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An iterative inversion of weighted Radon transforms along hyperplanes

F.O. Goncharov*

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Abstract

We propose iterative inversion algorithms for weighted Radon transforms R_W along hyperplanes in $\mathbb{R}^3$. More precisely, expanding the weight $W = W(x,\theta), x \in \mathbb{R}^3, \theta \in \mathbb{S}^2$, into the series of spherical harmonics in θ and assuming that the zero order term $w_{0,0}(x) \neq 0, x \in \mathbb{R}^3$, we reduce the inversion of R_W to solving a linear integral equation. In addition, under the assumption that the even part of W in θ (i.e., $\frac{1}{2}(W(x,\theta) + W(x,-\theta))$) is close to $w_{0,0}$, the aforementioned linear integral equation can be solved by the method of successive approximations. Approximate inversions of R_W are also given. Our results can be considered as an extension to 3D of two-dimensional results of Kunyansky (1992), Novikov (2014), Guillement, Novikov (2014). In our studies we are motivated, in particular, by problems of emission tomographies in 3D. In addition, we generalize our results to the case of dimension $n > 3$.

1 Introduction

We consider the weighted Radon transforms R_W defined by the formula

$$R_W f(s,\theta) = \int\limits_{x\theta=s} W(x,\theta) f(x)\, dx,\ (s,\theta) \in \mathbb{R} \times \mathbb{S}^2,\ x \in \mathbb{R}^3, \tag{1.1}$$

where $W = W(x,\theta)$ is the weight, $f = f(x)$ is a test function.

In this work we assume that

$$W \in C(\mathbb{R}^3 \times \mathbb{S}^2) \cap L^\infty(\mathbb{R}^3 \times \mathbb{S}^2), \tag{1.2}$$

$$w_{0,0}(x) \stackrel{def}{=} \frac{1}{4\pi} \int\limits_{\mathbb{S}^2} W(x,\theta)\, d\theta,\ w_{0,0}(x) \neq 0,\ x \in \mathbb{R}^3, \tag{1.3}$$

$$f \in L^\infty(\mathbb{R}^3),\ \operatorname{supp} f \subset D, \tag{1.4}$$

where W and f are complex-valued, $d\theta$ is an element of standard measure on $\mathbb{S}^2$, D is an open bounded domain (which is fixed apriori).

If $W \equiv 1$, then R_W is reduced to the classical Radon transform along hyperplanes in $\mathbb{R}^3$ introduced in [Rad17]; see also, e.g., [Nat86], [Den2016].

For known results on the aforementioned transforms R_W with non-constant W we refer to [Qui83], [Bey84], [BQ87], [GN16]. In particular, in [Qui83] it was shown that R_W is injective on $L^p_0(\mathbb{R}^3)$, $p \geq 2$ (L^p functions on $\mathbb{R}^3$ with compact support) if $W \in C^2$ and is real-valued, strictly positive and satisfies the strong symmetry assumption of rotation invariancy (see [Qui83] for details). On the other hand, in [BQ87] it was also proved that R_W is injective if W is real-analytic and strictly positive.

Besides, in [Bey84] the inversion of R_W is reduced to solving a Fredholm type linear integral equation in the case of infinitely smooth strictly positive W with the symmetry $W(x,\theta) = W(x,-\theta)$.

In turn, the work [GN16] extends to the case of weighted Radon transforms along hyperplanes in multidimensions the two-dimensional Chang approximate inversion formula (see [Cha78]) and the related two-dimensional result of [Nov11]. In particular, [GN16] describes all W for which such Chang-type formulas are simultaneously explicit and exact in multidimensions.

We recall that inversion methods for R_W admit tomographical applications in the framework of the scheme described as follows (see [GN16]).

*CMAP, Ecole Polytechnique, CNRS, Université Paris-Saclay, 91128, Palaiseau, France; email: fedor.goncharov.ol@gmail.com

It is well-known that in many tomographies measured data are modeled by weighted ray transforms $P_w f$ defined by the formula

$$P_w f(x,\alpha) = \int\limits_{\mathbb{R}} w(x+\alpha t,\alpha) f(x+\alpha t)\, dt,\ (x,\alpha)\in T\mathbb{S}^2, \tag{1.5}$$

$$T\mathbb{S}^2 = \{(x,\alpha)\in\mathbb{R}^3\times\mathbb{S}^2 : x\alpha = 0\},$$

where f is an object function defined on $\mathbb{R}^3$, w is the weight function defined on $\mathbb{R}^3\times\mathbb{S}^2$, and $T\mathbb{S}^2$ can be considered as the set of all rays (oriented straight lines) in $\mathbb{R}^3$, see, e.g., [Cha78], [Nat86], [Kun92], [GuNo14].

In particular, in the single photon emission computed tomography (SPECT) the weight w is given by the formulas (see, e.g., [Nat86]):

$$w(x,\theta) = \exp\big(-\int\limits_0^\infty a(x+t\theta)\, dt\big),\ (x,\theta)\in\mathbb{R}^3\times\mathbb{S}^2, \tag{1.6}$$

where $a = a(x)$ is the attenuation coefficient.

Moreover, in [GN16] (Section 3) it was shown that if $P_w f$ are given for all rays parallel to some fixed plane Σ in $\mathbb{R}^3$ then $R_W f$ with appropriate W can be obtained by the explcit formulas from $P_w f$ and w (in a similair way with the case $w\equiv 1$, $W\equiv 1$, see Chapter 2, formula (1.1) of [Nat86] and also [Gra91], [Den2016]). Therefore, reconstruction of f from data modeled by $P_w f$, defined by (1.5) and restricted to all rays parallel to Σ, can be reduced to reconstruction of f from $R_W f$, defined by (1.1). In [GN16] it was aslo indicated that the reduction from $P_w f$ to $R_W f$ with subsequent reconstruction of f from $R_W f$ and W can drastically reduce the impact of the random noise in the initial data modeled as $P_w f$.

In the present work we continue studies of [GN16], on one hand, and of [Kun92], [Nov14], [GuNo14], on the other hand. In particular, we extend the two-dimensional results of [Kun92], [Nov14], [GuNo14] to the case of weighted Radon transforms along hyperplanes in multidimensions. In particular, under assumptions (1.2), (1.3), expanding $W = W(x,\theta)$ into the series of spherical harmonics in θ we reduce the reconstruction of f to solving a linear integral equation (see Section 3). In particular, if the even part of W in θ (i.e., $\tilde{W}(x,\theta) = \frac{1}{2}(W(x,\theta)+W(x,-\theta))$) is close to $w_{0,0}$, then such linear integral equation can be solved by the method of successive approximations (see Subsections 3.1, 3.3 for details).

Note that our linear integral equation is very different from the aforementioned linear integral equation of [Bey84] (in particular, in our conditions on $\tilde{W}$, ensuring the applicability of the method of successive approximations).

Note also that in [Cha78], [Kun92], [Nov14], [GuNo14] the two-dimensional prototype of our inversion approach was developed in view of its numerical efficiency in problems of emission tomographies, including good stability to strong random noise in the emission data.

In more details our results can be sketched as follows.

We use the following expansion for W:

$$W(x,\theta(\gamma,\phi)) = \sum_{k=0}^{\infty}\sum_{n=-k}^{k} w_{k,n}(x) Y_k^n(\gamma,\phi),\ x\in\mathbb{R}^3, \tag{1.7}$$

$$Y_k^n(\gamma,\phi) \stackrel{def}{=} p_k^{|n|}(\cos\gamma)e^{in\phi},\ k\in\mathbb{N}\cup\{0\},\ n=\overline{-k,k}, \tag{1.8}$$

$$\theta(\gamma,\phi) = (\sin\gamma\cos\phi,\ \sin\gamma\sin\phi,\ \cos\gamma)\in\mathbb{S}^2,\ \gamma\in[0,\pi],\ \phi\in[0,2\pi], \tag{1.9}$$

where $p_k^n(x)$, $x\in[-1,1]$, are the associated semi-normalized Legendre polynomials. Polynomials p_k^n are well-known in literature (see e.g. [SW16]) and are defined using the ordinary Legendre polynomials p_k by the formulas:

$$p_k^n(x) = (-1)^n\sqrt{\frac{(k-n)!}{(k+n)!}}(1-x^2)^{n/2}\frac{d^n}{dx^n}(p_k(x)),\ n,k\in\mathbb{N}\cup\{0\}, \tag{1.10}$$

$$p_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}[(x^2-1)^k],\ x\in[-1,1], \tag{1.11}$$

see also [SW16], [ZT79] for other properties of the associated Legendre polynomials. In addition, coefficients $w_{k,n}$ in (1.7) are defined by the formulas:

$$w_{k,n}(x) = c(k,n)\int\limits_0^{2\pi} d\phi\, e^{-in\phi}\int\limits_0^{\pi} W(x,\theta(\gamma,\phi))p_k^{|n|}(\cos\gamma)\sin\gamma\, d\gamma, \tag{1.12}$$

$$c(k,n) = \frac{(2k+1)}{4\pi},\ k\in\mathbb{N}\cup\{0\},\ n=0,\pm1,\cdots,\pm k.$$

Under assumption (1.2), for each fixed x, series (1.7) converges in $L^2(\mathbb{S}^2)$; see, e.g., [SW16] (Chapter 4), [Mor98] (Chapter 2), [ZT79].

We consider also

$$\sigma_{\widetilde{W},D,m} = \frac{1}{2\pi\sqrt{2}} \sum_{k=1}^{m} \sum_{n=-2k}^{2k} \sup_{x\in D} \left| \frac{w_{2k,n}(x)}{w_{0,0}(x)} \right| \text{ for } m \in \mathbb{N}, \tag{1.13}$$

$$\sigma_{\widetilde{W},D,m} = 0 \text{ for } m = 0,$$

$$\sigma_{\widetilde{W},D,\infty} = \lim_{m\to\infty} \sigma_{\widetilde{W},D,m}, \tag{1.14}$$

$$W_N(x,\theta(\gamma,\phi)) = \sum_{k=0}^{N} \sum_{n=-k}^{k} w_{k,n}(x) Y_k^n(\gamma,\phi), \tag{1.15}$$

$$\widetilde{W}_N(x,\theta(\gamma,\phi)) = \sum_{k=0}^{[N/2]} \sum_{n=-2k}^{2k} w_{2k,n}(x) Y_{2k}^n(\gamma,\phi), \tag{1.16}$$

$$x \in \mathbb{R}^3,\ \gamma \in [0,\pi],\ \phi \in [0,2\pi],$$

where coefficients $w_{k,n}$ are defined in (1.12), $[N/2]$ denotes the integer part of $N/2$.

Our expansion (1.7) and the related formulas are motivated by their two-dimensional prototypes of [Kun92], [Nov14], [GuNo14].

In the present article we obtained, in particular, the following results under assumptions (1.2), (1.3), (1.4):

1. If $\sigma_{\widetilde{W},D,\infty} < 1$, then R_W is injective and, in addition, the inversion of R_W is given via formulas (3.2), (3.3); see Subsection 3.1 for details.

2. If $\sigma_{\widetilde{W},D,\infty} \geq 1$, then f can be approximately reconstructed from $R_W f$ as $f \approx (R_{\widetilde{W}_N})^{-1} R_W f$, where $R_{\widetilde{W}_N}$ is defined according to (1.1) for $\widetilde{W}_N$ defined by (1.16) for $N = 2m$, where m is chosen as the largest while condition $\sigma_{\widetilde{W},D,m} < 1$ holds. More precisely, approximate inversion of $R_W f$ is given via the formulas (3.7), (3.8); see Subsection 3.2 for details.

   In addition, if $W = W_N$ defined by (1.15) and $\sigma_{\widetilde{W},D,m} < 1$, $m = [N/2]$, then R_{W_N} is injective and invertible by formula (3.17); see Subsection 3.3 for details.

In addition, in these results assumptions (1.2), (1.3) can be relaxed as follows:

$$W \in L^\infty(\mathbb{R}^3 \times \mathbb{S}^2), \tag{1.17}$$

$$w_{0,0} \geq c > 0 \text{ on } \mathbb{R}^3, \tag{1.18}$$

where $w_{0,0}$ is defined (1.3), c is some positive constant.

Prototypes of these results for the weighted Radon transforms in 2D were obtained in [Kun92], [Nov14], [GuNo14].

The present work also continues studies of [GN16], where approximate inversion of R_W was realized as $(R_{W_N})^{-1}$ for $N = 0$ or by other words as an approximate Chang-type inversion formula. We recall that the original two-dimensional Chang formula ([Cha78]) is often used as an efficient first approximation in the framework of slice-by-slice reconstructions in the single photon emission computed tomography.

In Section 2 we give some notations and preliminary results.

The main results of the present work are presented in detail in Section 3.

In Section 4 we generalize results of Sections 2, 3 for the case of dimension $n > 3$.

Proofs of results of Sections 2, 3, 4 are presented in Sections 5, 6.

2 Some preliminary results

2.1 Some formulas for R and R^{-1}

We recall that for the classical Radon transform R (formula (1.1) for $W \equiv 1$) the following identity holds (see [Nat86], Theorem 1.2, p.13):

$$R(f *_{\mathbb{R}^3} g) = Rf *_{\mathbb{R}} Rg, \tag{2.1}$$

where $*_{\mathbb{R}^3}, *_{\mathbb{R}}$ denote the 3D and 1D convolutions (respectively), f, g are test functions.

The classical Radon inversion formula is defined as follows (see, e.g., [Nat86]):

$$R^{-1}q(x) = -\frac{1}{8\pi^2}\int\limits_{\mathbb{S}^2} q^{(2)}(x\theta,\theta)d\theta,\ x\in\mathbb{R}^3, \tag{2.2}$$

$$q^{(2)}(s,\theta) = \frac{d^2}{ds^2}q(s,\theta),\ (s,\theta)\in\mathbb{R}\times\mathbb{S}^2,$$

where q is a test function on $\mathbb{R}\times\mathbb{S}^2$.

In addition, from the Projection theorem (see [Nat86], Theorem 1.1, p.11) it follows that:

$$R^{-1}q(x) \stackrel{def}{=} \frac{1}{(2\pi)^{5/2}}\int\limits_{\mathbb{R}}\frac{\rho^2}{2}d\rho\int\limits_{\mathbb{S}^2}\hat{q}(\rho,\omega)e^{i\rho(x\omega)}d\omega,\ x\in\mathbb{R}^3, \tag{2.3}$$

$$\hat{q}(s,\theta) \stackrel{def}{=} \frac{1}{\sqrt{2\pi}}\int\limits_{\mathbb{R}} q(t,\theta)e^{-its}\,dt,\ (s,\theta)\in\mathbb{R}\times\mathbb{S}^2, \tag{2.4}$$

where $q(t,\theta)$ is a test function on $\mathbb{R}\times\mathbb{S}^2$.

For the case of $\hat{q}$ even (i.e., $\hat{q}(s,\theta)=\hat{q}(-s,-\theta)$, $(s,\theta)\in\mathbb{R}\times\mathbb{S}^2$, where $\hat{q}$ is defined in (2.4)), formulas (2.3), (2.4) can be rewritten as follows:

$$R^{-1}q = \frac{1}{2\pi}\mathcal{F}[\hat{q}] = \frac{1}{2\pi}\mathcal{F}^{-1}[\hat{q}], \tag{2.5}$$

where $\mathcal{F}[\cdot],\mathcal{F}^{-1}[\cdot]$ denote the Fourier transform and its inverse in 3D, respectively, and they are defined by the following formulas (in spherical coordinates):

$$\mathcal{F}[q](\xi) \stackrel{def}{=} \frac{1}{(2\pi)^{3/2}}\int\limits_0^{+\infty}\rho^2 d\rho\int\limits_{\mathbb{S}^2} q(\rho,\omega)e^{-i\rho(\xi\omega)}d\omega, \tag{2.6}$$

$$\mathcal{F}^{-1}[q](\xi) \stackrel{def}{=} \frac{1}{(2\pi)^{3/2}}\int\limits_0^{+\infty}\rho^2 d\rho\int\limits_{\mathbb{S}^2} q(\rho,\omega)e^{i\rho(\xi\omega)}d\omega,\ \xi\in\mathbb{R}^3, \tag{2.7}$$

where $q(\rho,\omega)$ is a test-function on $[0,+\infty)\times\mathbb{S}^2$ (identified with $\mathbb{R}^3$).

2.2 Symmetrization of W

Let

$$A_W f = R^{-1}R_W f, \tag{2.8}$$

where R_W is defined in (1.1), f is a test function, satisfying assumptions of (1.4).

Let

$$\widetilde{W}(x,\theta) \stackrel{def}{=} \frac{1}{2}(W(x,\theta)+W(x,-\theta)),\ x\in\mathbb{R}^3,\theta\in\mathbb{S}^2. \tag{2.9}$$

The following formulas hold:

$$A_W f = R^{-1}R_{\widetilde{W}}f, \tag{2.10}$$

$$R_{\widetilde{W}}f(s,\theta) = \frac{1}{2}(R_W f(s,\theta)+R_W f(-s,-\theta)),\ (s,\theta)\in\mathbb{R}\times\mathbb{S}^2, \tag{2.11}$$

$$\widetilde{W}(x,\theta(\gamma,\phi)) = \sum_{k=0}^{\infty}\sum_{n=-2k}^{2k} w_{2k,n}(x)Y_{2k}^n(\gamma,\phi), \tag{2.12}$$

$$x\in\mathbb{R}^3,\ \gamma\in[0,\pi],\ \phi\in[0,2\pi].$$

Identity (2.10) is proved in [GN16] in 3D, where $\widetilde{W}$ is denoted as W_{sym}.

Identity (2.12) follows from (1.7), (1.8), (2.9) and the following identities:

$$p_k^n(-x) = (-1)^{n+k}p_k^n(x),\ x\in[-1,1], \tag{2.13}$$

$$e^{in(\phi+\pi)} = (-1)^n e^{in\phi},\ k\in\mathbb{N}\cup\{0\},\ n=\overline{-k,k}. \tag{2.14}$$

Note also that $\widetilde{W}_N$ defined by (1.16) is the approximation of $\widetilde{W}$ defined by (2.9) and

$$\widetilde{W}_N(x,\cdot)\xrightarrow[N\to\infty]{L^2(\mathbb{S}^2)}\widetilde{W}(x,\cdot)\text{ for each fixed }x\in\mathbb{R}^3. \tag{2.15}$$

Using formulas (2.11)-(2.12) we reduce the inversion of R_W to the inversion of $R_{\widetilde{W}}$ defined by (1.1) for $W = \widetilde{W}$.

In our work $A_W f$ (or, more precisely, $(w_{0,0})^{-1} A_W f$) is used as the *initial point* for our iterative inversion algorithms (see Section 3).

Note that the simmetrization $\widetilde{W}$ of W arises in (2.10).

In addition, prototypes of (2.8), (2.10), (2.12) for the two-dimensional case can be found in [Kun92], [Nov11].

2.3 Operators $Q_{\widetilde{W},D,m}$ and numbers $\sigma_{\widetilde{W},D,m}$

Let

$$c \stackrel{def}{=} \inf_{x \in D} |w_{0,0}(x)| > 0, \tag{2.16}$$

where the inequality follows from the continuity of W on $\overline{D}$ (closure of D) and assumption (1.3).

Let D be the domain of (1.4), and χ_D denote the characteristic function of D, i.e.

$$\chi_D \equiv 1 \text{ on } D,\ \chi_D \equiv 0 \text{ on } \mathbb{R}^3 \backslash D. \tag{2.17}$$

Let

$$Q_{\widetilde{W},D,m} u(x) \stackrel{def}{=} R^{-1}(R_{\widetilde{W},D,m} u)(x),\ m \in \mathbb{N} \tag{2.18}$$

$$Q_{\widetilde{W},D,m} u(x) = 0 \text{ for } m = 0,$$

$$Q_{\widetilde{W},D,\infty} u(x) \stackrel{def}{=} R^{-1}(R_{\widetilde{W},D,\infty} u)(x), \tag{2.19}$$

where

$$R_{\widetilde{W},D,m} u(s, \theta(\gamma,\phi)) \stackrel{def}{=} \int\limits_{x\theta = s} \left(\sum_{k=1}^{m} \sum_{n=-2k}^{2k} \frac{w_{2k,n}(x)}{w_{0,0}(x)} Y_{2k}^n(\gamma,\phi) \right) \chi_D(x) u(x)\, dx, \tag{2.20}$$

$$R_{\widetilde{W},D,\infty} u(s, \theta(\gamma,\phi)) \stackrel{def}{=} \lim_{m\to\infty} R_{\widetilde{W},D,m} u(s, \theta(\gamma,\phi))$$

$$= \int\limits_{x\theta = s} \left(\sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} \frac{w_{2k,n}(x)}{w_{0,0}(x)} Y_{2k}^n(\gamma,\phi) \right) \chi_D(x) u(x)\, dx, \tag{2.21}$$

$$x \in \mathbb{R}^3,\ s \in \mathbb{R},\ \theta(\gamma,\phi) \in \mathbb{S}^2,$$

where Y_k^n are defined in (1.8), R^{-1} is defined by (2.3) (or (2.2)), u is a test function, $w_{0,0}, w_{2k,n}$ are the Fourier-Laplace coefficients defined by (1.12) and $w_{2k,n}/w_{0,0}, \chi_D$ are considered as multiplication operators on $\mathbb{R}^3$. Note also that $R_{\widetilde{W},D,\infty}(w_{0,0} f) + R(w_{0,0} f) = R_{\widetilde{W}} f$ under assumptions (1.2)-(1.4).

Let

$$d_{2k,n}(x) \stackrel{def}{=} R^{-1}(\delta(\cdot) Y_{2k}^n)(x),\ x \in \mathbb{R}^3,\ k \in \mathbb{N},\ n = \overline{-2k, 2k}, \tag{2.22}$$

where $\delta = \delta(s)$ denotes the 1D Dirac delta function. In (2.22) the action of R^{-1} on the generalized functions is defined by formula (2.3).

Lemma 2.1. *Let $d_{2k,n}$ be defined by* (2.22). *Then*

$$d_{2k,n}(x(r,\gamma,\phi)) = \frac{(-1)^k \sqrt{2}\Gamma(\frac{3}{2}+k)}{\pi\Gamma(k)} \frac{Y_{2k}^n(\gamma,\phi)}{r^3}, r > 0, \tag{2.23}$$

where $\Gamma(\cdot)$ is the Gamma-function, $x(r,\gamma,\phi)$ is defined by the identity:

$$x(r,\gamma,\phi) = (r\sin\gamma\cos\phi, r\sin\gamma\sin\phi,\ r\cos\gamma) \in \mathbb{R}^3,\ \gamma \in [0,\pi],\ \phi \in [0,2\pi],\ r \geq 0. \tag{2.24}$$

In addition, the following inequality holds:

$$|\mathcal{F}[d_{2k,n}](\xi)| \leq \frac{1}{2\pi\sqrt{2}},\ \xi \in \mathbb{R}^3, \tag{2.25}$$

where $\mathcal{F}[\cdot]$ is the Fourier transform, defined in (2.6).

The following lemma gives some useful expressions for operators $Q_{\widetilde{W},D,m}$, $Q_{\widetilde{W},D,\infty}$ defined in (2.18), (2.19).

Lemma 2.2. *Let operators $Q_{\widetilde{W},D,m}, Q_{\widetilde{W},D,\infty}$ be defined by* (2.18), (2.19)*, respectively, and u be a test function satisfying* (1.4)*. Then*

$$Q_{\widetilde{W},D,m}u = \sum_{k=1}^{m}\sum_{n=-2k}^{2k} d_{2k,n} *_{\mathbb{R}^3} \frac{w_{2k,n}}{w_{0,0}}u, \tag{2.26}$$

$$Q_{\widetilde{W},D,\infty}u = \sum_{k=1}^{\infty}\sum_{n=-2k}^{2k} d_{2k,n} *_{\mathbb{R}^3} \frac{w_{2k,n}}{w_{0,0}}u, \tag{2.27}$$

where coefficients $w_{k,n}$ are defined in (1.12)*, $d_{2k,n}$ is defined by* (2.23) *(or equivalently by* (2.22)*).*

Remark 2.1. Convolution terms in the right-hand side of (2.26), (2.27) are well defined functions in $L^2(\mathbb{R}^3)$. This follows from identity (2.23) and the Calderón-Zygmund theorem for convolution-type operators with singular kernels (see [Kna05], p.83, Theorem 3.26).

The following lemma shows that $Q_{\widetilde{W},D,m}$, $Q_{\widetilde{W},D,\infty}$ are well-defined operators in $L^2(\mathbb{R}^3)$.

Lemma 2.3. *Operator $Q_{\widetilde{W},D,m}$ defined by* (2.26) *(or equivalently by* (2.18)*) is a linear bounded operator in $L^2(\mathbb{R}^3)$ and the following estimate holds:*

$$\|Q_{\widetilde{W},D,m}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \le \sigma_{\widetilde{W},D,m}, \tag{2.28}$$

where $\sigma_{\widetilde{W},D,m}$ is defined by (1.13)*.*

Operator $Q_{\widetilde{W},D,\infty}$ defined by (2.27) *(or equivalently by* (2.19)*), is a linear bounded operator in $L^2(\mathbb{R}^3)$ and the following estimate holds:*

$$\|Q_{\widetilde{W},D,\infty}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \le \sigma_{\widetilde{W},D,\infty}, \tag{2.29}$$

where $\sigma_{\widetilde{W},D,\infty}$ is defined by (1.14)

Lemma 2.4. *Let*

$$\sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\left\|\frac{w_{2k,n}}{w_{0,0}}\right\|_{L^2(D)} < +\infty, \tag{2.30}$$

where $w_{k,n}$ are defined in (1.12)*. Then*

$$R^{-1}R_W f \in L^2(\mathbb{R}^3). \tag{2.31}$$

In addition, the following formula holds:

$$R^{-1}R_W f = w_{0,0}f + \sum_{k=1}^{\infty}\sum_{n=-2k}^{2k} d_{2k,n} *_{\mathbb{R}^3} w_{2k,n}f = (I + Q_{\widetilde{W},D,\infty})(w_{0,0}f), \tag{2.32}$$

where f satisfies (1.4)*, operator R^{-1} is defined by* (2.3) *and $Q_{\tilde{W},D,\infty}$ is given by* (2.27)*.*

In particular, if $W = W_N$, $N \in \mathbb{N}\cup\{0\}$, then the following analog of (2.32) *holds:*

$$R^{-1}R_W f = w_{0,0}f + \sum_{k=1}^{m}\sum_{n=-2k}^{2k} d_{2k,n} *_{\mathbb{R}^3} w_{2k,n}f = (I + Q_{\widetilde{W},D,m})(w_{0,0}f),\ m = [N/2], \tag{2.33}$$

where $Q_{\widetilde{W},D,m}$ is given by (2.26)*.*

3 Main results

3.1 Case of $\sigma_{\widetilde{W},D,\infty} < 1$

Let

$$\sigma_{\widetilde{W},D,\infty} < 1, \tag{3.1}$$

where $\sigma_{\widetilde{W},D,\infty}$ is defined by (1.14).

Inequality (3.1) and estimate (2.29) in Lemma 2.3 imply that operator $I + Q_{\widetilde{W},D,\infty}$ is continuosly invertible in $L^2(\mathbb{R}^3)$ and the following identity holds (in the sense of the operator norm in $L^2(\mathbb{R}^3)$):

$$(I + Q_{\widetilde{W},D,\infty})^{-1} = I + \sum_{j=1}^{\infty}(-Q_{\widetilde{W},D,\infty})^j, \tag{3.2}$$

where I is the identity operator in $L^2(\mathbb{R}^3)$.

Theorem 3.1. *Let conditions* (1.2)-(1.4), (3.1) *be fulfilled. Then* R_W *defined by* (1.1) *is injective and the following exact inversion formula holds:*

$$f = (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,\infty})^{-1}R^{-1}R_W f, \tag{3.3}$$

where $w_{0,0}$ *is defined in* (1.3), R^{-1} *is defined in* (2.3), *operator* $(I + Q_{\widetilde{W},D,\infty})^{-1}$ *is given in* (3.2).

Remark 3.1. Formula (3.3) can be considered as the following linear integral equation for the $w_{0,0}f$:

$$w_{0,0}f + Q_{\widetilde{W},D,\infty}(w_{0,0}f) = R^{-1}R_W f. \tag{3.4}$$

Inequality (3.1) and identity (3.2) imply that equation (3.4) can be solved by the method of successive approximations.

One can see that, under conditions (1.2)-(1.4) and (3.1), Theorem 3.1 gives an exact inversion of R_W. However, condition (3.1) is not always fulfilled in practice; see [GuNo14] for related numerical analysis in 2D. If condition (3.1) is not fulfilled, then, approximating W by finite Fourier series, in a similar way with [Cha78], [Kun92], [Nov14], [GuNo14], we suggest approximate inversion of R_W; see Subsections 3.2, 3.3.

3.2 Case of $1 \leq \sigma_{\widetilde{W},D,\infty} < +\infty$

Let

$$\sigma_{\widetilde{W},D,m} < 1, \text{ for some } m \in \mathbb{N} \cup \{0\}, \tag{3.5}$$

$$\sigma_{\widetilde{W},D,\infty} < +\infty, \tag{3.6}$$

where $\sigma_{\widetilde{W},D,m}$ is defined by (1.13), $\sigma_{\widetilde{W},D,\infty}$ is defined by (1.14).

Inequality (3.5) and estimate (2.28) in Lemma 2.3 imply that $I+Q_{\widetilde{W},D,m}$ is continuously invertible in $L^2(\mathbb{R}^3)$ and the following identity holds (in the sense of the operator norm in $L^2(\mathbb{R}^3)$):

$$(I + Q_{\widetilde{W},D,m})^{-1} = I + \sum_{j=1}^{\infty}(-Q_{\widetilde{W},D,m})^j, \tag{3.7}$$

where I is the identity operator in $L^2(\mathbb{R}^3)$.

Theorem 3.2. *Let conditions* (1.2)-(1.4), (3.5), (3.6) *be fulfilled. Then*

$$f \approx f_m \stackrel{def}{=} (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,m})^{-1}R^{-1}R_W f, \tag{3.8}$$

$$f = f_m - (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,m})^{-1}R^{-1}R_{\delta W_m} f, \tag{3.9}$$

$$\|f - f_m\|_{L^2(D)} \leq \frac{\|f\|_\infty}{2\pi\sqrt{2}c(1 - \sigma_{\widetilde{W},D,m})} \sum_{k=m+1}^{\infty} \sum_{n=-2k}^{2k} \|w_{2k,n}\|_{L^2(D)} < +\infty, \tag{3.10}$$

where

$$\delta W_m(x, \theta(\gamma, \phi)) \stackrel{def}{=} W(x, \theta(\gamma, \phi)) - \sum_{k=0}^{2m+1} \sum_{n=-k}^{k} w_{k,n}(x) Y_k^n(\gamma, \phi), \tag{3.11}$$

$$x \in \mathbb{R}^3,\ \gamma \in [0, \pi],\ \phi \in [0, 2\pi],\ m \in \mathbb{N} \cup \{0\}, \tag{3.12}$$

$w_{0,0}$ *is defined in* (1.3), $\theta(\gamma, \phi)$ *is defined in* (1.9), Y_k^n *are defined in* (1.8), *operator* $(I + Q_{\widetilde{W},D,m})^{-1}$ *is given in* (3.7), *constant* c *is defined in* (2.16).

Remark 3.2. Formula (3.8) can be considered as the following linear integral equation for $w_{0,0}f_m$:

$$w_{0,0}f_m + Q_{\widetilde{W},D,m}(w_{0,0}f_m) = R^{-1}R_W f. \tag{3.13}$$

Inequality (3.5) and identity (3.7) imply that equation (3.13) is solvable by the method of successive approximations.

Note also that condition (3.6) can be relaxed to the following one:

$$\sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} \left\| \frac{w_{2k,n}}{w_{0,0}} \right\|_{L^2(D)} < +\infty. \tag{3.14}$$

Formula (3.8) is an extension to 3D of the Chang-type two-dimensional inversion formulas in [Cha78], [Nov14], [GuNo14]. In addition, formula (3.8) is an extension of the approximate inversion formula in [GN16], where this formula was given for $m = 0$.

If (3.5) is fulfilled for some $m \geq 1$, then f_m is a refinement of the Chang-type approximation f_0 and, more generally, f_j is a refinement of f_i for $0 \leq i < j \leq m$. In addition, $f_j = f_i$ if $w_{2k,n} \equiv 0$ for $i < k \leq j$, $n = \overline{-2k, 2k}$. Thus, we propose the following approximate reconstruction of f from $R_W f$:

(i) find maximal m such that (3.5) is still efficiently fulfilled,

(ii) approximately reconstruct f by f_m using (3.8).

3.3 Exact inversion for finite Fourier series weights

Let

$$W = W_N,\ N \in \mathbb{N} \cup \{0\}, \tag{3.15}$$

where W_N is defined by (1.15).

Suppose that

$$\sigma_{\widetilde{W},D,m} < 1 \text{ for } m = [N/2], \tag{3.16}$$

where $\sigma_{\widetilde{W},D,m}$ is defined by (1.13).

Theorem 3.3. *Let conditions* (1.2)-(1.4), (3.15), (3.16) *be fulfilled. Then* R_W *defined by* (1.1) *is injective and the following exact inversion formula holds:*

$$f = (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,m})^{-1} R^{-1} R_W f, \tag{3.17}$$

where $w_{0,0}$ *is defined in* (1.2), $(I + Q_{\widetilde{W},D,m})^{-1}$ *is given in* (3.7), R^{-1} *is defined by* (2.3).

Remark 3.3. Formula (3.17) can be considered as the following linear integral equation for $w_{0,0}f$:

$$w_{0,0}f + Q_{\widetilde{W},D,m}(w_{0,0}f) = R^{-1}R_W f. \tag{3.18}$$

Identity (3.16) implies that (3.18) can be solved by the method of successive approximations.

Remark 3.4. Note that Theorems 3.1, 3.2, 3.3 remain valid under assumptions (1.17), (1.18) in place of (1.2), (1.3). This follows from the fact that in the proofs in Sections 5, 6 it is required only existence of integral transforms, given by operators R_W, R^{-1}, $\mathcal{F}[\cdot]$, $\mathcal{F}^{-1}[\cdot]$, $Q_{\widetilde{W},D,\infty}$, $Q_{\widetilde{W},D,m}$, their compositions (see Subsections 2.1, 2.3) and of uniform upper bound on $(w_{0,0})^{-1}$ on D.

3.4 Additional comments

There are different classes of weights W for which the results of Subsections 3.1, 3.2, 3.3 can be used. For example, condition (3.1) is satisfied for the weights W of the following form:

$$W(x,\theta) = c + V(x,\theta),\ (x,\theta) \in \mathbb{R}^3 \times \mathbb{S}^2, \tag{3.19}$$

where

$$\begin{aligned} &c \text{ - is some positive constant,} \\ &V \in C^3(\mathbb{R}^3 \times \mathbb{S}^2),\ \|V\|_{C^3(D\times\mathbb{S}^2)} \leq M(c,D), \end{aligned} \tag{3.20}$$

where M is some positive constant depending only on c and on D.

On the other hand, all weights W admitting the finite Fourier series expansions (i.e., $W = W_N$ of (1.15) for some $N \in \mathbb{N} \cup \{0\}$) are dense in the space $C(\overline{D} \times \mathbb{S}^2)$ (and also in $L^2(D \times \mathbb{S}^2)$), where $\overline{D} = D \cup \partial D$. In addition, for such "finite" $W_N \in C(\mathbb{R}^3 \times \mathbb{S}^2)$ the sense of each of conditions (3.1), (3.5), (3.6), (3.14), (3.16) is especially clear. Moreover, condition (3.6) is always satisfied, for example, if $W_N \in C(\mathbb{R}^3 \times \mathbb{S}^2)$. In addition, even if (3.5) is not satisfied for the whole W_N, one can consider such a cutoff W_m of W_N, $m < N$, so that (3.5) holds and, therefore, results of Subsection 3.2 can be applied.

Finally, we recall that in many cases even the zero order approximation $W \approx W_0 = w_{0,0}$ can be practically efficient in view of results presented in [Cha78], [Nov11], [GuNo14], [GN16]. Therefore, we expect that inversion algorithms of Subsections 3.2, 3.3 based on higher order finite Fourier approximations of $W \approx W_N$ may be even considerably more efficient in related tomographies.

4 Generalization to multidimensions

Definition (1.1) and assumptions (1.2)-(1.4) are naturally extended as follows to the case of dimension $n > 3$:

$$R_W f(s,\theta) = \int\limits_{x\theta=s} W(x,\theta) f(x)\, dx,\ (s,\theta) \in \mathbb{R}\times\mathbb{S}^{n-1},\ x\in\mathbb{R}^n, \tag{4.1}$$

$$W \in L^\infty(\mathbb{R}^n\times\mathbb{S}^{n-1}), \tag{4.2}$$

$$w_{0,0}(x) \overset{def}{=} \frac{1}{|\mathbb{S}^{n-1}|}\int\limits_{\mathbb{S}^{n-1}} W(x,\theta)\, d\theta,\ w_{0,0}\ge c>0 \text{ on } D, \tag{4.3}$$

$$f\in L^\infty(\mathbb{R}^n),\ \operatorname{supp} f\subset D, \tag{4.4}$$

where $|\mathbb{S}^{n-1}|$ denotes the standard Euclidean volume of $\mathbb{S}^{n-1}$, c is a constant, D is an open bounded domain in $\mathbb{R}^n$.

For the weight W we consider the Fourier-Laplace expansion:

$$W(x,\theta) = \sum_{k=0}^{\infty}\sum_{i=0}^{a_{k,n}-1} w_{k,i}(x) Y_k^i(\theta),\ x\in\mathbb{R}^n,\ \theta\in\mathbb{S}^{n-1}, \tag{4.5}$$

where

$$w_{k,i}(x) = \|Y_k^i\|^{-2}_{L^2(\mathbb{S}^{n-1})}\int\limits_{\mathbb{S}^{n-1}} W(x,\theta)\overline{Y_k^i(\theta)}\, d\theta, \tag{4.6}$$

$$a_{k,n+1} = \frac{(n+k)!}{k!n!} - \frac{(n+k-2)!}{(k-2)!n!},\ n,k\ge 2;\ a_{0,n}=1,\ a_{1,n}=n, \tag{4.7}$$

where $\{Y_k^i \mid k=\overline{0,\infty}, i=\overline{0,a_{k,n}-1}\}$ is the Fourier-Laplace basis of harmonics on $\mathbb{S}^{n-1}$, $\overline{Y_k^i}$ denotes the complex conjugate of Y_k^i ; see [SW16], [Mor98]. In the present work we choose the basis $\{Y_k^i\}$ as in [Hig87] without normalizing constants ${}_nc_L^l$ (i.e., $\{Y_k^i\}$ are the products of the Schmidt semi-normalized Legendre polynomials with one complex exponent and without any additional constants).

In dimension $n > 3$, formulas (1.13), (1.14), (2.18), (2.19), are rewritten as follows:

$$\sigma_{\widetilde{W},D,m} \overset{def}{=} (2\pi\sqrt{2})^{(1-n)/2}\sum_{k=1}^{m}\sum_{i=0}^{a_{2k,n}-1}\sup_{x\in D}\left|\frac{w_{2k,i}(x)}{w_{0,0}(x)}\right|, \tag{4.8}$$

$$\sigma_{\widetilde{W},D,\infty} \overset{def}{=} \lim_{m\to+\infty}\sigma_{\widetilde{W},D,m} \tag{4.9}$$

$$Q_{\widetilde{W},D,m}u(x) \overset{def}{=} R^{-1}(R_{\widetilde{W},D,m}u)(x),\ m\in\mathbb{N} \tag{4.10}$$

$$Q_{\widetilde{W},D,m}u(x) = 0 \text{ for } m=0,$$

$$Q_{\widetilde{W},D,\infty}u(x) \overset{def}{=} R^{-1}(R_{\widetilde{W},D,\infty}u)(x), \tag{4.11}$$

where

$$R_{\widetilde{W},D,m}u(s,\theta) \overset{def}{=} \int\limits_{x\theta=s}\left(\sum_{k=1}^{m}\sum_{i=0}^{a_{2k,n}-1}\frac{w_{2k,i}(x)}{w_{0,0}(x)}Y_{2k}^i(\theta)\right)\chi_D(x)u(x)\, dx, \tag{4.12}$$

$$R_{\widetilde{W},D,\infty}u(s,\theta) \overset{def}{=} \lim_{m\to\infty} R_{\widetilde{W},D,m}u(s,\theta), \tag{4.13}$$

$$x\in\mathbb{R}^n,\ s\in\mathbb{R},\ \theta\in\mathbb{S}^{n-1},$$

where R^{-1} is defined further in (4.14).

Under assumptions (4.2), (4.4), for each fixed x, series of (4.5) converges in $L^2(\mathbb{S}^{n-1})$; see, e.g., [SW16] (Chapter 4), [Mor98] (Chapter 2), [ZT79].

Formula (2.3) is extended as follows:

$$R^{-1}q(x) = (2\pi)^{1/2-n}\int\limits_{\mathbb{R}}\frac{|\rho|^{n-1}}{2}\, d\rho\int\limits_{\mathbb{S}^{n-1}}\hat{q}(\rho,\theta)e^{i\rho(x\theta)}d\theta,\ x\in\mathbb{R}^n, \tag{4.14}$$

where $q(s,\theta)$ is a test function on $\mathbb{R}\times\mathbb{S}^{n-1}$, $\hat{q}(s,\theta)$ is defined as in (2.4) (with $\mathbb{S}^{n-1}$ in place of $\mathbb{S}^2$).

The Fourier transforms, defined in (2.6), (2.7), are extended as follows:

$$\mathcal{F}[q](\xi) \stackrel{def}{=} (2\pi)^{-n/2} \int\limits_0^{+\infty} \rho^{n-1} d\rho \int\limits_{\mathbb{S}^{n-1}} q(\rho,\omega) e^{-i\rho(\xi\omega)} d\omega, \tag{4.15}$$

$$\mathcal{F}^{-1}[q](\xi) \stackrel{def}{=} (2\pi)^{-n/2} \int\limits_0^{+\infty} \rho^{n-1} d\rho \int\limits_{\mathbb{S}^{n-1}} q(\rho,\omega) e^{i\rho(\xi\omega)} d\omega,\ \xi \in \mathbb{R}^n, \tag{4.16}$$

where $q(\rho,\omega)$ is a test function on $[0,+\infty) \times \mathbb{S}^{n-1}$ (identified with $\mathbb{R}^n$).

In dimension $n > 3$, formulas (2.8)-(2.12) remain valid with Y_k^m defined in (1.8) replaced by basis of spherical harmonics $\{Y_k^i\}$ on $\mathbb{S}^{n-1}$. In particular, the following multidimensional analog of (2.12) holds:

$$Y_k^i(-\theta) = (-1)^k Y_k^i(\theta),\ \theta \in \mathbb{S}^{n-1},\ k \in \mathbb{N} \cup \{0\},\ i = \overline{0, a_{k,n} - 1}, \tag{4.17}$$

where $a_{k,n}$ is defined by (4.7). Identity (4.17) reflects the fact that $Y_k^i(\theta) = Y_k^i(\theta_1, \theta_2, \cdots, \theta_n)$, $\theta = (\theta_1, \theta_2, \cdots, \theta_n) \in \mathbb{S}^{n-1}$, $i = \overline{0, a_{k,n} - 1}$ is a homogenous polynomial of degree k, see, e.g., [SW16], [Mor98].

Formula (2.22) is now rewritten as follows:

$$d_{2k,i}(x) \stackrel{def}{=} R^{-1}(\delta(\cdot) Y_{2k}^i)(x),\ x \in \mathbb{R}^n,\ i = \overline{0, a_{k,n} - 1}. \tag{4.18}$$

Results of Lemma 2.1 remain valid with formula (2.23) replaced by the following one:

$$d_{2k,i}(r,\theta) = c(k,n) \frac{(-1)^k Y_{2k}^i(\theta)}{r^n},\ r > 0,\ \theta \in \mathbb{S}^{n-1}, \tag{4.19}$$

where

$$c(k,n) = \frac{\sqrt{2}\pi^{(1-n)/2} \Gamma(k + \frac{1}{2}) \Gamma(k + \frac{n}{2})}{\Gamma(k) \Gamma(k + \frac{n-1}{2})} \cdot \left(\frac{\Gamma(k+1)}{\Gamma(k + \frac{1}{2})} \right)^{n-2}, \tag{4.20}$$

$\Gamma(\cdot)$ is the Gamma function.

In addition, inequality (2.25) is rewritten as follows:

$$|\mathcal{F}[d_{2k,i}](\xi)| \le (2\pi\sqrt{2})^{(1-n)/2},\ \xi \in \mathbb{R}^n, \tag{4.21}$$

where $\mathcal{F}[\cdot]$ is the Fourier transform defined in (4.15). The constant $c(k,n)$ in (4.20) is obtained using formulas (4.14), (4.18) and Theorems 1, 2 of [Gon16].

The results of Lemma 2.2 remain valid in the case of dimension $n > 3$, with (2.26), (2.27) rewritten as follows:

$$Q_{\widetilde{W},D,m} u = \sum_{k=1}^{m} \sum_{i=0}^{a_{k,n}-1} d_{2k,i} *_{\mathbb{R}^n} \frac{w_{2k,i}}{w_{0,0}} u, \tag{4.22}$$

$$Q_{\widetilde{W},D,\infty} u = \sum_{k=1}^{\infty} \sum_{i=0}^{a_{k,n}-1} d_{2k,i} *_{\mathbb{R}^n} \frac{w_{2k,i}}{w_{0,0}} u, \tag{4.23}$$

where coefficients $w_{2k,i}$, $w_{0,0}$ are defined in (4.6), $a_{k,n}$ is defined in (4.7), $d_{2k,i}$ is defined in (4.19), $*_{\mathbb{R}^n}$ denotes the convolution in $\mathbb{R}^n$.

The results of Lemma 2.3 remain valid with $\mathbb{R}^3$ replaced by $\mathbb{R}^n$, $n > 3$, where we use definitions (4.8), (4.10), (4.11).

Assumption (2.30) in Lemma 2.4 is rewritten now as follows:

$$\sum_{k=1}^{\infty} \sum_{i=0}^{a_{k,n}-1} \left\| \frac{w_{2k,i}}{w_{0,0}} \right\|_{L^2(D)} < +\infty. \tag{4.24}$$

Under assumption (4.24), property (2.31) of Lemma 2.4 remains valid in dimension $n > 3$. In particular, formula (2.32) is rewritten now as follows:

$$R^{-1} R_W f = w_{0,0} f + \sum_{k=1}^{\infty} \sum_{i=0}^{a_{k,n}-1} d_{2k,i} *_{\mathbb{R}^n} w_{2k,i} f, \tag{4.25}$$

where R^{-1} is defined in (4.14), f is a test function satisfying (4.4).

Using formulas and notations from (4.5)-(4.23) we obtain straightforward extensions of Theorems 3.1, 3.2, 3.3.

- The result of Theorem 3.1 remains valid in dimension $n > 3$, under assumptions (4.2)-(4.4) and under condition (3.1), where $w_{0,0}$ is defined in (4.3), R^{-1} is defined in (4.14), $\sigma_{\widetilde{W},D,\infty}$ is defined in (4.9), $Q_{\widetilde{W},D,\infty}$ is defined in (4.11).

- The result of Theorem 3.2 remains valid in dimension $n > 3$, under assumptions (4.2)-(4.4) and under conditions (3.5), (3.6), where $w_{0,0}$ is defined in (4.3), R^{-1} is defined in (4.14), $\sigma_{\widetilde{W},D,m}$ is defined in (4.8), $Q_{\widetilde{W},D,m}$ is defined in (4.10) and where formulas (3.9)-(3.11) are rewritten as follows:

$$\|f - f_m\|_{L^2(D)} \leq \frac{\|f\|_\infty}{(2\pi\sqrt{2})^{(n-1)/2} c(1 - \sigma_{\widetilde{W},D,m})} \sum_{k=m+1}^{\infty} \sum_{i=0}^{a_{k,n}-1} \|w_{2k,i}\|_{L^2(D)} < +\infty, \tag{4.26}$$

$$\delta W_m(x,\theta) \stackrel{def}{=} W(x,\theta) - \sum_{k=0}^{2m+1} \sum_{i=0}^{a_{k,n}-1} w_{k,i}(x) Y_k^i(\theta), \tag{4.27}$$

$$x \in \mathbb{R}^n,\ \theta \in \mathbb{S}^{n-1}.$$

- The result of Theorem 3.3 remains valid in dimension $n > 3$, under assumptions (4.2)-(4.4) and under conditions (3.15), (3.16), where $w_{0,0}$ is defined in (4.3), R^{-1} is defined in (4.14), $\sigma_{\widetilde{W},D,m}$ is defined in (4.8), $Q_{\widetilde{W},D,m}$ is defined in (4.10).

The related proofs are the straightforward extensions to the case of dimension $n > 3$ of proofs in Section 6 for $n = 3$.

5 Proofs of Lemma 2.1, 2.2, 2.3, 2.4

5.1 Proof of Lemma 2.1

We consider $x(r,\gamma,\phi)$ defined by (2.24) and $\omega(\gamma,\phi) = x(1,\gamma,\phi)$ (i.e., $\omega \in \mathbb{S}^2$).

Identity (2.24) implies the following expression for the scalar product $(x\omega)$:

$$(x\omega) = (x(r,\tilde\gamma,\tilde\phi), \omega(\gamma,\phi)) = r(\cos\gamma\cos\tilde\gamma + \sin\gamma\sin\tilde\gamma\cos(\phi - \tilde\phi)), \tag{5.1}$$

where $\gamma, \tilde\gamma \in [0,\pi]$, $\phi, \tilde\phi \in [0, 2\pi]$, $r \geq 0$.

From formulas (1.8), (2.3), (2.22), (5.1) it follows that

$$\begin{aligned}
(2\pi)^{5/2} d_{2k,n}(x(r,\tilde\gamma,\tilde\phi)) &= \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_{\mathbb{S}^2} e^{i\rho(x\omega(\gamma,\phi))} Y_{2k}^n(\gamma,\phi)\, d\omega(\gamma,\phi) \\
&= \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_0^\pi \sin(\gamma)\, p_{2k}^{|n|}(\cos\gamma)\, d\gamma \int_0^{2\pi} e^{i\rho(x\omega(\gamma,\phi)) + in\phi} d\phi \\
&= e^{in\tilde\phi} \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_0^\pi \sin(\gamma) p_{2k}^{|n|}(\cos\gamma) e^{i\rho r\cos\gamma\cos\tilde\gamma}\, d\gamma \int_0^{2\pi} e^{i\rho r\sin\gamma\sin\tilde\gamma\cos(\phi-\tilde\phi) + in(\phi-\tilde\phi)} d\phi \\
&= e^{in\tilde\phi} \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_0^\pi \sin(\gamma) p_{2k}^{|n|}(\cos\gamma) e^{i\rho r\cos\gamma\cos\tilde\gamma}\, d\gamma \int_0^{2\pi} e^{i\rho r\sin\gamma\sin\tilde\gamma\cos\phi + in\phi} d\phi \\
&= 2\pi e^{in(\tilde\phi - \pi/2)} (-1)^n \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_0^\pi \sin(\gamma) p_{2k}^{|n|}(\cos\gamma) e^{i\rho r\cos\gamma\cos\tilde\gamma} J_n(\rho r\sin\gamma\sin\tilde\gamma)\, d\gamma,
\end{aligned} \tag{5.2}$$

where J_n is the n-th standard Bessel function of the first kind; see e.g. [Tem11]. In (5.2) we used the well known formula for the Bessel function J_n:

$$J_n(t) \stackrel{def}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\phi - t\sin\phi} d\phi = \frac{(-1)^n e^{in\pi/2}}{2\pi} \int_0^{2\pi} e^{in\phi + t\cos\phi} d\phi.$$

The integral in $d\gamma$ in the right-hand side of (5.2) was considered in [NPF+06], where the following exact analytic solution was given:

$$\int_0^\pi \sin(\gamma) p_{2k}^{|n|}(\cos\gamma) e^{i\rho r\cos\gamma\cos\tilde{\gamma}} J_n(\rho r\sin\gamma\sin\tilde{\gamma})\, d\gamma = 2i^{2k-n} p_{2k}^{|n|}(\cos\tilde{\gamma}) j_{2k}(\rho r), \tag{5.3}$$

where j_{2k} is the standard spherical Bessel function of order $2k$; see e.g. [Tem11].

From identities (5.2), (5.3) it follows that:

$$\begin{aligned} d_{2k,n}(x(r,\tilde{\gamma},\tilde{\phi})) &= \frac{(-1)^k}{(2\pi)^{3/2}} p_{2k}^{|n|}(\cos\tilde{\gamma}) e^{in\tilde{\phi}} \int_{\mathbb{R}} \rho^2 j_{2k}(\rho r)\, d\rho \\ &= \frac{4\sqrt{\pi}(-1)^k \Gamma(\frac{3}{2}+k)}{(2\pi)^{3/2}\Gamma(k)} \frac{p_{2k}^{|n|}(\cos\tilde{\gamma}) e^{in\tilde{\phi}}}{r^3},\ r>0. \end{aligned} \tag{5.4}$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition (1.8) and identity (5.4) imply formula (2.23).

Formulas (2.5), (2.13), (2.14), (2.22) imply that

$$(2\pi) d_{2k,n}(x(r,\tilde{\gamma},\tilde{\phi})) = \mathcal{F}^{-1}[Y_{2k}^n](x(r,\tilde{\gamma},\tilde{\phi})),\ r>0,\ \tilde{\gamma}\in[0,\pi],\ \tilde{\phi}\in[0,2\pi], \tag{5.5}$$

where $\mathcal{F}^{-1}[\cdot]$ is defined in (2.7).

Due to invertibility of the Fourier transform defined in (2.6) and identity (5.5) the following identity holds:

$$2\pi\mathcal{F}[d_{2k,n}](\xi) = \mathcal{F}\mathcal{F}^{-1}[Y_{2k}^n](\xi/|\xi|) = Y_{2k}^n(\xi/|\xi|),\ \xi\in\mathbb{R}^3\backslash\{0\}. \tag{5.6}$$

For Y_k^n defined in (1.8) the following inequality holds (see, e.g., [Loh98]):

$$|Y_k^n(\gamma,\phi)| \le \frac{1}{\sqrt{2}},\ \gamma\in[0,\pi],\ \phi\in[0,2\pi]. \tag{5.7}$$

Identities (5.5) and inequality (5.7) imply (2.25).

Note that $|\mathcal{F}[d_{2k,n}](\xi)|$, is uniformly bounded by $1/(2\pi\sqrt{2})$ everywhere except only at one point $\xi=0$, where direction $\xi/|\xi|\in\mathbb{S}^2$ is not defined. However, point $\xi=0$ is of Lebesgue measure zero and $\mathcal{F}[d_{2k,n}]$ can be defined with any value at the origin in $\mathbb{R}^3$.

Lemma 2.1 is proved.

5.2 Proof of Lemma 2.2

From identity (2.18) it follows that

$$\begin{aligned} Q_{\widetilde{W},D,m}u &= R^{-1}\left(\sum_{k=1}^m \sum_{n=-2k}^{2k} Y_{2k}^n R\left(\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right)\right) \\ &= R^{-1}\left(\sum_{k=1}^m \sum_{n=-2k}^{2k} (\delta(\cdot)Y_{2k}^n) *_{\mathbb{R}} R\left(\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right)\right) \\ &= R^{-1}\left(\sum_{k=1}^m \sum_{n=-2k}^{2k} R(d_{2k,n}) *_{\mathbb{R}} R\left(\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right)\right) \end{aligned} \tag{5.8}$$

where $*_{\mathbb{R}}$ denotes the 1D convolution, $\delta=\delta(s)$ is the 1D Dirac delta function, $d_{2k,n}$ is defined by (2.22).

Identities (2.1), (5.8) imply (2.26).

For the operator $Q_{\widetilde{W},D,\infty}$ defined by (2.19) we proceed according to identity (5.8) with $m\to+\infty$. Identites (2.1), (5.8) and linearity of operator R^{-1} defined by (2.3) imply (2.27).

Lemma 2.2 is proved.

5.3 Proof of Lemma 2.3

From formula (2.26) and the fact that the Fourier transform defined in (2.6) does not change the L^2-norm we obtain:

$$\begin{aligned} \|Q_{\widetilde{W},D,m}u\|_{L^2(\mathbb{R}^3)} &\le \sum_{k=1}^m \sum_{n=-2k}^{2k} \left\| d_{2k,n} *_{\mathbb{R}^3} \frac{w_{2k,n}}{w_{0,0}}\chi_D u \right\|_{L^2(\mathbb{R}^3)} \\ &= \sum_{k=1}^m \sum_{n=-2k}^{2k} \left\| \mathcal{F}[d_{2k,n}]\mathcal{F}\left[\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right] \right\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{5.9}$$

From inequalities (2.25), (5.9) we obtain:

$$\|Q_{\widetilde{W},D,m}u\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{m}\sum_{n=-2k}^{2k}\left|\left|\mathcal{F}\left(\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right)\right|\right|_{L^2(\mathbb{R}^3)}$$
$$= \frac{1}{2\pi\sqrt{2}}\sum_{k=1}^{m}\sum_{n=-2k}^{2k}\left|\left|\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right|\right|_{L^2(\mathbb{R}^3)}$$
$$\leq \sigma_{\widetilde{W},D,m}\|u\|_{L^2(D)}, \tag{5.10}$$

where $\sigma_{\widetilde{W},D,m}$ is defined by (1.13).

Inequality (5.10) implies (2.28).

Estimate (2.29) follows from definition (2.19), formula (2.27), linearity of operator R^{-1} defined by (2.2) and inequalities (5.9), (5.10) for $m \to +\infty$.

Lemma 2.3 is proved.

5.4 Proof of Lemma 2.4

From formulas (1.1), (1.7) it follows that

$$R_W f(s,\theta(\gamma,\phi)) = \sum_{k=0}^{\infty}\sum_{n=-k}^{k} Y_k^n(\gamma,\phi)R(w_{k,n}f)(s,\theta(\gamma,\phi)), \tag{5.11}$$
$$s \in \mathbb{R},\ \gamma \in [0,\pi],\ \phi \in [0,2\pi], \tag{5.12}$$

where $\theta(\gamma,\phi)$ is defined in (1.9), $Y_k^n(\gamma,\phi)$ are defined by (1.8), $w_{k,n}$ are defined in (1.12).

Formula (2.32) follows from formulas (2.8), (2.10), (2.12), (2.19) and formula (2.27) in Lemma 2.2, where test function u is replaced by $w_{0,0}u$.

From inequality (2.25), formulas (2.1), (2.32) and the fact that the Fourier tranfsform defined in (2.6) does not change the L^2-norm we obtain:

$$\|R^{-1}R_W f\|_{L^2(\mathbb{R}^3)} \leq \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\|d_{2k,n} *_{\mathbb{R}^3} w_{2k,n}f\|_{L^2(\mathbb{R}^3)}$$
$$= \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\|\mathcal{F}[d_{2k,n}]\mathcal{F}[w_{2k,n}f]\|_{L^2(\mathbb{R}^3)} \tag{5.13}$$
$$\leq \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \frac{\|f\|_\infty}{2\pi\sqrt{2}}\sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\|w_{2k,n}\|_{L^2(D)},$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm.

From assumption (2.30) and formula (2.16) it follows that

$$\sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\|w_{2k,n}\|_{L^2(D)} < +\infty. \tag{5.14}$$

Assumptions (1.2)-(1.4) and inequalities (5.13), (5.14) imply that

$$\|R^{-1}R_W f\|_{L^2(\mathbb{R}^3)} \leq \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \frac{\|f\|_\infty}{2\pi\sqrt{2}}\sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\|w_{2k,n}\|_{L^2(D)} < +\infty. \tag{5.15}$$

Lemma 2 is proved.

6 Proofs of Theorems 3.1, 3.2, 3.3

6.1 Proof of Theorem 3.1

Property (2.31) and identity (2.32) of Lemma 2.4 follow from assumption (3.1).

Formulas (2.31), (2.32), (3.1), (3.2) imply formula (3.3).

The injectivity of R_W follows from formula (3.3).

Theorem 3.1 is proved.

6.2 Proof of Theorem 3.3

Property (2.31) and identity (2.33) of Lemma 2.4 follow from assumption (3.16).

Formulas (2.31), (2.33), (3.7), (3.16) imply formula (3.17).

The injectivity of R_W follows from formula (3.17).

Theorem 3.3 is proved.

6.3 Proof of Theorem 3.2

Inequality (2.30) follows from assumption (3.6). Hence, formulas (2.31), (2.32) (in Lemma 2.4) hold.

Assumptions (1.2)-(1.4), (3.5) and inequality (2.28) from Lemma 2.3 imply that $f_m \in L^2(\mathbb{R}^3)$, where f_m is defined in (3.8).

We split expansion (1.7) of weight W defined by (1.2) in the following way:

$$W(x,\theta) = W_{N+1}(x,\theta) + \delta W_m(x,\theta),\ \theta \in \mathbb{S}^2,\ x \in \mathbb{R}^3,\ m = [N/2], \tag{6.1}$$

where W_{N+1} is defined by (1.15), $[N/2]$ denotes the integer part of $N/2$, δW_m is defined by (3.11).

From (1.2)-(1.4) and from (6.1) it follows that

$$R_W f = R_{W_{N+1}} f + R_{\delta W_m} f, \tag{6.2}$$

where $R_W f, R_{W_{N+1}} f, R_{\delta W_m} f$ are defined by (1.1) for the case of weights $W, W_{N+1}, \delta W_m$ defined in (1.2), (1.12), (3.11), respectively.

Identity (6.2) implies that

$$R^{-1} R_W f = R^{-1} R_{W_N} f + R^{-1} R_{\delta W_m} f, \tag{6.3}$$

where R^{-1} is defined by (2.3). Inequality (2.30) for the cases of weights W, W_N, δW_m, follows from assumption (3.6). Therefore, Lemma 2.4 holds for W, W_N, δW_m and, in particular, from (2.31) we have that:

$$R^{-1} R_{W_N} f \in L^2(\mathbb{R}^3),\ R^{-1} R_{\delta W_m} f \in L^2(\mathbb{R}^3). \tag{6.4}$$

Theorem 3.3 for $W = W_N$, $N = 2m$ holds by assumption (3.5). Therefore, from formula (3.17) we obtain:

$$f = (w_{0,0})^{-1} (I + Q_{\widetilde{W},D,m})^{-1} R^{-1} R_{W_N} f, \tag{6.5}$$

where operator $Q_{\widetilde{W},D,m}$ is defined in (2.18) for m arising in (3.5).

From (3.8), (6.3), (6.4), (6.5) it follows that:

$$\begin{aligned} f_m &= (w_{0,0})^{-1} (I + Q_{\widetilde{W},D,m})^{-1} R^{-1} R_W f \\ &= (w_{0,0})^{-1} (I + Q_{\widetilde{W},D,m})^{-1} R^{-1} R_{W_N} f \\ &+ (w_{0,0})^{-1} (I + Q_{\widetilde{W},D,m})^{-1} R^{-1} R_{\delta W_m} f \\ &= f + (w_{0,0})^{-1} (I + Q_{\widetilde{W},D,m})^{-1} R^{-1} R_{\delta W_m} f. \end{aligned} \tag{6.6}$$

Formula (3.9) directly follows from (6.6).

Inequality (3.5) and identities (3.7), (3.9) imply the following inequality:

$$\|f - f_m\|_{L^2(\mathbb{R}^3)} \le \frac{1}{c} \|(I + Q_{\widetilde{W},D,m})^{-1}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \cdot \|R^{-1} R_{\delta W_m} f\|_{L^2(\mathbb{R}^3)}, \tag{6.7}$$

where c is defined in (2.16), $Q_{\widetilde{W},D,m}$ is defined by (2.18) for m in (3.5).

From (2.28) of Lemma 2.3 and from identity (3.7) it follows that:

$$\|(I + Q_{\widetilde{W},D,m})^{-1}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \le 1 + \sum_{j=1}^{\infty} \|Q_{\widetilde{W},D,m}\|^j_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \le \frac{1}{1 - \sigma_{\widetilde{W},D,m}}. \tag{6.8}$$

From formulas (2.32), (3.11) and according to (5.13) it follows that

$$\|R^{-1} R_{\delta W_m} f\|_{L^2(\mathbb{R}^3)} \le \frac{\|f\|_\infty}{2\pi\sqrt{2}} \sum_{k=m+1}^{\infty} \sum_{n=-2k}^{2k} \|w_{2k,n}\|_{L^2(D)}, \tag{6.9}$$

where R^{-1} is defined by (2.3), $w_{2k,n}$ are defined by (1.12).

Putting the estimates (6.8), (6.9) in the right-hand side of (6.7) we obtain (3.10).

Theorem 3.2 is proved.

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