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On a multivariate renewal-reward process involving time delays and discounting: Applications to IBNR process and infinite server queues

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Abstract:

This paper considers a particular renewal-reward process with multivariate discounted rewards (inputs) where the arrival epochs are adjusted by adding some random delays. Then this accumulated reward can be regarded as multivariate discounted *Incurring But Not Reported* (IBNR) claims in actuarial science and some important quantities studied in queueing theory such as the number of customers in $G/G/\infty$ queues with correlated batch arrivals. We study the long term behavior of this process as well as its moments. Asymptotic expressions and bounds for the quantities of our interest, and also convergence result for the distribution of this process after renormalization, are studied, when interarrival times and time delays are light tailed. Next, assuming exponentially distributed delays, we derive some explicit and numerically feasible expressions for the limiting joint moments. In such case, for an infinite server queues with renewal arrival process, we obtain limiting results on the expectation of the workload, and the covariance of queue size and workload. Finally, some queueing theoretic applications are provided.

AMS 2000 subject classifications: Primary 60G50, 60K30, 62P05, 60K25.

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1. Introduction and notation

Many situations in which processes restart probabilistically at renewal instants and there are non-negative rewards associated with each renewal epoch, are well described by a multivariate renewal-reward process. For example, a multivariate reward function can be viewed as an accumulated cost from different types of properties or infrastructures caused by a single catastrophe event, which is of interest in actuarial science and reliability analysis. The asymptotic distribution and the asymptotic expansion for the covariance function of the rewards were studied by [19] and [1] who extended the result of [4] to multivariate case. In the context of actuarial science, much research about the aggregate discounted claims has been done on its moment under renewal claim arrival processes. For example, [12], [13], [14], and [15] analyzed the renewal process, and [28] looked at the dependent renewal process.

In this paper, we assume that there are time lags added to the original arrival times of renewal process. These delayed renewal epochs allow us to study the quantities related to infinite server queues with correlated batch arrivals and multivariate *Incurred But Not Reported* (IBNR) claims where there is a delay in reporting or payment for claims. Furthermore, rewards are accumulated at a discounted value. A direct application to some problems in infinite server queues includes the case, for example, when the bulk size random variable is multivariate (i.e. correlated) and the service time distribution is dependent on the type of input, in which case a multivariate reward function incorporating time delays up to time t (with zero discounting factor) is essentially the number of customers in the system up to time t . In the infinite server queues with multiple batch Markovian arrival streams, a time-dependent matrix joint generating function of the number of customers in the system was derived by [17]. For the univariate case, IBNR claim count with batch arrivals was considered by [8] and the total discounted IBNR claim amount was studied by [11]. For the multivariate case, [27] provided expressions for joint moments of multivariate IBNR claims which are computable recursively. For the number of IBNR claims, a direct relation to the number of customers in the infinite server queues with batch arrivals is well known as discussed in the literature, e.g. [10], [11], [24], [25], [26]. The transient behavior of a distribution of the number of customer in various multichannel bulk queues was studied in [5]. See also [3] for example.

Let us introduce the model more precisely. We shall suppose that the batch arrival process $\{N_t\}_{t \geq 0}$ is a renewal process with a sequence of independent and identically distributed (iid) positive continuous random variables (rv)s $(T_i)_{i \in \mathbb{N}}$ representing the arrival time of the i th batch with $T_0 \equiv 0$. Let $\tau_i = T_i - T_{i-1}$ be the interarrival time of the i th batch with a common probability density function (pdf) f , distribution F , Laplace transform $\mathcal{L}^\tau(u) = \mathbb{E}[e^{-u\tau_1}]$ for $u \geq 0$, and renewal function $t \mapsto m(t) := \mathbb{E}[N_t]$. Each batch arrival contains several (k) types of inputs which may simultaneously occur from the same renewal event (e.g. [19], [27]). Let us denote the j -type of input from the i th batch as $X_{i,j}$ where $\{(X_{i,1}, \dots, X_{i,k})\}_{i \in \mathbb{N}}$ is a sequence of iid random vectors. A vector for multivariate input variables is denoted as $X = (X_1, X_2, \dots, X_k)$. Here multivariate input values are assumed to be dependent on the occurrence time and/or the adjusted time by adding a random delay. This time delay for the j -type of input from i th batch is denoted by $L_{i,j}$ where $(L_{i,j})_{i \in \mathbb{N}}$ is a sequence of iid rvs with a common cumulative distribution function W_j , and such that $(L_{i,j})_{i,j \in \mathbb{N}}$ is a sequence of independent rvs. For the sake of simplicity let us assume a constant force of interest δ to discount input values to time 0, and define the following discounted compound delayed process

$$Z(t) = Z(t, \delta) = (Z_1(t), \dots, Z_k(t)), \quad t \geq 0, \quad (1)$$

where

$$Z_j(t) := \sum_{i=1}^{N_t} e^{-\delta(T_i + L_{i,j})} X_{i,j} \mathbb{1}_{\{T_i + L_{i,j} > t\}} = \sum_{i=1}^{\infty} e^{-\delta(T_i + L_{i,j})} X_{i,j} \mathbb{1}_{\{T_i \leq t < T_i + L_{i,j}\}}, \quad j \in \{1, \dots, k\}. \quad (2)$$

Here, we can interpret the process $\{Z(t)\}_{t \geq 0}$ in two different ways. The first one is related to actuarial science: we suppose that an aggregate claim amounts (or claim severities) $X_{i,j}$ in branch $j \in \{1, \dots, k\}$ of an insurance company is caused by the event arriving at time T_i . Instead of being dealt with immediately, they are (within a batch) subject to a delay $L_{i,j}$ before being treated. $Z_j(t)$ then represents the discounted total claim amounts of such incurred but not reported claims in branch j . The second one is related to queueing theory. Let us consider

a single queue containing batches of k types of customers in an infinite-server model. Here customers arrive according to a renewal process $\{N_t\}_{t \geq 0}$ with corresponding arrival times $(T_i)_{i \in \mathbb{N}}$. At each arrival instant T_i a batch of correlated customers $(X_{i,1}, \dots, X_{i,k})$ arrive in the system, with $X_{i,j} \in \mathbb{N}$. For each customer of class $j \in \{1, \dots, k\}$ (of which number is $X_{i,j}$) the service time $L_{i,j}$ is the same. A random sequence $(X_{i,1}, \dots, X_{i,k})$ for $i \in \mathbb{N}$, is iid and distributed as (X_1, \dots, X_k) . The service times $(L_{i,j})_{i \in \mathbb{N}, j=1, \dots, k}$ are thus assumed to be independent, although $L_{i,1}, \dots, L_{i,k}$ possibly have different distributions, i.e. service times are different according to the type of customer class. For example, if $\delta = 0$, the model is reduced to that of a $G/G/\infty$ queue with multiple types of customer classes in a batch. As an illustration, let us look at the particular case where (X_1, \dots, X_k) follows a multinomial distribution with parameters $M \in \mathbb{N}$ and probability vector (p_1, \dots, p_k) where $p_j \geq 0$ and $\sum_{j=1}^k p_j = 1$. This models a situation where at each instant T_i exactly M customers arrive, each of which belongs to class j with probability p_j . X_j is the number of customers of class j in this batch. See Figure 1. The simplest scenario is when $M = 1$, where each customer arrives according to renewal

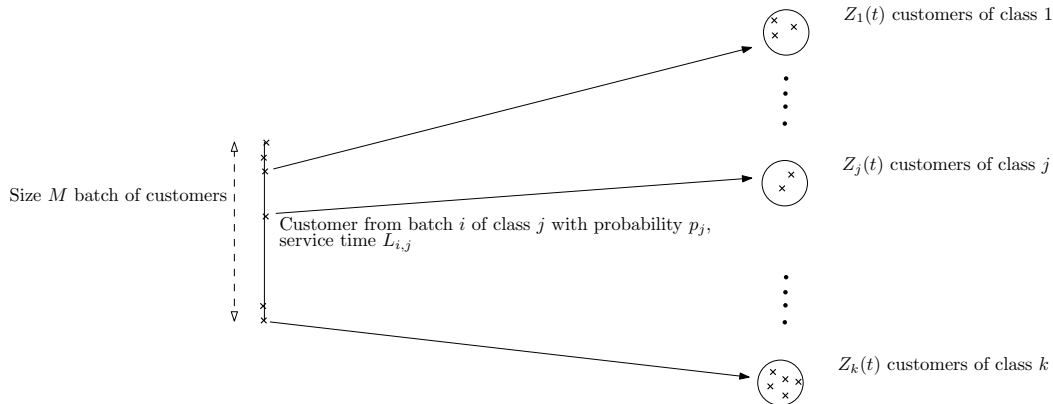


FIG 1. The $G/G/\infty$ queue with multinomial distributed classes batches (X_1, \dots, X_k) .

process $\{N_t\}_{t \geq 0}$, and belongs to class j with probability p_j . Because of both those two, either actuarial or queueing, interpretations, we will refer the $L_{i,j}$'s as either "delay" or "service" times, without ambiguity.

We note that it is usually difficult to derive a distribution for this compound renewal process $Z(t)$ since there is no concrete representation for an inversion of the complicated moment generating function (mgf) for this quantity in a general arrival process $\{N_t\}_{t \geq 0}$. In this sense, it is appealing to study the long term behavior of the process in terms of its moment and distribution. From [27], explicit expressions for the joint moments of $Z(t) = (Z_1(t), \dots, Z_k(t))$ are computable recursively. However, an analytic expression of the lower moment which appears in its integral term, is required for the calculation of the higher moment. Also, it requires an explicit form of the renewal density $m(t)$ for evaluation of this moment. Therefore, our objective here is to develop simpler approximation methods such as asymptotics and bound results for the joint moments of $Z(t)$. To the best of our knowledge, these kinds of approximation approaches have never been developed in the analysis of a multivariate renewal-reward process with discounted inputs and time delays. Also, a relationship between multivariate discounted IBNR claim process and quantities studied in infinite server queues with correlated batch arrivals and a discounting factor is firstly exploited in this paper. Moreover, we shall also consider the case with exponential time delays in a general arrival process and provide

asymptotic results for the joint moments. In this case, when interarrival times are light tailed and single input case, we are able to quantify the approximation precision by providing many terms for the asymptotics for the first order moment of our process. We note that this approach was previously found in [4, Lemma 1] where 2-term asymptotics for a general renewal reward process without delays was provided, see also [19] and [1] for an expansion of the covariance. In particular, some asymptotic results regarding queueing theoretic applications such as the workload in a $G/M/\infty$ queue, are obtained.

In most cases in this paper, we suppose that the discounted factor δ is real and non negative because of its discounting role. However it has to be pointed out that, mathematically speaking, Definitions (1) and (2) can in some cases be extended to some *complex* δ , as will be the case in Section 5 where $\delta \in \mathbb{C}$ is needed for technical purposes. It will also be convenient to define the process $\tilde{Z}(t) = \tilde{Z}(t, \delta) = (\tilde{Z}_1(t), \dots, \tilde{Z}_k(t)) = e^{\delta t} Z(t)$, i.e.

$$\tilde{Z}_j(t) = \sum_{i=1}^{N_t} e^{\delta(t-T_i-L_{i,j})} X_{i,j} \mathbb{1}_{\{T_i+L_{i,j}>t\}}, \quad j \in \{1, \dots, k\}. \quad (3)$$

Although $\tilde{Z}(t)$ does not have a direct actuarial or queueing interpretation, it will turn out that most results will concern this process rather than $Z(t)$.

Notation. For $n = (n_1, \dots, n_k) \in \mathbb{N}^k$, the n th joint moments for $Z(t)$ and its mgf are respectively denoted as

$$\begin{aligned} M_n(t) &= \mathbb{E} \left[\prod_{j=1}^k Z_j^{n_j}(t) \right], \quad t \geq 0, \quad n = (n_1, \dots, n_k) \in \mathbb{N}^k, \\ \psi(s, t) &= \mathbb{E} \left[e^{\langle s, Z(t) \rangle} \right], \quad s = (s_1, \dots, s_k) \in \mathbb{R}^k, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the euclidian scalar product. For notational convenience, we let, for all $n = (n_1, \dots, n_k) \in \mathbb{N}^k$ and $t \geq 0$,

$$\eta_n := \sum_{i=1}^k n_i,$$

$$\tilde{M}_n(t) := e^{\eta_n \delta t} M_n(t) = \mathbb{E} \left[\prod_{j=1}^k \tilde{Z}_j^{n_j}(t) \right], \quad (4)$$

$$\tilde{\psi}(s, t) = \mathbb{E} \left[e^{\langle s, \tilde{Z}(t) \rangle} \right], \quad s = (s_1, \dots, s_k) \in \mathbb{R}^k. \quad (5)$$

We define the natural partial order on set \mathbb{N}^k as follows. We say that two vectors ℓ and n in \mathbb{N}^k verify $\ell < n$ if $\ell_i \leq n_i$ for all $i = 1, \dots, k$ and $\ell_i < n_i$ for (at least) an i , i.e. $\eta_n > \eta_\ell$. Let us introduce, for all $n \in \mathbb{N}^k$,

$$C_{\ell, n} := \{j = 1, \dots, k \mid \ell_j < n_j\} \subset \{1, \dots, k\}.$$

We will denote by $n(i) \in \mathbb{N}^k$ the vector of which j th entry is $\delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta function.

It is convenient to introduce function $t \mapsto \varphi_{\ell, n}(t)$ for $\ell < n$:

$$\varphi_{\ell, n}(t) = \mathbb{E} \left[e^{(\eta_n - \eta_\ell) \delta (t - \tau_1)} \tilde{M}_\ell(t - \tau_1) \prod_{j \in C_{\ell, n}} \bar{\omega}_{(n_j - \ell_j) \delta, j}(t - \tau_1) \cdot \mathbb{1}_{[\tau_1 < t]} \right], \quad (6)$$

where

$$\bar{\omega}_{\delta,i}(t) = \int_t^\infty e^{-\delta y} dW_i(y). \quad (7)$$

Following [27], we define $\tilde{b}_n(t)$ by

$$\tilde{b}_n(t) = \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] \varphi_{\ell,n}(t). \quad (8)$$

Throughout the paper, $\mathcal{E}(\mu)$ denotes an exponential distribution with a mean $1/\mu$. We denote $|A|$ as the cardinal of A for any finite set A .

We assume that vector X admits joint moments of all order. Lastly, we denote assumptions **(A1)**, **(A1')** and **(A2)** by:

- (A1)** Density $f(\cdot)$ of interarrival time τ_1 is bounded,
- (A1')** interarrival time τ_1 is light tailed: $\exists R > 0$, $\int_0^\infty e^{Rx} dF(x) = \mathbb{E}[e^{R\tau_1}] < +\infty$,
- (A2)** $\exists M > 0$ such that $\forall j = 1, \dots, k$, $0 \leq X_j \leq M$ a.s.

An important consequence of **(A1)** is the following result, of which proof is given at the beginning of Section 7.

Lemma 1. *If **(A1)** holds then the associated renewal function $m : t \geq 0 \mapsto \mathbb{E}[N_t]$ admits a density $u(t)$, and this latter verifies*

$$u(t) = \frac{d}{dt} m(t) = \sum_{j=0}^{\infty} f^{*(j)}(t). \quad (9)$$

Besides, this density is upper bounded: There exists $C > 0$ such that

$$u(t) \leq C, \quad \forall t \geq 0. \quad (10)$$

Structure of paper. For ease of presentation, all main results are given in Sections 2, 3, 4 and 5, and all the proofs in Section 7. Section 2 recalls the results from [27] that are used throughout the paper, with some immediate applications when interarrivals are exponentially distributed. Section 3 addresses the general case where interarrival and delays have a general distribution, in which case one proves convergence of moments of $\tilde{Z}(t)$ (Proposition 4) as well as convergence in distribution when **(A1)** and **(A2)** holds (Theorem 8). Section 4 addresses the case where delays are exponentially distributed (Theorem 10). Particular focus is made in Section 5 on the case $k = 1$ with exponentially distributed delays: we first give an asymptotic expansion for $\tilde{M}_1(t)$ as $t \rightarrow \infty$ when **(A1')** holds (Theorem 14). In the subsequent subsection, this result is utilized to obtain asymptotic moments for the workload of the $G/M/\infty$ queue when **(A1)** and **(A1')** hold (Theorem 18). In both those latter sections, we compare the results to the existing queueing literature, particularly those from Takács [21]. Finally, in Section 6, an attempt is made to put some emphasis on the fact that the generality of the model yields interesting applications.

2. Renewal equations: General and Exponential interarrival times

The aim of this section is to briefly review the results obtained in [27] that will be the starting point of most of the results in the present paper, and to recover some particular results when claims arrive according to a Poisson process. Following notation in Section 3.3 of [27], we let for all $t \geq 0$ and $s = (s_1, \dots, s_k) \in \mathbb{R}^k$,

$$\begin{aligned} M_{t,X}^*(s) &:= \mathbb{E} \left[\exp \left(\sum_{j=1}^k s_j e^{-\delta L_{ij} X_{ij}} \mathbb{1}_{[L_{ij} > t]} \right) \right] \\ &= \int_0^\infty \cdots \int_0^\infty \mathbb{E} \left[\exp \left(\sum_{j=1}^k s_j e^{-\delta v_j} X_{ij} \mathbb{1}_{[v_j > t]} \right) \right] dW_1(v_1) \cdots dW_k(v_k). \end{aligned}$$

From Theorem 3 in [27], The mgf $\psi(s, t)$ of $Z(t)$ satisfies

$$\psi(s, t) = \mathbb{E} \left[\prod_{i=1}^{N_t} M_{t,X}^*(e^{-\delta T_i} s) \right].$$

Also, $\tilde{M}_n(t)$ defined in (4) satisfies

$$\begin{aligned} \tilde{M}_{n(i)}(t) &= \mathbb{E}[X_i] \cdot \int_0^t e^{\delta(t-y)} \bar{\omega}_{\delta,i}(t-y) dm(y), \quad i = 1, \dots, k, \\ \tilde{M}_n(t) &= \tilde{b}_n(t) + \tilde{M}_n \star F(t), \quad t \geq 0, \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}, \end{aligned} \quad (11)$$

where $\bar{\omega}_{\delta,i}(t)$ and $\tilde{b}_n(t)$ are respectively given by (7) and (8). It is standard that the solution to (11) is given by $\tilde{M}_n(t) = \int_0^t \tilde{b}_n(t-y) dm(y)$ for all $t \geq 0$. However, as pointed out in [27], this solution is hardly explicit in practice because $\tilde{b}_n(\cdot)$ depends on $\tilde{M}_\ell(\cdot)$, $\ell < n$. So, the expression for $\tilde{M}_n(t)$ at time t depends on the whole trajectory of $\tilde{M}_\ell(y)$, $\ell < n$, for $y \in [0, t]$. Not only that, but the renewal function $t \mapsto m(t)$ is not always explicit.

Corollary 2. *The mgf $\tilde{\psi}(s, t)$ of $\tilde{Z}(t)$ satisfies the integral-renewal equation*

$$\tilde{\psi}(s, t) = \bar{F}(t) + \int_0^t M_{t-y,X}^*(e^{\delta(t-y)} s) \tilde{\psi}(s, t-y) dF(y), \quad t \geq 0, \quad (12)$$

for all $s \in \mathbb{R}^k$.

Proof. The renewal equation (12) is obtained thanks to relation $\tilde{\psi}(s, t) = \psi(e^{\delta t} s, t)$ and by using (2) as well as a classical renewal argument. \square

An interesting application of the above corollary is that a closed form expression for $\tilde{\psi}(s, t)$ when arrivals occur according to a Poisson process.

Proposition 3 (Poisson arrival and general delay). *If $\tau_1 \sim \mathcal{E}(\lambda)$ then one has the following expression*

$$\tilde{\psi}(s, t) = \exp \left[\lambda \int_0^t \left(M_{v,X}^*(e^{\delta v} s) - 1 \right) dv \right], \quad t \geq 0, \quad s \in \mathbb{R}^k. \quad (13)$$

Then, the mgf of $Z(t)$ is obtained explicitly by $\psi(s, t) = \tilde{\psi}(e^{-\delta t} s, t)$.

Proof. When $\tau_1 \sim \mathcal{E}(\lambda)$, renewal equation (12) reads, up to a change of variable $y := t - y$ in the integral,

$$\tilde{\psi}(s, t) = e^{-\lambda t} + \int_0^t M_{y,X}^*(e^{\delta y} s) \tilde{\psi}(s, y) \lambda e^{-\lambda(t-y)} dy, \quad t \geq 0,$$

which, derived with respect to t , yields the linear differential equation

$$\partial_t \tilde{\psi}(s, t) = \lambda \left[-1 + M_{t,X}^*(e^{\delta t} s) \right] \tilde{\psi}(s, t)$$

of which solution is given by (13). Note that the above differential equation is also available in a similar form in [17, Theorem 3.1]. \square

Two remarks are to be deduced from Proposition 3. First, since the density function of the exponential distribution is upper bounded, Condition (A1) is fulfilled, and thus one has from upcoming Theorem 8, Section 3, that $\tilde{Z}(t)$ converges in distribution towards some light tailed random vector \mathcal{Z}_∞ . Thus, it is immediate from (13) that the mgf of \mathcal{Z}_∞ is, when $\tau_1 \sim \mathcal{E}(\lambda)$, given by

$$\mathbb{E} \left[e^{\langle s, \mathcal{Z}_\infty \rangle} \right] = \lim_{t \rightarrow \infty} \tilde{\psi}(s, t) = \exp \left[\lambda \int_0^\infty \left(M_{v,X}^*(e^{\delta v} s) - 1 \right) dv \right], \quad s \in \mathbb{R}^k.$$

Second, one is able to recover some well known result in the $M/G/\infty$ queue by setting $\delta = 0$. For example, when $k = 1$ and $X = X_1$, one computes that $M_{t,X}^*(s) = 1 + (e^s - 1)\bar{W}(t)$, and (13) reduces to

$$\tilde{\psi}(s, t) = \psi(s, t) = \exp \left[\lambda \int_0^t \bar{W}(v) dv \cdot [e^s - 1] \right]$$

recovering that the distribution of the number of customers in an infinite server queue with Poisson arrivals of intensity λ is Poisson distributed with parameter $\lambda \int_0^t \bar{W}(v) dv$ at time t , see Theorem 1 p.160 of [21]. When $\delta = 0$, (13) in Proposition 3 is similar to the results obtained in Section 3.1 of [16], concerning infinite server queues with Poisson arrivals.

3. General results: Convergence of joint moments and distribution

We are interested in the general limiting behaviour of processes $Z(t)$ and $\tilde{Z}(t)$ when arrivals and delays have a general distribution. It may be difficult to compute its distribution in all generality, however some information may be obtained if we add some extra assumption on the arrival process $\{N_t\}_{t \geq 0}$. Our first immediate result is convergence of joint moments of $\tilde{Z}(t)$:

Proposition 4. *One finds the following asymptotic result for the joint moments of $Z(t)$, for all $n \in \mathbb{N}^k$:*

$$\lim_{t \rightarrow \infty} \tilde{M}_n(t) = \chi_n \iff M_n(t) \sim \chi_n e^{-\eta_n \delta t}, \quad t \rightarrow \infty,$$

where

$$0 \leq \chi_n := \frac{\int_0^\infty \tilde{b}_n(t) dt}{\mathbb{E}[\tau_1]} < +\infty, \tag{14}$$

and $\tilde{b}_n(t)$ is given by (8).

Proof. See Section 7.1. □

A direct consequence of Proposition 4 when $k = 1$ with [27, Example 3] yields the result for the first moment in the following corollary.

Corollary 5 (Single type of input, arbitrary time delays). *The mean of $\tilde{Z}_1(t) = \tilde{Z}(t, \delta)$ in (3) for $k = 1$ with arbitrary time lag distribution L is asymptotically obtained as*

$$\mathbb{E}[\tilde{Z}_1(t)] \sim \chi_1, \quad t \rightarrow \infty,$$

where

$$\chi_1 = \frac{\mathbb{E}[X_1]\mathbb{E}[L]\tilde{w}_{1,1}(\delta)}{\mathbb{E}[\tau_1]}, \tag{15}$$

and $\tilde{w}_{1,1}(\delta) = \int_0^\infty e^{-\delta x} \overline{W}_1(x) dx / \mathbb{E}[L]$. This is a generalization of Corollary 3 in [27] in which it is assumed that $X_i = 1$ and $\delta = 0$.

Remark 6 (Little’s law revisited). Expression (15) gives an interesting interpretation in a queueing context. Let us suppose here (without loss of generality) that $X_1 = 1$ (i.e. customers do not arrive in batches). (15) reads

$$\lim_{t \rightarrow \infty} \mathbb{E}[\tilde{Z}(t, \delta)] = \chi_1 = \frac{\mathbb{E}[L]\tilde{w}_{1,1}(\delta)}{\mathbb{E}[\tau_1]}. \tag{16}$$

When $\delta = 0$, $\tilde{Z}(t, \delta) = \tilde{Z}(t, 0)$ is the number of customers at time t in an infinite server queues; In that case $\tilde{w}_{1,1}(\delta) = 1$ and (16) is just a rephrasing of Little’s law which says that the limiting expected number of customers in the queue is equal to the arrival rate multiplied by the mean service time. When $\delta > 0$, we notice that $\mathbb{E}[L_1]\tilde{w}_{1,1}(\delta) = \Pr(L > E_\delta) / \delta$ where $E_\delta \sim \mathcal{E}(\delta)$ is a r.v. independent from everything, so that (16) reads

$$\lim_{t \rightarrow \infty} \mathbb{E}[\tilde{Z}(t, \delta)] = \frac{1}{\mathbb{E}[\tau_1]} \frac{\Pr(L > E_\delta)}{\delta} = \frac{1}{\mathbb{E}[\tau_1]} \Pr(L > E_\delta) \mathbb{E}[E_\delta]. \tag{17}$$

The asymptotic expression in (17) implies that the limiting expected number of customers of which residual service time is no more than horizon $E_\delta \sim \mathcal{E}(\delta)$ is equal to the arrival rate multiplied by the expected horizon time, multiplied by the proportion of customers of which service time did exceed this horizon E_δ . So, (17) can be seen as a generalization of Little’s Law in the $G/G/\infty$ context.

In the following we address the limiting behaviour in distribution of process $\{\tilde{Z}(t)\}_{t \geq 0}$ under (A1).

First of all, one sees, looking back at Proposition 4, that coefficients χ_n , $n \in \mathbb{N}^k$ are in general not directly computable, as the function $t \mapsto \tilde{b}_n(t)$ in the integral (14) does not have an easy expression, and are defined recursively in the function of $t \mapsto \tilde{M}_\ell(t)$, $\ell < n$. We thus provide in the following easily computable bounds for the χ_n ’s and a uniform upper bound in t for $\tilde{M}_n(t)$ if we impose that (A1) holds.

Proposition 7 (Upper bounds for the joint moments). *Let us suppose that (A1) holds. One has the following bounds for all $n \in \mathbb{N}^k$:*

$$\chi_n \leq \frac{1}{C\mathbb{E}(\tau_1)} R_n, \tag{18}$$

$$\tilde{M}_n(t) \leq R_n, \quad \forall t \geq 0, \tag{19}$$

where $(R_n)_{n \in \mathbb{N}^k}$ is defined recursively by

$$\begin{aligned} R_{n(i)} &= C \mathbb{E}[X_i] \delta^{-1} \{1 - \mathbb{E}[e^{-\delta L_i}]\}, \quad i = 1, \dots, k, \\ R_n &= C \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] \min_{i \in C_{\ell, n}} \mathbb{E}[L_i] \cdot R_\ell, \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}, \end{aligned} \quad (20)$$

Here, the constant C is the upper bound for renewal density $u(t)$ in Lemma 1.

Proof. See Section 7.2. □

Proposition 7 is useful for two reasons. First, we remark that coefficients R_n , $n \in \mathbb{N}^k$, in (20) can be easily computed because R_n is a linear function of the R_ℓ , $\ell < n$, and only involves the joint moments of $X = (X_1, \dots, X_k)$, the Laplace transform of the L_1, \dots, L_k as well as their expectations. So, simple bounds are available, which is useful since it is not possible in general to compute the distribution (or even moments) of the process. Second, Proposition 4 reads that $\tilde{M}_n(t)$ converges towards χ_n . Since $\tilde{M}_n(t)$ is the joint moments of \mathbb{R}^k valued process $\{\tilde{Z}(t)\}_{t \geq 0}$, this suggests in turn that this process converges in distribution. As convergence of moments does not always implies convergence in distribution, we give some sufficient conditions such that this latter holds, and we prove it thanks to the bounds obtained in Proposition 7.

Theorem 8. *Let us suppose that (A1) and (A2) hold. Then one has the result of convergence in distribution for $\tilde{Z}(t)$:*

$$\tilde{Z}(t) = e^{\delta t} Z(t) \xrightarrow{\mathcal{D}} \mathcal{Z}_\infty, \quad t \rightarrow \infty,$$

where $\mathcal{Z}_\infty = (\mathcal{Z}_{\infty,1}, \dots, \mathcal{Z}_{\infty,k}) = \mathcal{Z}_\infty(\delta)$ is a light tailed vector valued rv with the joint moments

$$\mathbb{E} \left[\prod_{i=1}^k \mathcal{Z}_{\infty,i}^{n_i} \right] = \chi_n = \chi_n(\delta)$$

given by (14) for $n \in \mathbb{N}^k$.

Proof. See Section 7.3. □

4. Joint moments with exponential delays

Let us note that Theorem 8 holds for general interarrival times τ_i that satisfy (A1), and general time delays L_j 's. The boundedness assumption (A2) is relatively not too restrictive and is fulfilled in many cases, for example when, in a queueing scenario, only one customer arrives at each time T_i . The aim of this subsection is to prove that the χ_n 's are explicit when the L_j are exponentially distributed. We suppose for simplicity that all L_j 's for $j = 1, \dots, k$, have same distribution $\mathcal{E}(\mu)$, for some $\mu > 0$. Note that we may obtain similar results in the following for more general cases such as a mixture or a combination of exponential distributions, but the following expressions would only be more complicated.

Some notations are necessary in this section. Let $\mathcal{L}_n^M(u)$ and $\mathcal{L}_n^b(u)$ for $u \geq 0$ and $n \in \mathbb{N}^k$, be the Laplace transforms of $\tilde{M}_n(\cdot)$ and $\tilde{b}_n(\cdot)$ respectively

$$\mathcal{L}_n^M(u) := \int_0^\infty e^{-uy} \tilde{M}_n(y) dy, \quad \mathcal{L}_n^b(u) := \int_0^\infty e^{-uy} \tilde{b}_n(y) dy.$$

Note that these Laplace transforms exist (i.e. the integrals converge) respectively when $u > 0$ and $u \geq 0$ since $\tilde{M}_n(y)$ converges to some finite limit χ_n as $y \rightarrow \infty$, and $\tilde{b}_n(\cdot)$ is integrable (as proved in Proposition 4). The following lemma gives a recursive expression of $\mathcal{L}_n^b(u)$.

Lemma 9. *When time delays L_j are $\mathcal{E}(\mu)$ distributed, the Laplace transform of $\tilde{b}_n(\cdot)$ in (8) is obtained as*

$$\mathcal{L}_{n(i)}^b(u) = \mathbb{E}[X_i] \frac{\mu}{(\mu + \delta)(\mu + u)} \mathcal{L}^\tau(u), \quad i = 1, \dots, k, \quad (21)$$

and

$$\mathcal{L}_n^b(u) = B_{\mathbf{0},n} \frac{\mathcal{L}^\tau(u)}{u + |C_{\mathbf{0},n}|\mu} + \sum_{\mathbf{0} < \ell < n} B_{\ell,n} \frac{\mathcal{L}^\tau(u)}{1 - \mathcal{L}^\tau(u + |C_{\ell,n}|\mu)} \mathcal{L}_n^b(u + |C_{\ell,n}|\mu), \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}, \quad (22)$$

where $\mathbf{0}$ is a zero vector in \mathbb{N}^k ,

$$B_{\ell,n} := \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] \prod_{j \in C_{\ell,n}} \frac{\mu}{\mu + (n_j - \ell_j)\delta}, \quad (23)$$

and we recall that $C_{\ell,n} = \{j = 1, \dots, k \mid \ell_j < n_j\} \subset \{1, \dots, k\}$.

Proof. See Section 7.4. □

The following shows that the χ_n 's can be computed in function of coefficients $(D_\ell(j))_{j \in \mathbb{N}} = (\mathcal{L}_n^b(j\mu))_{j \in \mathbb{N}}$ which are defined recursively.

Theorem 10. *Let us denote $D_n(j) := \mathcal{L}_n^b(j\mu)$ for $j \in \mathbb{N}$ and $n \in \mathbb{N}^k$. When time delays L_j are $\mathcal{E}(\mu)$ distributed, the joint moments $\chi_n = \chi_n(\delta)$ for $n \in \mathbb{N}^k$ of $\mathcal{Z}_\infty = \mathcal{Z}_\infty(\delta)$ (the limiting distribution of $e^{\delta t} Z(t)$), are given by*

$$\chi_{n(i)} = \frac{\mathbb{E}[X_i]}{\mathbb{E}[\tau_1]} \left(\frac{1}{\mu + \delta} \right), \quad i = 1, \dots, k, \quad (24)$$

$$\chi_n = \frac{1}{\mathbb{E}[\tau_1]} \left(B_{\mathbf{0},n} \frac{1}{|C_{\mathbf{0},n}|\mu} + \sum_{\mathbf{0} < \ell < n} B_{\ell,n} \frac{1}{1 - \mathcal{L}^\tau(|C_{\ell,n}|\mu)} D_\ell(|C_{\ell,n}|) \right), \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}, \quad (25)$$

where $D_n(j)$'s for $j \in \mathbb{N}$ and $n \in \mathbb{N}^k$ are obtained recursively as:

$$D_{n(i)}(j) = \mathbb{E}[X_i] \frac{\mu}{(\mu + \delta)([j + 1]\mu)} \mathcal{L}^\tau(j\mu), \quad i = 1, \dots, k, \quad (26)$$

$$D_n(j) = B_{\mathbf{0},n} \frac{\mathcal{L}^\tau(j\mu)}{[j + |C_{\mathbf{0},n}|\mu]} + \sum_{\mathbf{0} < \ell < n} B_{\ell,n} \frac{\mathcal{L}^\tau(j\mu)}{1 - \mathcal{L}^\tau([j + |C_{\ell,n}|\mu])} D_\ell([j + |C_{\ell,n}|]), \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}, \quad (27)$$

with $B_{\ell,n}$ in (23).

Proof. From (14), using (21) and (22) when $u = 0$, we find (24) and (25) respectively. In addition, (26) and (27) are obtainable by setting $u = j\mu$ in (21) and (22) respectively. □

We remark that a close look at (25) and (27) reveals that computation of the *infinite* sequences $(D_\ell(j))_{j \in \mathbb{N}}$ for all $\ell < n$ is not needed to obtain χ_n . Since $|C_{\ell,n}|$ is bounded by k , it is not hard to see that one needs to compute (recursively) $D_\ell(j)$ for $\ell < n$ and for $j \leq k\eta_n$ (i.e. only for a finite number of j 's). Moreover, the values of $D_n(j)$ may be stored in memory while computing the successive χ_n as η_n increases, and thus one does not need to recompute them each time. Hence the algorithm (25) is relatively not too costly. Some special cases for $k = 1$ and $k = 2$ are given in the following.

Corollary 11 (Single type of input, exponential time delays). *The r -th moment of $Z_1(t)$ in (2) for $k = 1$ with exponential delay is asymptotically obtained as*

$$\mathbb{E}[Z_1^r(t)] \sim \chi_r e^{-r\delta t}, \quad t \rightarrow \infty, \quad r \in \mathbb{N},$$

where

$$\chi_1 = \frac{\mathbb{E}[X_1]}{\mathbb{E}[\tau_1]} \left(\frac{1}{\mu + \delta} \right), \quad (28)$$

and

$$\chi_r = \frac{1}{\mathbb{E}[\tau_1]} \left(\mathbb{E}[X_1^r] \frac{1}{\mu + r\delta} + \sum_{\ell=1}^{r-1} \binom{r}{\ell} \mathbb{E}[X_1^{r-\ell}] \frac{\mu}{\mu + (r-\ell)\delta} \frac{D_\ell(1)}{1 - \mathcal{L}^\tau(\mu)} \right), \quad r = 2, 3, \dots, \quad (29)$$

and $D_\ell(1)$ recursively available from the formulas (26) and (27) respectively given by

$$D_1(j) = \mathbb{E}[X_1] \frac{\mu}{(\mu + \delta)([j+1]\mu)} \mathcal{L}^\tau(j\mu)$$

and

$$D_n(j) = \mathbb{E}[X_1^n] \frac{\mu}{\mu + n\delta} \frac{\mathcal{L}^\tau(j\mu)}{[j+1]\mu} + \sum_{\ell=1}^{n-1} \binom{n}{\ell} \mathbb{E}[X_1^{n-\ell}] \frac{\mu}{\mu + (n-\ell)\delta} \frac{\mathcal{L}^\tau(j\mu)}{1 - \mathcal{L}^\tau([j+1]\mu)} D_\ell(j+1), \quad n = 2, 3, \dots$$

Proof. When $n(1) = 1$ and $n(i) = 0$ for $i \neq 1$ together with $\eta_n = r$, $\eta_\ell = \ell$, and $|C_{\ell,n}| = 1$, from Proposition 4 and Theorem 10, the result follows. \square

Remark 12. When $\delta = 0$ and $X_1 = 1$, the model in Corollary 11 reduces to the classical $G/M/\infty$ queue, which was extensively studied by Takács [21, Chapter 3, Section 3]. More precisely, the results in this corollary are to be compared to [21, Theorem 2 p.166], [18, Theorem 2] and [20, Corollary of Theorem 1] where the approach is different and the distribution of the asymptotic queue level is derived but requires an infinite sum involving so called binomial moments.

Next, we compute the covariance of $Z_1(t)$ and $Z_2(t)$ when $k = 2$. We thus let $n = (n_1, n_2) = (1, 1)$ (i.e. $\ell = (\ell_1, \ell_2) \in \{(0, 0), (0, 1), (1, 0)\}$). From (8) and (6), we have

$$\begin{aligned} \tilde{b}_n(t) &= \sum_{\ell_1, \ell_2 \setminus (\ell_1, \ell_2) < (n_1, n_2)} \binom{n_1}{\ell_1} \binom{n_2}{\ell_2} \mathbb{E} \left[\prod_{j=1}^2 X_j^{n_j - \ell_j} \right] \varphi_{\ell,n}(t) \\ &= \mathbb{E}[X_1 X_2] \varphi_{(0,0),n}(t) + \mathbb{E}[X_1] \varphi_{(0,1),n}(t) + \mathbb{E}[X_2] \varphi_{(1,0),n}(t), \end{aligned} \quad (30)$$

where $\varphi_{(0,0),n}(t) = \mathbb{E}[e^{2\delta(t-\tau_1)}\bar{\omega}_{\delta,1}(t-\tau_1)\bar{\omega}_{\delta,2}(t-\tau_1)\mathbb{1}_{\{\tau_1 < t\}}]$ because of $\tilde{M}_{(0,0)}(t-\tau_1) = 1$, $\varphi_{(0,1),n}(t) = \mathbb{E}[e^{\delta(t-\tau_1)}\tilde{M}_{(0,1)}(t-\tau_1)\bar{\omega}_{\delta,1}(t-\tau_1)\mathbb{1}_{\{\tau_1 < t\}}]$, and $\varphi_{(1,0),n}(t) = \mathbb{E}[e^{\delta(t-\tau_1)}\tilde{M}_{(1,0)}(t-\tau_1)\bar{\omega}_{\delta,2}(t-\tau_1)\mathbb{1}_{\{\tau_1 < t\}}]$. As shown previously, (30) is simplified when L_i for $i = 1, 2$, is exponentially distributed. In this case, the joint expectation and the covariance of $Z_1(t)$ and $Z_2(t)$ are presented in the following.

Corollary 13 (Two types of inputs, exponential time delays). *The joint mean of two types of $Z_1(t)$ and $Z_2(t)$ in (2) where the time delay of type- i input L_i for $i = 1, 2$ is $\mathcal{E}(\mu)$ distributed, is asymptotically given by*

$$\mathbb{E}[Z_1(t)Z_2(t)] \sim \chi_n e^{-2\delta t}, \quad t \rightarrow \infty,$$

where, $n = (1, 1)$, and

$$\chi_n = \frac{1}{\mathbb{E}[\tau_1]} \frac{\mu}{(\mu + \delta)^2} \left[\frac{\mathbb{E}[X_1 X_2]}{2} + \frac{\mathbb{E}[X_1]\mathbb{E}[X_2]}{2} \frac{\mathcal{L}^\tau(\mu)}{1 - \mathcal{L}^\tau(\mu)} \right]. \quad (31)$$

Consequently, the covariance is given by

$$\text{Cov}[Z_1(t), Z_2(t)] \sim \xi_n e^{-2\delta t}, \quad t \rightarrow \infty,$$

where $\xi_n = \chi_n - \frac{\mathbb{E}[X_1]\mathbb{E}[X_2]}{\mathbb{E}[\tau_1]^2(\mu+\delta)^2}$ with χ_n given in (31).

Proof. From Theorem 10 when $n = (n_1, n_2) = (1, 1)$ (i.e $|C_\ell| = 1$ when $\ell = (\ell_1, \ell_2) \in \{(1, 0), (0, 1)\}$), we have

$$\chi_n = \frac{1}{\mathbb{E}[\tau_1]} \left[B_{(0,0),(1,1)} \frac{1}{2\mu} + B_{(1,0),(1,1)} \frac{D_{(1,0)}(1)}{1 - \mathcal{L}^\tau(\mu)} + B_{(0,1),(1,1)} \frac{D_{(0,1)}(1)}{1 - \mathcal{L}^\tau(\mu)} \right]. \quad (32)$$

But from (23), B 's are given by

$$B_{(0,0),(1,1)} = \mathbb{E}[X_1 X_2] \left(\frac{\mu}{\mu + \delta} \right)^2, \quad B_{(1,0),(1,1)} = \mathbb{E}[X_2] \frac{\mu}{\mu + \delta}, \quad B_{(0,1),(1,1)} = \mathbb{E}[X_1] \frac{\mu}{\mu + \delta}.$$

Also, $D_{n(i)}(1)$ for $i = 1, 2$ is available from (26) as $D_{n(i)}(1) = \mathbb{E}[X_i] \frac{\mu}{(\mu+\delta)(2\mu)} \mathcal{L}^\tau(\mu)$. Combining results given above, (32) is expressed thanks to (31) and the asymptotics of $\mathbb{E}(Z_1(t))$ and $\mathbb{E}(Z_2(t))$ obtained in Corollary 11. \square

It is noted that the form given in Theorem 3 of [27] was not suitable to derive the asymptotic behavior of $Z_1(t)$. A comment therein reveals only that this quantity is asymptotically close to zero. Hence Corollary 11 is useful for calculating higher moments of $Z_1(t)$ in any order for a large t when time delays are exponentially distributed.

An interesting consequence of Lemma 9 is that the expression of $\mathcal{L}_n^M(u)$ for all $u > 0$ can be obtained recursively thanks to relation $\mathcal{L}_n^M(u) = \frac{\mathcal{L}_n^b(u)}{1 - \mathcal{L}^\tau(u)}$ (which stems from the renewal equation (11), and was used in the proof of the lemma, see Section 7.4) as well as the recursive relation (22). As in the computation of $(D_\ell(j))_{j \in \mathbb{N}}$, this requires computing $\mathcal{L}_\ell^b(u + j\mu)$ for a finite number of j 's and $\ell < n$ only. This enables to obtain the Laplace transform in y of the mgf of $\tilde{Z}(y)$ thanks to the formula

$$\int_0^\infty e^{-uy} \tilde{\psi}(s, y) dy = \sum_{n \in \mathbb{N}^k} \prod_{i=1}^k \frac{s_i^{n_i}}{n_i!} \mathcal{L}_n^M(u), \quad u > 0, \quad s \in \mathbb{N}^k,$$

and gives some information on the transient behaviour of $\tilde{Z}(t)$, see Theorem 3 p.168 of [21] (which deals with the case of $k = 1$ in the current model), for a comparable result.

5. Single input with exponential delays

We further narrow the scope of Section 4 and consider here the particular case where delays are exponentially distributed and $k = 1$. As we deal with a one dimensional process, we drop a subscript j in $L_{i,j}$ which represents the service time for the j type of input (i.e. write L_i for $i \in \mathbb{N}$), and denote by L for the generic service time. The same way we write X instead of X_1 and $W(t)$ for $W_1(t)$. The first subsection gives some information on the rate of convergence of the first moment of $\tilde{Z}(t)$. As a result, some limiting behaviour result of the workload in an infinite server queues context is available.

5.1. High order expansions

Recalling that $k = 1$, we study in this subsection how fast the first moment $\tilde{M}_1(t) = \mathbb{E}[e^{\delta t} Z(t)]$ converges to χ_1 given in Proposition 4 when $t \rightarrow \infty$. As $\tilde{M}_1(t)$ satisfies the renewal equation (11), using its solution it may be expressed as

$$\tilde{M}_1(t) = \int_0^t \tilde{b}_1(t-s) dm(s), \tag{33}$$

and from Proposition 4, recall that

$$\tilde{M}_1(t) \rightarrow \chi_1 = \frac{\int_0^\infty \tilde{b}_1(t) dt}{\mathbb{E}[\tau_1]}, \quad t \rightarrow \infty, \tag{34}$$

where $\chi_1 = \{\mathbb{E}[X]\mathbb{E}[L]\tilde{w}(\delta)\}/\mathbb{E}[\tau_1]$ and $\tilde{w}(\delta) = \int_0^\infty e^{-\delta x} \overline{W}(x) dx / \mathbb{E}[L]$ as given in (15), Corollary 5. From [6], we use the result of higher order expansions for the function $v(x)$ which is related to the renewal function as

$$v(x) := m(x) - \frac{x}{\mathbb{E}[\tau_1]} - \frac{\mathbb{E}[\tau_1^2]}{2\mathbb{E}[\tau_1]^2}, \tag{35}$$

where F here is non-lattice (as it admits a density) which we suppose is light tailed, i.e. there exists $R > 0$ such that (A1') holds. It admits the following expression

$$v(x) = \sum_{j=1}^N \gamma_j e^{-z_j x} + o(e^{-z_N x}), \tag{36}$$

where z_j 's are the solution of $\mathbb{E}(e^{z_j \tau_1}) = 1$ which are in the range of $0 \leq \text{Re}(z_j) \leq R$ for some $R > 0$ and ordered as $\text{Re}(z_j) \leq \text{Re}(z_{j+1})$. In order for (36) to hold, we in addition require all roots z_1, \dots, z_N to be of multiplicity 1, i.e. such that $\frac{\partial}{\partial z} \mathbb{E}(e^{z \tau_1})|_{z=z_j} \neq 0$ (the condition is not necessary but it enables us to avoid some technicalities later), in which case one has

$$\gamma_j = -\frac{1}{z_j \frac{\partial}{\partial z} \mathbb{E}(e^{z \tau_1})|_{z=z_j}}, \quad j = 1, \dots, N,$$

see [6, Theorem 3]. Although they are complex, the z_j 's actually come in pair as one sees that if z_j verifies $\mathbb{E}(e^{z_j \tau_1}) = 1$ then so does \bar{z}_j , so that one checks that the right-hand side of (36) is in fact real. Furthermore, in the following result we need to write $o(e^{-z_N x})$ term in (36) in the form of

$$o(e^{-z_N x}) = \eta(x) e^{-z_N x} \tag{37}$$

for some function $\eta(x)$ such that $\lim_{x \rightarrow \infty} \eta(x) = 0$.

Theorem 14. *Let us assume that time delays L_i are $\mathcal{E}(\mu)$ distributed and that **(A1')** holds, and define*

$$A = -\mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \left[\frac{\mathbb{E}[\tau_1^2]}{2\mathbb{E}[\tau_1]^2} + \sum_{k=1}^N \gamma_k \frac{\mu}{z_k - \mu} + \mu \int_0^\infty \eta(s) e^{(\mu - z_N)s} ds \right] \mathcal{L}^\tau(-\mu), \quad (38)$$

where $\eta(x)$ is defined by (37), and

$$B_k = \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \left[\gamma_k \frac{z_k}{z_k - \mu} \right] \mathcal{L}^\tau(-z_k). \quad (39)$$

Then $\tilde{M}_1(t)$ in (4) satisfies the following high order expansions

$$\tilde{M}_1(t) = \chi_1 + A^* e^{-\mu t} + \sum_{k=1}^N B_k e^{-z_k t} + o(e^{-z_N t}), \quad (40)$$

where $A^* = A - \frac{\mathbb{E}[X]}{\mathbb{E}[\tau_1]} \cdot \frac{1}{\mu + \delta} \mathcal{L}^\tau(-\mu)$ with A in (38) and B_k in (39).

Proof. See Section 7.5. □

Note that in expansion (40) the B_k 's are explicit. On the other hand, A in (38) features an integral involving function $x \mapsto \eta(x)$ which is not explicit in general. This means that (40) is explicit only if we truncate the expansion to the i_0 th term where $i_0 = \max\{j = 1, \dots, N \mid \operatorname{Re}(z_j) < \mu\}$. We may write the expansion in this way, however we prefer to keep a form as general as possible. Besides, we point out on a similar note that an expansion akin to (40) was provided in [4, Lemma 1] for a general renewal reward process in the particular context where there is no time delay, under the weaker assumption that interarrival times and rewards admit a moment of order 1.

Remark 15 (Dependence of (40) in δ). Upon inspecting (38) and (39) one notices that

$$|A^*|, |B_k| \leq \frac{M}{\mu + \delta}, \quad k = 1, \dots, N,$$

for all $\delta \geq 0$, where $M > 0$ is a constant independent from δ . On further analysis, one also checks that when δ is *complex* and verifies $|\delta| < \mu$ then

$$|A^*|, |B_k| \leq \frac{M}{\mu - |\delta|}, \quad k = 1, \dots, N. \quad (41)$$

In particular this inequality also holds when δ is *negative* and larger than $-\mu$. Hence, from (41), it is shown that that $\tilde{M}_1(t)$ and χ_1 are defined for such a complex δ . This is particularly going to be the case in Section 5.2. Concerning the term $o(e^{-z_N t})$ in (40), one carefully checks from the proof of Theorem 14 that

$$|o(e^{-z_N t})| \leq \frac{1}{\mu - |\delta|} \zeta(t) e^{-\operatorname{Re}(z_N)t}, \quad (42)$$

when $\delta \in \mathbb{C}$, $|\delta| < \mu$, for some function $\zeta(\cdot)$ independent from δ verifying $\lim_{t \rightarrow \infty} \zeta(t) = 0$.

5.2. Asymptotics for the workload of the $G/M/\infty$ queue

An interesting application of the previous study of the one dimensional discounted compound delayed process $Z(t)$ is that we are able to find results concerning the asymptotic behaviour of the workload $D(t)$ of the infinite server queue when $k = 1$. This $D(t)$ represents the time needed to empty the queue at time t if there is no arrival afterwards. The distribution of this quantity was derived in [3, Section 3] for an $M/G/\infty$, but no results seem to have been obtained for a general arrival process with exponential service times, i.e. in $G/M/\infty$. In particular, [3] derives the distribution of the transient workload $D(t)$ in the case of Poisson arrivals with inhomogeneous intensity. The workload has the following expression

$$D(t) := \sum_{i=1}^{\infty} (T_i + L_i - t) \mathbb{1}_{\{T_i \leq t < T_i + L_i\}},$$

which is obtained from $\tilde{Z}(t, \delta) := e^{\delta t} Z(t)$ as:

$$D(t) = -\frac{\partial}{\partial \delta} \tilde{Z}(t, \delta) \Big|_{\delta=0}, \tag{43}$$

We assume in this section that all $X_{i,1}$ for $i \in \mathbb{N}$, are equal to one. In that case, $Z(t)$ in (2) is, when $\delta = 0$, the size of this infinite server queue at time t . A sample path of $D(t)$

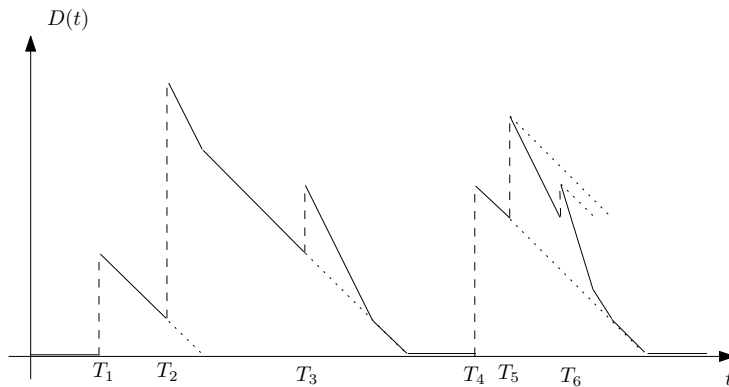


FIG 2. Sample path of workload for the $G/G/\infty$ queue.

is depicted in Figure 2. Let us note that $D(t)$ is also the sum of the residual times for all services to be completed at time t . From an actuarial point of view, $D(t)$ may be interpreted as the remaining time before all current claims have been reported. We are interested in the limiting expectation of workload and the covariance of queue size and workload. We thus need to study the two first moments of $\tilde{Z}(t, \delta)$, i.e. quantities $\tilde{M}_{n(1)}(t, \delta) = \tilde{M}_1(t, \delta) = \mathbb{E}[\tilde{Z}(t, \delta)]$ and $\tilde{M}_{2n(1)}(t, \delta) = \tilde{M}_2(t, \delta) = \mathbb{E}[\tilde{Z}(t, \delta)^2]$, where we here underline the dependence on δ .

Here we assume that service time L is $\mathcal{E}(\mu)$ distributed, i.e.

$$\mathbb{E}[e^{xL}] = \frac{\mu}{\mu - x}, \quad \forall x \in (-\infty, \mu), \tag{44}$$

so that this queue is the $G/M/\infty$, and that interarrival times are light tailed, i.e. Condition (A1') holds for some $R > 0$. To begin, two lemmas are first required. We need to define for $r > 0$, the disc D_r centered at 0 with the radius r , included in \mathbb{C} , by

$$D_r := \{z \in \mathbb{C} \mid |z| \leq r\}.$$

Lemma 16. *Let $a < \mu$ and let us suppose that **(A1)** and **(A1')** holds. For all $t > 0$, $\tilde{M}_1(t, \delta)$ and $\tilde{M}_2(t, \delta)$ are respectively defined on D_a and $D_{a/2}$. Furthermore, $\delta \mapsto \tilde{M}_1(t, \delta)$ and $\delta \mapsto \tilde{M}_2(t, \delta)$ are analytic on those sets, hence a fortiori at $\delta = 0$.*

Note that one implication of the above lemma is that quantities $\tilde{M}_1(t, \delta)$ and $\tilde{M}_2(t, \delta)$ (and, hence $\tilde{Z}(t, \delta)$) are defined for some complex values of δ , and in particular for negative values (not only for $\delta \geq 0$). This is especially handy to express the workload as (43) and to be able to define analyticity of $\tilde{M}_1(t, \delta)$ and $\tilde{M}_2(t, \delta)$ at $\delta = 0$, which is needed to differentiate with respect to δ at 0.

Proof. See Section 7.6. □

Lemma 17. *Let us suppose that **(A1)** and **(A1')** hold and let $a < \mu$. $\delta \mapsto \tilde{M}_1(t, \delta)$ and $\delta \mapsto \tilde{M}_2(t, \delta)$ uniformly converge to $\delta \mapsto \chi_1(\delta)$ and $\delta \mapsto \chi_2(\delta)$ respectively on D_a and $D_{a/2}$ as $t \rightarrow +\infty$.*

Proof. See Section 7.6. □

Now we are ready to provide some results for the long term behaviour of the expected workload, and the covariance function of the workload and the queue size in the following.

Theorem 18. *Let us suppose that **(A1)** and **(A1')** hold. In the $G/M/\infty$ queue, the limiting expected workload is given by*

$$\lim_{t \rightarrow \infty} \mathbb{E}[D(t)] = \frac{1}{\mu^2 \mathbb{E}[\tau_1]} = \frac{\mathbb{E}[L^2]}{2\mathbb{E}[\tau_1]}, \quad (45)$$

and the limiting covariance of the workload and queue size is given by

$$\lim_{t \rightarrow \infty} \text{Cov}[D(t), Z_1(t, 0)] = \frac{1}{\mu^2 \mathbb{E}[\tau_1]} \left[1 + \frac{\mathcal{L}^\tau(\mu)}{1 - \mathcal{L}^\tau(\mu)} - \frac{1}{\mu \mathbb{E}[\tau_1]} \right]. \quad (46)$$

Proof. See Section 7.6. □

Remark 19. When $k = 1$, utilizing (43), it is possible to get an expression of the expected workload and covariance of workload and queue size at time t in the $M/G/\infty$ queue as well. This is done thanks to the (easily verified) relations

$$\begin{aligned} \mathbb{E}[D(t)] &= -\frac{1}{s} \frac{\partial}{\partial \delta} \tilde{\psi}(s, t) \Big|_{\delta=0, s=0}, \\ \text{Cov}[D(t), Z_1(t, 0)] &= \left[\frac{\partial}{\partial s} \left[-\frac{1}{s} \frac{\partial}{\partial \delta} \tilde{\psi}(s, t) \right] - \left[-\frac{1}{s} \frac{\partial}{\partial \delta} \tilde{\psi}(s, t) \right] \cdot \left[\frac{\partial}{\partial s} \tilde{\psi}(s, t) \right] \right] \Big|_{\delta=0, s=0}, \end{aligned}$$

where $\tilde{\psi}(s, t)$ is given by (13) with $M_{t,X}^*(s) = \bar{W}(t) + \int_t^\infty e^{se^{-\delta v}} dW(s)$. Contrarily to Theorem 18, justification of the above formulas is much easier as one does not have to justify interchange of expectation and derivation with respect to δ , which is the core step in the proof of Theorem 18, and is done with the help of Lemmas 16 and 17.

6. Applications

6.1. Queues with different service times within a batch

The queueing model introduced in Section 1 features a queue where customers arrive in a batch of size $X_{i,j}$ with class j at time T_i . Each customer in this batch has the same service time $L_{i,j}$ within the same class j for $j = 1, 2, \dots, k$. One may argue that this scenario is not much realistic since each customer $\ell \in \{1, \dots, X_{i,j}\}$ may have different service times $L_{ij\ell}$, where $(L_{ij\ell})_{ij\ell \in \mathbb{N}^3}$ are independent rv, with $(L_{ij\ell})_{\ell \in \mathbb{N}}$ identically distributed for all $i \in \mathbb{N}$, $j = 1, \dots, k$, so that customers within a batch get different service times.

It can be shown that this (more realistic) situation is essentially expressed in the form of our model by constructing a "larger" vector $X = (X_1, \dots, X_k)$ in the following. For illustration purposes, recall the situation depicted in Figure 1 where M customers arrive in a batch, but let us consider it with customer $\ell \in \{1, \dots, X_{i,j}\}$ having service time $L_{ij\ell}$ instead of $L_{i,j}$. In other words, let such a sequence $(L_{ij\ell})_{(i,j,\ell) \in \mathbb{N}^3}$ be given and let $p_{j\ell}$ the probability of having a customer of class $j \in \{1, \dots, k\}$ and delay $L_{j\ell}$, for some generic random matrix $(L_{j\ell})_{j=1, \dots, k, \ell=1, \dots, M}$. This situation is then modelled thanks to the one described in Section 1 by considering a vector $X = (X_{j,\ell})_{j=1, \dots, k, \ell=1, \dots, M}$ of length kM (written as a matrix) such that

$$X = (X_{j\ell})_{j=1, \dots, k, \ell=1, \dots, M} \\ \sim \mathcal{D}((Y_{j\ell})_{j=1, \dots, k, \ell=1, \dots, M} | Y_{j\ell} \in \{0, 1\}, \forall (j, \ell) \in \{1, \dots, k\} \times \{1, \dots, M\}).$$

where $(Y_{j\ell})_{j=1, \dots, k, \ell=1, \dots, M}$ be a matrix with distribution $\mathcal{M}(M, (p_{j\ell})_{j=1, \dots, k, \ell=1, \dots, M})$, i.e. a random vector of length kM with a multinomial distribution with parameter M and probability vector $(p_{11}, \dots, p_{1M}, p_{21}, \dots, p_{2M}, \dots, p_{k1}, \dots, p_{kM})$.

6.2. Infinite server queues in tandem

To furthermore illustrate the versatility of the present model, let us now consider the following two infinite server queues in tandem setup. We suppose that each batch arriving at time T_i contains $X_{i,j}$ customers where there are k classes of customers. Once customer of class $j \in \{1, \dots, k\}$ arrives in the first queue, he is served during a *deterministic* time $L_{i,j}^1$. Upon completion of the service, i.e. after leaving the first queue, he is then directly sent to the second queue (again with an infinite number of servers) where it is served during a time $L_{i,j}^2$. This kind of successive treatments of queues is easily observed in the claims payment process in actuarial science. In general, there are time delays between the time of incurral of the claim and the time of receipt of payment. Of course, for the insurers, they are concerned with the time from receipt of notification of the claim until approval or payment. It is natural that each stage for one claim has different processing times (i.e. different distribution for time delays). See [22] for detailed discussion related to the insight of queueing theoretic tools into the claims payment process.

Again, for illustration purposes, consider the case where there are M customers that arrive in a batch at time T_i . Within the same type of class j of which size $X_{i,j}$, all have the same service times $L_{i,j}^1$ and $L_{i,j}^2$. Certainly, as explained in Section 6.1, different service times within a batch may also be available. We assume that $(L_{i,j}^2)_{i \in \mathbb{N}, j=1, \dots, k}$ are independent, and that, as usual, $L_{i,1}^2, \dots, L_{i,k}^2$ have different distribution for each class; In the same vein, service times

$L_{i,1}^1, \dots, L_{i,k}^1$ in the first queue are all deterministic but are different for each class. This is represented in Figure 3. We are interested in the number of customers of class j in the second

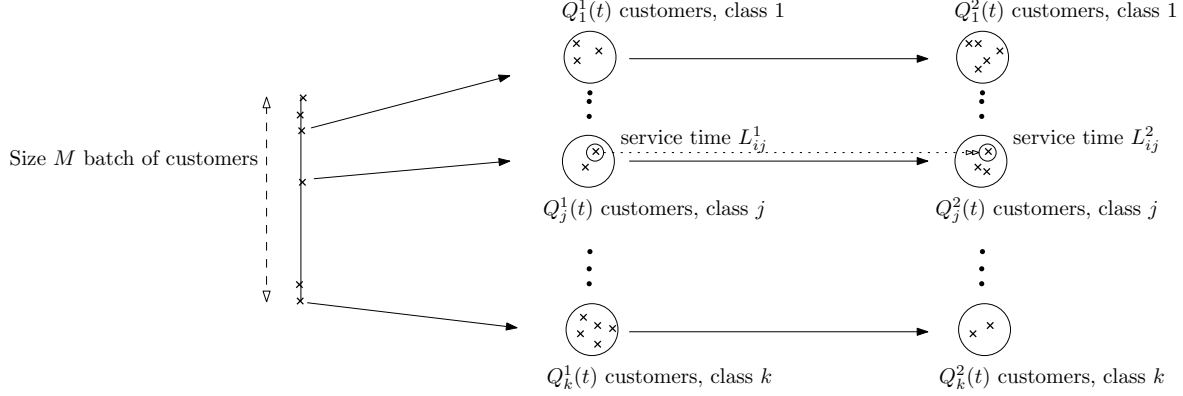


FIG 3. *Two queues in tandem.*

queue at time t which is denoted by $Q_j^2(t)$. It is not hard to see that one has the expression

$$Q_j^2(t) = \sum_{i=1}^{\infty} X_{i,j} \mathbb{1}_{\{T_i + L_{i,j}^1 \leq t < T_i + L_{i,j}^1 + L_{i,j}^2\}}, \quad j = 1, \dots, k, \quad (47)$$

where $X = (X_1, \dots, X_k) \sim \mathcal{M}(M, p_1, \dots, p_k)$. Let us then introduce, for $i \in \mathbb{N}$, the \mathbb{N}^{2k} sized vector

$$X' = (X'_{i,1}, \dots, X'_{i,2k}) := (X_{i,1}, \dots, X_{i,k}, X_{i,1}, \dots, X_{i,k})$$

(i.e. vector X concatenated with itself) as well as the $[0, +\infty)^{2k}$ sized vector

$$L'_{i,j} = \begin{cases} L_{i,j}^1, & j = 1, \dots, k, \\ L_{i,j-k}^1 + L_{i,j-k}^2, & j = k+1, \dots, 2k. \end{cases} \quad (48)$$

One important remark is that, since the L_{ij}^1 's are deterministic, the sequence $(L'_{ij})_{i \in \mathbb{N}, j=1, \dots, 2k}$ has *independent* components. Hence, this model can be expressed under the setting of our model as described in Section 1 but with X' and $(L'_{i,j})_{i \in \mathbb{N}, j=1, \dots, 2k}$ in lieu of X and $(L_{i,j})_{i \in \mathbb{N}, j=1, \dots, k}$. To be specific, let us define the \mathbb{N}^{2k} valued process $Z(t) = (Z_1(t), \dots, Z_{2k}(t))$ with

$$Z_j(t) = \sum_{i=1}^{\infty} X'_{i,j} \mathbb{1}_{\{T_i \leq t < T_i + L'_{i,j}\}}, \quad j = 1, \dots, 2k,$$

where $L'_{i,j}$ are defined in (48). Then, one has in particular $Z_j(t) = Q_j^1(t)$ for $j = 1, \dots, k$. One also notes from (47) that $Q_j^2(t) = Z_{j+k}(t) - Z_j(t)$ for $j = 1, \dots, k$. The mgf of $Q^2(t) = (Q_1^2(t), \dots, Q_k^2(t))$ can then be expressed in terms of the mgf of $Z(t)$ by

$$\mathbb{E} \left[e^{\langle s, Q^2(t) \rangle} \right] = \mathbb{E} \left[e^{\langle (-s, s), Z(t) \rangle} \right], \quad s = (s_1, \dots, s_k) \in \mathbb{R}^k, \quad (49)$$

where $(-s, s) := (-s_1, \dots, -s_k, s_1, \dots, s_k) \in \mathbb{R}^{2k}$. The consequence of (49) is that

- If arrivals occur according to a Poisson process with intensity λ , then the mgf of $Q^2(t)$ is **explicit** thanks to Proposition 3 that

$$\mathbb{E}\left[e^{\langle s, Q^2(t) \rangle}\right] = \exp\left[\lambda \int_0^t (M_{v, X'}^*((-s, s)) - 1) dv\right].$$

- If arrival processes are general but satisfy **(A1)** then less information is available on the transient distribution, however one has from Theorem 8 that $Q^2(t)$ converges in distribution to some light tailed random vector as $t \rightarrow \infty$, and that some simple bounds on the joint moments of this limiting random vector are available from Proposition 7.

7. Proofs

Proof of Lemma 1. When τ_1 admits a density $f(\cdot)$ then density $t \mapsto u(t)$ of renewal function $t \mapsto m(t)$ satisfies a renewal equation of the form

$$u(x) = f(x) + \int_0^x u(y)f(x-y)dy, \quad x \geq 0, \quad (50)$$

(e.g. see Equation (3.6) of [7]). Since **(A1)** holds, by [7, Lemma p.359] (50) admits a unique solution bounded on finite intervals given by (9). Also, the derivative $m'(t) = u(t)$ verifies $\lim_{t \rightarrow \infty} m'(t) = 1/\mathbb{E}[\tau_1]$, see [7, Theorem 2 p.367], and is thus bounded above by some constant C . \square

7.1. Proof of Proposition 4

Since $\tilde{M}_n(t)$ satisfies the renewal equation in (11), asymptotics result in (14) is a direct consequence of Smith's renewal theorem, provided that we prove that $\int_0^\infty \tilde{b}_n(y)dy$ or equivalently $\int_0^\infty \varphi_{\ell, n}(y)dy$ is finite for all $n \in \mathbb{N}^k$ and $\ell < n$. We shall demonstrate this by induction on $n \in \mathbb{N}^k$. First, consider the case of $n = n(i)$ for some $i \in \{1, \dots, k\}$. From Example 3 in [27] one has

$$b_n(t) = \mathbb{E}[X_i] \cdot \int_0^t e^{-\delta y} \bar{\omega}_{\delta, i}(t-y) dF(y) = \mathbb{E}[X_i] \cdot \bar{\omega}_{\delta, i} \star H_\delta(t), \quad (51)$$

where $\bar{\omega}_{\delta, i}(t)$ is given in (7) and $H_\delta(t) = \int_0^t e^{-\delta y} dF(y)$. But $\int_0^\infty e^{\delta z} \bar{\omega}_{\delta, i}(z) dz = \int_0^\infty e^{\delta z} \int_z^\infty e^{-\delta y} dW_i(y) dz = \delta^{-1} \{1 - \mathbb{E}[e^{-\delta L_i}]\}$, the following integration yields

$$\begin{aligned} \int_0^\infty \tilde{b}_n(y) dy &= \int_0^\infty e^{\delta y} b_n(y) dy = \mathbb{E}[X_i] \cdot \int_0^\infty e^{\delta y} \int_0^y e^{-\delta x} \bar{\omega}_{\delta, i}(y-x) dF(x) dy \\ &= \mathbb{E}[X_i] \cdot \int_0^\infty e^{-\delta x} \int_x^\infty e^{\delta y} \bar{\omega}_{\delta, i}(y-x) dy dF(x) \\ &= \mathbb{E}[X_i] \cdot \delta^{-1} \{1 - \mathbb{E}[e^{-\delta L_i}]\} < \infty, \end{aligned} \quad (52)$$

or equivalently

$$\int_0^\infty \tilde{b}_n(y) dy = \mathbb{E}[X_i] \mathbb{E}[L_i] \int_0^\infty e^{-\delta x} \frac{\bar{W}_i(x)}{\mathbb{E}[L_i]} dx = \mathbb{E}[X_i] \mathbb{E}[L_i] \tilde{w}_{1, i}(\delta),$$

where $w_{1, i}(x)$ is an equilibrium pdf of L_i defined as $w_{1, i}(x) = \bar{W}_i(x)/\mathbb{E}[L_i]$ and its Laplace transform is $\tilde{w}_{1, i}(s) = \int_0^\infty e^{-sx} w_{1, i}(x) dx$.

Moreover, recall Equation (36) in [27]

$$M_n(t) = \mathbb{E}[X_i] \cdot \int_0^t e^{-\delta y} \bar{\omega}_{\delta,i}(t-y) dm(y) = \mathbb{E}[X_i] \cdot e^{-\delta t} \int_0^t e^{\delta(t-y)} \bar{\omega}_{\delta,i}(t-y) dm(y).$$

By Smith's theorem, it satisfies

$$M_n(t) \sim \frac{\mathbb{E}[X_i]}{\mathbb{E}[\tau_1]} \left[\int_0^\infty e^{\delta y} \bar{\omega}_{\delta,i}(y) dy \right] e^{-\delta t}, \quad t \rightarrow \infty.$$

In other words, one identifies

$$\chi_n = \chi_{n(i)} = \frac{\mathbb{E}[X_i]}{\mathbb{E}[\tau_1]} \left[\int_0^\infty e^{\delta y} \bar{\omega}_{\delta,i}(y) dy \right].$$

We now assume for all $\ell < n$ that $\tilde{M}_\ell(t) \rightarrow \chi_\ell < +\infty$ as $t \rightarrow \infty$ with χ_ℓ defined as in (14). Hence $t \mapsto \tilde{M}_\ell(t)$ is bounded for all $\ell < n$ by some constant $K_\ell = \sup_{t \geq 0} \tilde{M}_\ell(t)$. Hence simple algebraic computation results in the upper bound for (6) as

$$\begin{aligned} \varphi_{\ell,n}(t) &\leq K_\ell \mathbb{E} \left[e^{(\eta_n - \eta_\ell)\delta(t - \tau_1)} \prod_{j \in \mathcal{C}_{\ell,n}} \bar{\omega}_{(n_j - \ell_j)\delta,j}(t - \tau_1) \cdot \mathbb{1}_{[\tau_1 < t]} \right] \\ &= K_\ell \mathbb{E} \left[e^{(\eta_n - \eta_\ell)\delta(t - \tau_1)} \prod_{j \in \mathcal{C}_{\ell,n}} \left[\int_{t - \tau_1}^\infty e^{-(n_j - \ell_j)\delta y} dW_j(y) \right] \cdot \mathbb{1}_{[\tau_1 < t]} \right] \\ &\leq K_\ell \mathbb{E} \left[e^{(\eta_n - \eta_\ell)\delta(t - \tau_1)} \prod_{j \in \mathcal{C}_{\ell,n}} \left[e^{-(n_j - \ell_j)\delta(t - \tau_1)} \bar{W}_j(t - \tau_1) \right] \cdot \mathbb{1}_{[\tau_1 < t]} \right] \\ &= K_\ell \mathbb{E} \left[\prod_{j \in \mathcal{C}_{\ell,n}} \bar{W}_j(t - \tau_1) \cdot \mathbb{1}_{[\tau_1 < t]} \right]. \end{aligned}$$

Then integrating $\varphi_\ell(t)$ from 0 and ∞ yields

$$\int_0^\infty \varphi_{\ell,n}(t) dt \leq K_\ell \mathbb{E} \left[\int_0^\infty \prod_{j \in \mathcal{C}_{\ell,n}} \left[\bar{W}_j(t - \tau_1) \right] \cdot \mathbb{1}_{[\tau_1 < t]} dt \right] = K_\ell \int_0^\infty \prod_{j \in \mathcal{C}_{\ell,n}} \bar{W}_j(t) dt,$$

and by Holder's inequality, one finds

$$\begin{aligned} \int_0^\infty \varphi_{\ell,n}(t) dt &\leq K_\ell \prod_{j \in \mathcal{C}_{\ell,n}} \left[\int_0^\infty \bar{W}_j(t)^{|\mathcal{C}_{\ell,n}|} dt \right]^{1/|\mathcal{C}_{\ell,n}|}, \\ &\leq K_\ell \prod_{j \in \mathcal{C}_{\ell,n}} \left[\int_0^\infty \bar{W}_j(t) dt \right]^{1/|\mathcal{C}_{\ell,n}|} = K_\ell \prod_{j \in \mathcal{C}_{\ell,n}} \mathbb{E}[L_j]^{1/|\mathcal{C}_{\ell,n}|} \\ &\leq K_\ell \min_{j \in \mathcal{C}_{\ell,n}} \mathbb{E}[L_j] < \infty, \end{aligned} \tag{53}$$

where $|\mathcal{C}_{\ell,n}|$ denotes the cardinal of set $\mathcal{C}_{\ell,n}$. Hence from (8) we deduce that $\int_0^\infty \tilde{b}_n(y) dy$ is also finite, and the induction is complete.

7.2. Proof of Proposition 7

Since $m(t)$ admits $u(t)$ as a density, one has from (11) that $\tilde{M}_n(t) = \int_0^t \tilde{b}_n(y)u(t-y)dy$, and in turn, from Lemma 1 we arrive at the following upper bound

$$\tilde{M}_n(t) \leq C \int_0^\infty \tilde{b}_n(y)dy. \quad (54)$$

Combining (8) and (53) yields the following upper bound

$$\int_0^\infty \tilde{b}_n(y)dy \leq \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] K_\ell \min_{j \in C_{\ell,n}} \mathbb{E}[L_j],$$

where we recall that $K_\ell = \sup_{t \geq 0} \tilde{M}_\ell(t)$ (see the proof of Proposition 4). Thus the above inequality together with (14) and (54) yields (18) and (19) respectively with $(R_n)_{n \in \mathbb{N}^k}$ defined in (20), provided we initialize value of R_n when $n = n(i)$ for $i \in \{1, \dots, k\}$. This is done by again using upper bound (54) and remembering that $\int_0^\infty \tilde{b}_n(y)dy$ is obtained by (52) when $n = n(i)$.

7.3. Proof of Theorem 8

Let $P(x_1, \dots, x_k) = \sum_{\eta_n \leq K} a_n x_1^{n_1} \cdots x_k^{n_k}$ be a nonnegative polynomial in the variables $x_1 \dots x_k$ of degree K . One has then that $\sum_{\eta_n \leq K} a_n \mathbb{E} \left[\prod_{i=1}^k (\tilde{Z}_i(t))^{n_i} \right] = \mathbb{E} \left[P(\tilde{Z}_1(t), \dots, \tilde{Z}_k(t)) \right] \geq 0$ for all t , which, from Proposition 4, yields $\sum_{\eta_n \leq K} a_n \chi_n \geq 0$ as $t \rightarrow \infty$. By the Riesz-Haviland theorem (see [9]), we deduce that sequence $(\chi_n)_{n \in \mathbb{N}^k}$ is a sequence of moments associated to some random variable $\mathcal{Z}_\infty = (\mathcal{Z}_{\infty,1}, \dots, \mathcal{Z}_{\infty,k})$.

Next we shall show that the mgf of $\tilde{Z}(t)$ exists and converges to the mgf of \mathcal{Z}_∞ as $t \rightarrow \infty$. To this end, we note that the mgfs of $\tilde{Z}(t)$ and of \mathcal{Z}_∞ respectively defined by (5) and denoted by $\psi_\infty(s)$ verify

$$\tilde{\psi}(s, t) = \sum_{n \in \mathbb{N}^k} \prod_{i=1}^k \frac{s_i^{n_i}}{n_i!} \tilde{M}_n(t), \quad (55)$$

$$\psi_\infty(s) = \mathbb{E} [e^{\langle s, \mathcal{Z}_\infty \rangle}] = \sum_{n \in \mathbb{N}^k} \prod_{i=1}^k \frac{s_i^{n_i}}{n_i!} \chi_n, \quad (56)$$

for $t \geq 0$ and $s = (s_1, \dots, s_k) \in \mathbb{R}^k$ in the neighborhood of $(0, \dots, 0)$. To let $t \rightarrow \infty$ in (55) and apply the dominated convergence theorem, we need to show that $\tilde{M}_n(t)$ is bounded such as

$$\tilde{M}_n(t) \leq U_n := (M m_L e^k)^{n_m} \prod_{i=1}^k n_i!, \quad \forall n \in \mathbb{N}^k, \quad \forall t \geq 0, \quad (57)$$

where $m_L := \max(1, C \cdot \max_{i=1, \dots, k} \mathbb{E}[L_i])$, so that, since

$$\sum_{n \in \mathbb{N}^k} \prod_{i=1}^k \frac{|s_i|^{n_i}}{n_i!} U_n = \prod_{i=1}^k \left(\sum_{n_i=0}^{\infty} |s_i| M m_L e^k |^{n_i} \right)$$

converges for

$$s = (s_1, \dots, s_k) \in J := \left[-\frac{1}{Mm_L e^k}, \frac{1}{Mm_L e^k} \right]^k,$$

the dominated convergence theorem yields $\tilde{\psi}(s, t) \rightarrow \psi_\infty(s)$ when $t \rightarrow \infty$ for $s \in J$.

Hence, we shall prove (57) by induction. Recall that in Proposition 7, we have already proved $\tilde{M}_n(t) \leq R_n$ where R_n is defined in (20). Thus we shall essentially show that $R_n \leq U_n$ for all $n \in \mathbb{N}^k$, so that (57) holds. We start by $n = n(i)$ for $i \in \{1, \dots, k\}$. In this case, upper bound (19) with (20) yields

$$R_{n(i)} = C\mathbb{E}[X_i]\delta^{-1} \left\{ 1 - \mathbb{E} \left[e^{-\delta L_i} \right] \right\} \leq CM\mathbb{E}[L_i] \leq Mm_L = U_{n(i)},$$

where the first inequality is due to $\delta^{-1} \{1 - \mathbb{E}[e^{-\delta L_i}]\} = \int_0^\infty e^{-\delta x} \bar{W}_i(x) dx \leq \int_0^\infty \bar{W}_i(x) dx$. Let us now suppose that n is such that $R_\ell \leq U_\ell$ for all $\ell < n$. Using (20) as well as the induction assumption we get

$$R_n \leq C \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] \min_{i \in C_{\ell, n}} \mathbb{E}[L_i] \cdot U_\ell \leq m_L \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} M^{\eta_n - \eta_\ell} \cdot U_\ell. \quad (58)$$

But, $\ell < n$ implies $\eta_n - \eta_\ell \geq 1$ and m_L and e are larger than 1, the following inequality is valid

$$m_L M^{\eta_n - \eta_\ell} \leq m_L^{\eta_n - \eta_\ell} M^{\eta_n - \eta_\ell} (e^k)^{\eta_n - \eta_\ell - 1} = (m_L M e^k)^{\eta_n - \eta_\ell} e^{-k}.$$

Substituting the above inequality and $U_\ell = (Mm_L e^k)^{\eta_\ell} \prod_{i=1}^k \ell_i!$ into (58), the right-hand side of (58) is now bounded by

$$\begin{aligned} R_n &\leq \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} (m_L M e^k)^{\eta_n - \eta_\ell} e^{-k} (Mm_L e^k)^{\eta_\ell} \prod_{i=1}^k \ell_i! \\ &= (Mm_L e^k)^{\eta_n} \left[\sum_{\ell < n} \prod_{i=1}^k \frac{n_i!}{(n_i - \ell_i)!} \right] e^{-k} = (Mm_L e^k)^{\eta_n} \left[\prod_{i=1}^k n_i! \right] \left[\sum_{\ell < n} \prod_{i=1}^k \frac{1}{(n_i - \ell_i)!} \right] e^{-k} \\ &= U_n \left[\sum_{\ell < n} \prod_{i=1}^k \frac{1}{(n_i - \ell_i)!} \right] e^{-k}. \end{aligned} \quad (59)$$

We then conclude by noticing that

$$\begin{aligned} \sum_{\ell < n} \prod_{i=1}^k \frac{1}{(n_i - \ell_i)!} &\leq \sum_{\ell_i \leq n_i, i \in \{1, \dots, k\}} \prod_{i=1}^k \frac{1}{(n_i - \ell_i)!} \\ &= \prod_{i=1}^k \left[\sum_{\ell_i=1}^{n_i} \frac{1}{(n_i - \ell_i)!} \right] = \prod_{i=1}^k \left[\sum_{\ell_i=1}^{n_i} \frac{1}{\ell_i!} \right] \leq \prod_{i=1}^k \left[\sum_{\ell_i=1}^{\infty} \frac{1}{\ell_i!} \right] = e^k, \end{aligned}$$

which, plugged into (59), yields $R_n \leq U_n$. Therefore, by the dominated convergence theorem, $\tilde{\psi}(s, t)$ in (55) converges to $\psi_\infty(s)$ in (56) as $t \rightarrow \infty$.

In conclusion, since $\tilde{M}_n(t)$ and χ_n are bounded as shown in Proposition 7, the mgfs of $e^{\delta t} Z(t)$ in (55) and Z_∞ in (56) exist. Also, we have shown that $\tilde{\psi}(s, t) \rightarrow \psi_\infty(s)$ when $t \rightarrow \infty$ for $s \in J$ in some neighborhood of $(0, \dots, 0)$. Hence, $e^{\delta t} Z(t)$ converges to Z_∞ in distribution.

7.4. Proof of Lemma 9

When $n = n(i)$ and $i \in \{1, \dots, k\}$, we may obtain an expression of $\mathcal{L}_n^b(s)$ by using the expression of $b_n(t)$ in Example 3 in [27], and applying similar idea as applied in (52). We now turn to proving (22). Since L_j 's are all $\mathcal{E}(\mu)$ distributed, $\varphi_{\ell,n}(t) = \varphi_{\ell,n}(t)$ given by (6) simplifies to

$$\varphi_{\ell,n}(t) = \mathbb{E} \left[\tilde{M}_\ell(t - \tau_1) \left\{ \prod_{j \in C_{\ell,n}} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right\} e^{-|C_{\ell,n}|\mu(t-\tau_1)} \cdot \mathbb{1}_{[\tau_1 < t]} \right].$$

Then using Fubini's theorem to interchange the expectation with the integration as well as a change of variable $t := t - \tau_1$, it follows that

$$\begin{aligned} \int_0^\infty e^{-ut} \varphi_{\ell,n}(t) dt &= \left[\prod_{j \in C_{\ell,n}} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right] \mathbb{E} \left[\int_{\tau_1}^\infty e^{-ut} \tilde{M}_\ell(t - \tau_1) e^{-|C_{\ell,n}|\mu(t-\tau_1)} dt \right] \\ &= \left[\prod_{j \in C_{\ell,n}} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right] \mathbb{E} \left[e^{-u\tau_1} \int_0^\infty e^{-ut} \tilde{M}_\ell(t) e^{-|C_{\ell,n}|\mu t} dt \right] \\ &= \left[\prod_{j \in C_{\ell,n}} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right] \mathcal{L}^\tau(u) \mathcal{L}_\ell^M(u + |C_{\ell,n}|\mu). \end{aligned} \quad (60)$$

If $\ell = \mathbf{0}$, where $\mathbf{0}$ is a zero vector in \mathbb{N}^k , then $\tilde{M}_\ell(t) = 1$ hence $\mathcal{L}_\ell^M(u + |C_{\mathbf{0},n}|\mu) = \frac{1}{u + |C_{\mathbf{0},n}|\mu}$, and we get

$$\int_0^\infty e^{-ut} \varphi_{\mathbf{0},n}(t) dt = \left[\prod_{j \in C_{\mathbf{0},n}} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right] \frac{\mathcal{L}^\tau(u)}{u + |C_{\mathbf{0},n}|\mu} = \left[\prod_{j=1}^k \frac{\mu}{\mu + n_j\delta} \right] \frac{\mathcal{L}^\tau(u)}{u + |C_{\mathbf{0},n}|\mu}.$$

In the case $\ell > \mathbf{0}$, let us now observe that taking Laplace transforms in renewal equation (11) satisfied by $\tilde{M}_n(\cdot)$ yields the following classical relation between $\mathcal{L}_n^M(u)$ and $\mathcal{L}_n^b(u)$:

$$\mathcal{L}_n^M(u) = \frac{\mathcal{L}_n^b(u)}{1 - \mathcal{L}^\tau(u)}, \quad \forall u > 0, \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\},$$

so that (60) leads to

$$\int_0^\infty e^{-ut} \varphi_{\ell,n}(t) dt = \left[\prod_{j \in C_{\ell,n}} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right] \frac{\mathcal{L}^\tau(u)}{1 - \mathcal{L}^\tau(u + |C_{\ell,n}|\mu)} \mathcal{L}_\ell^b(u + |C_{\ell,n}|\mu).$$

With the above result, the Laplace transform of (8) becomes (22).

7.5. Proof of Theorem 14

Substituting (35) into (33) for $dm(s)$ yields

$$\tilde{M}_1(t) = \frac{1}{\mathbb{E}[\tau_1]} \int_0^t \tilde{b}_1(t-s) ds + \int_0^t \tilde{b}_1(t-s) dv(x).$$

A change of variable $s := t - s$ in the first integral and a subtraction of χ_1 in (34) on both sides result in

$$\tilde{M}_1(t) - \chi_1 = -\frac{1}{\mathbb{E}[\tau_1]} \int_t^\infty \tilde{b}_1(s) ds + \int_0^t \tilde{b}_1(t-s) dv(x). \quad (61)$$

Let

$$I_1(t) = -\frac{1}{\mathbb{E}[\tau_1]} \int_t^\infty \tilde{b}_1(s) ds, \quad I_2(t) = \int_0^t \tilde{b}_1(t-s) dv(s), \quad (62)$$

then (61) is essentially a sum of $I_1(t)$ and $I_2(t)$. In the sequel, we shall separately study the asymptotic behaviors of $I_1(t)$ and $I_2(t)$ when $t \rightarrow \infty$. First it is convenient to introduce the following quantity and its asymptotic result as it will be often utilized in the later analysis.

$$\begin{aligned} \mathbb{E}[\mathbb{1}_{\{\tau_1 \geq t\}} e^{-\mu_i(t-\tau_1)}] &= e^{-\mu_i t} \int_t^\infty e^{\mu_i s} dF(x) \\ &= e^{-\mu_i t} \int_t^\infty e^{(\mu_i - R)s} e^{Rs} dF(s) \leq e^{-\mu_i t} \int_t^\infty e^{(\mu_i - R)t} e^{Rs} dF(s) \\ &\leq e^{-Rt} \int_t^\infty e^{Rs} dF(x) = o(e^{-Rt}), \end{aligned} \quad (63)$$

where the second last inequality is due to the assumption on $\mu_i < R$ for all i 's and the last result is due to $\mathbb{E}[e^{R\tau_1}] = \mathcal{L}^\tau(-R) < \infty$ by **(A1')**.

We begin to analyze $I_1(t)$ in (62) when $t \rightarrow \infty$. From (51) and (7) we may write

$$\int_t^\infty \tilde{b}_1(z) dz = \mathbb{E}[X] \cdot \mathbb{E} \left[\int_t^\infty e^{\delta(z-\tau_1)} \mathbb{1}_{\{\tau_1 < z\}} \int_{z-\tau_1}^\infty e^{-\delta s} dW(s) dz \right]. \quad (64)$$

When we assume that L_j 's are $\mathcal{E}(\mu)$ distributed for $\mu > 0$, then the second integral on the above equation is simplified as

$$\int_{z-\tau_1}^\infty e^{-\delta s} dW(s) = \frac{\mu}{\mu + \delta} e^{-(\mu + \delta)(z-\tau_1)}. \quad (65)$$

As $\mathbb{1}_{\{\tau_1 \geq t\}} + \mathbb{1}_{\{\tau_1 < t\}} = 1$, inserting these two indicator functions in (64) together with (65) results in

$$\int_t^\infty \tilde{b}_1(z) dz = \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathbb{E} \left[(\mathbb{1}_{\{\tau_1 < t\}} + \mathbb{1}_{\{\tau_1 \geq t\}}) \int_t^\infty \mathbb{1}_{\{\tau_1 < z\}} e^{-\mu_i(z-\tau_1)} dz \right].$$

For the case of $\tau_1 < t$, as $z > t$ and $\tau_1 < z$, the above expectation is reduced to

$$\begin{aligned} \mathbb{E} \left[\mathbb{1}_{\{\tau_1 < t\}} \int_t^\infty \mathbb{1}_{\{\tau_1 < z\}} e^{-\mu_i(z-\tau_1)} dz \right] &= \frac{1}{\mu} \mathbb{E}[\mathbb{1}_{\{\tau_1 < t\}} e^{-\mu t - \tau_1}] \\ &= \frac{1}{\mu} \mathbb{E}[(1 - \mathbb{1}_{\{\tau_1 \geq t\}}) e^{-\mu(t-\tau_1)}] \\ &= \frac{1}{\mu} \left\{ e^{-\mu t} \mathcal{L}^\tau(-\mu) - \mathbb{E}[\mathbb{1}_{\{\tau_1 \geq t\}} e^{-\mu(t-\tau_1)}] \right\} \\ &= \frac{1}{\mu} e^{-\mu t} \mathcal{L}^\tau(-\mu) + o(e^{-Rt}), \end{aligned}$$

where the last line is obtained by applying (63). On the other hand, when $\tau_1 \geq t$,

$$\begin{aligned} \mathbb{E}\left[\mathbb{1}_{\{\tau_1 \geq t\}} \int_t^\infty \mathbb{1}_{\{\tau_1 < z\}} e^{-\mu(z-\tau_1)} dz\right] &= \mathbb{E}\left[\mathbb{1}_{\{\tau_1 \geq t\}} \int_{\tau_1}^\infty e^{-\mu(z-\tau_1)} dz\right] \\ &= \frac{1}{\mu} \Pr(\tau_1 \geq t), \end{aligned}$$

and note that, using Chernoff's inequality, $\Pr(\tau_1 \geq t) \leq \mathbb{E}(e^{R\tau_1})e^{-Rt} = o(e^{-z_N t})$ because of $\mathbb{E}(e^{R\tau_1}) < \infty$ (by condition **(A1')**) and $\operatorname{Re}(z_N) < R$. Hence combining the above results using the fact that an $o(e^{-Rt})$ is a fortiori an $o(e^{-z_N t})$, it follows that

$$I_1(t) = -\frac{1}{\mathbb{E}[\tau_1]} \int_t^\infty \tilde{b}_1(s) ds = -\frac{\mathbb{E}[X]}{\mathbb{E}[\tau_1]} \cdot \frac{1}{\mu + \delta} \mathcal{L}^\tau(-\mu) e^{-\mu t} + o(e^{-z_N t}). \quad (66)$$

We now turn to $I_2(t)$ in (62). As $\tilde{b}_1(0) = 0$, applying integration by parts for Stieltjes integrals on the right side of $I_2(t)$ yields

$$I_2(t) = \int_0^t \tilde{b}_1(t-s) dv(x) = \tilde{b}_1(t)v(0^-) + \int_0^t v(s) \tilde{b}'_1(t-s) ds. \quad (67)$$

But $v(0^-) = -\mathbb{E}[\tau_1^2]/(2\mathbb{E}[\tau_1]^2)$ and using a similar reasoning applied to (63) we get

$$\tilde{b}_1(t) = \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \mathbb{E}[\mathbb{1}_{\{\tau_1 < t\}} e^{-\mu(t-\tau_1)}] = \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \mathcal{L}^\tau(-\mu) e^{-\mu t} + o(e^{-Rt}), \quad (68)$$

i.e.

$$\tilde{b}_1(t)v(0^-) = -\frac{\mathbb{E}[X_i]\mathbb{E}[\tau_1^2]}{2\mathbb{E}[\tau_1]^2} \frac{\mu}{\mu + \delta} \mathcal{L}^\tau(-\mu) e^{-\mu t} + o(e^{-z_N t}), \quad t \rightarrow \infty. \quad (69)$$

Also we have $\tilde{b}_1(t) = \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} e^{-\mu t} \int_0^t e^{\mu s} dF(s)$ and then $\tilde{b}'_1(t) = -\mu \tilde{b}_1(t) + \mathbb{E}[X] \frac{\mu}{\mu + \delta} f(t)$. Thus

$$\begin{aligned} \int_0^t e^{-z_k s} \tilde{b}'_1(t-s) dx &= e^{-z_k t} \int_0^t e^{z_k s} \tilde{b}'_1(s) ds \\ &= e^{-z_k t} \int_0^t e^{z_k s} \left[-\mu \tilde{b}_1(s) + \mathbb{E}[X] \frac{\mu}{\mu + \delta} f(s) \right] ds, \quad k = 1, \dots, N. \end{aligned} \quad (70)$$

On the first term of the above equation, from (68) it follows that

$$\begin{aligned} e^{-z_k t} \int_0^t e^{z_k s} \tilde{b}_1(s) ds &= \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \left(\frac{1}{z_k - \mu} \right) \mathbb{E}\left[\mathbb{1}_{\{\tau_1 < t\}} \{e^{-\mu(t-\tau_1)} - e^{-z_k(t-\tau_1)}\}\right] \\ &= \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \left(\frac{1}{z_k - \mu} \right) \{e^{-\mu t} \mathcal{L}^\tau(-\mu) - e^{-z_k t} \mathcal{L}^\tau(-z_k)\} + o(e^{-Rt}), \end{aligned} \quad (71)$$

for $k = 1, \dots, N$. Next, on the second term, one has

$$\begin{aligned} e^{-z_k t} \int_0^t e^{z_k s} f(s) ds &= e^{-z_k t} \mathcal{L}^\tau(-z_k) - e^{-z_k t} \int_t^\infty e^{z_k s} f(s) ds \\ &= e^{-z_k t} \mathcal{L}^\tau(-z_k) + o(e^{-z_N t}) \end{aligned} \quad (72)$$

since

$$\begin{aligned} \left| e^{-z_k t} \int_t^\infty e^{z_k s} f(s) ds \right| &= \left| e^{-z_k t} \int_t^\infty e^{(z_k - R)s} e^{Rs} f(s) ds \right| \leq e^{-\operatorname{Re}(z_k)t} \int_t^\infty e^{(\operatorname{Re}(z_k) - R)s} e^{Rs} f(s) ds \\ &\leq e^{-\operatorname{Re}(z_k)t} e^{(\operatorname{Re}(z_k) - R)t} \int_t^\infty e^{Rs} f(s) ds = e^{-Rt} \int_0^\infty e^{Rs} f(s) ds = o(e^{-z_N t}). \end{aligned}$$

Then using (36) and (70) with (71) and (72), and since an $o(e^{-Rt})$ is a fortiori an $o(e^{-z_N t})$, the second term of (67) (except for the term involving $o(e^{-z_N x})$ in $v(x)$ in (36)) is now given by

$$\begin{aligned} &\int_0^t [v(s) - o(e^{-z_N s})] \tilde{b}'_1(t-s) ds \\ &= \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \left[\sum_{k=1}^N \gamma_k \left(\frac{\mu}{z_k - \mu} \right) \left\{ e^{-z_k t} \mathcal{L}^\tau(-z_k) - e^{-\mu t} \mathcal{L}^\tau(-\mu) \right\} + \gamma_k e^{-z_k t} \mathcal{L}^\tau(-z_k) \right] + o(e^{-z_N t}) \\ &= \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \left[\sum_{k=1}^N \gamma_k \left(\frac{z_k}{z_k - \mu} e^{-z_k t} \mathcal{L}^\tau(-z_k) - \frac{\mu}{z_k - \mu} e^{-\mu t} \mathcal{L}^\tau(-\mu) \right) \right] + o(e^{-z_N t}). \quad (73) \end{aligned}$$

Recall that function $\eta(\cdot)$ is defined by (37). Then, putting the expression for $\tilde{b}'_1(t)$ into the integral, it follows that

$$\begin{aligned} \int_0^t o(e^{-z_N s}) \tilde{b}'_1(t-s) ds &= \int_0^t \eta(s) e^{-z_N s} \tilde{b}'_1(t-s) ds \\ &= \int_0^t \eta(s) e^{-z_N s} \left[-\mu \tilde{b}_1(t-s) + \mathbb{E}[X] \frac{\mu}{\mu + \delta} f(t-s) \right] ds. \quad (74) \end{aligned}$$

We start by considering $\int_0^t \eta(s) e^{-z_N s} f(t-s) ds$ which can be written as

$$\int_0^t \eta(t-s) e^{-z_N(t-s)} f(s) ds = e^{-z_N t} \int_0^\infty \eta(t-s) \mathbb{1}_{\{0 < s < t\}} e^{z_N s} f(s) ds.$$

The fact that $\int_0^\infty |e^{z_N s} f(s) ds| = \int_0^\infty e^{(\operatorname{Re}(z_N))s} f(s) ds$ is convergent implies, by dominated convergence,

$$\int_0^\infty \eta(t-s) \mathbb{1}_{\{0 < s < t\}} e^{z_N s} f(s) ds \longrightarrow 0, \quad t \rightarrow \infty.$$

Consequently,

$$\int_0^t \eta(s) e^{-z_N s} f(t-s) ds = o(e^{-z_N t}), \quad t \rightarrow \infty. \quad (75)$$

Now we turn our attention to the first term of (74) involving $\int_0^t \eta(s) e^{-z_N s} \tilde{b}_1(t-s) ds$. Writing from (51) (see also (68))

$$\tilde{b}_1(t) = \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \mathbb{E}[\mathbb{1}_{\{\tau_1 < t\}} e^{-\mu(t-\tau_1)}] = \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \mathcal{L}^\tau(-\mu) e^{-\mu t} - \mathbb{E}[X] \cdot \frac{\mu}{\mu + \delta} \mathbb{E}[\mathbb{1}_{\{\tau_1 \geq t\}} e^{-\mu(t-\tau_1)}],$$

we then split $\int_0^t \eta(s)e^{-zNs}\tilde{b}_1(t-s)ds$ into two parts, namely $\mathbb{E}[X] \cdot \frac{\mu}{\mu+\delta} \mathcal{L}^\tau(-\mu) \int_0^t \eta(s)e^{-zNs}e^{-\mu(t-s)}ds$ and $\mathbb{E}[X] \cdot \frac{\mu}{\mu+\delta} \int_0^t \eta(s)e^{-zNs} \mathbb{E}[\mathbb{1}_{\{\tau_1 \geq t-s\}} e^{-\mu((t-s)-\tau_1)}] ds$. The first term is expressed as

$$\begin{aligned} & \mathbb{E}[X] \cdot \frac{\mu}{\mu+\delta} \mathcal{L}^\tau(-\mu) \int_0^t \eta(s)e^{-zNs}e^{-\mu(t-s)}ds \\ &= \mathbb{E}[X] \cdot \frac{\mu}{\mu+\delta} \mathcal{L}^\tau(-\mu) \left[\int_0^\infty \eta(s)e^{-zNs}e^{\mu s}ds \right] e^{-\mu t} - \mathbb{E}[X] \cdot \frac{\mu}{\mu+\delta} \mathcal{L}^\tau(-\mu) \left[\int_t^\infty \eta(s)e^{-zNs}e^{\mu s}ds \right] e^{-\mu t} \\ &= \mathbb{E}[X] \cdot \frac{\mu}{\mu+\delta} \mathcal{L}^\tau(-\mu) \left[\int_0^\infty \eta(s)e^{-zNs}e^{\mu s}ds \right] e^{-\mu t} + o(e^{-zNt}), \end{aligned} \quad (76)$$

where the latter term $o(e^{-zNt})$ being again justified as in (63). Now (63) implies that the second term verifies, by dominated convergence

$$\begin{aligned} & \mathbb{E}[X] \cdot \frac{\mu}{\mu+\delta} \int_0^t \eta(s)e^{-zNs} \mathbb{E}[\mathbb{1}_{\{\tau_1 \geq t-s\}} e^{-\mu((t-s)-\tau_1)}] ds \\ &= \mathbb{E}[X] \cdot \frac{\mu}{\mu+\delta} e^{-zNt} \int_0^t \eta(t-s)e^{zNs} \mathbb{E}[\mathbb{1}_{\{\tau_1 \geq s\}} e^{-\mu(s-\tau_1)}] ds = o(e^{-zNt}). \end{aligned} \quad (77)$$

Gathering (76) and (77) thus yields

$$\int_0^t \eta(s)e^{-zNs}\tilde{b}_n(t-s)ds = \mathbb{E}[X] \cdot \frac{\mu}{\mu+\delta} \mathcal{L}^\tau(-\mu) \left[\int_0^\infty \eta(s)e^{(\mu-zN)s}ds \right] e^{-\mu t} + o(e^{-zNt}). \quad (78)$$

Then from (73) and (74) with (75) and (78) we get

$$\begin{aligned} \int_0^t v(s)\tilde{b}'_1(t-s)ds &= \mathbb{E}[X] \cdot \frac{\mu}{\mu+\delta} \left[\sum_{k=1}^N \gamma_k \left(\frac{z_k}{z_k-\mu} e^{-z_k t} \mathcal{L}^\tau(-z_k) - \frac{\mu}{z_k-\mu} e^{-\mu t} \mathcal{L}^\tau(-\mu) \right) \right] \\ &\quad - \mathbb{E}[X] \cdot \frac{\mu^2}{\mu+\delta} \mathcal{L}^\tau(-\mu) \left[\int_0^\infty \eta(s)e^{(\mu-zN)s}ds \right] e^{-\mu t} + o(e^{-zNt}), \quad t \rightarrow \infty. \end{aligned}$$

Hence the above result together with (69) allows us to have an expression for (67) as

$$I_2(t) = Ae^{-\mu t} + \sum_{k=1}^N B_k e^{-z_k t} + o(e^{-zNt}), \quad (79)$$

where A and B_k for $k = 1, \dots, N$ are defined by (38) and (39). As a result, combining (66) and (79) leads the theorem.

7.6. Proof of Theorem 18

Proof of Lemma 16. We shall start by proving the properties for $\tilde{M}_1(t, \delta)$, as those for $\tilde{M}_2(t, \delta)$ are a bit more technical but follow in a similar way. Let us write

$$\tilde{M}_1(t, \delta) = \sum_{i=1}^{\infty} \psi_i(t, \delta), \quad \psi_i(t, \delta) := \mathbb{E}[e^{-\delta(T_i+L_i-t)} \mathbb{1}_{\{T_i \leq t < T_i+L_i\}}], \quad i \in \mathbb{N}. \quad (80)$$

We first start by proving that $\psi_i(t, \delta)$ is defined and analytic on set D_a . Indeed, inequality

$$\left| \delta^j \frac{(-1)^j}{j!} (T_i + L_i - t)^j \mathbb{1}_{\{T_i \leq t < T_i+L_i\}} \right| \leq a^j \frac{1}{j!} L_i^j, \quad j \in \mathbb{N}, \quad \delta \in D_a, \quad (81)$$

coupled with the fact that $\sum_{j=0}^{\infty} \mathbb{E} \left[a^j \frac{1}{j!} L_i^j \right] = \mathbb{E}[e^{aL}] = \frac{\mu}{\mu-a} < +\infty$ by (44), yields that

$$\sum_{j=0}^{\infty} \delta^j \mathbb{E} \left[\frac{(-1)^j}{j!} (T_i + L_i - t)^j \mathbb{1}_{\{T_i \leq t < T_i + L_i\}} \right]$$

is a convergent series on $\delta \in D_a$ and that $\delta \mapsto \psi_i(t, \delta)$ is analytic on that set for all $t \geq 0$, and admits the above power series expansion in δ . Now one checks easily, by independence of L_i and T_i ,

$$\psi_i(t, \delta) \leq \mathbb{E}[e^{aL_i} \mathbb{1}_{\{T_i \leq t\}}] = \mathbb{E}[e^{aL}] \Pr[T_i \leq t], \quad \forall \delta \in D_a, \quad (82)$$

with $\sum_{i=1}^{\infty} \mathbb{E}[e^{aL}] \Pr[T_i \leq t] = \mathbb{E}[e^{aL}] m(t) < +\infty$. This yields that for all $t \geq 0$, series $\sum_{i=1}^{\infty} \psi_i(t, \delta)$ converges normally on $\delta \in D_a$. Thus for all $t \geq 0$, $\delta \mapsto \tilde{M}_1(t, \delta)$ is thus analytic as the uniform limit of an analytic sequence of functions on compact set D_a .

We then move on $\tilde{M}_2(t, \delta)$. Similar to (80), one has

$$\tilde{M}_2(t, \delta) = \sum_{r,j=1}^{\infty} \pi_{r,j}(t, \delta), \quad \pi_{r,j}(t, \delta) := \mathbb{E}[e^{-\delta(T_r + L_r - t)} \mathbb{1}_{\{T_r \leq t < T_r + L_r\}} e^{-\delta(T_j + L_j - t)} \mathbb{1}_{\{T_j \leq t < T_j + L_j\}}].$$

The analog of (81) is

$$\left| \delta^p \frac{(-1)^p}{p!} [(T_r + L_r - t) + (T_j + L_j - t)]^p \mathbb{1}_{\{T_r \leq t < T_r + L_r\}} \mathbb{1}_{\{T_j \leq t < T_j + L_j\}} \right| \leq (a/2)^p \frac{1}{p!} [L_r + L_j]^p, \\ r \in \mathbb{N}, j \in \mathbb{N}, \quad \delta \in D_{a/2},$$

with $\sum_{p=0}^{\infty} (a/2)^p \frac{1}{p!} [L_r + L_j]^p = \mathbb{E}[e^{a(L_r + L_j)/2}] \leq \mathbb{E}[e^{aL}]$ (by Jensen's inequality), a finite quantity, so that $\delta \in D_{a/2} \mapsto \pi_{r,j}(t, \delta)$ is analytic. The analog of (82) is

$$\pi_{r,j}(t, \delta) \leq \mathbb{E} \left[e^{a(L_r + L_j)/2} \mathbb{1}_{\{T_r \leq t\}} \mathbb{1}_{\{T_j \leq t\}} \right], \quad r \in \mathbb{N}, j \in \mathbb{N}, \delta \in D_{a/2}, \quad (83)$$

with, again thanks to Jensen's inequality as well as independence of (L_r, L_j) from (T_r, T_j) ,

$$\sum_{r,j=1}^{\infty} \mathbb{E}[e^{a(L_r + L_j)/2} \mathbb{1}_{\{T_r \leq t\}} \mathbb{1}_{\{T_j \leq t\}}] \leq \mathbb{E}[e^{aL}] \sum_{r,j=1}^{\infty} \mathbb{E} \left[\mathbb{1}_{\{T_r \leq t\}} \mathbb{1}_{\{T_j \leq t\}} \right] = \mathbb{E}[e^{aL}] \mathbb{E}[N_t^2] < +\infty.$$

Hence, from (83), $\sum_{r,j=1}^{\infty} \pi_{r,j}(t, \delta) = \tilde{M}_2(t, \delta)$ converges normally on $\delta \in D_{a/2}$, and is analytic on this set by the same argument as $\delta \mapsto \tilde{M}_1(t, \delta)$. Note that we used that N_t admits the second moment, a fact that holds because $\mathbb{E}[\tau_1^2] < +\infty$, see e.g. [2, Chapter V.6]. \square

Prior to proving Lemma 17, we prove a few upper bounds concerning $\tilde{M}_1(t, \delta)$. First, we note that deriving $\tilde{b}_1(t) = \frac{\mu}{\mu+\delta} e^{-\mu t} \int_0^t e^{\mu s} dF(s)$ yields $\tilde{b}'_1(t) = -\mu \tilde{b}_1(t) + \frac{\mu}{\mu+\delta} f(t)$. Besides, since (A1) holds, a density $u(t) = m'(t)$ of renewal function exists and is bounded by above by $C > 0$ thanks to Lemma 1. Both these facts entail, deriving (33), the following

$$\left| \tilde{M}'_1(t) \right| = \left| \int_0^t \tilde{b}'_1(t-s) m'(s) ds + \tilde{b}_1(0) m'(t) \right| = \left| \int_0^t \tilde{b}'_1(t-s) m'(s) ds \right|$$

as $f(\cdot)$ is a density, so that $\tilde{b}_1(0) = 0$. Then one finds

$$\begin{aligned} \left| \tilde{M}'_1(t) \right| &\leq \mu \int_0^t \left| \tilde{b}_1(t-s)m'(s) \right| ds + \left| \frac{\mu}{\mu+\delta} \right| \int_0^t |f(t-s)m'(s)| ds \\ &\leq \mu C \int_0^\infty \left| \tilde{b}_1(s) \right| ds + \left| \frac{\mu}{\mu+\delta} \right| C \int_0^\infty f(s) ds \\ &\leq C \left[\left| \frac{\mu}{\mu+\delta} \right| + \left| \frac{\mu}{\mu+\delta} \right| \right] \leq \frac{2C\mu}{\mu-|\delta|}, \end{aligned} \quad (84)$$

where the last line is due to the fact that $f(\cdot)$ is a density, and $\int_0^\infty |\tilde{b}_1(s)| ds \leq C \left| \frac{\mu}{\mu+\delta} \right|$ from (52).

Proof of Lemma 17. We again start with $\tilde{M}_1(t, \delta)$. The key to is to use expansions for $\tilde{M}_1(t) = \tilde{M}_1(t, \delta)$ in Theorem 14 and particularly the dependence of this expansion in δ as discussed in Remark 15. Indeed, an immediate consequence of (41) and (42) from Remark 15 is that

$$\begin{aligned} \left| \tilde{M}_1(t, \delta) - \chi_1(\delta) \right| &\leq \frac{M^*}{\mu - |\delta|} \left[e^{-\mu t} + \sum_{k=1}^N e^{-\operatorname{Re}(z_k)t} + \zeta(t) e^{-\operatorname{Re}(z_N)t} \right] \\ &\leq \frac{M^*}{\mu - a} \left[e^{-\mu t} + \sum_{k=1}^N e^{-\operatorname{Re}(z_k)t} + \zeta(t) e^{-\operatorname{Re}(z_N)t} \right], \quad \forall \delta \in D_a, \end{aligned}$$

for some constant M^* independent from δ and t , which implies uniform convergence of $\tilde{M}_1(t, \delta)$ as $t \rightarrow \infty$ towards $\chi_1(\delta)$ on $\delta \in D_a$.

We then move on to $\tilde{M}_2(t, \delta)$. Relation (8) when $k = 1$, $X_j = 1$, $L \sim \mathcal{E}(\mu)$, along with (6) and (7) yields the following expression

$$\tilde{b}_2(t) = \tilde{b}_2(t, \delta) = \varphi_0(t) + 2\varphi_1(t), \quad (85)$$

$$\varphi_0(t) = \varphi_{0,2}(t, \delta) = \frac{\mu}{\mu + 2\delta} \mathbb{E}[e^{-\mu(t-\tau_1)} \mathbb{1}_{\{\tau_1 < t\}}] = \frac{\mu}{\mu + 2\delta} \int_0^t e^{-\mu(t-s)} f(s) ds, \quad (86)$$

$$\varphi_1(t) = \varphi_{1,2}(t, \delta) = \frac{\mu}{\mu + \delta} \mathbb{E}[\tilde{M}_1(t - \tau_1, \delta) e^{-\mu(t-\tau_1)} \mathbb{1}_{\{\tau_1 < t\}}] = \frac{\mu}{\mu + \delta} \int_0^t \tilde{M}_1(t-s, \delta) e^{-\mu(t-s)} f(s) ds. \quad (87)$$

Differentiating (86) and (87) with respect to t results in

$$\varphi'_0(t) = \frac{\mu}{\mu + 2\delta} \left[-\mu \int_0^t e^{-\mu(t-s)} f(s) ds + f(t) \right], \quad (88)$$

$$\varphi'_1(t) = \frac{\mu}{\mu + \delta} \left[\int_0^t \left(\tilde{M}'_1(t-s, \delta) - \mu \tilde{M}_1(t-s, \delta) \right) e^{-\mu(t-s)} f(s) ds + f(t) \right]. \quad (89)$$

We are also going to need the following upper bounds for $\varphi_0(t, \delta)$, $\varphi_1(t, \delta)$, obtained from (86), (87), and the fact that $t \mapsto \tilde{M}_1(t, \delta)$ is uniformly bounded in $\delta \in D_{a/2}$ by some constant \tilde{C} (independent from δ and t , a consequence of the fact that it converges uniformly on that set):

$$\begin{aligned} |\varphi_0(t, \delta)| &\leq \left| \frac{\mu}{\mu + 2\delta} \right| e^{-\mu t} \int_0^\infty e^{\mu s} f(s) ds \leq \frac{\mu}{\mu - a} C_0 e^{-\mu t}, \quad \delta \in D_{a/2}, \quad (90) \\ |\varphi_1(t, \delta)| &\leq \left| \frac{\mu}{\mu + \delta} \right| \tilde{C} e^{-\mu t} \int_0^\infty e^{\mu s} f(s) ds \leq \frac{\mu}{\mu - a/2} C_1 e^{-\mu t}, \quad \delta \in D_{a/2}, \end{aligned}$$

for some constants C_0 and C_1 independent from $\delta \in D_{a/2}$ and t . We also wish to obtain similar bounds for $\varphi'_0(t, \delta)$ and $\varphi'_1(t, \delta)$. The following upper bound for $\varphi'_0(t, \delta)$ is easily obtained thanks to (88):

$$|\varphi'_0(t, \delta)| \leq \left| \frac{\mu}{\mu + 2\delta} \right| \left[\mu e^{-\mu t} \int_0^\infty e^{\mu s} f(s) ds + f(t) \right] \leq \frac{\mu}{\mu - a} [C_0^* e^{-\mu t} + f(t)], \quad \delta \in D_{a/2}, \quad (91)$$

for some constant C_0^* . As to $\varphi'_1(t, \delta)$, the fact that $t \mapsto \tilde{M}_1(t, \delta)$ and $t \mapsto \tilde{M}'_1(t, \delta)$ are uniformly bounded in $\delta \in D_{a/2}$ respectively by \tilde{C} and $2C \frac{\mu}{\mu - a/2}$ (thanks to (84)), easily yields from (89)

$$|\varphi'_1(t, \delta)| \leq \frac{\mu}{\mu - a/2} [C_1^* e^{-\mu t} + f(t)], \quad \delta \in D_{a/2},$$

for some constant $C_1^* > 0$. Getting back to our original concern of showing that $\tilde{M}_2(t, \delta)$ converges uniformly, we first note that, in view of (33) and (85), it is clear that it is necessary and sufficient to prove that

$$\delta \mapsto \int_0^t \varphi_l(t-s, \delta) dm(s), \quad l = 0, 1,$$

converges uniformly on $\delta \in D_{a/2}$ as $t \rightarrow \infty$ towards $\frac{1}{\mathbb{E}[\tau_1]} \int_0^\infty \varphi_l(s, \delta) ds$ for $l = 0, 1$. Details will be given only for $l = 0$ as similar proof is applicable for $l = 1$. The starting point is the following decomposition, already used in Relation (61) in Section 7.5:

$$\begin{aligned} \int_0^t \varphi_0(t-s, \delta) dm(s) - \frac{1}{\mathbb{E}[\tau_1]} \int_0^\infty \varphi_0(s, \delta) ds &= -\frac{1}{\mathbb{E}[\tau_1]} \int_t^\infty \varphi_0(s, \delta) ds + \int_0^t \varphi_0(t-s, \delta) dv(x) \\ &:= I_1(t, \delta) + I_2(t, \delta). \end{aligned} \quad (92)$$

Thus, in view of (92), it suffices to prove that $I_1(t, \delta)$ and $I_2(t, \delta)$ uniformly converge towards 0 as $t \rightarrow \infty$ on $\delta \in D_{a/2}$. Uniform convergence of $I_1(t, \delta)$ is obtained thanks to (90) that entail:

$$\sup_{\delta \in D_{a/2}} |I_1(t, \delta)| \leq \frac{1}{\mathbb{E}[\tau_1]} \frac{1}{\mu - a} C_0 e^{-\mu t} \rightarrow 0, \quad t \rightarrow \infty.$$

As to $I_2(t, \delta)$, performing an integration by parts as in (67) yields

$$I_2(t, \delta) = \varphi_0(t, \delta) v(0^-) + \int_0^t v(s) \varphi'_0(t-s, \delta) ds.$$

The first term on the right-hand side uniformly converges to 0 on $\delta \in D_{a/2}$ thanks to (90). As to the second term, we use the inequality (91) to get

$$\left| \int_0^t v(s) \varphi'_0(t-s, \delta) ds \right| \leq \int_0^t |v(s)| |\varphi'_0(t-s, \delta)| ds \leq \frac{\mu}{\mu - a} \int_0^t |v(s)| [C_0^* e^{-\mu(t-s)} + f(t-s)] ds, \quad \delta \in D_{a/2}. \quad (93)$$

Note that $\int_0^t |v(s)| e^{-\mu(t-s)} ds$ tends to zero by dominated convergence theorem, as $\int_0^\infty |v(s)| ds$ is finite (a direct consequence of expansion (36)). Also, the light tailed assumption in (44) for τ_1 entails that for all $j = 1, \dots, N$ one has $\int_0^t e^{-z_j s} f(t-s) ds = e^{-z_j t} \int_0^t e^{z_j s} f(s) ds \rightarrow 0$ as $t \rightarrow \infty$. Similarly, $\int_0^t \eta(s) e^{-z_j s} f(t-s) ds \rightarrow 0$ where $\eta(x)$ is defined by (37). Hence

$\int_0^t |v(s)|f(t-s)ds$ tends to zero as $t \rightarrow \infty$. Then, from (93) $I_2(t, \delta)$ uniformly converges to 0 on $\delta \in D_{a/2}$. Thus, all in all, $\tilde{M}_2(t, \delta)$ converges uniformly on $\delta \in D_{a/2}$ towards $\chi_2(\delta)$. \square

Proof of Theorem 18. Since $0 \leq -\frac{\partial}{\partial \delta} \tilde{Z}(t, \delta) \Big|_{\delta=0} = D(t) \leq \sum_{i=1}^{N(t)} L_i$ is integrable, it is possible to exchange derivation with respect to δ and expectation and one has for all $t > 0$

$$-\frac{\partial}{\partial \delta} \tilde{M}_1(t, \delta) \Big|_{\delta=0} = -\frac{\partial}{\partial \delta} \mathbb{E}[\tilde{Z}(t, \delta)] \Big|_{\delta=0} = -\mathbb{E} \left[\frac{\partial}{\partial \delta} \tilde{Z}(t, \delta) \Big|_{\delta=0} \right] = \mathbb{E}[D(t)]. \quad (94)$$

The main point in the proof is to be able to pass to the limit in (94) as $t \rightarrow \infty$. To do this, we use the fact that we proved in Lemma 16 that $\delta \mapsto \tilde{M}_{n(1)}(t, \delta)$ is analytic on the set D_a where $a < \mu$ is arbitrary. Since by Lemma 17, $\tilde{M}_1(t, \delta)$ uniformly converges towards $\chi_1(\delta)$ on this set, a standard result in complex analysis states that the limiting function $\delta \mapsto \chi_1(\delta)$ is analytic on the same set, hence in particular at $\delta = 0$ (which is known from its expression (28)), but, more importantly, that one can interchange derivation and passage to the limit, i.e.

$$\lim_{t \rightarrow \infty} \frac{\partial}{\partial \delta} \tilde{M}_1(t, \delta) \Big|_{\delta=0} = \frac{\partial}{\partial \delta} \left[\lim_{t \rightarrow \infty} \tilde{M}_1(t, \delta) \right] \Big|_{\delta=0} = \frac{\partial}{\partial \delta} \chi_1(\delta) \Big|_{\delta=0}.$$

Expression of $\chi_1(\delta)$ in the case $k = 1$ is given in Corollary 11, Expression (28) with $X_j = 1$, yielding (45).

Let us move on to the covariance of $D(t)$ and queue size $Z_1(t, 0)$. One has $-\frac{\partial}{\partial \delta} [Z_1(t, \delta)]^2 \Big|_{\delta=0} = 2D(t)Z_1(t, 0)$, and since the latter is integrable due to $D(t)Z_1(t, 0) \leq \left(\sum_{i=1}^{N(t)} L_i\right) N(t)$, as in (94), interchanging expectation and derivation results in

$$-\frac{\partial}{\partial \delta} \tilde{M}_2(t, \delta) \Big|_{\delta=0} = 2\mathbb{E}[D(t)Z_1(t, 0)].$$

The same argument of analyticity of $\delta \mapsto \tilde{M}_2(t, \delta)$ on $\delta \in D_{a/2}$ in Lemma 16, coupled with the fact that uniform convergence result as $t \rightarrow \infty$ in Lemma 17 yields that $\lim_{t \rightarrow \infty} \frac{\partial}{\partial \delta} \tilde{M}_2(t, \delta) \Big|_{\delta=0} = \frac{\partial}{\partial \delta} \chi_2(\delta) \Big|_{\delta=0}$. Now the fact that $\lim_{t \rightarrow \infty} \tilde{M}_1(t, 0) = \chi_1(0)$ and $\lim_{t \rightarrow \infty} \frac{\partial}{\partial \delta} \tilde{M}_1(t, \delta) \Big|_{\delta=0} = \frac{\partial}{\partial \delta} \chi_1(\delta) \Big|_{\delta=0}$ implies

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Cov}[D(t), Z_1(t, 0)] &= \lim_{t \rightarrow \infty} \mathbb{E}[D(t)Z_1(t, 0)] - \mathbb{E}[D(t)]\mathbb{E}[Z_1(t, 0)] \\ &= -\frac{1}{2} \frac{\partial}{\partial \delta} \chi_2(\delta) \Big|_{\delta=0} + \chi_1(0) \cdot \frac{\partial}{\partial \delta} \chi_1(\delta) \Big|_{\delta=0}. \end{aligned} \quad (95)$$

Expression (29) with $X_j = 1$ yields $\chi_2(\delta) = \frac{1}{\mathbb{E}[\tau_1]} \left(\frac{1}{\mu+2\delta} + \frac{\mu}{(\mu+\delta)^2} \frac{\mathcal{L}^\tau(\mu)}{1-\mathcal{L}^\tau(\mu)} \right)$, and in turn,

$$\frac{\partial}{\partial \delta} \chi_2(\delta) \Big|_{\delta=0} = -\frac{1}{\mathbb{E}[\tau_1]} \left(\frac{2}{(\mu+2\delta)^2} + \frac{2\mu}{(\mu+\delta)^3} \frac{\mathcal{L}^\tau(\mu)}{1-\mathcal{L}^\tau(\mu)} \right) \Big|_{\delta=0} = -\frac{2}{\mu^2 \mathbb{E}[\tau_1]} \left(1 + \frac{\mathcal{L}^\tau(\mu)}{1-\mathcal{L}^\tau(\mu)} \right).$$

Hence, substitution of the above expression together with $\chi_1(\delta)$ obtained previously into (95) yields (46) for the limiting covariance. \square

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