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# An iterative inversion of weighted Radon transforms along hyperplanes

F.O. Goncharov\*

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## Abstract

We propose iterative inversion algorithms for weighted Radon transforms  $R_W$  along hyperplanes in  $\mathbb{R}^3$ . More precisely, expanding the weight  $W = W(x, \theta)$ ,  $x \in \mathbb{R}^3$ ,  $\theta \in \mathbb{S}^2$ , into the series of spherical harmonics in  $\theta$  and assuming that the zero order term  $w_{0,0}(x) \neq 0$ ,  $x \in \mathbb{R}^3$ , we reduce the inversion of  $R_W$  to solving a linear integral equation. In addition, under the assumption that the even part of  $W$  in  $\theta$  (i.e.,  $\frac{1}{2}(W(x, \theta) + W(x, -\theta))$ ) is close to  $w_{0,0}$ , the aforementioned linear integral equation can be solved by the method of successive approximations. Approximate inversions of  $R_W$  are also given. Our results can be considered as an extension to 3D of two-dimensional results of Kunyansky (1992), Novikov (2014), Guillement, Novikov (2014). In our studies we are motivated, in particular, by problems of emission tomographies in 3D. In addition, we generalize our results to the case of dimension  $n > 3$ .

## 1 Introduction

We consider the weighted Radon transforms  $R_W$  defined by the formula

$$R_W f(s, \theta) = \int_{x\theta=s} W(x, \theta) f(x) dx, \quad (s, \theta) \in \mathbb{R} \times \mathbb{S}^2, \quad x \in \mathbb{R}^3, \quad (1.1)$$

where  $W = W(x, \theta)$  is the weight,  $f = f(x)$  is a test function.

In this work we assume that

$$W \in C(\mathbb{R}^3 \times \mathbb{S}^2) \cap L^\infty(\mathbb{R}^3 \times \mathbb{S}^2), \quad (1.2)$$

$$w_{0,0}(x) \stackrel{def}{=} \frac{1}{4\pi} \int_{\mathbb{S}^2} W(x, \theta) d\theta, \quad w_{0,0}(x) \neq 0, \quad x \in \mathbb{R}^3, \quad (1.3)$$

$$f \in L^\infty(\mathbb{R}^3), \quad \text{supp } f \subset D, \quad (1.4)$$

where  $W$  and  $f$  are complex-valued,  $d\theta$  is element of standard measure on  $\mathbb{S}^2$ ,  $D$  is an open bounded domain (which is fixed a priori).

If  $W \equiv 1$ , then  $R_W$  is reduced to the classical Radon transform along hyperplanes in  $\mathbb{R}^3$  introduced in [Rad17]; see also, e.g., [Nat86], [Den2016].

For known results on the aforementioned transforms  $R_W$  with non-constant  $W$  we refer to [Qui83], [Bey84], [BQ87], [GN16]. In particular, in [Qui83] it was shown that  $R_W$  is injective on  $L_0^p(\mathbb{R}^3)$ ,  $p \geq 2$  ( $L^p$  functions on  $\mathbb{R}^3$  with compact support) if  $W \in C^2$  and is real-valued, strictly positive and satisfies the strong symmetry assumption of rotation invariancy (see [Qui83] for details). On the other hand, in [BQ87] it was also proved that  $R_W$  is injective if  $W$  is real-analytic and strictly positive.

Besides, in [Bey84] the inversion of  $R_W$  is reduced to solving a Fredholm type linear integral equation in the case of infinitely smooth strictly positive  $W$  with the symmetry  $W(x, \theta) = W(x, -\theta)$ .

In turn, [GN16] extends to the case of weighted Radon transforms along hyperplanes in multidimensions the two-dimensional Chang approximate inversion formula (see [Cha78]) and the related two-dimensional result of [Nov11]. In particular, [GN16] describes all  $W$  for which such Chang-type formulas are simultaneously explicit and exact in multidimensions.

We recall that inversion methods for  $R_W$  admit tomographical applications in the framework of the scheme described as follows (see [GN16]).

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It is well-known that in many tomographies measured data are modeled by weighted ray transforms  $P_w f$  defined by the formula

$$P_w f(x, \alpha) = \int_{\mathbb{R}} w(x + \alpha t, \alpha) f(x + \alpha t) dt, \quad (x, \alpha) \in TS^2, \quad (1.5)$$

$$TS^2 = \{(x, \alpha) \in \mathbb{R}^3 \times \mathbb{S}^2 : x\alpha = 0\},$$

where  $f$  is an object function defined on  $\mathbb{R}^3$ ,  $w$  is the weight function defined on  $\mathbb{R}^3 \times \mathbb{S}^2$ , and  $TS^2$  can be considered as the set of all rays (oriented straight lines) in  $\mathbb{R}^3$ , see, e.g., [Cha78], [Nat86], [Kun92], [GuNo14].

In addition, in [GN16] (Section 3) it was shown that if  $P_w f$  are given for all rays parallel to some fixed plane  $\Sigma$  in  $\mathbb{R}^3$  then  $R_W f$  with appropriate  $W$  can be obtained by the explicit formulas from  $P_w f$  and  $w$  (in a similar way with the case  $w \equiv 1$ ,  $W \equiv 1$ , see Chapter 2, formula (1.1) of [Nat86] and also [Gra91], [Den2016]). Therefore, reconstruction of  $f$  from data modeled by  $P_w f$ , defined by (1.5) and restricted to all rays parallel to  $\Sigma$ , can be reduced to reconstruction of  $f$  from  $R_W f$ , defined by (1.1). In [GN16] it was also indicated that the reduction from  $P_w f$  to  $R_W f$  with subsequent reconstruction of  $f$  from  $R_W f$  and  $W$  can drastically reduce the impact of the random noise in the initial data modeled as  $P_w f$ .

In the present work we continue studies of [GN16], on one hand, and of [Kun92], [Nov14], [GuNo14], on the other hand. In particular, we extend to the case of weighted Radon transforms along hyperplanes in multidimensions the two-dimensional results of [Kun92], [Nov14], [GuNo14]. In particular, under the assumptions (1.2), (1.3), expanding  $W = W(x, \theta)$  into the series of spherical harmonics in  $\theta$  we reduce the reconstruction of  $f$  to solving a linear integral equation (see Section 3). In particular, if the even part of  $W$  in  $\theta$  (i.e.,  $\tilde{W}(x, \theta) = \frac{1}{2}(W(x, \theta) + W(x, -\theta))$ ) is close to  $w_{0,0}$ , then such linear integral equation can be solved by the method of successive approximations (see Subsections 3.1, 3.3 for details).

Note that our linear integral equation is very different from the aforementioned linear integral equation of [Bey84] (in particular, in our conditions on  $\tilde{W}$ , ensuring the applicability of the method of successive approximations).

Note also that in [Cha78], [Kun92], [Nov14], [GuNo14] the two-dimensional prototype of our inversion approach was developed in view of its numerical efficiency in problems of emission tomographies, including good stability to strong random noise in the emission data.

In more details our results can be sketched as follows.

We use the following expansion for  $W$ :

$$W(x, \theta(\gamma, \phi)) = \sum_{k=0}^{\infty} \sum_{n=-k}^k w_{k,n}(x) Y_k^n(\gamma, \phi), \quad x \in \mathbb{R}^3, \quad (1.6)$$

$$Y_k^n(\gamma, \phi) \stackrel{def}{=} p_k^{|n|}(\cos \gamma) e^{in\phi}, \quad k \in \mathbb{N} \cup \{0\}, \quad n = \overline{-k, k}, \quad (1.7)$$

$$\theta(\gamma, \phi) = (\cos \gamma, \sin \gamma \cos \phi, \sin \gamma \sin \phi) \in \mathbb{S}^2 \subset \mathbb{R}^3, \quad \gamma \in [0, \pi], \quad \phi \in [0, 2\pi], \quad (1.8)$$

where  $p_k^n(x)$ ,  $x \in [-1, 1]$ , are the associated Legendre polynomials. Polynomials  $p_k^n$  are well-known in literature (see e.g. [SW16]) and are defined using the ordinary Legendre polynomials  $p_k$  by the formulas:

$$p_k^n(x) = (-1)^n (1-x^2)^{n/2} \frac{d^n}{dx^n} (p_k(x)), \quad n, k \in \mathbb{N} \cup \{0\}, \quad (1.9)$$

$$p_k(x) = \frac{1}{2^k k!} \frac{d^k}{dx^k} [(x^2 - 1)^k], \quad x \in [-1, 1], \quad (1.10)$$

see also [SW16], [ZT79] for other properties of the associated Legendre polynomials. In addition, coefficients  $w_{k,n}$  in (1.6) are defined by the formulas:

$$w_{k,n}(x) = c(k, n) \int_0^{2\pi} d\phi e^{-in\phi} \int_0^\pi W(x, \theta(\gamma, \phi)) p_k^{|n|}(\cos \gamma) \sin \gamma d\gamma, \quad (1.11)$$

$$c(k, n) = \frac{(2k+1)(k-|n|)!}{4\pi(k+|n|)!}, \quad k \in \mathbb{N} \cup \{0\}, \quad n = 0, \pm 1, \dots, \pm k.$$

Under assumption (1.2), for each fixed  $x$ , series (1.6) converge in  $L^2(\mathbb{S}^2)$ ; see e.g. [SW16] (Chapter 4), [Mor98] (Chapter 2), [ZT79].

We consider also

$$\sigma_{\widetilde{W},D,m} = \frac{1}{2\pi} \sum_{k=1}^m \sum_{n=-2k}^{2k} \sup_{x \in D} \left| \frac{w_{2k,n}(x)}{w_{0,0}(x)} \right| \text{ for } m \in \mathbb{N}, \quad (1.12)$$

$$\sigma_{\widetilde{W},D,m} = 0 \text{ for } m = 0,$$

$$\sigma_{\widetilde{W},D,\infty} = \lim_{m \rightarrow \infty} \sigma_{W,D,m}, \quad (1.13)$$

$$W_N(x, \theta(\gamma, \phi)) = \sum_{k=0}^N \sum_{n=-k}^k w_{k,n}(x) Y_k^n(\gamma, \phi), \quad (1.14)$$

$$\widetilde{W}_N(x, \theta(\gamma, \phi)) = \sum_{k=0}^{[N/2]} \sum_{n=-2k}^{2k} w_{2k,n}(x) Y_{2k}^n(\gamma, \phi), \quad (1.15)$$

$$x \in \mathbb{R}^3, \gamma \in [0, \pi], \phi \in [0, 2\pi],$$

where coefficients  $w_{k,n}$  are defined in (1.11),  $[N/2]$  denotes the integer part of  $N/2$ .

Our expansion (1.6) and the related formulas are motivated by their two-dimensional prototypes of [Kun92], [Nov14], [GuNo14].

In the present article we obtained, in particular, the following results under assumptions (1.2), (1.3), (1.4):

1. If  $\sigma_{\widetilde{W},D,\infty} < 1$ , then  $R_W$  is injective and, in addition, the inversion of  $R_W$  is given via formulas (3.2), (3.3); see Subsection 3.1 for details.
2. If  $\sigma_{\widetilde{W},D,\infty} \geq 1$ , then  $f$  can be approximately reconstructed from  $R_W f$  as  $f \approx (R_{\widetilde{W}_N})^{-1} R_W f$ , where  $R_{\widetilde{W}_N}$  is defined according to (1.1) for  $\widetilde{W}_N$  defined by (1.15) for  $N = 2m$ , where  $m$  is chosen as the largest while condition  $\sigma_{W,D,m} < 1$  holds. More precisely, approximate inversion of  $R_W f$  is given via the formulas (3.7), (3.8); see Subsection 3.2 for details.

In addition, if  $W = W_N$  defined by (1.14) and  $\sigma_{\widetilde{W},D,m} < 1$ ,  $m = [N/2]$ , then  $R_{W_N}$  is injective and invertible by formula (3.17); see Subsection 3.3 for details.

In addition, in these results assumptions (1.2), (1.3) can be relaxed as follows:

$$W \in L^\infty(\mathbb{R}^3 \times \mathbb{S}^2), \quad (1.16)$$

$$w_{0,0} \geq c > 0 \text{ on } \mathbb{R}^3, \quad (1.17)$$

where  $w_{0,0}$  is defined (1.3),  $c$  is some positive constant.

Prototypes of these results for the weighted Radon transforms in 2D were obtained in [Kun92], [Nov14], [GuNo14].

The present work also continues studies of [GN16], where approximate inversion of  $R_W$  was realized as  $(R_{W_N})^{-1}$  for  $N = 0$  or by other words as an approximate Chang-type inversion formula. We recall that the original two-dimensional Chang formula ([Cha78]) is often used as an efficient first approximation in the framework of slice-by-slice reconstructions in the single photon emission computed tomography.

In Section 2 we give some notations and preliminary results.

The main results of the present work are presented in detail in Section 3.

In Section 4 we generalize results of Sections 2, 3 for the case of dimension  $n > 3$ .

Proofs of results of Sections 2, 3, 4 are presented in Sections 5, 6.

## 2 Some preliminary results

### 2.1 Some formulas for $R$ and $R^{-1}$

We recall that for the classical Radon transform  $R$  (formula (1.1) for  $W \equiv 1$ ) the following identity holds (see [Nat86], Theorem 1.2, p.13):

$$R(f *_\mathbb{R}^3 g) = Rf *_\mathbb{R} Rg, \quad (2.1)$$

where  $*_{\mathbb{R}^3}, *_\mathbb{R}$  denote the 3D and 1D convolutions (respectively),  $f, g$  are test functions.

The classical Radon inversion formula is defined as follows (see, e.g., [Nat86]):

$$R^{-1}q(x) = -\frac{1}{8\pi^2} \int_{\mathbb{S}^2} q^{(2)}(x\theta, \theta) d\theta, \quad x \in \mathbb{R}^3, \quad (2.2)$$

$$q^{(2)}(s, \theta) = \frac{d^2}{ds^2} q(s, \theta), \quad (s, \theta) \in \mathbb{R} \times \mathbb{S}^2,$$

where  $q$  is a test function on  $\mathbb{R} \times \mathbb{S}^2$ .

In addition, from the Projection theorem (see [Nat86], Theorem 1.1, p.11) it follows that:

$$R^{-1}q(x) \stackrel{def}{=} \frac{1}{(2\pi)^{5/2}} \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_{\mathbb{S}^2} \hat{q}(\rho, \omega) e^{i\rho(x\omega)} d\omega, \quad x \in \mathbb{R}^3, \quad (2.3)$$

$$\hat{q}(s, \theta) \stackrel{def}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} q(t, \theta) e^{-its} dt, \quad (s, \theta) \in \mathbb{R} \times \mathbb{S}^2, \quad (2.4)$$

where  $q(t, \theta)$  is a test function on  $\mathbb{R} \times \mathbb{S}^2$ .

For the case of  $\hat{q}$  even (i.e.,  $\hat{q}(s, \theta) = \hat{q}(-s, -\theta)$ ,  $(s, \theta) \in \mathbb{R} \times \mathbb{S}^2$ , where  $\hat{q}$  is defined in (2.4)), formulas (2.3), (2.4) can be rewritten as follows:

$$R^{-1}q = \frac{1}{2\pi} \mathcal{F}[\hat{q}] = \frac{1}{2\pi} \mathcal{F}^{-1}[\hat{q}], \quad (2.5)$$

where  $\mathcal{F}[\cdot], \mathcal{F}^{-1}[\cdot]$  denote the Fourier transform and its inverse in 3D, respectively, and are defined by the following formulas (in spherical coordinates):

$$\mathcal{F}[q](\xi) \stackrel{def}{=} \frac{1}{(2\pi)^{3/2}} \int_0^{+\infty} \rho^2 d\rho \int_{\mathbb{S}^2} q(\rho, \omega) e^{-i\rho(\xi\omega)} d\omega, \quad (2.6)$$

$$\mathcal{F}^{-1}[q](\xi) \stackrel{def}{=} \frac{1}{(2\pi)^{3/2}} \int_0^{+\infty} \rho^2 d\rho \int_{\mathbb{S}^2} q(\rho, \omega) e^{i\rho(\xi\omega)} d\omega, \quad \xi \in \mathbb{R}^3, \quad (2.7)$$

where  $q(\rho, \omega)$  is a test-function on  $[0, +\infty) \times \mathbb{S}^2$  (identified with  $\mathbb{R}^3$ ).

## 2.2 Symmetrization of $W$

Let

$$A_W f = R^{-1} R_W f, \quad (2.8)$$

where  $R_W$  is defined in (1.1),  $f$  is a test function, satisfying assumptions of (1.4).

Let

$$\widetilde{W}(x, \theta) \stackrel{def}{=} \frac{1}{2} (W(x, \theta) + W(x, -\theta)), \quad x \in \mathbb{R}^3, \theta \in \mathbb{S}^2. \quad (2.9)$$

The following formulas hold:

$$A_W f = R^{-1} R_{\widetilde{W}} f, \quad (2.10)$$

$$R_{\widetilde{W}} f(s, \theta) = \frac{1}{2} (R_W f(s, \theta) + R_W f(-s, -\theta)), \quad (s, \theta) \in \mathbb{R} \times \mathbb{S}^2, \quad (2.11)$$

$$\widetilde{W}(x, \theta(\gamma, \phi)) = \sum_{k=0}^{\infty} \sum_{n=-2k}^{2k} w_{2k,n}(x) Y_{2k}^n(\gamma, \phi), \quad (2.12)$$

$$x \in \mathbb{R}^3, \gamma \in [0, \pi], \phi \in [0, 2\pi].$$

Identity (2.10) is proved in [GN16] in 3D, where  $\widetilde{W}$  is denoted as  $W_{sym}$ .

Identity (2.12) follows from (1.6), (1.7), (2.9) and the following identities:

$$p_k^n(-x) = (-1)^{n+k} p_k^n(x), \quad x \in [-1, 1], \quad (2.13)$$

$$e^{in(\phi+\pi)} = (-1)^n e^{in\phi}, \quad k \in \mathbb{N} \cup \{0\}, n = \overline{-k, k}. \quad (2.14)$$

Note also that  $\widetilde{W}_N$  defined by (1.15) is the approximation of  $\widetilde{W}$  defined by (2.9) and

$$\widetilde{W}_N(x, \cdot) \xrightarrow[N \rightarrow \infty]{L^2(\mathbb{S}^2)} \widetilde{W}(x, \cdot) \text{ for each fixed } x \in \mathbb{R}^3. \quad (2.15)$$

Using formulas (2.11)-(2.12) we reduce inversion of  $R_W$  to the inversion of  $R_{\widetilde{W}}$  defined by (1.1) for  $W = \widetilde{W}$ .

In our work  $A_W f$  (or, more precisely,  $(w_{0,0})^{-1} A_W f$ ) is used as the *initial point* for our iterative inversion algorithms (see Section 3).

Note that the symmetrization  $\widetilde{W}$  of  $W$  arises in (2.10).

In addition, prototypes of (2.8), (2.10), (2.12) for the two-dimensional case can be found in [Kun92], [Nov11].

### 2.3 Operators $Q_{W,D,m}$ and numbers $\sigma_{W,D,m}$

Let

$$c \stackrel{\text{def}}{=} \inf_{x \in D} |w_{0,0}(x)| > 0, \quad (2.16)$$

where the inequality follows from the continuity of  $W$  on  $\overline{D}$  (closure of  $D$ ) and assumption (1.3).

Let  $D$  be the domain of (1.4), and  $\chi_D$  denote the characteristic function of  $D$ , i.e.

$$\chi_D \equiv 1 \text{ on } D, \chi_D \equiv 0 \text{ on } \mathbb{R}^3 \setminus D. \quad (2.17)$$

Let

$$Q_{\widetilde{W},D,m} u(x) \stackrel{\text{def}}{=} R^{-1}(R_{\widetilde{W},D,m} u)(x), \quad m \in \mathbb{N} \quad (2.18)$$

$$Q_{\widetilde{W},D,m} u(x) = 0 \text{ for } m = 0,$$

$$Q_{\widetilde{W},D,\infty} u(x) \stackrel{\text{def}}{=} R^{-1}(R_{\widetilde{W},D,\infty} u)(x), \quad (2.19)$$

where

$$R_{\widetilde{W},D,m} u(s, \theta(\gamma, \phi)) \stackrel{\text{def}}{=} \int_{x\theta=s} \left( \sum_{k=1}^m \sum_{n=-2k}^{2k} \frac{w_{2k,n}(x)}{w_{0,0}(x)} Y_{2k}^n(\gamma, \phi) \right) \chi_D(x) u(x) dx, \quad (2.20)$$

$$\begin{aligned} R_{\widetilde{W},D,\infty} u(s, \theta(\gamma, \phi)) &\stackrel{\text{def}}{=} \lim_{m \rightarrow \infty} R_{\widetilde{W},D,m} u(s, \theta(\gamma, \phi)) \\ &= \int_{x\theta=s} \left( \sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} \frac{w_{2k,n}(x)}{w_{0,0}(x)} Y_{2k}^n(\gamma, \phi) \right) \chi_D(x) u(x) dx, \end{aligned} \quad (2.21)$$

$$x \in \mathbb{R}^3, s \in \mathbb{R}, \theta(\gamma, \phi) \in \mathbb{S}^2,$$

where  $Y_k^n$  are defined in (1.7),  $R^{-1}$  is defined by (2.3) (or (2.2)),  $u$  is a test function,  $w_{0,0}, w_{2k,n}$  are the Fourier-Laplace coefficients defined by (1.11) and  $w_{2k,n}/w_{0,0}, \chi_D$  are considered as multiplication operators on  $\mathbb{R}^3$ . Note also that  $R_{\widetilde{W},D,\infty} f = R_{\widetilde{W}} f$  under assumptions (1.2)-(1.4).

Let

$$d_{2k,n}(x) \stackrel{\text{def}}{=} R^{-1}(\delta(\cdot) Y_k^n)(x), \quad x \in \mathbb{R}^3, k \in \mathbb{N}, n = \overline{-2k, 2k}, \quad (2.22)$$

where  $\delta = \delta(s)$  denotes the 1D Dirac delta function. In (2.22) the action of  $R^{-1}$  on the generalized functions is defined by formula (2.3).

**Lemma 2.1.** *Let  $d_{2k,n}$  be defined by (2.22). Then*

$$d_{2k,n}(x(r, \gamma, \phi)) = \frac{(-1)^k \sqrt{2} \Gamma(\frac{3}{2} + k)}{\pi \Gamma(k)} \frac{Y_{2k}^n(\gamma, \phi)}{r^3}, \quad r > 0, \quad (2.23)$$

where  $\Gamma(\cdot)$  is the Gamma-function,  $x(r, \gamma, \phi)$  is defined by the identity:

$$x(r, \gamma, \phi) = (r \cos \gamma, r \sin \gamma \cos \phi, r \sin \gamma \sin \phi) \in \mathbb{R}^3, \quad \gamma \in [0, \pi], \phi \in [0, 2\pi], r \geq 0. \quad (2.24)$$

In addition, the following inequality holds:

$$|\mathcal{F}[d_{2k,n}](\xi)| \leq 1/2\pi, \quad \xi \in \mathbb{R}^3, \quad (2.25)$$

where  $\mathcal{F}[\cdot]$  is the Fourier transform, defined in (2.6).

The following lemma gives some useful expressions for operators  $Q_{\widetilde{W},D,m}, Q_{\widetilde{W},D,\infty}$  defined in (2.18), (2.19).

**Lemma 2.2.** *Let operators  $Q_{\widetilde{W},D,m}, Q_{\widetilde{W},D,\infty}$  be defined by (2.18), (2.19), respectively, and  $u$  be a test function satisfying (1.4). Then*

$$Q_{\widetilde{W},D,m} u = \sum_{k=1}^m \sum_{n=-2k}^{2k} d_{2k,n} \ast_{\mathbb{R}^3} \frac{w_{2k,n}}{w_{0,0}} u, \quad (2.26)$$

$$Q_{\widetilde{W},D,\infty} u = \sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} d_{2k,n} \ast_{\mathbb{R}^3} \frac{w_{2k,n}}{w_{0,0}} u, \quad (2.27)$$

where coefficients  $w_{k,n}$  are defined in (1.11),  $d_{2k,n}$  is defined by (2.23) (or equivalently by (2.22)),  $\ast_{\mathbb{R}^3}$  denotes the convolution in  $3D$ .

*Remark 2.1.* Convolution terms in the right-hand side of (2.26), (2.27) are well defined functions in  $L^2(\mathbb{R}^3)$ . This follows from identity (2.23) and the Calderón-Zygmund theorem for convolution-type operators with singular kernels (see [Kna05], p.83, Theorem 3.26).

The following lemma shows that  $Q_{\widetilde{W},D,m}$ ,  $Q_{\widetilde{W},D,\infty}$  are well-defined operators in  $L^2(\mathbb{R}^3)$ .

**Lemma 2.3.** *Operator  $Q_{\widetilde{W},D,m}$  defined by (2.26) (or equivalently by (2.18)) is a linear bounded operator in  $L^2(\mathbb{R}^3)$  and the following estimate holds:*

$$\|Q_{\widetilde{W},D,m}\|_{L^2(\mathbb{R}^3)\rightarrow L^2(\mathbb{R}^3)} \leq \sigma_{\widetilde{W},D,m}, \quad (2.28)$$

where  $\sigma_{\widetilde{W},D,m}$  is defined by (1.12).

If  $\sigma_{\widetilde{W},D,\infty} < +\infty$ , where  $\sigma_{\widetilde{W},D,\infty}$  is defined by (1.13), then  $Q_{\widetilde{W},D,\infty}$ , defined by (2.27) (or equivalently by (2.19)), is a linear bounded operator in  $L^2(\mathbb{R}^3)$  and the following estimate holds:

$$\|Q_{\widetilde{W},D,\infty}\|_{L^2(\mathbb{R}^3)\rightarrow L^2(\mathbb{R}^3)} \leq \sigma_{\widetilde{W},D,\infty}. \quad (2.29)$$

**Lemma 2.4.** *Let*

$$\sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} \left\| \frac{w_{2k,n}}{w_{0,0}} \right\|_{L^2(D)} < +\infty, \quad (2.30)$$

where  $w_{k,n}$  are defined in (1.11). Then

$$R^{-1}R_W f \in L^2(\mathbb{R}^3). \quad (2.31)$$

In addition, the following formula holds:

$$R^{-1}R_W f = w_{0,0}f + \sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} d_{2k,n} *_{\mathbb{R}^3} w_{2k,n} f = (I + Q_{\widetilde{W},D,\infty})(w_{0,0}f), \quad (2.32)$$

where  $f$  satisfies (1.4), operator  $R^{-1}$  is defined by (2.3) and  $Q_{\widetilde{W},D,\infty}$  is given by (2.27).

### 3 Main results

#### 3.1 Case of $\sigma_{\widetilde{W},D,\infty} < 1$

Let

$$\sigma_{\widetilde{W},D,\infty} < 1, \quad (3.1)$$

where  $\sigma_{\widetilde{W},D,\infty}$  is defined by (1.13).

Inequality (3.1) and upper bound (2.29) in Lemma 2.3 imply that operator  $I + Q_{\widetilde{W},D,\infty}$  is continuously invertible in  $L^2(\mathbb{R}^3)$  and the following identity holds (in the sense of the operator norm in  $L^2(\mathbb{R}^3)$ ):

$$(I + Q_{\widetilde{W},D,\infty})^{-1} = I + \sum_{j=1}^{\infty} (-Q_{\widetilde{W},D,\infty})^j, \quad (3.2)$$

where  $I$  is the identity operator in  $L^2(\mathbb{R}^3)$ .

**Theorem 3.1.** *Let conditions (1.2)-(1.4), (3.1) be fulfilled. Then  $R_W$ , defined by (1.1), is injective and the following exact inversion formula holds:*

$$f = (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,\infty})^{-1}R^{-1}R_W f, \quad (3.3)$$

where  $w_{0,0}$  is defined in (1.3),  $R^{-1}$  is defined in (2.2), operator  $(I + Q_{\widetilde{W},D,\infty})^{-1}$  is given in (3.2).

*Remark 3.1.* Formula (3.3) can be considered as the following linear integral equation for the  $w_{0,0}f$ :

$$w_{0,0}f + Q_{\widetilde{W},D,\infty}(w_{0,0}f) = R^{-1}R_W f. \quad (3.4)$$

Inequality (3.1) and identity (3.2) imply that equation (3.4) can be solved by the method of successive approximations.

One can see that, under conditions (1.2)-(1.4) and (3.1), Theorem 3.1 gives an exact inversion of  $R_W$ . However, condition (3.1) is not always fulfilled in practice; see [GuNo14] for related numerical analysis in 2D. If condition (3.1) is not fulfilled, then, approximating  $W$  by finite Fourier series, in a similar way with [Cha78], [Kun92], [Nov14], [GuNo14] we suggest approximate inversion of  $R_W$ ; see Subsections 3.2, 3.3.

### 3.2 Case of $1 \leq \sigma_{\widetilde{W},D,\infty} < +\infty$

Let

$$\sigma_{\widetilde{W},D,m} < 1, \text{ for some } m \in \mathbb{N} \cup \{0\}, \quad (3.5)$$

$$\sigma_{\widetilde{W},D,\infty} < +\infty, \quad (3.6)$$

where  $\sigma_{\widetilde{W},D,m}$  is defined by (1.12),  $\sigma_{\widetilde{W},D,\infty}$  is defined by (1.13).

Inequality (3.5) and upper bound (2.28) in Lemma 2.3 imply that  $I + Q_{\widetilde{W},D,m}$  is continuously invertible in  $L^2(\mathbb{R}^3)$  and the following identity holds (in the sense of the operator norm in  $L^2(\mathbb{R}^3)$ ):

$$(I + Q_{\widetilde{W},D,m})^{-1} = I + \sum_{j=1}^{\infty} (-Q_{\widetilde{W},D,m})^j, \quad (3.7)$$

where  $I$  is the identity operator in  $L^2(\mathbb{R}^3)$ .

**Theorem 3.2.** *Let conditions (1.2)-(1.4), (3.5), (3.6) be fulfilled. Then*

$$f \approx f_m \stackrel{\text{def}}{=} (w_{0,0})^{-1} (I + Q_{\widetilde{W},D,m})^{-1} R^{-1} R_W f, \quad (3.8)$$

$$f = f_m - (w_{0,0})^{-1} (I + Q_{\widetilde{W},D,m})^{-1} R^{-1} R_{\delta W_m} f, \quad (3.9)$$

$$\|f - f_m\|_{L^2(D)} \leq \frac{\|f\|_{\infty}}{2\pi c(1 - \sigma_{\widetilde{W},D,m})} \sum_{k=m+1}^{\infty} \sum_{n=-2k}^{2k} \|w_{2k,n}\|_{L^2(D)} < +\infty, \quad (3.10)$$

where

$$\delta W_m(x, \theta(\gamma, \phi)) \stackrel{\text{def}}{=} W(x, \theta(\gamma, \phi)) - \sum_{k=0}^{2m+1} \sum_{n=-k}^k w_{k,n}(x) Y_k^n(\gamma, \phi), \quad (3.11)$$

$$x \in \mathbb{R}^3, \gamma \in [0, \pi], \phi \in [0, 2\pi], m \in \mathbb{N} \cup \{0\}, \quad (3.12)$$

$w_{0,0}$  is defined in (1.3),  $\theta(\gamma, \phi)$  is defined in (1.8),  $Y_k^n$  are defined in (1.7), operator  $(I + Q_{\widetilde{W},D,m})^{-1}$  is given in (3.7), constant  $c$  is defined in (2.16).

*Remark 3.2.* Formula (3.8) can be considered as the following linear integral equation for  $w_{0,0}f$ :

$$w_{0,0}f + Q_{\widetilde{W},D,m}(w_{0,0}f) = R^{-1} R_W f. \quad (3.13)$$

Inequality (3.5) and identity (3.7) imply that equation (3.13) is solvable by the method of successive approximations.

Note also that condition (3.6) can be relaxed to the following one:

$$\sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} \left\| \frac{w_{2k,n}}{w_{0,0}} \right\|_{L^2(D)} < +\infty, \quad (3.14)$$

where  $w_{2k,n}$  are defined in (1.11).

Formula (3.8) is an extension to 3D of the Chang-type two-dimensional inversion formulas in [Cha78], [Nov14], [GuNo14]. In addition, formula (3.8) is an extension of approximate inversion formula in [GN16], where this formula was given for  $m = 0$ .

If (3.5) is fulfilled for some  $m \geq 1$ , then  $f_m$  is a refinement of the Chang-type approximation  $f_0$  and, more generally,  $f_j$  is a refinement of  $f_i$  for  $0 \leq i < j \leq m$ . In addition,  $f_j = f_i$  if  $w_{2k,n} \equiv 0$  for  $i < k \leq j$ ,  $n = -2k, 2k$ . Thus, we propose the following approximate reconstruction of  $f$  from  $R_W f$ :

(i) find maximal  $m$  such that (3.5) is still efficiently fulfilled,

(ii) approximately reconstruct  $f$  by  $f_m$  using (3.8).

### 3.3 Exact inversion for finite Fourier series weights

Let

$$W = W_N, N \in \mathbb{N} \cup \{0\}, \quad (3.15)$$

where  $W_N$  is defined by (1.14).

Suppose that

$$\sigma_{\widetilde{W},D,m} < 1 \text{ for } m = [N/2], \quad (3.16)$$

where  $\sigma_{\widetilde{W},D,m}$  is defined by (1.12).

**Theorem 3.3.** *Let conditions (1.2)-(1.4), (3.15), (3.16) be fulfilled. Then  $R_W$  defined by (1.1) is injective and the following exact inversion formula holds:*

$$f = (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,m})^{-1}R^{-1}R_W f, \quad (3.17)$$

where  $w_{0,0}$  is defined in (1.2),  $(I + Q_{\widetilde{W},D,m})^{-1}$  is given in (3.7),  $R^{-1}$  is defined by (2.2).

*Remark 3.3.* Formula (3.17) can be considered as the following linear integral equation for  $w_{0,0}f$ :

$$w_{0,0}f + Q_{\widetilde{W},D,m}(w_{0,0}f) = R^{-1}R_W f. \quad (3.18)$$

Identity (3.16) imply that (3.18) can be solved by the method of successive approximations.

*Remark 3.4.* Note that Theorems 3.1, 3.2, 3.3 remain valid under assumptions (1.16), (1.17) in place of (1.2), (1.3). This follows from the fact that in the proofs in Sections 5, 6 it is required only existence of integral transforms, given by operators  $R_W$ ,  $R^{-1}$ ,  $\mathcal{F}[\cdot]$ ,  $\mathcal{F}^{-1}[\cdot]$ ,  $Q_{W,D,\infty}$ ,  $Q_{W,D,m}$  and their compositions (see Subsections 2.1, 2.3) and of uniform upper bound on  $(w_{0,0})^{-1}$ .

## 4 Generalization to multidimensions

Definition (1.1) and assumptions (1.2)-(1.4) are naturally extended as follows to the case of dimension  $n > 3$ :

$$R_W f(s, \theta) = \int_{x\theta=s} W(x, \theta) f(x) dx, \quad (s, \theta) \in \mathbb{R} \times \mathbb{S}^{n-1}, \quad x \in \mathbb{R}^n, \quad (4.1)$$

$$W \in L^\infty(\mathbb{R}^n \times \mathbb{S}^{n-1}), \quad (4.2)$$

$$w_{0,0}(x) \stackrel{def}{=} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} W(x, \theta) d\theta, \quad w_{0,0} \geq c > 0, \quad (4.3)$$

$$f \in L^\infty(\mathbb{R}^n), \quad \text{supp } f \subset D, \quad (4.4)$$

where  $|\mathbb{S}^{n-1}|$  denotes the standard Euclidean volume of  $\mathbb{S}^{n-1}$ ,  $c$  is some positive constant,  $D$  is an open bounded domain in  $\mathbb{R}^n$ .

For the weight  $W$  we consider the Fourier-Laplace expansion:

$$W(x, \theta) = \sum_{k=0}^{\infty} \sum_{i=0}^{a_{k,n}-1} w_{k,i}(x) Y_k^i(\theta), \quad x \in \mathbb{R}^n, \quad \theta \in \mathbb{S}^{n-1}, \quad (4.5)$$

where

$$w_{k,i}(x) = \|Y_k^i\|_{L^2(\mathbb{S}^{n-1})}^{-2} \int_{\mathbb{S}^{n-1}} W(x, \theta) \overline{Y_k^i(\theta)} d\theta, \quad (4.6)$$

$$a_{k,n+1} = \frac{(n+k)!}{k!n!} - \frac{(n+k-2)!}{(k-2)!n!}, \quad n, k \geq 2; \quad a_{0,n} = 1, \quad a_{1,n} = n, \quad (4.7)$$

where  $\{Y_k^i \mid k = \overline{0, \infty}, i = \overline{0, a_{k,n}-1}\}$  is the Fourier-Laplace basis of harmonics on  $\mathbb{S}^{n-1}$ ,  $\overline{Y_k^i}$  denotes the complex conjugate of  $Y_k^i$ ; see [SW16], [Mor98]. In the present work we choose the basis  $Y_k^i$  as in [Hig87] without normalizing constants  ${}_n c_L^i$  (i.e.,  $\{Y_k^i\}$  are the products of the Legendre polynomials with one complex exponent and without any additional constants).

In dimension  $n > 3$ , formulas (1.12), (1.13), (2.18), (2.19), are rewritten as follows:

$$\sigma_{\widetilde{W},D,m} \stackrel{def}{=} (2\pi)^{(1-n)/2} \sum_{k=1}^m \sum_{i=0}^{a_{2k,n}-1} \sup_{x \in D} \left| \frac{w_{2k,i}(x)}{w_{0,0}(x)} \right|, \quad (4.8)$$

$$\sigma_{\widetilde{W},D,\infty} \stackrel{def}{=} \lim_{m \rightarrow +\infty} \sigma_{\widetilde{W},D,m} \quad (4.9)$$

$$Q_{\widetilde{W},D,m} u(x) \stackrel{def}{=} R^{-1}(R_{\widetilde{W},D,m} u)(x), \quad m \in \mathbb{N} \quad (4.10)$$

$$Q_{\widetilde{W},D,m} u(x) = 0 \quad \text{for } m = 0,$$

$$Q_{\widetilde{W},D,\infty} u(x) \stackrel{def}{=} R^{-1}(R_{\widetilde{W},D,\infty} u)(x), \quad (4.11)$$

where

$$R_{\widetilde{W},D,m} u(s, \theta) \stackrel{def}{=} \int_{x\theta=s} \left( \sum_{k=1}^m \sum_{i=0}^{a_{2k,n}-1} \frac{w_{2k,i}(x)}{w_{0,0}(x)} Y_{2k}^i(\theta) \right) \chi_D(x) u(x) dx, \quad (4.12)$$

$$R_{\widetilde{W},D,\infty} u(s, \theta) \stackrel{def}{=} \lim_{m \rightarrow \infty} R_{\widetilde{W},D,m} u(s, \theta), \quad (4.13)$$

$x \in \mathbb{R}^n, s \in \mathbb{R}, \theta \in \mathbb{S}^{n-1},$

where  $R^{-1}$  is defined further in (4.14).

Under assumptions (4.2), (4.4) series of (4.5) converge in  $L^2(\mathbb{S}^{n-1})$ ; see e.g. [SW16] (Chapter 4), [Mor98] (Chapter 2), [ZT79].

Formula (2.3) is extended as follows:

$$R^{-1}q(x) = (2\pi)^{1/2-n} \int_{\mathbb{R}} \frac{|\rho|^{n-1}}{2} \int_{\mathbb{S}^{n-1}} \hat{q}(\rho, \theta) e^{i\rho(x\theta)} d\theta, \quad x \in \mathbb{R}^n, \quad (4.14)$$

where  $q(s, \theta)$  is a test function on  $\mathbb{R} \times \mathbb{S}^{n-1}$ ,  $\hat{q}(s, \theta)$  is defined as in (2.4) (with  $\mathbb{S}^{n-1}$  in place of  $\mathbb{S}^2$ ).

The Fourier transforms, defined in (2.6), (2.7), are extended as follows:

$$\mathcal{F}[q](\xi) \stackrel{def}{=} (2\pi)^{-n/2} \int_0^{+\infty} \rho^{n-1} d\rho \int_{\mathbb{S}^{n-1}} q(\rho, \omega) e^{-i\rho(\xi\omega)} d\omega, \quad (4.15)$$

$$\mathcal{F}^{-1}[q](\xi) \stackrel{def}{=} (2\pi)^{-n/2} \int_0^{+\infty} \rho^{n-1} d\rho \int_{\mathbb{S}^{n-1}} q(\rho, \omega) e^{i\rho(\xi\omega)} d\omega, \quad \xi \in \mathbb{R}^n, \quad (4.16)$$

where  $q(\rho, \omega)$  is a test function on  $[0, +\infty) \times \mathbb{S}^{n-1}$  (identified with  $\mathbb{R}^n$ ).

In dimension  $n > 3$ , formulas (2.8)-(2.12) remain valid with  $Y_k^m$  defined in (1.7) replaced by general basis of spherical harmonics  $\{Y_k^i\}$  on  $\mathbb{S}^{n-1}$ . In particular, the following multidimensional analog of formula (2.12) holds:

$$Y_k^i(-\theta) = (-1)^k Y_k^i(\theta), \quad \theta \in \mathbb{S}^{n-1}, \quad k \in \mathbb{N} \cup \{0\}, \quad i = \overline{0, a_{k,n} - 1}, \quad (4.17)$$

where  $a_{k,n}$  is defined by (4.7). Identity (4.17) reflects the fact that  $Y_k^i(\theta) = Y_k^i(\theta_1, \theta_2, \dots, \theta_n)$ ,  $\theta = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{S}^{n-1}$ ,  $i = \overline{0, a_{k,n} - 1}$  is a homogenous polynomial of degree  $k$ , see e.g. [SW16], [Mor98].

Formula (2.22) is now rewritten as follows:

$$d_{2k,i}(x) \stackrel{def}{=} R^{-1}(\delta(\cdot) Y_{2k}^i)(x), \quad x \in \mathbb{R}^n, \quad i = \overline{0, a_{k,n} - 1}. \quad (4.18)$$

Results of Lemma 2.1 remain valid with formula (2.23) replaced by the following one:

$$d_{2k,i}(r, \theta) = c(k, n) \frac{(-1)^k Y_{2k}^i(\theta)}{r^n}, \quad r > 0, \quad \theta \in \mathbb{S}^{n-1}, \quad (4.19)$$

where

$$c(k, n) = \frac{\sqrt{2}\pi^{(1-n)/2} \Gamma(k + \frac{1}{2}) \Gamma(k + \frac{n}{2})}{\Gamma(k) \Gamma(k + \frac{n-1}{2})} \cdot \left( \frac{\Gamma(k+1)}{\Gamma(k + \frac{1}{2})} \right)^{n-2}, \quad (4.20)$$

$\Gamma(\cdot)$  is the Gamma function.

In addition, inequality (2.25) is rewritten as follows:

$$|\mathcal{F}[d_{2k,i}](\xi)| \leq (2\pi)^{(1-n)/2}, \quad \xi \in \mathbb{R}^n, \quad (4.21)$$

where  $\mathcal{F}[\cdot]$  is the Fourier transform defined in (4.15). The constant  $c(k, n)$  in (4.20) is obtained using formulas (4.14), (4.18) and Theorems 1, 2 in [Gon16].

The results of Lemma 2.2 remain valid in the case of dimension  $n > 3$ , with formulas (2.26), (2.27) rewritten as follows:

$$Q_{\widetilde{W},D,m} u = \sum_{k=1}^m \sum_{i=0}^{a_{k,n}-1} d_{2k,i} *_{\mathbb{R}^n} \frac{w_{2k,i}}{w_{0,0}} u, \quad (4.22)$$

$$Q_{\widetilde{W},D,\infty} u = \sum_{k=1}^{\infty} \sum_{i=0}^{a_{k,n}-1} d_{2k,i} *_{\mathbb{R}^n} \frac{w_{2k,i}}{w_{0,0}} u, \quad (4.23)$$

where coefficients  $w_{2k,i}$ ,  $w_{0,0}$  are defined in (4.6),  $a_{k,n}$  is defined in (4.7),  $d_{2k,i}$  is defined in (4.19),  $*_{\mathbb{R}^n}$  denotes the convolution in  $\mathbb{R}^n$ .

The result of Lemma 2.3 remains valid with  $\mathbb{R}^3$  replaced by  $\mathbb{R}^n$ ,  $n > 3$ , where we use definitions (4.8), (4.10), (4.11).

Assumption (2.30) in Lemma 2.4 is rewritten now as follows:

$$\sum_{k=1}^{\infty} \sum_{i=0}^{a_{k,n}-1} \left\| \frac{w_{2k,i}}{w_{0,0}} \right\|_{L^2(D)} < +\infty. \quad (4.24)$$

Under assumption (4.24), property (2.31) of Lemma 2.4 remains valid in dimension  $n > 3$ . In particular, formula (2.32) is rewritten as follows:

$$R^{-1}R_W f = w_{0,0}f + \sum_{k=1}^{\infty} \sum_{i=0}^{a_{k,n}-1} d_{2k,i} *_{\mathbb{R}^n} w_{2k,i}f, \quad (4.25)$$

where  $R^{-1}$  is defined in (4.14),  $f$  is a test function satisfying (4.4),  $d_{2k,i}$  is now defined in (4.19).

Using formulas and notations from (4.5)-(4.23) we obtain straightforward extensions of Theorems 3.1, 3.2, 3.3.

- The result of Theorem 3.1 remains valid in dimension  $n > 3$ , under assumptions (4.2)-(4.4) and under condition (3.1), where  $w_{0,0}$  is defined in (4.3),  $R^{-1}$  is defined in (4.14),  $\sigma_{\widetilde{W},D,\infty}$  is defined in (4.9),  $Q_{\widetilde{W},D,\infty}$  is defined in (4.11).
- The result of Theorem 3.2 remains valid in dimension  $n > 3$ , under assumptions (4.2)-(4.4) and under conditions (3.5), (3.6), where  $w_{0,0}$  is defined in (4.3),  $R^{-1}$  is defined in (4.14),  $\sigma_{\widetilde{W},D,m}$  is defined in (4.8),  $Q_{\widetilde{W},D,m}$  is defined in (4.10) and where formulas (3.9)-(3.11) are rewritten as follows:

$$\|f - f_m\|_{L^2(D)} \leq \frac{\|f\|_{\infty}}{(2\pi)^{(n-1)/2} c(1 - \sigma_{\widetilde{W},D,m})} \sum_{k=m+1}^{\infty} \sum_{i=0}^{a_{k,n}-1} \|w_{2k,i}\|_{L^2(D)} < +\infty, \quad (4.26)$$

$$\delta W_m(x, \theta) \stackrel{def}{=} W(x, \theta) - \sum_{k=0}^{2m+1} \sum_{i=0}^{a_{k,n}-1} w_{k,i}(x) Y_k^i(\theta), \quad (4.27)$$

$$x \in \mathbb{R}^n, \theta \in \mathbb{S}^{n-1}.$$

- The result of Theorem 3.3 remains valid in dimension  $n > 3$ , under assumptions (4.2)-(4.4) and under conditions (3.15), (3.16), where  $w_{0,0}$  is defined in (4.3),  $R^{-1}$  is defined in (4.14),  $\sigma_{\widetilde{W},D,m}$  is defined in (4.8),  $Q_{\widetilde{W},D,m}$  is defined in (4.10).

The related proofs are the straightforward extensions to the case of dimension  $n > 3$  of proofs in Section 6 for  $n = 3$ .

## 5 Proofs of Lemma 2.1, 2.2, 2.3, 2.4

### 5.1 Proof of Lemma 2.1

We consider  $x(r, \gamma, \phi)$  defined by (2.24) and  $\omega(\gamma, \phi) = x(1, \gamma, \phi)$  (i.e.,  $\omega \in \mathbb{S}^2$ ).

Identity (2.24) implies the following expression for the scalar product  $(x\omega)$  in spherical coordinates in  $\mathbb{R}^3$ :

$$(x\omega) = (x(r, \tilde{\gamma}, \tilde{\phi}), \omega(\gamma, \phi)) = r(\cos \gamma \cos \tilde{\gamma} + \sin \gamma \sin \tilde{\gamma} \cos(\phi - \tilde{\phi})), \quad (5.1)$$

where  $\gamma, \tilde{\gamma} \in [0, \pi]$ ,  $\phi, \tilde{\phi} \in [0, 2\pi]$ ,  $r \geq 0$ .

From formulas (1.7), (2.3), (2.22), (5.1) it follows that

$$\begin{aligned}
(2\pi)^{5/2}d_{2k,n}(x(r, \tilde{\gamma}, \tilde{\phi})) &= \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_{\mathbb{S}^2} e^{i\rho(x\omega(\gamma, \phi))} Y_{2k}^n(\gamma, \phi) d\omega(\gamma, \phi) \\
&= \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_0^\pi \sin(\gamma) p_k^{|n|}(\cos \gamma) d\gamma \int_0^{2\pi} e^{i\rho(x\omega(\gamma, \phi)) + in\phi} d\phi \\
&= e^{in\tilde{\phi}} \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_0^\pi \sin(\gamma) p_k^{|n|}(\cos \gamma) d\gamma \int_0^{2\pi} e^{i\rho r \sin \gamma \sin \tilde{\gamma} \cos(\phi - \tilde{\phi}) + in(\phi - \tilde{\phi})} d\phi \\
&= e^{in\tilde{\phi}} \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_0^\pi \sin(\gamma) p_k^{|n|}(\cos \gamma) d\gamma \int_0^{2\pi} e^{i\rho r \sin \gamma \sin \tilde{\gamma} \cos \phi + in\phi} d\phi \\
&= 2\pi e^{in(\tilde{\phi} - \pi/2)} (-1)^n \int_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int_0^\pi \sin(\gamma) p_{2k}^{|n|}(\cos \gamma) e^{i\rho r \cos \gamma \cos \tilde{\gamma}} J_n(\rho r \sin \gamma \sin \tilde{\gamma}) d\gamma, \quad (5.2)
\end{aligned}$$

where  $J_n$  is the  $n$ -th standard Bessel function of the first kind; see e.g. [Tem11]. In (5.2) we used the well known formula for the Bessel function  $J_n$ :

$$J_n(t) \stackrel{def}{=} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\phi - t \sin \phi} d\phi = \frac{(-1)^n e^{in\pi/2}}{2\pi} \int_0^{2\pi} e^{in\phi + t \cos \phi} d\phi.$$

The integral in  $d\gamma$  in the right-hand side of (5.2) was considered in [NPF+06], where the following exact analytic solution was given:

$$\int_0^\pi \sin(\gamma) p_{2k}^{|n|}(\cos \gamma) e^{i\rho r \cos \gamma \cos \tilde{\gamma}} J_n(\rho r \sin \gamma \sin \tilde{\gamma}) d\gamma = 2i^{2k-n} p_{2k}^{|n|}(\cos \tilde{\gamma}) j_{2k}(\rho r), \quad (5.3)$$

where  $j_{2k}$  is the standard spherical Bessel function of order  $2k$ ; see e.g. [Tem11].

From identities (5.2), (5.3) it follows that:

$$\begin{aligned}
d_{2k,n}(x(r, \tilde{\gamma}, \tilde{\phi})) &= \frac{(-1)^k}{(2\pi)^{3/2}} p_{2k}^{|n|}(\cos \tilde{\gamma}) e^{in\tilde{\phi}} \int_{\mathbb{R}} \rho^2 j_{2k}(\rho r) d\rho \\
&= \frac{4\sqrt{\pi}(-1)^k \Gamma(\frac{3}{2} + k)}{(2\pi)^{3/2} \Gamma(k)} \frac{p_{2k}^{|n|}(\cos \tilde{\gamma}) e^{in\tilde{\phi}}}{r^3}, \quad r > 0. \quad (5.4)
\end{aligned}$$

where  $\Gamma(\cdot)$  is the Gamma function.

Defenition (1.7) and identity (5.4) imply formula (2.23).

Formulas (2.5), (2.13), (2.14), (2.22) imply that

$$(2\pi)d_{2k,n}(x(r, \tilde{\gamma}, \tilde{\phi})) = \mathcal{F}^{-1}[Y_{2k}^n](x(r, \tilde{\gamma}, \tilde{\phi})), \quad r > 0, \quad \tilde{\gamma} \in [0, \pi], \quad \tilde{\phi} \in [0, 2\pi], \quad (5.5)$$

where  $\mathcal{F}^{-1}[\cdot]$  is defined in (2.7).

From the invertibility of the Fourier transform defined in (2.6) and identity (5.5) the following identity holds:

$$2\pi \mathcal{F}[d_{2k,n}] = \mathcal{F} \mathcal{F}^{-1}[Y_{2k}^n] = Y_{2k}^n. \quad (5.6)$$

For  $Y_k^n$  defined in (1.7) the following inequality holds (see e.g. [SW16]):

$$|Y_k^n(\gamma, \phi)| \leq 1, \quad \gamma \in [0, \pi], \quad \phi \in [0, 2\pi]. \quad (5.7)$$

Identities (5.5) and inequality (5.7) imply (2.25).

Note that  $|\mathcal{F}[d_{2k,n}](\xi)|$ ,  $\xi \in \mathbb{R}^3$ , is uniformly bounded by  $1/2\pi$  except only one point  $\xi = 0$ , where direction  $\xi/|\xi| \in \mathbb{S}^2$  is not defined. However, point  $\xi = 0$  is of Lebesgue measure zero and  $\mathcal{F}[d_{2k,n}]$  can be defined with any value at the origin in  $\mathbb{R}^3$ .

Lemma 2.1 is proved.

## 5.2 Proof of Lemma 2.2

From identity (2.18) it follows that

$$\begin{aligned}
Q_{\widetilde{W},D,m}u &= R^{-1} \left( \sum_{k=1}^m \sum_{n=-2k}^{2k} Y_{2k}^n R \left( \frac{w_{2k,n}}{w_{0,0}} \chi_D u \right) \right) \\
&= R^{-1} \left( \sum_{k=1}^m \sum_{n=-2k}^{2k} (\delta(\cdot) Y_{2k}^n) *_{\mathbb{R}} R \left( \frac{w_{2k,n}}{w_{0,0}} \chi_D u \right) \right) \\
&= R^{-1} \left( \sum_{k=1}^m \sum_{n=-2k}^{2k} R(d_{2k,n}) *_{\mathbb{R}} R \left( \frac{w_{2k,n}}{w_{0,0}} \chi_D u \right) \right)
\end{aligned} \tag{5.8}$$

where  $*_{\mathbb{R}}$  denotes the 1D convolution,  $\delta = \delta(s)$  is the 1D Dirac delta function,  $d_{2k,n}$  is defined by (2.22).

Identities (2.1), (5.8) imply (2.26).

For the operator  $Q_{\widetilde{W},D,\infty}$  defined by (2.19) we proceed according to identity (5.8) with  $m \rightarrow +\infty$ . Identities (2.1), (5.8) and linearity of operator  $R^{-1}$  defined by (2.2) imply (2.27).

Lemma 2.2 is proved.

## 5.3 Proof of Lemma 2.3

From formula (2.26) and the fact that the Fourier transform defined in (2.6) does not change the  $L^2$ -norm we obtain:

$$\begin{aligned}
\|Q_{\widetilde{W},D,m}u\|_{L^2(\mathbb{R}^3)} &\leq \sum_{k=1}^m \sum_{n=-2k}^{2k} \left\| d_{2k,n} *_{\mathbb{R}^3} \frac{w_{2k,n}}{w_{0,0}} \chi_D u \right\|_{L^2(\mathbb{R}^3)} \\
&= \sum_{k=1}^m \sum_{n=-2k}^{2k} \left\| \mathcal{F}[d_{2k,n}] \mathcal{F} \left[ \frac{w_{2k,n}}{w_{0,0}} \chi_D u \right] \right\|_{L^2(\mathbb{R}^3)}.
\end{aligned} \tag{5.9}$$

From inequalities (2.25), (5.9) we obtain:

$$\begin{aligned}
\|Q_{\widetilde{W},D,m}u\|_{L^2(\mathbb{R}^3)} &\leq \frac{1}{2\pi} \sum_{k=1}^m \sum_{n=-2k}^{2k} \left\| \mathcal{F} \left( \frac{w_{2k,n}}{w_{0,0}} \chi_D u \right) \right\|_{L^2(\mathbb{R}^3)} \\
&= \frac{1}{2\pi} \sum_{k=1}^m \sum_{n=-2k}^{2k} \left\| \frac{w_{2k,n}}{w_{0,0}} \chi_D u \right\|_{L^2(\mathbb{R}^3)} \\
&\leq \sigma_{\widetilde{W},D,m} \|u\|_{L^2(D)},
\end{aligned} \tag{5.10}$$

where  $\sigma_{\widetilde{W},D,m}$  is defined by (1.12).

Inequality (5.10) implies (2.28).

Estimate (2.29) follows from definition (2.19), formula (2.27), linearity of operator  $R^{-1}$  defined by (2.2) and inequalities (5.9), (5.10) for  $m \rightarrow +\infty$ .

Lemma 2.3 is proved.

## 5.4 Proof of Lemma 2.4

From formulas (1.1), (1.6) it follows that

$$R_W f(s, \theta(\gamma, \phi)) = \sum_{k=0}^{\infty} \sum_{n=-k}^k Y_k^n(\gamma, \phi) R(w_{k,n} f)(s, \theta(\gamma, \phi)), \tag{5.11}$$

$$s \in \mathbb{R}, \gamma \in [0, \pi], \phi \in [0, 2\pi], \tag{5.12}$$

where  $\theta(\gamma, \phi)$  is defined in (1.8),  $Y_k^n(\gamma, \phi)$  are defined by (1.7),  $w_{k,n}$  are defined in (1.11).

Formula (2.32) follows from formulas (2.8), (2.10), (2.12), (2.19) and formula (2.27) in Lemma 2.2, where test function  $u$  is replaced by  $w_{0,0}u$ .

From inequality (2.25), formulas (2.1), (2.32) and the fact that the Fourier transform defined in (2.6) does not change the  $L^2$ -norm we obtain:

$$\begin{aligned}
\|R^{-1}R_W f\|_{L^2(\mathbb{R}^3)} &\leq \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} \|d_{2k,n} *_{\mathbb{R}^3} w_{2k,n}f\|_{L^2(\mathbb{R}^3)} \\
&= \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} \|\mathcal{F}[d_{2k,n}]\mathcal{F}[w_{2k,n}f]\|_{L^2(\mathbb{R}^3)} \\
&\leq \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \frac{\|f\|_{\infty}}{2\pi} \sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} \|w_{2k,n}\|_{L^2(D)},
\end{aligned} \tag{5.13}$$

where  $\|\cdot\|_{\infty}$  denotes the  $L^{\infty}$ -norm,  $\mathcal{F}[\cdot]$  is defined in (2.6).

From assumption (2.30) and formula (2.16) it follows that

$$\sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} \|w_{2k,n}\|_{L^2(D)} < +\infty. \tag{5.14}$$

Assumptions (1.2)-(1.4) and inequalities (5.13), (5.14) imply that

$$\|R^{-1}R_W f\|_{L^2(\mathbb{R}^3)} \leq \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \frac{\|f\|_{\infty}}{2\pi} \sum_{k=1}^{\infty} \sum_{n=-2k}^{2k} \|w_{2k,n}\|_{L^2(D)} < +\infty. \tag{5.15}$$

Lemma 2 is proved.

## 6 Proofs of Theorems 3.1, 3.2, 3.3

### 6.1 Proof of Theorem 3.1

From formulas (2.19), (2.27), (2.32) we obtain:

$$R^{-1}R_W f = (I + Q_{\widetilde{W},D,\infty})(w_{0,0}f), \tag{6.1}$$

where  $I$  is the identity operator in  $L^2(\mathbb{R}^3)$ ,  $w_{0,0}$  is defined in (1.3).

By assumption (3.1) property (2.31) of Lemma 2.4 holds. Such property with identity (3.2) and formula (6.1) imply formula (3.3).

The injectivity of  $R_W$  follows from formula (3.3).

Theorem 3.1 is proved.

### 6.2 Proof of Theorem 3.3

From formulas (2.18), (2.26), (2.32) we obtain:

$$R^{-1}R_W f = (I + Q_{\widetilde{W},D,m})(w_{0,0}f), \tag{6.2}$$

where  $I$  is the identity operator in  $L^2(\mathbb{R}^3)$ ,  $w_{0,0}$  is defined in (1.3).

By assumption (3.16) property (2.31) of Lemma 2.4 holds. Such property with identity (3.7) and formula (6.2) imply formula (3.17).

The injectivity of  $R_W$  follows from formula (3.17).

Theorem 3.3 is proved.

### 6.3 Proof of Theorem 3.2

By assumption (3.6) inequality (2.30) holds. Hence, formulas (2.31), (2.32) (in Lemma 2.4) hold.

Assumptions (1.2)-(1.4), (3.5) and inequality (2.28) from Lemma 2.3 imply that  $f_m \in L^2(\mathbb{R}^3)$ , where  $f_m$  is defined in (3.8).

We split expansion (1.6) of weight  $W$  defined by (1.2) in the following way:

$$W(x, \theta) = W_{N+1}(x, \theta) + \delta W_m(x, \theta), \quad \theta \in \mathbb{S}^2, x \in \mathbb{R}^3, m = [N/2], \tag{6.3}$$

where  $W_{N+1}$  is defined by (1.14),  $[N/2]$  denotes the integer part of  $N/2$ ,  $\delta W_m$  is defined by (3.11).

From (1.2)-(1.4) and from (6.3) it follows that

$$R_W f = R_{W_{N+1}} f + R_{\delta W_m} f, \quad (6.4)$$

where  $R_W f, R_{W_{N+1}} f, R_{\delta W_m} f$  are defined by (1.1) for the case of weights  $W, W_{N+1}, \delta W_m$  defined in (1.2), (1.11), (3.11), respectively.

Identity (6.4) implies that

$$R^{-1} R_W f = R^{-1} R_{W_N} f + R^{-1} R_{\delta W_m} f, \quad (6.5)$$

where  $R^{-1}$  is defined by (2.2). By assumption (3.6) inequality (2.30) holds for the cases of weights  $W, W_N, \delta W_m$ , respectively. Therefore, Lemma 2.4 holds for weights  $W, W_N, \delta W_m$  and, in particular, from (2.31) we have that:

$$R^{-1} R_{W_N} f \in L^2(\mathbb{R}^3), R^{-1} R_{\delta W_m} f \in L^2(\mathbb{R}^3). \quad (6.6)$$

By assumption (3.5) Theorem 3.3 holds for  $W = W_N, N = 2m$ . Therefore, from formula (3.17) we obtain:

$$f = (w_{0,0})^{-1} (I + Q_{\widetilde{W}, D, m})^{-1} R^{-1} R_{W_N} f, \quad (6.7)$$

where operator  $Q_{\widetilde{W}, D, m}$  is defined in (2.18) for  $m$  arising in (3.5).

From (3.8), (6.5), (6.6), (6.7) it follows that:

$$\begin{aligned} f_m &= (w_{0,0})^{-1} (I + Q_{\widetilde{W}, D, m})^{-1} R^{-1} R_W f \\ &= (w_{0,0})^{-1} (I + Q_{\widetilde{W}, D, m})^{-1} R^{-1} R_{W_N} f \\ &\quad + (w_{0,0})^{-1} (I + Q_{\widetilde{W}, D, m})^{-1} R^{-1} R_{\delta W_m} f \\ &= f + (w_{0,0})^{-1} (I + Q_{\widetilde{W}, D, m})^{-1} R^{-1} R_{\delta W_m} f. \end{aligned} \quad (6.8)$$

Formula (3.9) directly follows from (6.8).

Inequality (3.5) and identities (3.7), (3.9) imply the following inequality:

$$\|f - f_m\|_{L^2(\mathbb{R}^3)} \leq \frac{1}{c} \|(I + Q_{\widetilde{W}, D, m})^{-1}\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \cdot \|R^{-1} R_{\delta W_m} f\|_{L^2(\mathbb{R}^3)}, \quad (6.9)$$

where  $c$  is defined in (2.16),  $Q_{\widetilde{W}, D, m}$  is defined by (2.18) for  $m$  in (3.5).

From (2.28) of Lemma 2.3 and from identity (3.7) it follows that:

$$\|(I + Q_{\widetilde{W}, D, m})^{-1}\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \leq 1 + \sum_{j=1}^{\infty} \|Q_{\widetilde{W}, D, m}^j\|_{L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)} \leq \frac{1}{1 - \sigma_{\widetilde{W}, D, m}}. \quad (6.10)$$

From formulas (2.32), (3.11) and according to (5.13) it follows that

$$\|R^{-1} R_{\delta W_m} f\|_{L^2(\mathbb{R}^3)} \leq \frac{\|f\|_{\infty}}{2\pi} \sum_{k=m+1}^{\infty} \sum_{n=-2k}^{2k} \|w_{2k,n}\|_{L^2(D)}, \quad (6.11)$$

where  $R^{-1}$  is defined by (2.2),  $w_{2k,n}$  are defined by (1.11).

Putting the estimates (6.10), (6.11) in the right-hand side of (6.9) we obtain (3.10).

Theorem 3.2 is proved.

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