Iterative inversion of weighted Radon transforms in 3D

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Abstract

We propose iterative inversion algorithms for weighted Radon transforms in 3D. Our results can be considered as extension to 3D of two-dimensional results of Kunyansky (1992), Novikov (2014), Guillement, Novikov (2014). In addition, we generalize our results to the case of dimension $n > 3$.

1 Introduction

Generalized Radon transforms arise in many domains of pure and applied mathematics; see e.g. [BQ87], [Dea07], [DB84], [GGV14], [GN16], [Gra91], [Kun92].

In the present article we consider the weighted Radon transforms R_W defined by the formula

$$R_W f(s,\theta) = \int\limits_{x\theta=s} W(x,\theta) f(x)\, dx,\ (s,\theta) \in \mathbb{R}\times\mathbb{S}^2,\ x \in \mathbb{R}^3, \tag{1.1}$$

where $W = W(x,\theta)$ is the weight, $f = f(x)$ is a test function.

In this work we assume that

$$W \in C(\mathbb{R}^3\times\mathbb{S}^2) \cap L^\infty(\mathbb{R}^3\times\mathbb{S}^2), \tag{1.2}$$

$$w_{0,0}(x) \stackrel{def}{=} \frac{1}{4\pi}\int\limits_{\mathbb{S}^2} W(x,\theta)\, d\theta,\ w_{0,0}(x) \neq 0,\ x \in \mathbb{R}^3, \tag{1.3}$$

where W is complex-valued weight, $d\theta$ is element of standard measure on $\mathbb{S}^2$. If $W \equiv 1$, then R_W is reduced to the classical Radon transform $\mathbb{R}^3$, see e.g. [Rad17], [Nat86].

For the test function we assume that

$$f \in L^\infty(\mathbb{R}^3),\ \operatorname{supp} f \subset D, \tag{1.4}$$

where D is an open bounded domain (which is fixed apriori).

Let

$$c \stackrel{def}{=} \inf_{x\in D} |w_{0,0}(x)| > 0, \tag{1.5}$$

where D is the domain in (1.4) (inequality (1.5) holds by assumptions (1.2)-(1.4)).

We recall that inversion methods for R_W admit tomographical applications in the framework of the scheme described as follows.

It is well-known that in many tomographies measured data are modeled by weighted ray transforms $P_w f$ defined by the formula

$$P_w f(x,\alpha) = \int\limits_{\mathbb{R}} w(x+\alpha t,\alpha) f(x+\alpha t)\, dt,\ (x,\alpha) \in T\mathbb{S}^2, \tag{1.6}$$

$$T\mathbb{S}^2 = \{(x,\alpha) \in \mathbb{R}^3\times\mathbb{S}^2 : x\alpha = 0\},$$

where f is an object function defined on $\mathbb{R}^3$, w is the weight function defined on $\mathbb{R}^3\times\mathbb{S}^2$, and $T\mathbb{S}^2$ can be considered as the set of all rays (oriented straight lines) in $\mathbb{R}^3$, see e.g. [Cha78], [Nat86], [Kun92].

In addition, in [GN16] (Section 3) it was shown that if $P_w f$ are given for all rays parallel to some fixed plane Σ in $\mathbb{R}^3$ then $R_W f$ with appropriate W can be obtained by the explcit formulas from $P_w f$ and w (in a similair way with the case $w \equiv 1$, see [Nat86], Chapter 2, Formula (1.1); see also [Gra91]). Therefore, reconstruction of f from data modeled by $P_w f$, defined by (1.6) and restricted to all rays parallel to Σ, can be reduced to

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reconstruction of f from $R_W f$, defined by (1.1). In [GN16] it was aslo indicated that the reduction from $P_w f$ to $R_W f$ with subsequent reconstruction of f from $R_W f$ can drastically reduce the impact of the noise in the initial data.

In the present article we use the expansion for weight W:

$$W(x,\theta(\gamma,\phi)) = \sum_{k=0}^{\infty}\sum_{n=-k}^{k} w_{k,n}(x) Y_k^n(\gamma,\phi),\ x\in\mathbb{R}^3, \tag{1.7}$$

$$Y_k^n(\gamma,\phi) \stackrel{def}{=} p_k^{|n|}(\cos\gamma)e^{in\phi}, \tag{1.8}$$

$$\theta(\gamma,\phi) = (\cos\gamma,\ \sin\gamma\cos\phi,\ \sin\gamma\sin\phi)\in\mathbb{S}^2\subset\mathbb{R}^3,\ \gamma\in[0,\pi],\ \phi\in[0,2\pi], \tag{1.9}$$

where $p_k^n(x)$, $x\in[-1,1]$, $k\in\mathbb{N}\cup\{0\}$, $n=\overline{-k,k}$ are the associated Legendre polynomials. Polynomials $p_k^n(x)$ are well-known in literature (see e.g. [SW16]) and are defined using ordinary Legendre polynomials $p_k(x)$ by the formulas:

$$p_k^n(x) = (-1)^n(1-x^2)^{n/2}\frac{d^n}{dx^n}(p_k(x)),\ n,k\in\mathbb{N}\cup\{0\}, \tag{1.10}$$

$$p_k^{-n}(x) = (-1)^n\frac{(k-n)!}{(k+n)!}p_k^n(x),\ p_k^n(x)\equiv 0 \text{ if } |n|>k, \tag{1.11}$$

$$p_k(x) = \frac{1}{2^k k!}\frac{d^k}{dx^k}[(x^2-1)^k],\ x\in[-1,1],\ k\in\mathbb{N}\cup\{0\}, \tag{1.12}$$

see also [SW16], [ZT79] for other properties of associated Legendre polynomials. In addition, coefficients $w_{k,n}(x)$ in (1.7) are defined by the formulas:

$$w_{k,n}(x) = c(k,n)\int_0^{2\pi} d\phi\, e^{-in\phi}\int_0^{\pi} W(x,\theta(\gamma,\phi))p_k^{|n|}(\cos\gamma)\sin\gamma\, d\gamma, \tag{1.13}$$

$$c(k,n) = \frac{(2k+1)}{4\pi}\frac{(k-|n|)!}{(k+|n|)!},\ k\in\mathbb{N}\cup\{0\},\ n=0,\pm1,\cdots,\pm k,$$

where function $\theta(\gamma,\phi)$ is defined in (1.9).

Under assumption (1.2), for each fixed x, series (1.7) converge in $L^2(\mathbb{S}^2)$; see e.g. [SW16] (Chapter 4), [Mor98] (Chapter 2), [ZT79].

We consider also

$$\sigma_{\widetilde{W},D,m} = \frac{1}{2\pi}\sum_{k=1}^{m}\sum_{n=-2k}^{2k}\sup_{x\in D}\left|\frac{w_{2k,n}(x)}{w_{0,0}(x)}\right| \text{ for } m\in\mathbb{N}, \tag{1.14}$$

$$\sigma_{\widetilde{W},D,m} = 0 \text{ for } m=0,$$

$$\sigma_{\widetilde{W},D,\infty} = \lim_{m\to\infty}\sigma_{W,D,m}, \tag{1.15}$$

$$W_N(x,\theta(\gamma,\phi)) = \sum_{k=0}^{N}\sum_{n=-k}^{k} w_{k,n}(x)Y_k^n(\gamma,\phi), \tag{1.16}$$

$$\widetilde{W}_N(x,\theta(\gamma,\phi)) = \sum_{k=0}^{[N/2]}\sum_{n=-2k}^{2k} w_{2k,n}(x)Y_{2k}^n(\gamma,\phi), \tag{1.17}$$

$$x\in\mathbb{R}^3,\ \gamma\in[0,\pi],\ \phi\in[0,2\pi], \tag{1.18}$$

where coefficients $w_{k,n}(x)$ are defined in (1.13), $Y_k^n(\gamma,\phi)$ are defined by (1.8), $[N/2]$ denotes the integer part of $N/2$.

In the present article we obtained, in particular, the following results under assumptions (1.2), (1.3), (1.4):

1. If $\sigma_{W,D,\infty}<1$, then R_W is injective and, in addition, the inversion of $R_W f$ is given via formulas (3.2), (3.3); see Section 3 for details.

2. If $\sigma_{W,D,\infty}\geq 1$, then f can be approximately reconstructed from $R_W f$ as $f\approx(R_{\widetilde{W}_N})^{-1}R_W f$, where $R_{\widetilde{W}_N}$ is defined according to (1.1) for $\widetilde{W}_N$ defined by (1.17) for $N=2m$, where m is chosen as the largest while condition $\sigma_{W,D,m}<1$ holds. More precisely, approximate inversion of $R_W f$ is given via the formulas (3.7), (3.8).

   In addition, if $W=W_N$ defined by (1.16) and $\sigma_{W,D,m}<1$, $m=[N/2]$, where $[N/2]$ denotes the integer part of $N/2$, then R_{W_N} is injective and invertible by formula (3.17); see Section 3 for details.

Prototypes of these results for the weighted Radon transforms in 2D were obtained in [Kun92], [Nov14], [GuNo14].

The present work also continues the studies of [GN16], where approximate inversion of R_W was realized as $(R_{W_N})^{-1}$ for $N = 0$ or by other words as an approximate Chang-type inversion formula.

In Section 2 we give some notations and preliminary results.

Main results of the present work are presented in detail in Section 3.

In Section 4 we generalize results of Sections 2, 3 for the case of dimension $n \geq 3$.

Proofs of results of Sections 2, 3, 4 are presented in Sections 5, 6.

2 Some preliminary results

2.1 Some formulas for R and R^{-1}

We recall that for the classical Radon transform R (formula (1.1) for the case of $W \equiv 1$) the following identity holds (see [Nat86], Theorem 1.2, p.13):

$$R(f *_{\mathbb{R}^3} g) = Rf *_{\mathbb{R}} Rg, \tag{2.1}$$

where $*_{\mathbb{R}^3}, *_{\mathbb{R}}$ denote the 3D and 1D convolutions (respectively), f, g are test functions.

The classical Radon inversion formula is defined as follows (see e.g. [Nat86]):

$$R^{-1}q(x) = -\frac{1}{8\pi^2} \int\limits_{\mathbb{S}^{n-1}} q^{(2)}(x\theta, \theta) d\theta,\ x \in \mathbb{R}^3, \tag{2.2}$$

$$q^{(2)}(s, \theta) = \frac{d^2}{ds^2} q(s, \theta), s \in \mathbb{R},\ \theta \in \mathbb{S}^2,$$

where q is a test function on $\mathbb{R} \times \mathbb{S}^2$.

In addition, from the Projection Theorem (see [Nat86], Theorem 1.1, p.11) it follows that:

$$R^{-1}q(x) \stackrel{def}{=} \frac{1}{(2\pi)^{5/2}} \int\limits_{\mathbb{R}} \frac{\rho^2}{2} d\rho \int\limits_{\mathbb{S}^2} \hat{q}(\rho, \omega) e^{i\rho(x\omega)} d\omega,\ x \in \mathbb{R}^3, \tag{2.3}$$

$$\hat{q}(s, \theta) \stackrel{def}{=} \frac{1}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} q(t, \theta) e^{-its}\, dt,\ (s, \theta) \in \mathbb{R} \times \mathbb{S}^2, \tag{2.4}$$

where $q(t, \theta)$ is a test function on $\mathbb{R} \times \mathbb{S}^2$.

For the case of $\hat{q}$ even (i.e. $\hat{q}(s, \theta) = \hat{q}(-s, -\theta)$, $(s, \theta) \in \mathbb{R} \times \mathbb{S}^2$, where $\hat{q}$ is defined in (2.4)), formulas (2.3), (2.4) can be rewritten as follows:

$$R^{-1}q = \frac{1}{2\pi} \mathcal{F}[\hat{q}] = \frac{1}{2\pi} \mathcal{F}^{-1}[\hat{q}], \tag{2.5}$$

where $\mathcal{F}[\cdot], \mathcal{F}^{-1}[\cdot]$ denote the Fourier transform and its inverse in 3D, respectively, and are defined by the formulas (in spherical coordinates):

$$\mathcal{F}[q](\xi) \stackrel{def}{=} \frac{1}{(2\pi)^{3/2}} \int\limits_0^{+\infty} \rho^2 d\rho \int\limits_{\mathbb{S}^2} q(\rho, \omega) e^{-i\rho(\xi\omega)} d\omega, \tag{2.6}$$

$$\mathcal{F}^{-1}[q](\xi) \stackrel{def}{=} \frac{1}{(2\pi)^{3/2}} \int\limits_0^{+\infty} \rho^2 d\rho \int\limits_{\mathbb{S}^2} q(\rho, \omega) e^{i\rho(\xi\omega)} d\omega,\ \xi \in \mathbb{R}^3, \tag{2.7}$$

where $q(\rho, \omega)$ is a test-function on $[0, +\infty) \times \mathbb{S}^2$ (identified with $\mathbb{R}^3$).

2.2 Symmetrization of W

Let

$$A_W f = R^{-1} R_W f, \tag{2.8}$$

where R_W is defined in (1.1), f is a test function, satisfying assumptions (1.4).

Let

$$\widetilde{W}(x, \theta) \stackrel{def}{=} \frac{1}{2}(W(x, \theta) + W(x, -\theta)),\ x \in \mathbb{R}^3, \theta \in \mathbb{S}^2. \tag{2.9}$$

The following formulas hold:

$$\frac{1}{2}(R_W f(s,\theta) + R_W f(-s,-\theta)) = R_{\widetilde{W}} f(s,\theta),\ (s,\theta) \in \mathbb{R}\times\mathbb{S}^2, \tag{2.10}$$

$$A_W f = R^{-1} R_{\widetilde{W}} f, \tag{2.11}$$

$$\widetilde{W}(x,\theta(\gamma,\phi)) = \sum_{k=0}^{\infty}\sum_{n=-2k}^{2k} w_{2k,n}(x) Y_{2k}^n(\gamma,\phi), \tag{2.12}$$

$$x\in\mathbb{R}^3,\ \gamma\in[0,\pi],\ \phi\in[0,2\pi],$$

where $w_{k,n}$ are defined in (1.13), $\theta(\gamma,\phi)$ is defined in (1.9), $Y_k^n(\gamma,\phi)$ are defined in (1.8). Identity (2.11) is proved in [GN16] in 3D, where $\widetilde{W}$ is denoted as W_{sym}.

Identity (2.12) follows from (1.8) and the following identities:

$$p_k^n(-x) = (-1)^{n+k} p_k^n(x),\ x\in[-1,1], \tag{2.13}$$

$$e^{in(\phi+\pi)} = (-1)^n e^{in\phi} \tag{2.14}$$

where $p_k^n(x)$, $k\in\mathbb{N}\cup\{0\}$, $n=\overline{-k,k}$ are the associated Legendre polynomials, defined by identites (1.10), (1.11), (1.12) (see e.g. [SW16]).

Note also that $\widetilde{W}_N$ defined by (1.17) is the approximation of $\widetilde{W}$ defined by (2.9) and

$$\widetilde{W}_N(x,\cdot) \xrightarrow[N\to\infty]{L^2(\mathbb{S}^2)} \widetilde{W}(x,\cdot) \text{ for each fixed } x\in\mathbb{R}^3. \tag{2.15}$$

Using formulas (2.10)-(2.12) we reduce inversion of R_W to the inversion of $R_{\widetilde{W}}$ defined by (1.1) for $W=\widetilde{W}$. Prototypes of (2.8), (2.11), (2.12) for the two-dimensional case can be found in [Kun92], [Nov11].

2.3 Operators $Q_{W,D,m}$ and numbers $\sigma_{W,D,m}$

Let D be the domain of (1.4), and χ_D denote the characteristic function of D, i.e.

$$\chi_D \equiv 1 \text{ on } D,\ \chi_D \equiv 0 \text{ on } \mathbb{R}^3\backslash D. \tag{2.16}$$

Let

$$Q_{\widetilde{W},D,m} u(x) \stackrel{def}{=} R^{-1}(R_{\widetilde{W},D,m} u)(x),\ m\in\mathbb{N} \tag{2.17}$$

$$Q_{\widetilde{W},D,m} u(x) = 0 \text{ for } m=0,$$

$$Q_{\widetilde{W},D,\infty} u(x) \stackrel{def}{=} R^{-1}(R_{\widetilde{W},D,\infty} u)(x), \tag{2.18}$$

where

$$R_{\widetilde{W},D,m} u(s,\theta(\gamma,\phi)) \stackrel{def}{=} \int\limits_{x\theta=s} \left(\sum_{k=1}^{m}\sum_{n=-2k}^{2k} \frac{w_{2k,n}(x)}{w_{0,0}(x)} Y_{2k}^n(\gamma,\phi)\right) \chi_D(x) u(x)\, dx, \tag{2.19}$$

$$R_{\widetilde{W},D,\infty} u(s,\theta(\gamma,\phi)) \stackrel{def}{=} \lim_{m\to\infty} R_{\widetilde{W},D,m} u(s,\theta(\gamma,\phi))$$

$$= \int\limits_{x\theta=s} \left(\sum_{k=1}^{\infty}\sum_{n=-2k}^{2k} \frac{w_{2k,n}(x)}{w_{0,0}(x)} Y_{2k}^n(\gamma,\phi)\right) \chi_D(x) u(x)\, dx, \tag{2.20}$$

$$x\in\mathbb{R}^3,\ s\in\mathbb{R},\ \theta(\gamma,\phi)\in\mathbb{S}^2,$$

where R^{-1} is defined by (2.2) or (2.3), u is a test function, $Y_k^n(\gamma,\phi)$ are defined in (1.8), $w_{0,0}, w_{2k,n}$ are the Fourier-Laplace coefficients defined by (1.13) and $w_{2k,n}/w_{0,0}, \chi_D$ are considered as multiplication operators on $\mathbb{R}^3$. Note that $R_{\widetilde{W},D,\infty} f = R_{\widetilde{W}} f$ under assumption (1.4).

Let

$$d_{2k,n}(x) \stackrel{def}{=} R^{-1}(\delta(\cdot) Y_k^n)(x),\ x\in\mathbb{R}^3,\ k\in\mathbb{N},\ n=\overline{-2k,2k}, \tag{2.21}$$

where R^{-1} is defined by (2.3), $\delta=\delta(s)$ denotes the 1D Dirac delta function, Y_k^n is defined in (1.8).

Lemma 1. *Let $d_{2k,n}$ be defined by* (2.21). *Then*

$$d_{2k,n}(x(r,\gamma,\phi)) = \frac{(-1)^k\sqrt{2}\Gamma(\frac{3}{2}+k)}{\pi\Gamma(k)} \frac{Y_{2k}^n(\gamma,\phi)}{r^3}, r>0, \tag{2.22}$$

where $\Gamma(\cdot)$ *is the Gamma-function,* $x(r,\gamma,\phi)$ *is defined by the identity:*

$$x(r,\gamma,\phi) = (r\cos\gamma, r\sin\gamma\cos\phi, r\sin\gamma\sin\phi) \in \mathbb{R}^3,\ \gamma \in [0,\pi],\ \phi \in [0,2\pi],\ r \geq 0. \tag{2.23}$$

In addition, the following inequality holds:

$$|\mathcal{F}[d_{2k,n}](\xi)| \leq 1/2\pi,\ \xi \in \mathbb{R}^3, \tag{2.24}$$

where $\mathcal{F}[\cdot]$ *is the Fourier transform defined in* (2.6).

The following lemma gives some useful expressions for operators $Q_{\widetilde{W},D,m}$, $Q_{\widetilde{W},D,\infty}$ defined in (2.17), (2.18).

Lemma 2. *Let operators* $Q_{\widetilde{W},D,m}, Q_{\widetilde{W},D,\infty}$ *be defined by* (2.17), (2.18), *respectively, and* u *be a test function satisfying* (1.4). *Then*

$$Q_{\widetilde{W},D,m}u = \sum_{k=1}^{m}\sum_{n=-2k}^{2k} d_{2k,n} *_{\mathbb{R}^3} \frac{w_{2k,n}}{w_{0,0}}u, \tag{2.25}$$

$$Q_{\widetilde{W},D,\infty}u = \sum_{k=1}^{\infty}\sum_{n=-2k}^{2k} d_{2k,n} *_{\mathbb{R}^3} \frac{w_{2k,n}}{w_{0,0}}u, \tag{2.26}$$

where coefficients $w_{k,n}$ *are defined in* (1.13), $d_{2k,n}$ *is defined by* (2.21), $*_{\mathbb{R}^3}$ *denotes the convolution in 3D.*

Remark 1. Convolution terms in the right hand-side of (2.25), (2.26) are well defined functions in $L^2(\mathbb{R}^3)$. This follows from identity (2.22) and Calderón-Zygmund Theorem for convolution-type operators with singular kernels (see [Kna05], p.83, Theorem 3.26).

The following lemma shows that the operators $Q_{\widetilde{W},D,m}$, $Q_{\widetilde{W},D,\infty}$ are well-defined in $L^2(\mathbb{R}^3)$.

Lemma 3. *Operator* $Q_{\widetilde{W},D,m}$, *defined by* (2.17), *is a linear bounded operator in* $L^2(\mathbb{R}^3)$ *and the following estimate holds:*

$$\|Q_{\widetilde{W},D,m}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \leq \sigma_{\widetilde{W},D,m}, \tag{2.27}$$

where $\sigma_{\widetilde{W},D,m}$ *is defined by* (1.14).

If $\sigma_{\widetilde{W},D,\infty} < +\infty$, *where* $\sigma_{\widetilde{W},D,\infty}$ *is defined by* (1.15), *then* $Q_{\widetilde{W},D,\infty}$, *defined by* (2.18), *is a linear bounded operator in* $L^2(\mathbb{R}^3)$ *and the following estimate holds:*

$$\|Q_{\widetilde{W},D,\infty}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \leq \sigma_{\widetilde{W},D,\infty}. \tag{2.28}$$

Lemma 4. *Let*

$$\sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\left|\left|\frac{w_{2k,n}}{w_{0,0}}\right|\right|_{L^2(D)} < +\infty, \tag{2.29}$$

where $w_{k,n}$ *are defined in* (1.13). *Then*

$$R^{-1}R_W f \in L^2(\mathbb{R}^3). \tag{2.30}$$

In addition, the following formula holds:

$$R^{-1}R_W f = w_{0,0}f + \sum_{k=1}^{\infty}\sum_{n=-2k}^{2k} d_{2k,n} *_{\mathbb{R}^3} w_{2k,n}f, \tag{2.31}$$

where R_W *is defined by* (1.1), W *satisfies* (1.2), (1.3), f *satisfies* (1.4), *operator* R^{-1} *is defined by* (2.3), $d_{2k,n}$ *is defined by* (2.21).

3 Main results

3.1 Case of $\sigma_{\widetilde{W},D,\infty} < 1$

Let

$$\sigma_{\widetilde{W},D,\infty} < 1, \tag{3.1}$$

where $\sigma_{\widetilde{W},D,\infty}$ is defined by (1.15).

Inequality (3.1) and upper bound (2.28) in Lemma 3 imply that operator $I+Q_{\widetilde{W},D,\infty}$ is continuosly invertible in $L^2(\mathbb{R}^3)$ and the following identity holds (in the sense of the operator norm in $L^2(\mathbb{R}^3)$):

$$(I + Q_{\widetilde{W},D,\infty})^{-1} = I + \sum_{j=1}^{\infty}(-Q_{\widetilde{W},D,\infty})^j, \tag{3.2}$$

where I is the identity operator in $L^2(\mathbb{R}^3)$, operator $Q_{\widetilde{W},D,\infty}$ is defined in (2.18).

Theorem 1. *Let conditions* (1.2)-(1.4), (3.1) *be fulfilled. Then* R_W, *defined by* (1.1), *is injective and the following exact inversion formula holds:*

$$f = (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,\infty})^{-1}R^{-1}R_W f, \tag{3.3}$$

where $w_{0,0}$ *is defined in* (1.3), *operator* $(I + Q_{\widetilde{W},D,\infty})^{-1}$ *is defined by* (3.2), R^{-1} *is defined in* (2.2).

Remark 2. Formula (3.3) can be considered as the following Fredholm's second kind linear integral equation for the $w_{0,0}f$:

$$w_{0,0}f + Q_{\widetilde{W},D,\infty}(w_{0,0}f) = R^{-1}R_W f. \tag{3.4}$$

Inequality (3.1) and identity (3.2) imply that equation (3.4) can be solved by the method of successive approximations.

3.2 Case of $1 \leq \sigma_{\widetilde{W},D,\infty} < +\infty$

Let

$$\sigma_{\widetilde{W},D,m} < 1, \text{ for some } m \in \mathbb{N} \cup \{0\}, \tag{3.5}$$

$$\sigma_{\widetilde{W},D,\infty} < +\infty. \tag{3.6}$$

Inequality (3.5) and upper bound (2.27) in Lemma 3 imply that $I + Q_{\widetilde{W},D,m}$ is continuously invertible in $L^2(\mathbb{R}^3)$ and the following identity holds (in the sense of the operator norm in $L^2(\mathbb{R}^3)$):

$$(I + Q_{\widetilde{W},D,m})^{-1} = I + \sum_{j=1}^{\infty}(-Q_{\widetilde{W},D,m})^j, \tag{3.7}$$

where I is the identity operator in $L^2(\mathbb{R}^3)$, operator $Q_{\widetilde{W},D,m}$ is defined in (2.17).

Theorem 2. *Let conditions* (1.2)-(1.4), (3.5), (3.6) *be fulfilled. Then*

$$f \approx f_m \stackrel{def}{=} (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,m})^{-1}R^{-1}R_W f, \tag{3.8}$$

$$f = f_m - (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,m})^{-1}R^{-1}R_{\delta W_m} f, \tag{3.9}$$

$$\|f - f_m\|_{L^2(D)} \leq \frac{\|f\|_\infty}{2\pi c(1 - \sigma_{\widetilde{W},D,m})} \sum_{k=m+1}^{\infty}\sum_{n=-2k}^{2k} \|w_{2k,n}\|_{L^2(D)} < +\infty, \tag{3.10}$$

where

$$\delta W_m(x, \theta(\gamma,\phi)) \stackrel{def}{=} W(x, \theta(\gamma,\phi)) - \sum_{k=0}^{2m+1}\sum_{n=-k}^{k} w_{k,n}(x)Y_k^n(\gamma,\phi), \tag{3.11}$$

$$x \in \mathbb{R}^3,\ \gamma \in [0,\pi],\ \phi \in [0, 2\pi],\ m \in \mathbb{N} \cup \{0\}, \tag{3.12}$$

function $\theta(\gamma,\phi)$ *is defined in* (1.9), *coefficients* $w_{k,n}$ *are defined in* (1.13), Y_k^n *are defined in* (1.8), $\sigma_{\widetilde{W},D,m}$ *is defined in* (1.14), *operator* $(I + Q_{\widetilde{W},D,m})^{-1}$ *is defined in* (3.7), *constant* c *is defined in* (1.5).

Remark 3. Formula (3.8) can be considered as the following Fredholm's second kind linear integral equation for $w_{0,0}f$:

$$w_{0,0}f + Q_{\widetilde{W},D,m}(w_{0,0}f) = R^{-1}R_W f. \tag{3.13}$$

Inequality (3.5) and identity (3.7) imply that equation (3.13) is solvable by the method of successive approximations.

Note also that condition (3.6) can be relaxed to the following one:

$$\sum_{k=1}^{\infty}\sum_{n=-2k}^{2k} \left\| \frac{w_{2k,n}}{w_{0,0}} \right\|_{L^2(D)} < +\infty, \tag{3.14}$$

where $w_{2k,n}$ are defined in (1.13).

Formula (3.8) is an extension to 3D of the Chang-type two-dimensional inversion formulas in [Cha78], [Nov14], [GuNo14]. In addition, formula (3.8) is an extension of approximate inversion formula in [GN16], where this formula was given for $m = 0$.

If (3.5) is fulfilled for some $m \geq 1$, then f_m is a refinement of the Chang-type approximation f_0 and, more generally, f_j is a refinement of f_i for $0 \leq i < j \leq m$. In addition, $f_j = f_i$ if $w_{2k,n} \equiv 0$ for $i < k \leq j$, $n = \overline{-2k, 2k}$. Thus, we propose the following approximate reconstruction of f from $R_W f$:

(i) find maximal m such that (3.5) is still efficiently fulfilled,

(ii) approximately reconstruct f by f_m using (3.8).

3.3 Exact inversion for finite Fourier series weights

Let

$$W = W_N,\ N \in \mathbb{N} \cup \{0\}, \tag{3.15}$$

where W_N is defined by (1.16).

Suppose that

$$\sigma_{\widetilde{W},D,m} < 1 \text{ for } m = [N/2], \tag{3.16}$$

where $\sigma_{\widetilde{W},D,m}$ is defined by (1.14), $[N/2]$ denotes the integer part of $N/2$.

Theorem 3. *Let conditions* (1.2)-(1.4), (3.15), (3.16) *be fulfilled. Then* R_W *defined by* (1.1) *is injective and the following exact inversion formula holds:*

$$f = (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,m})^{-1} R^{-1} R_W f, \tag{3.17}$$

where $w_{0,0}$ *is defined in* (1.2), $(I + Q_{\widetilde{W},D,m})^{-1}$ *is defined by* (3.7), R^{-1} *is defined by* (2.2).

Remark 4. Formula (3.17) can be considered as the following Fredholm's second type linear integral equation for $w_{0,0} f$:

$$w_{0,0} f + Q_{\widetilde{W},D,m}(w_{0,0} f) = R^{-1} R_W f. \tag{3.18}$$

Identity (3.16) imply that (3.18) can be solved by the method of successive approximations.

4 Generalization for multidimensions

Formulas (1.1)-(1.4) admit straightforward extensions to the case of dimension $n > 3$. In particular, in this case $R_W f$ is defined as corresponding integrals along $(n-1)$-dimensional hyperplanes in $\mathbb{R}^n$; see e.g. [GN16], [BQ87].

For weight W we consider Fourier-Laplace expansion:

$$W(x, \theta) = \sum_{k=0}^{\infty} \sum_{i=1}^{a_{k,n}} w_k^i(x) Y_k^i(\theta),\ x \in \mathbb{R}^n,\ \theta \in \mathbb{S}^{n-1}, \tag{4.1}$$

where

$$w_k^i(x) = \|Y_k^i\|^{-2}_{L^2(\mathbb{S}^{n-1})} \int\limits_{\mathbb{S}^{n-1}} W(x, \theta) \overline{Y_k^i(\theta)}\, d\theta, \tag{4.2}$$

$$a_{k,n+1} = \frac{(n+k)!}{k!n!} - \frac{(n+k-2)!}{(k-2)!n!},\ n, k \geq 2;\ a_{0,n} = 1,\ a_{1,n} = n, \tag{4.3}$$

where $\{Y_k^i \,|\, k = \overline{0, \infty}, i = \overline{1, a_{k,n}}\}$ is the Fourier-Laplace basis of harmonics on $\mathbb{S}^{n-1}$, $\overline{Y_k^i}$ denotes the complex conjugate of Y_k^i ; see [SW16], [Mor98]. In this work we choose basis Y_k^i as in [Hig87] without normalizing constants ${}_n c_L^l$ (i.e. $\{Y_k^i\}$ are the products of Legendre polynomials with one complex exponent and without any additional constants).

Formula (1.3) can be rewritten now as follows:

$$w_0^1(x) \stackrel{def}{=} \frac{1}{\mathrm{vol}(\mathbb{S}^{n-1})} \int\limits_{\mathbb{S}^{n-1}} W(x, \theta)\, d\theta,\ x \in \mathbb{R}^n,\ w_0^1(x) \neq 0,\ x \in \mathbb{R}^n, \tag{4.4}$$

where $\mathrm{vol}(\mathbb{S}^{n-1})$ denotes the standard Euclidean volume of $\mathbb{S}^{n-1}$.

Under assumptions (1.2), (1.4) for dimension $n > 3$, series (4.1) converge in $L^2(\mathbb{S}^{n-1})$; see e.g. [SW16] (Chapter 4), [Mor98] (Chapter 2), [ZT79].

Formula (2.3) is extended as follows:

$$R^{-1}q(x) = (2\pi)^{1/2-n} \int\limits_{\mathbb{R}} \frac{|\rho|^{n-1}}{2} \int\limits_{\mathbb{S}^{n-1}} \hat{q}(\rho,\theta) e^{i\rho(x\theta)} d\theta,\ x \in \mathbb{R}^n, \tag{4.5}$$

where $q(s,\theta)$ is a test function on $\mathbb{R} \times \mathbb{S}^{n-1}$, $\hat{q}(s,\theta)$ is defined as in (2.4) (with $\mathbb{S}^2$ replaced by $\mathbb{S}^{n-1}$).

Fourier transforms defined in (2.6), (2.7) are rewritten now as follows:

$$\mathcal{F}[q](\xi) \stackrel{def}{=} (2\pi)^{-n/2} \int\limits_0^{+\infty} \rho^{n-1} d\rho \int\limits_{\mathbb{S}^{n-1}} q(\rho,\omega) e^{-i\rho(\xi\omega)} d\omega, \tag{4.6}$$

$$\mathcal{F}^{-1}[q](\xi) \stackrel{def}{=} (2\pi)^{-n/2} \int\limits_0^{+\infty} \rho^{n-1} d\rho \int\limits_{\mathbb{S}^{n-1}} q(\rho,\omega) e^{i\rho(\xi\omega)} d\omega,\ \xi \in \mathbb{R}^n, \tag{4.7}$$

where $q(\rho,\omega)$ is a test function on $[0,+\infty) \times \mathbb{S}^{n-1}$ (identified with $\mathbb{R}^n$).

In dimension $n > 3$, formulas (2.8)-(2.12) remain valid with Y_k^m defined in (1.8) replaced by general basis of spherical harmonics $\{Y_k^i\}$ on $\mathbb{S}^{n-1}$. In particular, multidimensional analog of formula (2.12) (with $x \in \mathbb{R}^n$, $\theta \in \mathbb{S}^{n-1}$) holds by the following property of harmonic $Y_k^i(\theta)$:

$$Y_k^i(-\theta) = (-1)^k Y_k^i(\theta),\ \theta \in \mathbb{S}^{n-1},\ k \in \mathbb{N} \cup \{0\},\ i = \overline{1, a_{k,n}}, \tag{4.8}$$

where $a_{k,n}$ is defined by (4.3). Identity (4.8) reflects the fact that $Y_k^i(\theta) = Y_k^i(\theta_1, \theta_2, \cdots, \theta_n)$, $\theta = (\theta_1, \theta_2, \cdots, \theta_n) \in \mathbb{S}^{n-1}$, $i = \overline{1, a_{k,n}}$ is a homogenous polynomial of degree k, see e.g. [SW16], [Mor98].

The extensions of (1.14), (1.15), (2.17), (2.18), to the case of dimension $n > 3$, are defined as follows:

$$\sigma_{\widetilde{W},D,m} \stackrel{def}{=} (2\pi)^{(1-n)/2} \sum_{k=1}^{m} \sum_{i=1}^{a_{2k,n}} \sup_{x \in D} \left| \frac{w_{2k}^i(x)}{w_0^1(x)} \right|, \tag{4.9}$$

$$\sigma_{\widetilde{W},D,\infty} \stackrel{def}{=} \lim_{m \to +\infty} \sigma_{\widetilde{W},D,m} \tag{4.10}$$

$$Q_{\widetilde{W},D,m} u(x) \stackrel{def}{=} R^{-1}(R_{\widetilde{W},D,m} u)(x),\ m \in \mathbb{N} \tag{4.11}$$

$$Q_{\widetilde{W},D,m} u(x) = 0 \text{ for } m = 0,$$

$$Q_{\widetilde{W},D,\infty} u(x) \stackrel{def}{=} R^{-1}(R_{\widetilde{W},D,\infty} u)(x), \tag{4.12}$$

where

$$R_{\widetilde{W},D,m} u(s,\theta) \stackrel{def}{=} \int\limits_{x\theta = s} \left(\sum_{k=1}^{m} \sum_{i=1}^{a_{2k,n}} \frac{w_{2k}^i(x)}{w_0^1(x)} Y_{2k}^i(\theta) \right) \chi_D(x) u(x)\, dx, \tag{4.13}$$

$$R_{\widetilde{W},D,\infty} u(s,\theta) \stackrel{def}{=} \lim_{m \to \infty} R_{\widetilde{W},D,m} u(s,\theta), \tag{4.14}$$

$$x \in \mathbb{R}^n,\ s \in \mathbb{R},\ \theta \in \mathbb{S}^{n-1},$$

where R^{-1} is defined in (4.5), coefficients w_k^i are defined in (4.2).

Formula (2.21) is now rewritten as follows:

$$d_{2k,i}(x) \stackrel{def}{=} R^{-1}(\delta(\cdot) Y_{2k}^i)(x),\ x \in \mathbb{R}^n,\ i = \overline{1, a_{k,n}}, \tag{4.15}$$

where R^{-1} is defined in (4.5), Y_{2k}^i is the spherical harmonic on $\mathbb{S}^{n-1}$.

Results of Lemma 1 remain valid with formula (2.22) replaced by the following one:

$$d_{2k,i}(r,\theta) = c(k,n) \frac{(-1)^k Y_{2k}^i(\theta)}{r^n},\ r > 0,\ \theta \in \mathbb{S}^{n-1}, \tag{4.16}$$

where

$$c(k,n) = \frac{\sqrt{2}\pi^{(1-n)/2} \Gamma(k + \frac{1}{2}) \Gamma(k + \frac{n}{2})}{\Gamma(k) \Gamma(k + \frac{n-1}{2})} \cdot \left(\frac{\Gamma(k+1)}{\Gamma(k + \frac{1}{2})} \right)^{n-2}, \tag{4.17}$$

$\Gamma(\cdot)$ is the Gamma function, and inequality (2.24) is rewritten as follows:

$$|\mathcal{F}[d_{2k,i}](\xi)| \le (2\pi)^{(1-n)/2},\ \xi \in \mathbb{R}^n, \tag{4.18}$$

where $\mathcal{F}[\cdot]$ is the Fourier transform defined in (4.6), $d_{2k,i}$ is defined in (4.15). Constant $c(k,n)$ in (4.17) is obtained using formulas (4.5), (4.15) and Theorems 1, 2 in [Gon16].

Results of Lemma 2 remain valid, for the case of dimension $n > 3$, with formulas (2.25), (2.26) rewritten as follows:

$$Q_{\widetilde{W},D,m}u = \sum_{k=1}^{m}\sum_{i=1}^{a_{k,n}} d_{2k,i} *_{\mathbb{R}^n} \frac{w_{2k}^i}{w_0^1}u, \tag{4.19}$$

$$Q_{\widetilde{W},D,\infty}u = \sum_{k=1}^{\infty}\sum_{i=1}^{a_{k,n}} d_{2k,i} *_{\mathbb{R}^n} \frac{w_{2k}^i}{w_0^1}u, \tag{4.20}$$

where coefficients $w_{2k}^i(x), w_0^1(x)$ are defined in (4.2), $a_{k,n}$ is defined in (4.3), $d_{2k,i}$ is defined in (4.16), $*_{\mathbb{R}^n}$ denotes the convolution in $\mathbb{R}^n$.

Results of Lemma 3 remain valid with $\mathbb{R}^3$ replaced by $\mathbb{R}^n$, $n > 3$, where we use definitions (4.9), (4.11), (4.12).

Assumption (2.29) in Lemma 4 is rewritten now as follows:

$$\sum_{k=1}^{\infty}\sum_{i=1}^{a_{k,n}} \left|\left|\frac{w_{2k}^i}{w_0^1}\right|\right|_{L^2(D)} < +\infty, \tag{4.21}$$

where coefficients w_k^i are defined in (4.2), $a_{k,n}$ is defined in (4.3), D is the open bounded domain defined according to (1.4) in $\mathbb{R}^n$. Under assumption (4.21), result (2.30) of Lemma 4 remains valid, with $\mathbb{R}^3$ replaced by $\mathbb{R}^n$ and with the following analog of formula (2.31), for the case of dimension $n > 3$:

$$R^{-1}R_W f = w_0^1 f + \sum_{k=1}^{\infty}\sum_{i=1}^{a_{k,n}} d_{2k,i} *_{\mathbb{R}^n} w_{2k}^i f, \tag{4.22}$$

where R^{-1} is defined in (4.5), f is a test function satisfying (1.4) (with $\mathbb{R}^3$ replaced by $\mathbb{R}^n$), $a_{k,n}$ is defined in (4.3), $d_{2k,i}$ is defined in (4.16).

Under assumption (3.1) (with $\sigma_{\widetilde{W},D,\infty}$ defined in (4.10)), results of Theorem 1 remain valid in dimension $n > 3$.

Under assumptions (3.5), (3.6) (with $\sigma_{\widetilde{W},D,m}, \sigma_{\widetilde{W},D,\infty}$ defined in (4.9), (4.10); operators $Q_{\widetilde{W},D,m}, Q_{\widetilde{W},D,\infty}$ defined in (4.11), (4.12); operator R^{-1} defined in (4.5); coefficients $w_{k,n}$ replaced by w_k^i, defined in (4.2)) results of Theorem 2 remain valid in the case of dimension $n > 3$.

Under assumption (3.16) (with $\sigma_{\widetilde{W},D,m}$ defined in (4.9)) results of Theorem 3 also remain valid.

5 Proofs of Lemma 1, 2, 3, 4

5.1 Proof of Lemma 1

We consider $x(r,\gamma,\phi)$ defined by (2.23) and $\omega(\gamma,\phi) = x(1,\gamma,\phi)$ (i.e. $\omega \in \mathbb{S}^2$).

Identity (2.23) implies the following expression of the scalar product $(x\omega)$ in spherical coordinates in $\mathbb{R}^3$:

$$(x\omega) = (x(r,\tilde\gamma,\tilde\phi),\omega(\gamma,\phi)) = r(\cos\gamma\cos\tilde\gamma + \sin\gamma\sin\tilde\gamma\cos(\phi-\tilde\phi)), \tag{5.1}$$

where $\gamma,\tilde\gamma \in [0,\pi]$, $\phi,\tilde\phi \in [0,2\pi]$, $r \ge 0$.

From formulas (1.8), (2.3), (2.21), (5.1) it follows that

$$(2\pi)^{5/2} d_{2k,n}(x(r,\tilde\gamma,\tilde\phi)) = \int\limits_{\mathbb{R}} \frac{\rho^2}{2}d\rho \int\limits_{\mathbb{S}^2} e^{i\rho(x\omega(\gamma,\phi))} Y_{2k}^n(\gamma,\phi)\, d\omega(\gamma,\phi) \tag{5.2}$$

$$= \int\limits_{\mathbb{R}} \frac{\rho^2}{2}d\rho \int\limits_0^{\pi} \sin(\gamma)\, p_k^{|n|}(\cos\gamma)\, d\gamma \int\limits_0^{2\pi} e^{i\rho(x,\omega(\gamma,\phi))+in\phi} d\phi$$

$$= e^{in(\tilde\phi-\pi/2)} \int\limits_{\mathbb{R}} \frac{\rho^2}{2}d\rho \int\limits_0^{\pi} \sin(\gamma) p_{2k}^{|n|}(\cos\gamma) e^{i\rho r\cos\gamma\cos\tilde\gamma} \int\limits_0^{2\pi} e^{in\phi+\rho r\sin\gamma\sin\tilde\gamma\sin\phi} d\phi$$

$$= 2\pi e^{in(\tilde\phi-\pi/2)}(-1)^n \int\limits_{\mathbb{R}} \frac{\rho^2}{2}d\rho \int\limits_0^{\pi} \sin(\gamma) p_{2k}^{|n|}(\cos\gamma) e^{i\rho r\cos\gamma\cos\tilde\gamma} J_n(\rho r\sin\gamma\sin\tilde\gamma)\, d\gamma, \tag{5.3}$$

where $J_n(\cdot)$ is the n-th standard Bessel function of the first kind; see e.g. [Tem11].

Integrand in $d\gamma$ in the right hand-side of (5.3) was considered in [NPF+06], where the following exact analytic solution was given:

$$\int_0^\pi \sin(\gamma) p_{2k}^{|n|}(\cos\gamma) e^{i\rho r\cos\gamma\cos\tilde{\gamma}} J_n(\rho r\sin\gamma\sin\tilde{\gamma})\,d\gamma = 2i^{2k-n} p_{2k}^{|n|}(\cos\tilde{\gamma}) j_{2k}(\rho r), \tag{5.4}$$

where j_{2k} is the standard spherical Bessel function of order $2k$; see e.g. [Tem11].

From identities (5.3), (5.4) it follows that:

$$\begin{aligned} d_{2k,n}(x(r,\tilde{\gamma},\tilde{\phi})) &= \frac{(-1)^k}{(2\pi)^{3/2}} p_{2k}^{|n|}(\cos\tilde{\gamma}) e^{in\tilde{\phi}} \int_{\mathbb{R}} \rho^2 j_{2k}(\rho r)\,d\rho \\ &= \frac{4\sqrt{\pi}(-1)^k\Gamma(\frac{3}{2}+k)}{(2\pi)^{3/2}\Gamma(k)} \frac{p_{2k}^{|n|}(\cos\tilde{\gamma}) e^{in\tilde{\phi}}}{r^3},\ r>0. \end{aligned} \tag{5.5}$$

where $\Gamma(\cdot)$ is the Gamma function.

Defenition (1.8) and identity (5.5) imply formula (2.22).

Formulas (2.5), (2.13), (2.14), (2.21) imply that

$$(2\pi) d_{2k,n}(x(r,\tilde{\gamma},\tilde{\phi})) = \mathcal{F}^{-1}[Y_{2k}^n](x(r,\tilde{\gamma},\tilde{\phi})),\ r>0,\ \tilde{\gamma}\in[0,\pi],\ \tilde{\phi}\in[0,2\pi], \tag{5.6}$$

where $\mathcal{F}^{-1}[\cdot]$ is defined in (2.7), Y_k^n is defined in (1.8).

From the invertibility of the Fourier transform defined in (2.6), (2.7) and identity (5.6) the following identity holds:

$$2\pi\mathcal{F}[d_{2k,n}] = \mathcal{F}\mathcal{F}^{-1}[Y_{2k}^n] = Y_{2k}^n. \tag{5.7}$$

For Y_k^n defined in (1.8) the following inequality holds (see e.g. [SW16]):

$$|Y_k^n(\gamma,\theta)| \le 1,\ \gamma\in[0,\pi],\ \phi\in[0,2\pi]. \tag{5.8}$$

Identities (5.6) and inequality (5.8) imply (2.24).

Note that $|\mathcal{F}[d_{2k,n}](\xi)|$, $\xi\in\mathbb{R}^3$, is uniformly bounded by $1/2\pi$ except only one point $\xi=0$, where direction $\xi/|\xi|\in\mathbb{S}^2$ is not defined. However, point $\xi=0$ is of Lebesgue measure zero and $\mathcal{F}[d_{2k,n}]$ can be defined with any value at the origin in $\mathbb{R}^3$.

Lemma 1 is proved.

5.2 Proof of Lemma 2

From identity (2.17) it follows that

$$\begin{aligned} Q_{\widetilde{W},D,m}u &= R^{-1}\left(\sum_{k=1}^{m}\sum_{n=-2k}^{2k} Y_{2k}^n R\left(\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right)\right) \\ &= R^{-1}\left(\sum_{k=1}^{m}\sum_{n=-2k}^{2k} (\delta(\cdot)Y_{2k}^n) *_{\mathbb{R}} R\left(\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right)\right) \\ &= R^{-1}\left(\sum_{k=1}^{m}\sum_{n=-2k}^{2k} R(d_{2k,n}) *_{\mathbb{R}} R\left(\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right)\right) \end{aligned} \tag{5.9}$$

where $*_{\mathbb{R}}$ denotes the 1D convolution, $\delta=\delta(s)$ is the 1D-Dirac delta function, $d_{2k,n}$ is defined by (2.21).

Identities (2.1), (5.9) imply (2.25).

For operator $Q_{\widetilde{W},D,\infty}$ defined by (2.18) we proceed according to identity (5.9) with $m\to+\infty$. Linearity of operator R^{-1} defined by (2.2) and identites (2.1), (5.9) imply (2.26).

Lemma 2 is proved.

5.3 Proof of Lemma 3

From formula (2.25) and the fact that Fourier transform defined in (2.6) does not change the L^2-norm we obtain:

$$\begin{aligned} \|Q_{\widetilde{W},D,m}u\|_{L^2(\mathbb{R}^3)} &\le \sum_{k=1}^{m}\sum_{n=-2k}^{2k} \left|\left| d_{2k,n} *_{\mathbb{R}^3} \frac{w_{2k,n}}{w_{0,0}}\chi_D u\right|\right|_{L^2(\mathbb{R}^3)} \\ &= \sum_{k=1}^{m}\sum_{n=-2k}^{2k} \left|\left| \mathcal{F}[d_{2k,n}]\mathcal{F}\left[\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right]\right|\right|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{5.10}$$

From inequalities (2.24), (5.10) we obtain:

$$\|Q_{\widetilde{W},D,m}u\|_{L^2(\mathbb{R}^3)} \le \frac{1}{2\pi}\sum_{k=1}^{m}\sum_{n=-2k}^{2k}\left\|\mathcal{F}\left(\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right)\right\|_{L^2(\mathbb{R}^3)}$$
$$= \frac{1}{2\pi}\sum_{k=1}^{m}\sum_{n=-2k}^{2k}\left\|\frac{w_{2k,n}}{w_{0,0}}\chi_D u\right\|_{L^2(\mathbb{R}^3)}$$
$$\le \sigma_{\widetilde{W},D,m}\|u\|_{L^2(D)}, \tag{5.11}$$

where $\sigma_{\widetilde{W},D,m}$ is defined by (1.14).

Inequality (5.11) implies (2.27).

Estimate (2.28) follows from definition (2.18), formula (2.26), linearity of operator R^{-1} defined by (2.2), inequalities (5.10), (5.11) for $m \to +\infty$.

Lemma 3 is proved.

5.4 Proof of Lemma 4

From formulas (1.1), (1.7) it follows that

$$R_W f(s,\theta(\gamma,\phi)) = \sum_{k=0}^{\infty}\sum_{n=-k}^{k} Y_k^n(\gamma,\phi) R[w_{k,n}f](s,\theta(\gamma,\phi)), \tag{5.12}$$

$$s \in \mathbb{R},\ \gamma \in [0,\pi],\ \phi \in [0,2\pi], \tag{5.13}$$

where $\theta(\gamma,\phi)$ is defined in (1.9), $Y_k^n(\gamma,\phi)$ are defined by (1.8), $w_{k,n}$ are defined in (1.13).

Formula (2.31) follows from formulas (2.8), (2.11), (2.12), (2.18) and formula (2.26) in Lemma 2, where test function u is replaced by $w_{0,0}u$.

From inequality (2.24), formulas (2.1), (2.31) and the fact that Fourier tranfsform defined in (2.6) does not change the L^2-norm we obtain:

$$\|R^{-1}R_W f\|_{L^2(\mathbb{R}^3)} \le \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\|d_{2k,n} *_{\mathbb{R}^3} w_{2k,n}f\|_{L^2(\mathbb{R}^3)}$$
$$= \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\|\mathcal{F}[d_{2k,n}]\mathcal{F}[w_{2k,n}f]\|_{L^2(\mathbb{R}^3)} \tag{5.14}$$
$$\le \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \frac{\|f\|_\infty}{2\pi}\sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\|w_{2k,n}\|_{L^2(D)},$$

where $\|\cdot\|_\infty$ denotes the L^∞-norm, $\mathcal{F}[\cdot]$ is defined in (2.6).

From assumption (2.29) and formula (1.5) it follows that

$$\sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\|w_{2k,n}\|_{L^2(D)} < +\infty. \tag{5.15}$$

Inequalities (5.14), (5.15) and formulas (1.2), (1.3), (1.4) imply that

$$\|R^{-1}R_W f\|_{L^2(\mathbb{R}^3)} \le \|w_{0,0}f\|_{L^2(\mathbb{R}^3)} + \frac{\|f\|_\infty}{2\pi}\sum_{k=1}^{\infty}\sum_{n=-2k}^{2k}\|w_{2k,n}\|_{L^2(D)} < +\infty. \tag{5.16}$$

Lemma 2 is proved.

6 Proofs of Theorems 1, 2, 3

6.1 Proof of Theorem 1

From formulas (2.18), (2.26), (2.31) we obtain:

$$R^{-1}R_W f = (I + Q_{\widetilde{W},D,\infty})(w_{0,0}f), \tag{6.1}$$

where I is the identity operator in $L^2(\mathbb{R}^3)$, $w_{0,0}$ is defined in (1.3).

Assumption (3.1) implies that identity (2.30) holds (by Lemma 4).

Formula (3.3) follows from assumption (3.1), inequality (2.28) in Lemma 3 and formula (6.1).

Theorem 1 is proved.

6.2 Proof of Theorem 3

From formulas (2.17), (2.25), (2.31) we obtain:

$$R^{-1}R_W f = (I + Q_{\widetilde{W},D,m})(w_{0,0}f), \tag{6.2}$$

where I is the identity operator in $L^2(\mathbb{R}^3)$, $w_{0,0}$ is defined in (1.3).

Assumption (3.16) implies that identity (2.30) holds (by Lemma 4).

Formula (3.17) follows from assumption (3.16), inequality (2.27) in Lemma 3 and formula (6.2).

Theorem 3 is proved.

6.3 Proof of Theorem 2

By assumption (3.6) inequality (2.29) holds. Hence, formulas (2.30), (2.31) (in Lemma 4) hold.

Formulas (1.2)-(1.4), assumption (3.5) and inequality (2.27) from Lemma 3 imply that $f_m \in L^2(\mathbb{R}^3)$, where f_m is defined in (3.8).

We split expansion (1.7) of weight W defined by (1.2) in the following way:

$$W(x,\theta) = W_{N+1}(x,\theta) + \delta W_m(x,\theta),\ \theta \in \mathbb{S}^2,\ x \in \mathbb{R}^3,\ m = [N/2], \tag{6.3}$$

where W_{N+1} is defined by (1.16), $[N/2]$ denotes the integer part of $N/2$, δW_m is defined by (3.11).

From formulas (1.2)-(1.4) it follows that

$$R_W f = R_{W_{N+1}} f + R_{\delta W_m} f, \tag{6.4}$$

where $R_W f, R_{W_{N+1}} f, R_{\delta W_m} f$ are defined by (1.1) for the case of weights $W, W_{N+1}, \delta W_m$ defined in (1.2), (1.13), (3.11), respectively.

Identities (2.8), (2.11), (6.3), (6.3) imply that

$$R^{-1}R_W f = R^{-1}R_{W_N} f + R^{-1}R_{\delta W_m} f, \tag{6.5}$$

where R^{-1} is defined by (2.2). By assumption (3.6) inequality (2.29) holds for the cases of weights W, W_N, δW_m, respectively. Therefore, Lemma 4 holds for weights W, W_N, δW_m and, in particular, from identity (2.30) we obtain:

$$R^{-1}R_{W_N} f \in L^2(\mathbb{R}^3),\ R^{-1}R_{\delta W_m} f \in L^2(\mathbb{R}^3). \tag{6.6}$$

By assumption (3.5) Theorem 3 holds for $W = W_N$, $N = 2m$. Therefore, from formula (3.17) we obtain:

$$f = (w_{0,0})^{-1}(I + Q_{\widetilde{W},D,m})^{-1}R^{-1}R_{W_N} f, \tag{6.7}$$

where operator $Q_{\widetilde{W},D,m}$ is defined in (2.17) for m in (3.5).

Formula (3.9) follows from formula (3.8), identities (6.5), (6.6), (6.7).

Identities (3.7), (3.9) and inequality (3.5) imply the following sequence of inequalities:

$$\|f - f_m\|_{L^2(\mathbb{R}^3)} \le \frac{1}{c}\|(I + Q_{\widetilde{W},D,m})^{-1}\| \cdot \|R^{-1}R_{\delta W_m} f\|_{L^2(\mathbb{R}^3)} \le \frac{1}{c(1 - \sigma_{\widetilde{W},D,m})}\|R^{-1}R_{\delta W_m} f\|_{L^2(\mathbb{R}^3)} \tag{6.8}$$

where c is defined in (1.5), $Q_{\widetilde{W},D,m}$ is defined by (2.17) for m in (3.5).

From formulas (2.31), (3.11) and according to (5.14) it follows that

$$\|R^{-1}R_{\delta W_m} f\|_{L^2(\mathbb{R}^3)} \le \frac{\|f\|_\infty}{2\pi} \sum_{k=m+1}^{\infty} \sum_{n=-2k}^{2k} \|w_{2k,n}\|_{L^2(D)}, \tag{6.9}$$

where R^{-1} is defined by (2.2), $w_{2k,n}$ are defined by (1.13).

Identities (6.8), (6.9) imply (3.10).

Theorem 2 is proved.

7 Aknowledgments

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