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Multi-scale arithmetization of the linear transformations

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Abstract—A constructive non-standard interpretation of a multi-scale affine transformation scheme is exposed. It is based on the Ω -numbers of Laugwitz and Schmieden and on the discrete model of the real line of Reeb and Harthong. In this setting, the non-standard version of the Euclidean affine transformation gives rise to a sequence of so-called *quasi-linear* transformations over integer spaces, allowing integer-only computations.

Index Terms—affine transformation, discretization, multigrid convergence, Omega-number, Harthong-Reeb line, digital geometry, Nonstandard analysis, Constructive mathematics



1 INTRODUCTION

In the past decade, a series of papers [1], [2], [3], [4], [5], [6] was devoted to the arithmetization of the continuum with the goal to solve numerical problem using only integers. In particular, a multi-scale arithmetization of the Euler scheme for solving differential equations was developed and used to produce discrete representations of continuous functions and curves. These works are rooted on the one hand in the studies of G. Reeb and J Harthong on nonstandard analysis [7] that led to a discrete model of \mathbb{R} , the *Harthong-Reeb line*. This model, though it requires to dispose of an infinitely large number (which was axiomatically assumed), allowed Reveillès [8], [9] to construct a discretization scheme which resulted in its seminal work on the digital line. On the other hand, Laugwitz and Schmieden [10], [11] proposed a constructive definition of infinitely large numbers: the Ω -numbers. Mixing the Harthong-Reeb line with the Ω -numbers was done in the PhD thesis of Chollet [12] and gave rise to the Ω -arithmetization scheme.

In the present paper, we aim at using the multi-scale discretization scheme of Chollet to obtain a multi-scale discretization of the affine transformations between Euclidean spaces. In her PhD thesis [13], Jacob-Da Col, which was a student of Reveillès, studied the properties

of an arithmetization of the affine transformations of \mathbb{R}^2 , called *Quasi-affine transformations* (QAT). Then, the study was extended to the nD -spaces [14], [15], [16], [17], [18], [19], [20]. Nevertheless, in these works, the QAT are studied *per se* and do not result from a discretization scheme, *a fortiori* a multi-scale one. Here, we exhibit the link between an Ω -arithmetization scheme derived from the one of Chollet *et al.* and sequences of QATs and prove the convergence of the scheme toward the Euclidean transformation.

The article is organized as follows. In Sec. 2, we introduce the Ω -numbers and the Harthong-Reeb line. We recall their properties and the skeleton of the Ω -arithmetization scheme. The section ends with the definition of the QATs. Section 3 links sequences of QATs with the affine transformations between spaces over the Harthong-Reeb line. Section 4 concludes the article.

2 BACKGROUND

2.1 Notations

Below are the notations used throughout the article.

x, y, z : reals.

a, b, c : rationals.

p, q, r : integers.

$\lfloor x \rfloor$: floor of x , that is $\max\{p \in \mathbb{N} \mid p \leq x\}$.

\div : integer division : $p \div q = \lfloor p/q \rfloor$.

Sequences are indexed by \mathbb{N} and noted $\langle^n s \rangle$ where $^n s$ is the $(n+1)$ th term of the sequence.

i : $x \in \mathbb{R} \mapsto \langle^n x \rangle \in \mathbb{R}^{\mathbb{N}}$ where $^n x = x$ for any n is the canonical injection of \mathbb{R} into $\mathbb{R}^{\mathbb{N}}$.

$\alpha, \beta, \gamma, \omega$: increasing sequence of positive integers.

O : big O (Landau notation)

o : little o (Landau notation)

$\langle E \rangle_\alpha$: sequences in E ($E \subseteq \mathbb{R}$) that are in $O(\alpha)$

($\langle E \rangle_1$ stands for $\langle E \rangle_{i(1)}$).

\mathcal{HR}_ω : the Harthong-Reeb line related to ω (see page 2).

u, v, w : elements of \mathcal{HR}_ω .

d, d' : space dimension.

f : affine transform between \mathbb{R}^d and $\mathbb{R}^{d'}$ whose matrix is denoted f .

g : quasi-affine transform between \mathbb{Z}^d and $\mathbb{Z}^{d'}$ whose matrix is denoted g (see page 3).

ℓ : linear map between \mathcal{HR}_ω -modules whose matrix is denoted ℓ .

i, j : indices for the matrices.

The coefficients of the matrix m are denoted $m_{i,j}$ (m_j for a column matrix). The anonymous matrix whose coefficients is an expression $m_{i,j}$ is denoted $(m_{i,j})_{i,j}$ ($(m_j)_j$ for a column matrix).

Vectors are in bold and often identified with their matrix representation. So, we can write: $f(\mathbf{x}) = f\mathbf{x}$ and $\ell(\mathbf{u}) = \ell\mathbf{u}$.

No difference is made between sequence matrices and matrix sequences. For instance, the sequence of matrices $\langle^n f \rangle$ is a matrix whose coefficients are the sequences $\langle^n f_{i,j} \rangle$.

Given an equivalent relation,

$- [x]$ denotes the equivalent class of x ,

$- \dot{y}$ denotes an element picked in the equivalent class y .

\mathbb{Q}_{lim} : a quotient space of the rational sequences (see page 2).

\mathbb{Q}_{Cau} : subspace of \mathbb{Q}_{lim} which contains the projections of the Cauchy rational sequences.

π, ρ : element of \mathbb{Q}_{lim} .

If C is a collection (sequence, vector, matrix or equivalence class) of elements in a set X and h a map from X to some set Y , $h(C)$ denotes the collection obtained by applying h to each

term of C . When C is an equivalence class, it is assumed that h is invariant under the equivalence relation.

Alike, the operations $+$, \times , $/$, \div , the relations $<$, \leq and so on are termwise on sequences, vectors and matrices (but not necessarily on equivalence classes).

$\psi_\alpha : s \mapsto \lfloor \alpha \times s \rfloor$, or $s \mapsto \llbracket \alpha \times \dot{s} \rrbracket$ depending whether s is a number sequence or a number sequence equivalence class and $\varphi_\alpha : s \mapsto s/\alpha$ or $s \mapsto \dot{s}/\alpha$. Note that $\psi_\alpha \circ \varphi_\alpha$ is the identity map of $\mathbb{Z}^{\mathbb{N}}$.

2.2 Ω -rationals and the Harthong-Reeb line

Let ω be an increasing sequence of positive integers. Such a sequence is called an *infinitely large number*. Using the Landau notations, we consider the subsets $O(\omega)$ and $o(\omega)$ of $\mathbb{Z}^{\mathbb{N}}$. Then, the Harthong-Reeb line, \mathcal{HR}_ω is the quotient space $O(\omega)/o(\omega)$. We denote by $=_\omega$ the corresponding equivalence relation. The Harthong-Reeb line is equipped with the following operations.

$$u + v \stackrel{\text{def}}{=} [\dot{u} + \dot{v}] \quad ;$$

$$u \times_\omega v \stackrel{\text{def}}{=} [(\dot{u} \times \dot{v}) \div \omega] \quad .$$

All these operations are well-defined. The tuple $(\mathcal{HR}_\omega, +, \times)$ is a commutative ring [3], [5], [6].

Let us now consider the subsets $O(1)$ and $o(1)$ of the rational sequences. We denote by \mathbb{Q}_{lim} the quotient space $O(1)/o(1)$ of the bounded sequences of rational up to the sequences converging toward 0. In \mathbb{Q}_{lim} , we set:

$$\pi + \rho \stackrel{\text{def}}{=} [\dot{\pi} + \dot{\rho}] \quad ;$$

$$\pi \times \rho \stackrel{\text{def}}{=} [\dot{\pi} \times \dot{\rho}] \quad .$$

It was proved in [3], [5] that \mathcal{HR}_ω and \mathbb{Q}_{lim} are isomorph via the map $\varphi_\omega : \mathcal{HR}_\omega \rightarrow \mathbb{Q}_{\text{lim}}$ defined by $\varphi_\omega(u) = [\dot{u}/\omega]$ whose converse is the map $\psi_\omega : \mathbb{Q}_{\text{lim}} \rightarrow \mathcal{HR}_\omega$ defined by $\psi_\omega(\pi) = \llbracket \omega \times \dot{\pi} \rrbracket$. Both maps are well-defined. The restriction of \mathbb{Q}_{lim} to the classes of the Cauchy sequences is noted \mathbb{Q}_{Cau} . It is plain that the limit operator provides an isomorphism from $(\mathbb{Q}_{\text{Cau}}, +, \times, \leq)$ to $(\mathbb{R}, +, \times, \leq)$. It should be noted that neither \mathcal{HR}_ω nor \mathbb{Q}_{lim} are fields in the classical sense since many of their elements do not have any inverse.

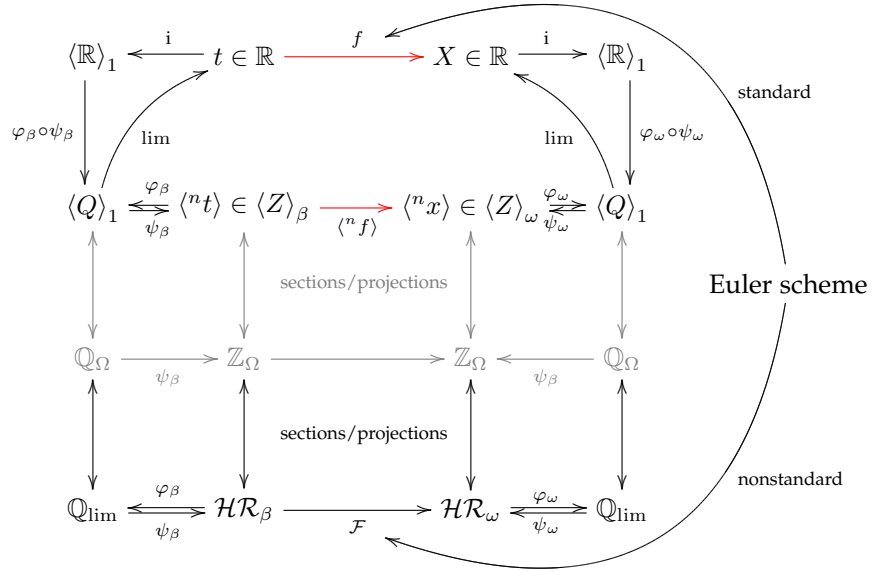


Fig. 1. The skeleton of the arithmetization scheme described in [5] which link the Euclidean function f with a sequence of integer functions $\langle^n f\rangle$. Basically, the integer function $\langle^n f\rangle$ is the arithmetized expression at the scale n of the Euclidean function f . The sequence $\langle^n f\rangle$ itself is obtained as a section of the function \mathcal{F} which translate f when \mathbb{R} is replaced by the Harthong-Reeb line. The function \mathcal{F} is computed using a non-standard version of the Euler scheme for solving differential equations.

2.3 Ω -arithmetization scheme

The ω -arithmetization scheme provides a way to digitize a real function which is the solution of a Cauchy problem. The differential equation is solved using the Euler scheme through non-standard analysis, substituting $\mathcal{H}\mathcal{R}_\omega$ to \mathbb{R} and infinitely small steps to standard steps. Then, going back to the very definition of the Harthong-Reeb line and taking any section of the non-standard solution, one obtains a sequence of integer functions that converges in some sense toward the real function to digitize. The commutative diagram of Fig. 1 summarizes the relation between a real continuous function and the corresponding multi-scale integer function in the Ω -arithmetization scheme as exposed in [5] except that many of the maps exposed in the commutative diagram are implicit in the paper. The sets \mathbb{Z}_Ω and \mathbb{Q}_Ω are defined by $\mathbb{Z}_\Omega = \mathbb{Z}^\mathbb{N}/\sim$ and $\mathbb{Q}_\Omega = \mathbb{Q}^\mathbb{N}/\sim$ where \sim is the equivalence relation on the real sequences defined by $\langle^n x \rangle \sim \langle^n y \rangle$ iff ${}^n x = {}^n y$ for any n greater than some threshold (formally, $\langle^n x \rangle \sim \langle^n y \rangle \iff \exists N, \forall n \geq N, {}^n x = {}^n y$). In the present paper, for simplicity, we jump over these spaces to directly refer to \mathbb{Q}_{\lim} , or \mathbb{Q}_{Cau} , and $\mathcal{H}\mathcal{R}_\omega$ or $\mathcal{H}\mathcal{R}_\beta$.

2.4 Quasi affine transformations

Let $f: \mathbb{Z}^d \rightarrow \mathbb{Z}^{d'}$ be an affine transformation whose linear part is f_0 and r be a positive integer. The QAT (quasi affine transformation) $\lfloor f \rfloor_r$ is defined by

$$\lfloor f \rfloor_r : \mathbf{p} \in \mathbb{Z}^d \longrightarrow f(\mathbf{p}) \div r \in \mathbb{Z}^{d'}$$

A QLT (quasi linear transformation) is a QAT that leaves invariant the origin: $\lfloor f \rfloor_r$ is a QLT if $f(\mathbf{p}) = f_0(\mathbf{p}) + \mathbf{q}$ where $0 \leq \mathbf{q} < r$ coordinate-wise.

Given a QLT g , the triple (r, f_0, \mathbf{v}) defining g that has the smallest r is called the *canonical representative* of g and r is called the *scale* of g .

The reader interested by the properties of QATs is referred back to the articles cited in the introduction.

In the following section, we show that Ω -arithmetization scheme extends well to the linear transformations between Euclidean spaces, involving sequences of QLTs

3 Ω -ARITHMETIZATION OF AFFINE TRANSFORMATIONS

Let us consider again the commutative diagram presented in Fig. 1. In this section, we want to generalize it by replacing the rings by modules

over these rings, the Euclidean function by a linear transformation between \mathbb{R}^d and $\mathbb{R}^{d'}$. Since affine transformations are easily characterized by their matrices, we don't need to use any numerical algorithm like the Euler scheme. *A priori*, we want to keep the possibility to use distinct grids in the source and the destination spaces. So we will again use two infinitely large numbers, denoted α and β . Practically, for a linear transformation over \mathbb{R}^d , the sequence α can be viewed as a sequence of resolutions — or a sequence of finer and finer grids — while the sequence β allows to quantify the matrix coefficients more and more accurately. Nevertheless, $\alpha \neq \beta$ implies generally that the rings \mathcal{HR}_α and \mathcal{HR}_β are distinct. Thus, we need to extend the inner product of \mathcal{HR}_ω to an outer product. Firstly, we observe that one has the following property.

Lemma 1. *Let α, β, ω be three infinitely large numbers such that $\omega = \beta^2 \div \alpha$. Then, for any $u \in \mathcal{HR}_\alpha, v \in \mathcal{HR}_\omega$,*

$$\psi_\beta \circ \varphi_\omega(v) \times_\beta \psi_\beta \circ \varphi_\alpha(u) = [(i \times \dot{v}) \div \beta] .$$

Proof:

$$\begin{aligned} & \psi_\beta \circ \varphi_\omega(v) \times_\beta \psi_\beta \circ \varphi_\alpha(u) \\ &= \left[\left[\beta \frac{\dot{v}}{\omega} \right] \right] \times_\beta \left[\left[\beta \frac{\dot{u}}{\alpha} \right] \right] \\ &= \left[\left(\left[\beta \frac{\dot{v}}{\omega} \right] \times \left[\beta \frac{\dot{u}}{\alpha} \right] \right) \div \beta \right] \\ &= \left[\frac{\beta \dot{v} \dot{u}}{\beta^2 + o(\beta^2)} + O\left(\frac{\dot{v}}{\omega}\right) + O\left(\frac{\dot{u}}{\alpha}\right) \right] \\ &= [\dot{v} \dot{u} \div \beta] \\ & \quad \text{for } \dot{v} \in O(\omega) \text{ and } \dot{u} \in O(\alpha) \\ &= v \times_\alpha^\omega u . \end{aligned}$$

□

Then, given two infinitely large numbers α, β such that $\omega = \beta^2 \div \alpha$ is also an infinitely large number, that is, $\alpha \in o(\beta^2)$, we define the outer product \times_α^ω as follows:

$$\begin{aligned} & \mathcal{HR}_\omega \times \mathcal{HR}_\alpha \longrightarrow \mathcal{HR}_\beta \\ & (v, u) \longmapsto v \times_\alpha^\omega u \stackrel{\text{def}}{=} [(i \times \dot{v}) \div \beta] . \end{aligned}$$

From Lemma1, because the maps ψ and φ are morphisms, we readily derive that this product

is well-defined and satisfies the following two properties:

$$\begin{aligned} & \forall u_1, u_2 \in \mathcal{HR}_\alpha, \forall v \in \mathcal{HR}_\beta, \\ & v \times_\alpha^\omega (u_1 + u_2) = v \times_\alpha^\omega u_1 + v \times_\alpha^\omega u_2 ; \end{aligned}$$

$$\begin{aligned} & \forall \lambda \in \mathcal{HR}_\alpha, \forall u \in \mathcal{HR}_\alpha, \forall v \in \mathcal{HR}_\beta, \\ & v \times_\alpha^\omega (\lambda u) = \psi_\beta \circ \varphi_\alpha(\lambda)(v \times_\alpha^\omega u) . \end{aligned}$$

Provided $\alpha \in o(\beta^2)$, this outer product allows to define a matrix product from $\mathcal{M}_{d',d}(\mathcal{HR}_\omega) \times \mathcal{M}_{d,1}(\mathcal{HR}_\alpha)$ to $\mathcal{M}_{d',1}(\mathcal{HR}_\beta)$ by setting $\omega = \beta^2 \div \alpha$. Note that this matrix product is a shortcut for a succession of morphisms. Indeed, thanks to the pair of converse isomorphisms (φ, ψ) , the Harthong-Reeb lines \mathcal{HR}_α and \mathcal{HR}_β are bound to \mathcal{HR}_ω via the following isomorphisms:

$$\begin{aligned} & \psi_\omega \circ \varphi_\alpha: \mathcal{HR}_\alpha \longrightarrow \mathcal{HR}_\omega \\ & x \longmapsto \left[\left[\omega \frac{\dot{x}}{\alpha} \right] \right] \\ & \psi_\omega \circ \varphi_\beta: \mathcal{HR}_\beta \longrightarrow \mathcal{HR}_\omega \\ & x \longmapsto \left[\left[\omega \frac{\dot{x}}{\beta} \right] \right] . \end{aligned}$$

Then, using the outer matrix product is equivalent to apply $\psi_\omega \circ \varphi_\alpha$ on the vector coordinates, then to perform the inner matrix product and finally to apply $\psi_\beta \circ \varphi_\omega$ on the resulting coordinates.

Eventually, the proposed arithmetization scheme for the Euclidean linear transformations is defined in the commutative diagram of Fig. 3. In the scheme, the starting point is the $d' \times d$ matrix f with real coefficients. These coefficients yield non standard versions in \mathcal{HR}_ω which results in a matrix l and a map ℓ between the modules $\mathcal{HR}_\omega^\alpha$ and $\mathcal{HR}_\beta^{d'}$. We say that ℓ is an Ω -linear transformation (Ω -LT) derived from the matrix l . In the following section, we study the relation between the map ℓ and the Euclidean linear map f .

3.1 Ω -LT

In the commutative diagram of Fig 2, following the external arrows from \mathbb{R}^e to \mathcal{HR}_γ^e , where

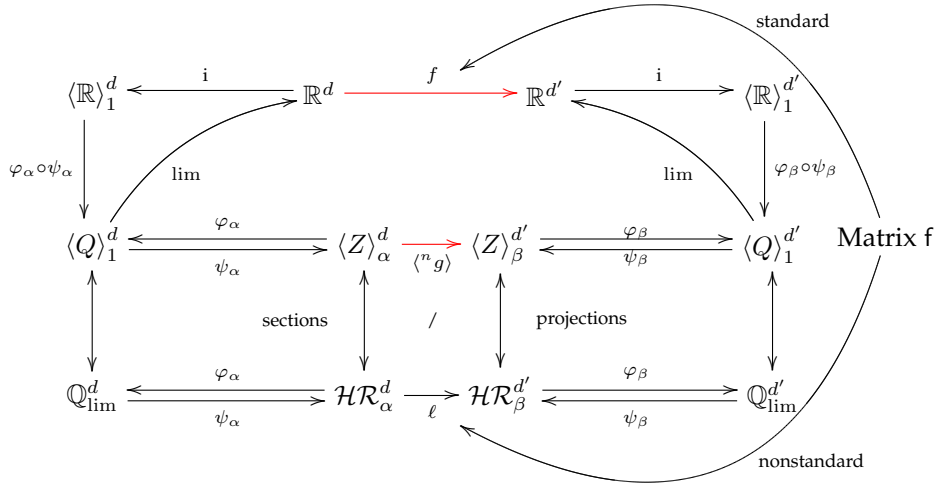


Fig. 2. The Ω -arithmeticization scheme which associates to the linear transformation f the sequence of quasi-affine transformations $\langle^n g\rangle$ from $\langle \mathbb{Z} \rangle_\alpha^d$ to $\langle \mathbb{Z} \rangle_\beta^{d'}$ which is a section of a non-standard translation of the matrix f of f .

$(e, \gamma) \in \{(d, \alpha), (d', \beta)\}$, we observe that, for any real vector \mathbf{x} ,

$$\begin{aligned} \psi_\gamma([\varphi_\gamma \circ \psi_\gamma(\mathbf{i}(\mathbf{x}))]) &= [\psi_\gamma \circ \varphi_\gamma \circ \psi_\gamma(\mathbf{i}(\mathbf{x}))] \\ &= [\psi_\gamma(\mathbf{i}(\mathbf{x}))] \\ &= [[\gamma \mathbf{x}]] . \end{aligned}$$

We denote by ξ_γ^{HR} the function $\psi_\gamma \circ \mathbf{i}$ from \mathbb{R}^e to \mathcal{HR}_γ^e that maps \mathbf{x} to $[[\gamma \mathbf{x}]]$. It is plain that $\xi_\gamma^{\text{HR}}: \mathbb{R} \rightarrow \mathcal{HR}_\gamma^e$ is a ring morphism since the maps \mathbf{i} , φ , ψ and the projection map of the equivalence relation are all ring morphisms. Furthermore, it is easy to show that ξ_γ^{HR} is an injection whose image is $\psi_\gamma(\mathbb{Q}_{\text{Cau}}^e)$ and whose converse maps a vector $\mathbf{u} \in \psi_\gamma(\mathbb{Q}_{\text{Cau}}^e)$ to the real vector $\lim \dot{\rho}$ where $\rho = \varphi_\gamma(\mathbf{u})$. Note that the choice of the representative $\dot{\rho}$ does not matter since, by definition of \mathbb{Q}_{lim} , all the sequences belonging to the class ρ share the same limit if any. The converse map can be extended to \mathcal{HR}_γ^e by the function $\chi_\gamma^{\text{HR}}: \mathcal{HR}_\gamma^e \rightarrow \mathbb{R}^e$ defined by $\chi_\gamma^{\text{HR}}(\mathbf{u}) = \lim \sup(\dot{\rho})$ where $\rho = \varphi_\gamma(\mathbf{u})$. Again, we observe that the choice of the representative $\dot{\rho}$ does not matter since two sequences whose difference tends toward zero have the same upper limit. We have $\chi_\gamma^{\text{HR}} \circ \xi_\gamma^{\text{HR}} = \text{id}$ where id is the identity map. The converse map $\xi_\gamma^{\text{HR}} \circ \chi_\gamma^{\text{HR}}$ is neither injective nor surjective. Nevertheless, its restriction to $\xi_\gamma^{\text{HR}}(\mathbb{R}^d)$ is the identity map.

We are now able to specify the relation between the linear map ℓ and the Ω -LT ℓ in the commutative diagram of Fig 2.

Proposition 1. *Let f be a linear transformation between \mathbb{R}^d and $\mathbb{R}^{d'}$ whose matrix is $f = (f_{i,j})$ and α, β be two infinitely large numbers such that $\alpha \in o(\beta^2)$. Then, setting $\omega = \beta^2 \div \alpha$, the Ω -LT ℓ from \mathcal{HR}_α^d to $\mathcal{HR}_\beta^{d'}$ whose matrix is $\mathfrak{l} = (\xi_\omega^{\text{HR}}(f_{i,j}))$ is such that*

$$\xi_\beta^{\text{HR}} \circ f = \ell \circ \xi_\alpha^{\text{HR}} .$$

In other words, one has the commutative diagram:

$$\begin{array}{ccc} \mathbf{x} & \xrightarrow{f} & \mathbf{y} = \mathbf{f}\mathbf{x} \\ \xi_\alpha^{\text{HR}} \downarrow & & \downarrow \xi_\beta^{\text{HR}} \\ \mathbf{u} & \xrightarrow{\ell} & \mathbf{v} = \mathfrak{l}\mathbf{u} \end{array}$$

Proof: Since ξ_β^{HR} is a ring morphism, we have

$$\begin{aligned} \xi_\beta^{\text{HR}}(\mathbf{f}\mathbf{x}) &= \left(\xi_\beta^{\text{HR}} \left(\sum f_{i,j} \times x_j \right) \right)_j \\ &= \left(\sum \xi_\beta^{\text{HR}}(f_{i,j}) \times_\beta \xi_\beta^{\text{HR}}(x_j) \right)_j \\ &= \left(\sum \xi_\omega^{\text{HR}}(f_{i,j}) \times_\alpha^\omega \xi_\alpha^{\text{HR}}(x_j) \right)_j \\ &\quad \text{(from Lemma 1)} \\ &= \mathfrak{l}\mathbf{u} . \end{aligned}$$

□

With some restriction on the underlying Harthong-Reeb lines, Prop. 1 allows to compute the Ω -LT ℓ from the linear map f and conversely.

Corollary 1. *With the notations of Prop. 1, one has*

$$f = \chi_\beta \circ \ell \circ \xi_\alpha^{\text{HR}} ;$$

$$\ell_{\uparrow \xi_\alpha^{\text{HR}}(\mathbb{R}^d)} = \xi_\beta^{\text{HR}} \circ f \circ \chi_\alpha .$$

Proof: In Prop. 1, we have $\mathbf{u} = \xi_\alpha^{\text{HR}}(\mathbf{x})$, then $\mathbf{u} \in \xi_\alpha^{\text{HR}}(\mathbb{R}^d)$. Since the restriction of χ_α to $\xi_\alpha^{\text{HR}}(\mathbb{R}^d)$ is the inverse function of ξ_α^{HR} , we conclude straightforwardly. \square

In the following section we study the relation between the map ℓ and the sequence of maps $\langle {}^n g \rangle$ in the diagram of Fig. 2. We will show that the maps ${}^n g$ are QLTs.

3.2 Ω -LTs and QATs

Forgetting the scheme illustrated in Fig. 2, we now focus on the understanding of the Ω -LTs *per se*, that is going back to the very definition of \mathcal{HR}_ω .

Firstly, we show that the definition of the multiplication in \mathcal{HR}_ω has a natural extension with matrices.

Lemma 2. *Let α, β, ω be three infinitely large numbers such that $\alpha \in o(\beta^2)$ and $\omega = \beta^2 \div \alpha$. Let $\mathbf{w} \in \mathcal{M}_{d',d}(\mathcal{HR}_\omega)$ et $\mathbf{u} \in \mathcal{M}_{d,1}(\mathcal{HR}_\alpha)$. Then,*

$$\mathbf{w}\mathbf{u} = [(\dot{\mathbf{w}}\dot{\mathbf{u}}) \div \beta] ,$$

where $\dot{\mathbf{w}}$ is any matrix $(\dot{w}_{i,j})_{i,j}$, $\dot{\mathbf{u}}$ is any vector $(\dot{u}_j)_j$.

Proof: Starting from the matrix product definition, we have

$$\begin{aligned} \mathbf{w}\mathbf{u} &= \left(\sum_j \mathbf{w}_{i,j} \times_\alpha^\omega \mathbf{u}_j \right)_i \\ &= \left(\sum_j [(\dot{w}_{i,j} \times \dot{u}_j) \div \beta] \right)_i \\ &= \left(\left[\sum_j (\dot{w}_{i,j} \times \dot{u}_j) \div \beta \right] \right)_i \\ &\quad \text{for } + \text{ respects } =_\beta \\ &= [(\dot{\mathbf{w}}\dot{\mathbf{u}}) \div \beta] . \end{aligned}$$

\square

It appears that Lemma 2 is the key to introduce the relation between Ω -LTs defined over non-standard spaces and quasi-linear transformations over integer spaces. Let $\ell: \mathcal{HR}_\alpha^d \rightarrow \mathcal{HR}_\beta^{d'}$ be a Ω -LT whose matrix is noted \mathfrak{l} ($\mathfrak{l} \in$

$\mathcal{M}_{d',d}(\mathcal{HR}_\omega)$ where $\omega = \beta^2 \div \alpha$). Let $\langle {}^n \mathfrak{l} \rangle$ be a sequence of integer matrices such that $[\langle {}^n \mathfrak{l}_{i,j} \rangle] = \mathfrak{l}_{i,j}$. We write $\dot{\ell}$, or $\langle {}^n \dot{\ell} \rangle$, for the sequence of QLTs whose canonical representatives are $({}^n \beta, {}^n \mathfrak{l}, \mathbf{0})$ and, given a sequence $\mathbf{p} = \langle {}^n \mathbf{p} \rangle$ of integer vectors, we denote by $\dot{\ell}(\mathbf{p})$ the sequence $\langle {}^n \dot{\ell}({}^n \mathbf{p}) \rangle$. Then, from Lemma 2, we derive the next assertion that links Ω -LTs and sequences of QLTs.

Corollary 2. *Let $\ell: \mathcal{HR}_\alpha^d \rightarrow \mathcal{HR}_\beta^{d'}$ be a Ω -LT. Then, for any vector $\mathbf{u} \in \mathcal{HR}_\alpha^d$, one has*

$$\ell(\mathbf{u}) = [\dot{\ell}(\dot{\mathbf{u}})] .$$

Proof: Let \mathfrak{l} and \mathbf{u} be the matrices arising from ℓ and \mathbf{u} .

On the one hand, the n th element of the sequence $\dot{\ell}(\dot{\mathbf{u}})$ is the vector ${}^n \dot{\ell}({}^n \dot{\mathbf{u}})$ which, by definition of ${}^n \dot{\ell}$, has matrix $({}^n \mathfrak{l} \dot{\mathbf{u}}) \div {}^n \beta$.

On the other hand, from Lemma 2, the vector $\ell(\mathbf{u})$ has matrix $[(\dot{\mathbf{u}}) \div \beta]$ and, since $+$, \times and \div are termwise operations in $\mathbb{Z}^{\mathbb{N}}$, the n th element of the sequence $(\dot{\mathbf{u}}) \div \beta$ is $({}^n \dot{\mathbf{u}}) \div {}^n \beta$. \square

The following theorem characterizes the relationship between the Ω -LT from \mathcal{HR}_α^d to $\mathcal{HR}_\beta^{d'}$ and the quasi-linear transformations from \mathbb{Z}^d to $\mathbb{Z}^{d'}$.

Theorem 1. *Let ℓ be a map from \mathcal{HR}_α^d to $\mathcal{HR}_\beta^{d'}$ where $\alpha \in o(\beta^2)$. If ℓ is an Ω -LT with matrix \mathfrak{l} , then, for any sequence $\langle {}^n g \rangle$ of QLTs with canonical representatives $({}^n \gamma, {}^n \mathbf{g}, {}^n \mathbf{p})$ such that $\langle {}^n \gamma \rangle =_\beta \beta$, $\langle \|\mathbf{g}\|_\infty \rangle \in O(\omega)$ where $\omega = \beta^2 \div \alpha$ and $[\langle {}^n \mathbf{g} \rangle] = \mathfrak{l}$, one has, for any $\mathbf{u} \in \mathcal{HR}_\alpha^d$, $\ell(\mathbf{u}) = [\langle {}^n g({}^n \dot{\mathbf{u}}) \rangle]$. Conversely, ℓ is an Ω -LT if there exists a sequence of QLTs $\langle {}^n g \rangle$ with scales ${}^n \gamma$ such that $\langle {}^n \gamma \rangle =_\omega \beta$, $\langle \|\mathbf{g}\|_\infty \rangle \in O(\omega)$ where $\omega = \beta^2 \div \alpha$ and, for any $\mathbf{u} \in \mathcal{HR}_\alpha^d$, there exists $\dot{\mathbf{u}}$ such that $\ell(\mathbf{u}) = [\langle {}^n g({}^n \dot{\mathbf{u}}) \rangle]$.*

Proof: Firstly, we assume that ℓ is an Ω -LT. Let $\mathfrak{l} \in \mathcal{M}_{d',d}(\mathcal{HR}_\omega)$ be the matrix of ℓ . Let $\langle {}^n g \rangle$ be a sequence of QLTs whose canonical representatives are $({}^n \gamma, {}^n \mathbf{g}, {}^n \mathbf{p})$ where $\langle {}^n \gamma \rangle =_\beta \beta$ and $[\langle {}^n \mathbf{g} \rangle] = \mathfrak{l}$. We derive from Corollary 2 that, for any vector $\mathbf{u} \in \mathcal{HR}_\alpha^d$, $\ell(\mathbf{u})$ has a representative whose n th term is $({}^n \mathbf{g} \dot{\mathbf{u}}) \div {}^n \beta$. Then, to achieve the first part of the proof, it suffices to show

that the sequences $\langle ({}^n\mathbf{g}^n\dot{\mathbf{u}}) \div {}^n\beta \rangle$ and $\langle {}^ng({}^n\dot{\mathbf{u}}) \rangle$ are equivalent for the relation $=_\beta$.

$$\begin{aligned} |({}^n\mathbf{g}({}^n\dot{\mathbf{u}}) \div {}^n\beta) - {}^ng({}^n\dot{\mathbf{u}})| &= \\ &|({}^n\mathbf{g}({}^n\dot{\mathbf{u}}) \div {}^n\beta) - (({}^n\mathbf{g}({}^n\dot{\mathbf{u}}) + {}^n\mathbf{p}) \div {}^n\gamma)| \\ &\leq 1 + \left| \frac{{}^n\mathbf{g}({}^n\dot{\mathbf{u}})}{{}^n\beta} - \frac{{}^n\mathbf{g}({}^n\dot{\mathbf{u}}) + {}^n\mathbf{p}}{{}^n\gamma} \right| \\ &\leq 1 + \left| \frac{{}^n\mathbf{g}({}^n\dot{\mathbf{u}})}{{}^n\beta} \right| \left| 1 - \frac{{}^n\beta}{{}^n\gamma} \right| + \left| \frac{{}^n\mathbf{p}}{{}^n\gamma} \right|. \end{aligned}$$

Since, for any integer n , ng is a QLT, we derive that

$$\begin{aligned} |({}^n\mathbf{g}({}^n\dot{\mathbf{u}}) \div {}^n\beta) - {}^ng({}^n\dot{\mathbf{u}})| & \\ &\leq 2 + \left| \frac{{}^n\mathbf{g}({}^n\dot{\mathbf{u}})}{{}^n\beta} \right| \left| 1 - \frac{{}^n\beta}{{}^n\gamma} \right|. \quad (1) \end{aligned}$$

As $\mathbf{u} \in \mathcal{HR}_\alpha^d$, $\|{}^n\dot{\mathbf{u}}\|_\infty$ is in $\mathcal{O}(\alpha)$ and by hypothesis, $\langle \|{}^n\mathbf{g}\|_\infty \rangle \in \mathcal{O}(\beta^2 \div \alpha)$. Then, in Eq. (1), the ratio ${}^ng({}^n\dot{\mathbf{u}})/{}^n\beta$ is in $\mathcal{O}(\beta)$. Moreover, by hypothesis, the ratio ${}^n\beta/{}^n\gamma$ tends toward 1. Therefore, the difference between $({}^n\mathbf{g}^n\dot{\mathbf{u}}) \div {}^n\beta$ and ${}^ng({}^n\dot{\mathbf{u}})$ is in $\mathcal{o}(\beta)$, that is, $\langle ({}^n\mathbf{g}^n\dot{\mathbf{u}}) \div {}^n\beta \rangle =_\beta \langle {}^ng({}^n\dot{\mathbf{u}}) \rangle$.

Conversely, we assume a sequence of QLTs $\langle {}^ng \rangle$ with scales ${}^n\gamma$ such that $\langle {}^n\gamma \rangle =_\beta \beta$, $\langle \|{}^n\mathbf{g}\|_\infty \rangle \in \mathcal{O}(\beta^2 \div \alpha)$ and, for any $\mathbf{u} \in \mathcal{HR}_\alpha^d$, there exists $\dot{\mathbf{u}}$ such that $\ell(\mathbf{u}) = \langle [{}^ng({}^n\dot{\mathbf{u}})] \rangle$. Let ng be the matrix of the canonical representative of ng . Since, as shown in the first part of the proof, the sequences $\langle ({}^n\mathbf{g}^n\dot{\mathbf{u}}) \div {}^n\beta \rangle$ and $\langle {}^ng({}^n\dot{\mathbf{u}}) \rangle$ are equivalent for the relation $=_\beta$, we derive that, for any \mathbf{u} there exists $\dot{\mathbf{u}}$ such that $\ell(\mathbf{u}) = \langle [({}^n\mathbf{g}^n\dot{\mathbf{u}}) \div {}^n\beta] \rangle$. Actually, since $\langle \|{}^n\mathbf{g}\|_\infty \rangle \in \mathcal{O}(\beta^2 \div \alpha)$, it is plain that the latter equality is true for any representative of \mathbf{u} . Then,

$$\begin{aligned} \ell(\mathbf{u} + \mathbf{v}) &= \langle [({}^n\mathbf{g}({}^n\dot{\mathbf{u}} + {}^n\dot{\mathbf{v}}) \div {}^n\beta) \rangle \\ &= \langle [({}^n\mathbf{g}({}^n\dot{\mathbf{u}}) + {}^n\mathbf{g}({}^n\dot{\mathbf{v}})) \div {}^n\beta] \rangle \\ &= \langle [({}^n\mathbf{g}({}^n\dot{\mathbf{u}}) \div {}^n\beta + {}^n\mathbf{g}({}^n\dot{\mathbf{v}}) \div {}^n\beta + {}^nR) \rangle, \end{aligned}$$

where ${}^nR \in \{0, 1\}^d$. Thus,

$$\begin{aligned} \ell(\mathbf{u} + \mathbf{v}) &= \langle [({}^n\mathbf{g}({}^n\dot{\mathbf{u}}) \div {}^n\beta) \rangle + \langle [({}^n\mathbf{g}({}^n\dot{\mathbf{v}}) \div {}^n\beta) \rangle \\ &= \ell(\mathbf{u}) + \ell(\mathbf{v}). \end{aligned}$$

Alike, we prove that $\ell(\lambda\mathbf{u}) = \psi_\beta \circ \varphi_\alpha(\lambda) \ell(\mathbf{u})$ for any $\lambda \in \mathcal{HR}_\alpha$.

On the one hand,

$$\begin{aligned} \ell(\lambda\mathbf{u}) &= \ell(\langle [({}^n\lambda {}^n\dot{\mathbf{u}} \div {}^n\alpha) \rangle]) \\ &= \langle [({}^n\mathbf{g}({}^n\lambda {}^n\dot{\mathbf{u}} \div {}^n\alpha) \div {}^n\beta) \rangle] \\ &= \left\langle \left\langle \frac{{}^n\lambda}{{}^n\alpha {}^n\beta} {}^n\mathbf{g}({}^n\dot{\mathbf{u}}) + \mathcal{O}\left(\frac{\beta}{\alpha}\right) \right\rangle \right\rangle \\ &= \left\langle \left\langle \frac{{}^n\lambda}{{}^n\alpha {}^n\beta} {}^n\mathbf{g}({}^n\dot{\mathbf{u}}) \right\rangle \right\rangle. \end{aligned}$$

On the other hand,

$$\begin{aligned} \psi_\beta \circ \varphi_\alpha(\lambda) \ell(\mathbf{u}) &= \left[\left[\beta \frac{\lambda}{\alpha} \right] \right] \times_\beta \langle [({}^n\mathbf{g}({}^n\dot{\mathbf{u}}) \div {}^n\beta) \rangle] \\ &= \left\langle \left\langle \frac{{}^n\lambda}{{}^n\alpha {}^n\beta} {}^n\mathbf{g}({}^n\dot{\mathbf{u}}) + \mathcal{O}(1) \right\rangle \right\rangle \\ &= \left\langle \left\langle \frac{{}^n\lambda}{{}^n\alpha {}^n\beta} {}^n\mathbf{g}({}^n\dot{\mathbf{u}}) \right\rangle \right\rangle. \end{aligned}$$

Eventually, we conclude that ℓ is an Ω -LT. \square

In Sec. 3.2, we have shown that sequences of QLTs arise from the Ω -LTs between \mathcal{HR}_α^d and \mathcal{HR}_β^d and in Sec. 3.1, it was established that those Ω -LTs are in correspondance with the Euclidean linear maps. Thereby, we can achieve the Ω -arithmetization scheme for the Euclidean linear maps. This the purpose of the next section.

3.3 Convergence

Corollary 1 establishes a point wise convergence property of some sequences of maps — involving Ω -LTs — towards the Euclidean linear transformations. The main result of this section, Theorem 2 enforces the latter result by showing that the convergence is uniform on any compact subset of \mathbb{R}^n and by describing the convergence speed. Furthermore, rather than appealing to Ω -LT, the result is now expressed in terms of sequences of QLTs.

Before stating and proving our theorem, let us assess the relations between Euclidean linear maps, QLTs and Ω -LT through a new commutative diagram (Fig 3). Besides, in the same way we defined in Sec. 3.1 the maps

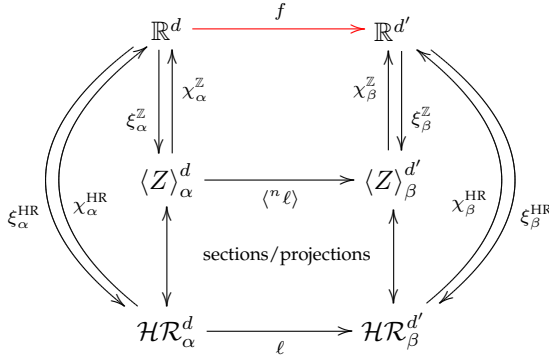


Fig. 3. The skeleton of the Ω -arithmetization scheme of an Euclidean linear map f . The Ω -LT ℓ has for matrix the non-standard version (over the ring \mathcal{HR}_ω where $\omega = \beta^2 \div \alpha$) of the matrix of f . The sections of the Ω -LT for the involved equivalence relations give raise to sequences of QLTs $\langle^n \ell \rangle$ which uniformly converge, up to a rescaling, toward the linear map f .

$\xi_\gamma^{HR} : \mathbb{R}^e \rightarrow \mathcal{HR}_\gamma^e$ and $\chi_\gamma^{HR} : \mathcal{HR}_\gamma^e \rightarrow \mathbb{R}^e$ ($e \geq 1$), we now define, for any infinitely large number ω , the maps $\xi_\omega^Z : \mathbb{R}^e \rightarrow \langle Z \rangle_\omega^e$ and $\chi_\omega^Z : \langle Z \rangle_\omega^e \rightarrow \mathbb{R}^e$ by $\xi_\omega^Z(\mathbf{x}) = \lfloor \omega \mathbf{x} \rfloor$ and $\chi_\omega^Z(\langle^n p \rangle) = \limsup \langle^n p / \omega \rangle$. These maps have the same properties as ξ_γ^{HR} and χ_γ^{HR} : ξ_γ^Z is an injective linear map and $\chi_\gamma^Z \circ \xi_\gamma^Z$ is the identity map of \mathbb{R}^e . Furthermore, the restriction of $\xi_\gamma^Z \circ \chi_\gamma^Z$ to $\xi_\gamma^Z(\mathbb{R}^e)$ is the identity map of $\xi_\gamma^Z(\mathbb{R}^e)$.

Theorem 2. Let $f: \mathbb{R}^d \rightarrow \mathbb{R}^{d'}$ be a linear transformation, f its matrix and $K \subset \mathbb{R}^d$ a compact set. Given α, β, ω three infinitely large numbers such that $\alpha \in o(\beta^2)$ and $\omega = \beta^2 \div \alpha$, we consider the sequence $\langle^n g \rangle$ where ${}^n g$ is the QLT whose canonical representation is $({}^n \beta, \lfloor {}^n \omega f \rfloor, \mathbf{0})$. Then, for any $\mathbf{x} \in K$,

$$\left\| f(\mathbf{x}) - \frac{{}^n g(\lfloor {}^n \alpha \mathbf{x} \rfloor)}{{}^n \beta} \right\|_\infty \leq \frac{\|f\|_\infty}{{}^n \alpha} + \frac{(\|f\|_\infty + 1 + d)(\|\mathbf{x}\|_\infty + 1)}{{}^n \omega} + \frac{1}{{}^n \beta}.$$

Proof: We set $\mathbf{p} = {}^n(\xi_\alpha^Z(\mathbf{x})) = \lfloor {}^n \alpha \mathbf{x} \rfloor$ and $\mathbf{g} = {}^n(\xi_\omega^Z(f)) = \lfloor {}^n \omega f \rfloor$. Moreover, we write $\alpha, \beta,$

ω instead of ${}^n \alpha, {}^n \beta$ and ${}^n \omega$. Then,

$$\begin{aligned} f(\mathbf{x}) - \frac{{}^n g(\mathbf{p})}{\beta} &= f \mathbf{x} - \frac{\lfloor \frac{1}{\beta} \mathbf{g} \mathbf{p} \rfloor}{\beta} \\ &= \left(f \mathbf{x} - \frac{f \mathbf{p}}{\alpha} \right) + \left(\frac{f \mathbf{p}}{\alpha} - \frac{\lfloor \frac{1}{\beta} \mathbf{g} \mathbf{p} \rfloor}{\beta} \right) \\ &= \frac{1}{\alpha} f (\alpha \mathbf{x} - \mathbf{p}) + \frac{1}{\alpha \omega} (\omega f - \mathbf{g}) \mathbf{p} + \\ &\quad \left(\frac{1}{\alpha \omega} \mathbf{g} \mathbf{p} - \frac{1}{\beta} \lfloor \frac{1}{\beta} \mathbf{g} \mathbf{p} \rfloor \right) \\ &= \frac{1}{\alpha} f (\alpha \mathbf{x} - \mathbf{p}) + \frac{1}{\alpha \omega} (\omega f - \mathbf{g}) \mathbf{p} + \\ &\quad \left(\frac{\beta^2}{\alpha \omega} - 1 \right) \left(\frac{1}{\beta^2} \mathbf{g} \mathbf{p} \right) + \frac{1}{\beta} \left(\frac{1}{\beta} \mathbf{g} \mathbf{p} - \lfloor \frac{1}{\beta} \mathbf{g} \mathbf{p} \rfloor \right). \end{aligned}$$

The four terms of the latter sum in the right hand side of the above equation can easily be upper bounded in infinite norm.

- $\|(1/\alpha)f(\alpha \mathbf{x} - \mathbf{p})\|_\infty \leq (1/\alpha) \|f\|_\infty \|\alpha \mathbf{x} - \mathbf{p}\|_\infty \leq (1/\alpha) \|f\|_\infty$,
for $\mathbf{p} = \lfloor \alpha \mathbf{x} \rfloor$;
- $\|(1/\alpha \omega)(\omega f - \mathbf{g}) \mathbf{p}\|_\infty \leq (1/\alpha \omega) \|\omega f - \mathbf{g}\|_\infty \|\mathbf{p}\|_\infty \leq (d/\alpha \omega) \|\mathbf{p}\|_\infty$,

for the coefficients of the matrix $\omega f - \mathbf{g}$ are fractional parts: $(\omega f - \mathbf{g})_{i,j} = \omega f_{i,j} - \lfloor \omega f_{i,j} \rfloor$. Since $\|\mathbf{p}\|_\infty < \alpha \|\mathbf{x}\|_\infty + 1 \leq \alpha(\|\mathbf{x}\|_\infty + 1)$, it follows that:

- $\|(1/\alpha \omega)(\omega f - \mathbf{g}) \mathbf{p}\|_\infty \leq (d/\omega)(\|\mathbf{x}\|_\infty + 1)$;
- By definition of ω , there exists r , $0 < r < \alpha$, such that $\beta^2 = \alpha \omega + r$. Furthermore, $\|\mathbf{g}\|_\infty < \omega(\|f\|_\infty + 1)$ and $\|\mathbf{p}\|_\infty < \alpha(\|\mathbf{x}\|_\infty + 1)$. Then, $\|((\beta^2/\alpha \omega) - 1)((1/\beta^2) \mathbf{g} \mathbf{p})\|_\infty$ is bounded from above by $(1/\omega)(\|f\|_\infty + 1)(\|\mathbf{x}\|_\infty + 1)$.
- $(1/\beta) \|(1/\beta) \mathbf{g} \mathbf{p} - \lfloor (1/\beta) \mathbf{g} \mathbf{p} \rfloor\|_\infty$ is clearly bounded from above by $1/\beta$.

Putting together all these upper bounds, we obtain the desired inequality. \square

4 CONCLUSION

Via a constructive non-standard version of the real numbers, we have obtained a multi-scale

arithmetization scheme of the Euclidean affine transformations. In this framework, an infinite sequence of quasi-linear transformations is the translation of a single Euclidean affine map. Depending, on the choices of the infinitely large numbers α , β , ω , the obtained sequence can be more or less obvious. Nevertheless, the merit of this approach is to bring an insight on the properties of discrete computations of affine maps and to open a new field of study. Indeed, it is well known that computer affine transformation implementations are not one-to-one, do not preserve fixed points, topology and so on. But the no-standard model show that such implementations operate on sections of equivalence classes instead of equivalence classes. Then, for instance, an Euclidean vector may have a singular preimage while its section on a computer have a non-singular preimage which contains the points of the equivalence class that are exposed by the section (preimage structures are studied in the papers [17], [20] among others). Then, the next task in the proposed framework is to revisit the properties of the quasi-linear transformations under this multi-scale and non-standard interpretation.

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