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COMPACT PERTURBATIONS OF CONTROLLED SYSTEMS

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ABSTRACT. In this article we study the controllability properties of general compactly perturbed exactly controlled linear systems with admissible control operators. Firstly, we show that approximate and exact controllability are equivalent properties for such systems. This unifies previous results available in the literature and that were established separately so far. Then, and more importantly, we provide for the perturbed system a complete characterization of the set of reachable states in terms of the Fattorini-Hautus test. The results rely on the Peetre lemma.

1. Introduction and main result. In this work we study the exact controllability property of general compactly perturbed controlled linear systems using a compactness-uniqueness approach. This technique has been introduced for the very first time in the pioneering work [16] to establish the exponential decay of the solution to some hyperbolic equations. On the other hand, the first controllability results using this method were obtained in [18] for a plate equation and then in [19] for a wave equation perturbed by a bounded potential. Whether one wants to establish a stability result or a controllability result, one is lead in both cases to prove estimates, energy estimates or observability inequalities. For a perturbed system, a general procedure is to start by the known estimate satisfied by the unperturbed system and to try to derive the desired estimate, up to some “lower order terms” that we would like to remove. The compactness-uniqueness argument then reduces the task of absorbing these additional terms to a unique continuation property for the perturbed system. We should point out that, despite the numerous applications of this flexible method to successfully establish the controllability of systems governed by partial differential equations (see e.g. [18, 19, 2, 11, 4, 8, 12], etc.), no systematic treatment has been provided so far, by which we mean that there is no abstract result available in the literature that covers all type of systems, regardless the nature of the PDE we are considering (wave, plate, etc.). This will be the first

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point of the present paper to fill this gap (Theorem 1.1 below). Then, and more importantly, we considerably improve this result by establishing an explicit characterization of the set of reachable states (Theorem 1.2 below). This characterization is given in terms of the Fattorini-Hautus test - a far weaker kind of unique continuation property than the approximate controllability. Our result shows in particular that this test is actually sufficient to ensure the exact controllability of the perturbed system (Corollary 1.3 below). We illustrate this result by answering to a recent open problem introduced in [8] where the Fattorini-Hautus test plays a key role (Proposition 3.1 below). The proofs of the main results of this article are based on the Peetre Lemma, introduced in [15], which is in fact the root of compactness-uniqueness methods. Finally, let us mention that in the work [8] the authors used a compactness-uniqueness argument to establish the null-controllability of some heat equation, which is a controllability property that does not enter in our framework. Therefore, it would be very interesting to see if general results similar to the ones of the present work hold as well for null-controllability property.

Let us now introduce some notations and recall some basic facts about the controllability of abstract linear evolution equations. We refer to the excellent textbook [17] for the proof of the statements below. Let H and U be two (real or complex) Hilbert spaces, let $A : D(A) \subset H \rightarrow H$ be the generator of a C_0 -semigroup $(S_A(t))_{t \geq 0}$ on H and let $B \in \mathcal{L}(U, D(A^*)')$. For $T \geq 0$ let $\Phi_T \in \mathcal{L}(L^2(0, +\infty; U), D(A^*)')$ be the input map of (A, B) , that is the linear operator defined for every $u \in L^2(0, +\infty; U)$ by

$$\Phi_T u = \int_0^T S_A(T-s)Bu(s) ds.$$

We assume that B is admissible for A , which means that $\text{Im } \Phi_T \subset H$ for some (and hence all) $T > 0$. From this assumption it follows that $\Phi_T \in \mathcal{L}(L^2(0, +\infty; U), H)$. Its adjoint $\Phi_T^* \in \mathcal{L}(H, L^2(0, +\infty; U))$ is the unique continuous linear extension to H of the map $z \in D(A^*) \mapsto B^* S_A(T-\cdot)^* z \in L^2(0, +\infty; U)$, where $B^* S_A(T-\cdot)^* z$ is extended by zero outside $(0, T)$ (in particular, $\Phi_T^* z(t) = 0$ for a.e. $t > T$). Let us now consider the abstract evolution system

$$\begin{cases} \frac{d}{dt}y &= Ay + Bu, \quad t \in (0, T), \\ y(0) &= y^0, \end{cases} \quad (1)$$

where $T > 0$ is the time of control, $y^0 \in H$ is the initial data, y is the state and $u \in L^2(0, T; U)$ is the control. Since B is admissible for A , system (1) is well-posed: for every $y^0 \in H$ and every $u \in L^2(0, T; U)$, there exists a unique solution $y \in C^0([0, T]; H)$ to system (1) given by the Duhamel formula

$$y(t) = S_A(t)y^0 + \Phi_t u, \quad \forall t \geq 0.$$

The regularity of the solution allows us to consider control problems for the system (1). We say that the system (1) or (A, B) is:

- exactly controllable in time T if, for every $y^0, y^1 \in H$, there exists $u \in L^2(0, T; U)$ such that the corresponding solution y to the system (1) satisfies $y(T) = y^1$.
- approximately controllable in time T if, for every $\varepsilon > 0$ and every $y^0, y^1 \in H$, there exists $u \in L^2(0, T; U)$ such that the corresponding solution y to the system (1) satisfies $\|y(T) - y^1\|_H \leq \varepsilon$.

Clearly, exact controllability in time T implies approximate controllability in the same time. The set $\text{Im } \Phi_T$ (*resp.* $\overline{\text{Im } \Phi_T}$) is called the set of exactly (*resp.* approximately) reachable states in time T . Therefore, (A, B) is exactly (*resp.* approximately) controllable in time T if, and only if, $\text{Im } \Phi_T = H$ (*resp.* $\overline{\text{Im } \Phi_T} = H$). It is also well-known that the controllability has a dual concept named observability. More precisely, (A, B) is exactly controllable in time T if, and only if, there exists $C > 0$ such that

$$\|z\|_H^2 \leq C \int_0^T \|\Phi_T^* z(t)\|_U^2 dt, \quad \forall z \in H, \quad (2)$$

and (A, B) is approximately controllable in time T if, and only if,

$$\left(\Phi_T^* z(t) = 0, \quad \text{a.e. } t \in (0, T) \right) \implies z = 0, \quad \forall z \in H. \quad (3)$$

Let us now state the main results of this paper. The first one simply unifies previous results available in the literature under a general semigroup setting:

Theorem 1.1. *Let H and U be two (real or complex) Hilbert spaces. Let $A_0 : D(A_0) \subset H \rightarrow H$ be the generator of a C_0 -semigroup on H and let us consider $B \in \mathcal{L}(U, D(A_0)')$ an admissible control operator for A_0 . Let $K \in \mathcal{L}(H)$ and let us form the unbounded operator $A_K = A_0 + K$ with $D(A_K) = D(A_0)$ ¹. We assume that:*

- (i) *There exists $T_0 > 0$ such that (A_0, B) is exactly controllable in time T_0 .*
- (ii) *K is compact.*
- (iii) *(A_K, B) is approximately controllable in time T_0 .*

Then, (A_K, B) is exactly controllable in time T_0 .

The second and most important result of the present paper shows that we can even give a very precise characterization of the reachable states for the perturbed system, if we allow the time of control to be slightly longer:

Theorem 1.2. *Let H and U be complex Hilbert spaces and let A_0, A_K and B be defined as in Theorem 1.1. For $T > 0$ let $\Phi_T \in \mathcal{L}(L^2(0, +\infty; U), H)$ be the input map of (A_K, B) . Let σ_F be the set given by*

$$\sigma_F = \{ \lambda \in \mathbb{C}, \quad \ker(\lambda - A_K^*) \cap \ker B^* \neq \{0\} \},$$

and for every $\lambda \in \mathbb{C}$ let E_λ be the subspace of H defined by

$$E_\lambda = \left\{ z \in \bigcup_{m=1}^{+\infty} \ker(\lambda - A_K^*)^m, \quad B^*(\lambda - A_K^*)^m z = 0, \quad \forall m \in \mathbb{N} \right\}.$$

Then, under the assumptions (i) and (ii) of Theorem 1.1, the set σ_F is finite, E_λ is finite dimensional for every $\lambda \in \mathbb{C}$, and we have

$$\text{Im } \Phi_T = \left(\bigoplus_{\lambda \in \sigma_F} E_\lambda \right)^\perp, \quad \forall T > T_0.$$

This second result shows in particular that the approximate controllability assumption (iii) of Theorem 1.1 can be weakened to the Fattorini-Hautus test:

¹ A_K is then the generator of a C_0 -semigroup on H and B is also admissible for A_K , see below.

Corollary 1.3. *Let H and U be complex Hilbert spaces and let A_0, A_K and B be defined as in Theorem 1.1. Then, under the assumptions (i) and (ii) of Theorem 1.1, and if (A_K, B) satisfies the Fattorini-Hautus test:*

$$\ker(\lambda - A_K^*) \cap \ker B^* = \{0\}, \quad \forall \lambda \in \mathbb{C}, \quad (4)$$

then (A_K, B) is exactly controllable in time T for every $T > T_0$.

Remark 1.4. It follows from Corollary 1.3 that, if (A_0, B) and (A_K, B) are two systems satisfying the Fattorini-Hautus test, and K is compact, then

$$\begin{aligned} & \inf \{T > 0, (A_0, B) \text{ is exactly controllable in time } T\} \\ &= \inf \{T > 0, (A_K, B) \text{ is exactly controllable in time } T\}. \end{aligned}$$

In other words, both systems share the same minimal time of control.

Remark 1.5. In many applications the spaces H and U are real Hilbert spaces. To apply Theorem 1.2 (and Corollary 1.3) in such a framework, we first introduce the complexified spaces $\hat{H} = H + iH$ and $\hat{U} = U + iU$ and we define the complexified operators \hat{A}_K and \hat{B} by $\hat{A}_K(y_1 + iy_2) = A_K y_1 + iA_K y_2$ for $y_1, y_2 \in D(A_K)$ and $\hat{B}(u_1 + iu_2) = Bu_1 + iBu_2$ for $u_1, u_2 \in U$. Splitting the evolution system described by (\hat{A}_K, \hat{B}) into real and imaginary parts, we readily see that (\hat{A}_K, \hat{B}) is (exactly or approximately) controllable in time T if, and only if, so is (A_K, B) . Then, we check the Fattorini-Hautus test for (\hat{A}_K, \hat{B}) . In the sequel we shall keep the same notation to denote the operators and their extensions.

Corollary 1.3 shows that, in order to prove the exact controllability of a compactly perturbed system which is known to be exactly controllable, it is (necessary and) sufficient to only check the Fattorini-Hautus test (4). This result has been established in a particular case in [4, Theorem 5] for a perturbed Euler-Bernoulli equation with distributed controls. The Fattorini-Hautus test appears for the very first time in [7, Corollary 3.3] and it is also sometimes misleadingly known as the Hautus test in finite dimension, despite it has been introduced earlier by Fattorini, moreover in a much larger setting. In a complete abstract control theory framework, it is the sharpest sufficient condition one can hope for since it is always a necessary condition for the exact, null or approximate controllability, to hold in some time. This is easily seen through the dual characterizations (2) or (3) since $S_A(t)^* z = e^{\lambda t} z$ for $z \in \ker(\lambda - A^*)$. It is also nowadays well-known that this condition characterizes the approximate controllability of a large class of systems generated by analytic semigroups (see [7, 1, 14]). Surprisingly enough, Corollary 1.3 shows that it may as well characterize the exact controllability property for some systems. In practice, the Fattorini-Hautus test can be checked by various techniques, such as Carleman estimates for stationary systems (see e.g. [4, 1]) or through a spectral analysis when this later technique is not available (see e.g. [14, 3, 5] or the example of Section 3 below).

Let us mention that it is not clear when the Fattorini-Hautus test (4) remains sufficient to obtain the exact controllability of the perturbed system in time T_0 . Therefore, both Theorem 1.1 and Corollary 1.3 are important. Obviously, Corollary 1.3 is a stronger result if we do not look for the best time. However, it may very well happen that the conservation of the time T_0 is required to apply some other results, as for instance in [5] where the authors fundamentally need it to stabilize a perturbed hyperbolic equation in finite time.

Finally, let us point out that in this work we do not request any spectral properties whatsoever on the operators A_0 or A_K , contrary to the papers [10, 13] where the existence of a Riesz basis of generalized eigenvectors or related spectral properties are required.

The rest of this paper is organized as follows. In Section 2, we prove the main results of this work. In Section 3, we show that our results can easily produce new results for the controllability of PDEs. Finally, we have included in Appendix A a proof of an estimate that is needed in the proof of our main results (especially for unbounded admissible control operators).

2. Proofs of the results. The proofs of Theorem 1.1 and Theorem 1.2 both rely on the Peetre Lemma (see [15, Lemma 3]):

Lemma 2.1. *Let H_1, H_2, H_3 be three Banach spaces. Let $L \in \mathcal{L}(H_1, H_2)$ and $P \in \mathcal{L}(H_1, H_3)$ be two linear bounded operators. We assume that P is compact and that there exists $\alpha > 0$ such that*

$$\alpha \|z\|_{H_1} \leq \|Lz\|_{H_2} + \|Pz\|_{H_3}, \quad \forall z \in H_1. \quad (5)$$

Then, $\text{Im } L$ is closed and $\ker L$ is finite dimensional.

Remark 2.2. If $\ker L = \{0\}$, it is well-known (see e.g. [15, Lemma 4]) that the conclusion of Lemma 2.1 implies that there exists $\beta > 0$ such that

$$\beta \|z\|_{H_1} \leq \|Lz\|_{H_2}, \quad \forall z \in H_1.$$

In other words, the compact term in (5) can be cancelled.

Let us denote by $(S_{A_0}(t))_{t \geq 0}$ (*resp.* $(S_{A_K}(t))_{t \geq 0}$) the C_0 -semigroup generated by A_0 (*resp.* A_K). For $T > 0$ let $\Psi_T \in \mathcal{L}(L^2(0, +\infty; U), H)$ (*resp.* $\Phi_T \in \mathcal{L}(L^2(0, +\infty; U), H)$) be the input map of (A_0, B) (*resp.* (A_K, B)). Assume now that (A_0, B) is exactly controllable in time T_0 . Then, for every $T \geq T_0$, there exists $C > 0$ such that, for every $z \in H$,

$$\|z\|_H^2 \leq C \int_0^T \|\Psi_T^* z(t)\|_U^2 dt,$$

so that

$$\|z\|_H^2 \leq 2C \left(\int_0^T \|\Phi_T^* z(t)\|_U^2 dt + \int_0^T \|\Psi_T^* z(t) - \Phi_T^* z(t)\|_U^2 dt \right).$$

To prove that (A_K, B) is exactly controllable in time T , we would like to get rid of the last term in the right-hand side of the previous inequality. Therefore, we would like to apply Lemma 2.1 to the operators $L = \Phi_T^*$ and $P = \Psi_T^* - \Phi_T^*$. Note that both operators are bounded linear operators since B is admissible for both A_0 and A_K (see below). To apply Lemma 2.1, we have to check that $\Psi_T^* - \Phi_T^*$ is compact.

Lemma 2.3. *The operator $\Psi_T^* - \Phi_T^* \in \mathcal{L}(H, L^2(0, +\infty; U))$ is compact for every $T > 0$.*

For the proof of this lemma we need to recall the following estimate (see Appendix A): there exists $C > 0$ such that, for every $f \in C^1([0, T]; H)$, we have

$$\int_0^T \left\| B^* \int_0^t S_{A_0}(t-s)^* f(s) ds \right\|_U^2 dt \leq C \|f\|_{L^2(0, T; H)}^2. \quad (6)$$

This estimate holds because B is admissible for A_0 (for bounded operators $B \in \mathcal{L}(U, H)$ it is a straightforward consequence of the Cauchy-Schwarz inequality). Using the dual characterization of admissibility (see (21) below) and combining (6) with the identity (7) below we see that, if B is admissible for A_0 , then B is admissible for A_K as well.

Proof of Lemma 2.3. Let us first compute $\Psi_T^* - \Phi_T^*$. To this end, we recall the integral equation satisfied by semigroups of boundedly perturbed operators (see e.g. [6, Corollary III.1.7]), valid for every $z \in H$ and $t \in [0, T]$:

$$S_{A_K}(t)^* z = S_{A_0}(t)^* z + \int_0^t S_{A_0}(t-s)^* Fz(s) ds, \quad (7)$$

where we introduced $F \in \mathcal{L}(H, L^2(0, T; H))$ defined for every $z \in H$ and every $s \in (0, T)$ by

$$Fz(s) = K^* S_{A_K}(s)^* z.$$

Note that $Fz \in C^1([0, T]; H)$ for $z \in D(A_K^*)$. Thus, we have $\int_0^t S_{A_0}(t-s)^* Fz(s) ds \in D(A_0^*)$ for every $t \in (0, T)$ if $z \in D(A_K^*)$. This shows that each term in (7) actually belongs to $D(A_0^*)$ if $z \in D(A_0^*) = D(A_K^*)$. Therefore, we can apply B^* to obtain the following expression for $\Psi_T^* - \Phi_T^*$:

$$(\Psi_T^* - \Phi_T^*)z(t) = -B^* \int_0^t S_{A_0}(t-s)^* Fz(s) ds,$$

for every $z \in D(A_0^*)$ and a.e. $t \in (0, T)$. Using now (6) there exists $C > 0$ such that

$$\|(\Psi_T^* - \Phi_T^*)z\|_{L^2(0, T; U)} \leq C \|Fz\|_{L^2(0, T; H)},$$

for every $z \in D(A_K^*)$, and thus for every $z \in H$ by density. To conclude the proof it only remains to show that F is compact. Since H is a Hilbert space, we will prove that, if $(z_n)_n \subset H$ is such that $z_n \rightarrow 0$ weakly in H as $n \rightarrow +\infty$, then $Fz_n \rightarrow 0$ strongly in $L^2(0, T; H)$ as $n \rightarrow +\infty$. Since $z_n \rightarrow 0$ weakly in H as $n \rightarrow +\infty$, using the strong (and therefore weak) continuity of semigroups on H , we obtain

$$S_{A_K}(s)^* z_n \xrightarrow[n \rightarrow +\infty]{} 0 \quad \text{weakly in } H, \quad \forall s \in [0, T].$$

Since K^* is compact, we obtain

$$K^* S_{A_K}(s)^* z_n \xrightarrow[n \rightarrow +\infty]{} 0 \quad \text{strongly in } H, \quad \forall s \in [0, T].$$

On the other hand, by the classical semigroup estimate, $(K^* S_{A_K}(s)^* z_n)_n$ is clearly uniformly bounded in H with respect to s and n . Therefore, the Lebesgue's dominated convergence theorem applies, so that $Fz_n \rightarrow 0$ strongly in $L^2(0, T; H)$ as $n \rightarrow +\infty$. This shows that F is compact. \square

The proof of Theorem 1.1 is now a direct consequence of Lemma 2.1 and 2.3.

Proof of Theorem 1.1. The assumptions of Lemma 2.1 are satisfied for $L = \Phi_{T_0}^*$ and $P = \Psi_{T_0}^* - \Phi_{T_0}^*$. Therefore, $\text{Im } \Phi_{T_0}$ is closed (we recall that $\text{Im } \Phi_{T_0}$ is closed if, and only if, so is $\text{Im } \Phi_{T_0}^*$) and it follows from the very definitions of the notions of controllability that (A_K, B) is then exactly controllable in time T_0 if, and only if, (A_K, B) is approximately controllable in time T_0 . \square

Note that so far we have used only the first part of the conclusion of Lemma 2.1. For the proof of Theorem 1.2 we need the following general result:

Lemma 2.4. *Let H and U be two complex Hilbert spaces. Let $A : D(A) \subset H \rightarrow H$ be the generator of a C_0 -semigroup on H and let $B \in \mathcal{L}(U, D(A^*))'$ be admissible for A . For $T > 0$ let $\Phi_T \in \mathcal{L}(L^2(0, +\infty; U), H)$ be the input map of (A, B) . Let σ_F be the set given by*

$$\sigma_F = \{\lambda \in \mathbb{C}, \quad \ker(\lambda - A^*) \cap \ker B^* \neq \{0\}\},$$

and for every $\lambda \in \mathbb{C}$ let E_λ be the subspace of H defined by

$$E_\lambda = \left\{ z \in \bigcup_{m=1}^{+\infty} \ker(\lambda - A^*)^m, \quad B^*(\lambda - A^*)^m z = 0, \quad \forall m \in \mathbb{N} \right\}.$$

Assume that there exists $T_0 > 0$ such that

$$\dim \ker \Phi_{T_0}^* < +\infty. \quad (8)$$

Then, the set σ_F is finite, E_λ is finite dimensional for every $\lambda \in \mathbb{C}$, and we have

$$\ker \Phi_T^* = \bigoplus_{\lambda \in \sigma_F} E_\lambda, \quad \forall T > T_0. \quad (9)$$

Remark 2.5. From the proof of Lemma 2.4 below we easily see that the equality (9) remains valid for $T = T_0$ too if $\ker \Phi_{T_0}^* \subset D(A^*)$ (in addition to (8)).

Remark 2.6. In the finite dimensional case $H = \mathbb{C}^n$ and $U = \mathbb{C}^m$ ($n, m \in \mathbb{N}^*$) we recover the well-known fact that $\text{Im } \Phi_T = \text{Im}(B|AB| \cdots |A^{n-1}B)$ for every $T > 0$.

Proof of Lemma 2.4. Let us first prove that, for every $T > 0$ and $\lambda \in \mathbb{C}$, we have

$$\ker \Phi_T^* \supset E_\lambda. \quad (10)$$

Let then $z \in E_\lambda$. Thus, $z \in D((A^*)^\infty)$ and there exists $m \in \mathbb{N}^*$ such that

$$(\lambda - A^*)^m z = 0, \quad (11)$$

and

$$B^*(\lambda - A^*)^r z = 0, \quad \forall r \in \{0, \dots, m-1\}. \quad (12)$$

Thanks to (11) we have, for every $t \geq 0$,

$$S_A(t)^* z = e^{\lambda t} \sum_{r=0}^{m-1} \frac{t^r}{r!} (A^* - \lambda)^r z.$$

Applying B^* and using (12) we obtain that $z \in \ker \Phi_T^*$. This establishes (10). Since the sum $\sum_{\lambda \in \sigma_F} E_\lambda$ is clearly a direct sum, (10) implies that

$$\ker \Phi_T^* \supset \bigoplus_{\lambda \in \sigma_F} E_\lambda, \quad \forall T > 0. \quad (13)$$

In particular, by (8) we obtain that σ_F is finite and that E_λ is finite dimensional for every $\lambda \in \sigma_F$.

Let us now prove the reverse inclusion for $T > T_0$. First note that

$$\ker \Phi_T^* \subset \ker \Phi_{T'}^*, \quad \forall T \geq T'. \quad (14)$$

From now on, T is fixed such that $T > T_0$. Let $\varepsilon \in (0, T - T_0]$ so that $T - \varepsilon \geq T_0$ and thus, by (14) and (8),

$$\dim \ker \Phi_{T-\varepsilon}^* < +\infty. \quad (15)$$

The key point is to establish that

$$\ker \Phi_T^* \subset D(A^*). \quad (16)$$

We follow the ideas of the proof of [4, Theorem 5]. Let $z \in \ker \Phi_T^*$. We have to show that, for any sequence $t_n > 0$ with $t_n \rightarrow 0$ as $n \rightarrow +\infty$, the sequence

$$u_n = \frac{(S_A(t_n))^* z - z}{t_n}$$

converges in H as $n \rightarrow +\infty$. Let $N \in \mathbb{N}$ be large enough so that $t_n < \varepsilon$ for every $n \geq N$. Let us first show that

$$u_n \in \ker \Phi_{T-\varepsilon}^*, \quad \forall n \geq N. \quad (17)$$

To this end, observe that, for $n \geq N$, we have

$$\int_0^{T-\varepsilon} \|\Phi_{T-\varepsilon}^* S_A(t_n)^* z(t)\|_U^2 dt = \int_{\varepsilon-t_n}^{T-t_n} \|\Phi_T^* z(t)\|_U^2 dt,$$

(this is true for $z \in D(A^*)$ and thus for $z \in H$ by density and continuity of Φ_T^* for $T > 0$). This shows that $S_A(t_n)^* z \in \ker \Phi_{T-\varepsilon}^*$ for $n \geq N$ since $z \in \ker \Phi_T^*$. Thus, we have (17).

Let now $\mu \in \rho(A^*)$ be fixed and let us introduce the following norm on $\ker \Phi_{T-\varepsilon}^*$:

$$\|z\|_{-1} = \|(\mu - A^*)^{-1} z\|_H.$$

Since $(\mu - A^*)^{-1} z \in D(A^*)$, we have

$$(\mu - A^*)^{-1} u_n = \frac{S_A(t_n)^* - \text{Id}}{t_n} (\mu - A^*)^{-1} z \xrightarrow{n \rightarrow +\infty} A^* (\mu - A^*)^{-1} z \quad \text{in } H.$$

Therefore, $(u_n)_{n \geq N}$ is a Cauchy sequence in $\ker \Phi_{T-\varepsilon}^*$ for the norm $\|\cdot\|_{-1}$. Since $\ker \Phi_{T-\varepsilon}^*$ is finite dimensional (see (15)), all the norms are equivalent on $\ker \Phi_{T-\varepsilon}^*$. Thus, $(u_n)_{n \geq N}$ is then a Cauchy sequence for the usual norm $\|\cdot\|_H$ as well and, as a result, converges for this norm. This shows that $z \in D(A^*)$ and establishes (16).

Next, observe that

$$\ker \Phi_T^* \subset \ker B^*. \quad (18)$$

Indeed, if $z \in \ker \Phi_T^*$, then $z \in D(A^*)$ as we have just seen, so that the map $t \in [0, T] \mapsto B^* S_A(t)^* z \in U$ is continuous and we can take $t = 0$ in the definition of $\ker \Phi_T^*$ to obtain that $B^* z = 0$.

Finally, let us prove that $\ker \Phi_T^*$ is stable by A^* . Let $z \in \ker \Phi_T^*$, that is

$$\Phi_T^* z(t) = 0, \quad \text{a.e. } t \in (0, T).$$

Since $z \in D(A^*)$ we can differentiate this identity to obtain (see e.g. [17, Proposition 4.3.4])

$$\Phi_T^* A^* z(t) = 0, \quad \text{a.e. } t \in (0, T),$$

that is $A^* z \in \ker \Phi_T^*$.

Consequently, the restriction M of A^* to $\ker \Phi_T^*$ is a linear operator from the finite dimensional space $\ker \Phi_T^*$ into itself. Assume that $\ker \Phi_T^* \neq \{0\}$ (otherwise (9) is clear from (13)). Therefore, M is triangularizable in $\ker \Phi_T^*$ (here we use that H is a complex Hilbert space). In other words, $\ker \Phi_T^*$ is the direct sum of the root subspaces of M : for every $\lambda \in \sigma(M)$, there exists $m(\lambda) \in \mathbb{N}^*$ such that

$$\ker \Phi_T^* = \bigoplus_{\lambda \in \sigma(M)} \ker (\lambda - M)^{m(\lambda)}.$$

Finally, thanks to (18) we have $\sigma(M) \subset \sigma_F$ and $\ker (\lambda - M)^{m(\lambda)} \subset E_\lambda$ for every $\lambda \in \sigma(M)$. \square

Let us now conclude this section with the proof of Theorem 1.2.

Proof of Theorem 1.2. The assumptions of Lemma 2.1 are satisfied for $L = \Phi_T^*$ and $P = \Psi_T^* - \Phi_T^*$ for every $T \geq T_0$. Therefore, for every $T \geq T_0$, we have

$$(\ker \Phi_T^*)^\perp = \text{Im } \Phi_T, \quad \dim \ker \Phi_T^* < +\infty.$$

Applying now Lemma 2.4 we obtain the desired conclusion. \square

3. An example. Our results, especially Corollary 1.3, potentially have a lot of applications. In this section we focus on a recent open problem introduced in [8, Section 4].

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open bounded subset with boundary $\partial\Omega$ of class C^2 and let $\omega \subset \Omega$ be a non empty open subset. Let $T > 0$. We consider the following wave equation with non local spatial term:

$$\begin{cases} y_{tt} - \Delta y = \int_{\Omega} k_2(\xi)y(t, \xi) d\xi + \mathbb{1}_{\omega}(x)u(t, x) & \text{in } (0, T) \times \Omega, \\ y = 0 & \text{on } (0, T) \times \partial\Omega, \\ y(0) = y_1^0, \quad y_t(0) = y_2^0 & \text{in } \Omega. \end{cases} \quad (19)$$

In (19), (y_1^0, y_2^0) is the initial data, y is the state and u is the control. $\mathbb{1}_{\omega}$ denotes the function that is equal to 1 in ω and 0 outside. The kernel k_2 is assumed to be in $L^2(\Omega)$. Clearly, such kernels do not in general satisfy the strong analyticity assumption (3) of [8].

Let us recast (19) as a first-order abstract evolution system (1). The state space H and the control space U are

$$H = \begin{matrix} H_0^1(\Omega) \\ \times \\ L^2(\Omega) \end{matrix}, \quad U = L^2(\Omega).$$

The operator $A_K : D(A_K) \subset H \rightarrow H$ is

$$A_K \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \Delta y_1 + \int_{\Omega} k_2(\xi)y_1(\xi) d\xi \end{pmatrix}, \quad D(A_K) = \begin{matrix} H^2(\Omega) \cap H_0^1(\Omega) \\ \times \\ H_0^1(\Omega) \end{matrix},$$

and the control operator $B : U \rightarrow H$ is

$$Bu = \begin{pmatrix} 0 \\ \mathbb{1}_{\omega}u \end{pmatrix}.$$

Clearly, A_K splits up into $A_K = A_0 + K$, where $A_0 : D(A_0) \subset H \rightarrow H$ is given by

$$A_0 \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ \Delta y_1 \end{pmatrix}, \quad D(A_0) = D(A_K),$$

and $K : H \rightarrow H$ is given by

$$K \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \int_{\Omega} k_2(\xi)y_1(\xi) d\xi \end{pmatrix}.$$

It is well-known that A_0 is the generator of a C_0 -group on H . On the other hand, by the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, it is clear that K is compact. Finally, observe that B is bounded and thus admissible. Therefore, the assumptions (i) and (ii) of Theorem 1.1 are satisfied.

There is a little subtlety though. Indeed, as usual, we identify $L^2(\Omega)$ with its adjoint. Therefore, we cannot identify H with its adjoint as well. The results of this paper still remain valid in such a framework but we need to distinguish between H and its dual H' :

$$H' = \begin{matrix} L^2(\Omega) \\ \times \\ H^{-1}(\Omega) \end{matrix},$$

equipped with the duality product

$$\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle_{H', H} = \langle z_1, y_2 \rangle_{L^2(\Omega)} - \langle z_2, y_1 \rangle_{H^{-1}(\Omega), H_0^1(\Omega)},$$

for $(z_1, z_2) \in H'$ and $(y_1, y_2) \in H$. Then, we can check that

$$A_K^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} -z_2 \\ -\Delta z_1 - k_2 \int_{\Omega} z_1(\xi) d\xi \end{pmatrix}, \quad D(A_K^*) = \begin{matrix} H_0^1(\Omega) \\ \times \\ L^2(\Omega) \end{matrix},$$

(where $\langle \Delta z, y \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = -\langle \nabla z, \nabla y \rangle_{L^2(\Omega)}$ for $z, y \in H_0^1(\Omega)$) and

$$B^* \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \mathbb{1}_{\omega} z_1.$$

We can now state the following simple (but new) consequence of Corollary 1.3:

Proposition 3.1. *Assume that the wave equation (A_0, B) is exactly controllable in time $T_0 > 0$. If $k_2 \neq 0$ in ω , then (19) is exactly controllable in time T for every $T > T_0$.*

Proof. As mentioned before, to apply Corollary 1.3 we only have to check the Fattorini-Hautus test (4) corresponding to (19). Let then $\lambda \in \mathbb{C}$ and $z_1 \in H_0^1(\Omega)$, $z_2 \in L^2(\Omega)$, be such that

$$\begin{cases} -z_2 = \lambda z_1 & \text{in } L^2(\Omega), \\ -\Delta z_1 - k_2 \int_{\Omega} z_1(\xi) d\xi = \lambda z_2 & \text{in } H^{-1}(\Omega), \\ \mathbb{1}_{\omega} z_1 = 0 & \text{in } L^2(\Omega), \end{cases} \quad (20)$$

and let us show that this implies that $(z_1, z_2) = 0$ in H' . Plugging the first equation into the second one gives

$$-\Delta z_1 - k_2 \int_{\Omega} z_1(\xi) d\xi = -\lambda^2 z_1 \quad \text{in } H^{-1}(\Omega).$$

Using the third condition of (20) we obtain that

$$k_2 \int_{\Omega} z_1(\xi) d\xi = 0 \quad \text{in } H^{-1}(\omega).$$

The assumption $k_2 \neq 0$ in ω then implies that the constant $\int_{\Omega} z_1(\xi) d\xi$ is equal to zero. Coming back to (20) we see that $z_1 \in H_0^1(\Omega)$ satisfies

$$\begin{cases} -\Delta z_1 = \mu z_1 & \text{in } H^{-1}(\Omega), \\ \mathbb{1}_{\omega} z_1 = 0 & \text{in } L^2(\Omega), \end{cases}$$

with $\mu = -\lambda^2$. As it is well-known, this implies that $z_1 = 0$ in Ω . Coming back to the first equation of (20) we obtain that $z_2 = 0$ in Ω as well. \square

Appendix A. Proof of the estimate (6). This appendix is devoted to a proof of the estimate (6) that is used in the proof of Lemma 2.3. It is largely inspired by [9, Proposition 3.3].

Let us recall our framework. H and U are two Hilbert spaces. $A : D(A) \subset H \rightarrow H$ is the generator of a C_0 -semigroup $(S_A(t))_{t \geq 0}$ on H and $B \in \mathcal{L}(U, D(A^*)')$ is admissible for A . Let us recall the following dual characterization of admissibility: B is admissible for A if, and only if, for some (and hence all) $T > 0$, there exists $\beta > 0$ such that

$$\int_0^T \|B^* S_A(T-t)^* z\|_U^2 dt \leq \beta \|z\|_H^2, \quad \forall z \in D(A^*). \quad (21)$$

Let us now introduce for $n \in \mathbb{N}$ large enough ($n > \omega_0$, where $\omega_0 \in \mathbb{R}$ is the growth bound of A) the Yosida-like approximations $C_n \in \mathcal{L}(H, U)$ defined by

$$C_n z = n B^* (n - A^*)^{-1} z, \quad \forall z \in H.$$

Let us recall that (see e.g. [6, Lemma II.3.4])

$$n(n - A^*)^{-1} z \xrightarrow{n \rightarrow +\infty} z \quad \text{in } H, \quad \forall z \in H. \quad (22)$$

This implies in particular that

$$C_n z \xrightarrow{n \rightarrow +\infty} B^* z \quad \text{in } U, \quad \forall z \in D(A^*), \quad (23)$$

since for every $z \in D(A^*)$ we have

$$\begin{aligned} & \|C_n z - B^* z\|_U \\ & \leq \|B^*\|_{\mathcal{L}(D(A^*), U)} (\|n(n - A^*)^{-1} A^* z - A^* z\|_H + \|n(n - A^*)^{-1} z - z\|_H). \end{aligned}$$

For $f \in L^2(0, T; H)$, let us denote by $S_A^* * f \in L^2(0, T; H)$ the function defined for every $t \in (0, T)$ by

$$(S_A^* * f)(t) = \int_0^t S_A(t-s)^* f(s) ds.$$

Using the Cauchy-Schwarz inequality we have

$$\int_0^T \|C_n(S_A^* * f)(t)\|_U^2 dt \leq T \int_0^T \int_0^t \|B^* S_A(t-s)^* n(n - A^*)^{-1} f(s)\|_U^2 ds dt.$$

Using Fubini's theorem we obtain

$$\int_0^T \|C_n(S_A^* * f)(t)\|_U^2 dt \leq T \int_0^T \int_0^t \|B^* S_A(t-s)^* n(n - A^*)^{-1} f(s)\|_U^2 dt ds.$$

Using now the admissibility of B (see (21)) this gives

$$\int_0^T \|C_n(S_A^* * f)(t)\|_U^2 dt \leq T\beta \int_0^T \|n(n - A^*)^{-1} f(s)\|_H^2 ds. \quad (24)$$

Let us now remark that $S_A^* * f \in L^2(0, T; D(A^*))$ for $f \in C^1([0, T]; H)$. Using then (23) and (22) (and the uniform boundedness principle), we see that the Lebesgue's dominated convergence theorem applies and that we can pass to the limit $n \rightarrow +\infty$ in (24) to finally obtain the desired estimate.

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