k \in \{f, m\}$ ):

$$\mathbf{u}_{\varepsilon,k}^{[3]}(\mathbf{x}) = \mathbf{u}^{(0)}(\mathbf{x}) + \varepsilon \mathbf{V}_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 \mathbf{Z}_k^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 \mathbf{V}_k^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^2 \mathbf{v}^{(2)}(\mathbf{x}) + \varepsilon^3 \mathbf{Z}_k^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^3 \mathbf{V}_k^{(2)}\left(\frac{\mathbf{x}}{\varepsilon}\right) + \varepsilon^4 \mathbf{Z}_k^{(2)}\left(\frac{\mathbf{x}}{\varepsilon}\right). \quad (7.18)$$

Then  $\mathbf{d}_{3,k}^\varepsilon$  ( $k \in \{f, m\}$ ) solves the following problem:

$$\begin{aligned} \operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_f(\mathbf{d}_{3,f}^\varepsilon)(\mathbf{x}) &= -\operatorname{div}_{\mathbf{x}}(\mathbf{E}_f : \boldsymbol{\varepsilon}(\mathbf{u}^{(0)}))(\boldsymbol{\varepsilon}\mathbf{y}) - \boldsymbol{\gamma}_0^{(0)}(\mathbf{y}) \\ &\quad - \varepsilon \boldsymbol{\gamma}_0^{(1)}(\mathbf{y}) - \varepsilon^2 \boldsymbol{\gamma}_0^{(2)}(\mathbf{y}) \text{ in } \Omega_f^\varepsilon, \end{aligned} \quad (7.19a)$$

$$\operatorname{div}_{\mathbf{x}} \boldsymbol{\sigma}_2(\mathbf{d}_{3,m}^\varepsilon)(\mathbf{x}) = \mathbf{0} \text{ in } \Omega_m^\varepsilon, \quad (7.19b)$$

$$\mathbf{d}_{3,f}^\varepsilon(\mathbf{x}) = \mathbf{d}_{3,m}^\varepsilon(\mathbf{x}) \text{ on } \Gamma^\varepsilon, \quad (7.19c)$$

$$\begin{aligned} \boldsymbol{\sigma}_f(\mathbf{d}_{3,f}^\varepsilon)(\mathbf{x}) \cdot \mathbf{n}_f &= -\boldsymbol{\sigma}_m(\mathbf{d}_{3,m}^\varepsilon)(\mathbf{x}) \cdot \mathbf{n}_m \\ + (\mathbf{E}_m - \mathbf{E}_f) : [\boldsymbol{\varepsilon}(\mathbf{u}^{(0)})(\boldsymbol{\varepsilon}\mathbf{y}) - \boldsymbol{\omega}_0^{(0)}(\mathbf{y}) - \varepsilon \boldsymbol{\omega}_0^{(1)}(\mathbf{y}) - \varepsilon^2 \boldsymbol{\omega}_0^{(2)}(\mathbf{y})] \cdot \mathbf{n}_f \\ + \varepsilon^2 (\mathbf{E}_m - \mathbf{E}_f) : [\boldsymbol{\varepsilon}(\mathbf{v}^{(2)})(\boldsymbol{\varepsilon}\mathbf{y}) - \boldsymbol{\omega}_2^{(0)}(\mathbf{y})] \cdot \mathbf{n}_f \text{ on } \Gamma^\varepsilon, \end{aligned} \quad (7.19d)$$

$$\begin{aligned} \mathbf{d}_{3,m}^\varepsilon(\mathbf{x}) &= \varepsilon \left( \varepsilon \frac{\mathbf{a}_0^{(1)}(\theta)}{|\mathbf{x}|} - \mathbf{V}_m^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) \right) \\ - \varepsilon^2 \mathbf{Z}_m^{(0)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^2 \mathbf{V}_m^{(1)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^3 \mathbf{V}_m^{(2)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^4 \mathbf{Z}_m^{(2)}\left(\frac{\mathbf{x}}{\varepsilon}\right) \text{ on } \Gamma. \end{aligned} \quad (7.19e)$$

## 7.6 Fourth-order expansion

Now, we need to expand  $\mathbf{V}_m^{(1)} + \mathbf{Z}_m^{(0)}$  at  $\infty$  since both functions have a factor of  $\varepsilon^2$ :

$$\mathbf{V}_m^{(1)}(\mathbf{y}) + \mathbf{Z}_m^{(0)}(\mathbf{y}) = \sum_{i=1}^K \frac{\mathbf{a}_1^{(i)}(\theta)}{|\mathbf{y}|^{(i)}} + O\left(\frac{1}{|\mathbf{y}|^{K+1}}\right) = \sum_{i=1}^K \varepsilon^{(i)} \frac{\mathbf{a}_1^{(i)}(\theta)}{|\mathbf{x}|^i} + O(\varepsilon^{K+1}); \quad \mathbf{x} \in \Gamma. \quad (7.20)$$

The previous expansion is proven in Section (9) in the isotropic case.

As done for the Poisson problem, we go there to the fourth order to deduce the general form of functions  $\boldsymbol{\psi}$ . So with  $\mathbf{v}^{(3)}$  solution of Problem (7.14) for  $\boldsymbol{\psi}^{(3)} = -\frac{\mathbf{a}_1^{(1)}(\theta)}{|\mathbf{x}|} - \frac{\mathbf{a}_0^{(2)}(\theta)}{|\mathbf{x}|^2}$  we expand  $\boldsymbol{\varepsilon}(\mathbf{v}^{(3)})(\boldsymbol{\varepsilon}\mathbf{y})$ :

$$\boldsymbol{\varepsilon}(\mathbf{v}^{(3)})(\boldsymbol{\varepsilon}\mathbf{y}) = \sum_{i=0}^K \varepsilon^i \boldsymbol{\omega}_3^{(i)}(\mathbf{y}) + O(\varepsilon^{K+1}). \quad (7.21)$$

Finally, with  $\mathbf{V}^{(3)}$  solution of Problem (7.4) for  $\boldsymbol{\varphi}^{(3)}(\mathbf{y}) = (\mathbf{E}_m - \mathbf{E}_f) : (\boldsymbol{\omega}_0^{(3)}(\mathbf{y}) + \boldsymbol{\omega}_2^{(1)}(\mathbf{y}) + \boldsymbol{\omega}_3^{(0)}(\mathbf{y})) \cdot \mathbf{n}_f$  and  $\mathbf{Z}^{(3)}$  solving Problem (7.5) for  $\boldsymbol{\kappa}^{(3)}(\mathbf{y}) = \boldsymbol{\gamma}_0^{(3)}(\mathbf{y})$  we write the fourth-order expansion ( $k \in \{f, m\}$ ):

$$\mathbf{d}_{4,k}^\varepsilon(\mathbf{x}) = \mathbf{d}_{3,k}^\varepsilon(\mathbf{x}) - \varepsilon^{(3)} \mathbf{v}^{(3)}(\mathbf{x}) - \varepsilon^4 \mathbf{V}_k^{(3)}\left(\frac{\mathbf{x}}{\varepsilon}\right) - \varepsilon^5 \mathbf{Z}_k^{(3)}\left(\frac{\mathbf{x}}{\varepsilon}\right). \quad (7.22)$$

And we obtain the following equations for  $\mathbf{d}_{4,k}^\varepsilon(\mathbf{x})$  ( $k \in \{f, m\}$ ):

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma}_f(\mathbf{d}_{4,f}^\varepsilon)(\mathbf{x}) &= -\operatorname{div}_{\mathbf{x}}(\mathbf{E}_f : \boldsymbol{\varepsilon}(\mathbf{u}^{(0)}))(\boldsymbol{\varepsilon}\mathbf{y}) - \boldsymbol{\gamma}_0^{(0)}(\mathbf{y}) - \varepsilon\boldsymbol{\gamma}_0^{(1)}(\mathbf{y}) \\ &\quad - \varepsilon^2\boldsymbol{\gamma}_0^{(2)}(\mathbf{y}) - \varepsilon^3\boldsymbol{\gamma}_0^{(3)}(\mathbf{y}) \text{ in } \Omega_f^\varepsilon, \end{aligned} \quad (7.23a)$$

$$\operatorname{div} \boldsymbol{\sigma}_m(\mathbf{d}_{4,m}^\varepsilon)(\mathbf{x}) = \mathbf{0} \text{ in } \Omega_m^\varepsilon, \quad (7.23b)$$

$$\mathbf{d}_{4,f}^\varepsilon(\mathbf{x}) = \mathbf{d}_{4,m}^\varepsilon(\mathbf{x}) \text{ on } \Gamma^\varepsilon, \quad (7.23c)$$

$$\begin{aligned} \boldsymbol{\sigma}_f(\mathbf{d}_{4,f}^\varepsilon)(\mathbf{x}) \cdot \mathbf{n}_f &= -\boldsymbol{\sigma}_m(\mathbf{d}_{4,m}^\varepsilon)(\mathbf{x}) \cdot \mathbf{n}_m \\ +(\mathbf{E}_m - \mathbf{E}_f) : [(\boldsymbol{\varepsilon}(\mathbf{u}^{(0)}(\boldsymbol{\varepsilon}\mathbf{y})) - \boldsymbol{\omega}_0^{(0)}(\mathbf{y}) - \varepsilon\boldsymbol{\omega}_0^{(1)}(\mathbf{y}) - \varepsilon^2\boldsymbol{\omega}_0^{(2)}(\mathbf{y}) - \varepsilon^3\boldsymbol{\omega}_0^{(3)}(\mathbf{y})) \cdot \mathbf{n}_f \\ + \varepsilon^2(\mathbf{E}_m - \mathbf{E}_f) : [(\boldsymbol{\varepsilon}(\mathbf{v}^{(2)}(\boldsymbol{\varepsilon}\mathbf{y})) - \boldsymbol{\omega}_2^{(0)}(\mathbf{y})) - \varepsilon\boldsymbol{\omega}_2^{(1)}(\mathbf{y})) \cdot \mathbf{n}_f \\ + \varepsilon^3(\mathbf{E}_m - \mathbf{E}_f) : [\boldsymbol{\varepsilon}(\mathbf{v}^{(3)}(\boldsymbol{\varepsilon}\mathbf{y})) - \boldsymbol{\omega}_3^{(0)}(\mathbf{y})) \cdot \mathbf{n}_f, \end{aligned} \quad (7.23d)$$

$$\begin{aligned} \mathbf{d}_{4,m}^\varepsilon(\mathbf{x}) &= \varepsilon(\varepsilon^2 \frac{\mathbf{a}_0^{(2)}(\boldsymbol{\theta})}{|\mathbf{x}|} + \varepsilon \frac{\mathbf{a}_0^{(1)}(\boldsymbol{\theta})}{|\mathbf{x}|} - \mathbf{V}_m^{(0)}(\frac{\mathbf{x}}{\varepsilon})) \\ + \varepsilon^2(\varepsilon \frac{\mathbf{a}_1^{(1)}(\boldsymbol{\theta})}{|\mathbf{x}|} - \mathbf{V}_m^{(1)}(\frac{\mathbf{x}}{\varepsilon}) - \mathbf{Z}_m^{(0)}(\frac{\mathbf{x}}{\varepsilon})) - \varepsilon^3\mathbf{V}_m^{(2)}(\frac{\mathbf{x}}{\varepsilon}) \text{ on } \Gamma. \end{aligned} \quad (7.23e)$$

## 7.7 The general case

We can update our residual at the order  $n$ ,  $n \geq 3$ , as following ( $k \in \{f, m\}$ ):

$$\mathbf{d}_{n,k}^\varepsilon(\mathbf{x}) = \mathbf{d}_{n-1,k}^\varepsilon(\mathbf{x}) - \varepsilon^{n-1}\mathbf{v}_k^{(n-1)}(\mathbf{x}) - \varepsilon^n\mathbf{V}_k^{(n-1)}(\frac{\mathbf{x}}{\varepsilon}) - \varepsilon^{n+1}\mathbf{Z}_k^{(n-1)}(\frac{\mathbf{x}}{\varepsilon}). \quad (7.24)$$

With  $\mathbf{v}^{(n-1)}$  solves Problem (7.14) for  $\boldsymbol{\psi}^{(n-1)} = -\sum_{p+q=n-2} \frac{\mathbf{a}_q^{(p)}(\boldsymbol{\theta})}{|\mathbf{x}|^p}$ , ( $p \geq 1$ ),  $\mathbf{V}^{(n-1)}$  the solution of Problem (7.4) with  $\boldsymbol{\varphi}^{(n-1)}(\mathbf{y}) = (\mathbf{E}_m - \mathbf{E}_f) : \sum_{p+q=n-1} \boldsymbol{\omega}_q^{(p)}(\mathbf{y}) \cdot \mathbf{n}_f$ , ( $q \neq 1$ ) and  $\mathbf{Z}^{(n-1)}$  solving Problem (7.5) for  $\boldsymbol{\kappa}^{(n-1)}(\mathbf{y}) = \boldsymbol{\gamma}_0^{(n-1)}(\mathbf{y})$ . We are now able to build an expansion  $\mathbf{u}_\varepsilon^{[K]}(\mathbf{x})$  to the order ‘ $K$ ’, ( $K \geq 3$ ) ( $k \in \{f, m\}$ ):

$$\mathbf{u}_{\varepsilon,k}^{[K]}(\mathbf{x}) = \mathbf{u}^{(0)}(\mathbf{x}) + \sum_{i=2}^{K-1} \varepsilon^i \mathbf{v}^{(i)}(\mathbf{x}) + \sum_{i=1}^K \varepsilon^i \mathbf{V}_k^{(i-1)}(\frac{\mathbf{x}}{\varepsilon}) + \sum_{i=1}^K \varepsilon^{i+1} \mathbf{Z}_k^{(i-1)}(\frac{\mathbf{x}}{\varepsilon}). \quad (7.25)$$

We obtain the following equations for  $\mathbf{d}_{K,k}^\varepsilon$  ( $k \in \{f, m\}$ ):

$$\operatorname{div} \boldsymbol{\sigma}_f(\mathbf{d}_{K,f}^\varepsilon)(\mathbf{x}) = O_{H^1(\Omega^\varepsilon; \mathbb{R}^2)}(\varepsilon^{K+1}) \text{ in } \Omega_f^\varepsilon, \quad (7.26a)$$

$$\operatorname{div} \boldsymbol{\sigma}_m(\mathbf{d}_{K,m}^\varepsilon)(\mathbf{x}) = \mathbf{0} \text{ in } \Omega_m^\varepsilon, \quad (7.26b)$$

$$\mathbf{d}_{K,f}^\varepsilon(\mathbf{x}) = \mathbf{d}_{K,m}^\varepsilon(\mathbf{x}) \text{ on } \Gamma^\varepsilon, \quad (7.26c)$$

$$\boldsymbol{\sigma}_f(\mathbf{d}_{K,f}^\varepsilon)(\mathbf{x}) \cdot \mathbf{n}_f = -\boldsymbol{\sigma}_m(\mathbf{d}_{K,m}^\varepsilon)(\mathbf{x}) \cdot \mathbf{n}_m + O_{L^2(\Gamma^\varepsilon; \mathbb{R}^2)}(\varepsilon^{K+\frac{1}{2}}) \text{ on } \Gamma^\varepsilon, \quad (7.26d)$$

$$\mathbf{d}_{K,m}^\varepsilon(\mathbf{x}) = O_{H^{\frac{1}{2}}(\Gamma; \mathbb{R}^2)}(\varepsilon^K) \text{ on } \Gamma. \quad (7.26e)$$

## 8 Error estimate for the linear elasticity problem

In this section we establish an error estimate for the studied problem by increasing the  $H^1(\Omega^\varepsilon; \mathbb{R}^2)$  norm by powers of  $\varepsilon$  (characteristic dimension of the inclusion). First we begin by considering

the following problem for a given  $\mathbf{g} \in H^{-\frac{1}{2}}(\Gamma^\varepsilon; \mathbb{R}^2)$ ,  $\mathbf{h} \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^2)$  and  $\mathbf{f} \in L^2(\Omega_f^\varepsilon; \mathbb{R}^2)$ :

$$\operatorname{div} \boldsymbol{\sigma}_f(\mathbf{z}_f)(\mathbf{x}) = -\mathbf{f}(\mathbf{x}) \quad \text{in } \Omega_f^\varepsilon, \quad (8.1a)$$

$$\operatorname{div} \boldsymbol{\sigma}_m(\mathbf{z}_m)(\mathbf{x}) = \mathbf{0} \quad \text{in } \Omega_m^\varepsilon, \quad (8.1b)$$

$$\mathbf{z}_f(\mathbf{x}) = \mathbf{z}_m(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (8.1c)$$

$$\boldsymbol{\sigma}_f(\mathbf{z}_f)(\mathbf{x}) \cdot \mathbf{n}_f = -\boldsymbol{\sigma}_m(\mathbf{z}_m)(\mathbf{x}) \cdot \mathbf{n}_m + \mathbf{g}(\mathbf{x}) \quad \text{on } \Gamma^\varepsilon, \quad (8.1d)$$

$$\mathbf{z}_m(\mathbf{x}) = \mathbf{h}(\mathbf{x}) \quad \text{on } \Gamma. \quad (8.1e)$$

By passing to a weak form we can write: Find  $\mathbf{z} \in H^1(\Omega^\varepsilon; \mathbb{R}^2)$  with  $z = h$  on  $\Gamma$  such that:

$$\int_{\Omega_f^\varepsilon} \boldsymbol{\varepsilon}(\mathbf{z}) : \mathbf{E}_f : \boldsymbol{\varepsilon}(\boldsymbol{\omega}) \, d\mathbf{x} + \int_{\Omega_m^\varepsilon} \boldsymbol{\varepsilon}(\mathbf{z}) : \mathbf{E}_m : \boldsymbol{\varepsilon}(\boldsymbol{\omega}) \, d\mathbf{x} + \int_{\Gamma^\varepsilon} \mathbf{g} \cdot \boldsymbol{\omega} \, d\sigma = \int_{\Omega_f^\varepsilon} \mathbf{f} \cdot \boldsymbol{\omega} \, d\mathbf{x} \quad (8.2)$$

for all  $\boldsymbol{\omega} \in H^1(\Omega)$  such as  $\boldsymbol{\omega} = 0$  on  $\Gamma$ .

**Lemma 8.1.** *The existence and the uniqueness of the solution of Problem (8.1) in a weak form is guaranteed thanks to Lax Milgram theorem and there is a constant  $C > 0$  independent of  $\varepsilon$  such that:*

$$\|\mathbf{z}\|_{H^1(\Omega^\varepsilon)} \leq C(\|\mathbf{h}\|_{H^{\frac{1}{2}}(\Gamma; \mathbb{R}^2)} + \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\Gamma^\varepsilon; \mathbb{R}^2)} + \|\mathbf{f}\|_{L^2(\Gamma^\varepsilon; \mathbb{R}^2)}).$$

*Proof.* The surjectivity of the trace operator guarantee the existence of a continuous lifting operator  $\mathbf{H} \in H^1(\Omega^\varepsilon; \mathbb{R}^2)$  of  $\mathbf{h}$  (see [14]):

Then  $\bar{\mathbf{z}} = \mathbf{z} - \mathbf{H}$  solves the following equation for all  $\boldsymbol{\omega} \in H^1(\Omega^\varepsilon; \mathbb{R}^2)$  such that  $\boldsymbol{\omega} = 0$  on  $\Gamma$ .

$$\begin{aligned} & \int_{\Omega_f^\varepsilon} \boldsymbol{\varepsilon}(\bar{\mathbf{z}}) : \mathbf{E}_f : \boldsymbol{\varepsilon}(\boldsymbol{\omega}) \, d\mathbf{x} + \int_{\Omega_m^\varepsilon} \boldsymbol{\varepsilon}(\bar{\mathbf{z}}) : \mathbf{E}_m : \boldsymbol{\varepsilon}(\boldsymbol{\omega}) \, d\mathbf{x} \\ & + \int_{\Omega_f^\varepsilon} \boldsymbol{\varepsilon}(\bar{\mathbf{z}}) : \mathbf{E}_f : \boldsymbol{\varepsilon}(\boldsymbol{\omega}) \, d\mathbf{x} + \int_{\Omega_m^\varepsilon} \boldsymbol{\varepsilon}(\bar{\mathbf{z}}) : \mathbf{E}_m : \boldsymbol{\varepsilon}(\boldsymbol{\omega}) \, d\mathbf{x} + \int_{\Gamma^\varepsilon} \mathbf{g} \cdot \boldsymbol{\omega} \, d\sigma = \int_{\Omega_f^\varepsilon} \mathbf{f} \cdot \boldsymbol{\omega} \, d\mathbf{x}. \end{aligned} \quad (8.3)$$

With  $a(\mathbf{u}, \mathbf{v}) = \int_{\Omega_f^\varepsilon} \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbf{E}_f : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\mathbf{x} + \int_{\Omega_m^\varepsilon} \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbf{E}_m : \boldsymbol{\varepsilon}(\mathbf{v}) \, d\mathbf{x}$  the bilinear form associated to the weak formulation, it's obvious to demonstrate that  $a(\mathbf{u}, \mathbf{v})$  is coercive ( $\alpha \|\bar{\mathbf{z}}\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)}^2 \leq a(\bar{\mathbf{z}}, \bar{\mathbf{z}})$ ) (see [11], [14]). Using the coercivity of the bilinear form we obtain:

$$\alpha \|\bar{\mathbf{z}}\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)}^2 \leq - \int_{\Gamma^\varepsilon} \mathbf{g} \cdot \bar{\mathbf{z}} \, d\sigma - \int_{\Omega^\varepsilon} \boldsymbol{\sigma}(\mathbf{H}) : \boldsymbol{\varepsilon}(\bar{\mathbf{z}}) \, d\mathbf{x} + \int_{\Omega_f^\varepsilon} \mathbf{f} \cdot \bar{\mathbf{z}} \, d\mathbf{x}, \quad (8.4)$$

$$\alpha \|\bar{\mathbf{z}}\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)}^2 \leq \|\mathbf{g}\|_{H^{-\frac{1}{2}}(\Gamma^\varepsilon; \mathbb{R}^2)} \|\bar{\mathbf{z}}\|_{H^{\frac{1}{2}}(\Gamma; \mathbb{R}^2)} + \|\mathbf{f}\|_{L^2(\Omega^\varepsilon; \mathbb{R}^2)} \|\bar{\mathbf{z}}\|_{L^2(\Omega^\varepsilon; \mathbb{R}^2)}, \quad (8.5)$$

$$\alpha \|\bar{\mathbf{z}}\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)}^2 \leq C(\|\mathbf{g}\|_{H^{-\frac{1}{2}}(\Gamma^\varepsilon; \mathbb{R}^2)} + \|\mathbf{H}\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)} + \|\mathbf{f}\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)}) \|\bar{\mathbf{z}}\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)}. \quad (8.6)$$

The continuity of the lifting operator gives us that  $\|\mathbf{H}\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)} \leq C\|\mathbf{h}\|_{H^{\frac{1}{2}}(\Gamma; \mathbb{R}^2)}$  so we can write:

$$\alpha \|\bar{\mathbf{z}}\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)} \leq C(\|\mathbf{g}\|_{H^{-\frac{1}{2}}(\Gamma^\varepsilon; \mathbb{R}^2)} + \|\mathbf{h}\|_{H^{\frac{1}{2}}(\Gamma; \mathbb{R}^2)} + \|\mathbf{f}\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)}). \quad (8.7)$$

□

We can now enunciate the following result:

**Theorem 8.2.** *There is a constant  $C > 0$  independent of  $\varepsilon$  as:*

$$\|\mathbf{d}_K^\varepsilon\|_{H^1(\Omega^\varepsilon; \mathbb{R}^2)} \leq C\varepsilon^K \text{ for all } K \in \mathbb{N}.$$

*Proof.* With  $\mathbf{z} = \mathbf{d}_K^\varepsilon$  we have  $\mathbf{f} \in O_{H^1(\Omega^\varepsilon; \mathbb{R}^2)}(\varepsilon^{K+1})$ ,  $\mathbf{g} \in O_{L^2(\Gamma^\varepsilon; \mathbb{R}^2)}(\varepsilon^{K+\frac{1}{2}})$  and  $\mathbf{h} \in O_{H^{\frac{1}{2}}(\Gamma; \mathbb{R}^2)}(\varepsilon^K)$  using lemma 8.1 we obtain the announced error estimate.  $\square$

## 9 Expansion of the $V_m^{(p)}$ functions at infinity

Let us introduce the functions that generate the expansion of  $\mathbf{V}_m^p$  at infinity ( $\rho = \frac{\mu}{\lambda+\mu}$ ):

$$\psi_1^{(s)}(\theta) = \begin{pmatrix} \sin((s+1)\theta) \\ \cos((s+1)\theta) \end{pmatrix}. \quad (9.1)$$

$$\psi_2^{(s)}(\theta) = \begin{pmatrix} \cos((s+1)\theta) \\ -\sin((s+1)\theta) \end{pmatrix}. \quad (9.2)$$

$$\psi_3^{(s)}(\theta) = \begin{pmatrix} (1-s+2\rho)\sin((s-1)\theta) \\ -(1+s+2\rho)\cos((s-1)\theta) \end{pmatrix}. \quad (9.3)$$

$$\psi_4^{(s)}(\theta) = \begin{pmatrix} -(1-s+2\rho)\cos((s-1)\theta) \\ -(1+s+2\rho)\sin((s-1)\theta) \end{pmatrix}. \quad (9.4)$$

Thanks to [5]  $\mathbf{V}_m^{(p)}$  at infinity takes the following form:

$$\mathbf{V}_m^{(p)} = \sum_{s \in \mathbb{N}^*} r^{-s} (a(s)\psi_1^{(-s)} + b(s)\psi_2^{(-s)} + c(s)\psi_3^{(-s)} + d(s)\psi_4^{(-s)}).$$

## 10 Existence and uniqueness in unbounded domains for the linear elasticity problem

To ensure existence and uniqueness of  $\mathbf{V}^{(p)}$  functions the first thing to do is to transform the strong formulation (Problem (7.4)) into a weak one as follows:

$$\int_{\Omega^1} \boldsymbol{\varepsilon}(\mathbf{V}_f^{(p)}) : \mathbf{E}_f : \boldsymbol{\varepsilon}(\boldsymbol{\omega}) \, d\mathbf{x} + \int_{\Omega^\infty} \boldsymbol{\varepsilon}(\mathbf{V}_m^{(p)}) : \mathbf{E}_m : \boldsymbol{\varepsilon}(\boldsymbol{\omega}) \, d\mathbf{x} = \int_{\Gamma^1} \boldsymbol{\varphi} \cdot \mathbf{w} \, d\sigma ; \text{ for all } \mathbf{w} \in V. \quad (10.1)$$

Introducing space  $V = \{\mathbf{v} \in (W_{0,0}^{1,2})^2; \int_{\Gamma_{R_0}} \mathbf{v} = 0\}$  we can see that the bilinear form  $\int_{\Omega^1} \boldsymbol{\varepsilon}(\mathbf{V}_f^{(p)}) : \mathbf{E}_f : \boldsymbol{\varepsilon}(\boldsymbol{\omega}) \, d\mathbf{x} + \int_{\Omega^\infty} \boldsymbol{\varepsilon}(\mathbf{V}_m^{(p)}) : \mathbf{E}_m : \boldsymbol{\varepsilon}(\boldsymbol{\omega}) \, d\mathbf{x}$  is coercive on  $V$ . In addition the bilinear form  $a(\cdot, \cdot)$  and the linear one  $l(\mathbf{w}) = \int_{\Gamma^1} \boldsymbol{\varphi} \cdot \mathbf{w} \, d\sigma$  are obviously continuous on  $V$ . Then thanks to Lax Milgram theorem there is existence and uniqueness of  $\mathbf{V}^{(p)}$  functions in  $V$ . The existence and uniqueness of  $\mathbf{Z}^{(p)}$  functions is proven by the same way.

## 11 Conclusion and perspectives

Recall that the purpose of this work was to study the interaction between an inclusion of a small size compared to the size of the elastic body in which it operates. We were able to build an asymptotic expansion starting from the field without inclusion and correcting by additional functions that models the presence of the inclusion and its influence on the mechanical fields (asymptotic expansion at an arbitrary order  $K$  given by  $u_\varepsilon^{[K]}(\mathbf{x})$  in (2.24) for the Poisson problem and by the displacement  $\mathbf{u}_\varepsilon^{[K]}(\mathbf{x})$  in (7.25) for linear elasticity). The model had been validated by comparing the obtained result given by (6.10) with the one obtained by solving the problem using a variable-separation method (6.6) (for the Poisson problem and with a circular shape of the inclusion). Error estimates had been established in Sections (3) and (8). Existence and uniqueness in a weak form had been established for functions posed on infinite domains in Sections (5) and (10).

Natural perspectives of this work is the design of a numerical strategy first to compute a few terms of the expansion obtained and then to compute the influence of an inclusion in an arbitrary shaped body and the influence of several inclusions. Extension to large strain elasticity and to three-dimensional problem are also of great interest. Finally, the extension to inhomogeneous and anisotropic cases should be done without major difficulties.

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## Appendices

### A A Poincaré type inequality

We establish the existence and uniqueness of the profile functions  $V^{(k)}$  introduced previously in the asymptotic expansion at order ' $K$ ' given by (2.24). The  $V^{(k)}$  functions are defined on all  $\mathbb{R}^2$  (two-dimensional geometry) and are solutions to the family of Problems (2.4). For that, we will introduce weighted Sobolev spaces  $W_{\alpha,\beta}^{m,p}$  ( $W_{0,0}^{1,2}$  in our case) similar to classical ones  $W^{m,p}$  ( $H^1$  in our case) but with weights that describe the growth or the decay of functions at infinity. The idea of using weighted spaces arises naturally from Hardy's inequalities and will allow us to establish a Poincaré inequality relating to norms of functions and to that of their derivatives.

The use of these weights is necessary to obtain a Poincaré-type inequality and eliminates the drawbacks of spaces defined by the closure of  $D(\mathbb{R}^2)$  for the Dirichlet norm and which are not always spaces of distributions. The used weights arises naturally from Hardy's inequality or from a generalized Hardy's inequality (see [2]) and the classical ones are of the form  $\rho = (1 + |x|^2)^{\frac{1}{2}}$  but there is appearance of a logarithmic factor in our case.

#### A.1 Notations and functional setting

All functions and distributions here are defined on the two-dimensional real euclidean space  $\mathbb{R}^2$ .  $r = |x| = (x_1^2 + x_2^2)^{\frac{1}{2}}$  defines the distance of a point  $x = (x_1, x_2)$  to the origin. Recall that

$D(\mathbb{R}^2)$  denotes the space of indefinitely differentiable functions with compact support and  $D'(\mathbb{R}^2)$  its dual space called space of distributions. With  $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$  a multi-index  $D^\lambda = D_1^{\lambda_1} \dots D_n^{\lambda_n}$  is the differential operator of order  $|\lambda| = \lambda_1 + \dots + \lambda_n$ .  $L^2(\mathbb{R}^2)$  is the classical space of measurable functions for which  $(\int_{\mathbb{R}^2} |u|^2 d\mathbf{x}) < \infty$ . It is a Banach space for the norm  $\|u\| = (\int_{\mathbb{R}^2} |u|^2 d\mathbf{x})^{\frac{1}{2}}$ .

With  $\rho = (1 + r^2)^{\frac{1}{2}}$  we introduce the appropriate weighted Sobolev space to our case by:

$$W_{0,0}^{1,2} = \{u \in D'(\mathbb{R}^2), \rho^{-1}(\lg \rho)^{-1}u \in L^2(\mathbb{R}^2), \nabla u \in (L^2(\mathbb{R}^2))^2\}.$$

Which is a reflexive Banach space equipped with its natural norm:

$$\|u\|_{W_{0,0}^{1,2}(\mathbb{R}^2)} = (\|\rho^{-1}(\lg \rho)^{-1}u\|_{L^2(\mathbb{R}^2)} + \|\nabla u\|_{L^2(\mathbb{R}^2)})^{\frac{1}{2}}.$$

We also define the semi-norm:

$$|u|_{W_{0,0}^{1,2}(\mathbb{R}^2)} = \|\nabla u\|_{L^2(\mathbb{R}^2)}$$

With  $r = |x|$  we set  $\lg(r) = \ln(2 + r^2)$ ,  $B_R = B(0, R)$  the open ball with center 0 and radius  $R$  and  $B'_R = (\overline{B_R})^c$  the exterior of  $\overline{B_R}$ .

Finally, we define  $P_0$  as the space containing constants.

## A.2 An intermediate result

The following result is an intermediate result to prove the equivalence of the norm and the semi-norm.

**Lemma A.1.** *For any large enough real number  $R$ , there exists a constant  $C_R$  such that:*

$$\forall \varphi \in D(B'_R), \|\varphi\|_{W_{0,0}^{1,2}(B'_R)} \leq C_R |\varphi|_{W_{0,0}^{1,2}(B'_R)}.$$

*Proof.* Let  $\varphi$  belongs to  $D'_R$ . First, observe that, owing to the support of  $\varphi$ , all integrals in the norm and semi-norm are taken on  $B'_R$  instead of  $\mathbb{R}^2$ . Hence, since the origin is in the interior of  $B_R$ , we can use  $r$  and  $\ln r$  instead of  $\rho(r)$  and  $\lg(r)$  in the expression of the norm and seminorm. Then using  $\frac{\partial \varphi}{\partial r} = \nabla \varphi \cdot \frac{\mathbf{x}}{r}$  we can write :

$$\left| \frac{\partial \varphi}{\partial r} \right|^2 \leq 2 \sum_{i=1}^2 \left| \frac{\partial \varphi}{\partial x_i} \right|^2, \tag{A.1}$$

Let  $\theta$  stand for the ‘‘angular variable’’, we have:

$$\varphi(r, \theta) = \int_R^r \varphi(t, \theta) dt.$$

Now, assuming that  $R$  is large enough, we apply the generalized Hardy’s inequality (see [2]) (with  $\gamma = -2$ ) to the function  $r \rightarrow \varphi(r, \theta)$ . Integrating with respect to  $\theta$  and applying (A.1) we obtain:

$$\|r^{-1} \ln r^{-1} \varphi\|_{L^2(B'_R)} \leq C \|D\varphi\|_{L^2(B'_R)}.$$

□

The needed result can be proven now.



### A.3 A Poincaré-type inequality

**Theorem A.2.** *The semi-norm  $|\cdot|_{W_{0,0}^{1,2}(\mathbb{R}^2)}$  defines on  $W_{0,0}^{1,2}(\mathbb{R}^2)/P_0$  a norm which is equivalent to the quotient norm.*

*Proof.* It is clear that  $|u|_{W_{0,0}^{1,2}(\mathbb{R}^2)}$  is a norm on  $W_{0,0}^{1,2}(\mathbb{R}^2)/P_0$ , and that:

$$\forall u \in W_{0,0}^{1,2}(\mathbb{R}^2), |u|_{W_{0,0}^{1,2}(\mathbb{R}^2)} \leq \|u\|_{W_{0,0}^{1,2}(\mathbb{R}^2)}. \quad (\text{A.2})$$

Thus we only have to prove that:

$$\exists c > 0 : \forall \dot{u} \in W_{0,0}^{1,2}(\mathbb{R}^2)/P_0, \|\dot{u}\|_{W_{0,0}^{1,2}(\mathbb{R}^2)/P_0} \leq c|u|_{W_{0,0}^{1,2}(\mathbb{R}^2)}. \quad (\text{A.3})$$

The proof proceeds in two steps. The first step consists in eliminating the quotient norm by choosing an adequate representative of the class of  $\dot{u}$ . To this end, we fix a bounded open domain of  $\mathbb{R}^2$ , with positive measure, say  $O$ , and we choose the representative  $U$  of  $\dot{u}$  in  $W_{0,0}^{1,2}(\mathbb{R}^2)$  that satisfies the system of equations:

$$\forall \mu \in P_0, \int_O U \mu d\mathbf{x} = 0. \quad (\text{A.4})$$

It is easy to see that (A.4) determines  $U$  uniquely and obviously:

$$\|\dot{u}\|_{W_{0,0}^{1,2}(\mathbb{R}^2)/P_0} \leq \|U\|_{W_{0,0}^{1,2}(\mathbb{R}^2)}. \quad (\text{A.5})$$

Therefore, the second step consists in proving that there exists a constant  $C$  such that the following bound holds for all  $U$  in  $W_{0,0}^{1,2}(\mathbb{R}^2)$  satisfying (A.4):

$$\|U\|_{W_{0,0}^{1,2}(\mathbb{R}^2)} \leq C|U|_{W_{0,0}^{1,2}(\mathbb{R}^2)}. \quad (\text{A.6})$$

We shall prove it by contradiction. If (A.6) is not true, there exists a sequence  $(U_\nu)$  of elements of  $W_{0,0}^{1,2}(\mathbb{R}^2)$  satisfying (A.5) and such that:

$$\|U_\nu\|_{W_{0,0}^{1,2}(\mathbb{R}^2)} = 1 \text{ and } |U_\nu|_{W_{0,0}^{1,2}(\mathbb{R}^2)} \rightarrow 0. \quad (\text{A.7})$$

Hence the sequence  $(U_\nu)$  is bounded in  $W_{0,0}^{1,2}(\mathbb{R}^2)$  and since this is a reflexive Banach space, we can extract a subsequence, still denoted by  $(U_\nu)$ , that converges weakly to an element  $U_*$  of  $W_{0,0}^{1,2}(\mathbb{R}^2)$  and it is easy to check from this weak convergence that  $U_*$  also satisfies (A.5). But since  $|u|_{W_{0,0}^{1,2}(\mathbb{R}^2)}$  tends to 0, the lower semi-continuity of the norm implies that  $|u|_{W_{0,0}^{1,2}(\mathbb{R}^2)} = 0$ . Thus,  $U_*$  is a constant (polynomial of  $P_0$ ) and the fact that  $U_*$  satisfies (A.5) implies that  $U_* = 0$ .

Now, we need a strong convergence to conclude by contradiction, but we cannot use a standard compactness argument on an unbounded domain. Instead, we shall derive a strong convergence via an adequate partition of unity that will enable us to consider separately a bounded domain where the topologies of  $W_{0,0}^{1,2}$  and  $H^1$  coincide and the exterior of a ball, where Lemma A.1 can be applied.

Let  $R$  denote a real number, large enough to apply the generalized Hardy's inequality. Let  $\varphi$  and  $\psi$  be two functions of  $C^\infty(\mathbb{R}^2)$  such that:

$$\forall \mathbf{x} \in \mathbb{R}^2, (\varphi + \psi)(\mathbf{x}) = 1, 0 \leq \varphi(\mathbf{x}) \leq 1, \text{supp}(\varphi) \subset \overline{B}_{R+1}, \text{supp}(\psi) \subset B'_R. \quad (\text{A.8})$$

Since for fixed  $R$ ,  $W_{0,0}^{1,2}(B_{R+1})$  is isomorphic to  $R$ ,  $H^1(B_{R+1})$ , we have that  $U_\nu$  converges weakly to 0 in  $H^1(B_{R+1})$ . And since  $H^1(B_{R+1})$  is compactly embedded into  $L^2(B_{R+1})$ , it follows that

$$U_\nu \rightarrow 0 \text{ strongly in } L^2(B_{R+1}), \quad (\text{A.9})$$

Since in addition  $|U_\nu|_{W_{0,0}^{1,2}(\mathbb{R}^2)}$  tends to 0, it follows that

$$U_\nu \rightarrow 0 \text{ strongly in } W^{1,2}(B_{R+1}), \quad (\text{A.10})$$

so that

$$\varphi U_\nu \rightarrow 0 \text{ strongly in } W_{0,0}^{1,2}(B_{R+1}). \quad (\text{A.11})$$

Now, let us examine the behavior of  $\psi U_\nu$ . For fixed  $\nu$ , let  $(\theta_j)$  be a sequence of functions of  $D(\mathbb{R}^2)$  that tends to  $U_{nu}$  in  $W_{0,0}^{1,2}(\mathbb{R}^2)$ . Then  $\psi\theta_j$  belongs to  $D(B'_R)$  and we can apply to it lemma (A.1)

$$\|\psi\theta_j\|_{W_{0,0}^{1,2}(\mathbb{R}^2)} \leq C_R |\psi\theta_j|_{W_{0,0}^{1,2}(B'_R)}. \quad (\text{A.12})$$

Then, letting  $j$  tend to infinity and using the fact that  $\psi$  is identically one outside  $B_{R+1}$ , we obtain:

$$\|\psi U_\nu\|_{W_{0,0}^{1,2}(\mathbb{R}^2)} \leq C_R |\psi U_\nu|_{W_{0,0}^{1,2}(B'_R)} \leq C_R (|\psi U_\nu|_{W_{0,0}^{1,2}(B'_R \cap B_{R+1})}^p + |U_\nu|_{W_{0,0}^{1,2}(B'_{R+1})}^p)^{\frac{1}{p}}. \quad (\text{A.13})$$

Next, let  $\nu$  tend to infinity and observe that  $\psi U_\nu$  tends to zero strongly in  $W_{0,0}^{1,2}(B'_R \cap B_{R+1})$  because  $B'_R \cap B_{R+1}$  is bounded and  $W_{0,0}^{1,2}(B'_R \cap B_{R+1})$  is isomorphic to  $W^{1,2}(B'_R \cap B_{R+1})$ ; we derive that:

$$\psi U_\nu \rightarrow 0 \text{ strongly in } W_{0,0}^{1,2}(B'_R). \quad (\text{A.14})$$

Since  $U_\nu = \varphi U_\mu + \psi U_\mu$ , we obtain:

$$U_\mu \rightarrow 0 \text{ strongly in } W_{0,0}^{1,2}(\mathbb{R}^2), \quad (\text{A.15})$$

which contradicts the assumption (A.7) that:

$$\|U_\mu\|_{W_{0,0}^{1,2}(\mathbb{R}^2)} = 1.$$

□

## References

- [1] H. AMMARI AND H. KANG, *Reconstruction of small inhomogeneities from boundary measurements*, vol. 1846 of Lecture Notes in Mathematics, Springer-Verlag, Berlin Heidelberg, 2004.
- [2] C. AMROUCHE, V. GIRAULT, AND J. GIROIRE, *Weighted Sobolev spaces for Laplace's equation in  $\mathbb{R}^n$* , J. Math. Pures Appl., 73 (1994), pp. 579–606.
- [3] A. BENDALI AND K. LEMRABET, *The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation*, SIAM J. Appl. Math., 56 (1996), pp. 1664–1693.

- [4] E. BERETTA, E. BONNETIER, E. FRANCI, AND A. L. MAZZUCATO, *An asymptotic formula for the displacement field in the presence of small anisotropic elastic inclusions*, Inverse Problems and Imaging, 6 (2012), pp. 1–23.
- [5] H. BREZIS, *Progress in nonlinear differential equations and their applications*, Université Pierre et Marie Curie, Paris and Rutgers University, New Brunswick, N.J., 22.
- [6] Y. CAPDEBOSCQ AND M. VOGELIUS, *A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction*, Math. Modelling Num. Anal., 37 (2003), pp. 159–173.
- [7] ———, *A review of some recent work on impedance imaging for inhomogeneities of low volume fraction*, Contemp. Math., 362 (2004), pp. 69–87.
- [8] C. S. CHAND AND H. D. CONWAY, *A parametric study of the complex variable method for analyzing the stresses in an infinite plate containing a rigid rectangular inclusion*, Int. J. Solids Structures, 4 (1968), pp. 1057–1066.
- [9] M. DAMBRINE AND G. VIAL, *Influence of a boundary perforation on a Dirichlet energy*, Control and Cybernetics, 34 (2005), pp. 117–136.
- [10] ———, *A multiscale correction method for local singular perturbations of the boundary*, Math. Model. Numer. Anal., 41 (2007), pp. 111–127.
- [11] A. ERN AND J.-L. GUERMOND, *Theory and Practice of Finite Elements*, vol. 159 of Applied Mathematical Sciences, Springer-Verlag, New York, 2004.
- [12] J. D. ESHELBY, *The determination of the elastic field of an ellipsoidal inclusion, and related problems*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 241 (1957), pp. 376–396.
- [13] ———, *The elastic field outside an ellipsoidal inclusion*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 252 (1959), pp. 561–569.
- [14] A. FORTIN AND A. GARON, *Les Eléments Finis : de la Théorie à la Pratique*, Université Laval, Ecole Polytechnique de Montréal, 2011.
- [15] G. GEYMONAT, F. KRASUCKI, AND S. LENCI, *Mathematical analysis of a bonded joint with a soft thin adhesive*, Math. Mech. Solids, 4 (1999), pp. 201–225.
- [16] D. J. HANSEN, C. POIGNARD, AND M. S. VOGELIUS, *Asymptotically precise norm estimates of scattering from a small circular inhomogeneity*, Appl. Anal., 86 (2007), pp. 433–458.
- [17] A. M. IL'IN, *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems*, vol. 102 of Translations of Mathematical Monographs, Amer. Math. Soc., Providence, RI, 1992.
- [18] E. KRÖNER, *Statistical modelling*, in Modelling Small Deformations of Polycrystals, J. Gittus and J. Zarka, eds., Springer, Dordrecht, the Netherlands, 1986, pp. 229–291.
- [19] ———, *Modified green functions in the theory of heterogeneous and/or anisotropic linearly elastic media*, in Micromechanics and Inhomogeneity: The Toshio Mura 65th Anniversary Volume, G. J. Weng, M. Taya, and H. Abé, eds., Springer, New York, NY, 1990, pp. 197–211.

- [20] T. LEWINSKI AND J. SOKOLOWSKI, *Topological derivative for nucleation of non-circular voids. the Neumann problem*, in Differential Geometric Methods in the Control of Partial Differential Equations, R. Gulliver, W. Littman, and R. Triggiani, eds., vol. 268 of Contemporary Mathematics, Amer. Math. Soc., Providence, RI, 2000, pp. 341–361.
- [21] S. LI, R. SAUER, AND G. WANG, *A circular inclusion in a finite domain I. the Dirichlet-Eshelby problem*, Acta Mechanica, 179 (2005), pp. 67–90.
- [22] E. LIGNON, *Modélisation multi-échelles de nappes fibrées en compression*, PhD thesis, Ecole Polytechnique, 2011.
- [23] P. MAZILU, *On the theory of linear elasticity in statically homogeneous media*, Rev. Roum. Math. Pur. Appl., 17 (1972), pp. 261–273.
- [24] V. MAZ'YA, S. NAZAROV, AND B. PLAMENEVSKIJ, *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains. Volume 1*, vol. 111 of Operator Theory: Advances and Applications, Birkhauser Verlag, Basel, 2000.
- [25] ———, *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains. Volume 2*, vol. 112 of Operator Theory: Advances and Applications, Birkhauser Verlag, Basel, 2000.
- [26] E. SÁNCHEZ-PALENCIA, *Problèmes de perturbations liés aux phénomènes de conduction à travers des couches minces de grande résistivité*, J. Math. Pures Appl. (9), 53 (1974), pp. 251–269.
- [27] Y. SUN AND Y. PENG, *Analytic solutions for the problems of an inclusion of arbitrary shape embedded in a half space*, Applied Mathematics and Computation, 140 (2003), pp. 105–113.
- [28] G. VIAL, *Analyse multi-échelle et conditions aux limites approchées pour un problème avec couche mince dans un domaine à coin*, PhD thesis, Université de Rennes 1, 2003.