# On the persistence of H'older regular patches of density for the inhomogeneous Navier-Stokes equations

Raphaël Danchin, Xin Zhang

# ON THE PERSISTENCE OF HÖLDER REGULAR PATCHES OF DENSITY FOR THE INHOMOGENEOUS NAVIER-STOKES EQUATIONS

RAPHAËL DANCHIN AND XIN ZHANG

Abstract. In our recent work dedicated to the Boussinesq equations [14], we established the persistence of solutions with piecewise constant temperature along interfaces with Hölder regularity. We here address the same problem for the inhomogeneous Navier-Stokes equations satisfied by a viscous incompressible and inhomogeneous fluid. We establish that, indeed, in the slightly inhomogeneous case, patches of densities with $\mathcal{C}^{1,\varepsilon}$ regularity propagate for all time.

As in [14], our result follows from the conservation of Hölder regularity along vector fields moving with the flow. The proof of that latter result is based on commutator estimates involving para-vector fields, and multiplier spaces. The overall analysis is more complicated than in [14] however, since the coupling between the mass and velocity equations in the inhomogeneous Navier-Stokes equations is *quasilinear* while it is linear for the Boussinesq equations.

## Introduction

We are concerned with the following *inhomogeneous incompressible Navier-Stokes equations* in the whole space $\mathbb{R}^N$ with $N \geq 2$:

$$(INS) \qquad \begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) - \mu \Delta u + \nabla P = 0, \\ \operatorname{div} u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases}$$

Above, the unknowns $(\rho, u, P) \in \mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{R}$ stand for the density, velocity vector field and pressure, respectively, and the so-called viscosity coefficient $\mu$ is a positive constant.

There is an important literature dedicated to the mathematical analysis of System $(INS)$. The global existence of finite energy weak solutions with no vacuum (i.e. $\rho > 0$) has been established in the seventies (see the monograph [3] and the references therein), then extended by Simon in [23] in the vacuum case. Similar results have been obtained shortly after by Lions in the more general case where the viscosity is density-dependent (see [21]).

Among the numerous open questions raised by Lions in [21], the so-called *density patch problem* is a particularly challenging one. The question is whether, assuming that $\rho_0 = \mathbb{1}_{\mathcal{D}_0}$ for some domain $\mathcal{D}_0$ of $\mathbb{R}^2$ and that $\sqrt{\rho_0}\, u_0$ is in $L^2(\mathbb{R}^2)$, it is true that we have

$$\rho(t) = \mathbb{1}_{\mathcal{D}_t} \quad \text{for all} \quad t \geq 0 \tag{0.1}$$

for some domain $\mathcal{D}_t$ *with the same regularity as the initial one.* Although the renormalized solutions theory of Di Perna and Lions [15] for transport equations ensures that we do have (0.1) with $\mathcal{D}_t$ being the image of $\mathcal{D}_0$ by the volume preserving (generalized) flow of $u$, the weak solution framework does not give much information on the regularity of the patch $\mathcal{D}_t$ for positive times.

The present paper aims at making one more step toward solving Lions' question, by considering the case where

$$\rho_0 = \eta_1 \mathbb{1}_{\mathcal{D}_0} + \eta_2 \mathbb{1}_{\mathcal{D}_0^c}, \tag{0.2}$$

*Key words and phrases.* Inhomogeneous Navier-Stokes equations; $\mathcal{C}^{1,\varepsilon}$ density patch; Striated regularity.

for some simply connected bounded domain $\mathcal{D}_0$ of class $\mathcal{C}^{1,\varepsilon}$, and positive constants $\eta_1$ and $\eta_2$ *close to one another*.

That issue has been considered recently in [19, 20] by Liao and Zhang in the 2-D case (see also [18] for the 3-D case), first assuming that $|\eta_1-\eta_2|$ is small then in the more challenging case where $\eta_1$ and $\eta_2$ are *any* positive real numbers. Under suitable striated-type regularity assumptions for the initial velocity, the authors proved the all-time persistence of high Sobolev regularity of patches of density.

Before giving more insight into our main results, let us briefly recall how Liao and Zhang's proof goes. As in the pioneering work by Chemin [7] dedicated to the vortex patches problem for the 2-D incompressible Euler equations, the regularity of the interfaces is described by means of one (or several) tangent vector fields that evolve according to the flow of the velocity field. More precisely, let us assume that the boundary $\partial\mathcal{D}_0$ of the initial patch $\mathcal{D}_0$ is the level set $f_0^{-1}(\{0\})$ of some function $f_0:\mathbb{R}^2\to\mathbb{R}$ that does not degenerate in a neighborhood of $\partial\mathcal{D}_0$. Then the vector field $X_0:=\nabla^\perp f_0$ is tangent to $\partial\mathcal{D}_0$. Now, if we denote by $\psi$ the flow associated to the velocity field $u$, that is the solution to the (integrated) ordinary differential equation

$$\psi(t,x)=x+\int_0^t u\big(\tau,\psi(\tau,x)\big)\,d\tau, \tag{0.3}$$

then the boundary of $\mathcal{D}_t:=\psi(t,\mathcal{D}_0)$ coincides with $f_t^{-1}(\{0\})$ where $f_t:=f_0\circ\psi_t^{-1}$ and $\psi_t:=\psi(t,\cdot)$, and we have

$$\rho(t,\cdot)=\eta_1\mathbb{1}_{\mathcal{D}_t}+\eta_2\mathbb{1}_{\mathcal{D}_t^c}. \tag{0.4}$$

Note that the tangent vector field $X_t:=\nabla^\perp f_t$ coincides with the evolution of the initial vector field $X_0$ along the flow of $u$, that is[1] :

$$X(t,\cdot):=(\partial_{X_0}\psi)\circ\psi_t^{-1}, \tag{0.5}$$

and thus satisfies, at least formally, the transport equation

$$\begin{cases}\partial_t X+u\cdot\nabla X=\partial_X u,\\ X|_{t=0}=X_0.\end{cases} \tag{0.6}$$

Consequently, the problem of persistence of regularity for the patch reduces to that of the vector field $X$ solution to (0.6). In their outstanding work, Liao and Zhang justified that heuristics in the case of high Sobolev regularity, first if $\eta_1$ and $\eta_2$ are close to one another [19], and next assuming only that $\eta_1$ and $\eta_2$ are positive [20]. More precisely, the function $f_0$ is assumed to be in $W^{k,p}(\mathbb{R}^2)$ for some integer number $k\geq 3$ and real number $p$ in $]2,4[$, and the initial velocity field $u_0$, to satisfy the following *striated regularity* property along the vector field $X_0:=\nabla^\perp f_0$:

$$\partial_{X_0}^\ell u_0\in H^{s-\frac{\varepsilon\ell}{k}}\quad\text{for all}\quad \ell\in\{0,\cdots,k\}\quad\text{and for some}\quad 0<\varepsilon<s<1.$$

Note however that the minimal regularity requirement in [19, 20] is that $f$ is in $W^{3,p}$ for some $p\in]2,4f[$. In terms of Hölder inequality, this means (using Sobolev embedding), that the boundary of the patch must be at least in $\mathcal{C}^{2,\varepsilon}$ for some $\varepsilon>0$.

In order to propagate lower order Hölder regularity, one may take advantage of the recent results by Huang, Paicu and Zhang in [16] (see also [13]). Indeed, there, for small enough $\delta>0$, the authors construct global unique solutions with flow in $\mathcal{C}^{1,\delta}$ whenever the initial density is close enough (for the $L^\infty$ norm) to some positive constant, and $u_0$ is in the Besov space $\dot B^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N)\cap\dot B^{\frac{N}{p}+\delta-1}_{p,1}(\mathbb{R}^N)$ (see the definition below in (1.3)). This clearly allows to propagate $\mathcal{C}^{1,\varepsilon}$ interfaces, but only for $\varepsilon\leq\delta$, because the maximal value of $\varepsilon$ is limited by the

[1]For any vector field $Y=Y^k(x)\partial_k$ and function $f$ in $\mathcal{C}^1(\mathbb{R}^N;\mathbb{R})$, we denote by $\partial_Y f$ the *directional derivative* of $f$ along $Y$, that is, with the Einstein summation convention, $\partial_Y f:=Y^k\partial_k f=Y\cdot\nabla f$.

global regularity assumption on $u_0$ although Liao and Zhang's results mentioned above (as well as those of Chemin [7] in the context of Euler equations) suggest that only tangential regularity is needed to propagate the regularity of the patch.

## 1. Results

Our goal here is to propagate the $\mathcal{C}^{1,\varepsilon}$ Hölder regularity of the patch, within a *critical* regularity framework. By critical, we mean that we strive for a solution space having the same scaling invariance by time and space dilations as $(INS)$ itself, namely:

$$(\rho,u,P)(t,x)\to(\rho,\lambda u,\lambda^2P)(\lambda^2t,\lambda x)\quad\text{and}\quad(\rho_0,u_0)(x)\to(\rho_0,\lambda u_0)(\lambda x). \tag{1.1}$$

Working with critical regularity is by now a classical approach for the homogeneous Navier-Stokes equations (that is $\rho$ is a positive constant in $(INS)$) in the whole space $\mathbb{R}^N$ (see e.g. [4, 17] and the references therein) and that it is also relevant in the inhomogeneous situation (see in particular the work by the first author in [10] devoted to the well-posedness issue in critical homogeneous Besov spaces, and its generalization to more general Besov spaces performed by Abidi in [1] and Abidi and Paicu in [2]).

In all those works however, the regularity requirements for the density are much too strong to consider piecewise constant functions. That difficulty has been by-passed in a joint work of the first author with P.B. Mucha [11], where well-posedness has been established in a critical regularity framework that allows for initial densities that are discontinuous along a $\mathcal{C}^1$ interface (see the comments below Theorem 1.1).

Before writing out the statement we are referring to and giving the main results of the present paper, we need to introduce some notations. In all the paper, we agree that $A\lesssim B$ means $A\leq CB$ for some harmless "constant" $C$, the meaning of which may be guessed from the context. For $T\in]0,+\infty[$, $p\in[1,+\infty]$ and $E$ a Banach space, the notation $L^p_T(E)$ designates the space of $L^p$ functions on $]0,T[$ with values in $E$, and $L^p(\mathbb{R}_+;E)$ corresponds to the case $T=+\infty$. For simplicity, we keep the same notation for vector or matrix-valued functions.

Next, let us recall the definition of Besov spaces (following e.g. [4, Section 2.2]). To this end, consider two smooth radial functions $\chi$ and $\varphi$ supported in $\{\xi\in\mathbb{R}^N:|\xi|\leq 4/3\}$ and $\{\xi\in\mathbb{R}^N:3/4\leq|\xi|\leq 8/3\}$, respectively, and satisfying

$$\sum_{j\in\mathbb{Z}}\varphi(2^{-j}\xi)=1,\ \forall\,\xi\in\mathbb{R}^N\backslash\{0\},\quad \chi(\xi)+\sum_{j\geq 0}\varphi(2^{-j}\xi)=1,\ \forall\,\xi\in\mathbb{R}^N. \tag{1.2}$$

Next, let us introduce the following Fourier truncation operators:

$$\dot\Delta_j:=\varphi(2^{-j}D),\ \dot S_j:=\chi(2^{-j}D),\ \forall j\in\mathbb{Z};\qquad \Delta_j:=\varphi(2^{-j}D),\ \forall j\geq 0,\quad \Delta_{-1}:=\chi(D).$$

For all triplet $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$, the homogeneous Besov space $\dot B^s_{p,r}(\mathbb{R}^N)$ (just denoted by $\dot B^s_{p,r}$ if the value of the dimension is clear from the context) is defined by

$$\dot B^s_{p,r}(\mathbb{R}^N):=\big\{\,u\in\mathcal{S}'_h(\mathbb{R}^N):\|u\|_{\dot B^s_{p,r}}:=\big\|2^{js}\|\dot\Delta_ju\|_{L^p}\big\|_{\ell^r(\mathbb{Z})}<\infty\,\big\}, \tag{1.3}$$

where $\mathcal{S}'_h(\mathbb{R}^N)$ is the subspace of tempered distributions $\mathcal{S}'(\mathbb{R}^N)$ defined by

$$\mathcal{S}'_h(\mathbb{R}^N):=\big\{\,u\in\mathcal{S}'(\mathbb{R}^N):\lim_{j\to-\infty}\dot S_ju=0\,\big\}\cdot$$

We shall also use sometimes the following inhomogeneous Besov spaces:

$$B^s_{p,r}(\mathbb{R}^N):=\big\{\,u\in\mathcal{S}'(\mathbb{R}^N):\|u\|_{B^s_{p,r}}:=\big\|2^{js}\|\Delta_ju\|_{L^p}\big\|_{\ell^r(\mathbb{N}\cup\{-1\})}<\infty\,\big\}. \tag{1.4}$$

Throughout the paper, we agree that the notation $b^s_{p,r}(\mathbb{R}^N)$ designates both $B^s_{p,r}(\mathbb{R}^N)$ and $\dot B^s_{p,r}(\mathbb{R}^N)$.

It is well-known that the family of Besov spaces contains more classical items like the Sobolev or Hölder spaces. For instance $\dot B^s_{2,2}(\mathbb{R}^N)$ coincides with the homogeneous Sobolev space $\dot H^s(\mathbb{R}^N)$ and we have

$$B^s_{\infty,\infty}(\mathbb{R}^N)=\mathcal{C}^{0,s}(\mathbb{R}^N)=L^\infty(\mathbb{R}^N)\cap\dot B^s_{\infty,\infty}(\mathbb{R}^N)\quad\text{if}\quad s\in]0,1[. \tag{1.5}$$

To emphasize that latter connection between Hölder and Besov spaces, we shall often use the notation $\dot{\mathscr{C}}^s:=\dot B^s_{\infty,\infty}$ (or $\mathscr{C}^s:=B^s_{\infty,\infty}$) for *any* $s\in\mathbb{R}$.

When investigating evolutionary equations in critical Besov spaces, it is wise to use the following *tilde homogeneous Besov spaces* first introduced by Chemin in [8]: for any $t\in]0,+\infty]$ and $(s,p,r,\gamma)\in\mathbb{R}\times[1,+\infty]^3$, we set

$$\widetilde{L}^\gamma_t(\dot B^s_{p,r}):=\Big\{u\in\mathcal{S}'(]0,t[\times\mathbb{R}^N):\lim_{j\to-\infty}\dot S_ju=0\quad\text{in}\quad L^\gamma_t(L^\infty)\quad\text{and}\quad\|u\|_{\widetilde{L}^\gamma_t(\dot B^s_{p,r})}<\infty\Big\},$$

where

$$\|u\|_{\widetilde{L}^\gamma_t(\dot B^s_{p,r})}:=\big\|2^{js}\|\dot\Delta_ju\|_{L^\gamma_t(L^p)}\big\|_{\ell^r(\mathbb{Z})}<\infty.$$

The index $t$ will be omitted if it is equal to $+\infty$, and we shall denote

$$\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^s_{p,r}):=\widetilde{L}^\infty(\mathbb{R}_+;\dot B^s_{p,r})\cap\mathcal{C}(\mathbb{R}_+;\dot B^s_{p,r}).$$

Finally, we shall make use of *multiplier spaces* associated to couples $(E,F)$ of Banach spaces included in the set of tempered distributions. The definition goes as follows:

**Definition.** *Let $E$ and $F$ be two Banach spaces embedded in $\mathcal{S}'(\mathbb{R}^N)$. The* multiplier space *$\mathcal{M}(E\to F)$ (simply denoted by $\mathcal{M}(E)$ if $E=F$) is the set of those functions $\varphi$ satisfying $\varphi u\in F$ for all $u$ in $E$ and, additionally,*

$$\|\varphi\|_{\mathcal{M}(E\to F)}:=\sup_{\substack{u\in E\\ \|u\|_E\leq1}}\|\varphi u\|_F<\infty. \tag{1.6}$$

It goes without saying that $\|\cdot\|_{\mathcal{M}(E\to F)}$ is a norm on $\mathcal{M}(E\to F)$ and that one may restrict the supremum in (1.6) to any *dense* subset of $E$.

The following result that has been proved in [11] is the starting point of our analysis.

**Theorem 1.1.** *Let $p\in[1,2N[$ and $u_0$ be a divergence-free vector field with coefficients in $\dot B^{\frac Np-1}_{p,1}$. Assume that $\rho_0$ belongs to the multiplier space $\mathcal{M}\big(\dot B^{\frac Np-1}_{p,1}\big)$. There exist two constants $c$ and $C$ depending only on $p$ and on $N$ such that if*

$$\|\rho_0-1\|_{\mathcal{M}\left(\dot B^{\frac Np-1}_{p,1}\right)}+\mu^{-1}\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}\leq c$$

*then System $(INS)$ in $\mathbb{R}^N$ with $N\geq2$ has a unique solution $(\rho,u,\nabla P)$ satisfying*

- $\rho\in L^\infty\Big(\mathbb{R}_+;\mathcal{M}\big(\dot B^{\frac Np-1}_{p,1}\big)\Big)$,
- $u\in\widetilde{\mathcal{C}}_b\big(\mathbb{R}_+;\dot B^{\frac Np-1}_{p,1}\big)$,
- $(\partial_tu,\nabla^2u,\nabla P)\in L^1\big(\mathbb{R}_+;\dot B^{\frac Np-1}_{p,1}\big)$.

*Furthermore, the following inequality is fulfilled:*

$$\|u\|_{\widetilde{L}^\infty\left(\mathbb{R}^+;\dot B^{\frac Np-1}_{p,1}\right)}+\|\partial_tu,\mu\nabla^2u,\nabla P\|_{L^1\left(\mathbb{R}^+;\dot B^{\frac Np-1}_{p,1}\right)}\leq C\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}. \tag{1.7}$$

A similar result (only local in time) may be proved for large $u_0$. However the smallness condition on $\rho_0-1$ is still needed, and whether one can extend Theorem 1.1 to the case of *large* density variations and *critical* velocity fields is totally open.

By classical embedding, having $\nabla^2 u$ in $L^1(\mathbb{R}_+;\dot{B}^{\frac{N}{p}-1}_{p,1})$ implies that $\nabla u$ is in $L^1(\mathbb{R}_+;\mathcal{C}_b)$. Therefore the flow $\psi$ of $u$ is in $\mathcal{C}^1$. Now, it has been observed in [11] that for any uniformly $\mathcal{C}^1$ bounded domain $\mathcal{D}_0$, the function $\mathbb{1}_{\mathcal{D}_0}$ belongs to $\mathcal{M}\big(\dot{B}^s_{p,1}\big)$ whenever $-1+\frac{1}{p}<s<\frac{1}{p}$. Therefore, one may deduce from Theorem 1.1 that if $\rho_0$ is given by (0.2), if $u_0$ is in $\dot{B}^{\frac{N}{p}-1}_{p,1}$ for some $N-1<p<2N$ and if

$$\|u_0\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}+|\eta_2-\eta_1| \quad \text{is small enough}$$

then System $(INS)$ admits a unique global solution in the above regularity class with $\rho(t,\cdot)$ given by (0.4) and $\mathcal{D}_t=\psi(t,\mathcal{D}_0)$ in $\mathcal{C}^1$ for all time $t\geq 0$.

The present paper aims at propagating $\mathcal{C}^{1,\varepsilon}$ regularity of density patches for any $\varepsilon\in]0,1[$ and *within a critical regularity framework.* For simplicity, we shall focus on simply connected bounded domains $\mathcal{D}_0$, and $\mathcal{C}^{1,\varepsilon}$ regularity thus means that there exists some open neighborhood $V_0$ of $\mathcal{D}_0$ and a function $f_0:\mathbb{R}^N\to\mathbb{R}$ of class $\mathcal{C}^{1,\varepsilon}$ such that

$$\mathcal{D}_0=f_0^{-1}(\{0\})\cap V_0 \quad \text{and} \quad \nabla f_0 \ \text{ does not vanish on } \ V_0. \tag{1.8}$$

As the viscosity coefficient $\mu$ will be fixed once and for all, we shall set it to 1 for notational simplicity. Likewise, we shall assume the reference density at infinity to be 1.

Our main statement of propagation of Hölder regularity of density patches for $(INS)$ in the plane reads as follows.

**Theorem 1.2.** *Let $\mathcal{D}_0$ be a simply connected bounded domain of $\mathbb{R}^2$ satisfying* (1.8) *for some $\varepsilon$ in $]0,1[$. There exists a constant $\eta_0$ depending only on $\mathcal{D}_0$ so that for all $\eta\in]-\eta_0,\eta_0[$ if the initial density is given by*

$$\rho_0:=(1+\eta)\mathbb{1}_{\mathcal{D}_0}+\mathbb{1}_{\mathcal{D}_0^c}, \tag{1.9}$$

*and the divergence free vector-field $u_0\in L^2$ has vorticity $\omega_0:=\partial_1 u_0^2-\partial_2 u_0^1$ with zero average and such that*

$$\omega_0=\widetilde{\omega}_0\,\mathbb{1}_{\mathcal{D}_0} \tag{1.10}$$

*for some small enough function $\widetilde{\omega}_0$ with Hölder regularity, then System $(INS)$ has a unique solution $(\rho,u,\nabla P)$ with the properties listed in Theorem 1.1 for some suitable $p\in]1,4[$.*

*In addition, if we denote by $\psi$ the flow of $u$ then for all $t\geq 0$, we have*

$$\rho(t,\cdot):=(1+\eta)\mathbb{1}_{\mathcal{D}_t}+\mathbb{1}_{\mathcal{D}_t^c} \quad \text{with} \quad \mathcal{D}_t:=\psi(t,\mathcal{D}_0), \tag{1.11}$$

*and $\mathcal{D}_t$ remains a simply connected bounded domain of class $\mathcal{C}^{1,\varepsilon}$.*

**Remark 1.3.** *We need the initial vorticity to be mean free, in order to guarantee that $u_0$ belongs to some homogeneous Besov space $\dot{B}^{\frac{2}{p}-1}_{p,1}$. It is no longer needed in dimension 3 (see Theorem 2.2 below).*

*Of course, there are many other examples of initial velocities for which propagation of $\mathcal{C}^{1,\varepsilon}$ patches holds true : an obvious one is when $u_0$ has critical regularity and vanishes on a neighborhood of $\mathcal{D}_0$.*

**Remark 1.4.** *Our method would allow us to consider large initial vorticities as in* (1.10). *However, we would end up with a local-in-time result only.*

As in [19, 20], Theorem 1.2 will come up as a consequence of a much more general result of persistence of geometric structures for $(INS)$. To give the exact statement, we need to introduce for $(\sigma,p,T)\in\mathbb{R}\times[1,\infty]\times]0,\infty]$, the space

$$\dot{E}^\sigma_p(T):=\big\{(v,\nabla Q): v\in\widetilde{\mathcal{C}}_b\big([0,T[;\dot{B}^{\frac{N}{p}-1+\sigma}_{p,1}\big),\ (\partial_t v,\nabla^2 v,\nabla Q)\in L^1_T\big(\dot{B}^{\frac{N}{p}-1+\sigma}_{p,1}\big)\big\},$$

endowed with the norm

$$\|(v,\nabla Q)\|_{\dot E^\sigma_p(T)} := \|v\|_{\widetilde{L}^\infty_T\left(\dot B^{\frac Np+\sigma-1}_{p,1}\right)} + \|(\partial_t v,\nabla^2 v,\nabla Q)\|_{L^1_T\left(\dot B^{\frac Np+\sigma-1}_{p,1}\right)}.$$

For notational simplicity, we shall omit $\sigma$ or $T$ in the notation $\dot E^\sigma_p(T)$ whenever $\sigma$ is zero or $T=\infty$. For instance, $\dot E_p := \dot E^0_p(\infty)$.

**Theorem 1.5.** *Let $\varepsilon$ be in $]0,1[$ and $p$ satisfy*

$$\frac N2 < p < \min\Big\{\frac{N}{1-\varepsilon}, 2N\Big\}. \tag{1.12}$$

*Let $u_0$ be a divergence-free vector field with coefficients in $\dot B^{\frac Np-1}_{p,1}$. Assume that the initial density $\rho_0$ is bounded and belongs to the multiplier space $\mathcal{M}\big(\dot B^{\frac Np-1}_{p,1}\big)\cap\mathcal{M}\big(\dot B^{\frac Np+\varepsilon-2}_{p,1}\big)$. There exists a constant $c$ depending only on $p$ and on $N$ such that if*

$$\|\rho_0-1\|_{\mathcal{M}\left(\dot B^{\frac Np-1}_{p,1}\right)\cap\mathcal{M}\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)\cap L^\infty} + \|u_0\|_{\dot B^{\frac Np-1}_{p,1}} \le c, \tag{1.13}$$

*then System $(INS)$ has a unique solution $(\rho,u,\nabla P)$ with*

$$\rho\in L^\infty\Big(\mathbb{R}_+;L^\infty\cap\mathcal{M}\big(\dot B^{\frac Np-1}_{p,1}\big)\cap\mathcal{M}\big(\dot B^{\frac Np+\varepsilon-2}_{p,1}\big)\Big)\quad\text{and}\quad(u,\nabla P)\in\dot E_p.$$

*Moreover, for any vector field $X_0$ with $\mathcal{C}^{0,\varepsilon}$ regularity (assuming in addition that $\varepsilon>2-\frac Np$ if $\operatorname{div}X_0\not\equiv0$), if the following striated-type conditions are fulfilled*

$$\partial_{X_0}\rho_0\in\mathcal{M}\big(\dot B^{\frac Np-1}_{p,1}\to\dot B^{\frac Np+\varepsilon-2}_{p,1}\big)\quad\text{and}\quad\partial_{X_0}u_0\in\dot B^{\frac Np+\varepsilon-2}_{p,1},$$

*then System (0.6) in $\mathbb{R}^N$ has a unique global solution $X\in C_w(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$, and we have*

$$\partial_X\rho\in L^\infty\Big(\mathbb{R}_+;\mathcal{M}\big(\dot B^{\frac Np-1}_{p,1}\to\dot B^{\frac Np+\varepsilon-2}_{p,1}\big)\Big)\quad\text{and}\quad(\partial_X u,\partial_X\nabla P)\in\dot E^{\varepsilon-1}_p.$$

Some comments are in order:

- The divergence-free property on $X_0$ is conserved during the evolution because if one takes the divergence of (0.6), and remember that $\operatorname{div}u=0$, then we get

$$\begin{cases}\partial_t\operatorname{div}X+u\cdot\nabla\operatorname{div}X=0,\\ \operatorname{div}X|_{t=0}=\operatorname{div}X_0.\end{cases} \tag{1.14}$$

- In the case $\operatorname{div}X_0\neq0$, the additional constraint on $(\varepsilon,p)$ is due to the fact that the product of a general $\mathcal{C}^{0,\varepsilon}$ function with a $\dot B^{\frac Np-2}_{p,1}$ distribution need not be defined if the sum of regularity coefficients, namely $\varepsilon+\frac Np-2$, is negative.
- The vector field $X$ given by (0.6) has the Finite Propagation Speed Property. Indeed, from the definitions of the flow and of the space $\dot E_p$, and from the embedding of $\dot B^{\frac Np}_{p,1}(\mathbb{R}^N)$ in $\mathcal{C}_b(\mathbb{R}^N)$, we readily get for all $t\ge0$ and $x\in\mathbb{R}^N$,

$$\big|\psi(t,x)-x\big|\lesssim\sqrt t\,\|u\|_{L^2_t(\dot B^{\frac Np}_{p,1})}\le C\sqrt t\,\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}.$$

  Therefore, if the initial vector field $X_0$ is supported in the set $K_0$ then $X(t)$ is supported in some set $K_t$ such that

$$\operatorname{diam}(K_t)\le\operatorname{diam}(K_0)+C\sqrt t\,\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}.$$

- One can prove a similar result (only local in time even in the 2D case) if we remove the smallness assumption on $u_0$. Moreover, we expect our method to be appropriate for handling Hölder regularity $\mathcal{C}^{k,\varepsilon}$ if making suitable assumptions on $\partial^j_{X_0}\rho_0$ and $\partial^j_{X_0}u_0$ for $j=0,\cdots,k$. We refrained from writing out here this generalization to keep the presentation as elementary as possible.

We end this section with a short presentation of the main ideas of the proof of Theorem 1.5. The starting point is Theorem 1.1 that provides us with a global solution $(\rho,u,\nabla P)$ with $\rho\in L^\infty\Big(\mathbb{R}_+;\mathcal{M}\big(\dot B^{\frac{N}{p}-1}_{p,1}\big)\Big)$ and $(u,\nabla P)\in\dot E_p$. The flow $\psi$ of $u$ is thus in $\mathcal{C}^1$. Our main task is to prove that $X(t,\cdot)$ remains in $\mathcal{C}^{0,\varepsilon}$ for all time. Now, (0.6) ensures that

$$X(t,x)=X_0\big(\psi_t^{-1}(x)\big)+\int_0^t\partial_X u\Big(t',\psi_{t'}\big(\psi_t^{-1}(x)\big)\Big)dt'.$$

Because $\psi_t$ is a $\mathcal{C}^1$ diffeomorphism of $\mathbb{R}^N$, it thus suffices to show that $\partial_X u$ is in $L^1_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$.

Note that Equation (0.6) exactly states that $[D_t,\partial_X]=0$, where $D_t:=\partial_t+u\cdot\nabla$ stands for the material derivative associated to $u$. Therefore differentiating the mass and momentum equations of $(INS)$ along $X$, we discover that

$$D_t\partial_X\rho=0 \tag{1.15}$$

and that

$$\rho D_t\partial_X u+\partial_X\rho D_t u-\partial_X\Delta u+\partial_X\nabla P=0. \tag{1.16}$$

On the one hand, Equation (1.15) implies that any (reasonable) regularity assumption for $\rho$ along $X$ is conserved through the evolution. On the other hand, even though (1.16) has some similarities with the Stokes system, it is not clear that it does have the same smoothing properties, as its coefficients have very low regularity. One of the difficulties lies in the product of the discontinuous function $\rho$ with $D_t\partial_X u$, as having only $\partial_X u$ in $\mathcal{C}^{0,\varepsilon}$ suggests that $D_t\partial_X u$ has *negative* regularity.

Our strategy is to assume that $\rho$ belongs to some multiplier space corresponding to the space to which $D_t\partial_X u$ is expected to belong. As our flow is $\mathcal{C}^1$, propagating multiplier informations turns out to be rather straightforward (see Lemma A.3). Thanks to this new viewpoint, one can avoid using the tricky energy estimates and iterated differentiation along vector fields (requiring higher regularity of the patch) that were the cornerstone of the work by Liao and Zhang. In fact, *under the smallness assumption* (1.13) which, unfortunately, forces the fluid to have small density variations, we succeed in closing the estimates via only one differentiation along $X$. This makes the proof rather elementary and allows us to propagate low Hölder regularity.

However, even with the above viewpoint, whether one can differentiate terms like $\Delta u$ or $\nabla P$ along $X$ within our critical regularity framework is not totally clear. In fact, as in our recent work [14] dedicated to the incompressible Boussinesq system, we shall resort to elementary paradifferential calculus (first introduced by Bony in [5]).

Let us briefly recall how it works. Fix some suitably large integer $N_0$ and introduce the following *paraproduct* and *remainder* operators:

$$\dot T_u v:=\sum_{j\in\mathbb{Z}}\dot S_{j-N_0}u\dot\Delta_j v\quad\text{and}\quad\dot R(u,v)\equiv\sum_{j\in\mathbb{Z}}\dot\Delta_j u\widetilde{\dot\Delta}_j v:=\sum_{\substack{j\in\mathbb{Z}\\|j-k|\leq N_0}}\dot\Delta_j u\dot\Delta_k v.$$

It is clear that, formally, any product may be decomposed as follows:

$$uv=\dot T_u v+\dot T_v u+\dot R(u,v). \tag{1.17}$$

To overcome the problem with the definition (and estimates) of $\partial_X\Delta u$ and $\partial_X\nabla P$, the idea is to change the vector field $X$ to the para-vector field operator $\dot{\mathcal{T}}_X\cdot:=\dot T_{X^k}\partial_k\cdot$. This is justified

because in our regularity framework $\dot{\mathcal{T}}_X$ turns out to be *the principal part* of operator $\partial_X$. Typically, $X$ will act on $\nabla P$ or on $\Delta u$ which are in $L^1\bigl(\mathbb{R}_+;\dot{B}^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N)\bigr)$. Now, suppose that $(X,f)\in\bigl(\dot{\mathscr{C}}^\varepsilon(\mathbb{R}^N)\bigr)^N\times\dot{B}^{\frac{N}{p}-1}_{p,1}(\mathbb{R}^N)$ with $(\varepsilon,p)\in]0,1[\times[1,+\infty]$ such that

$$\frac{N}{p}\in\,]\,1-\varepsilon,\,2\,[\ \text{ if }\ \operatorname{div}X=0,\quad\text{and}\quad\frac{N}{p}\in\,]\,2-\varepsilon,\,2\,[\ \text{ otherwise.}\tag{1.18}$$

Then, by virtue of Bony's decomposition (1.17), we have

$$(\dot{\mathcal{T}}_X-\partial_X)f=\dot{T}_{\partial_k f}X^k+\partial_k\dot{R}(f,X^k)-\dot{R}(f,\operatorname{div}X).$$

Taking advantage of classical continuity results for operators $\dot{T}$ and $\dot{R}$ (see [4]), we discover that, under Condition (1.18), we have

$$\|(\dot{\mathcal{T}}_X-\partial_X)f\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}}\lesssim\|f\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}.\tag{1.19}$$

Now, incising the term $\partial_X u$ by the scalpel $\dot{\mathcal{T}}_X$ in (1.16) and applying $\dot{\mathcal{T}}_X$ to the third equation of $(INS)$ yield

$$\begin{cases}\rho D_t\dot{\mathcal{T}}_Xu-\Delta\dot{\mathcal{T}}_Xu+\nabla\dot{\mathcal{T}}_XP=g,\\ \operatorname{div}\dot{\mathcal{T}}_Xu=\operatorname{div}\,(\dot{T}_{\partial_k X}u^k-\dot{T}_{\operatorname{div}X}u),\\ \dot{\mathcal{T}}_Xu|_{t=0}=\dot{\mathcal{T}}_{X_0}u_0\end{cases}\tag{1.20}$$

with

$$\begin{aligned}g:=-\rho[\dot{\mathcal{T}}_X,D_t]u+[\dot{\mathcal{T}}_X,\Delta]u-[\dot{\mathcal{T}}_X,\nabla]P+(\partial_X-\dot{\mathcal{T}}_X)(\Delta u-\nabla P)\\ -\partial_X\rho D_tu+\rho(\dot{\mathcal{T}}_X-\partial_X)D_tu.\end{aligned}\tag{1.21}$$

This surgery leading to (1.20) is quite effective for three reasons. Firstly, all the commutator terms in (1.21) are under control (see the Appendix). More importantly, as $D_t\partial_X u$ and $D_t\dot{\mathcal{T}}_Xu$ are in the same Besov space, we can still use the multiplier type regularity on the density that we pointed out before. Lastly, Condition (1.13) ensures that $(\dot{\mathcal{T}}_X-\partial_X)u$ is indeed a (small) remainder term.

Of course, the divergence free condition need not be satisfied by $\dot{\mathcal{T}}_Xu$. We shall thus further modify the above Stokes-like equation so as to enter in the standard maximal regularity theory. Then, under the smallness condition (1.13), one can close the estimates involving striated regularity along $X$, globally in time.

The rest of the paper unfolds as follows. In the next section, we show that Theorem 1.5 entails a general (but not so explicit) result of persistence of Hölder regularity for patches of density in any dimension, under suitable striated regularity assumptions for the velocity. We shall then obtain Theorem 1.2, and an analogous result in dimension $N=3$. Section 3 is devoted to the proof of all-time persistence of striated regularity (that is Theorem 1.5). Some technical results pertaining to commutators and multiplier spaces are postponed in appendix.

## 2. The density patch problem

This section is devoted to the proof of results of persistence of regularity for patches of constant densities, taking Theorem 1.5 for granted. Throughout this section we shall use repeatedly the fact (proved in e.g. see [11, Lemma A.7]) that for any (not necessarily bounded) domain $\mathcal{D}$ of $\mathbb{R}^N$ with uniform $\mathcal{C}^1$ boundary, we have

$$\mathbb{1}_{\mathcal{D}}\in\mathcal{M}\bigl(\dot{B}^s_{p,r}(\mathbb{R}^N)\bigr)\quad\text{whenever}\quad(s,p,r)\in]\frac{1}{p}-1,\frac{1}{p}[\times]1,\infty[\times[1,\infty].$$

From that property, we deduce that if $(\varepsilon,p)\in]0,1[\times]N-1,\frac{N-1}{1-\varepsilon}[$, then the density $\rho_0$ given by (1.9) belongs to $\mathcal{M}\big(\dot{B}^{\frac{N}{p}-1}_{p,r}(\mathbb{R}^N)\big)\cap\mathcal{M}\big(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,r}(\mathbb{R}^N)\big)$.

As a start, let us give a result of persistence of regularity, under rather general hypotheses.

**Proposition 2.1.** *Assume that $\rho_0$ is given by (1.9) with small enough $\eta$ and some domain $\mathcal{D}_0$ satisfying (1.8). Let $u_0$ be a small enough divergence free vector field with coefficients in $\dot{B}^{\frac{N}{p}-1}_{p,1}$ for some $N-1<p<\min\big\{\frac{N-1}{1-\varepsilon},2N\big\}$. Consider a family $(X_{\lambda,0})_{\lambda\in\Lambda}$ of $\mathcal{C}^{0,\varepsilon}$ divergence free vector fields tangent to $\mathcal{D}_0$ and such that $\partial_{X_{\lambda,0}}u_0\in\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}$ for all $\lambda\in\Lambda$.*

*Then the unique solution $(\rho,u,\nabla P)$ of $(INS)$ given by Theorem 1.1 satisfies the following additional properties:*

- *$\rho(t,\cdot)$ is given by (1.11),*
- *all the time-dependent vector fields $X_\lambda$ solutions to (0.6) with initial data $X_{\lambda,0}$ are in $L^\infty_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$ and remain tangent to the patch for all time.*

*Proof.* As pointed out at the beginning of this section, our assumptions on $p$ ensure that $\rho_0$ is in $\mathcal{M}(\dot{B}^{\frac{N}{p}-1}_{p,1})\cap\big(\dot{B}^{\frac{N}{p}-2+\varepsilon}_{p,1}\big)$, and (1.13) is fulfilled if $\eta$ and $u_0$ are small enough. Of course, $\partial_{X_{\lambda,0}}\rho_0\equiv0$ for all $\lambda\in\Lambda$ because the vector fields $X_{\lambda,0}$ are tangent to the boundary. Therefore, applying Theorem 1.5 ensures that all the vector fields $X_\lambda$ are in $L^\infty_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$. Now, if we consider a level set function $f_0$ in $\mathcal{C}^{1,\varepsilon}$ associated to $\mathcal{D}_0$ (see (1.8)) then the function $f_t:=f_0\circ\psi_t$ is associated to the transported domain $\mathcal{D}_t=\psi_t(\mathcal{D}_0)$, and easy computations show that

$$D_t\nabla f=-\nabla u\cdot\nabla f\quad\text{with}\quad(\nabla u)_{ij}=\partial_i u^j. \tag{2.1}$$

Therefore, as $X_\lambda$ satisfies (0.6), we have

$$D_t(X_\lambda\cdot\nabla f)=(D_tX_\lambda)\cdot\nabla f+X_\lambda\cdot(D_t\nabla f)=0,$$

which ensures that $X_\lambda$ remains tangent to the patch for all time. □

2.1. **The two-dimensional case.** Here we prove Theorem 1.2. So we assume that $\omega_0=\widetilde{\omega}_0\mathbb{1}_{\mathcal{D}_0}$ for some small enough function $\widetilde{\omega}_0$ that can be taken compactly supported and in the nonhomogeneous Besov space $B^\alpha_{\infty,1}(\mathbb{R}^2)$ for some $\alpha\in]0,\varepsilon[$, with no loss of generality. As we assumed that $u_0$ has some decay at infinity, it may be computed from $\omega_0$ through the following Biot-Savart law:

$$u_0=(-\Delta)^{-1}\nabla^\perp\omega_0.$$

We claim that $u_0$ belongs to all spaces $\dot{B}^{\frac{2}{p}-1}_{p,1}(\mathbb{R}^2)$ with $p>1$. Indeed, let us write that

$$u_0=\dot{S}_0u_0+(\mathrm{Id}-\dot{S}_0)u_0.$$

Because $\omega_0$ is bounded, compactly supported and mean free, it is obvious that $\dot{S}_0\omega_0$ is smooth, in all Lebesgue spaces and also mean free. Biot-Savart law thus ensures that $\dot{S}_0u_0$ belongs to all Lebesgue spaces $L^q$ with $q>1$ (as it is smooth and behaves like $\mathcal{O}(|x|^{-2})$ at infinity, due to the mean free property, see e.g. [22, p. 92]). Hence for any $1<q<2$ and $p\geq q$, one may write

$$\|\dot{S}_0u_0\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\lesssim\|\dot{S}_0u_0\|_{\dot{B}^{\frac{2}{p}-\frac{2}{q}}_{p,\infty}}\lesssim\|\dot{S}_0u_0\|_{L^q}\leq C_{\omega_0}.$$

As regards the high frequency part of $u_0$, because the Fourier multiplier $(\mathrm{Id}-\dot{S}_0)\nabla^\perp(-\Delta)^{-1}$ is homogeneous of degree $-1$ away from a neighborhood of 0, we have

$$\begin{aligned}\|(\mathrm{Id}-\dot{S}_0)u_0\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}&=\|(\mathrm{Id}-\dot{S}_0)\nabla^\perp(-\Delta)^{-1}\omega_0\|_{\dot{B}^{\frac{2}{p}-1}_{p,1}}\\&\lesssim\|(\mathrm{Id}-\dot{S}_0)\omega_0\|_{\dot{B}^{\frac{2}{p}-2}_{p,1}}\lesssim\|(\mathrm{Id}-\dot{S}_0)\omega_0\|_{L^p}\lesssim\|\omega_0\|_{L^1\cap L^\infty}.\end{aligned}$$

Next, consider the divergence free vector field $X_0 = \nabla^\perp f_0$ where $f_0$ is given by (1.8) and is (with no loss of generality) compactly supported. If it is true that

$$\partial_{X_0} u_0 \in \dot B^{\frac{2}{p}-2+\varepsilon}_{p,1} \quad \text{for some} \quad 1 < p < \min\Big(\frac{1}{1-\varepsilon}, 4\Big), \tag{2.2}$$

then one can apply Proposition 2.1 which ensures that the transported vector field $X_t$ remains in $\mathcal{C}^{0,\varepsilon}$ for all $t \geq 0$. Now, it is classical that we have $X_t = (\nabla f_t)^\perp$ with $f_t = f_0 \circ \psi_t$. Hence $\mathcal{D}_t$ has a $\mathcal{C}^{1,\varepsilon}$ boundary.

Let us establish (2.2). Of course, by embedding, we have $X_0$ in $B^\alpha_{\infty,1}$. Now, (1.19) ensures that for any $p \geq 1$ satisfying $\frac{2}{p} + \varepsilon - 1 > 0$,

$$\|\dot{\mathcal{T}}_{X_0} u_0 - \partial_{X_0} u_0\|_{\dot B^{\frac{2}{p}+\varepsilon-2}_{p,1}} \lesssim \|u_0\|_{\dot B^{\frac{2}{p}-1}_{p,1}} \|X_0\|_{\mathcal{C}^\varepsilon}. \tag{2.3}$$

From Biot-Savart law, we get

$$\dot{\mathcal{T}}_{X_0} u_0 = \dot{\mathcal{T}}_{X_0}(-\Delta)^{-1}\nabla^\perp \omega_0 = (-\Delta)^{-1}\nabla^\perp \dot{\mathcal{T}}_{X_0}\omega_0 + [\dot{\mathcal{T}}_{X_0}, (-\Delta)^{-1}\nabla^\perp]\omega_0,$$

whence using Lemma B.1,

$$\|\dot{\mathcal{T}}_{X_0} u_0 - (-\Delta)^{-1}\nabla^\perp \dot{\mathcal{T}}_{X_0}\omega_0\|_{\dot B^\alpha_{p,1}} \lesssim \|X_0\|_{\dot B^\alpha_{\infty,1}} \|\omega_0\|_{L^p}. \tag{2.4}$$

Next, we notice that

$$\dot{\mathcal{T}}_{X_0}\omega_0 - \operatorname{div}(X_0\omega_0) = -\operatorname{div}\big(\dot T_{\omega_0} X_0 + \dot R(\omega_0, X_0)\big).$$

Therefore, taking advantage of standard continuity results for $\dot T$ and $\dot R$, we have

$$\|\dot{\mathcal{T}}_{X_0}\omega_0 - \operatorname{div}(X_0\omega_0)\|_{\dot B^{\alpha-1}_{p,1}} \lesssim \|\omega_0\|_{L^p} \|X_0\|_{\dot B^\alpha_{\infty,1}} \quad \text{for all} \quad p \geq 1. \tag{2.5}$$

Finally, because $X_0$ and $\widetilde\omega_0$ are compactly supported and in $B^\alpha_{\infty,1}$, Proposition A.2 and obvious embedding ensure that

$$X_0 \quad \text{and} \quad \widetilde\omega_0 \quad \text{are in} \quad \dot B^\alpha_{p,1} \cap L^\infty.$$

Hence, remembering that $\operatorname{div}(X_0\omega_0) = \operatorname{div}(X_0\,\widetilde\omega_0\,\mathbb{1}_{\mathcal{D}_0})$, that $\operatorname{div} X_0 = 0$ and that $\partial_{X_0}\mathbb{1}_{\mathcal{D}_0} = 0$, Corollary B.5 implies that $\operatorname{div}(X_0\omega_0)$ belongs to $\dot B^{\alpha-1}_{p,1}$.

Putting (2.3), (2.4) and (2.5) together, we conclude that (2.2) is fulfilled provided the Lebesgue index $p$ defined by

$$\alpha = \frac{2}{p} - 2 + \varepsilon \tag{2.6}$$

is in $]1, \min(4, \frac{1}{1-\varepsilon})[$. As $0 < \alpha < \varepsilon$, this is indeed the case. This completes the proof of Theorem 1.2. □

2.2. **The three-dimensional case.** As a second application of Proposition 2.1, we now want to generalize Theorem 1.2 to the three-dimensional case. Our result reads as follows.

**Theorem 2.2.** *Let $\mathcal{D}_0$ be a $\mathcal{C}^{1,\varepsilon}$ simply connected bounded domain of $\mathbb{R}^3$ with $\varepsilon \in ]0,1[$ and $\rho_0$ be given by* (1.9) *for some small enough $\eta$. Assume that the initial velocity $u_0$ has coefficients in $\mathcal{S}'_h(\mathbb{R}^3)$ and vorticity*[2]

$$\Omega_0 := \nabla \wedge u_0 = \widetilde\Omega_0 \mathbb{1}_{\mathcal{D}_0},$$

*for some small enough $\widetilde\Omega_0$ in $\mathcal{C}^{0,\delta}(\mathbb{R}^3;\mathbb{R}^3)$ ($\delta \in ]0,\varepsilon[$) with $\operatorname{div}\widetilde\Omega_0 = 0$ and $\widetilde\Omega_0 \cdot \vec n_{\mathcal{D}_0}|_{\partial\mathcal{D}_0} \equiv 0$ (here $\vec n_{\mathcal{D}_0}$ denotes the outwards unit normal of the domain $\mathcal{D}_0$).*

[2]For any point $Y \in \mathbb{R}^3$, we set $X \wedge Y := (X^2Y^3 - X^3Y^2,\, X^3Y^1 - X^1Y^3,\, X^1Y^2 - X^2Y^1)$ where $X$ stands for an element of $\mathbb{R}^3$ or for the $\nabla$ operator.

*There exists a unique solution $(\rho, u, \nabla P)$ to System $(INS)$ with the properties listed in Theorem 1.1 for some suitable $p$ satisfying*

$$2 < p < \min\biggl(\frac{2}{1-\varepsilon}, 6\biggr). \tag{2.7}$$

*Furthermore, for all $t \geq 0$, we have (1.11) and $\mathcal{D}_t$ remains a simply connected bounded domain of class $\mathcal{C}^{1,\varepsilon}$.*

*Proof.* Without loss of generality, one may assume that $\widetilde{\Omega}_0$ is compactly supported (as multiplying it by a cut-off function with value 1 on $\mathcal{D}_0$ will not change $\Omega_0$). Like in the 2D case, we first have check that $u_0$ satisfies the assumptions of Proposition 2.1. As it is divergence free and decays at infinity (recall that $u_0 \in \mathcal{S}'_h$), it is given by the Biot-Savart law:

$$u_0 = (-\Delta)^{-1}\nabla \wedge \Omega_0, \quad \text{with } \Omega_0 = \widetilde{\Omega}_0\, \mathbb{1}_{\mathcal{D}_0}. \tag{2.8}$$

Let us first check that $u_0$ belongs to $\dot{B}^{\frac{3}{p}-1}_{p,1}$ for some $p$ satisfying Condition (2.7). Recall that the characteristic function of any bounded domain with $\mathcal{C}^1$ regularity belongs to all Besov spaces $B^{\frac{1}{q}}_{q,\infty}$ with $1 \leq q \leq \infty$ (see e.g. [24]). Hence combining Proposition A.1 and the embedding (A.1) gives

$$\mathbb{1}_{\mathcal{D}_0} \in\ \mathcal{E}' \cap b^{\frac{1}{q}}_{q,\infty} \hookrightarrow \dot{B}^{\frac{3}{q}-2}_{q,1}, \quad \text{for any } q \in ]1,\infty[ \text{ and } b \in \{B, \dot{B}\}. \tag{2.9}$$

Now, using Bony's decomposition and standard continuity results for operators $\dot{R}$ and $\dot{T}$, we discover that

$$\widetilde{\Omega}_0 \in \mathscr{C}^{\delta}_c \hookrightarrow \mathcal{M}\bigl(\dot{B}^{\frac{3}{q}-2}_{q,1}\bigr) \quad \text{for any } q \in \Bigl]\frac{3}{2}, \frac{3}{2-\delta}\Bigr[.$$

Hence the definition of Multiplier space and (2.9) yield

$$\Omega_0 = \widetilde{\Omega}_0\, \mathbb{1}_{\mathcal{D}_0} \in \dot{B}^{\frac{3}{q}-2}_{q,1} \quad \text{for any } q \in \Bigl]\frac{3}{2}, \frac{3}{2-\delta}\Bigr[. \tag{2.10}$$

As $u_0$ is in $\mathcal{S}'_h$ and $(-\Delta^{-1})^{-1}\nabla\wedge$ in (2.8) is a homogeneous multiplier of degree $-1$, one can conclude that

$$u_0 \in \dot{B}^{\frac{3}{q}-1}_{q,1} \hookrightarrow \dot{B}^{\frac{3}{p}-1}_{p,1}, \quad \text{for any } p \geq q.$$

Note that for any value of $\delta$ in $]0,1[$, one can find some $p$ satisfying (2.7).

Next, we consider some (compactly supported) level set function $f_0$ associated to $\partial\mathcal{D}_0$, and the three $\mathcal{C}^{0,\varepsilon}$ vector-fields $X_{k,0} := e_k \wedge \nabla f_0$ with $(e_1, e_2, e_3)$ being the canonical basis of $\mathbb{R}^3$. It is clear that those vector-fields are divergence free and tangent to $\partial\mathcal{D}_0$. Let us check that we have $\partial_{X_{k,0}} u_0 \in \dot{B}^{\frac{3}{p}-2+\varepsilon}_{p,1}$ for some $p$ satisfying (2.7). As in the two-dimensional case, this will follow from Biot-Savart law and the special structure of $\Omega_0$. Indeed, from (1.19) and $\operatorname{div} X_{k,0} = 0$, we have

$$\|\dot{\mathcal{T}}_{X_{k,0}} u_0 - \partial_{X_{k,0}} u_0\|_{\dot{B}^{\frac{3}{p}+\varepsilon-2}_{p,1}} \lesssim \|u_0\|_{\dot{B}^{\frac{3}{p}-1}_{p,1}} \|X_0\|_{\dot{\mathscr{C}}^{\varepsilon}}, \ \ \forall\, p \in \Bigl]\frac{3}{2}, \frac{3}{1-\varepsilon}\Bigr[.$$

Then (2.8) yields

$$\dot{\mathcal{T}}_{X_{k,0}} u_0 = \dot{\mathcal{T}}_{X_{k,0}} (-\Delta)^{-1}\nabla \wedge \Omega_0 = (-\Delta)^{-1}\nabla \wedge \dot{\mathcal{T}}_{X_{k,0}} \Omega_0 + [\dot{\mathcal{T}}_{X_{k,0}}, (-\Delta)^{-1}\nabla\wedge]\, \Omega_0.$$

Thanks to Lemma B.1 and homogeneity of $(-\Delta^{-1})^{-1}\nabla\wedge$, it is thus sufficient to verify that $\dot{\mathcal{T}}_{X_{k,0}}\Omega_0$ belongs to $\dot{B}^{\frac{3}{p}+\varepsilon-3}_{p,1}$ for some $p$ satisfying (2.7). In fact, from the decomposition

$$\dot{\mathcal{T}}_{X_{k,0}}\Omega_0 - \operatorname{div}(X_{k,0}\Omega_0) = -\operatorname{div}\bigl(\dot{T}_{\Omega_0} X_{k,0} + \dot{R}(\Omega_0, X_{k,0})\bigr),$$

and continuity results for $\dot R$ and $\dot T$, we get

$$\|\dot{\mathcal{T}}_{X_{k,0}}\Omega_0 - \operatorname{div}(X_{k,0}\Omega_0)\|_{\dot B^{\frac{3}{q}+\varepsilon-3}_{q,1}} \lesssim \|\Omega_0\|_{\dot B^{\frac{3}{q}-2}_{q,1}}\|X_{k,0}\|_{\dot{\mathscr{C}}^\varepsilon}, \text{ for any } q\in\Big]\frac{3}{2},\frac{3}{2-\varepsilon}\Big[.$$

Thus, remembering (2.10) and $0<\delta<\varepsilon$, we have to choose some $p$ satisfying (2.7), such that the following standard embedding holds

$$\dot B^{\frac{3}{q}+\varepsilon-3}_{q,1} \hookrightarrow \dot B^{\frac{3}{p}+\varepsilon-3}_{p,1} \text{ for some } q\in\Big]\frac{3}{2},\frac{3}{2-\delta}\Big[ \text{ with } q\le p. \tag{2.11}$$

Now, because $\partial_{X_{k,0}}\mathbb{1}_{\mathcal{D}_0}\equiv 0$ and $\widetilde\Omega_0$ is in $B^{\delta_\star}_{\infty,1}$ for all $0<\delta_\star<\delta$, Corollary B.5 yields,

$$\partial_{X_{k,0}}\Omega_0 = \operatorname{div}(X_{k,0}\otimes\Omega_0) = \operatorname{div}(X_{k,0}\otimes\widetilde\Omega_0\,\mathbb{1}_{\mathcal{D}_0}) \in \dot B^{\delta_\star-1}_{q,1} \quad\text{for all}\quad q\ge 1.$$

One can thus conclude that $\partial_{X_{k,0}}u_0 \in \dot B^{\frac{3}{p}-2+\varepsilon}_{p,1}$ for any index $p$ satisfying $p\ge q$ with $q$ satisfying Condition (2.11) and $\frac{3}{q}+\varepsilon-2=\delta^*\in]0,\delta[$.

As one can require in addition $p$ to fulfill (2.7), Proposition 2.1 applies with the family $(X_{k,0})_{1\le k\le 3}$. Denoting by $(X_k)_{1\le k\le 3}$ the corresponding family of divergence free vector fields in $\mathcal{C}^{0,\varepsilon}$ given by (0.6) with initial data $X_{0,k}$, and introducing $Y_1:=X_3\wedge X_1$, $Y_2:=X_3\wedge X_1$ and $Y_3=X_1\wedge X_2$, we discover that for $\alpha=1,2,3$,

$$\begin{cases} \partial_t Y_\alpha + u\cdot\nabla Y_\alpha = -\nabla u\cdot Y_\alpha,\\ (Y_\alpha)|_{t=0} = \partial_\alpha f_0\,\nabla f_0. \end{cases} \tag{2.12}$$

From (2.1), it is clear that the time-dependent vector field $\big(\partial_\alpha f_0(\psi_t^{-1})\big)\,\nabla f_t$ also satisfies (2.12), hence we have, by uniqueness, $Y_\alpha(t,\cdot)=\big((\partial_\alpha f_0)(\psi_t^{-1})\big)\nabla f_t$. So finally,

$$\big|\nabla f_0\circ\psi_t^{-1}\big|^2\,\nabla f_t = \sum_{\alpha=1}^3 Y_\alpha(t,\cdot)\,\partial_\alpha f_0\circ\psi_t^{-1}.$$

As $\psi_t^{-1}$ is $\mathcal{C}^1$ and as both $Y_\alpha$ and $\nabla f_0$ are in $\mathcal{C}^{0,\varepsilon}$, one can conclude that $\nabla f_t$ is $\mathcal{C}^{0,\varepsilon}$ in some neighborhood of $\partial\mathcal{D}_0$. Therefore $\mathcal{D}_t$ remains of class $\mathcal{C}^{1,\varepsilon}$ for all time. □

**Remark 2.3.** *In the 3-D case, the mean free assumption on initial vorticity is not required, but one cannot consider constant vortex patterns as in the 2-D case. Let us also emphasize that, as for the Boussinesq system studied in* [14]*, a similar statement may be proved in higher dimension.*

## 3. The proof of persistence of striated regularity

That section is devoted to the proof of Theorem 1.5. The first step is to apply Theorem 1.1. From it, we get a unique global solution $(\rho,u,\nabla P)$ with $\rho\in\mathcal{C}_b\big(\mathbb{R}_+;\mathcal{M}(\dot B^{\frac{N}{p}-1}_{p,1})\big)$ and $(u,\nabla P)\in\dot E_p$, satisfying (1.7). Because the product of functions maps $\dot B^{\frac{N}{p}-1}_{p,1}\times\dot B^{\frac{N}{p}}_{p,1}$ to $\dot B^{\frac{N}{p}-1}_{p,1}$, we deduce that the material derivative $D_tu=\partial_tu+u\cdot\nabla u$ is also bounded by the right-hand side of (1.7). So finally,

$$\|(u,\nabla P)\|_{\dot E_p} + \|D_tu\|_{L^1_t\left(\dot B^{\frac{N}{p}-1}_{p,1}\right)} \le C\|u_0\|_{\dot B^{\frac{N}{p}-1}_{p,1}}. \tag{3.1}$$

In order to complete the proof of the theorem, it is only a matter of showing that the additional multiplier and striated regularity properties are conserved for all positive times. In fact, we shall mainly concentrate on the proof of a priori estimates for the corresponding norms, just explaining at the end of this section how a suitable regularization process allows to make it rigorous.

3.1. **Bounds involving multiplier norms.** As already pointed out in the introduction, because $\nabla u$ is in $L^1(\mathbb{R}_+;\dot{B}^{\frac{N}{p}}_{p,1})$ and $\dot{B}^{\frac{N}{p}}_{p,1}$ is embedded in $\mathcal{C}_b$, the flow $\psi$ of $u$ is $\mathcal{C}^1$ and we have for all $t\geq 0$, owing to (1.7),

$$\|\nabla\psi_t^{\pm1}\|_{L^\infty}\leq\exp\biggl(\int_0^t\|\nabla u\|_{L^\infty}\,d\tau\biggr)\leq C \tag{3.2}$$

for a suitably large universal constant $C$.

Now, from the mass conservation equation and (1.15), we gather that

$$\rho(t,\cdot)=\rho_0\circ\psi_t^{-1}\quad\text{and}\quad(\partial_X\rho)(t,\cdot)=(\partial_{X_0}\rho_0)\circ\psi_t^{-1}.$$

Hence $\|\rho(t,\cdot)\|_{L^\infty}$ is time independent. Furthermore, Lemma A.3 and Condition (1.12) guarantee that for all $t\in\mathbb{R}_+$,

$$\|\rho(t)-1\|_{\mathcal{M}\bigl(\dot{B}^{\frac{N}{p}-1}_{p,1}\bigr)}\leq C\|\rho_0-1\|_{\mathcal{M}\bigl(\dot{B}^{\frac{N}{p}-1}_{p,1}\bigr)}, \tag{3.3}$$

$$\|\rho(t)-1\|_{\mathcal{M}\bigl(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\bigr)}\leq C\|\rho_0-1\|_{\mathcal{M}\bigl(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\bigr)}, \tag{3.4}$$

$$\|(\partial_X\rho)(t)\|_{\mathcal{M}\bigl(\dot{B}^{\frac{N}{p}-1}_{p,1}\to\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\bigr)}\leq C\|\partial_{X_0}\rho_0\|_{\mathcal{M}\bigl(\dot{B}^{\frac{N}{p}-1}_{p,1}\to\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\bigr)}. \tag{3.5}$$

3.2. **Estimates for the striated regularity.** Recall that $\dot{\mathcal{T}}_X u$ satisfies the Stokes-like system (1.20). As $\dot{\mathcal{T}}_X u$ need not be divergence free, to enter into the standard theory, we set

$$v:=\dot{\mathcal{T}}_X u-w\quad\text{with}\quad w:=\dot{T}_{\partial_k X}u^k-\dot{T}_{\operatorname{div}X}u.$$

Denoting $\widetilde{g}:=g-\rho u\cdot\nabla\dot{\mathcal{T}}_X u-(\rho\partial_t w-\Delta w)$ with $g$ defined in (1.21), we see that $v$ satisfies:

$$(S)\qquad\begin{cases}\rho\partial_t v-\Delta v+\nabla\dot{\mathcal{T}}_X P=\widetilde{g},\\ \operatorname{div}v=0,\\ v|_{t=0}=v_0.\end{cases}$$

We shall decompose the proof of a priori estimates for striated regularity into three steps. The first one is dedicated to bounding $\widetilde{g}$ (which mainly requires the commutator estimates of the appendix). In the second step, we take advantage of the smoothing effect of the heat flow so as to estimate $v$. In the third step, we revert to $\dot{\mathcal{T}}_X u$ and eventually bound $X$.

*First step: bounds of $\widetilde{g}$.* Recall that $\widetilde{g}:=g-\rho u\cdot\nabla\dot{\mathcal{T}}_X u-(\rho\partial_t w-\Delta w)$ with

$$g=-\rho[\dot{\mathcal{T}}_X,D_t]u+[\dot{\mathcal{T}}_X,\Delta]u-[\dot{\mathcal{T}}_X,\nabla]P+(\partial_X-\dot{\mathcal{T}}_X)(\Delta u-\nabla P)-\partial_X\rho D_t u+\rho(\dot{\mathcal{T}}_X-\partial_X)D_t u.$$

The first term of $g$ may be bounded according to Proposition B.3 and to the definition of multiplier spaces. We get, under assumption (1.18),

$$\begin{aligned}\|\rho[\dot{\mathcal{T}}_X,D_t]u\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}}\lesssim\|\rho\|_{\mathcal{M}\bigl(\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}\bigr)}\Bigl(&\|u\|_{\dot{\mathscr{C}}^{-1}}\|\dot{\mathcal{T}}_X u\|_{\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}}\\&+\|u\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}\|\dot{\mathcal{T}}_X u\|_{\dot{\mathscr{C}}^{\varepsilon-2}}+\|u\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}\|u\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\|X\|_{\dot{\mathscr{C}}^{\varepsilon}}\Bigr).\end{aligned} \tag{3.6}$$

Next, thanks to the commutator estimates in Lemma B.1, we have

$$\|[\dot{\mathcal{T}}_X,\Delta]u\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}}\lesssim\|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}}\|\nabla u\|_{\dot{B}^{\frac{N}{p}}_{p,1}}, \tag{3.7}$$

$$\|[\dot{\mathcal{T}}_X,\nabla]P\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}}\lesssim\|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}}\|\nabla P\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}. \tag{3.8}$$

Bounding the fourth term of $g$ stems from (1.19): we have

$$\|(\dot{\mathcal{T}}_X-\partial_X)(\Delta u-\nabla P)\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}}\lesssim\|(\Delta u,\nabla P)\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}}\|X\|_{\dot{\mathscr{C}}^{\varepsilon}}. \tag{3.9}$$

Then the definition of multiplier spaces yields

$$\|\partial_X\rho D_t u\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}\lesssim\|\partial_X\rho\|_{\mathcal M\left(\dot B^{\frac Np-1}_{p,1}\to\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\|D_tu\|_{\dot B^{\frac Np-1}_{p,1}}. \tag{3.10}$$

Finally, using again (1.19) and the definition of multiplier spaces, we may write

$$\|\rho(\dot{\mathcal T}_X-\partial_X)D_tu\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}\lesssim\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\|X\|_{\dot{\mathscr C}^\varepsilon}\|D_tu\|_{\dot B^{\frac Np-1}_{p,1}}. \tag{3.11}$$

Putting together (3.6) – (3.11) and integrating with respect to time, we end up with

$$\begin{aligned}\|g\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\int_0^t\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\big(\|u\|_{\dot{\mathscr C}^{-1}}\|\dot{\mathcal T}_Xu\|_{\dot B^{\frac Np+\varepsilon}_{p,1}}+\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|\dot{\mathcal T}_Xu\|_{\dot{\mathscr C}^{\varepsilon-2}}\big)\,dt'\\ +\int_0^t\|X\|_{\dot{\mathscr C}^\varepsilon}\Big(\big(\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|u\|_{\dot B^{\frac Np-1}_{p,1}}+\|D_tu\|_{\dot B^{\frac Np-1}_{p,1}}\big)\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}+\|(\nabla^2u,\nabla P)\|_{\dot B^{\frac Np-1}_{p,1}}\Big)dt'\\ +\int_0^t\|\partial_X\rho\|_{\mathcal M\left(\dot B^{\frac Np-1}_{p,1}\to\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\|D_tu\|_{\dot B^{\frac Np-1}_{p,1}}\,dt'.\end{aligned} \tag{3.12}$$

Bounding the second term of $\widetilde g$ is obvious : taking advantage of Bony's decomposition (1.17) and remembering that $\frac Np+\varepsilon>1$ and that $\operatorname{div}u=0$, we get

$$\begin{aligned}\|\rho u\cdot\nabla\dot{\mathcal T}_Xu\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\int_0^t\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\big(\|u\|_{\dot{\mathscr C}^{-1}}\|\dot{\mathcal T}_Xu\|_{\dot B^{\frac Np+\varepsilon}_{p,1}}\\ +\|u\|_{\dot B^{\frac Np+1}_{p,1}}\|\dot{\mathcal T}_Xu\|_{\dot{\mathscr C}^{\varepsilon-2}}\big)\,dt'.\end{aligned} \tag{3.13}$$

To bound the last term of $\widetilde g$, we use the decomposition

$$\rho\partial_tw-\Delta w=\rho(W_1+W_2)+W_3,$$

with

$$W_1:=\dot T_{\partial_kX}\partial_tu^k-\dot T_{\operatorname{div}X}\partial_tu,\quad W_2:=\dot T_{\partial_k\partial_tX}u^k-\dot T_{\operatorname{div}\partial_tX}u,\quad W_3:=\Delta\big(\dot T_{\operatorname{div}X}u-\dot T_{\partial_kX}u^k\big).$$

Continuity results for the paraproduct and the definition of $\mathcal M\big(\dot B^{\frac Np+\varepsilon-2}_{p,1}\big)$ ensure that

$$\|\rho W_1\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\int_0^t\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\|\nabla X\|_{\dot{\mathscr C}^{\varepsilon-1}}\|\partial_tu\|_{\dot B^{\frac Np-1}_{p,1}}\,dt', \tag{3.14}$$

$$\|\rho W_2\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\int_0^t\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\|\partial_tX\|_{\dot{\mathscr C}^{\varepsilon-2}}\|u\|_{\dot B^{\frac Np+1}_{p,1}}\,dt', \tag{3.15}$$

$$\|W_3\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\int_0^t\|\nabla X\|_{\dot{\mathscr C}^{\varepsilon-1}}\|u\|_{\dot B^{\frac Np+1}_{p,1}}\,dt'. \tag{3.16}$$

To estimate $\partial_tX$ in (3.15), we use the fact that

$$\partial_tX=-u\cdot\nabla X+\partial_Xu=-\operatorname{div}(u\otimes X)+\partial_Xu.$$

Hence using (1.17), and continuity results for the remainder and paraproduct operators, we get under Condition (1.18),

$$\|\partial_tX\|_{\dot{\mathscr C}^{\varepsilon-2}}\lesssim\|u\|_{\dot B^{\frac Np-1}_{p,1}}\|X\|_{\dot{\mathscr C}^\varepsilon}+\|\partial_Xu\|_{\dot{\mathscr C}^{\varepsilon-2}}.$$

Therefore, taking advantage of (1.19) yields

$$\|\rho W_2\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\int_0^t\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}(\|X\|_{\dot{\mathscr C}^\varepsilon}\|u\|_{\dot B^{\frac Np-1}_{p,1}}+\|\dot{\mathcal T}_Xu\|_{\dot{\mathscr C}^{\varepsilon-2}})\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\,dt'. \tag{3.17}$$

Combining (3.14), (3.15) and (3.17), we eventually obtain

$$\|\rho\partial_t w-\Delta w\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\int_0^t\|\dot{\mathcal T}_X u\|_{\dot{\mathscr C}^{\varepsilon-2}}\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\,dt' \tag{3.18}$$
$$+\int_0^t\|X\|_{\dot{\mathscr C}^\varepsilon}\Big(\big(\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\|u\|_{\dot B^{\frac Np-1}_{p,1}}+1\big)\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}+\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\|\partial_t u\|_{\dot B^{\frac Np-1}_{p,1}}\Big)\,dt'.$$

Putting together estimate (3.12), (3.13) and (3.18), we eventually obtain

$$\|\widetilde g\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\int_0^t\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\big(\|u\|_{\dot{\mathscr C}^{-1}}\|\dot{\mathcal T}_X u\|_{\dot B^{\frac Np+\varepsilon}_{p,1}}+\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|\dot{\mathcal T}_X u\|_{\dot{\mathscr C}^{\varepsilon-2}}\big)\,dt' \tag{3.19}$$
$$+\int_0^t\|X\|_{\dot{\mathscr C}^\varepsilon}\big(\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|u\|_{\dot B^{\frac Np-1}_{p,1}}+\|(\partial_t u,D_t u)\|_{\dot B^{\frac Np-1}_{p,1}}\big)\|\rho\|_{\mathcal M\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\,dt'$$
$$+\int_0^t\|X\|_{\dot{\mathscr C}^\varepsilon}\|(\nabla^2u,\nabla P)\|_{\dot B^{\frac Np-1}_{p,1}}\,dt'+\int_0^t\|\partial_X\rho\|_{\mathcal M\left(\dot B^{\frac Np-1}_{p,1}\to\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\|D_t u\|_{\dot B^{\frac Np-1}_{p,1}}\,dt'.$$

*Second step: bounds of $v$.* We now want to bound $v$ in $\widetilde L^\infty_t\big(\dot B^{\frac Np+\varepsilon-2}_{p,1}\big)\cap L^1_t\big(\dot B^{\frac Np+\varepsilon}_{p,1}\big)$, knowing (3.19). This will follow from the smoothing properties of the heat flow. More precisely, introduce the projector $\mathbb P$ over divergence-free vector fields, and apply $\mathbb P\dot\Delta_j$ (with $j\in\mathbb Z$) to the equation $(S)$. We get

$$\begin{cases}\partial_t\dot\Delta_j v-\Delta\dot\Delta_j v=\mathbb P\dot\Delta_j(\widetilde g+(1-\rho)\partial_t v)\\ \dot\Delta_j v|_{t=0}=\dot\Delta_j v_0.\end{cases}$$

Lemma 2.1 in [8] implies that if $p\in[1,\infty]$,

$$\|\dot\Delta_j v(t)\|_{L^p}\le e^{-ct2^{2j}}\|\dot\Delta_j v_0\|_{L^p}+C\int_0^t e^{-c(t-t')2^{2j}}\|\dot\Delta_j(\widetilde g+(1-\rho)\partial_t v)(t')\|_{L^p}\,dt'.$$

Therefore, taking the supremum over $j\in\mathbb Z$, using the fact that

$$\partial_t v=\Delta v+\mathbb P\big(\widetilde g+(1-\rho)\partial_t v\big)$$

and that $\mathbb P:\dot B^{\frac Np+\varepsilon-2}_{p,1}\to\dot B^{\frac Np+\varepsilon-2}_{p,1}$, we find that

$$\|v\|_{\widetilde L^\infty_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}+\|v\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon}_{p,1}\right)}+\|\partial_t v\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)} \tag{3.20}$$
$$\lesssim\|v_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}+\|\widetilde g\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}+\|(1-\rho)\partial_t v\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}.$$

The smallness condition (1.13) combined with Inequality (3.4) ensure that the last term of (3.20) may be absorbed by the left-hand side, and we thus end up with

$$\|v\|_{\widetilde L^\infty_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)\cap L^1_t\left(\dot B^{\frac Np+\varepsilon}_{p,1}\right)}+\|\partial_t v\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\|v_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}+\|\widetilde g\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}.$$

Next, we use the fact that by definition of $v_0$,

$$\begin{aligned}v_0&=\dot{\mathcal T}_{X_0}u_0-\dot T_{\partial_k X_0}u_0^k+\dot T_{\operatorname{div}X_0}u_0\\&=\partial_{X_0}u_0-\dot T_{\partial_k u_0}X_0^k-\partial_k\dot R(X_0^k,u_0)+\dot R(\operatorname{div}X_0,u_0)-\dot T_{\partial_k X_0}u_0^k+\dot T_{\operatorname{div}X_0}u_0.\end{aligned}$$

Hence continuity results for the paraproduct yield, under Condition (1.18),

$$\|v_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}\lesssim\|\partial_{X_0}u_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr C}^\varepsilon}\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}.$$

Thus

$$\|v\|_{\widetilde{L}^\infty_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)\cap L^1_t\left(\dot B^{\frac Np+\varepsilon}_{p,1}\right)}+\|\partial_t v\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim \|\partial_{X_0}u_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}\\ +\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}+\|\widetilde g\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}. \tag{3.21}$$

*Third step: bounds for striated regularity.* Remembering that

$$\dot{\mathcal{T}}_X u=v+w\quad\text{with}\quad w=\dot T_{\partial_k X}u^k-\dot T_{\operatorname{div}X}u,$$

it is now easy to bound the following quantity:

$$\mathscr{H}(t):=\|\dot{\mathcal{T}}_X u\|_{\widetilde{L}^\infty_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}+\|\dot{\mathcal{T}}_X u\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon}_{p,1}\right)}+\|\nabla\dot{\mathcal{T}}_X P\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}.$$

Indeed, we have

$$\nabla\dot{\mathcal{T}}_X P=(\mathrm{Id}-\mathbb{P})(\widetilde g-\rho\partial_t v), \tag{3.22}$$

and thus $\|\nabla\dot{\mathcal{T}}_X P\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}$ may be bounded by the right-hand side of (3.21). Note also that continuity results for paraproduct operators guarantee that

$$\|w\|_{\widetilde{L}^\infty_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\|u\|_{\widetilde{L}^\infty_t(\dot B^{\frac Np-1}_{p,1})}\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)},$$

$$\|w\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon}_{p,1}\right)}\lesssim\int_0^t\|u\|_{\dot B^{\frac Np+1}_{p,1}}\|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}}\,dt'.$$

Hence we have

$$\mathscr{H}(t)\lesssim\|\partial_{X_0}u_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}+\|\widetilde g\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\\ +\|u\|_{\widetilde{L}^\infty_t(\dot B^{\frac Np-1}_{p,1})\cap L^1_t(\dot B^{\frac Np+1}_{p,1})}\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}. \tag{3.23}$$

Because $X$ satisfies (0.6), standard Hölder estimates for transport equations imply that

$$\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}\le\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}+\int_0^t\|\nabla u\|_{L^\infty}\|X\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'+\int_0^t\|\partial_X u\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'.$$

Now, recall that

$$\partial_X u-\dot{\mathcal{T}}_X u=\dot T_{\partial_k u}X^k+\dot R(\partial_k u,X^k)$$

whence, using standard continuity results for operators $\dot T$ and $\dot R$, and embedding,

$$\|\dot{\mathcal{T}}_X u-\partial_X u\|_{\dot{\mathscr{C}}^\varepsilon}\lesssim\|\dot{\mathcal{T}}_X u-\partial_X u\|_{\dot B^{\frac Np+\varepsilon}_{p,1}}\lesssim\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}. \tag{3.24}$$

Therefore we have

$$\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}\le\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}+\int_0^t\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'+\|\dot{\mathcal{T}}_X u\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon}_{p,1}\right)}. \tag{3.25}$$

Then, using (3.1) and plugging the above inequality in (3.23), we get

$$\mathscr{H}(t)\lesssim\|\partial_{X_0}u_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}+\|\widetilde g\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\\ +\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}\biggl(\|\dot{\mathcal{T}}_X u\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon}_{p,1}\right)}+\int_0^t\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'\biggr).$$

Choosing $c$ small enough in (1.13), we see that the first term of the second line may be absorbed by the left-hand side. Therefore, setting

$$\mathscr{K}(t):=\mathscr{H}(t)+\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}$$

and using again (3.25) and the smallness of $u_0$,

$$\mathscr{K}(t)\lesssim \|\partial_{X_0}u_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}+\|\widetilde{g}\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}+\int_0^t\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}\,dt'.$$

In order to close the estimates, it suffices to bound $\widetilde{g}$ by means of (3.19). Then the above inequality becomes, after using (3.4) and (3.5),

$$\begin{aligned}\mathscr{K}(t)\lesssim{}&\|\partial_{X_0}u_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\\
&+\int_0^t\|\rho\|_{\mathcal{M}\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\bigl(\|u\|_{\dot{\mathscr{C}}^{-1}}\|\dot{\mathcal{T}}_Xu\|_{\dot B^{\frac Np+\varepsilon}_{p,1}}+\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|\dot{\mathcal{T}}_Xu\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\bigr)\,dt'\\
&+\int_0^t\|X\|_{\dot{\mathscr{C}}^\varepsilon}\bigl(\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|u\|_{\dot B^{\frac Np-1}_{p,1}}+\|(\partial_tu,D_tu)\|_{\dot B^{\frac Np-1}_{p,1}}\bigr)\|\rho_0\|_{\mathcal{M}\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\,dt'\\
&+\int_0^t\|X\|_{\dot{\mathscr{C}}^\varepsilon}\|(\nabla^2u,\nabla P)\|_{\dot B^{\frac Np-1}_{p,1}}\,dt'+\int_0^t\|\partial_{X_0}\rho_0\|_{\mathcal{M}\left(\dot B^{\frac Np-1}_{p,1}\to\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\|D_tu\|_{\dot B^{\frac Np-1}_{p,1}}\,dt'.\end{aligned}$$

The smallness of $\rho_0$ and $u_0$ implies that the second line may be absorbed by the l.h.s. Therefore using the bounds for $\partial_tu$ and $D_tu$ in (3.1), we eventually get

$$\begin{aligned}\mathscr{K}(t)\lesssim{}&\|\partial_{X_0}u_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}+\|\rho_0\|_{\mathcal{M}\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}\biggl(1+\int_0^t\|\nabla u\|_{\dot B^{\frac Np}_{p,1}}\|X\|_{\dot{\mathscr{C}}^\varepsilon}\,d\tau\biggr)\\
&+\int_0^t\|X\|_{\dot{\mathscr{C}}^\varepsilon}\|(\nabla^2u,\nabla P)\|_{\dot B^{\frac Np-1}_{p,1}}\,dt'+\|\partial_{X_0}\rho_0\|_{\mathcal{M}\left(\dot B^{\frac Np-1}_{p,1}\to\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\int_0^t\|D_tu\|_{\dot B^{\frac Np-1}_{p,1}}\,d\tau.\end{aligned}$$

It is now easy to conclude by means of Gronwall lemma and (3.1). Using once again the smallness of $u_0$, we get

$$\begin{aligned}\mathscr{K}(t)\lesssim{}&\|\partial_{X_0}u_0\|_{\dot B^{\frac Np+\varepsilon-2}_{p,1}}+\|X_0\|_{\dot{\mathscr{C}}^\varepsilon}\\
&+\bigl(\|\partial_{X_0}\rho_0\|_{\mathcal{M}\left(\dot B^{\frac Np-1}_{p,1}\to\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}+\|\rho_0\|_{\mathcal{M}\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\bigr)\|u_0\|_{\dot B^{\frac Np-1}_{p,1}}.\end{aligned}\tag{3.26}$$

From (3.24), we gather that $\partial_Xu$ is bounded by the right-hand side of (3.26). Next, in order to control the whole nonhomogeneous Hölder norm of $X$, it suffices to remember that

$$\|X\|_{\mathcal{C}^{0,\varepsilon}}=\|X\|_{L^\infty}+\|X\|_{\dot{\mathscr{C}}^\varepsilon}$$

and that Relation (0.5) together with (3.2) directly yield

$$\|X_t\|_{L^\infty}\leq\|\partial_{X_0}\psi_t\|_{L^\infty}\leq C\|X_0\|_{L^\infty}.$$

Finally, to estimate $\partial_X\nabla P$, we use Inequality (1.19) and get

$$\|\partial_X\nabla P-\nabla\dot{\mathcal{T}}_XP\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}\lesssim\|X\|_{L^\infty_t(\dot{\mathscr{C}}^\varepsilon)}\|\nabla P\|_{L^1_t(\dot B^{\frac Np-1}_{p,1})}.$$

Therefore $\|\partial_X\nabla P\|_{L^1_t\left(\dot B^{\frac Np+\varepsilon-2}_{p,1}\right)}$ may be bounded like $\mathscr{K}(t)$.

3.3. **The regularization process.** In all the above computations, we implicitly assumed that $X$ and $\partial_Xu$ were in $L^\infty_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$ and $L^1_{loc}(\mathbb{R}_+;\mathcal{C}^{0,\varepsilon})$, respectively. However, Theorem 1.1 just ensures continuity of those vector-fields, not Hölder regularity.

To overcome that difficulty, one may smooth out the initial velocity (not the density, not to destroy the multiplier hypotheses) by setting for example $u_0^n:=\dot S_nu_0$. Then Condition (1.13) is satisfied by $(\rho_0,u_0^n)$ and, as in addition $u_0^n$ belongs to all Besov spaces $\dot B^{\frac Np-1}_{p,r}$ with

$\widetilde{p} \geq p$ and $r \geq 1$, one can apply[3] [13, Th. 1.1] for solving (INS) with initial data $(\rho_0, u_0^n)$. This provides us with a unique global solution $(\rho^n, u^n, \nabla P^n)$ which, among others, satisfies

$$\nabla u^n \in L^r(\mathbb{R}_+; \dot{B}^{\frac{N}{p}}_{\widetilde{p},r}) \quad \text{for all} \quad r \in ]1,\infty[ \quad \text{and} \quad \max\left(p, \frac{Nr}{3r-2}\right) \leq \widetilde{p} \leq \frac{Nr}{r-1}.$$

By taking $r$ sufficiently close to 1 and using embedding, we see that this implies that $\nabla u^n$ is in $L^1_{loc}(\mathbb{R}_+; \mathscr{C}^{0,\delta})$ for all $0 < \delta < 1$ and thus the corresponding flow $\psi^n$ is (in particular) in $\mathcal{C}^{1,\varepsilon}$. This ensures, thanks to (0.5), that $X^n$ is in $L^\infty_{loc}(\mathbb{R}_+; \mathcal{C}^{0,\varepsilon})$ and thus that $\partial_{X^n} u^n$ is in $L^1_{loc}(\mathbb{R}_+; \mathcal{C}^{0,\varepsilon})$.

From the previous steps and the fact that the data $(\rho_0, u_0^n)$ satisfy (1.13) uniformly, we get uniform bounds for $\rho^n$, $u^n$, $\nabla P^n$ and $X^n$, and standard arguments thus allow to show that $u^n$ tends to $u$ in $L^1_{loc}(\mathbb{R}_+; L^\infty)$ and thus $(\psi^n - \psi) \to 0$ in $L^\infty_{loc}(\mathbb{R}_+; L^\infty)$. Interpolating with the uniform bounds and using standard functional analysis arguments, one can eventually conclude that $X^n \to X$ in $L^\infty_{loc}(\mathbb{R}_+; \mathcal{C}^{0,\varepsilon'})$ for all $\varepsilon' < \varepsilon$ (and similar results for $(u^n)_{n\in\mathbb{N}}$) and that all the estimates of the previous steps are satisfied. The details are left to the reader. □

## Appendix A. Multiplier spaces

The following relationship between the nonhomogeneous Besov spaces $B^s_{p,r}(\mathbb{R}^N)$ and the homogeneous Besov spaces $\dot{B}^s_{p,r}(\mathbb{R}^N)$ for *compactly supported* functions or distributions has been established in [12, Section 2.1].

**Proposition A.1.** *Let $(p,r) \in [1,\infty]^2$ and $s > -\frac{N}{p'} := -N(1-\frac{1}{p})$ (or just $s \geq -\frac{N}{p'}$ if $r = \infty$). For any $u$ in the set $\mathcal{E}'(\mathbb{R}^N)$ of compactly supported distributions on $\mathbb{R}^N$, we have*

$$u \in B^s_{p,r}(\mathbb{R}^N) \Longleftrightarrow u \in \dot{B}^s_{p,r}(\mathbb{R}^N).$$

*Moreover, there exists a constant $C = C(s,p,r,N,\operatorname{Supp} u)$ such that*

$$C^{-1}\|u\|_{\dot{B}^s_{p,r}} \leq \|u\|_{B^s_{p,r}} \leq C\|u\|_{\dot{B}^s_{p,r}}.$$

A simple consequence of Proposition A.1 and of standard embeddings for nonhomogeneous Besov spaces is that for any $(s,p,r)$ as above, we have

$$\mathcal{E}'(\mathbb{R}^N) \cap \dot{B}^{s+\delta}_{p,r}(\mathbb{R}^N) \hookrightarrow \mathcal{E}'(\mathbb{R}^N) \cap \dot{B}^s_{p,r}(\mathbb{R}^N) \quad \text{for any} \quad \delta > 0. \tag{A.1}$$

We also used the following statement:

**Proposition A.2.** *Let $(p,s)$ be arbitrary in $[1,\infty]\times\mathbb{R}$. Then for all $u \in B^s_{\infty,1}(\mathbb{R}^N) \cap \mathcal{E}'(\mathbb{R}^N)$, we have $u \in B^s_{p,1}(\mathbb{R}^N)$ and there exists $C = C(s,p,\operatorname{Supp} u)$ such that*

$$\|u\|_{B^s_{p,1}} \leq C\|u\|_{B^s_{\infty,1}}.$$

*Proof.* Let $u$ be in $B^s_{\infty,1}(\mathbb{R}^N)$ with compact support, and fix some smooth cut-off function $\phi$ so that $\phi \equiv 1$ on $\operatorname{Supp} u$. Of course, being compactly and smooth, $\phi$ belongs to any nonhomogeneous Besov space. Then, using decomposition (1.17) and the fact that $u = \phi u$, one can write

$$u = T_\phi u + T_u \phi + R(u,\phi).$$

Because $\phi$ is in $L^p$ and $u$, in $B^s_{\infty,1}$, standard continuity results for the paraproduct ensure that $T_\phi u$ is in $B^s_{p,1}$. For the second term, we just use that $u$ is in, say, $B^{\min(0,s)}_{\infty,1}$ and $\phi$, in $B^{-\min(0,s)+s}_{p,1}$ hence $T_u\phi$ is in $B^s_{p,1}$. For the remainder term, we use for instance the fact that $\phi$ is in $B^{|s|+\frac{1}{2}}_{p,1}$. Putting all those informations together completes the proof. □

[3] That paper is dedicated to the half-space, but having the same result in the whole space setting is much easier.

The following result was the key to bounding the density terms in our study of $(INS)$.

**Lemma A.3.** *Let $(s, s_k, p, p_k, r, r_k) \in ]-1,1[^2\times[1,\infty]^4$ with $k=1,2$, and $Z:\mathbb{R}^N\to\mathbb{R}^N$ be a $\mathcal{C}^1$ measure preserving diffeomorphism such that $DZ$ and $DZ^{-1}$ are bounded. When we consider the homogeneous Besov space $\dot B^s_{p,r}(\mathbb{R}^N)$ or $\dot B^{s_k}_{p_k,r_k}(\mathbb{R}^N)$, we assume in addition that $s\in]-\frac{N}{p'},\frac{N}{p}[$ and $s_k\in]-\frac{N}{p_k'},\frac{N}{p_k}[$ for $k=1,2$. Then we have:*

(i). *If $b^s_{p,r}(\mathbb{R}^N)$ stands for $B^s_{p,r}(\mathbb{R}^N)$ or $\dot B^s_{p,r}(\mathbb{R}^N)$, then the mapping $u\mapsto u\circ Z$ is continuous on $b^s_{p,r}(\mathbb{R}^N)$: there is a positive constant $C_{Z,s,p,r}$ such that*

$$\|u\circ Z\|_{b^s_{p,r}}\leq C_{Z,s,p,r}\|u\|_{b^s_{p,r}}. \tag{A.2}$$

(ii). *If $b^{s_k}_{p_k,r_k}$ with $k=1,2$, denote the same type of Besov spaces, then the mapping $\varphi\mapsto\varphi\circ Z$ is continuous on $\mathcal{M}\big(b^{s_1}_{p_1,r_1}(\mathbb{R}^N)\to b^{s_2}_{p_2,r_2}(\mathbb{R}^N)\big)$, that is*

$$\|\varphi\circ Z\|_{\mathcal{M}\left(b^{s_1}_{p_1,r_1}\to b^{s_2}_{p_2,r_2}\right)}\leq C_{Z^{-1},1}C_{Z,2}\|\varphi\|_{\mathcal{M}\left(b^{s_1}_{p_1,r_1}\to b^{s_2}_{p_2,r_2}\right)}.$$

(iii). *We have the following equivalence for any $\varphi\in\mathcal{E}'(\mathbb{R}^N)$,*

$$\varphi\in\mathcal{M}\big(B^{s_1}_{p_1,r_1}(\mathbb{R}^N)\to B^{s_2}_{p_2,r_2}(\mathbb{R}^N)\big)\Longleftrightarrow\varphi\in\mathcal{M}\big(b^{s_1}_{p_1,r_1}(\mathbb{R}^N)\to b^{s_2}_{p_2,r_2}(\mathbb{R}^N)\big).$$

*Here $b^{s_1}_{p_1,r_1}$ and $b^{s_2}_{p_2,r_2}$ can be different type of Besov spaces but obey our convention on the index $s_k$ for homogeneous Besov space.*

*Proof.* Item (i) in the case $b=\dot B$ has been proved in [12, Lemma 2.1.1]. One may easily modify the proof to handle nonhomogeneous Besov spaces: use the finite difference characterization of [24, Page 98] if $s>0$, argue by duality if $s<0$ and interpolate for the case $s=0$. We get $C_{Z,s,p,r}\approx 1+\|DZ\|_{L^\infty}^{s+\frac{N}{r}}$ if $s>0$, and $C_{Z,s,p,r}\approx 1+\|DZ^{-1}\|_{L^\infty}^{-s+\frac{N}{r'}}$ if $s<0$.

Part (ii) is immediate according to (1.6) and (A.2). Indeed we may write:

$$\begin{aligned}
\|\varphi\circ Z\|_{\mathcal{M}\left(b^{s_1}_{p_1,r_1}\to b^{s_2}_{p_2,r_2}\right)} &= \sup_{\|u\|_{b^{s_1}_{p_1,r_1}}\leq 1}\|(\varphi\circ Z)\,u\|_{b^{s_2}_{p_2,r_2}}\\
&= \sup_{\|u\|_{b^{s_1}_{p_1,r_1}}\leq 1}\|(\varphi\,(u\circ Z^{-1}))\circ Z\|_{b^{s_2}_{p_2,r_2}}\\
&\leq C_{Z,2}\sup_{\|u\|_{b^{s_1}_{p_1,r_1}}\leq 1}\|\varphi\,(u\circ Z^{-1})\|_{b^{s_2}_{p_2,r_2}}\\
&\leq C_{Z,2}\|\varphi\|_{\mathcal{M}(b^{s_1}_{p_1,r_1}\to b^{s_2}_{p_2,r_2})}\sup_{\|u\|_{b^{s_1}_{p_1,r_1}}\leq 1}\|u\circ Z^{-1}\|_{b^{s_1}_{p_1,r_2}}\\
&\leq C_{Z^{-1},1}C_{Z,2}\|\varphi\|_{\mathcal{M}(b^{s_1}_{p_1,r_1}\to b^{s_2}_{p_2,r_2})}.
\end{aligned}$$

To prove the last item, it suffices to check that if $\varphi$ belongs to $\mathcal{E}'\cap\mathcal{M}\big(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2}\big)$, then $\varphi$ is also in the multiplier space between the general type Besov spaces. Take $u\in b^{s_1}_{p_1,r_1}$ with compact support, and some smooth and compactly supported nonnegative cut-off function $\psi$ satisfying $\psi\equiv 1$ on $\operatorname{Supp}\varphi$. Then from Proposition A.1 and (1.6), we have

$$\begin{aligned}
\|\varphi u\|_{b^{s_2}_{p_2,r_2}} &= \|\varphi\psi u\|_{b^{s_2}_{p_2,r_2}}\lesssim\|\varphi\psi u\|_{B^{s_2}_{p_2,r_2}}\\
&\lesssim\|\varphi\|_{\mathcal{M}\left(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2}\right)}\|\psi u\|_{B^{s_1}_{p_1,r_1}}\\
&\lesssim\|\varphi\|_{\mathcal{M}\left(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2}\right)}\|\psi u\|_{b^{s_1}_{p_1,r_1}}\\
&\lesssim\|\varphi\|_{\mathcal{M}\left(B^{s_1}_{p_1,r_1}\to B^{s_2}_{p_2,r_2}\right)}\|\psi\|_{\mathcal{M}\left(b^{s_1}_{p_1,r_1}\right)}\|u\|_{b^{s_1}_{p_1,r_1}}.
\end{aligned}$$

For the last inequality, we used $\mathcal{C}^\infty_c\hookrightarrow\mathcal{M}\big(b^{s_1}_{p_1,r_1}\big)$ (see [12, Corollary 2.1.1]). □

## Appendix B. Commutator Estimates

We here recall and prove some commutator estimates that were crucial in this paper. All of them strongly rely on continuity results in Besov spaces for the paraproduct and remainder operators, and on the following classical result (see e.g. [4, Section 2.10]).

**Lemma B.1.** *Let $A : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}$ be a smooth function, homogeneous of degree $m$. Let $(\varepsilon, s, p, r, r_1, r_2, p_1, p_2) \in ]0,1[\times\mathbb{R}\times[1,\infty]^6$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ and*

$$s - m + \varepsilon < \frac{N}{p} \quad \text{or} \quad \Big\{ s - m + \varepsilon < \frac{N}{p} \text{ and } r = 1 \Big\}.$$

*There exists a constant $C$ depending only on $s, \varepsilon, N$ and $A$ such that,*

$$\|[\dot T_g, A(D)]u\|_{\dot B^{s-m+\varepsilon}_{p,r}} \le C\|\nabla g\|_{\dot B^{\varepsilon-1}_{p_1,r_1}} \|u\|_{\dot B^{s}_{p_2,r_2}}.$$

If the integer $N_0$ in the definition of Bony's paraproduct and remainder is large enough (for instance $N_0 = 4$ does), then the following fundamental lemma holds.

**Lemma B.2** (Chemin-Leibniz Formula)**.** *Let $(\varepsilon, s, s_k, p, p_k, r, r_k) \in ]0,1[\times\mathbb{R}^2\times[1,\infty]^4$ for $k = 1, 2$ satisfying*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \quad \text{and} \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}.$$

(i). *If $s_2 < 0$ and $s_1 + s_2 + \varepsilon - 1 < \frac{N}{p}$ or $\{s_1 + s_2 + \varepsilon - 1 = \frac{N}{p}$ and $r = 1\}$, then we have*

$$\|\dot{\mathcal{T}}_X \dot T_g f - \dot T_g \dot{\mathcal{T}}_X f - \dot T_{\dot{\mathcal{T}}_X g} f\|_{\dot B^{s_1+s_2+\varepsilon-1}_{p,r}} \le C\|X\|_{\dot{\mathscr{C}}^\varepsilon} \|f\|_{\dot B^{s_1}_{p,r_1}} \|g\|_{\dot B^{s_2}_{\infty,r_2}}.$$

*The above inequality still holds in the limit case $s_2 = 0$, if one replaces $\|g\|_{\dot B^0_{\infty,r_2}}$ by $\|g\|_{\dot B^0_{\infty,r_2}\cap L^\infty}$.*

(ii). *If $s_1 + s_2 + \varepsilon - 1 \in ]0, \frac{N}{p}[$ or $\{s_1 + s_2 + \varepsilon - 1 = \frac{N}{p}$ and $r = 1\}$, then we have*

$$\|\dot{\mathcal{T}}_X \dot R(f,g) - \dot R(\dot{\mathcal{T}}_X f, g) - \dot R(f, \dot{\mathcal{T}}_X g)\|_{\dot B^{s_1+s_2+\varepsilon-1}_{p,r}} \le C\|X\|_{\dot{\mathscr{C}}^\varepsilon} \|f\|_{\dot B^{s_1}_{p_1,r_1}} \|g\|_{\dot B^{s_2}_{p_2,r_2}}.$$

*The above inequality still holds in the limit case $s_1 + s_2 + \varepsilon - 1 = 0$, $r = \infty$ and $\frac{1}{r_1} + \frac{1}{r_2} = 1$.*

*Proof.* This is a mere adaptation of [14] to the homogeneous framework. The proof is based on a generalized Leibniz formula for para-vector field operators which was derived by J.-Y. Chemin in [6]. More precisely, define the following Fourier multipliers

$$\dot\Delta_{k,j} := \varphi_k(2^{-j}D) \qquad \text{with } \varphi_k(\xi) := i\xi_k\varphi(\xi) \text{ for } k \in \{1, \cdots, N\} \text{ and } j \in \mathbb{Z}.$$

Then we have

$$\begin{aligned}\dot{\mathcal{T}}_X \dot T_g f &= \sum_{j\in\mathbb{Z}} (\dot S_{j-N_0} g \dot{\mathcal{T}}_X \dot\Delta_j f + \dot\Delta_j f \dot{\mathcal{T}}_X \dot S_{j-N_0} g) + \sum_{j\in\mathbb{Z}} (\dot T_{1,j} + \dot T_{2,j}) \\ &= \dot T_g \dot{\mathcal{T}}_X f + \dot T_{\dot{\mathcal{T}}_X g} f + \sum_{\substack{j\in\mathbb{Z}\\ \alpha=1,\dots,4}} \dot T_{\alpha,j},\end{aligned}$$

where

$$\begin{aligned}\dot T_{1,j} &:= \sum_{\substack{j\le j'\le j+1\\ j-N_0-1\le j''\le j'-N_0-1}} 2^{j'} \dot\Delta_{j''} X^k \big(\dot\Delta_{k,j'}(\dot\Delta_j f \dot S_{j-N_0} g) - \dot\Delta_{k,j'}\dot\Delta_j f \dot S_{j-N_0} g\big), \\ \dot T_{2,j} &:= \sum_{\substack{j'\le j-2\\ j'-N_0\le j''\le j-N_0-2}} 2^{j'} \dot\Delta_{j''} X^k (\dot\Delta_j f) \dot\Delta_{k,j'} \dot S_{j-N_0} g, \\ \dot T_{3,j} &:= \dot S_{j-N_0} g [\dot T_{X^k}, \dot\Delta_j] \partial_k f, \\ \dot T_{4,j} &:= \dot\Delta_j f [\dot T_{X^k}, \dot S_{j-N_0}] \partial_k g.\end{aligned}$$

Bounding $\dot{T}_{1,j}$ and $\dot{T}_{2,j}$ stems from the definition of Besov norms, and Lemmas 2.99, 2.100 of [4] allow to bound $\dot{T}_{3,j}$ and $\dot{T}_{4,j}$ provided $\varepsilon < 1$.

In order to prove the second item, let us set

$$A_{j,j'} := \big\{j - N_0 - 1, \cdots, j' - N_0 - 1\big\} \cup \big\{j' - N_0, \cdots, j - N_0 - 2\big\}.$$

We have

$$\begin{aligned}\dot{\mathcal{T}}_X \dot{R}(f,g) &= \sum_{j\in\mathbb{Z}} (\widetilde{\dot{\Delta}}_j g \dot{\mathcal{T}}_X \dot{\Delta}_j f + \dot{\Delta}_j f \dot{\mathcal{T}}_X \widetilde{\dot{\Delta}}_j g) + \sum_{j\in\mathbb{Z}} (\dot{R}_{1,j} + \dot{R}_{2,j}) \\ &= \dot{R}(\dot{\mathcal{T}}_X f, g) + \dot{R}(f, \dot{\mathcal{T}}_X g) + \sum_{\substack{j\in\mathbb{Z}\\ \alpha=1,\dots,4}} \dot{R}_{\alpha,j},\end{aligned}$$

where, denoting $\widetilde{\dot{\Delta}}_j := \dot{\Delta}_{j-N_0} + \cdots + \dot{\Delta}_{j+N_0}$,

$$\begin{aligned}\dot{R}_{1,j} := &\sum_{\substack{|j'-j|\le N_0+1\\ j''\in A_{j,j'}}} \operatorname{sgn}(j'-j+1) 2^{j'} \dot{\Delta}_{j''} X^k \big(\dot{\Delta}_{k,j'}(\dot{\Delta}_j f \widetilde{\dot{\Delta}}_j g) - \dot{\Delta}_j f \dot{\Delta}_{k,j'} \widetilde{\dot{\Delta}}_j g\big) \\ &\qquad + \sum_{\substack{j-1\le j'\le j\\ j'-N_0\le j''\le j-N_0}} 2^{j'} \dot{\Delta}_{j''} X^k (\dot{\Delta}_{k,j'} \dot{\Delta}_j f) \widetilde{\dot{\Delta}}_j g, \\ \dot{R}_{2,j} := &\sum_{\substack{j'\le j-N_0-2\\ j'-N_0\le j''\le j-N_0-2}} 2^{j'} \dot{\Delta}_{j''} X^k \dot{\Delta}_{k,j'} (\dot{\Delta}_j f \widetilde{\dot{\Delta}}_j g), \\ \dot{R}_{3,j} := &\widetilde{\dot{\Delta}}_j g [\dot{T}_{X^k}, \dot{\Delta}_j] \partial_k f, \\ \dot{R}_{4,j} := &\dot{\Delta}_j f [\dot{T}_{X^k}, \widetilde{\dot{\Delta}}_j] \partial_k g.\end{aligned}$$

Here again, bounding $\dot{R}_{1,j}$ and $\dot{R}_{2,j}$ follows from the definition of Besov norms, while Lemma 2.100 of [4] allows to bound $\dot{R}_{3,j}$ and $\dot{R}_{4,j}$. □

**Proposition B.3.** *Let $(\varepsilon, p)$ be in $]0,1[\times[1,\infty]$. Consider a couple of vector fields $(X, v)$ in the space*

$$\big(L^\infty_{loc}(\mathbb{R}_+; \dot{\mathscr{C}}^\varepsilon)\big)^N \times \big(L^\infty_{loc}(\mathbb{R}_+; \dot{B}^{\frac{N}{p}-1}_{p,1}) \cap L^1_{loc}(\mathbb{R}_+; \dot{B}^{\frac{N}{p}+1}_{p,1})\big)^N,$$

*satisfying* $\operatorname{div} v = 0$ *and the transport equation*

$$\text{(B.1)} \qquad \begin{cases} (\partial_t + v\cdot\nabla) X = \partial_X v, \\ X|_{t=0} = X_0. \end{cases}$$

*If in addition*

$$\text{(B.2)} \qquad \frac{N}{p} > 2 - \varepsilon, \quad \text{or} \quad \frac{N}{p} > 1 - \varepsilon \ \text{ and } \ \operatorname{div} X \equiv 0.$$

*then there exists a constant $C$ such that:*

$$\begin{aligned}\text{(B.3)} \quad \|[\dot{\mathcal{T}}_X, \partial_t + v\cdot\nabla] v\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \le C\big(&\|X\|_{\dot{\mathscr{C}}^\varepsilon} \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|v\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \\ &+ \|v\|_{\dot{\mathscr{C}}^{-1}} \|\dot{\mathcal{T}}_X v\|_{\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}} + \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big).\end{aligned}$$

*Proof.* This is essentially the proof of [14, Proposition A.5]. For the reader convenience, we here give a sketch of it. Because $\operatorname{div} v = 0$, we may write

$$\begin{aligned}[\dot{\mathcal{T}}_X, \partial_t + v^\ell \partial_\ell] v &= -v^\ell \partial_\ell \dot{T}_{X^k} \partial_k v - \dot{T}_{\partial_t X^k} \partial_k v + \dot{T}_{X^k} \partial_k (v^\ell \partial_\ell v) \\ &= -\dot{T}_{\partial_t X^k} \partial_k v + \partial_\ell \dot{\mathcal{T}}_X (v^\ell v) - \dot{\mathcal{T}}_{\partial_\ell X}(v^\ell v) - v^\ell \partial_\ell \dot{\mathcal{T}}_X v.\end{aligned}$$

Hence, decomposing $v^\ell v$ according to Bony's decomposition, we discover that

$$[\dot{\mathcal{T}}_X, \partial_t + v^\ell \partial_\ell] v = \sum_{\alpha=1}^{\alpha=5} \dot{R}_\alpha$$

with

$$\begin{aligned}
&\dot{R}_1 := -\dot{T}_{\partial_t X^k} \partial_k v, && \dot{R}_2 := \partial_\ell (\dot{\mathcal{T}}_X \dot{T}_{v^\ell} v + \dot{\mathcal{T}}_X \dot{T}_v v^\ell),\\
&\dot{R}_3 := \partial_\ell \dot{\mathcal{T}}_X \dot{R}(v^\ell, v), && \dot{R}_4 := -\dot{\mathcal{T}}_{\partial_\ell X}(v^\ell v),\\
&\dot{R}_5 := -v^\ell \partial_\ell \dot{\mathcal{T}}_X v.
\end{aligned}$$

Now, it suffices to check that all the terms $\dot{R}_\alpha$ may be bounded by the r.h.s. of (B.3).

• *Bound of $\dot{R}_1$:* From the equation (B.1), we have

$$\dot{R}_1 = \dot{T}_{v\cdot\nabla X^k} \partial_k v - \dot{T}_{\partial_X v^k} \partial_k v.$$

Hence using standard continuity results for the paraproduct, we deduce that

$$\|\dot{R}_1\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\nabla v\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \big(\|v\cdot\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-2}} + \|\partial_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}\big).$$

Keeping in mind (B.2), the last term may be bounded according to (1.19), after using the embedding $\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}(\mathbb{R}^N) \hookrightarrow \dot{\mathscr{C}}^{\varepsilon-2}(\mathbb{R}^N)$. We get

$$\|\partial_X v - \dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}} \lesssim \|\nabla v\|_{\dot{B}^{\frac{N}{p}-2}_{p,1}} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}}.$$

As for the first term, we use the fact $\operatorname{div} v = 0$ and the following decomposition

$$v\cdot\nabla X = \dot{\mathcal{T}}_v X + \dot{T}_{\partial_\ell X} v^\ell + \partial_\ell \dot{R}(v^\ell, X),$$

which allow to get, as long as (B.2) holds

$$\|\dot{R}_1\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\nabla v\|_{\dot{B}^{\frac{N}{p}}_{p,1}} (\|v\|_{\dot{B}^{\frac{N}{p}-1}_{p,1}} \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} + \|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}).$$

• *Bound of $\dot{R}_2$:* Due to Lemma B.2 (i) and continuity of paraproduct operator, we have

$$\|\dot{R}_2\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|v\|_{\dot{\mathscr{C}}^{-1}} + \|v\|_{\dot{\mathscr{C}}^{-1}} \|\dot{\mathcal{T}}_X v\|_{\dot{B}^{\frac{N}{p}+\varepsilon}_{p,1}} + \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}.$$

• *Bound of $\dot{R}_3$:* Applying Lemma B.2 (ii) and continuity of remainder operator under the condition $\frac{N}{p} + \varepsilon - 1 > 0$ yields

$$\|\dot{R}_3\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|X\|_{\dot{\mathscr{C}}^{\varepsilon}} \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|v\|_{\dot{\mathscr{C}}^{-1}} + \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}} \|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}.$$

• *Bound of $\dot{R}_4$:* From Bony decomposition (1.17), it is easy to get

$$\|v^l v\|_{\dot{B}^{\frac{N}{p}}_{p,1}} \lesssim \|v\|_{\dot{\mathscr{C}}^{-1}} \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}.$$

Hence

$$\|\dot{R}_4\|_{\dot{B}^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|\nabla X\|_{\dot{\mathscr{C}}^{\varepsilon-1}} \|v\|_{\dot{\mathscr{C}}^{-1}} \|v\|_{\dot{B}^{\frac{N}{p}+1}_{p,1}}.$$

• *Bound of $\dot R_5$:* Applying Bony decomposition and using that $\operatorname{div} v = 0$ and $\frac{N}{p} + \varepsilon > 1$ give

$$\|\dot R_5\|_{\dot B^{\frac{N}{p}+\varepsilon-2}_{p,1}} \lesssim \|v\|_{\dot{\mathscr{C}}^{-1}} \|\dot{\mathcal{T}}_X v\|_{\dot B^{\frac{N}{p}+\varepsilon}_{p,1}} + \|v\|_{\dot B^{\frac{N}{p}+1}_{p,1}} \|\dot{\mathcal{T}}_X v\|_{\dot{\mathscr{C}}^{\varepsilon-2}}.$$

Combining the above estimates for all $\dot R_\alpha$, with $\alpha = 1, \dots, 5$ yields (B.3). □

Another consequence of Lemma B.2 is the following estimate of $\operatorname{div}(Xfg)$:

**Proposition B.4.** *Let $(s,p,r)$ be in $]0,1[\times[1,\infty]^2$, and $\eta$, in $]0,1-s[$. Consider a bounded vector field $X$ and two bounded functions $f, g$ satisfying*

$$X \in \big(\dot B^s_{p,r}(\mathbb{R}^N) \cap \mathscr{C}^{s+\eta}(\mathbb{R}^N)\big)^N, \ \ (f,g) \in \dot B^s_{p,r}(\mathbb{R}^N) \times \dot B^{-\eta}_{p,r}(\mathbb{R}^N) \ \text{ and } \ \partial_X g \in \dot B^{s-1}_{p,r}(\mathbb{R}^N).$$

*If in addition $\operatorname{div} X$ belongs to $\mathcal{M}\big(\dot B^s_{p,r}(\mathbb{R}^N) \to \dot B^{s-1}_{p,r}(\mathbb{R}^N)\big)$, and there exists some $q \in [1,p[$ such that*

$$\operatorname{div} X \in \dot B^{s_{p,q}}_{q,r}(\mathbb{R}^N) \ \ \text{with} \ \ s_{p,q} := s - 1 + N\big(\frac{1}{q} - \frac{1}{p}\big) > 0, \tag{B.4}$$

*then we have $\operatorname{div}(Xfg) \in \dot B^{s-1}_{p,r}(\mathbb{R}^N)$, and the following estimate holds true:*

$$\begin{aligned}\|\operatorname{div}(Xfg)\|_{\dot B^{s-1}_{p,r}} \lesssim \|X\|_{\dot B^s_{p,r}\cap\mathscr{C}^{s+\eta}} \|f\|_{L^\infty\cap\dot B^s_{p,r}} \|g\|_{L^\infty\cap\dot B^{-\eta}_{p,r}} + \|f\|_{L^\infty} \|\partial_X g\|_{\dot B^{s-1}_{p,r}} \\ + \|\operatorname{div} X\|_{\dot B^{s_{p,q}}_{q,r}\cap\mathcal{M}(\dot B^s_{p,r}\to\dot B^{s-1}_{p,r})} \|g\|_{L^\infty} \|f\|_{\dot B^s_{p,r}\cap L^\infty}.\end{aligned}$$

*Proof.* In light of Bony's decomposition (1.17), and denoting $\dot T'_g f := \dot T_g f + \dot R(f,g)$, we can decompose $\operatorname{div}(Xfg)$ into

$$\operatorname{div}(Xfg) = \operatorname{div}\big(\dot T'_{fg} X + \dot T_X(fg)\big) = \sum_{\alpha=1}^4 \dot F_\alpha,$$

where

$$\dot F_1 := \operatorname{div}(\dot T'_{fg} X), \qquad \dot F_2 := \dot T_{\operatorname{div} X}(fg),$$
$$\dot F_3 := \dot{\mathcal{T}}_X \dot T'_g f, \qquad \dot F_4 := \dot{\mathcal{T}}_X \dot T_f g.$$

• *Bound of $\dot F_1$:* As $s > 0$, standard continuity results for $\dot R$ and $\dot R$ give

$$\|\dot F_1\|_{B^{s-1}_{p,r}} \lesssim \|\dot T'_{fg} X\|_{\dot B^s_{p,r}} \lesssim \|f\|_{L^\infty} \|g\|_{L^\infty} \|X\|_{\dot B^s_{p,r}}.$$

• *Bound of $\dot F_2$:* Thanks to continuity results for $\dot T'$, we have for $s < 1$,

$$\|\dot F_2\|_{\dot B^{s-1}_{p,r}} \lesssim \|\operatorname{div} X\|_{\dot B^{s-1}_{p,r}} \|f\|_{L^\infty} \|g\|_{L^\infty}.$$

• *Bound of $\dot F_3$:* Because $X$ and $g$ are in $L^\infty$ and $s > 0$, we readily have

$$\|\dot F_3\|_{\dot B^{s-1}_{p,r}} \lesssim \|X\|_{L^\infty} \|\dot T'_g f\|_{\dot B^s_{p,r}} \lesssim \|X\|_{L^\infty} \|g\|_{L^\infty} \|f\|_{\dot B^s_{p,r}}.$$

• *Bound of $\dot F_4$:* Because $0 < s < s+\eta < 1$, Lemma B.2 and continuity results for the paraproduct imply that

$$\begin{aligned}\|\dot{\mathcal{T}}_X \dot T_f g\|_{\dot B^{s-1}_{p,r}} &\lesssim \|X\|_{\dot{\mathscr{C}}^{s+\eta}} \|f\|_{L^\infty} \|g\|_{\dot B^{-\eta}_{p,r}} + \|\dot T_f \dot{\mathcal{T}}_X g\|_{\dot B^{s-1}_{p,r}} + \|\dot T_{\dot{\mathcal{T}}_X f} g\|_{\dot B^{s-1}_{p,r}} \\ &\lesssim \|X\|_{\dot{\mathscr{C}}^{s+\eta}} \|f\|_{L^\infty} \|g\|_{\dot B^{-\eta}_{p,r}} + \|f\|_{L^\infty} \|\dot{\mathcal{T}}_X g\|_{\dot B^{s-1}_{p,r}} + \|g\|_{L^\infty} \|\dot{\mathcal{T}}_X f\|_{\dot B^{s-1}_{p,r}}.\end{aligned}$$

To bound the last term, one may use the decomposition

$$\dot{\mathcal{T}}_X f = \operatorname{div}(\dot T_X f) - f \operatorname{div} X + \dot T_f \operatorname{div} X + \dot R(f, \operatorname{div} X).$$

Hence using continuity results for $\dot R$ and $\dot T$ and the fact that $(s_{p,q}, q)$ satisfies (B.4),

$$\|\dot{\mathcal{T}}_X f\|_{\dot B^{s-1}_{p,r}} \lesssim \|f\|_{\dot B^s_{p,r}} \big(\|X\|_{L^\infty} + \|\operatorname{div} X\|_{\mathcal{M}(\dot B^s_{p,r}\to\dot B^{s-1}_{p,r})}\big) + \|f\|_{L^\infty} \|\operatorname{div} X\|_{\dot B^{s_{p,q}}_{q,r}}.$$

Finally, to bound the term with $\dot{\mathcal{T}}_X g$, we use the fact that

$$\partial_X g - \dot{\mathcal{T}}_X g = \dot{T}_{\nabla g} \cdot X + \operatorname{div} \dot{R}(X, g) - \dot{R}(\operatorname{div} X, g),$$

whence

$$\|\partial_X g - \dot{\mathcal{T}}_X g\|_{\dot{B}^{s-1}_{p,r}} \lesssim \|g\|_{L^\infty} \big( \|X\|_{\dot{B}^s_{p,r}} + \|\operatorname{div} X\|_{\dot{B}^{s_{p,q}}_{q,r}} \big). \tag{B.5}$$

This completes the proof of the proposition. □

Proposition B.4 above reveals that the bounded function $g$ may behave like some element in $\mathcal{M}(\dot{B}^s_{p,r})$ under a suitable additional structure assumption. If in addition $g$ has compact support, then one can relax a bit the regularity of $X$ and $f$ to study $\partial_X(fg)$, and get the following generalization of [9, Lemma A.6].

**Corollary B.5.** *Consider a divergence-free vector field $X$ with coefficients in $B^\alpha_{\infty,1}$, and some function $f$ in $B^{\alpha'}_{\infty,1}$ with $0 < \alpha, \alpha' < 1$. Let $g \in L^\infty$ be compactly supported and satisfy $\partial_X g \in \dot{B}^{\min\{\alpha,\alpha'\}-1}_{p,1}$ for some $p \in [1, \infty]$. Then we have $\partial_X(fg) \in \dot{B}^{\min\{\alpha,\alpha'\}-1}_{p,1}$.*

*Proof.* Let $\psi \in \mathcal{C}^\infty_c$ be a cut-off function such that $\psi \equiv 1$ near $\operatorname{Supp} g$. Denote $(\widetilde{X}, \widetilde{f}) := (\psi X, \psi f)$. From Proposition A.2, we know that $\widetilde{f}$ and $\widetilde{X}$ are in $\dot{B}^{\min(\alpha,\alpha')}_{q,1} \cap L^\infty$ for any $q \in [1, \infty]$. It is also clear that $\partial_X(fg) = \partial_{\widetilde{X}}(\widetilde{f}g) \in \dot{B}^{\min\{\alpha,\alpha'\}-1}_{p,1}$. Hence applying Proposition B.4 gives the result. □

## Acknowledgement

Part of this work has been performed during a visit of the second author in Peking University. Zhifei Zhang and Chao Wang are warmly thanked for their welcome and stimulating discussions during this period. The first author is partially supported by ANR-15-CE40-0011. The second author is granted by the *Réseau de Recherche Doctoral de Mathématiques de l'Île de France* (RDM-IdF).

## References

[1] H. Abidi. Equation de Navier-Stokes avec densité et viscosité variables dans l'espace critique. *Revista Matemática Iberoamericana*, **23**(2):537–586, 2007.

[2] H. Abidi and M. Paicu. Existence globale pour un fluide inhomogène. *Annales de l'Institut Fourier*, **57**: 883–917, 2007.

[3] S. N. Antontsev, A. V. Kazhikhov and V. N. Monakhov. *Boundary value problems in mechanics of nonhomogeneous fluids*, volume 22 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1990.

[4] H. Bahouri, J.-Y. Chemin and R. Danchin. *Fourier analysis and nonlinear partial differential equations*, Grundlehren der mathematischen Wissenschaften, **343**, Springer (2011).

[5] J.-M. Bony. Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires. *Ann. Sci. École Norm. Sup. (4)*, **14**(2):209–246, 1981.

[6] J.-Y. Chemin. Calcul paradifférentiel précisé et applications à des équations aux dérivées partielles non semi-linéaires. *Duke Math. J*, **56**(3):431–469, 1988.

[7] J.-Y. Chemin. Persistance de structures géométriques dans les fluides incompressibles bidimensionnels. *Ann. Sci. École Norm. Sup. (4)*, **26**(4):517–542, 1993.

[8] J.-Y. Chemin. Théorèmes d'unicité pour le système de Navier-Stokes tridimensionnel. *Journal d'Analyse Mathématique*, **77**(1):27–50, 1999.

[9] R. Danchin. Persistance de structures géométriques et limite non visqueuse pour les fluides incompressibles en dimension quelconque. *Bulletin de la Société Mathématique de France*, **127**(2):179–228, 1999.

[10] R. Danchin. Density-dependent incompressible viscous fluids in critical spaces. *Proceedings of the Royal Society of Edinburgh-A-Mathematics*, **133**:1311–1334, 2003.

[11] R. Danchin and P.B. Mucha. A Lagrangian approach for the incompressible Navier-Stokes equations with variable density. *Communications on Pure and Applied Mathematics*, **65**(10):1458–1480, 2012.

[12] R. Danchin and P. B. Mucha. Critical functional framework and maximal regularity in action on systems of incompressible flows. *Mém. Soc. Math. Fr. (N.S.)*, **143**:vii+151, 2015.

[13] R. Danchin and P. Zhang. Inhomogeneous Navier-Stokes equations in the half-space, with only bounded density. *J. Funct. Anal.*, **267**:2371–2436 (2014).
[14] R. Danchin and X. Zhang. Global persistence of geometrical structures for the Boussinesq equation with no diffusion. http://arxiv.org/abs/1603.07479. To appear in *Communications in Partial Differential Equations*.
[15] R. Di Perna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces, *Inventiones Mathematicae*, **98**:511–547 (1989).
[16] J. Huang, M. Paicu and P. Zhang. Global well-posedness of incompressible inhomogeneous fluid systems with bounded density or non-lipschitz velocity. *Arch. Ration. Mech. Anal.*, **209**(2):631–682, 2013.
[17] P.-G. Lemarié-Rieusset. *The Navier-Stokes problem in the 21st century*. CRC Press, Boca Raton, FL, 2016.
[18] X. Liao and Y. Liu. On the global regularity of three dimensional density patch for inhomogeneous incompressible viscous flow. 2016. http://arxiv.org/abs/1606.05395.
[19] X. Liao and P. Zhang. On the global regularity of the two-dimensional density patch for inhomogeneous incompressible viscous flow. *Arch. Ration. Mech. Anal.*, **220**(3):937–981, 2016.
[20] X. Liao and P. Zhang. Global regularities of two-dimensional density patch for inhomogeneous incompressible viscous flow with general density. 2016. http://arxiv.org/abs/1604.07922.
[21] P.-L. Lions. *Mathematical topics in fluid mechanics. Vol. 1*, volume 3 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press, Oxford University Press, New York, 1996. Incompressible models, Oxford Science Publications.
[22] A. Majda and A. Bertozzi. Vorticity and incompressible flow. Cambridge Texts in Applied Mathematics, **27**. Cambridge University Press, Cambridge, 2002.
[23] J. Simon. Nonhomogeneous viscous incompressible fluids: existence of velocity, density, and pressure. *SIAM J. Math. Anal.*, **21**(5):1093–1117, 1990.
[24] H. Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.

*E-mail address*: raphael.danchin@u-pec.fr and xin.zhang@univ-paris-est.fr

Université Paris-Est, LAMA (UMR 8050), UPEMLV, UPEC, CNRS, 61 avenue du Général de Gaulle, 94010 Créteil Cedex 10.