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On a multivariate renewal-reward process involving time delays: Applications to IBNR process and infinite server queues

*L. Rabehasaina*¹ and *J.-K. Woo*²

Abstract

This paper considers a particular renewal-reward process with multivariate discounted rewards (inputs) where the arrival epochs are adjusted by adding some random delays. Then this accumulated reward can be regarded as multivariate discounted *Incurred But Not Reported* (IBNR) claims in actuarial science and some important quantities studied in queueing theory such as the number of customers in $G/G/\infty$ queues with correlated batch arrivals. We study the long term behavior of this process as well as its moments. Asymptotic expressions and bounds for the quantities of our interest, and also convergence result for the distribution of this process after renormalization, are studied, when interarrival times and time delays are light tailed. Next, assuming exponentially distributed delays, we derive some explicit and numerically feasible expressions for the limiting joint moments. Finally, in infinite server queues with renewal arrival process, we obtain the limiting joint moments as well as the limiting expectation of workload, and the limiting covariance of queue size and workload.

MSC classification: 60G50, 60K30, 62Pxx.

Keywords: Renewal-reward process, Multivariate rewards, Incurred But Not Reported (IBNR) claims, Infinite server queues, Workload, Convergence in distribution

1 Introduction and notation

Many situations in which processes restart probabilistically at renewal instants and there are non-negative rewards associated with each renewal epoch, are well described by a multivariate renewal-reward process. For example, a multivariate reward function can be viewed as an accumulated cost from different types of properties or infrastructures caused by a single catastrophe event, which is of interest in actuarial science and reliability analysis. The asymptotic distribution and covariance function of a multivariate reward function were studied by [16] who extended the result of [3] to multivariate case. In an insurance context, much research about the aggregate discounted claims has been done on its moment under renewal claim arrival processes. For example, [11], [12], [13], and [14] in renewal process, and [22] in the dependent renewal process.

In this paper, we assume that there are time lags added to the original arrival times of renewal process. These delayed renewal epochs allow us to study the quantities related to infinite server queues with correlated batch arrivals and multivariate *Incurred But Not Reported* (IBNR) claims where there is a delay in reporting or payment for claims. Furthermore, rewards are accumulated as a discounted value, which is useful to analyze the discounted multivariate IBNR claim amounts and workload of the queue (the required time to empty the queue). For an univariate case, IBNR claim count with batch arrivals was considered by [7] and the total discounted IBNR claim amount was studied by [10]. For the multivariate case, [21] provided expressions for joint moments of multivariate IBNR claims which are computable recursively. As mentioned previously, a direct application to some problems in infinite server queues is also available. For example, suppose that the bulk size random variable is multivariate (i.e. correlated) and service time distribution is dependent on the type of input, then a multivariate reward function incorporating time delays up to time t (with zero discounting factor) is essentially the number of customers in the system up to time t . In the infinite server queues with multiple batch Markovian arrival streams, a time-dependent matrix joint generating function of the number of customers in the system was derived by [15].

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We note that it is usually difficult to derive a distribution for this compound renewal input since for a general arrival process there is no concrete representation for an inversion of the complicated moment generating function for this quantity. In this sense, it is appealing to study the long term behavior of the process in terms of its moment and distribution. From [21], explicit expressions for the joint moments of multivariate aggregate discounted claims involving time delays in renewal process are computable recursively. However, an analytic expression of the lower moment which appears in its integral term, is required for a calculation of the higher moment. Therefore, our objective here is to develop simpler approximation methods such as asymptotics and bound results for the joint moment of a multivariate discounted reward function incorporating time delays. To the best of our knowledge, these kinds of approximation approaches have never been developed in the analysis of a multivariate renewal-reward process with discounted inputs and time delays (or interpreted as a multivariate discounted IBNR claim process in an actuarial context). Also, in a queueing context, a relationship between multivariate discounted IBNR claim process and quantities studied in infinite server queues with correlated batch arrivals and a discounting factor is firstly exploited in this paper. In particular, some asymptotic results regarding queueing theoretic applications such as the workload in $G/\cdot/\infty$ queue, are obtained. For the number of IBNR claims, a direct relation to the number of customers in the infinite server queues with batch arrivals is well known as discussed in the literature, e.g. [9], [10], [18], [19], [20]. The transient behavior of a distribution of the number of customer in various multichannel bulk queues was studied in [4]. See also [2] for example. Moreover, when interarrival times are light tailed, we are able to quantify the approximation precision by providing many terms for the asymptotics for the first order moment of our process. We note that this approach was previously found in [3, Lemma 1] where 2-terms asymptotics for a general renewal reward process without delays was provided, see also [16] for an expansion of the covariance.

In the following, we describe the model by calling renewal and reward as batch and input respectively. We shall suppose that the batch arrival process $\{N_t\}_{t \geq 0}$ is a renewal process with a sequence of independent and identically distributed (iid) positive continuous random variables (rv)s $\{T_i\}_{i=1}^{\infty}$ representing the arrival time of the i th batch with $T_0 \equiv 0$. Let $\tau_i = T_i - T_{i-1}$ be the interarrival time of the i th batch with a common probability density function (pdf) f , distribution F , and Laplace transform $\mathcal{L}^{\tau}(s) = E[e^{-s\tau_1}]$ for $s \geq 0$. Each batch arrival containing several (k) types of inputs which may simultaneously occur from the same renewal event (e.g. [16], [21]). Let us denote the j -type of input from the i th batch as $X_{i,j}$ where $\{X_{i,j}\}_{i=1}^{\infty}$ is a sequence of iid rvs. A vector for multivariate variables is denoted as $X = (X_1, X_2, \dots, X_k)$. Here multivariate input values are assumed to be dependent on the occurrence time and/or the adjusted time by adding a random delay. This time delay for the j -type of input from i th batch is denoted by $L_{i,j}$ where $\{L_{i,j}\}_{i=1}^{\infty}$ is a sequence of iid rvs with a common pdf w_j and distribution W_j . For the sake of simplicity let us assume a constant force of interest δ to discount input values to time 0. Now we define the following discounted compound delayed process

$$Z(t) = (Z_1(t), \dots, Z_k(t)) = Z^{\delta}(t), \quad t \geq 0, \quad (1)$$

where

$$Z_j(t) := \sum_{i=1}^{N_t} e^{-\delta(T_i + L_{i,j})} X_{i,j} \mathbf{1}_{\{T_i + L_{i,j} > t\}} = \sum_{i=1}^{\infty} e^{-\delta(T_i + L_{i,j})} X_{i,j} \mathbf{1}_{\{T_i \leq t < T_i + L_{i,j}\}}, \quad j \in \{1, \dots, k\}. \quad (2)$$

In most cases in this paper, we suppose that the discounted factor δ is real and non negative because this has some direct actuarial or queueing interpretation. However, it has to be pointed out that, mathematically speaking, Definitions (1) and (2) can in some cases be extended to some *complex* δ , as is the case in Section 3.2 where $\delta \in \mathbb{C}$ is needed for technical purposes.

Throughout the paper, we assume that vector X admits joint moments of all order. Let us denote **(A)** for the following assumption:

$$\textbf{(A)} \quad \text{Density } f(\cdot) \text{ is bounded.}$$

An important consequence of **(A)** is the following result, of which proof is given at the beginning of Section 5.

Lemma 1 *If (A) holds then the associated renewal function $m : t \geq 0 \mapsto \mathbb{E}[N_t]$ admits a density $u(t)$, and this latter verifies*

$$u(t) = \frac{d}{dt}m(t) = \sum_{j=0}^{\infty} f^{*(j)}(t). \quad (3)$$

Besides, this density is upper bounded: There exists $C > 0$ such that $u(t) \leq C, \forall t \geq 0$.

Not all results in this paper require Assumption (A) to hold. We refer to it only when it is needed in what follows.

Notation. The n th joint moment for $Z(t)$ is denoted as

$$M_n(t) = \mathbb{E} \left[\prod_{j=1}^k Z_j^{n_j}(t) \right], \quad t \geq 0, \quad n = (n_1, \dots, n_k) \in \mathbb{N}^k. \quad (4)$$

For notational convenience, we let, for all $n = (n_1, \dots, n_k) \in \mathbb{N}^k$ and $t \geq 0$,

$$\eta_n := \sum_{i=1}^k n_i,$$

$$\tilde{M}_n(t) := e^{\eta_n \delta t} M_n(t), \quad (5)$$

$$\tilde{b}_n(t) := e^{\eta_n \delta t} b_n(t). \quad (6)$$

We define the natural partial order on set \mathbb{N}^k as follows. We say that two vectors ℓ and n in \mathbb{N}^k verify $\ell < n$ if $\ell_i \leq n_i$ for all $i = 1, \dots, k$ and $\ell_i < n_i$ for (at least) an i , i.e. $\eta_n > \eta_\ell$. Let us introduce, for all $n \in \mathbb{N}^k$,

$$C_\ell = C_{\ell, n} := \{j = 1, \dots, k \mid \ell_j < n_j\} \subset \{1, \dots, k\}.$$

We will denote by $n(i) \in \mathbb{N}^k$ the vector of which j th entry is $\delta_{i,j}$ where $\delta_{i,j}$ is the Kronecker delta function.

It is convenient to introduce function $t \mapsto \varphi_\ell(t) = \varphi_{\ell, n}(t)$ for $\ell < n$:

$$\varphi_\ell(t) = \mathbb{E} \left[e^{(\eta_n - \eta_\ell) \delta (t - \tau_1)} \tilde{M}_\ell(t - \tau_1) \prod_{j \in C_\ell} \bar{\omega}_{(n_j - \ell_j) \delta, j}(t - \tau_1) \cdot 1_{[\tau_1 < t]} \right], \quad (7)$$

where

$$\bar{\omega}_{\delta, i}(t) = \int_t^\infty e^{-\delta y} dW_i(y). \quad (8)$$

Then using (7), $\tilde{b}_n(t)$ in (6) admits the following expression with the help of Equation (34) in [21]:

$$\tilde{b}_n(t) = \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] \varphi_{\ell, n}(t), \quad (9)$$

and one has that $\tilde{M}_n(t)$ defined in (5) satisfies the following renewal equation (a direct consequence of Theorem 3 in [21])

$$\tilde{M}_n(t) = \tilde{b}_n(t) + \tilde{M}_n \star F(t), \quad t \geq 0, \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}. \quad (10)$$

Lastly, throughout the paper, $\mathcal{E}(\mu_i)$ denotes an exponential distribution with a mean $1/\mu_i$.

Structure of paper. For ease of presentation, in Section 2 we provide our main results without proof. It includes (i) asymptotic behavior for the joint moment of $Z(t)$ given in (4), (ii) bounds for (5), (iii) higher order expansion for asymptotic case of (5) when $n = n(i)$ and exponential time delay, (iv) convergence form of $e^{\delta t} Z(t)$ in distribution, and (v) joint moments of the limiting distribution of $e^{\delta t} Z(t)$ in the case of exponential time delays. In Section 3, we focus on queueing theoretic applications involving some particular $G/G/\infty$ queue with correlated batch arrivals and determine the asymptotic expected workload and covariance of the workload and queue size in the $G/M/\infty$ queue. Section 4 presents limiting moments and covariances when $k = 1$ or 2 in (4), and we profit by this section to give some (re)interpretation of Little's Law. Finally, proofs of all main results and applications are presented in Section 5.

2 Main results

2.1 Asymptotics and bounds

In this section, we study some asymptotic behaviors and bounds for the joint moment of the process $Z(t)$ in (1) defined as (4).

Proposition 2 *One finds the following asymptotic result for the moment of discounted compound delayed process, for all $n \in \mathbb{N}^k$:*

$$M_n(t) \sim \chi_n e^{-\eta_n \delta t}, \quad t \rightarrow \infty,$$

where

$$\chi_n := \frac{\int_0^\infty \tilde{b}_n(t) dt}{\mathbb{E}[\tau_1]}, \quad (11)$$

and $\tilde{b}_n(t)$ is given by (9).

Proof. See Section 5.1. □

It turns out that coefficients χ_n , $n \in \mathbb{N}^k$ are in general not directly computable, as the function $t \mapsto \tilde{b}_n(t)$ in the integral (11) does not have an easy expression, and are defined recursively in the function of $t \mapsto \tilde{M}_\ell(t)$, $\ell < n$. We thus provide in the following easily computable bounds for the χ_n 's and a uniform upper bound in t for $\tilde{M}_n(t)$.

Proposition 3 *Let us suppose that (A) holds. One has the following bounds for all $n \in \mathbb{N}^k$:*

$$\chi_n \leq \frac{1}{\mathbb{E}(\tau_1)} R_n, \quad (12)$$

$$\tilde{M}_n(t) \leq R_n, \quad \forall t \geq 0, \quad (13)$$

where $(R_n)_{n \in \mathbb{N}^k}$ is defined recursively by

$$\begin{aligned} R_{n(i)} &= C \mathbb{E}[X_i] \delta^{-1} \{1 - \mathbb{E}[e^{-\delta L_i}]\}, \quad i = 1, \dots, k, \\ R_n &= C \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] \min_{i \in C_\ell} \mathbb{E}[L_i] \cdot R_\ell, \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}, \end{aligned} \quad (14)$$

Here, the constant C is the upper bound for renewal density $u(t)$ in Lemma 1.

Proof. See Section 5.2. □

We remark that coefficients R_n , $n \in \mathbb{N}^k$, in (14) are easily obtained because R_n is a linear function of the R_ℓ , $\ell < n$, and only involves the joint moments of $X = (X_1, \dots, X_k)$, the Laplace transform of the L_1, \dots, L_k as well as their expectations. Proposition 3 thus provides a uniform upper bound for $\tilde{M}_n(t)$, it however lacks the property that it does not say much what happens when t is small. The following bounds are established under the conditions that the interarrival time distribution τ_1 has either an increasing or a decreasing failure rate.

Proposition 4 (Transient bounds) *If τ_1 has an increasing failure rate (IFR), then one has the lower bound for all $n \in \mathbb{N}^k$:*

$$\tilde{M}_n(t) \geq h_n(t), \quad \forall t \geq 0,$$

Conversely, τ_1 has a decreasing failure rate (DFR), then one has the upper bound for all $n \in \mathbb{N}^k$:

$$\tilde{M}_n(t) \leq h_n(t), \quad \forall t \geq 0,$$

where $t \mapsto h_n(t)$ for $n \in \mathbb{N}^k$ is defined recursively by

$$\begin{aligned} h_{n(i)}(t) &= \mathbb{E}[X_i] \int_0^t \bar{\omega}_{\delta,i} \star H_\delta(y) dy, \quad i = 1, \dots, k, \\ h_n(t) &= \sum_{\ell < n} \binom{n_1}{\ell_1} \dots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] f(0+) \\ &\quad \times \left[\int_0^t \int_0^y e^{-\eta_m \delta z} h_\ell(y-z) e^{-\eta_\ell \delta (y-z)} \prod_{j \in C_\ell} \bar{\omega}_{(n_j - \ell_j) \delta, j}(y-z) dF(z) dy \right], \quad n \in \mathbb{N}^k \setminus \{n(i), i=1, \dots, k\}, \end{aligned} \quad (15)$$

where $f(0+) = \lim_{t \rightarrow 0+} f(t)$ and $H_\delta(t) = \int_0^t e^{-\delta y} dF(y)$ is the discounted interarrival distribution.

Proof. See Section 5.3. □

2.2 High order expansions

In this section, we consider the case of $n = n(i)$ for $i \in \{1, \dots, k\}$ to study how fast $\tilde{M}_n(t)$ converges to χ_n given in Proposition 2 when $t \rightarrow \infty$. As $\tilde{M}_n(t)$ satisfies the renewal equation (10), using its solution it may be expressed as

$$\tilde{M}_n(t) = \int_0^t \tilde{b}_n(t-s) dm(s), \quad (16)$$

and from Proposition 2, recall that

$$\tilde{M}_n(t) \longrightarrow \chi_n = \frac{\int_0^\infty \tilde{b}_n(t) dt}{\mathbb{E}[\tau_1]}, \quad (17)$$

where here $\chi_n = \chi_{n(i)} = \{\mathbb{E}[X_i] \mathbb{E}[L_i] \tilde{w}_{1,i}(\delta)\} / \mathbb{E}[\tau_1]$ and $\tilde{w}_{1,i}(\delta) = \int_0^\infty e^{-\delta x} \bar{W}_i(x) dx / \mathbb{E}[L_i]$ as given later in Corollary 18, Expression (41). From [5], we use the result of higher order expansions for the function $v(x)$ which is related to the renewal function as

$$v(x) := m(x) - \frac{x}{\mathbb{E}[\tau_1]} - \frac{\mathbb{E}[\tau_1^2]}{2\mathbb{E}[\tau_1]^2}, \quad (18)$$

where F here is non-lattice (as it admits a density) and is light tailed, i.e. there exists $R > 0$ such that

$$\int_0^\infty e^{Rx} dF(x) = \mathbb{E}[e^{R\tau_1}] < +\infty. \quad (19)$$

It admits the following expression

$$v(x) = \sum_{j=1}^N \gamma_j e^{-z_j x} + o(e^{-z_N x}), \quad (20)$$

where z_j 's are the solution of $\mathbb{E}(e^{z_j \tau_1}) = 1$ which are in the range of $0 \leq \text{Re}(z_j) \leq R$ for some $R > 0$ and ordered as $\text{Re}(z_j) \leq \text{Re}(z_{j+1})$. In order to hold (20), we in addition require all roots z_1, \dots, z_N to be of multiplicity 1, i.e. such that $\frac{\partial}{\partial z} \mathbb{E}(e^{z \tau_1})|_{z=z_j} \neq 0$ (the condition is not necessary but it enables us to avoid some technicalities later), in which case one has

$$\gamma_j = -\frac{1}{z_j \frac{\partial}{\partial z} \mathbb{E}(e^{z \tau_1})|_{z=z_j}}, \quad j = 1, \dots, N,$$

see [5, Theorem 3]. Although they are complex, the z_j 's actually come in pair as one sees that if z_j verifies $\mathbb{E}(e^{z_j \tau_1}) = 1$ then so does \bar{z}_j , so that one checks in (20) that the right-hand side is in fact real. Furthermore, in the following result we need to write $o(e^{-z_N x})$ term in (20) in the form of

$$o(e^{-z_N x}) = \eta(x) e^{-z_N x} \quad (21)$$

for some function $\eta(x)$ such that $\lim_{x \rightarrow \infty} \eta(x) = 0$.

Theorem 5 Let us assume that time delays L_i are $\mathcal{E}(\mu_i)$ distributed, and define

$$A_i = -\mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \left[\frac{\mathbb{E}[\tau_1^2]}{2\mathbb{E}[\tau_1]^2} + \sum_{k=1}^N \gamma_k \frac{\mu_i}{z_k - \mu_i} + \mu_i \int_0^\infty \eta(s) e^{(\mu_i - z_N)s} ds \right] \mathcal{L}^\tau(-\mu_i) \quad (22)$$

and

$$B_{k,i} = \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \left[\gamma_k \frac{z_k}{z_k - \mu_i} \right] \mathcal{L}^\tau(-z_k). \quad (23)$$

Then $\tilde{M}_n(t)$ in (5) satisfies the following high order expansions

$$\tilde{M}_n(t) = \chi_n + A_i^* e^{-\mu_i t} + \sum_{k=1}^N B_{k,i} e^{-z_k t} + o(e^{-z_N t}), \quad n = n(i), \quad (24)$$

where $A_i^* = A_i - \frac{\mathbb{E}[X_i]}{\mathbb{E}[\tau_1]} \cdot \frac{1}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i)$ with A_i in (22) and $B_{k,i}$ in (23).

Proof. See Section 5.4. □

Note that in expansion (24) the $B_{k,i}$'s are explicit. On the other hand, A_i in (22) features an integral involving function $x \mapsto \eta(x)$ which is not explicit in general. This means that (24) is explicit only if we truncate the expansion to the i_0 th term where $i_0 = \max\{j = 1, \dots, N \mid \operatorname{Re}(z_j) < \mu_i\}$. We may write the expansion in this way, however we prefer to keep a form as general as possible. Besides, we point out on a similar note that an expansion akin to (24) was provided in [3, Lemma 1] for a general renewal reward process in the particular context where there is no time delay, under the weaker assumption that interarrival times and rewards admit the moment of order 1.

Remark 6 (Dependence of (24) in δ) Upon inspecting (22) and (23) one notices that

$$|A_i^*|, |B_{k,i}| \leq \frac{M}{\mu_i + \delta}, \quad k = 1, \dots, N,$$

for all $\delta \geq 0$, where $M > 0$ is a constant independent from δ . On further analysis, one also checks that when δ is *complex* and verifies $|\delta| < \mu_i$ then

$$|A_i^*|, |B_{k,i}| \leq \frac{M}{\mu_i - |\delta|}, \quad k = 1, \dots, N. \quad (25)$$

In particular this inequality also holds when δ is *negative* and larger than $-\mu_i$. Hence, from (25), it is shown that that $\tilde{M}_n(t)$ and χ_n are defined for such a complex δ . This is particularly going to be the case in Section 3.2. Concerning the term $o(e^{-z_N t})$ in (24), one carefully checks from the proof of Theorem 5 that

$$|o(e^{-z_N t})| \leq \frac{1}{\mu_i - |\delta|} \zeta(t) e^{-\operatorname{Re}(z_N)t}, \quad (26)$$

when $\delta \in \mathbb{C}$, $|\delta| < \mu_i$, for some function $\zeta(\cdot)$ independent from δ verifying $\lim_{t \rightarrow \infty} \zeta(t) = 0$.

Remark 7 The exponential distribution assumption for L_i may seem a bit restrictive. In fact, the result in Theorem 5 can be similarly extended to the case of a combination of exponentials. For example, assume that $w_i(x) = \sum_{j=1}^2 p_{ij} \mu_{ij} e^{-\mu_{ij} x}$ where $\sum_{j=1}^2 p_{ij} = 1$. Then a key step in the proof of Theorem 5 in Section 5.4 is to be able to prove a similar structure for the discounted survival function of W_i . For instance, (59) becomes

$$\int_{z-\tau_1}^\infty e^{-\delta s} dW_i(s) = \sum_{j=1}^2 \frac{p_{ij} \mu_{ij}}{\mu_{ij} + \delta} e^{-(\mu_{ij} + \delta)(z - \tau_1)},$$

which is a combination of exponentials. It is thus not hard to be convinced that the rest of steps are similar, hence the details are omitted here for brevity.

2.3 Convergence in distribution of renormalized process

From the proof of Proposition 2, it is shown that $\tilde{M}_n(t)$ converges towards χ_n . Since $\tilde{M}_n(t)$ is the joint moments of \mathbb{R}^k valued process $\{e^{\delta t}Z(t)\}_{t \geq 0}$, convergence result suggests in turn that this process converges in distribution. Since convergence of moments does not always implies convergence in distribution, we give in this section some sufficient conditions such that this latter holds.

Theorem 8 *Let us suppose that **(A)** holds and that each rv X_j for $j = 1, \dots, k$, is a.s. bounded by some constant M . Then one has the result of convergence in distribution for $e^{\delta t}Z(t)$ given by*

$$e^{\delta t}Z(t) \xrightarrow{\mathcal{D}} \mathcal{Z}_\infty, \quad t \rightarrow \infty,$$

where $\mathcal{Z}_\infty = (\mathcal{Z}_{\infty,1}, \dots, \mathcal{Z}_{\infty,k}) = \mathcal{Z}_\infty(\delta)$ is a light tailed vector valued rv with the joint moments

$$\mathbb{E} \left[\prod_{i=1}^k \mathcal{Z}_{\infty,i}^{n_i} \right] = \chi_n = \chi_n(\delta)$$

given by (11) for $n \in \mathbb{N}^k$.

Proof. See Section 5.5. □

2.4 Exponentially distributed delays

Let us note that Theorem 8 actually holds for general light tailed interarrival times τ_i that satisfy **(A)**, and general time delays L_j 's. In practice, it is not easy to compute explicitly limiting moments χ_n for $n \in \mathbb{N}^k$, as given by (11), although they are obtainable recursively in principle. Hence, we shall now restrict to the case where the L_j 's are exponentially distributed. To make analysis simpler, we suppose that all L_j 's for $j = 1, \dots, k$, are all $\mathcal{E}(\mu)$ distributed for some $\mu > 0$. In the same spirit as in Remark 7, we may obtain similar results in the following for more general cases such as a mixture or a combination of exponentials.

To begin, some notations are introduced. Let $\mathcal{L}_n^M(s)$ and $\mathcal{L}_n^b(s)$ for $s \geq 0$ and $n \in \mathbb{N}^k$, be the Laplace transforms of $\tilde{M}_n(\cdot)$ and $\tilde{b}_n(\cdot)$ respectively

$$\mathcal{L}_n^M(s) := \int_0^\infty e^{-sy} \tilde{M}_n(y) dy, \quad \mathcal{L}_n^b(s) := \int_0^\infty e^{-sy} \tilde{b}_n(y) dy.$$

Note that these Laplace transforms exist (i.e. the integrals converge) respectively when $s > 0$ and $s \geq 0$ since $\tilde{M}_n(y)$ converges to some finite limit χ_n as $y \rightarrow \infty$, and $\tilde{b}_n(\cdot)$ is integrable (as proved in Proposition 2). The following lemma gives a recursive expression of $\mathcal{L}_n^b(s)$. We denote $|A|$ as the cardinal of A for any finite set A .

Lemma 9 *When time delays L_j are $\mathcal{E}(\mu)$ distributed, the Laplace transform of $\tilde{b}_n(\cdot)$ in (9) is obtained as*

$$\mathcal{L}_{n(i)}^b(s) = \mathbb{E}[X_i] \frac{\mu}{(\mu + \delta)(\mu + s)} \mathcal{L}^\tau(s), \quad i = 1, \dots, k, \quad (27)$$

and

$$\mathcal{L}_n^b(s) = B_{\mathbf{0},n} \frac{\mathcal{L}^\tau(s)}{s + |C_{\mathbf{0}}|\mu} + \sum_{\mathbf{0} < \ell < n} B_{\ell,n} \frac{\mathcal{L}^\tau(s)}{1 - \mathcal{L}^\tau(s + |C_\ell|\mu)} \mathcal{L}_\ell^b(s + |C_\ell|\mu), \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}, \quad (28)$$

where

$$B_{\ell,n} := \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] \prod_{j \in C_\ell} \frac{\mu}{\mu + (n_j - \ell_j)\delta}, \quad (29)$$

and we recall that $C_\ell = C_{\ell,n} = \{j = 1, \dots, k \mid \ell_j < n_j\} \subset \{1, \dots, k\}$.

Proof. See Section 5.6. □

Theorem 10 Let us denote $D_n(j) := \mathcal{L}_n^b(j\mu)$ for $j \in \mathbb{N}$ and $n \in \mathbb{N}^k$. When time delays L_j are $\mathcal{E}(\mu)$ distributed, the joint moments $\chi_n = \chi_n^\delta$ for $n \in \mathbb{N}^k$ of $\mathcal{Z}_\infty = \mathcal{Z}_\infty^\delta$ (the limiting distribution of $e^{\delta t}Z(t)$), are given by

$$\chi_{n(i)} = \frac{\mathbb{E}[X_i]}{\mathbb{E}[\tau_1]} \left(\frac{1}{\mu + \delta} \right), \quad i = 1, \dots, k, \quad (30)$$

and

$$\chi_n = \frac{1}{\mathbb{E}[\tau_1]} \left(B_{\mathbf{0},n} \frac{1}{|C_{\mathbf{0}}|\mu} + \sum_{\mathbf{0} < \ell < n} B_{\ell,n} \frac{1}{1 - \mathcal{L}^\tau(|C_\ell|\mu)} D_\ell(|C_\ell|) \right), \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}, \quad (31)$$

where $D_n(j)$'s for $j \in \mathbb{N}$ and $n \in \mathbb{N}^k$ are obtained recursively as:

$$D_{n(i)}(j) = \mathbb{E}[X_i] \frac{\mu}{(\mu + \delta)([j + 1]\mu)} \mathcal{L}^\tau(j\mu), \quad i = 1, \dots, k, \quad (32)$$

$$D_n(j) = B_{\mathbf{0},n} \frac{\mathcal{L}^\tau(j\mu)}{[j + |C_{\mathbf{0}}|\mu]} + \sum_{\mathbf{0} < \ell < n} B_{\ell,n} \frac{\mathcal{L}^\tau(j\mu)}{1 - \mathcal{L}^\tau([j + |C_\ell|\mu])} D_\ell([j + |C_\ell|]), \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}, \quad (33)$$

with $B_{\ell,n}$ in (29).

Proof. From (11), using (27) and (28) when $s = 0$, we find (30) and (31) respectively. In addition, (32) and (33) are obtainable by setting $s = j\mu$ in (27) and (28) respectively. □

We remark that a close look at (31) and (33) reveals that computation of the *infinite* sequences $(D_\ell(j))_{j \in \mathbb{N}}$ for all $\ell < n$ is not needed to obtain χ_n . Since $|C_\ell|$ is bounded by k , it is not hard to see that one needs to compute (recursively) $D_\ell(j)$ for $\ell < n$ and for $j \leq k\eta_n$ (i.e. only for a finite number of j 's). Moreover, the values of $D_n(j)$ may be stored in memory while computing the successive χ_n as η_n increases, and thus one does not need to recompute them each time. Hence the algorithm (31) is relatively not too costly.

3 Applications to infinite server queues

Now we consider the following application related to queueing theory. To begin, we restate the model assumptions described in Section 1 in terms of technical terms used in queueing theory. Let us consider a single queue containing batches of k types of customers in the infinite-server model. Here queues arrive according to a renewal process $\{N_i\}_{i \geq 0}$ with corresponding arrival times $\{T_i\}_{i=1}^\infty$. At each arrival instant T_i a batch of (correlated) customers $(X_{i,1}, \dots, X_{i,k})$ arrive in the system, with each customer within batch $j \in \{1, \dots, k\}$ having the same service times $L_{i,j}$. A random sequence $(X_{i,1}, \dots, X_{i,k})$ for $i \in \mathbb{N}$, is iid and distributed as (X_1, \dots, X_k) . In order to comply with the previous section we suppose furthermore that the X_j 's are upper bounded, i.e. there exists some $M \in \mathbb{N}$ such that all X_j 's have support included in $\{0, \dots, M\}$. Here, service times $(L_{i,j})_{i,j \in \mathbb{N}}$ are assumed to be independent, although $L_{i,1}, \dots, L_{i,k}$ possibly have different distributions, i.e. service times are different according to the type of customer class.

3.1 $G/G/\infty$ queue with correlated batch arrivals and customer classes

We are first interested in the process $Z(t) = Z^\delta(t) = \{(Z_1(t), \dots, Z_k(t))\}_{t \geq 0}$ defined in (1). Note in particular that when $\delta = 0$, $Z_j(t)$ is the number of customers of class $j \in \{1, \dots, k\}$ in the system at time t . When $\delta > 0$, $Z(t)$ has no real interpretation and can be seen as the number of customers of class j penalized with respect to their departure times through a discount with rate δ ; another interpretation of the rescaled process $e^{\delta t}Z(t)$ is given in upcoming Remark 13. Theorem 8 then reads as follows in this context:

Theorem 11 *Let us suppose that (A) holds. The following convergence holds for the discounted queue size:*

$$e^{\delta t} Z(t) \xrightarrow{\mathcal{D}} \mathcal{Z}_\infty, \quad t \rightarrow \infty,$$

where $\mathcal{Z}_\infty = \mathcal{Z}_\infty(\delta) = (\mathcal{Z}_{\infty,1}, \dots, \mathcal{Z}_{\infty,k})$ is a light tailed vector valued rv with joint moments $\mathbb{E} \left[\prod_{i=1}^k \mathcal{Z}_{\infty,i}^{n_i} \right] = \chi_n = \chi_n(\delta)$ given by (11) for $n \in \mathbb{N}^k$.

In particular, when $\delta = 0$, we obtain that the joint number of customers within different classes $(Z_1(t), \dots, Z_k(t))$ converges in distribution as $t \rightarrow \infty$ to a stationary regime \mathcal{Z}_∞ with joint moments given by $(\chi_n)_{n \in \mathbb{N}^k}$. \square

Example 12 *As an illustration, let us look at the particular case where (X_1, \dots, X_k) follows a multinomial distribution with parameters $M \in \mathbb{N}$ and probability vector (p_1, \dots, p_k) where $p_j \geq 0$ and $\sum_{j=1}^k p_j = 1$. This models a situation where at each instant T_i exactly M customers arrive, each of which belongs to class j with probability p_j . X_j is the number of customers of class j in this batch. See Figure 1. When $M = 1$, customers*

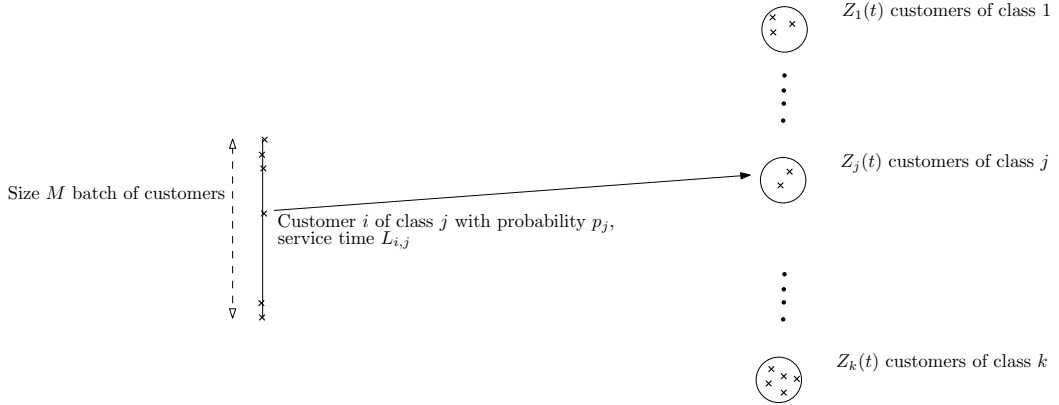


Figure 1: The $G/G/\infty$ queue with multinomial distributed classes batches (X_1, \dots, X_k) .

arrive according to renewal process $\{N_t\}_{t \geq 0}$, and each arriving customer belongs to class j with probability p_j .

Remark 13 (Another queueing interpretation in the case of $\delta > 0$) As pointed out at the beginning of this section, no direct interpretation of the vector valued process $\{Z(t)\}_{t \geq 0} = \{Z^\delta(t)\}_{t \geq 0}$ is available in a queueing context. One way to introduce a queueing interpretation is by using Fubini's theorem and noticing that for all $t \geq 0$ and $j = 1, \dots, k$,

$$\mathbb{E}[e^{\delta t} Z_j(t)] = \mathbb{E} \left[\sum_{i=1}^{\infty} e^{-\delta(T_i + L_{i,j} - t)} X_{i,j} 1_{\{T_i \leq t < T_i + L_{i,j}\}} \right] = \mathbb{E} \left[\sum_{i=1}^{\infty} 1_{\{T_i + L_{i,j} - t \leq E_\delta\}} X_{i,j} 1_{\{T_i \leq t < T_i + L_{i,j}\}} \right], \quad (34)$$

where E_δ is an $\mathcal{E}(\delta)$ distributed rv independent from everything. Since $T_i + L_{i,j} - t$ is the residual service time of the i th batch of customers of size $X_{i,j}$, (34) can then be interpreted as *the expected number of customers at time t of which residual service time does not exceed horizon E_δ* , where $\delta > 0$ is arbitrary. Thus, a direct consequence of Theorem 11 is that this expected number converges towards $\chi_{n(j)}(\delta) = \mathbb{E}[\mathcal{Z}_{\infty,j}(\delta)]$, see upcoming Remark 19 for another interesting insight on this convergence.

Similar to what observed at the beginning of Section 2.4 concerning Theorem 8, Theorem 11 holds for any light tailed interarrival (that satisfies (A)) and service times. However, computing the χ_n 's for $n \in \mathbb{N}^k$, is theoretically feasible but practically complicated, as explained just before Proposition 3. On the other hand, the case where the $L_{i,j}$ are *exponentially distributed*, i.e. when one considers the $G/M/\infty$ queue with multiple types of customer classes, is much more tractable and one may use the procedure given in Theorem 10 to compute χ_n 's much more easily.

3.2 Asymptotics for the workload of the $G/M/\infty$ queue

We now turn to the asymptotic behaviour of the workload $D(t)$ of the queue when $k = 1$, which is defined as the time needed to empty the queue at time t if there is no arrival afterwards. As we deal with one queue only, we drop a subscript in $L_{i,1}$ for the i th service time (i.e. write L_i for $i \in \mathbb{N}$), and denote by L for the generic service time. The workload has the following expression

$$D(t) := \sum_{i=1}^{\infty} (T_i + L_i - t) 1_{\{T_i \leq t < T_i + L_i\}},$$

and is obtained from $\tilde{Z}(t, \delta) := e^{\delta t} Z_1(t)$ as:

$$D(t) = -\frac{\partial}{\partial \delta} \tilde{Z}(t, \delta) \Big|_{\delta=0}, \quad (35)$$

where here $Z_1(t)$ is the first entry of process $Z(t)$ (i.e. (2) when $k = 1$). We assume in this subsection that all $X_{i,1}$ for $i \in \mathbb{N}$, are equal to one. In that case, $Z_1(t)$ in (2) is, when $\delta = 0$, the size of this infinite server queue at time t . A sample path of $D(t)$ is depicted in Figure 2. Let us note that $D(t)$ is also the sum of

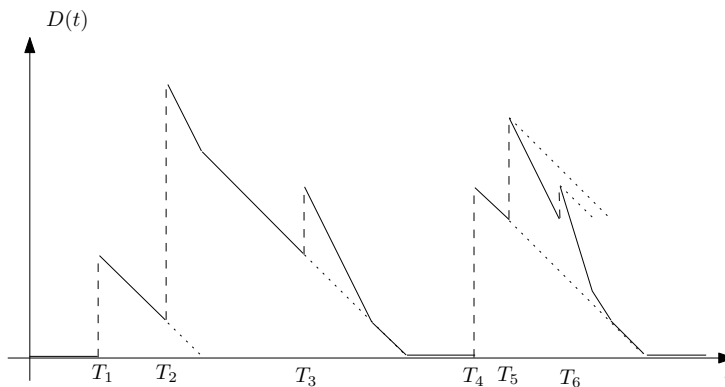


Figure 2: Sample path of workload for the $G/G/\infty$ queue.

the residual times for all services to be completed at time t . We are interested in the limiting expectation of workload and the covariance of queue size and workload. We thus need to study the two first moments of $\tilde{Z}(t, \delta)$, i.e. quantities $\tilde{M}_{n(1)}(t, \delta) = \mathbb{E}[\tilde{Z}(t, \delta)^n]$ and $\tilde{M}_{2n(1)}(t, \delta) = \mathbb{E}[\tilde{Z}(t, \delta)^{2n}]$, where here $n(1) = 1$ and $2n(1) = 2$ in the case $k = 1$, sticking with the notation introduced in Section 1. In the following, we write $n(1) = 1$ and $2n(1) = 2$ for notational convenience.

The main assumptions in this subsection are that service time L is $\mathcal{E}(\mu)$ distributed, i.e.

$$\mathbb{E}[e^{xL}] = \frac{\mu}{\mu - x}, \quad \forall x \in (-\infty, \mu), \quad (36)$$

so that this queue is the $G/M/\infty$, and that interarrival times are light tailed, i.e. Condition (19) holds for some $R > 0$. A few lemmas are first required. We need to define for $r > 0$, the disc D_r centered at 0 with the radius r , included in \mathbb{C} , by

$$D_r := \{z \in \mathbb{C} \mid |z| \leq r\}.$$

Lemma 14 *Let $a < \mu$. For all $t > 0$, $\tilde{M}_1(t, \delta)$ and $\tilde{M}_2(t, \delta)$ are respectively defined on D_a and $D_{a/2}$. Furthermore, $\delta \mapsto \tilde{M}_1(t, \delta)$ and $\delta \mapsto \tilde{M}_2(t, \delta)$ are analytic on those sets, hence a fortiori at $\delta = 0$.*

Note that one implication of the above lemma is that quantities $\tilde{M}_1(t, \delta)$ and $\tilde{M}_2(t, \delta)$ (and, hence $\tilde{Z}(t, \delta)$) are defined for some complex values of δ , and in particular for negative values (not only for $\delta \geq 0$). This is

especially handy to express the workload as (35) and to be able to define analyticity of $\tilde{M}_1(t, \delta)$ and $\tilde{M}_2(t, \delta)$ at $\delta = 0$, which is needed to differentiate with respect to δ at 0.

Proof. See Section 5.7. □

Lemma 15 *Let us suppose that (A) holds and let $a < \mu$. $\tilde{M}_1(t, \delta)$ and $\tilde{M}_2(t, \delta)$ uniformly converge to $\chi_1(\delta)$ and $\chi_2(\delta)$ respectively on D_a and $D_{a/2}$ as $t \rightarrow +\infty$.*

Proof. See Section 5.7. □

Now we are ready to provide some results for the long term behaviour of the expected workload, and the covariance function of the workload and the queue size in the following.

Theorem 16 *Let us suppose that (A) holds. The limiting expected workload for the $G/M/\infty$ queue is given by*

$$\lim_{t \rightarrow \infty} \mathbb{E}[D(t)] = \frac{1}{\mu^2 \mathbb{E}[\tau_1]} = \frac{\mathbb{E}[L^2]}{2\mathbb{E}[\tau_1]}. \quad (37)$$

The limiting covariance of the workload and queue size is given by

$$\lim_{t \rightarrow \infty} \text{Cov}[D(t), Z_1(t, 0)] = \frac{1}{\mu^2 \mathbb{E}[\tau_1]} \left[1 + \frac{\mathcal{L}^\tau(\mu)}{1 - \mathcal{L}^\tau(\mu)} - \frac{1}{\mu \mathbb{E}[\tau_1]} \right]. \quad (38)$$

Proof. See Section 5.7. □

4 Special cases

In this section, we use the results given in previous sections to obtain nice simple forms of asymptotic results for some special cases. The following two corollaries are the results when $k = 1$ in (2). The last corollary is the case of $k = 2$ which is useful to find the covariance of two types of inputs.

Corollary 17 (Single type of input, exponential time delays) *The r -th moment of discounted compound delayed process $Z_1(t)$ in (2) for $k = 1$ with exponential time lag is asymptotically obtained as*

$$\mathbb{E}[Z_1^r(t)] \sim \chi_r e^{-r\delta t}, \quad t \rightarrow \infty, \quad r \in \mathbb{N},$$

where

$$\chi_1 = \frac{\mathbb{E}[X_1]}{\mathbb{E}[\tau_1]} \left(\frac{1}{\mu + \delta} \right), \quad (39)$$

and

$$\chi_r = \frac{1}{\mathbb{E}[\tau_1]} \left(\mathbb{E}[X_1^r] \frac{1}{\mu + r\delta} + \sum_{\ell=1}^{r-1} \binom{r}{\ell} \mathbb{E}[X_1^{r-\ell}] \frac{\mu}{\mu + (r-\ell)\delta} \frac{D_\ell(1)}{1 - \mathcal{L}^\tau(\mu)} \right), \quad r = 2, 3, \dots, \quad (40)$$

and $D_\ell(1)$ recursively available from the formulas (32) and (33) respectively given by

$$D_1(j) = \mathbb{E}[X_1] \frac{\mu}{(\mu + \delta)([j+1]\mu)} \mathcal{L}^\tau(j\mu)$$

and

$$D_n(j) = \mathbb{E}[X_1^n] \frac{\mu}{\mu + n\delta} \frac{\mathcal{L}^\tau(j\mu)}{[j+1]\mu} + \sum_{\ell=1}^{n-1} \binom{n}{\ell} \mathbb{E}[X_1^{n-\ell}] \frac{\mu}{\mu + (n-\ell)\delta} \frac{\mathcal{L}^\tau(j\mu)}{1 - \mathcal{L}^\tau([j+1]\mu)} D_\ell([j+1]), \quad n = 2, 3, \dots$$

Proof. When $n(1) = 1$ and $n(i) = 0$ for $i \neq 1$ together with $\eta_n = r$, $\eta_\ell = \ell$, and $|C_\ell| = 1$, from Proposition 2 and Theorem 10, the result follows. \square

We remark that the form given in Theorem 3 of [21] was not suitable to derive asymptotic behavior of $Z_1(t)$. A comment therein reveals only that this quantity is asymptotically closed to zero. Hence Corollary 17 is useful for calculating higher moments of $Z_1(t)$ in any order for a large t when time delays are exponentially distributed.

For a general time lag distribution, a direct consequence of Proposition 2 when $k = 1$ with (5.1) yields the result for the first moment in the following corollary.

Corollary 18 (Single type of input, arbitrary time delays) *The mean of discounted compound delayed process $Z_1(t)$ in (2) for $k = 1$ with **arbitrary** time lag distribution is asymptotically obtained as*

$$\mathbb{E}[Z_1(t)] \sim \chi_1 e^{-\delta t}, \quad t \rightarrow \infty,$$

where

$$\chi_1 = \frac{\mathbb{E}[X_1]\mathbb{E}[L]\tilde{w}_{1,1}(\delta)}{\mathbb{E}[\tau_1]}, \quad (41)$$

and $\tilde{w}_{1,1}(\delta) = \int_0^\infty e^{-\delta x} \overline{W}_1(x) dx / \mathbb{E}[L_1]$. This is a generalization of Corollary 3 in [21] in which it is assumed that $X_i = 1$ and $\delta = 0$.

Remark 19 (Little's law revisited) Remark 13 as well as Expression (41) gives an interesting interpretation in a queuing context. Let us suppose here (without loss of generality) that $X_1 = 1$ (i.e. customers do not arrive in batches). Recall that we defined $\tilde{Z}(t, \delta) := e^{\delta t} Z_1(t)$, (41) then reads

$$\lim_{t \rightarrow \infty} \mathbb{E}[\tilde{Z}(t, \delta)] = \chi_1 = \frac{\mathbb{E}[L]\tilde{w}_{1,1}(\delta)}{\mathbb{E}[\tau_1]}. \quad (42)$$

When $\delta = 0$, $\tilde{Z}(t, \delta) = \tilde{Z}(t, 0)$ is the number of customers at time t in infinite server queues; In that case $\tilde{w}_{1,1}(\delta) = 1$ and (42) is just a rephrasing of Little's law which says that the limiting expected number of customers in the queue is equal to the arrival rate multiplied by the mean service time. When $\delta > 0$, the interpretation comes from (34): Noticing that $\mathbb{E}[L_1]\tilde{w}_{1,1}(\delta) = \Pr(L > E_\delta)/\delta$, (42) reads

$$\lim_{t \rightarrow \infty} \mathbb{E}[\tilde{Z}(t, \delta)] = \frac{1}{\mathbb{E}[\tau_1]} \frac{\Pr(L > E_\delta)}{\delta} = \frac{1}{\mathbb{E}[\tau_1]} \Pr(L > E_\delta) \mathbb{E}[E_\delta] \quad (43)$$

which says that the limiting expected number of customers of which residual service time is no more than horizon $E_\delta \sim \mathcal{E}(\delta)$ is equal to the arrival rate multiplied by the expected horizon time, multiplied by the proportion of customers of which service time did exceed this horizon E_δ . So, (43) can be seen as a generalization of Little's Law in the $G/G/\infty$ context.

Next, to compute the covariance for different types of discounted compound delayed process, the first joint moment of X_i and X_j for $i \neq j$ is needed. For notational convenience, let us denote arbitrary pair of claims as X_1 and X_2 . Suppose that $k = 2$ and $n_1 = n_2 = 1$ (i.e. $\ell = (\ell_1, \ell_2) \in \{(0, 0), (0, 1), (1, 0)\}$). From (9) and (7), we have

$$\begin{aligned} \tilde{b}_n(t) &= \sum_{\ell_1, \ell_2 \setminus (\ell_1, \ell_2) = (n_1, n_2)} \binom{n_1}{\ell_1} \binom{n_2}{\ell_2} \mathbb{E} \left[\prod_{j=1}^2 X_j^{n_j - \ell_j} \right] \varphi_\ell(t) \\ &= \mathbb{E}[X_1 X_2] \varphi_{(0,0)}(t) + \mathbb{E}[X_1] \varphi_{(0,1)}(t) + \mathbb{E}[X_2] \varphi_{(1,0)}(t), \end{aligned} \quad (44)$$

where $\varphi_{(0,0)}(t) = \mathbb{E}[e^{2\delta(t-\tau_1)} \overline{w}_{\delta,1}(t-\tau_1) \overline{w}_{\delta,2}(t-\tau_1) \cdot 1_{[\tau_1 < t]}]$ because of $\tilde{M}_{(0,0)}(t-\tau_1) = 1$, $\varphi_{(0,1)}(t) = \mathbb{E}[e^{\delta(t-\tau_1)} \tilde{M}_{(0,1)}(t-\tau_1) \overline{w}_{\delta,1}(t-\tau_1) \cdot 1_{[\tau_1 < t]}]$, and $\varphi_{(1,0)}(t) = \mathbb{E}[e^{\delta(t-\tau_1)} \tilde{M}_{(1,0)}(t-\tau_1) \overline{w}_{\delta,2}(t-\tau_1) \cdot 1_{[\tau_1 < t]}]$. As shown previously, (44) is simplified when L_i for $i = 1, 2$ is exponentially distributed. In this case, the joint expectation and the covariance of $Z_1(t)$ and $Z_2(t)$ are presented in the following.

Corollary 20 (Two types of inputs, exponential time delays) *The joint mean of two types of discounted compound delayed processes (2) where the time delay of type- i input L_i for $i = 1, 2$ is $\mathcal{E}(\mu)$ distributed, is asymptotically given by*

$$\mathbb{E}[Z_1(t)Z_2(t)] \sim \chi_n e^{-2\delta t}, \quad t \rightarrow \infty,$$

where

$$\chi_n = \frac{1}{\mathbb{E}[\tau_1]} \frac{\mu}{(\mu + \delta)^2} \left[\frac{\mathbb{E}[X_1 X_2]}{2} + \frac{\mathbb{E}[X_1] \mathbb{E}[X_2]}{2} \frac{\mathcal{L}^\tau(\mu)}{1 - \mathcal{L}^\tau(\mu)} \right]. \quad (45)$$

Consequently, the covariance is given by

$$\text{Cov}[Z_1(t), Z_2(t)] \sim \xi_n e^{-2\delta t}, \quad t \rightarrow \infty,$$

where $\xi_n = \chi_n - \frac{\mathbb{E}[X_1] \mathbb{E}[X_2]}{\mathbb{E}[\tau_1]^2 (\mu + \delta)^2}$ with χ_n given in (45).

Proof. From Theorem 10 when $n = (n_1, n_2) = (1, 1)$ (i.e. $|C_\ell| = 1$ when $\ell = (\ell_1, \ell_2) \in \{(1, 0), (0, 1)\}$), we have

$$\chi_n = \frac{1}{\mathbb{E}[\tau_1]} \left[B_{(0,0),(1,1)} \frac{1}{2\mu} + B_{(1,0),(1,1)} \frac{D_{(1,0)}(1)}{1 - \mathcal{L}^\tau(\mu)} + B_{(0,1),(1,1)} \frac{D_{(0,1)}(1)}{1 - \mathcal{L}^\tau(\mu)} \right]. \quad (46)$$

But from (29), B 's are given by

$$B_{(0,0),(1,1)} = \mathbb{E}[X_1 X_2] \left(\frac{\mu}{\mu + \delta} \right)^2, \quad B_{(1,0),(1,1)} = \mathbb{E}[X_2] \frac{\mu}{\mu + \delta}, \quad B_{(0,1),(1,1)} = \mathbb{E}[X_1] \frac{\mu}{\mu + \delta}.$$

Also, $D_{n(i)}(1)$ for $i = 1, 2$ is available from (32) as $D_{n(i)}(1) = \mathbb{E}[X_i] \frac{\mu}{(\mu + \delta)(2\mu)} \mathcal{L}^\tau(\mu)$. Combining results given above, (46) is expressed as (45). \square

5 Proofs

Proof of Lemma 1. When τ_1 admits a density $f(\cdot)$ then density $t \mapsto u(t)$ of renewal function $t \mapsto m(t)$ satisfies a renewal equation of the form

$$u(x) = f(x) + \int_0^x u(y) f(x - y) dy, \quad x \geq 0, \quad (47)$$

(e.g. see Equation (3.6) of [6]). Since **(A)** holds, by [6, Lemma p.359] (47) admits a unique solution bounded on finite intervals given by (3). Also, the derivative $m'(t) = u(t)$ verifies $\lim_{t \rightarrow \infty} m'(t) = 1/\mathbb{E}[\tau_1]$, see [6, Theorem 2 p.367], and is thus bounded above by some constant C . \square

5.1 Proof of Proposition 2

Since $\tilde{M}_n(t)$ satisfies the renewal equation in (10), asymptotics result in (11) is a direct consequence of Blackwell's renewal theorem, provided that we prove that $\int_0^\infty \tilde{b}_n(y) dy$ or equivalently $\int_0^\infty \varphi_{\ell,n}(y) dy$ is finite for all $n \in \mathbb{N}^k$ and $\ell < n$. We shall demonstrate this by induction on $n \in \mathbb{N}^k$. First, consider the case of $n = n(i)$ for some $i \in \{1, \dots, k\}$. From Example 3 in [21] one has

$$b_n(t) = \mathbb{E}[X_i] \cdot \int_0^t e^{-\delta y} \bar{\omega}_{\delta,i}(t - y) dF(y) = \mathbb{E}[X_i] \cdot \bar{\omega}_{\delta,i} \star H_\delta(t), \quad (48)$$

where $\bar{w}_{\delta,i}(t)$ is given in (8) and $H_\delta(t) = \int_0^t e^{-\delta y} dF(y)$. But $\int_0^\infty e^{\delta z} \bar{w}_{\delta,i}(z) dz = \int_0^\infty e^{\delta z} \int_z^\infty e^{-\delta y} dW_i(y) dz = \delta^{-1} \{1 - \mathbb{E}[e^{-\delta L_i}]\}$, the following integration yields

$$\begin{aligned} \int_0^\infty \tilde{b}_n(y) dy &= \int_0^\infty e^{\delta y} b_n(y) dy = \mathbb{E}[X_i] \cdot \int_0^\infty e^{\delta y} \int_0^y e^{-\delta x} \bar{w}_{\delta,i}(y-x) dF(x) dy \\ &= \mathbb{E}[X_i] \cdot \int_0^\infty e^{-\delta x} \int_x^\infty e^{\delta y} \bar{w}_{\delta,i}(y-x) dy dF(x) \\ &= \mathbb{E}[X_i] \cdot \delta^{-1} \{1 - \mathbb{E}[e^{-\delta L_i}]\} < \infty, \end{aligned} \quad (49)$$

or equivalently

$$\int_0^\infty \tilde{b}_n(y) dy = \mathbb{E}[X_i] \mathbb{E}[L_i] \int_0^\infty e^{-\delta x} \frac{\bar{W}_i(x)}{\mathbb{E}[L_i]} dx = \mathbb{E}[X_i] \mathbb{E}[L_i] \tilde{w}_{1,i}(\delta),$$

where $w_{1,i}(x)$ is an equilibrium pdf of L_i defined as $w_{1,i}(x) = \bar{W}_i(x)/\mathbb{E}[L_i]$ and its Laplace transform is $\tilde{w}_{1,i}(s) = \int_0^\infty e^{-sx} w_{1,i}(x) dx$.

Moreover, recall Equation (36) in [21]

$$M_n(t) = \mathbb{E}[X_i] \cdot \int_0^t e^{-\delta y} \bar{w}_{\delta,i}(t-y) dm(y) = \mathbb{E}[X_i] \cdot e^{-\delta t} \int_0^t e^{\delta(t-y)} \bar{w}_{\delta,i}(t-y) dm(y).$$

By Blackwell's theorem, it satisfies

$$M_n(t) \sim \frac{\mathbb{E}[X_i]}{\mathbb{E}[\tau_1]} \left[\int_0^\infty e^{\delta y} \bar{w}_{\delta,i}(y) dy \right] e^{-\delta t}, \quad t \rightarrow \infty.$$

In other words, one identifies

$$\chi_n = \chi_{n(i)} = \frac{\mathbb{E}[X_i]}{\mathbb{E}[\tau_1]} \left[\int_0^\infty e^{\delta y} \bar{w}_{\delta,i}(y) dy \right].$$

We now assume for all $\ell < n$ that $\tilde{M}_\ell(t) \rightarrow \chi_\ell < +\infty$ as $t \rightarrow \infty$ with χ_ℓ defined as in (11). Hence $t \mapsto \tilde{M}_\ell(t)$ is bounded for all $\ell < n$ by some constant $K_\ell = \sup_{t \geq 0} \tilde{M}_\ell(t)$. Hence simple algebraic computation results in the upper bound for (7) as

$$\begin{aligned} \varphi_\ell(t) &\leq K_\ell \mathbb{E} \left[e^{(\eta_n - \eta_\ell) \delta (t - \tau_1)} \prod_{j \in C_\ell} \bar{w}_{(n_j - \ell_j) \delta, j}(t - \tau_1) \cdot 1_{[\tau_1 < t]} \right] \\ &= K_\ell \mathbb{E} \left[e^{(\eta_n - \eta_\ell) \delta (t - \tau_1)} \prod_{j \in C_\ell} \left[\int_{t - \tau_1}^\infty e^{-(n_j - \ell_j) \delta y} dW_j(y) \right] \cdot 1_{[\tau_1 < t]} \right] \\ &\leq K_\ell \mathbb{E} \left[e^{(\eta_n - \eta_\ell) \delta (t - \tau_1)} \prod_{j \in C_\ell} \left[e^{-(n_j - \ell_j) \delta (t - \tau_1)} \bar{W}_j(t - \tau_1) \right] \cdot 1_{[\tau_1 < t]} \right] \\ &= K_\ell \mathbb{E} \left[\prod_{j \in C_\ell} \bar{W}_j(t - \tau_1) \cdot 1_{[\tau_1 < t]} \right]. \end{aligned}$$

Then integrating $\varphi_\ell(t)$ from 0 and ∞ yields

$$\int_0^\infty \varphi_\ell(t) dt \leq K_\ell \mathbb{E} \left[\int_0^\infty \prod_{j \in C_\ell} \left[\bar{W}_j(t - \tau_1) \right] \cdot 1_{[\tau_1 < t]} dt \right] = K_\ell \int_0^\infty \prod_{j \in C_\ell} \bar{W}_j(t) dt,$$

and by Holder's inequality, one finds

$$\begin{aligned}
\int_0^\infty \varphi_\ell(t) dt &\leq K_\ell \prod_{j \in C_\ell} \left[\int_0^\infty \overline{W}_j(t)^{|C_\ell|} dt \right]^{1/|C_\ell|}, \\
&\leq K_\ell \prod_{j \in C_\ell} \left[\int_0^\infty \overline{W}_j(t) dt \right]^{1/|C_\ell|} = K_\ell \prod_{j \in C_\ell} \mathbb{E}[L_j]^{1/|C_\ell|} \\
&\leq K_\ell \min_{j \in C_\ell} \mathbb{E}[L_j] < \infty,
\end{aligned} \tag{50}$$

where $|C_\ell|$ denotes the cardinal of set C_ℓ . Hence from (9) we deduce that $\int_0^\infty \tilde{b}_n(y) dy$ is also finite, and the induction is complete.

5.2 Proof of Proposition 3

Since $m(t)$ admits $u(t)$ as a density, one has from (10) that $\tilde{M}_n(t) = \int_0^t \tilde{b}_n(y) u(t-y) dy$, and in turn, from Lemma 1 we arrive at the following upper bound

$$\tilde{M}_n(t) \leq C \int_0^\infty \tilde{b}_n(y) dy. \tag{51}$$

Combining (9) and (50) yields the following upper bound

$$\int_0^\infty \tilde{b}_n(y) dy \leq \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] K_\ell \min_{j \in C_\ell} \mathbb{E}[L_j],$$

where we recall that $K_\ell = \sup_{t \geq 0} \tilde{M}_\ell(t)$ (see the proof of Proposition 2). Thus the above inequality together with (11) and (51) yields (12) and (13) respectively with $(R_n)_{n \in \mathbb{N}^k}$ defined in (14), provided we initialize value of R_n when $n = n(i)$ for $i \in \{1, \dots, k\}$. This is done by again using upper bound (51) and remembering that $\int_0^\infty \tilde{b}_n(y) dy$ is obtained by (49) when $n = n(i)$.

5.3 Proof of Proposition 4

Since $t \mapsto M_n(t)$ satisfies renewal equation in Theorem 3 of [21], one can write

$$M_n(t) = \int_0^t b_n(t-y) dm_{\eta_n \delta}(y), \quad t \geq 0, \tag{52}$$

where $m_\delta(y)$ is a discounted renewal function defined as $\sum_{j \geq 0} H_\delta^{(*)j}(y)$ with $H_\delta(t) = \int_0^t e^{-\delta y} dF(y)$. From (47), applying Theorem 3.1 of [17], one has lower and upper bounds for a renewal density $u(x)$ as $\alpha_L(x) \leq u(x) \leq \alpha_U(x)$ for $x \geq 0$ where $\alpha_L(t) = \inf_{y \in [0, t]} \alpha(y)$ and $\alpha_U(t) = \sup_{y \in [0, t]} \alpha(y)$ with $\alpha(y) := \frac{f(y)}{F(y)}$. Hence, if τ_1 is IFR, one has that $y \mapsto \alpha(y)$ is nondecreasing, i.e. $\alpha_L(t) = \lim_{t \rightarrow 0+} \alpha(t) = \alpha(0+) = f(0+)$ assuming that $F(0) = 0$, and it implies that

$$\frac{d}{dt} m_{\eta_n \delta}(t) \geq \alpha_L(t) e^{-\eta_n \delta t} = f(0) e^{-\eta_n \delta t}, \tag{53}$$

Substituting (53) into (52) results in

$$\begin{aligned}
M_n(t) &= \int_0^t b_n(t-y) \frac{d}{dy} m_{\eta_n \delta}(y) dy \geq \int_0^t b_n(t-y) dy \left[\inf_{y \in [0, t]} \frac{d}{dy} m_{\eta_n \delta}(y) \right] \\
&\geq \int_0^t b_n(t-y) dy [f(0) e^{-\eta_n \delta t}] = f(0) \int_0^t b_n(y) dy e^{-\eta_n \delta t}.
\end{aligned} \tag{54}$$

We then finish proving this Proposition by induction on $n \in \mathbb{N}^k$. For $n = n(i)$, one has a closed-form expression for $b_n(y)$ as $\mathbb{E}[X_i] \cdot \bar{\omega}_{\delta, i} \star H_\delta(y)$ from Example 3 of [21], hence the explicit expression for $h_n(t)$ is available in this case. Next, if lower bound $M_\ell(t) \geq h_\ell(t)e^{-\eta_\ell \delta t}$ is satisfied for all $\ell < n$ then, using expression for $t \mapsto b_n(t)$ given in Equation (34) of [21], one obtains lower bound

$$b_n(y) \geq \sum_{\ell < n} \binom{n_1}{\ell_1} \cdots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] \int_0^y e^{-\eta_n \delta z} h_\ell(y-z) e^{-\eta_\ell \delta (y-z)} \prod_{j \in C_\ell} \bar{\omega}_{(n_j - \ell_j) \delta, j}(y-z) dF(z),$$

and thus putting this lower bound into (54) yields (15) when $n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\}$.

On the other hand, if τ_1 is DFR then $\alpha_U(x) = f(0)$, and the proof is similar.

5.4 Proof of Theorem 5

Substituting (18) into (16) for $dm(s)$ yields

$$\tilde{M}_n(t) = \frac{1}{\mathbb{E}[\tau_1]} \int_0^t \tilde{b}_n(t-s) ds + \int_0^t \tilde{b}_n(t-s) dv(x).$$

A change of variable $s := t - s$ in the first integral and a subtraction of χ_n in (17) on both sides result in

$$\tilde{M}_n(t) - \chi_n = -\frac{1}{\mathbb{E}[\tau_1]} \int_t^\infty \tilde{b}_n(s) ds + \int_0^t \tilde{b}_n(t-s) dv(x). \quad (55)$$

Let

$$I_1(t) = -\frac{1}{\mathbb{E}[\tau_1]} \int_t^\infty \tilde{b}_n(s) ds, \quad I_2(t) = \int_0^t \tilde{b}_n(t-s) dv(s), \quad (56)$$

then (55) is essentially a sum of $I_1(t)$ and $I_2(t)$. In the sequel, we shall separately study the asymptotic behaviors of $I_1(t)$ and $I_2(t)$ when $t \rightarrow \infty$. First it is convenient to introduce the following quantity and its asymptotic result as it will be often utilized in the later analysis.

$$\begin{aligned} \mathbb{E}[1_{\{\tau_1 \geq t\}} e^{-\mu_i(t-\tau_1)}] &= e^{-\mu_i t} \int_t^\infty e^{\mu_i s} dF(x) \\ &= e^{-\mu_i t} \int_t^\infty e^{(\mu_i - R)s} e^{Rs} dF(s) \leq e^{-\mu_i t} \int_t^\infty e^{(\mu_i - R)t} e^{Rs} dF(s) \\ &\leq e^{-Rt} \int_t^\infty e^{Rs} dF(x) = o(e^{-Rt}), \end{aligned} \quad (57)$$

where the second last inequality is due to the assumption on $\mu_i < R$ for all i 's and the last result is due to $\mathbb{E}(e^{R\tau_1}) = \mathcal{L}^\tau(-R) < \infty$ by (19).

We begin to analyze $I_1(t)$ in (56) when $t \rightarrow \infty$. From (6) and (48) with (8) we may write

$$\int_t^\infty \tilde{b}_n(z) dz = \mathbb{E}[X_i] \cdot \mathbb{E} \left[\int_t^\infty e^{\delta(z-\tau_1)} 1_{\{\tau_1 < z\}} \int_{z-\tau_1}^\infty e^{-\delta s} dW_i(s) dz \right]. \quad (58)$$

When we assume that L_j 's are $\mathcal{E}(\mu_i)$ distributed for $\mu_i > 0$, then the second integral on the above equation is simplified as

$$\int_{z-\tau_1}^\infty e^{-\delta s} dW_i(s) = \frac{\mu_i}{\mu_i + \delta} e^{-(\mu_i + \delta)(z-\tau_1)}. \quad (59)$$

As $1_{\{\tau_1 \geq t\}} + 1_{\{\tau_1 < t\}} = 1$, inserting these two indicator functions in (58) together with (59) results in

$$\int_t^\infty \tilde{b}_n(z) dz = \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathbb{E} \left[(1_{\{\tau_1 < t\}} + 1_{\{\tau_1 \geq t\}}) \int_t^\infty 1_{\{\tau_1 < z\}} e^{-\mu_i(z-\tau_1)} dz \right]. \quad (60)$$

For the case of $\tau_1 < t$, as $z > t$ and $\tau_1 < z$, the above expectation is reduced to

$$\begin{aligned} \mathbb{E}\left[1_{\{\tau_1 < t\}} \int_t^\infty 1_{\{\tau_1 < z\}} e^{-\mu_i(z-\tau_1)} dz\right] &= \frac{1}{\mu_i} \mathbb{E}[1_{\{\tau_1 < t\}} e^{-\mu_i(t-\tau_1)}] \\ &= \frac{1}{\mu_i} \mathbb{E}[(1 - 1_{\{\tau_1 \geq t\}}) e^{-\mu_i(t-\tau_1)}] \\ &= \frac{1}{\mu_i} \left\{ e^{-\mu_i t} \mathcal{L}^\tau(-\mu_i) - \mathbb{E}[1_{\{\tau_1 \geq t\}} e^{-\mu_i(t-\tau_1)}] \right\} \\ &= \frac{1}{\mu_i} e^{-\mu_i t} \mathcal{L}^\tau(-\mu_i) + o(e^{-Rt}), \end{aligned}$$

where the last line is obtained by applying (57). On the other hand, when $\tau_1 \geq t$,

$$\begin{aligned} \mathbb{E}\left[1_{\{\tau_1 \geq t\}} \int_t^\infty 1_{\{\tau_1 < z\}} e^{-\mu_i(z-\tau_1)} dz\right] &= \mathbb{E}\left[1_{\{\tau_1 \geq t\}} \int_{\tau_1}^\infty e^{-\mu_i(z-\tau_1)} dz\right] \\ &= \frac{1}{\mu_i} \Pr(\tau_1 \geq t), \end{aligned}$$

and note that, using Chernoff's inequality, $\Pr(\tau_1 \geq t) \leq \mathbb{E}(e^{R\tau_1})e^{-Rt} = o(e^{-z_N t})$ because of $\mathbb{E}(e^{R\tau_1}) < \infty$ (by (19)) and $\text{Re}(z_N) < R$. Hence combining the above results using the fact that an $o(e^{-Rt})$ is a fortiori an $o(e^{-z_N t})$, it follows that

$$I_1(t) = -\frac{1}{\mathbb{E}[\tau_1]} \int_t^\infty \tilde{b}_n(s) ds = -\frac{\mathbb{E}[X_i]}{\mathbb{E}[\tau_1]} \cdot \frac{1}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) e^{-\mu_i t} + o(e^{-z_N t}). \quad (61)$$

We now turn to $I_2(t)$ in (56). As $\tilde{b}_n(0) = 0$, applying integration by parts for Stieltjes integrals on the right side of $I_2(t)$ yields

$$I_2(t) = \int_0^t \tilde{b}_n(t-s) dv(x) = \tilde{b}_n(t)v(0^-) + \int_0^t v(s) \tilde{b}'_n(t-s) ds. \quad (62)$$

But $v(0^-) = -\mathbb{E}[\tau_1^2]/(2\mathbb{E}[\tau_1]^2)$ and using a similar reasoning applied to (57) we get

$$\tilde{b}_n(t) = \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathbb{E}[1_{\{\tau_1 < t\}} e^{-\mu_i(t-\tau_1)}] = \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) e^{-\mu_i t} + o(e^{-Rt}), \quad (63)$$

i.e.

$$\tilde{b}_n(t)v(0^-) = -\frac{\mathbb{E}[X_i]\mathbb{E}[\tau_1^2]}{2\mathbb{E}[\tau_1]^2} \frac{\mu_i}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) e^{-\mu_i t} + o(e^{-z_N t}), \quad t \rightarrow \infty. \quad (64)$$

Also we have $\tilde{b}_n(t) = \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} e^{-\mu_i t} \int_0^t e^{\mu_i s} dF(s)$ and then $\tilde{b}'_n(t) = -\mu_i \tilde{b}_n(t) + \mathbb{E}[X_i] \frac{\mu_i}{\mu_i + \delta} f(t)$. Thus

$$\begin{aligned} \int_0^t e^{-z_k s} \tilde{b}'_n(t-s) dx &= e^{-z_k t} \int_0^t e^{z_k s} \tilde{b}'_n(s) ds \\ &= e^{-z_k t} \int_0^t e^{z_k s} \left[-\mu_i \tilde{b}_n(s) + \mathbb{E}[X_i] \frac{\mu_i}{\mu_i + \delta} f(s) \right] ds, \quad k = 1, \dots, N. \end{aligned} \quad (65)$$

On the first term of the above equation, from (63) it follows that

$$\begin{aligned} e^{-z_k t} \int_0^t e^{z_k s} \tilde{b}_n(s) ds &= \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \left(\frac{1}{z_k - \mu_i} \right) \mathbb{E}\left[1_{\{\tau_1 < t\}} \{e^{-\mu_i(t-\tau_1)} - e^{-z_k(t-\tau_1)}\}\right] \\ &= \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \left(\frac{1}{z_k - \mu_i} \right) \left\{ e^{-\mu_i t} \mathcal{L}^\tau(-\mu_i) - e^{-z_k t} \mathcal{L}^\tau(-z_k) \right\} + o(e^{-Rt}), \end{aligned} \quad (66)$$

for $k = 1, \dots, N$. Next, on the second term, one has

$$\begin{aligned} e^{-z_k t} \int_0^t e^{z_k s} f(s) ds &= e^{-z_k t} \mathcal{L}^\tau(-z_k) - e^{-z_k t} \int_t^\infty e^{z_k s} f(s) ds \\ &= e^{-z_k t} \mathcal{L}^\tau(-z_k) + o(e^{-z_N t}) \end{aligned} \quad (67)$$

since

$$\begin{aligned} \left| e^{-z_k t} \int_t^\infty e^{z_k s} f(s) ds \right| &= \left| e^{-z_k t} \int_t^\infty e^{(z_k - R)s} e^{Rs} f(s) ds \right| \leq e^{-\operatorname{Re}(z_k)t} \int_t^\infty e^{(\operatorname{Re}(z_k) - R)s} e^{Rs} f(s) ds \\ &\leq e^{-\operatorname{Re}(z_k)t} e^{(\operatorname{Re}(z_k) - R)t} \int_t^\infty e^{Rs} f(s) ds = e^{-Rt} \int_0^\infty e^{Rs} f(s) ds = o(e^{-z_N t}). \end{aligned}$$

Then using (20) and (65) with (66) and (67), and since an $o(e^{-Rt})$ is a fortiori an $o(e^{-z_N t})$, the second term of (62) (except for the term involving $o(e^{-z_N x})$ in $v(x)$ in (20)) is now given by

$$\begin{aligned} &\int_0^t [v(s) - o(e^{-z_N s})] \tilde{b}'_n(t-s) ds \\ &= \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \left[\sum_{k=1}^N \gamma_k \left(\frac{\mu_i}{z_k - \mu_i} \right) \{ e^{-z_k t} \mathcal{L}^\tau(-z_k) - e^{-\mu_i t} \mathcal{L}^\tau(-\mu_i) \} + \gamma_k e^{-z_k t} \mathcal{L}^\tau(-z_k) \right] + o(e^{-z_N t}) \\ &= \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \left[\sum_{k=1}^N \gamma_k \left(\frac{z_k}{z_k - \mu_i} e^{-z_k t} \mathcal{L}^\tau(-z_k) - \frac{\mu_i}{z_k - \mu_i} e^{-\mu_i t} \mathcal{L}^\tau(-\mu_i) \right) \right] + o(e^{-z_N t}). \end{aligned} \quad (68)$$

Recall that function $\eta(\cdot)$ is defined by (21). Then, putting the expression for $\tilde{b}'_n(t)$ into the integral, it follows that

$$\begin{aligned} \int_0^t o(e^{-z_N s}) \tilde{b}'_n(t-s) ds &= \int_0^t \eta(s) e^{-z_N s} \tilde{b}'_n(t-s) ds \\ &= \int_0^t \eta(s) e^{-z_N s} \left[-\mu_i \tilde{b}_n(t-s) + \mathbb{E}[X_i] \frac{\mu_i}{\mu_i + \delta} f(t-s) \right] ds. \end{aligned} \quad (69)$$

We start by considering $\int_0^t \eta(s) e^{-z_N s} f(t-s) ds$ which can be written as

$$\int_0^t \eta(t-s) e^{-z_N(t-s)} f(s) ds = e^{-z_N t} \int_0^\infty \eta(t-s) 1_{\{0 < s < t\}} e^{z_N s} f(s) ds.$$

The fact that $\int_0^\infty |e^{z_N s} f(s) ds| = \int_0^\infty e^{(\operatorname{Re}(z_N))s} f(s) ds$ is convergent implies, by dominated convergence,

$$\int_0^\infty \eta(t-s) 1_{\{0 < s < t\}} e^{z_N s} f(s) ds \longrightarrow 0, \quad t \rightarrow \infty.$$

Consequently,

$$\int_0^t \eta(s) e^{-z_N s} f(t-s) ds = o(e^{-z_N t}), \quad t \rightarrow \infty. \quad (70)$$

Now we turn our attention to the first term of (69) involving $\int_0^t \eta(s) e^{-z_N s} \tilde{b}_n(t-s) ds$. Writing from (48) (see also (63))

$$\tilde{b}_n(t) = \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathbb{E}[1_{\{\tau_1 < t\}} e^{-\mu_i(t-\tau_1)}] = \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) e^{-\mu_i t} - \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathbb{E}[1_{\{\tau_1 \geq t\}} e^{-\mu_i(t-\tau_1)}],$$

we then split $\int_0^t \eta(s) e^{-z_N s} \tilde{b}_n(t-s) ds$ into two parts, namely $\mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) \int_0^t \eta(s) e^{-z_N s} e^{-\mu_i(t-s)} ds$ and $\mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \int_0^t \eta(s) e^{-z_N s} \mathbb{E}[1_{\{\tau_1 \geq t-s\}} e^{-\mu_i((t-s)-\tau_1)}] ds$. The first term is expressed as

$$\begin{aligned} & \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) \int_0^t \eta(s) e^{-z_N s} e^{-\mu_i(t-s)} ds \\ &= \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) \left[\int_0^\infty \eta(s) e^{-z_N s} e^{\mu_i s} ds \right] e^{-\mu_i t} - \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) \left[\int_t^\infty \eta(s) e^{-z_N s} e^{\mu_i s} ds \right] e^{-\mu_i t} \\ &= \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) \left[\int_0^\infty \eta(s) e^{-z_N s} e^{\mu_i s} ds \right] e^{-\mu_i t} + o(e^{-z_N t}), \end{aligned} \quad (71)$$

where the latter term $o(e^{-z_N t})$ being again justified as in (57). Now (57) implies that the second term verifies, by dominated convergence

$$\begin{aligned} & \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \int_0^t \eta(s) e^{-z_N s} \mathbb{E}[1_{\{\tau_1 \geq t-s\}} e^{-\mu_i((t-s)-\tau_1)}] ds \\ &= \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} e^{-z_N t} \int_0^t \eta(t-s) e^{z_N s} \mathbb{E}[1_{\{\tau_1 \geq s\}} e^{-\mu_i(s-\tau_1)}] ds = o(e^{-z_N t}). \end{aligned} \quad (72)$$

Gathering (71) and (72) thus yields

$$\int_0^t \eta(s) e^{-z_N s} \tilde{b}_n(t-s) ds = \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) \left[\int_0^\infty \eta(s) e^{(\mu_i - z_N)s} ds \right] e^{-\mu_i t} + o(e^{-z_N t}). \quad (73)$$

Then from (68) and (69) with (70) and (73) we get

$$\begin{aligned} \int_0^t v(s) \tilde{b}'_n(t-s) ds &= \mathbb{E}[X_i] \cdot \frac{\mu_i}{\mu_i + \delta} \left[\sum_{k=1}^N \gamma_k \left(\frac{z_k}{z_k - \mu_i} e^{-z_k t} \mathcal{L}^\tau(-z_k) - \frac{\mu_i}{z_k - \mu_i} e^{-\mu_i t} \mathcal{L}^\tau(-\mu_i) \right) \right] \\ &\quad - \mathbb{E}[X_i] \cdot \frac{\mu_i^2}{\mu_i + \delta} \mathcal{L}^\tau(-\mu_i) \left[\int_0^\infty \eta(s) e^{(\mu_i - z_N)s} ds \right] e^{-\mu_i t} + o(e^{-z_N t}), \quad t \rightarrow \infty. \end{aligned}$$

Hence the above result together with (64) allows us to have an expression for (62) as

$$I_2(t) = A_i e^{-\mu_i t} + \sum_{k=1}^N B_{k,i} e^{-z_k t} + o(e^{-z_N t}), \quad (74)$$

where A_i and $B_{k,i}$ for $k = 1, \dots, N$ are defined by (22) and (23). As a result, combining (61) and (74) leads the theorem.

5.5 Proof of Theorem 8

Let $P(x_1, \dots, x_k) = \sum_{\eta_n \leq K} a_n x_1^{\eta_1} \cdots x_k^{\eta_k}$ be a nonnegative polynomial in the variables $x_1 \dots x_k$ of degree K . One has then that $\sum_{\eta_n \leq K} a_n \mathbb{E} \left[\prod_{i=1}^k (e^{\delta t} Z_i(t))^{\eta_i} \right] = \mathbb{E} [P(e^{\delta t} Z_1(t), \dots, e^{\delta t} Z_k(t))] \geq 0$ for all t , which, from Proposition 2, yields $\sum_{\eta_n \leq K} a_n \chi_n \geq 0$ as $t \rightarrow \infty$. By the Riesz-Haviland theorem (see [8]), we deduce that sequence $(\chi_n)_{n \in \mathbb{N}^k}$ is a sequence of moments associated to some random variable $\mathcal{Z}_\infty = (\mathcal{Z}_{\infty,1}, \dots, \mathcal{Z}_{\infty,k})$.

Next we shall show that the moment generating function (mgf) of $e^{\delta t} Z(t)$ exists and converges to the mgf of \mathcal{Z}_∞ as $t \rightarrow \infty$. To this end, we first introduce the mgfs of $e^{\delta t} Z(t)$ and of \mathcal{Z}_∞ denoted by $\tilde{\varphi}_t(q)$ and $\varphi_{\mathcal{Z}_\infty}(q)$ respectively in the following:

$$\tilde{\varphi}_t(q) := \mathbb{E} \left[e^{\langle q, e^{\delta t} Z(t) \rangle} \right] = \sum_{n \in \mathbb{N}^k} \prod_{i=1}^k \frac{q_i^{\eta_i}}{n_i!} \tilde{M}_n(t), \quad t \geq 0, \quad (75)$$

and

$$\varphi_{\mathcal{Z}_\infty}(q) = \mathbb{E}[e^{\langle q, \mathcal{Z}_\infty \rangle}] = \sum_{n \in \mathbb{N}^k} \prod_{i=1}^k \frac{q_i^{n_i}}{n_i!} \chi_n, \quad (76)$$

for $q = (q_1, \dots, q_k)$ in the neighborhood of $(0, \dots, 0)$. To apply the dominated convergence theorem to (75), we need to show that $\tilde{M}_n(t)$ is bounded such as

$$\tilde{M}_n(t) \leq U_n := (Mm_L e^k)^{\eta_n} \prod_{i=1}^k n_i!, \quad \forall n \in \mathbb{N}^k, \quad \forall t \geq 0, \quad (77)$$

where $m_L := \max(1, C \cdot \max_{i=1, \dots, k} \mathbb{E}[L_i])$, so that, since

$$\sum_{n \in \mathbb{N}^k} \prod_{i=1}^k \frac{|q_i|^{n_i}}{n_i!} U_n = \prod_{i=1}^k \left(\sum_{n_i=1}^{\infty} |q_i| M m_L e^k |^{n_i} \right)$$

converges for

$$q = (q_1, \dots, q_k) \in J := \left[-\frac{1}{M m_L e^k}, \frac{1}{M m_L e^k} \right]^k,$$

the dominated convergence theorem yields $\tilde{\varphi}_t(q) \rightarrow \varphi_{\mathcal{Z}_\infty}(q)$ when $t \rightarrow \infty$ for $q \in J$.

Hence, we shall prove (77) by induction. Recall that in Proposition 3, we have already proved $\tilde{M}_n(t) \leq R_n$ where R_n is defined in (14). Thus we shall essentially show that $R_n \leq U_n$ for all $n \in \mathbb{N}^k$, so that (77) holds. We start by $n = n(i)$ for $i \in \{1, \dots, k\}$. In this case, upper bound (13) with (14) yields

$$R_{n(i)} = C \mathbb{E}[X_i] \delta^{-1} \{1 - \mathbb{E}[e^{-\delta L_i}]\} \leq C M \mathbb{E}[L_i] \leq M m_L = U_{n(i)},$$

where the first inequality is due to $\delta^{-1} \{1 - \mathbb{E}[e^{-\delta L_i}]\} = \int_0^\infty e^{-\delta x} \overline{W}_i(x) dx \leq \int_0^\infty \overline{W}_i(x) dx$. Let us now suppose that n is such that $R_\ell \leq U_\ell$ for all $\ell < n$. Using (14) as well as the induction assumption we get

$$R_n \leq C \sum_{\ell < n} \binom{n_1}{\ell_1} \dots \binom{n_k}{\ell_k} \mathbb{E} \left[\prod_{j=1}^k X_j^{n_j - \ell_j} \right] \min_{i \in C_\ell} \mathbb{E}[L_i] \cdot U_\ell \leq m_L \sum_{\ell < n} \binom{n_1}{\ell_1} \dots \binom{n_k}{\ell_k} M^{\eta_n - \eta_\ell} \cdot U_\ell. \quad (78)$$

But, $\ell < n$ implies $\eta_n - \eta_\ell \geq 1$ and m_L and e are larger than 1, the following inequality is valid

$$m_L M^{\eta_n - \eta_\ell} \leq m_L^{\eta_n - \eta_\ell} M^{\eta_n - \eta_\ell} (e^k)^{\eta_n - \eta_\ell - 1} = (m_L M e^k)^{\eta_n - \eta_\ell} e^{-k}.$$

Substituting the above inequality and $U_\ell = (M m_L e^k)^{\eta_\ell} \prod_{i=1}^k \ell_i!$ into (78), the right-hand side of (78) is now bounded by

$$\begin{aligned} R_n &\leq \sum_{\ell < n} \binom{n_1}{\ell_1} \dots \binom{n_k}{\ell_k} (m_L M e^k)^{\eta_n - \eta_\ell} e^{-k} (M m_L e^k)^{\eta_\ell} \prod_{i=1}^k \ell_i! \\ &= (M m_L e^k)^{\eta_n} \left[\sum_{\ell < n} \prod_{i=1}^k \frac{n_i!}{(n_i - \ell_i)!} \right] e^{-k} = (M m_L e^k)^{\eta_n} \left[\prod_{i=1}^k n_i! \right] \left[\sum_{\ell < n} \prod_{i=1}^k \frac{1}{(n_i - \ell_i)!} \right] e^{-k} \\ &= U_n \left[\sum_{\ell < n} \prod_{i=1}^k \frac{1}{(n_i - \ell_i)!} \right] e^{-k}. \end{aligned} \quad (79)$$

We then conclude by noticing that

$$\begin{aligned} \sum_{\ell < n} \prod_{i=1}^k \frac{1}{(n_i - \ell_i)!} &\leq \sum_{\ell_i \leq n_i, i \in \{1, \dots, k\}} \prod_{i=1}^k \frac{1}{(n_i - \ell_i)!} \\ &= \prod_{i=1}^k \left[\sum_{\ell_i=1}^{n_i} \frac{1}{(n_i - \ell_i)!} \right] = \prod_{i=1}^k \left[\sum_{\ell_i=1}^{n_i} \frac{1}{\ell_i!} \right] \leq \prod_{i=1}^k \left[\sum_{\ell_i=1}^{\infty} \frac{1}{\ell_i!} \right] = e^k, \end{aligned}$$

which, plugged into (79), yields $R_n \leq U_n$. Therefore, by the dominated convergence theorem, $\tilde{\varphi}_t(q)$ in (75) converges to $\varphi_{\mathcal{Z}_\infty}(q)$ in (76) as $t \rightarrow \infty$.

Now it remains to show that the convergence in mgf implies the convergence in distribution. Since $\tilde{M}_n(t)$ and χ_n are bounded as shown in Proposition 3, the mgfs of $e^{\delta t}Z(t)$ in (75) and \mathcal{Z}_∞ in (76) exist. Also, we have shown that $\varphi_t(q) \rightarrow \varphi_{\mathcal{Z}_\infty}(q)$ when $t \rightarrow \infty$ for $q \in J$ in some neighborhood of $(0, \dots, 0)$. Hence, $e^{\delta t}Z(t)$ converges to \mathcal{Z}_∞ in distribution.

5.6 Proof of Lemma 9

When $n = n(i)$ and $i \in \{1, \dots, k\}$, we may obtain an expression of $\mathcal{L}_n^b(s)$ by using the expression of $b_n(t)$ in Example 3 in [21], and applying similar idea as applied in (49). We now turn to proving (28). Since L_j 's are all $\mathcal{E}(\mu)$ distributed, $\varphi_\ell(t) = \varphi_{\ell,n}(t)$ given by (7) simplifies to

$$\varphi_\ell(t) = \mathbb{E} \left[\tilde{M}_\ell(t - \tau_1) \left\{ \prod_{j \in C_\ell} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right\} e^{-|C_\ell|\mu(t - \tau_1)} \cdot \mathbf{1}_{[\tau_1 < t]} \right].$$

Then using Fubini's theorem to interchange the expectation with the integration as well as a change of variable $t := t - \tau_1$, it follows that

$$\begin{aligned} \int_0^\infty e^{-st} \varphi_\ell(t) dt &= \left[\prod_{j \in C_\ell} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right] \mathbb{E} \left[\int_{\tau_1}^\infty e^{-st} \tilde{M}_\ell(t - \tau_1) e^{-|C_\ell|\mu(t - \tau_1)} dt \right] \\ &= \left[\prod_{j \in C_\ell} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right] \mathbb{E} \left[e^{-s\tau_1} \int_0^\infty e^{-st} \tilde{M}_\ell(t) e^{-|C_\ell|\mu t} dt \right] \\ &= \left[\prod_{j \in C_\ell} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right] \mathcal{L}^\tau(s) \mathcal{L}_\ell^M(s + |C_\ell|\mu). \end{aligned} \quad (80)$$

If $\ell = \mathbf{0}$ where $\mathbf{0}$ is a zero vector in \mathbb{N}^k then $\tilde{M}_\ell(t) = 1$ hence $\mathcal{L}_\ell^M(s + |C_\mathbf{0}|\mu) = \frac{1}{s + |C_\mathbf{0}|\mu}$, and we get

$$\int_0^\infty e^{-st} \varphi_\mathbf{0}(t) dt = \left[\prod_{j \in C_\mathbf{0}} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right] \frac{\mathcal{L}^\tau(s)}{s + |C_\mathbf{0}|\mu} = \left[\prod_{j=1}^k \frac{\mu}{\mu + n_j\delta} \right] \frac{\mathcal{L}^\tau(s)}{s + |C_\mathbf{0}|\mu}.$$

In the case $\ell > \mathbf{0}$, let us now observe that taking Laplace transforms in renewal equation (10) satisfied by $\tilde{M}_n(\cdot)$ yields the following classical relation between $\mathcal{L}_n^M(s)$ and $\mathcal{L}_n^b(s)$:

$$\mathcal{L}_n^M(s) = \frac{\mathcal{L}_n^b(s)}{1 - \mathcal{L}^\tau(s)}, \quad \forall s > 0, \quad n \in \mathbb{N}^k \setminus \{n(i), i = 1, \dots, k\},$$

so that (80) leads to

$$\int_0^\infty e^{-st} \varphi_\ell(t) dt = \left[\prod_{j \in C_\ell} \frac{\mu}{\mu + (n_j - \ell_j)\delta} \right] \frac{\mathcal{L}^\tau(s)}{1 - \mathcal{L}^\tau(s + |C_\ell|\mu)} \mathcal{L}_\ell^b(s + |C_\ell|\mu).$$

With the above result, the Laplace transform of (9) becomes (28).

5.7 Proof of Theorem 16

Proof of Lemma 14. We shall start by proving the properties for $\tilde{M}_1(t, \delta)$, as those for $\tilde{M}_2(t, \delta)$ are a bit more technical but follow in a similar way. Let us write

$$\tilde{M}_1(t, \delta) = \sum_{i=1}^{\infty} \psi_i(t, \delta), \quad \psi_i(t, \delta) := \mathbb{E} [e^{-\delta(T_i + L_i - t)} \mathbf{1}_{\{T_i \leq t < T_i + L_i\}}], \quad i \in \mathbb{N}. \quad (81)$$

We first start by proving that $\psi_i(t, \delta)$ is defined and analytic on set D_a . Indeed, inequality

$$\left| \delta^j \frac{(-1)^j}{j!} [T_i + L_i - t]^j \mathbf{1}_{\{T_i \leq t < T_i + L_i\}} \right| \leq a^j \frac{1}{j!} L_i^j, \quad j \in \mathbb{N}, \quad \delta \in D_a, \quad (82)$$

coupled with the fact that $\sum_{j=0}^{\infty} \mathbb{E} \left(a^j \frac{1}{j!} L_i^j \right) = \mathbb{E}[e^{aL}] = \frac{\mu}{\mu - a} < +\infty$ by (36), yields that

$$\sum_{j=0}^{\infty} \delta^j \mathbb{E} \left[\frac{(-1)^j}{j!} [T_i + L_i - t]^j \mathbf{1}_{\{T_i \leq t < T_i + L_i\}} \right]$$

is a convergent series on $\delta \in D_a$ and that $\delta \mapsto \psi_i(t, \delta)$ is analytic on that set for all $t \geq 0$, and admits the above power series expansion in δ . Now one checks easily, by independence of L_i and T_i ,

$$\psi_i(t, \delta) \leq \mathbb{E}[e^{aL_i} \mathbf{1}_{\{T_i \leq t\}}] = \mathbb{E}[e^{aL}] \Pr[T_i \leq t], \quad \forall \delta \in D_a, \quad (83)$$

with $\sum_{i=1}^{\infty} \mathbb{E}[e^{aL}] \Pr[T_i \leq t] = \mathbb{E}[e^{aL}] m(t) < +\infty$. This yields that for all $t \geq 0$, series $\sum_{i=1}^{\infty} \psi_i(t, \delta)$ converges normally on $\delta \in D_a$. Thus for all $t \geq 0$, $\delta \mapsto \tilde{M}_1(t, \delta)$ is thus analytic as the uniform limit of an analytic sequence of functions on compact set D_a .

We then move on $\tilde{M}_2(t, \delta)$. Similar to (81), one has

$$\tilde{M}_2(t, \delta) = \sum_{r,j=1}^{\infty} \pi_{r,j}(t, \delta), \quad \pi_{r,j}(t, \delta) := \mathbb{E}[e^{-\delta(T_r + L_r - t)} \mathbf{1}_{\{T_r \leq t < T_r + L_r\}} e^{-\delta(T_j + L_j - t)} \mathbf{1}_{\{T_j \leq t < T_j + L_j\}}].$$

The analog of (82) is

$$\left| \delta^p \frac{(-1)^p}{p!} [(T_r + L_r - t) + (T_j + L_j - t)]^p \mathbf{1}_{\{T_r \leq t < T_r + L_r\}} \mathbf{1}_{\{T_j \leq t < T_j + L_j\}} \right| \leq (a/2)^p \frac{1}{p!} [L_r + L_j]^p, \quad r \in \mathbb{N}, \quad j \in \mathbb{N}, \quad \delta \in D_{a/2},$$

with $\sum_{p=0}^{\infty} (a/2)^p \frac{1}{p!} [L_r + L_j]^p = \mathbb{E}(e^{a(L_r + L_j)/2}) \leq \mathbb{E}(e^{aL})$ (by Jensen's inequality), a finite quantity, so that $\delta \in D_{a/2} \mapsto \pi_{r,j}(t, \delta)$ is analytic. The analog of (83) is

$$\pi_{r,j}(t, \delta) \leq \mathbb{E} \left[e^{a(L_r + L_j)/2} \mathbf{1}_{\{T_r \leq t\}} \mathbf{1}_{\{T_j \leq t\}} \right], \quad r \in \mathbb{N}, \quad j \in \mathbb{N}, \quad \delta \in D_{a/2}, \quad (84)$$

with, again thanks to Jensen's inequality as well as independence of (L_r, L_j) from (T_r, T_j) ,

$$\sum_{r,j=1}^{\infty} \mathbb{E}[e^{a(L_r + L_j)/2} \mathbf{1}_{\{T_r \leq t\}} \mathbf{1}_{\{T_j \leq t\}}] \leq \mathbb{E}(e^{aL}) \sum_{r,j=1}^{\infty} \mathbb{E}[\mathbf{1}_{\{T_r \leq t\}} \mathbf{1}_{\{T_j \leq t\}}] = \mathbb{E}(e^{aL}) \mathbb{E}(N(t)^2) < +\infty.$$

Hence, from (84), $\sum_{r,j=1}^{\infty} \pi_{r,j}(t, \delta) = \tilde{M}_2(t, \delta)$ converges normally on $\delta \in D_{a/2}$, and is analytic on this set by the same argument as $\delta \mapsto \tilde{M}_1(t, \delta)$. Note that we used that $N(t)$ admits the second moment, a fact that holds because $\mathbb{E}[\tau_1^2] < +\infty$, see e.g. [1, Chapter V.6]. \square

Prior to proving Lemma 15, we prove a few upper bounds concerning $\tilde{M}_1(t, \delta)$. First, we note that deriving $\tilde{b}_1(t) = \frac{\mu}{\mu + \delta} e^{-\mu t} \int_0^t e^{\mu s} dF(s)$ yields $\tilde{b}'_1(t) = -\mu \tilde{b}_1(t) + \frac{\mu}{\mu + \delta} f(t)$. Besides, since **(A)** holds, a density $u(t) = m'(t)$ of renewal function exists and is bounded by above by $C > 0$ thanks to Lemma 1. Both these facts entail, deriving (16), the following

$$\left| \tilde{M}'_1(t) \right| = \left| \int_0^t \tilde{b}'_1(t-s) m'(s) ds + \tilde{b}_1(0) m'(t) \right| = \left| \int_0^t \tilde{b}'_1(t-s) m'(s) ds \right|$$

as $f(\cdot)$ is a density, so that $\tilde{b}_1(0) = 0$. Then one finds

$$\begin{aligned} \left| \tilde{M}'_1(t) \right| &\leq \mu \int_0^t \left| \tilde{b}_1(t-s)m'(s) \right| ds + \left| \frac{\mu}{\mu+\delta} \right| \int_0^t |f(t-s)m'(s)| ds \\ &\leq \mu C \int_0^\infty \left| \tilde{b}_1(s) \right| ds + \left| \frac{\mu}{\mu+\delta} \right| C \int_0^\infty f(s) ds \\ &\leq C \left[\left| \frac{\mu}{\mu+\delta} \right| + \left| \frac{\mu}{\mu+\delta} \right| \right] \leq \frac{2C\mu}{\mu-|\delta|}, \end{aligned} \quad (85)$$

where the last line is due to the fact that $f(\cdot)$ is a density, and $\int_0^\infty |\tilde{b}_1(s)| ds \leq C \left| \frac{\mu}{\mu+\delta} \right|$ from (49).

Proof of Lemma 15. We again start with $\tilde{M}_1(t, \delta)$. The key to is to use expansions for $\tilde{M}_1(t) = \tilde{M}_1(t, \delta)$ in Theorem 5 and particularly the dependence of this expansion in δ as discussed in Remark 6. Indeed, an immediate consequence of (25) and (26) from Remark 6 is that

$$\begin{aligned} \left| \tilde{M}_1(t, \delta) - \chi_1(\delta) \right| &\leq \frac{M^*}{\mu-|\delta|} \left[e^{-\mu t} + \sum_{k=1}^N e^{-\operatorname{Re}(z_k)t} + \zeta(t) e^{-\operatorname{Re}(z_N)t} \right] \\ &\leq \frac{M^*}{\mu-a} \left[e^{-\mu t} + \sum_{k=1}^N e^{-\operatorname{Re}(z_k)t} + \zeta(t) e^{-\operatorname{Re}(z_N)t} \right], \quad \forall \delta \in D_a, \end{aligned}$$

for some constant M^* independent from δ and t , which implies uniform convergence of $\tilde{M}_1(t, \delta)$ as $t \rightarrow \infty$ towards $\chi_1(\delta)$ on $\delta \in D_a$.

We then move on to $\tilde{M}_2(t, \delta)$. Relation (9) when $k=1$, $X_j=1$, $L \sim \mathcal{E}(\mu)$, along with (7) and (8) yields the following expression

$$\tilde{b}_2(t) = \tilde{b}_2(t, \delta) = \varphi_0(t) + 2\varphi_1(t), \quad (86)$$

$$\varphi_0(t) = \varphi_0(t, \delta) = \frac{\mu}{\mu+2\delta} \mathbb{E}[e^{-\mu(t-\tau_1)} 1_{\{\tau_1 < t\}}] = \frac{\mu}{\mu+2\delta} \int_0^t e^{-\mu(t-s)} f(s) ds, \quad (87)$$

$$\varphi_1(t) = \varphi_1(t, \delta) = \frac{\mu}{\mu+\delta} \mathbb{E}[\tilde{M}_1(t-\tau_1, \delta) e^{-\mu(t-\tau_1)} 1_{\{\tau_1 < t\}}] = \frac{\mu}{\mu+\delta} \int_0^t \tilde{M}_1(t-s, \delta) e^{-\mu(t-s)} f(s) ds. \quad (88)$$

Differentiating (87) and (88) with respect to t results in

$$\varphi'_0(t) = \frac{\mu}{\mu+2\delta} \left[-\mu \int_0^t e^{-\mu(t-s)} f(s) ds + f(t) \right], \quad (89)$$

$$\varphi'_1(t) = \frac{\mu}{\mu+\delta} \left[\int_0^t \left(\tilde{M}'_1(t-s, \delta) - \mu \tilde{M}_1(t-s, \delta) \right) e^{-\mu(t-s)} f(s) ds + f(t) \right]. \quad (90)$$

We are also going to need the following upper bounds for $\varphi_0(t, \delta)$, $\varphi_1(t, \delta)$, obtained from (87), (88), and the fact that $t \mapsto \tilde{M}_1(t, \delta)$ is uniformly bounded in $\delta \in D_{a/2}$ by some constant \tilde{C} (independent from δ and t , a consequence of the fact that it converges uniformly on that set):

$$|\varphi_0(t, \delta)| \leq \left| \frac{\mu}{\mu+2\delta} \right| e^{-\mu t} \int_0^\infty e^{\mu s} f(s) ds \leq \frac{\mu}{\mu-a} C_0 e^{-\mu t}, \quad \delta \in D_{a/2}, \quad (91)$$

$$|\varphi_1(t, \delta)| \leq \left| \frac{\mu}{\mu+\delta} \right| \tilde{C} e^{-\mu t} \int_0^\infty e^{\mu s} f(s) ds \leq \frac{\mu}{\mu-a/2} C_1 e^{-\mu t}, \quad \delta \in D_{a/2}, \quad (92)$$

for some constants C_0 and C_1 independent from $\delta \in D_{a/2}$ and t . We also wish to obtain similar bounds for $\varphi'_0(t, \delta)$ and $\varphi'_1(t, \delta)$. The following upper bound for $\varphi'_0(t, \delta)$ is easily obtained thanks to (89):

$$|\varphi'_0(t, \delta)| \leq \left| \frac{\mu}{\mu+2\delta} \right| \left[\mu e^{-\mu t} \int_0^\infty e^{\mu s} f(s) ds + f(t) \right] \leq \frac{\mu}{\mu-a} [C_0^* e^{-\mu t} + f(t)], \quad \delta \in D_{a/2}, \quad (93)$$

for some constant C_0^* . As to $\varphi_1'(t, \delta)$, the fact that $t \mapsto \tilde{M}_1(t, \delta)$ and $t \mapsto \tilde{M}_1'(t, \delta)$ are uniformly bounded in $\delta \in D_{a/2}$ respectively by \tilde{C} and $2C \frac{\mu}{\mu - a/2}$ (thanks to (85)), easily yields from (90)

$$|\varphi_1'(t, \delta)| \leq \frac{\mu}{\mu - a/2} [C_1^* e^{-\mu t} + f(t)], \quad \delta \in D_{a/2},$$

for some constant $C_1^* > 0$. Getting back to our original concern of showing that $\tilde{M}_2(t, \delta)$ converges uniformly, we first note that, in view of (16) and (86), it is clear that it is necessary and sufficient to prove that

$$\delta \mapsto \int_0^t \varphi_l(t-s, \delta) dm(s), \quad l = 0, 1,$$

converges uniformly on $\delta \in D_{a/2}$ as $t \rightarrow \infty$ towards $\frac{1}{\mathbb{E}[\tau_1]} \int_0^\infty \varphi_l(s, \delta) ds$ for $l = 0, 1$. Details will be given only for $l = 0$ as similar proof is applicable for $l = 1$. The starting point is the following decomposition, already used in Relation (55) in Section 5.4:

$$\begin{aligned} \int_0^t \varphi_0(t-s, \delta) dm(s) - \frac{1}{\mathbb{E}[\tau_1]} \int_0^\infty \varphi_0(s, \delta) ds &= -\frac{1}{\mathbb{E}[\tau_1]} \int_t^\infty \varphi_0(s, \delta) ds + \int_0^t \varphi_0(t-s, \delta) dv(x) \\ &:= I_1(t, \delta) + I_2(t, \delta). \end{aligned} \quad (94)$$

Thus, in view of (94), it suffices to prove that $I_1(t, \delta)$ and $I_2(t, \delta)$ uniformly converge towards 0 as $t \rightarrow \infty$ on $\delta \in D_{a/2}$. Uniform convergence of $I_1(t, \delta)$ is obtained thanks to (91) that entail:

$$\sup_{\delta \in D_{a/2}} |I_1(t, \delta)| \leq \frac{1}{\mathbb{E}[\tau_1]} \frac{1}{\mu - a} C_0 e^{-\mu t} \rightarrow 0, \quad t \rightarrow \infty.$$

As to $I_2(t, \delta)$, performing an integration by parts as in (62) yields

$$I_2(t, \delta) = \varphi_0(t, \delta) v(0^-) + \int_0^t v(s) \varphi_0'(t-s, \delta) ds.$$

The first term on the right-hand side uniformly converges to 0 on $\delta \in D_{a/2}$ thanks to (91). As to the second term, we use the inequality (93) to get

$$\left| \int_0^t v(s) \varphi_0'(t-s, \delta) ds \right| \leq \int_0^t |v(s)| |\varphi_0'(t-s, \delta)| ds \leq \frac{\mu}{\mu - a} \int_0^t |v(s)| [C_0^* e^{-\mu(t-s)} + f(t-s)] ds, \quad \delta \in D_{a/2}. \quad (95)$$

Note that $\int_0^t |v(s)| e^{-\mu(t-s)} ds$ tends to zero by dominated convergence theorem, as $\int_0^\infty |v(s)| ds$ is finite (a direct consequence of expansion (20)). Also, the light tailed assumption in (36) for τ_1 entails that for all $j = 1, \dots, N$ one has $\int_0^t e^{-z_j s} f(t-s) ds = e^{-z_j t} \int_0^t e^{z_j s} f(s) ds \rightarrow 0$ as $t \rightarrow \infty$. Similarly, $\int_0^t \eta(s) e^{-z_j s} f(t-s) ds \rightarrow 0$ where $\eta(x)$ is defined by (21). Hence $\int_0^t |v(s)| f(t-s) ds$ tends to zero as $t \rightarrow \infty$. Then, from (95) $I_2(t, \delta)$ uniformly converges to 0 on $\delta \in D_{a/2}$. Thus, all in all, $\tilde{M}_2(t, \delta)$ converges uniformly on $\delta \in D_{a/2}$ towards $\chi_2(\delta)$. \square

Proof of Theorem 16. Since $0 \leq -\frac{\partial}{\partial \delta} \tilde{Z}(t, \delta) \Big|_{\delta=0} = D(t) \leq \sum_{i=1}^{N(t)} L_i$ is integrable, it is possible to exchange derivation with respect to δ and expectation and one has for all $t > 0$

$$-\frac{\partial}{\partial \delta} \tilde{M}_1(t, \delta) \Big|_{\delta=0} = -\frac{\partial}{\partial \delta} \mathbb{E}[\tilde{Z}(t, \delta)] \Big|_{\delta=0} = -\mathbb{E} \left[\frac{\partial}{\partial \delta} \tilde{Z}(t, \delta) \Big|_{\delta=0} \right] = \mathbb{E}[D(t)]. \quad (96)$$

The main point in the proof is to be able to pass to the limit in (96) as $t \rightarrow \infty$. To do this, we use the fact that we proved in Lemma 14 that $\delta \mapsto \tilde{M}_{n(1)}(t, \delta)$ is analytic on the set D_a where $a < \mu$ is arbitrary. Since by Lemma 15, $\tilde{M}_1(t, \delta)$ uniformly converges towards $\chi_1(\delta)$ on this set, a standard result in complex analysis states that the limiting function $\delta \mapsto \chi_1(\delta)$ is analytic on the same set, hence in particular at $\delta = 0$ (which is

known from its expression (39)), but, more importantly, that one can interchange derivation and passage to the limit, i.e.

$$\lim_{t \rightarrow \infty} \left. \frac{\partial}{\partial \delta} \tilde{M}_1(t, \delta) \right|_{\delta=0} = \frac{\partial}{\partial \delta} \left[\lim_{t \rightarrow \infty} \tilde{M}_1(t, \delta) \right] \Big|_{\delta=0} = \left. \frac{\partial}{\partial \delta} \chi_1(\delta) \right|_{\delta=0}.$$

Expression of $\chi_1(\delta)$ in the case $k = 1$ is given in Corollary 17, Expression (39) with $X_j = 1$, yielding (37).

Let us move on to the covariance of $D(t)$ and queue size $Z_1(t, 0)$. One has $-\left. \frac{\partial}{\partial \delta} [Z_1(t, \delta)]^2 \right|_{\delta=0} = 2D(t)Z_1(t, 0)$, and since the latter is integrable due to $D(t)Z_1(t, 0) \leq \left(\sum_{i=1}^{N(t)} L_i \right) N(t)$, as in (96), interchanging expectation and derivation results in

$$-\left. \frac{\partial}{\partial \delta} \tilde{M}_2(t, \delta) \right|_{\delta=0} = 2\mathbf{E}[D(t)Z_1(t, 0)].$$

The same argument of analyticity of $\delta \mapsto \tilde{M}_2(t, \delta)$ on $\delta \in D_{a/2}$ in Lemma 14, coupled with the fact that uniform convergence result as $t \rightarrow \infty$ in Lemma 15 yields that $\lim_{t \rightarrow \infty} \left. \frac{\partial}{\partial \delta} \tilde{M}_2(t, \delta) \right|_{\delta=0} = \left. \frac{\partial}{\partial \delta} \chi_2(\delta) \right|_{\delta=0}$. Now the fact that $\lim_{t \rightarrow \infty} \tilde{M}_1(t, 0) = \chi_1(0)$ and $\lim_{t \rightarrow \infty} \left. \frac{\partial}{\partial \delta} \tilde{M}_1(t, \delta) \right|_{\delta=0} = \left. \frac{\partial}{\partial \delta} \chi_1(\delta) \right|_{\delta=0}$ implies

$$\begin{aligned} \lim_{t \rightarrow \infty} \text{Cov}[D(t), Z_1(t, 0)] &= \lim_{t \rightarrow \infty} \mathbf{E}[D(t)Z_1(t, 0)] - \mathbf{E}[D(t)]\mathbf{E}[Z_1(t, 0)] \\ &= -\left. \frac{1}{2} \frac{\partial}{\partial \delta} \chi_2(\delta) \right|_{\delta=0} + \chi_1(0) \cdot \left. \frac{\partial}{\partial \delta} \chi_1(\delta) \right|_{\delta=0}. \end{aligned} \quad (97)$$

Expression (40) with $X_j = 1$ yields $\chi_2(\delta) = \frac{1}{\mathbf{E}[\tau_1]} \left(\frac{1}{\mu + 2\delta} + \frac{\mu}{(\mu + \delta)^2} \frac{\mathcal{L}^\tau(\mu)}{1 - \mathcal{L}^\tau(\mu)} \right)$, and in turn,

$$\left. \frac{\partial}{\partial \delta} \chi_2(\delta) \right|_{\delta=0} = -\frac{1}{\mathbf{E}[\tau_1]} \left(\frac{2}{(\mu + 2\delta)^2} + \frac{2\mu}{(\mu + \delta)^3} \frac{\mathcal{L}^\tau(\mu)}{1 - \mathcal{L}^\tau(\mu)} \right) \Big|_{\delta=0} = -\frac{2}{\mu^2 \mathbf{E}[\tau_1]} \left(1 + \frac{\mathcal{L}^\tau(\mu)}{1 - \mathcal{L}^\tau(\mu)} \right).$$

Hence, substitution of the above expression together with $\chi_1(\delta)$ obtained previously into (97) yields (38) for the limiting covariance. \square

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