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STARK UNITS IN POSITIVE CHARACTERISTIC

BRUNO ANGLÈS, TUAN NGO DAC, AND FLORIC TAVARES RIBEIRO

ABSTRACT. We show that the module of Stark units associated to a sign-normalized rank one Drinfeld module can be obtained from Anderson's equivariant A -harmonic series. We apply this to obtain a class formula à la Taelman and to prove a several variable log-algebraicity theorem, generalizing Anderson's log-algebraicity theorem. We also give another proof of Anderson's log-algebraicity theorem using shtukas and obtain various results concerning the module of Stark units for Drinfeld modules of arbitrary rank.

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INTRODUCTION

The power-series $\sum_{n \geq 1} \frac{z^n}{n}$ is log-algebraic:

$$\sum_{n \geq 1} \frac{z^n}{n} = -\log(1 - z).$$

This identity allows one to obtain the value of a Dirichlet L -series at $s = 1$ as an algebraic linear combination of logarithms of circular units. Inspired by examples of Carlitz [10] and Thakur [26], Anderson ([1], [2]) proved an analogue of this identity for a sign-normalized rank one Drinfeld A -module, known as Anderson's log-algebraicity theorem.

When $A = \mathbb{F}_q[\theta]$ (the genus 0 case), various works ([1, 2, 3, 6, 7, 9, 11, 15, 16, 17, 20, 21, 22, 23]) have revealed the importance of certain units in the study of special values of the Goss L -functions at $s = 1$. To give a simple example, the Carlitz module is considered to play the role of the multiplicative group \mathbb{G}_m over \mathbb{Z} , and Anderson ([1, 2]) showed that the images through the Carlitz exponential of some special units give algebraic elements which are the equivalent of the circular units. The special units constructed in such a way are then "log-algebraic". Recently, Taelman ([22, 23]) introduced the module of units attached to any Drinfeld module and proved a class formula which states that the special value of the Goss L -function attached to a Drinfeld module at $s = 1$ is the product of a regulator term arising from the module of units and an algebraic term arising from a certain class module. Also, deformations of Goss L -series values in Tate algebras are investigated by Pellarin and two of the authors ([6, 7, 9, 21]). For higher dimensional versions of Drinfeld modules, we refer the reader to [3, 11, 15, 16, 17, 20]. We should mention that all these works are based on a crucial fact that $\mathbb{F}_q[\theta]$ is a principal ideal domain, which is no longer true in general.

In the present paper, we develop a new method to deal with higher genus cases. We introduce Stark units attached to Drinfeld A -modules extending the previous work of two of the authors ([9]) and make a systematic study of these modules of Stark units. For a sign-normalized rank one Drinfeld module, we prove a direct link between the module of Stark units and Anderson's equivariant A -harmonic series, which is an analogue of Stark's conjectures. It allows us to obtain a class formula à la Taelman and a several variable log-algebraicity theorem in the general context.

Let us give now more precise statements of our results.

Let K/\mathbb{F}_q be a global function field (\mathbb{F}_q is algebraically closed in K), let A be the ring of elements of K which are regular outside a fixed place ∞ of K of degree $d_\infty \geq 1$. The completion K_∞ of K at the place ∞ has residue field \mathbb{F}_∞ and is endowed with the ∞ -adic valuation $v_\infty : K_\infty \rightarrow \mathbb{Z} \cup \{+\infty\}$. For $a \in A$, we set: $\deg a := -d_\infty v_\infty(a)$. We fix an algebraic closure \overline{K}_∞ of K_∞ , and still denote $v_\infty : \overline{K}_\infty \rightarrow \mathbb{Q} \cup \{+\infty\}$ the extension of v_∞ to \overline{K}_∞ . Let $\tau : \overline{K}_\infty \rightarrow \overline{K}_\infty$ be the \mathbb{F}_q -algebra homomorphism which sends x to x^q .

We choose a sign function $\text{sgn} : K_\infty^\times \rightarrow \mathbb{F}_\infty^\times$, that is, a group homomorphism such that $\text{sgn}|_{\mathbb{F}_\infty^\times} = \text{Id}_{\mathbb{F}_\infty^\times}$. Let $\phi : A \hookrightarrow \overline{K}_\infty \setminus \{0\}$ be a sign-normalized rank one Drinfeld module (see Section 3.2), i.e. there exists an integer $i(\phi) \in \mathbb{N}$ such that:

$$\forall a \in A, \quad \phi_a = a + \cdots + \text{sgn}(a)^{q^{i(\phi)}} \tau^{\deg a}.$$

Then, the exponential series attached to ϕ is the unique element $\exp_\phi \in \overline{K}_\infty\{\{\tau\}\}$, such that $\exp_\phi \equiv 1 \pmod{\tau}$, and:

$$\forall a \in A, \quad \exp_\phi a = \phi_a \exp_\phi.$$

If we write:

$$\exp_\phi = \sum_{i \geq 0} e_i(\phi) \tau^i,$$

with $e_i(\phi) \in \overline{K}_\infty$, then the field $H := K(e_i(\phi), i \in \mathbb{N})$ is a finite abelian extension of K which is unramified outside ∞ (see Section 3.2). Let B be the integral closure of A in H . For all $a \in A$, we have:

$$\phi_a \in B\{\tau\}.$$

For a non-zero ideal I of A , we define $\phi_I \in H\{\tau\}$ to be the monic element in $H\{\tau\}$ such that:

$$H\{\tau\}\phi_I = \sum_{a \in I} H\{\tau\}\phi_a.$$

In fact, $\phi_I \in B\{\tau\}$ and we denote its constant term by $\psi(I) \in B \setminus \{0\}$.

For simplicity, we will work over the abelian extension H/K . We should mention that the results presented below are still valid for any finite abelian extension E/K such that $H \subset E$.

Let $G = \text{Gal}(H/K)$. For a non-zero ideal I of A , we denote by $\sigma_I = (I, H/K) \in G$, where $(\cdot, H/K)$ is the Artin map. Let z be an indeterminate over K_∞ and let $\mathbb{T}_z(K_\infty)$ be the Tate algebra in the variable z with coefficients in K_∞ . Let's set:

$$H_\infty = H \otimes_K K_\infty,$$

and:

$$\mathbb{T}_z(H_\infty) = H \otimes_K \mathbb{T}_z(K_\infty).$$

Let $\tau : \mathbb{T}_z(H_\infty) \rightarrow \mathbb{T}_z(H_\infty)$ be the continuous $\mathbb{F}_q[z]$ -algebra homomorphism such that:

$$\forall x \in H_\infty, \quad \tau(x) = x^q.$$

We set:

$$\exp_{\tilde{\phi}} = \sum_{i \geq 0} e_i(\phi) z^i \tau^i \in H[z]\{\{\tau\}\}.$$

Then $\exp_{\tilde{\phi}}$ converges on $\mathbb{T}_z(H_\infty)$. Following [9], we introduce the module of z -units attached to ϕ/B :

$$U(\tilde{\phi}/B[z]) = \{f \in \mathbb{T}_z(H_\infty), \exp_{\tilde{\phi}}(f) \in B[z]\}.$$

We denote by $\text{ev} : \mathbb{T}_z(H_\infty) \rightarrow H_\infty$ the evaluation at $z = 1$. The module of Stark units attached to ϕ/B is defined by (see [9], Section 2):

$$U_{St}(\phi/B) = \text{ev}(U(\tilde{\phi}/B[z])) \subset H_\infty.$$

Then $U_{St}(\phi/B)$ is an A -lattice in H_∞ (see Theorem 2.7), i.e. $U_{St}(\phi/B)$ is an A -module which is discrete and cocompact in H_∞ . In fact, $U_{St}(\phi/B)$ is contained in the A -module of the Taelman module of units [22] defined by:

$$U(\phi/B) = \{x \in H_\infty, \exp_\phi(x) \in B\},$$

which is also an A -lattice in H_∞ . Following Taelman [22], the Taelman class module $H(\phi/B)$ is a finite A -module (via ϕ) defined by:

$$H(\phi/B) = \frac{H_\infty}{B + \exp_\phi(H_\infty)}.$$

Following Anderson [1], we introduce the following series (see Section 3.3):

$$\mathcal{L}(\phi/B; 1; z) = \sum_I \frac{z^{\deg I}}{\psi(I)} \sigma_I \in \mathbb{T}_z(H_\infty)[G],$$

where the sum runs through the non-zero ideals I of A . The equivariant A -harmonic series attached to ϕ/B is defined by:

$$\mathcal{L}(\phi/B) = \text{ev}(\mathcal{L}(\phi/B; 1; z)) \in H_\infty[G].$$

One of our main theorems states (see Theorem 3.8) that the module of Stark units $U_{St}(\phi/B)$ can be obtained from the equivariant A -harmonic series $\mathcal{L}(\phi/B)$, which is reminiscent of Stark's Conjectures ([25]):

Theorem A. We have:

$$U(\tilde{\phi}/B[z]) = \mathcal{L}(\phi/B; 1; z)B[z].$$

In particular,

$$U_{St}(\phi/B) = \mathcal{L}(\phi/B)B.$$

We will present several applications of this theorem.

Firstly, we apply Theorem A to obtain a class formula à la Taelman for ϕ/B , by a different method of Taelman's original one [23]. Roughly speaking, we introduce the Stark regulator (resp. the regulator defined by Taelman [23]) attached to ϕ/B by $[B : U_{St}(\phi/B)]_A \in \overline{K}_\infty^\times$ (resp. $[B : U(\phi/B)]_A \in \overline{K}_\infty^\times$) (see Section 2.3). We show (see Theorem 2.7):

Theorem B. We have:

$$\text{Fitt}_A \frac{U(\phi/B)}{U_{St}(\phi/B)} = \text{Fitt}_A H(\phi/B),$$

where, for a finite A -module M , $\text{Fitt}_A M$ is the Fitting ideal of M .

Observe that $\mathcal{L}(\phi/B)$ induces a K_∞ -linear map on H_∞ , and we denote by $\det_{K_\infty} \mathcal{L}(\phi/B)$ its determinant. We prove the following formula (see Theorem 3.6):

$$\det_{K_\infty} \mathcal{L}(\phi/B) = \zeta_B(1) := \prod_{\mathfrak{P}} \left(1 - \frac{1}{[\frac{B}{\mathfrak{P}}]_A}\right)^{-1} \in \overline{K}_\infty^\times,$$

where \mathfrak{P} runs through the maximal ideals of B . Note that $\zeta_B(1)$ is a special value at $s = 1$ of some zeta function $\zeta_B(s)$ introduced by Goss (see [19], Chapter 8). Therefore, Theorem A and Theorem B imply Taelman's class formula for ϕ/B (see Theorem 3.10):

Theorem C. We have:

$$\zeta_B(1) = [B : U_{St}(\phi/B)]_A = [B : U(\phi/B)]_A [H(\phi/B)]_A.$$

When the genus of K is zero and $d_\infty = 1$, Taelman's class formula, its higher dimensional versions, and its arithmetic consequences are now well-understood due to the recent works ([6], [8], [9], [11], [15], [16], [17], [22], [23]). All these works are based on the crucial fact that when $g = 0$ and $d_\infty = 1$, the ring A is a principal ideal domain (when A is not assumed to be principal, the existence of a class formula is still an open problem in general). Using the module of Stark units, we are able to overcome this difficulty, and Theorem C provides a large class of examples of Taelman's class formula when A is no longer principal. We refer the reader to Section 2.4 for a more detailed discussion.

Secondly, we apply Theorem A to prove a several variable log-algebraicity theorem, generalizing Anderson's log-algebraicity theorems (see Theorem 4.2). (The theorem below is valid for any finite abelian extension E/K , $H \subset E$, see Theorem 4.2 for the precise statement).

Theorem D. Let $n \geq 0$ and let X_1, \dots, X_n, z be $n + 1$ indeterminates over K . Let $\tau : K[X_1, \dots, X_n][[z]] \rightarrow K[X_1, \dots, X_n][[z]]$ be the continuous $\mathbb{F}_q[[z]]$ -algebra homomorphism for the z -adic topology such that $\forall x \in K[X_1, \dots, X_n], \tau(x) = x^q$. Then:

$$\forall b \in B, \quad \exp_{\tilde{\varphi}}\left(\sum_I \frac{\sigma_I(b)}{\psi(I)} \phi_I(X_1) \cdots \phi_I(X_n) z^{\deg I}\right) \in B[X_1, \dots, X_n, z],$$

where I runs through the non-zero ideals of A .

For $n \leq 1$ and $d_\infty = 1$, this theorem was due to G. Anderson ([1], Theorem 5.1.1 and [2], Theorem 3):

$$\forall b \in B, \quad \exp_{\tilde{\varphi}}\left(\sum_I \frac{\sigma_I(b)}{\psi(I)} z^{\deg I}\right) \in B[z],$$

$$\forall b \in B, \quad \exp_{\tilde{\varphi}}\left(\sum_I \frac{\sigma_I(b)}{\psi(I)} \phi_I(X) z^{\deg I}\right) \in B[X, z],$$

where the sum runs through the non-zero ideals of A . Again, this result is now well-understood when the genus of K is zero (and $d_\infty = 1$) due to the recent works of many people ([6], [7], [8], [9], [23], [28] Sections 8.9 and 8.10, and the forthcoming work of M. Papanikolas [20]). However, to our knowledge, Anderson's log-algebraicity remains quite mysterious for $g > 0$ until now.

Thirdly, we present an alternative approach to the previous several variable log-algebraicity theorem (Theorem D) via Drinfeld's correspondence between Drinfeld modules and shtukas. Using the shtuka function attached to ϕ/B via Drinfeld's correspondence, we introduce one variable versions of the previous objects, i.e. the modules of z -units and Stark units, the equivariant A -harmonic series and the L -series (see Section 4.2). We prove an analogue of Theorem A in this one variable context (see Theorem 4.9). More generally, we also obtain a several variable log-algebraicity theorem (see Corollary 4.10). In the case $g = 0$ and $d_\infty = 1$, we rediscover the Pellarin's L -series [21] and its several variable variants studied in [4], [6], [7], [9]. We deduce from this another proof of Theorem D (see Section 4.4).

Finally, we prove some results concerning the module of Stark units for Drinfeld modules of arbitrary rank in Section 2. In particular, Theorem B is still valid for any Drinfeld module.

1. NOTATION

Let K/\mathbb{F}_q be a global function field of genus g , where \mathbb{F}_q is a finite field of characteristic p , having q elements (\mathbb{F}_q is algebraically closed in K). We fix a place ∞ of K of degree d_∞ , and denote by A the ring of elements of K which are regular outside of ∞ . The completion K_∞ of K at the place ∞ has residue field \mathbb{F}_∞ and comes with the ∞ -adic valuation $v_\infty : K_\infty \rightarrow \mathbb{Z} \cup \{+\infty\}$. We fix an algebraic closure \overline{K}_∞ of K_∞ and still denote by $v_\infty : \mathbb{C}_\infty \rightarrow \mathbb{Q} \cup \{+\infty\}$ the extension of v_∞ to the completion \mathbb{C}_∞ of \overline{K}_∞ .

We will fix a uniformizer π of K_∞ . Set $\pi_1 = \pi$, and for $n \geq 2$, choose $\pi_n \in \overline{K}_\infty^\times$ such that $\pi_n^n = \pi_{n-1}$. If $z \in \mathbb{Q}$, $z = \frac{m}{n!}$ for some $m \in \mathbb{Z}, n \geq 1$, we set:

$$\pi^z := \pi_n^m.$$

Let $\overline{\mathbb{F}}_q$ be the algebraic closure of \mathbb{F}_q in \overline{K}_∞ , and let $U_\infty = \{x \in \overline{K}_\infty, v_\infty(x-1) > 0\}$. Then:

$$\overline{K}_\infty^\times = \pi^\mathbb{Q} \times \overline{\mathbb{F}}_q^\times \times U_\infty.$$

Therefore, if $x \in \overline{K}_\infty^\times$, one can write in a unique way:

$$x = \pi^{v_\infty(x)} \operatorname{sgn}(x)\langle x \rangle, \quad \operatorname{sgn}(x) \in \overline{\mathbb{F}}_q^\times, \langle x \rangle \in U_\infty.$$

Let $\mathcal{I}(A)$ be the group of non-zero fractional ideals of A . For $I \in \mathcal{I}(A), I \subset A$, we set:

$$\deg I := \dim_{\mathbb{F}_q} A/I.$$

Then, the function \deg on non-zero ideals of A extends into a group homomorphism:

$$\deg : \mathcal{I}(A) \rightarrow \mathbb{Z}.$$

Let's observe that, for $x \in K^\times$, we have:

$$\deg(x) := \deg(xA) = -d_\infty v_\infty(x).$$

Let $I \in \mathcal{I}(A)$, then there exists an integer $h \geq 1$ such that $I^h = xA$, $x \in K^\times$. We set:

$$\langle I \rangle := \langle x \rangle^{\frac{1}{h}} \in U_\infty.$$

Then one shows (see [19], Section 8.2) that the map $[\cdot] : \mathcal{I}(A) \rightarrow \overline{K}_\infty^\times, I \mapsto \langle I \rangle \pi^{-\frac{\deg I}{d_\infty}}$ is a group homomorphism such that:

$$\forall x \in K^\times, \quad [xA] = \frac{x}{\operatorname{sgn}(x)}.$$

Observe that:

$$\forall I \in \mathcal{I}(A), \quad \operatorname{sgn}([I]) = 1.$$

If M is a finite A -module, and $\operatorname{Fitt}_A(M)$ is the Fitting ideal of M , we set:

$$[M]_A := [\operatorname{Fitt}_A(M)].$$

Let's observe that, if $0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0$ is a short exact sequence of finite A -modules, then:

$$[M]_A = [M_1]_A [M_2]_A.$$

Let E/K be a finite extension, and let O_E be the integral closure of A in E . Let $\mathcal{I}(O_E)$ be the group of non-zero fractional ideals of O_E . We denote by $N_{E/K} :$

$\mathcal{I}(O_E) \rightarrow \mathcal{I}(A)$ the group homomorphism such that, if \mathfrak{P} is a maximal ideal of O_E and $P = \mathfrak{P} \cap A$, we have:

$$N_{E/K}(\mathfrak{P}) = P^{[\frac{O_E}{\mathfrak{P}} : \frac{A}{P}]}.$$

Note that, if $\mathfrak{P} = xO_E, x \in E^\times$, then:

$$N_{E/K}(\mathfrak{P}) = N_{E/K}(x)A,$$

where $N_{E/K} : E \rightarrow K$ also denotes the usual norm map.

2. STARK UNITS AND L -SERIES ATTACHED TO DRINFELD MODULES

2.1. L -series attached to Drinfeld modules.

Let E/K be a finite extension, and let O_E be the integral closure of A in E . Let $\tau : E \rightarrow E, x \mapsto x^q$. Let ρ be an Drinfeld A -module (or a *Drinfeld module* for short) of rank $r \geq 1$ defined over O_E , i.e. $\rho : A \hookrightarrow O_E\{\tau\}$ is an \mathbb{F}_q -algebra homomorphism such that:

$$\forall a \in A \setminus \{0\}, \quad \rho_a = \rho_{a,0} + \rho_{a,1}\tau + \cdots + \rho_{a,r \deg a} \tau^{r \deg a},$$

where $\rho_{a,0}, \dots, \rho_{a,r \deg a} \in O_E, \rho_{a,0} = a$, and $\rho_{a,r \deg a} \neq 0$.

Let \mathfrak{P} be a maximal ideal of O_E , we denote by $\rho(O_E/\mathfrak{P})$ the finite dimensional \mathbb{F}_q -vector space O_E/\mathfrak{P} equipped with the structure of A -module induced by ρ .

Proposition 2.1. *The following product converges to a principal unit in K_∞^\times (i.e. an element in $U_\infty \cap K_\infty^\times$):*

$$L_A(\rho/O_E) := \prod_{\mathfrak{P}} \frac{[\frac{O_E}{\mathfrak{P}}]_A}{[\rho(\frac{O_E}{\mathfrak{P}})]_A},$$

where \mathfrak{P} runs through the maximal ideals of O_E .

Proof. By [19], Remark 7.1.8.2, we have: $H_A \subset E$, where H_A/K is the maximal unramified abelian extension of K such that ∞ splits completely in H_A . Thus $N_{E/K}(\mathfrak{P})$ is a principal ideal. Observe that:

$$\text{Fitt}_A \frac{O_E}{\mathfrak{P}} = N_{E/K}(\mathfrak{P}).$$

Thus:

$$[\frac{O_E}{\mathfrak{P}}]_A = [N_{E/K}(\mathfrak{P})].$$

By [18], Theorem 5.1, there exists a unitary polynomial $P(X) \in A[X]$ of degree $r' \leq r$ such that:

$$\begin{aligned} N_{E/K}(\mathfrak{P}) &= P(0)A, \\ \text{Fitt}_A \rho(\frac{O_E}{\mathfrak{P}}) &= P(1)A, \\ v_\infty\left(\frac{(-1)^{r'} P(0)}{P(1)} - 1\right) &\geq \frac{\deg(N_{E/K}(\mathfrak{P}))}{r' d_\infty}. \end{aligned}$$

This last assertion comes from the fact that $P(X)$ is a power of the minimal polynomial over K of the Frobenius F of $\frac{O_E}{\mathfrak{P}}$ (see [18], Lemma 3.3), and that $K(F)/K$

is totally imaginary (i.e. there exists a unique place of $K(F)$ over ∞). By the properties of $[\cdot]$ (see Section 1), we have:

$$\frac{[\frac{O_E}{\mathfrak{P}}]_A}{[\rho(\frac{O_E}{\mathfrak{P}})]_A} = \frac{(-1)^{r'} P(0)}{P(1)}.$$

The proposition follows. \square

Remark 2.2. The element $L_A(\rho/O_E) \in K_\infty^\times$ is called *the L-series* attached to ρ/O_E . By the proof of Proposition 2.1, $L_A(\rho/O_E)$ depends on A, ρ and O_E , but not on the choice of π .

Let F/K be a finite extension with $F \subset E$, and such that there exists a unique place of F above ∞ (still denoted by ∞). Let A' be the integral closure of A in F , then A' is the set of elements in F which are regular outside ∞ . We assume that ρ extends into a Drinfeld A' -module: $\rho : A' \hookrightarrow O_E\{\tau\}$. Let $[\cdot]_{A'} : \mathcal{I}(A') \rightarrow \overline{K}_\infty^\times$ be the map constructed as in Section 1 with the help of the choice of a uniformizer $\pi' \in F_\infty^\times$. Let $N_{F_\infty/K_\infty} : F_\infty \rightarrow K_\infty$ be the usual norm map.

Corollary 2.3. *We have:*

$$N_{F_\infty/K_\infty}(L_{A'}(\rho/O_E)) = L_A(\rho/O_E).$$

Proof. Recall that:

$$L_{A'}(\rho/O_E) := \prod_{\mathfrak{P}} \frac{[\frac{O_E}{\mathfrak{P}}]_{A'}}{[\rho(\frac{O_E}{\mathfrak{P}})]_{A'}},$$

where \mathfrak{P} runs through the maximal ideals of O_E . Since N_{F_∞/K_∞} is continuous, we get by the proof of Proposition 2.1:

$$N_{F_\infty/K_\infty}(L_{A'}(\rho/O_E)) = \prod_{\mathfrak{P}} N_{F_\infty/K_\infty}\left(\frac{[\frac{O_E}{\mathfrak{P}}]_{A'}}{[\rho(\frac{O_E}{\mathfrak{P}})]_{A'}}\right).$$

Let \mathfrak{P} be a maximal ideal of O_E . Since $\frac{[\frac{O_E}{\mathfrak{P}}]_{A'}}{[\rho(\frac{O_E}{\mathfrak{P}})]_{A'}} \in F^\times$, we get:

$$N_{F_\infty/K_\infty}\left(\frac{[\frac{O_E}{\mathfrak{P}}]_{A'}}{[\rho(\frac{O_E}{\mathfrak{P}})]_{A'}}\right) = N_{F/K}\left(\frac{[\frac{O_E}{\mathfrak{P}}]_{A'}}{[\rho(\frac{O_E}{\mathfrak{P}})]_{A'}}\right).$$

But, observe that if M is a finite A' -module, we have:

$$N_{F/K}(\text{Fitt}_{A'} M) = \text{Fitt}_A M.$$

By the proof of Proposition 2.1, $\frac{[\frac{O_E}{\mathfrak{P}}]_{A'}}{[\rho(\frac{O_E}{\mathfrak{P}})]_{A'}}$ is a principal unit in F_∞^\times , and therefore

$N_{F/K}\left(\frac{[\frac{O_E}{\mathfrak{P}}]_{A'}}{[\rho(\frac{O_E}{\mathfrak{P}})]_{A'}}\right)$ is also a principal unit in K_∞^\times . Again, by the proof of Proposition 2.1, we get:

$$N_{F/K}\left(\frac{[\frac{O_E}{\mathfrak{P}}]_{A'}}{[\rho(\frac{O_E}{\mathfrak{P}})]_{A'}}\right) = \frac{[\frac{O_E}{\mathfrak{P}}]_A}{[\rho(\frac{O_E}{\mathfrak{P}})]_A}.$$

The corollary follows. \square

2.2. Stark units and the Taelman class module.

Let E/K be a finite extension of degree n , and let O_E be the integral closure of A in E . Set:

$$E_\infty = E \otimes_K K_\infty.$$

Let M be an A -module, $M \subset E_\infty$, we say that M is an A -lattice in E_∞ if M is discrete and cocompact in E_∞ . Observe that if M is an A -lattice in E_∞ , then there exist $e_1, \dots, e_n \in E_\infty$ (recall that $n = [E : K]$) such that $E_\infty = \bigoplus_{i=1}^n K_\infty e_i$, $N := \bigoplus_{i=1}^n A e_i \subset M$ and $\frac{M}{N}$ is a finite A -module. Note also that O_E is an A -lattice in E_∞ .

Let $\tau : E_\infty \rightarrow E_\infty, x \mapsto x^q$. Let $\rho : A \hookrightarrow O_E\{\tau\}$ be a Drinfeld module of rank $r \geq 1$. Then, there exist unique elements $\exp_\rho, \log_\rho \in E\{\{\tau\}\}$ such that

$$\begin{aligned} \exp_\rho, \log_\rho &\in 1 + E\{\{\tau\}\}\tau, \\ \forall a \in A, \quad \exp_\rho a &= \rho_a \exp_\rho, \\ \exp_\rho \log_\rho &= \log_\rho \exp_\rho = 1. \end{aligned}$$

The formal series \exp_ρ , and \log_ρ are respectively called *the exponential series* and *the logarithm series* associated to ρ/O_E . We will write:

$$\begin{aligned} \exp_\rho &= \sum_{i \geq 0} e_i(\rho) \tau^i, \\ \log_\rho &= \sum_{i \geq 0} l_i(\rho) \tau^i, \end{aligned}$$

with $e_i(\rho), l_i(\rho) \in E$. Moreover, \exp_ρ converges on E_∞ (see [19], proof of Theorem 4.6.9).

Definition 2.4. We define *the Taelman module of units* associated to ρ/O_E as follows:

$$U(\rho/O_E) = \{x \in E_\infty, \exp_\rho(x) \in O_E\}.$$

Then, as a consequence of [22], Theorem 1, the A -module $U(\rho/O_E)$ is an A -lattice in E_∞ .

Definition 2.5. We define *the Taelman class module* associated to ρ/O_E by:

$$H(\rho/O_E) = \frac{E_\infty}{O_E + \exp_\rho(E_\infty)}.$$

Note that $H(\rho/O_E)$ is an A -module via ρ , and by [22], Theorem 1, $H(\rho/O_E)$ is a finite A -module.

Let z be an indeterminate over K_∞ , and let $\mathbb{T}_z(K_\infty)$ be the Tate algebra in the variable z with coefficients in K_∞ . We set:

$$\mathbb{T}_z(E_\infty) = E \otimes_K \mathbb{T}_z(K_\infty).$$

Observe that $E_\infty \subset \mathbb{T}_z(E_\infty)$, and $\mathbb{T}_z(E_\infty)$ is a free $\mathbb{T}_z(K_\infty)$ -module of rank $[E : K]$. Let $\tau : \mathbb{T}_z(E_\infty) \rightarrow \mathbb{T}_z(E_\infty)$ be the continuous $\mathbb{F}_q[z]$ -algebra homomorphism such that:

$$\forall x \in E_\infty, \quad \tau(x) = x^q.$$

Let $\text{ev} : \mathbb{T}_z(E_\infty) \rightarrow E_\infty$ be the surjective E_∞ -algebra homomorphism given by:

$$\forall f \in \mathbb{T}_z(E_\infty), \quad \text{ev}(f) = f|_{z=1}.$$

We have: $\ker \text{ev} = (z - 1)\mathbb{T}_z(E_\infty)$, and:

$$\forall f \in \mathbb{T}_z(E_\infty), \quad \text{ev}(\tau(f)) = \tau(\text{ev}(f)).$$

Recall that:

$$\exp_\rho = \sum_{i \geq 0} e_i(\rho)\tau^i, \quad \text{with } e_i(\rho) \in E.$$

We set:

$$\exp_{\tilde{\rho}} = \sum_{i \geq 0} e_i(\rho)z^i\tau^i \in E[z]\{\{\tau\}\}.$$

Observe that $\exp_{\tilde{\rho}}$ converges on $\mathbb{T}_z(E_\infty)$, and:

$$\forall f \in \mathbb{T}_z(E_\infty), \quad \text{ev}(\exp_{\tilde{\rho}}(f)) = \exp_\rho(\text{ev}(f)).$$

Let $\tilde{\rho}: A \hookrightarrow O_E[z]\{\tau\}$ be the \mathbb{F}_q -algebra homomorphism given by:

$$\forall a \in A, \quad \tilde{\rho}a = a + \rho_{a,1}z\tau + \cdots + \rho_{a,r \deg a}z^{r \deg a}\tau^{r \deg a},$$

where $\rho_a = a + \rho_{a,1}\tau + \cdots + \rho_{a,r \deg a}\tau^{r \deg a}$. Then:

$$\forall a \in A, \quad \exp_{\tilde{\rho}}a = \tilde{\rho}a \exp_{\tilde{\rho}}.$$

Definition 2.6. *The module of z -units associated to ρ/O_E is defined by:*

$$U(\tilde{\rho}/O_E[z]) = \{f \in \mathbb{T}_z(E_\infty), \exp_{\tilde{\rho}}(f) \in O_E[z]\}.$$

And *the module of Stark units associated to ρ/O_E is defined by:*

$$U_{St}(\rho/O_E) := \text{ev}(U(\tilde{\rho}/O_E[z])).$$

Observe that $U_{St}(\rho/O_E) \subset U(\rho/O_E)$.

Theorem 2.7. *The A -module $U_{St}(\rho/O_E)$ is an A -lattice in E_∞ . Furthermore:*

$$\left[\frac{U(\rho/O_E)}{U_{St}(\rho/O_E)} \right]_A = [H(\rho/O_E)]_A.$$

Proof. This is a consequence of the proof of [9], Theorem 1. For the convenience of the reader, we give a sketch of the proof. Let's set:

$$H(\tilde{\rho}/O_E[z]) = \frac{\mathbb{T}_z(E_\infty)}{O_E[z] + \exp_{\tilde{\rho}}(\mathbb{T}_z(E_\infty))}.$$

Observe that $H(\tilde{\rho}/O_E[z])$ is an $A[z]$ -module via $\tilde{\rho}$, and furthermore $H(\tilde{\rho}/O_E[z])$ is a finite $\mathbb{F}_q[z]$ -module ([9], Proposition 2). Let's set:

$$V = \{x \in H(\tilde{\rho}/O_E[z]), (z - 1)x = 0\}.$$

Since $\ker \text{ev} = (z - 1)\mathbb{T}_z(E_\infty)$, the multiplication by $z - 1$ on $H(\tilde{\rho}/O_E)$ gives rise to an exact sequence of finite A -modules:

$$0 \rightarrow V \rightarrow H(\tilde{\rho}/O_E[z]) \rightarrow H(\tilde{\rho}/O_E[z]) \rightarrow H(\rho/O_E) \rightarrow 0.$$

Thus:

$$\text{Fitt}_A V = \text{Fitt}_A H(\rho/O_E).$$

Now, let's consider the homomorphism of $\mathbb{F}_q[z]$ -modules $\alpha: \mathbb{T}_z(E_\infty) \rightarrow \mathbb{T}_z(E_\infty)$ given by:

$$\forall x \in \mathbb{T}_z(E_\infty), \quad \alpha(x) = \frac{\exp_{\tilde{\rho}}(x) - \exp_\rho(x)}{z - 1}.$$

Observe that:

$$(z - 1)\alpha(U(\rho/O_E)) \subset O_E + \exp_{\tilde{\rho}}(\mathbb{T}_z(E_\infty)),$$

$$\forall a \in A, \forall x \in U(\rho(O_E)), \quad \alpha(ax) - \tilde{\rho}_a(\alpha(x)) \in O_E[z].$$

Thus α induces a homomorphism of A -modules:

$$\bar{\alpha} : U(\rho/O_E) \rightarrow V.$$

By [9], Proposition 3, this homomorphism is surjective and its kernel is precisely $U_{St}(\rho/O_E)$. The theorem follows. \square

2.3. Co-volumes.

Let V be a finite dimensional K_∞ -vector space of dimension $n \geq 1$. An A -lattice in V is a discrete and cocompact sub- A -module of V .

Lemma 2.8. *Let M, N be two A -lattices in V . Then there exists an isomorphism of K_∞ -vector spaces $\sigma : V \rightarrow V$ such that:*

$$\sigma(M) \subset N.$$

Proof. Since A is a Dedekind domain, there exist two non-zero ideals I, J of A , and two K_∞ -basis $\{e_1, \dots, e_n\}, \{f_1, \dots, f_n\}$ of V , such that:

$$\begin{aligned} M &= \bigoplus_{j=1}^{n-1} Ae_j \oplus Ie_n, \\ N &= \bigoplus_{j=1}^{n-1} Af_j \oplus Jf_n. \end{aligned}$$

Furthermore, M and N are isomorphic as A -modules if and only if I and J have the same class in the ideal class group $\text{Pic}(A)$ of A . Let $x \in I^{-1}J \setminus \{0\}$. Let $\sigma : V \rightarrow V$ such that:

$$\begin{aligned} \sigma(e_j) &= f_j, \quad j = 1, \dots, n-1, \\ \sigma(e_n) &= xf_n. \end{aligned}$$

Then:

$$\sigma(M) \subset N.$$

Note that if M and N are isomorphic A -modules then we can select $x \in K^\times$ such that $I^{-1}J = xA$ and in this case $\sigma(M) = N$. \square

Lemma 2.9. *Let M, N be two A -lattices in V . Let $\sigma_1, \sigma_2 : V \rightarrow V$ be two isomorphisms of K_∞ -vector spaces such that $\sigma_i(M) \subset N, i = 1, 2$. Then:*

$$\frac{\det_{K_\infty} \sigma_1}{\text{sgn}(\det_{K_\infty} \sigma_1)} \left[\frac{N}{\sigma_1(M)} \right]_A^{-1} = \frac{\det_{K_\infty} \sigma_2}{\text{sgn}(\det_{K_\infty} \sigma_2)} \left[\frac{N}{\sigma_2(M)} \right]_A^{-1}.$$

Proof. Let $\sigma = \sigma_1 \sigma_2^{-1}$. Since $\sigma(\sigma_2(M)) = \sigma_1(M) \subset N$, with $\sigma_2(M) \subset N$, we can find $a \in A$ with $\text{sgn } a = 1$ such that $a\sigma(N) \subset N$. Set $U = \frac{1}{a}\sigma_2(M) \cap N$. Then the multiplication by a induces an exact sequence of finite A -modules:

$$0 \longrightarrow \frac{U}{\sigma_2(M)} \longrightarrow \frac{N}{\sigma_2(M)} \xrightarrow{a} \frac{N}{\sigma_2(M)} \longrightarrow \frac{N}{aN} \longrightarrow 0$$

from which we deduce

$$\left[\frac{U}{\sigma_2(M)} \right]_A = \left[\frac{N}{aN} \right]_A = a^n.$$

And $a\sigma$ similarly induces an exact sequence of finite A -modules:

$$0 \longrightarrow \frac{U}{\sigma_2(M)} \longrightarrow \frac{N}{\sigma_2(M)} \xrightarrow{a\sigma} \frac{N}{\sigma_1(M)} \longrightarrow \frac{N}{a\sigma(N)} \longrightarrow 0.$$

We get:

$$\begin{aligned} \left[\frac{N}{\sigma_1(M)}\right]_A &= \left[\frac{N}{\sigma_2(M)}\right]_A \left[\frac{U}{\sigma_2(M)}\right]_A^{-1} \left[\frac{N}{a\sigma(N)}\right]_A = \left[\frac{N}{\sigma_2(M)}\right]_A a^{-n} \frac{\det_{K_\infty}(a\sigma)}{\operatorname{sgn}(\det_{K_\infty}(a\sigma))} \\ &= \left[\frac{N}{\sigma_2(M)}\right]_A \frac{\det_{K_\infty}(\sigma)}{\operatorname{sgn}(\det_{K_\infty}(\sigma))}. \end{aligned}$$

The lemma follows. \square

Let M, N be two A -lattices in V . By Lemma 2.8, there exists an isomorphism of K_∞ -vector spaces $\sigma : V \rightarrow V$ such that $\sigma(M) \subset N$, we set:

$$[M : N]_A = \frac{\det_{K_\infty} \sigma}{\operatorname{sgn}(\det_{K_\infty} \sigma)} \left[\frac{N}{\sigma(M)}\right]_A^{-1}.$$

By Lemma 2.9, this is well-defined. In particular, if M, N are two A -lattices in V such that $N \subset M$, then:

$$[M : N]_A = \left[\frac{M}{N}\right]_A.$$

If M, N, U are three A -lattices in V , we get:

$$[M : N]_A = [M : U]_A [U : N]_A.$$

Let F/K be a finite extension such that there exists a unique place of F above ∞ (still denoted by ∞). Let A' be the integral closure of A in F . We assume that V is also an F_∞ -vector space. Let $[\cdot]_{A'} : \mathcal{I}(A') \rightarrow \overline{K}_\infty^\times$ be the map constructed as in Section 1 with the help of the choice of a uniformizer $\pi' \in F_\infty^\times$. Let $N_{F_\infty/K_\infty} : F_\infty \rightarrow K_\infty$ be the usual norm map.

Lemma 2.10. *Let M, N be two A' -lattices in V . Then there exists an integer $m \geq 1$ such that $[M : N]_{A'}^m \in F_\infty^\times$, $[M : N]_A^m \in K_\infty^\times$, and:*

$$N_{F_\infty/K_\infty}([M : N]_{A'}^m) = [M : N]_A^m.$$

Proof. Let $\sigma : V \rightarrow V$ be an isomorphism of F_∞ -vector spaces such that $\sigma(M) \subset N$, and we set: $I' = \operatorname{Fitt}_{A'} \frac{N}{\sigma(M)}$. Then:

$$\operatorname{Fitt}_A \frac{N}{\sigma(M)} = N_{F/K}(I').$$

Let $m \geq 1$ be an integer such that:

$$I'^m = xA', \quad x \in A' \setminus \{0\}.$$

Then:

$$[M : N]_{A'}^m = \left(\frac{\det_{F_\infty} \sigma}{\operatorname{sgn}'(\det_{F_\infty} \sigma)}\right)^m \frac{\operatorname{sgn}'(x)}{x}.$$

Furthermore, we have:

$$\det_{K_\infty} \sigma = N_{F_\infty/K_\infty}(\det_{F_\infty} \sigma).$$

Thus:

$$[M : N]_A^m = \left(\frac{N_{F_\infty/K_\infty}(\det_{F_\infty} \sigma)}{\operatorname{sgn}(N_{F_\infty/K_\infty}(\det_{F_\infty} \sigma))}\right)^m \frac{\operatorname{sgn}(N_{F/K}(x))}{N_{F/K}(x)}.$$

Therefore:

$$N_{F_\infty/K_\infty}([M : N]_{A'}^m) \in [M : N]_A^m F_\infty^\times.$$

The lemma follows. \square

2.4. Regulator of Stark units and L -series.

Let E/K be a finite extension, $E \subset \mathbb{C}_\infty$. Recall that $E_\infty = E \otimes_K K_\infty$. If M is an A -lattice in E_∞ , then we call $[O_E : M]_A$ the A -regulator of M .

Definition 2.11. Let $\rho : A \hookrightarrow O_E\{\tau\}$ be a Drinfeld module of rank $r \geq 1$. We define the regulator of Stark units associated to ρ/O_E by $[O_E : U_{St}(\rho/O_E)]_A$.

Proposition 2.12. Let $\rho : A \hookrightarrow O_E\{\tau\}$ be a Drinfeld module of rank $r \geq 1$. We have:

$$[O_E : U_{St}(\rho/O_E)]_A \in U_\infty.$$

Furthermore, the regulator of Stark units relative to ρ/O_E depends on ρ, A and O_E , not on the choice of π .

Proof. Let $\theta \in A \setminus \mathbb{F}_q$, and let $L = \mathbb{F}_q(\theta), B = \mathbb{F}_q[\theta]$. Let $[\cdot]_B : \mathcal{I}(B) \rightarrow L_\infty^\times$ be the map as in Section 1 associated to the choice of $\frac{1}{\theta}$ as a uniformizer of L_∞ . Then, by Theorem 2.7, we have:

$$[O_E : U_{St}(\rho/O_E)]_B = [O_E : U(\rho/O_E)]_B [H(\rho/O_E)]_B.$$

Then, by [22], Theorem 2, we get:

$$[O_E : U_{St}(\rho/O_E)]_B \in 1 + \frac{1}{\theta} \mathbb{F}_q \left[\left[\frac{1}{\theta} \right] \right].$$

Now, by Lemma 2.10, there exists an integer $m \geq 1$ such that:

$$N_{K_\infty/L_\infty}([O_E : U_{St}(\rho/O_E)]_A^m) = [O_E : U_{St}(\rho/O_E)]_B^m.$$

This implies:

$$v_\infty([O_E : U_{St}(\rho/O_E)]_A) = 0.$$

Thus:

$$[O_E : U_{St}(\rho/O_E)]_A \in \overline{\mathbb{F}_q}^\times \times U_\infty.$$

But $\text{sgn}([O_E : U_{St}(\rho/O_E)]_A) = 1$, thus:

$$[O_E : U_{St}(\rho/O_E)]_A \in U_\infty.$$

Let π' be another uniformizer of K_∞ , and let $[\cdot]'_A : \mathcal{I}(A) \rightarrow \overline{K}_\infty^\times$ be the map as in Section 1 associated to π' . Then, by the above discussion, we get:

$$[O_E : U_{St}(\rho/O_E)]'_A \in U_\infty.$$

Again, by Lemma 2.10, there exists an integer $m' \geq 1$ such that:

$$([O_E : U_{St}(\rho/O_E)]'_A)^{m'} = [O_E : U_{St}(\rho/O_E)]_A^{m'}.$$

Since $[O_E : U_{St}(\rho/O_E)]'_A, [O_E : U_{St}(\rho/O_E)]_A \in U_\infty$, we get:

$$[O_E : U_{St}(\rho/O_E)]'_A = [O_E : U_{St}(\rho/O_E)]_A.$$

This concludes the proof of the proposition. \square

Let's set:

$$\alpha_A(\rho/O_E) := \frac{L_A(\rho/O_E)}{[O_E : U_{St}(\rho/O_E)]_A} \in \overline{K}_\infty^\times.$$

By Proposition 2.12 and Remark 2.2, $\alpha_A(\rho/O_E)$ depends on A, ρ , and O_E , not on the choice of π . Furthermore:

$$\alpha_A(\rho/O_E) \in U_\infty.$$

Let's also observe that, if p^k is the exact power of p dividing $|\text{Pic}(A)|$, then:

$$\alpha_A(\rho/O_E)^{p^k d_\infty} \in K_\infty^\times.$$

We have the fundamental result due to L. Taelman ([23], Theorem 1):

Theorem 2.13 (Taelman). *Assume that the genus of K is zero and $d_\infty = 1$. Then:*

$$\alpha_A(\rho/O_E) = 1.$$

Proof. Select $\theta \in A \setminus \mathbb{F}_q$ such that $v_\infty(\theta) = 1$. Then $A = \mathbb{F}_q[\theta]$. Let $[\cdot]_A : \mathcal{I}(A) \rightarrow K_\infty^\times$ be the map as in Section 1 associated to the choice of $\frac{1}{\theta}$ as a uniformizer of K_∞ . Then, by Proposition 2.12, Theorem 2.7 and [23], Theorem 1:

$$[O_E : U_{St}(\rho/O_E)]_A = [O_E : U(\rho/O_E)]_A [H(\rho/O_E)]_A = L_A(\rho/O_E).$$

This concludes the proof of the theorem. \square

Corollary 2.14.

1) *Let F/K be a finite extension, $F \subset E$, and such that there exists a unique place of F above ∞ (still denoted by ∞). Let A' be the integral closure of A in F . Let $N_{F_\infty/K_\infty} : F_\infty \rightarrow K_\infty$ be the usual norm map. Then, there exists an integer $k \geq 1$ such that $\alpha_{A'}(\rho/O_E)^k \in F_\infty^\times$, $\alpha_A(\rho/O_E)^k \in K_\infty^\times$, and:*

$$N_{F_\infty/K_\infty}(\alpha_{A'}(\rho/O_E)^k) = \alpha_A(\rho/O_E)^k.$$

In particular, $\alpha_{A'}(\rho/O_E) = 1 \Rightarrow \alpha_A(\rho/O_E) = 1$.

2) *If there exists an integer $m \geq 1$ such that $\alpha_A(\rho/O_E)^m \in K^\times$, then $\alpha_A(\rho/O_E) = 1$. In particular, if $\sigma(\alpha_A(\rho/O_E)) = \alpha_A(\rho^\sigma/\sigma(O_E))$ for all $\sigma \in \text{Aut}_K(\mathbb{C}_\infty)$, then $\alpha_A(\rho/O_E) = 1$.*

Proof.

1) The first assertion is a consequence of Corollary 2.3 and Lemma 2.10. If $\alpha_{A'}(\rho/O_E) = 1$, then there exists an integer $k \geq 1$ such that $\alpha_A(\rho/O_E)^k = 1$. But, since $\text{sgn}(\alpha_A(\rho/O_E)) = 1$, we get $\alpha_A(\rho/O_E) = 1$.

2) Let $x = \alpha_A(\rho/O_E)^m \in K^\times$. Let P be a maximal ideal of A , and select an integer $l \geq 1$ such that P^l is a principal ideal. Let $\theta \in A \setminus \mathbb{F}_q$ such that $P^l = \theta A$. Let $L = \mathbb{F}_q(\theta)$ and $B = \mathbb{F}_q[\theta]$. Then, by Taelman's Theorem (Theorem 2.13), we have:

$$\alpha_B(\rho/O_E) = 1.$$

Therefore, by 1), we have:

$$N_{K/L}(x) \in \mathbb{F}_q^\times.$$

Since P is the only maximal ideal of A above θB , we deduce that x is a P -adic unit. Since this is true for all maximal ideal of A , we get:

$$x \in \mathbb{F}_q^\times.$$

But, $\text{sgn}(\alpha_A(\rho/O_E)) = 1$, thus: $\alpha_A(\rho/O_E) = 1$.

Let's assume that $\sigma(\alpha_A(\rho/O_E)) = \alpha_A(\rho^\sigma/\sigma(O_E))$ for all $\sigma \in \text{Aut}_K(\mathbb{C}_\infty)$. Let $\sigma \in \text{Aut}_K(\mathbb{C}_\infty)$. Let \mathfrak{P} be a maximal ideal of O_E , then:

$$\begin{aligned} \left[\frac{\sigma(O_E)}{\sigma(\mathfrak{P})} \right]_A &= \left[\frac{O_E}{\mathfrak{P}} \right]_A, \\ \left[\rho^\sigma \left(\frac{\sigma(O_E)}{\sigma(\mathfrak{P})} \right) \right]_A &= \left[\rho \left(\frac{O_E}{\mathfrak{P}} \right) \right]_A. \end{aligned}$$

Thus:

$$L_A(\rho^\sigma/\sigma(O_E)) = L_A(\rho/O_E).$$

Observe that σ induces a K_∞ -algebra isomorphism:

$$E_\infty \simeq \sigma(E)_\infty.$$

Note that $\exp_{\tilde{\rho}} : E[[z]] \rightarrow E[[z]]$ is an $\mathbb{F}_q[[z]]$ -algebra isomorphism. Therefore:

$$U(\tilde{\rho}/O_E[[z]]) \subset E[[z]].$$

Thus:

$$U(\tilde{\rho}^\sigma/\sigma(O_E)[z]) = \sigma(U(\tilde{\rho}/O_E[[z]])).$$

By the definition of Stark units, we get:

$$U_{St}(\rho^\sigma/\sigma(O_E)) = \sigma(U_{St}(\rho/O_E)).$$

Thus:

$$[\sigma(O_E) : U_{St}(\rho^\sigma/\sigma(O_E))]_A = [O_E : U_{St}(\rho/O_E)]_A.$$

Therefore:

$$\alpha_A(\rho^\sigma/\sigma(O_E)) = \alpha_A(\rho/O_E).$$

We get:

$$\forall \sigma \in \text{Aut}_K(\mathbb{C}_\infty), \quad \sigma(\alpha_A(\rho/O_E)) = \alpha_A(\rho/O_E).$$

This implies that $\alpha_A(\rho/O_E)$ is algebraic over K and that there exists an integer $k \geq 0$ such that:

$$\alpha_A(\rho/O_E)^{p^k} \in K^\times.$$

Therefore:

$$\alpha_A(\rho/O_E) = 1.$$

□

We do not know whether $\alpha_A(\rho/O_E)$ is algebraic over K , and it might be too naive to expect that $\alpha_A(\rho/O_E) = 1$ in general. However, in the next section, we will prove that, if ϕ is a sign-normalized rank one Drinfeld module and E/K is a finite abelian extension such that $H \subset E$, then $\alpha_A(\phi/O_E) = 1$ (Theorem 3.10). L. Taelman informed us that the class formula ([23], Theorem 1) has been generalized by C. Debyr to the case where A is a principal ideal domain.

We also prove below that $\alpha_A(\phi/O_E)$ is invariant under isogeny, which could be considered as an analogue of the isogeny invariance of the Birch and Swinnerton-Dyer conjecture due to Tate [24]:

Theorem 2.15. *Let E/K be a finite extension and let $\rho, \phi : A \rightarrow O_E\{\tau\}$ be two Drinfeld A -modules such that there exists $u \in O_E\{\tau\} \setminus \{0\}$ with the following property:*

$$\forall a \in A, \quad \rho_a u = u \phi_a,$$

then:

$$\alpha_A(\rho/O_E) = \alpha_A(\phi/O_E).$$

Proof. Let \mathfrak{P} be a maximal ideal of O_E such that $u \not\equiv 0 \pmod{\mathfrak{P}}$. Then by [18], Theorem 3.5 and Theorem 5.1, we get:

$$[\rho(\frac{O_E}{\mathfrak{P}})]_A = [\phi(\frac{O_E}{\mathfrak{P}})]_A.$$

This implies that there exists an ideal $I \in \mathcal{I}(A)$ such that:

$$\frac{L_A(\rho/O_E)}{L_A(\phi/O_E)} = [I].$$

Let $\zeta \in O_E \setminus \{0\}$ be the constant coefficient of u . Then we have the following equality in $E\{\{\tau\}\}$:

$$\exp_\rho \zeta = u \exp_\phi.$$

Thus:

$$\exp_{\tilde{\rho}} \zeta = \tilde{u} \exp_{\tilde{\phi}},$$

where, if $u = \sum_{i=0}^m u_i \tau^i$, $u_i \in O_E$, $\tilde{u} = \sum_{i=0}^m u_i z^i \tau^i$. This implies that:

$$\zeta U(\tilde{\phi}/O_E[z]) \subset U(\tilde{\rho}/O_E[z]).$$

Therefore:

$$\zeta U_{St}(\phi/O_E) \subset U_{St}(\rho/O_E).$$

We get:

$$[O_E : \zeta U_{St}(\phi/O_E)]_A = [O_E : U_{St}(\phi/O_E)]_A [\frac{O_E}{\zeta O_E}]_A,$$

and:

$$[O_E : \zeta U_{St}(\phi/O_E)]_A = [O_E : U_{St}(\rho/O_E)]_A [\frac{U_{St}(\rho/O_E)}{\zeta U_{St}(\phi/O_E)}]_A.$$

Therefore, there exists an element $J \in \mathcal{I}(A)$ such that:

$$\frac{[O_E : U_{St}(\rho/O_E)]_A}{[O_E : U_{St}(\phi/O_E)]_A} = [J].$$

Finally, we get:

$$\frac{\alpha_A(\rho/O_E)}{\alpha_A(\phi/O_E)} = [IJ^{-1}].$$

Let $x = (\frac{\alpha_A(\rho/O_E)}{\alpha_A(\phi/O_E)})^{h(q^{d_\infty}-1)} \in K^\times$, where $h = |\text{Pic}(A)|$. Then, by Corollary 2.14, and Theorem 2.13, if $\theta \in A \setminus \mathbb{F}_q$, there exists an integer $k \geq 1$ such that:

$$N_{K_\infty/\mathbb{F}_q((\frac{1}{\theta}))}(x^k) = 1.$$

But, by Proposition 2.12, x is a principal unit in K_∞ , thus:

$$N_{K/\mathbb{F}_q(\theta)}(x) = 1.$$

The above equality being valid for any $\theta \in A \setminus \mathbb{F}_q$, by the proof of Corollary 2.14, we deduce that:

$$x = 1.$$

Since $\text{sgn}(\frac{\alpha_A(\rho/O_E)}{\alpha_A(\phi/O_E)}) = 1$, we get:

$$\frac{\alpha_A(\rho/O_E)}{\alpha_A(\phi/O_E)} = 1.$$

□

3. STARK UNITS ASSOCIATED TO SIGN-NORMALIZED RANK ONE DRINFELD
MODULES

3.1. Zeta functions.

In this section, we briefly recall the definition of some zeta functions ([19], Chapter 8).

Recall that if $I \in \mathcal{I}(A)$, we have set:

$$[I] = \langle I \rangle \pi^{-\frac{\deg I}{d_\infty}} \in \overline{K}_\infty^\times,$$

where $v_\infty(\langle I \rangle - 1) > 0$, and:

$$\forall x \in K^\times, \quad \langle xA \rangle = \frac{x}{\text{sgn}(x)} \pi^{-v_\infty(x)}.$$

Let $\mathbb{S}_\infty = \mathbb{C}_\infty^\times \times \mathbb{Z}_p$ be the Goss ‘‘complex plane’’. The group action of \mathbb{S}_∞ is written additively. Let $I \in \mathcal{I}(A)$ and $s = (x; y) \in \mathbb{S}_\infty$, we set:

$$I^s = \langle I \rangle^y x^{\deg I} \in \mathbb{C}_\infty^\times.$$

We have a natural injective group homomorphism: $\mathbb{Z} \rightarrow \mathbb{S}_\infty, j \mapsto s_j = (\pi^{-\frac{j}{d_\infty}}, j)$. Observe that:

$$\forall j \in \mathbb{Z}, \forall I \in \mathcal{I}(A), \quad I^{s_j} = [I]^j.$$

Let E/K be a finite extension, and let O_E be the integral closure of A in E . Let \mathfrak{J} be a non-zero ideal of E . We have:

$$\forall j \in \mathbb{Z}, \quad N_{E/K}(\mathfrak{J})^{s_j} = \left[\frac{O_E}{\mathfrak{J}} \right]_A^j.$$

Let $s \in \mathbb{S}_\infty$, then the following sum converges in \mathbb{C}_∞ ([19], Theorem 8.9.2):

$$\zeta_{O_E}(s) := \sum_{d \geq 0} \sum_{\substack{\mathfrak{J} \in \mathcal{I}(O_E), \mathfrak{J} \subset O_E, \\ \deg(N_{E/K}(\mathfrak{J})) = d}} N_{E/K}(\mathfrak{J})^{-s}.$$

The function $\zeta_{O_E} : \mathbb{S}_\infty \rightarrow \mathbb{C}_\infty$ is called the *zeta function attached to O_E and $[\cdot]_A$* . Observe that:

$$\forall j \in \mathbb{Z}, \quad \zeta_{O_E}(j) := \zeta_{O_E}(s_j) = \sum_{d \geq 0} \sum_{\substack{\mathfrak{J} \in \mathcal{I}(A), \mathfrak{J} \subset O_E, \\ \deg(N_{E/K}(\mathfrak{J})) = d}} \left[\frac{O_E}{\mathfrak{J}} \right]_A^{-j}.$$

In particular:

$$\zeta_{O_E}(1) = \prod_{\mathfrak{P}} \left(1 - \frac{1}{\left[\frac{O_E}{\mathfrak{P}} \right]_A} \right)^{-1} \in \overline{K}_\infty^\times,$$

where \mathfrak{P} runs through the maximal ideals of O_E .

Lemma 3.1. *Let H_A be the Hilbert class field of A , i.e. H_A/K is the maximal unramified abelian extension of A in which ∞ splits completely. If $H_A \subset E$, then the function $\zeta_{O_E}(\cdot)$ depends only on O_E and $\text{sgn}|_{K_\infty^\times}$.*

Proof. Let \mathfrak{P} be a maximal ideal of O_E . Let A' be the integral closure of A in H_A . Let $P' = \mathfrak{P} \cap A', P = \mathfrak{P} \cap A$. By class field theory, $P^{\left[\frac{A'}{P'} : \frac{A}{P} \right]}$ is a principal ideal. Thus:

$$N_{E/K}(\mathfrak{P}) = \theta A,$$

for some $\theta \in A \setminus \mathbb{F}_q$. Let $j \in \mathbb{N}, j \geq 1$. We have:

$$\left(1 - \frac{1}{\left[\frac{O_E}{\mathfrak{P}}\right]_A^j}\right)^{-1} = \frac{\frac{\theta^j}{\text{sgn}(\theta^j)}}{\frac{\theta^j}{\text{sgn}(\theta^j)} - 1}.$$

But, observe that:

$$\zeta_{O_E}(j) = \prod_{\mathfrak{P}} \left(1 - \frac{1}{\left[\frac{O_E}{\mathfrak{P}}\right]_A^j}\right)^{-1} \in U_\infty \cap K_\infty^\times.$$

The lemma is thus a consequence of [19], Theorem 8.7.1. \square

3.2. Background on sign-normalized rank one Drinfeld modules.

Let $\phi : A \rightarrow \overline{K}_\infty\{\tau\}$ be a rank one Drinfeld module such that there exists $i(\phi) \in \mathbb{N}$ with the following property:

$$\forall a \in A \setminus \{0\}, \quad \phi_a = a + \dots + \text{sgn}(a)q^{i(\phi)}\tau^{\deg a}.$$

Such a Drinfeld module ϕ is said to be *sign-normalized*. By [19], Theorem 7.2.15, there always exist sign-normalized rank one Drinfeld modules.

From now on, we will fix a sign-normalized rank one Drinfeld module $\phi : A \rightarrow \overline{K}_\infty\{\tau\}$.

Let I_K be the group of idèles of K . Let's consider the following subgroup of the idèles of K :

$$K^\times \ker \text{sgn} \Big|_{K_\infty^\times} \prod_{v \neq \infty} O_v^\times,$$

where for a place v of K , O_v denotes the valuation ring of the v -adic completion of K . By class field theory, there exists a unique finite abelian extension H/K such that the reciprocity map induces an isomorphism:

$$\frac{I_K}{K^\times \ker \text{sgn} \Big|_{K_\infty^\times} \prod_{v \neq \infty} O_v^\times} \simeq \text{Gal}(H/K).$$

The natural surjective homomorphism $I_K \rightarrow \mathcal{I}(A)$ induces an isomorphism given by the Artin map $(\cdot, H/K)$:

$$\frac{\mathcal{I}(A)}{\mathcal{P}_+(A)} \simeq \text{Gal}(H/K),$$

where $\mathcal{P}_+(A) = \{xA, x \in K, \text{sgn}(x) = 1\}$. Let H_A be the Hilbert class field of A , i.e. H_A/K corresponds to the following subgroup of the idèles of K :

$$K^\times K_\infty^\times \prod_{v \neq \infty} O_v^\times.$$

Then H/K is unramified outside ∞ , and H/H_A is totally ramified at the places of H_A above ∞ . Furthermore:

$$\text{Gal}(H/H_A) \simeq \frac{\mathbb{F}_\infty^\times}{\mathbb{F}_q^\times}.$$

If w is a place of H above ∞ , then the w -adic completion of H is isomorphic to:

$$K_\infty \left(((-1)^{d_\infty - 1} \pi)^{\frac{q-1}{q^{d_\infty - 1}}} \right).$$

We denote by B the integral closure of A in H and set $A' = B \cap H_A$. We observe that $\mathbb{F}_\infty \subset A'$.

We denote by G the Galois group $\text{Gal}(H/K)$. For $I \in \mathcal{I}(A)$, we set:

$$(3.1) \quad \sigma_I = (I, H/K) \in G.$$

By [19], Proposition 7.4.2 and Corollary 7.4.9, the subfield of \mathbb{C}_∞ generated by K and the coefficients of ϕ_a is H . Furthermore ([19], Lemma 7.4.5):

$$\forall a \in A, \quad \phi_a \in B\{\tau\}.$$

Let I be a non-zero ideal of A , and let's define ϕ_I to be the unitary element in $H\{\tau\}$ such that:

$$H\{\tau\}\phi_I = \sum_{a \in I} H\{\tau\}\phi_a.$$

We have:

$$\begin{aligned} \ker \phi_I &= \bigcap_{a \in I} \ker \phi_a, \\ \phi_I &\in B\{\tau\}, \\ \deg_\tau \phi_I &= \deg I. \end{aligned}$$

We write: $\phi_I = \phi_{I,0} + \dots + \phi_{I,\deg I} \tau^{\deg I}$ with $\phi_{I,\deg I} = 1$ and denote by $\psi(I) \in B \setminus \{0\}$ the constant coefficient $\phi_{I,0}$ of ϕ_I .

Lemma 3.2. *The map ψ extends uniquely into a map $\psi : \mathcal{I}(A) \rightarrow H^\times$ with the following properties:*

- 1) for all $I, J \in \mathcal{I}(A)$, $\psi(IJ) = \sigma_J(\psi(I)) \psi(J)$,
- 2) for all $I \in \mathcal{I}(A)$, $IB = \psi(I)B$,
- 3) for all $x \in K^\times$, $\psi(xA) = \frac{x}{\text{sgn}(x)^{q^{i(\phi)}}}$.

In particular, we have:

$$\forall x \in K^\times, \quad \sigma_{xA}(\psi(I)) = \text{sgn}(x)^{q^{i(\phi)} - q^{i(\phi) + \deg I}} \psi(I).$$

Proof. Let $I \in \mathcal{I}(A)$, select $a \in A$, $\text{sgn}(a) = 1$, such that $aI \subset A$. Let's set:

$$\psi(I) := \frac{\psi(aI)}{a} \in H^\times.$$

By [19], Theorem 7.4.8 and Theorem 7.6.2, the map $\psi : \mathcal{I}(A) \rightarrow H^\times$ is well-defined and satisfies the desired properties. \square

Note that the map ψ determines H and H_A :

Proposition 3.3. *We have:*

- 1) $H = K(\psi(I), I \in \mathcal{I}(A))$;
- 2) $H_A = K(\psi(I), I \in \mathcal{I}(A), \deg I \equiv 0 \pmod{d_\infty})$.

Proof.

1) Let $\sigma \in \text{Gal}(H/K(\psi(I), I \in \mathcal{I}(A)))$. Let $J \in \mathcal{I}(A)$ such that $\sigma = \sigma_J$. Then:

$$\forall I \in \mathcal{I}(A), \quad \sigma_I(\psi(J)) = \psi(J).$$

Therefore:

$$\psi(J) \in K^\times.$$

Since $JB = \psi(J)B$ (Lemma 3.2), we get that $J = xA$ for some $x \in K^\times$. Thus, for all $I \in \mathcal{I}(A)$, we get:

$$\text{sgn}(x)^{q^{i(\phi)} - q^{i(\phi) + \deg I}} = 1.$$

Since $\deg : \mathcal{I}(A) \rightarrow \mathbb{Z}$ is a surjective group homomorphism, this implies that $\text{sgn}(x) \in \mathbb{F}_q^\times$ and thus $J \in \mathcal{P}_+(A)$. Therefore $\sigma = 1$.

2) Set $E = K(\psi(I), I \in \mathcal{I}(A), \deg I \equiv 0 \pmod{d_\infty})$. Observe that:

$$\text{Gal}(H/H_A) = \{\sigma_{x_A}, x \in K^\times\}.$$

Thus:

$$K(\mathbb{F}_\infty) \subset E \subset H_A.$$

We also have:

$$\text{Gal}(H_A/K(\mathbb{F}_\infty)) = \{(I, H_A/K), I \in \mathcal{I}(A), \deg I \equiv 0 \pmod{d_\infty}\}.$$

Let $\sigma \in \text{Gal}(H_A/E)$. Then, there exists $J \in \mathcal{I}(A)$, $\deg J \equiv 0 \pmod{d_\infty}$, such that $\sigma = (J, H_A/K)$. But for all $I \in \mathcal{I}(A)$, $\deg I \equiv 0 \pmod{d_\infty}$, we have:

$$\psi(IJ) = \sigma(\psi(I))\psi(J) = \psi(I)\psi(J),$$

and therefore:

$$(I, H_A/K)(\psi(J)) = \psi(J).$$

This implies:

$$\psi(J) \in K(\mathbb{F}_\infty)^\times.$$

But:

$$JA[\mathbb{F}_\infty] = \psi(J)A[\mathbb{F}_\infty].$$

Thus J^{d_∞} is a principal ideal. But:

$$\psi(J^{d_\infty}) = \psi(J)^{d_\infty}.$$

In particular:

$$\psi(J)^{d_\infty \frac{q^{d_\infty} - 1}{q - 1}} \in K^\times.$$

Thus, if δ is the Frobenius in $\text{Gal}(K(\mathbb{F}_\infty)/K)$, there exists $\zeta \in \mathbb{F}_\infty^\times$ such that:

$$\delta(\psi(J)) = \zeta\psi(J).$$

Observe that:

$$N_{\mathbb{F}_\infty/\mathbb{F}_q}(\zeta) = 1.$$

Thus:

$$\zeta = \frac{\mu}{\delta(\mu)},$$

for some $\mu \in \mathbb{F}_\infty^\times$. This implies that:

$$\psi(J)\mu \in K^\times.$$

Therefore J is a principal ideal and thus $\sigma = 1$. □

We have the following crucial fact:

Proposition 3.4. *Let E/K be a finite extension such that $H \subset E$. Then:*

$$L_A(\phi/O_E) = \zeta_{O_E}(1).$$

Proof. Let \mathfrak{P} be a maximal ideal of O_E . Let $m = [\frac{O_E}{\mathfrak{P}} : \frac{A}{P}]$. Then:

$$N_{E/K}(\mathfrak{P}) = P^m.$$

Since $H \subset O_E$, by class field theory, we get:

$$P^m = \theta A, \quad \text{with } \theta \in A, \text{sgn}(\theta) = 1.$$

Since ϕ is a rank one Drinfeld module, it implies that:

$$\phi_\theta \equiv \tau^{m \deg P} \pmod{\mathfrak{P}}.$$

This implies that:

$$[\phi(\frac{O_E}{\mathfrak{P}})]_A = \theta - 1 = [\frac{O_E}{\mathfrak{P}}]_A - 1.$$

We get:

$$L_A(\phi/O_E) = \prod_{\mathfrak{P}} \frac{[\frac{O_E}{\mathfrak{P}}]_A}{[\frac{O_E}{\mathfrak{P}}]_A - 1} = \prod_{\mathfrak{P}} (1 - \frac{1}{[\frac{O_E}{\mathfrak{P}}]_A})^{-1} = \zeta_{O_E}(1).$$

□

3.3. Equivariant A -harmonic series: a detailed example.

We keep the notation of Section 3.2. Let z be an indeterminate over K_∞ , and recall that $\mathbb{T}_z(K_\infty)$ denotes the Tate algebra in the variable z with coefficients in K_∞ . Recall that:

$$\begin{aligned} H_\infty &= H \otimes_K K_\infty, \\ \mathbb{T}_z(H_\infty) &= H \otimes_K \mathbb{T}_z(K_\infty). \end{aligned}$$

For $n \in \mathbb{Z}$, we set:

$$Z_B(n; z) = \sum_{d \geq 0} \sum_{\substack{\mathfrak{J} \in \mathcal{I}(B), \mathfrak{J} \subset B, \\ \deg(N_{E/K}(\mathfrak{J}))=d}} [\frac{O_E}{\mathfrak{J}}]_A^{-n} z^d.$$

Then, by [19], Theorem 8.9.2, for all $n \in \mathbb{Z}$, $Z_B(n; \cdot)$ defines an entire function on \mathbb{C}_∞ , and:

$$\forall n \in \mathbb{N}, \quad Z_B(-n; z) \in A[z].$$

Observe that:

$$\forall n \in \mathbb{Z}, \quad Z_B(n; z) \in \mathbb{T}_z(K_\infty),$$

and:

$$\forall n \geq 1, \quad Z_B(n; z) = \prod_{\mathfrak{P}} (1 - \frac{z^{\deg(N_{H/K}(\mathfrak{P}))}}{[\frac{O_E}{\mathfrak{P}}]_A^n})^{-1} \in \mathbb{T}_z(K_\infty)^\times.$$

Finally, we note that:

$$Z_B(1; 1) = \zeta_B(1).$$

Recall that $G = \text{Gal}(H/K)$. Then $G \simeq \text{Gal}(H(z)/K(z))$ acts on $\mathbb{T}_z(H_\infty)$. We denote by $\mathbb{T}_z(H_\infty)[G]$ the non-commutative group ring where the commutation rule is given by:

$$\forall h, h' \in \mathbb{T}_z(H_\infty), \forall g, g' \in G, \quad hg.h'g' = hg(h')gg'.$$

Recall that for $I \in \mathcal{I}(A)$, we have set (3.1):

$$\sigma_I = (I, H/K) \in G.$$

Lemma 3.5. *Let $n \in \mathbb{Z}$. The following infinite sum converges in $\mathbb{T}_z(H_\infty)[G]$:*

$$\mathcal{L}(\phi/B; n; z) := \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(A), I \subset A, \\ \deg I = d}} \frac{z^{\deg I}}{\psi(I)^n} \sigma_I.$$

Furthermore, for all $n \geq 1$, we have:

$$\mathcal{L}(\phi/B; n; z) = \prod_P (1 - \frac{z^{\deg P}}{\psi(P)^n} \sigma_P)^{-1} \in (\mathbb{T}_z(H_\infty)[G])^\times$$

and for all $n \leq 0$:

$$\mathcal{L}(\phi/B; n; z) \in B[z][G].$$

Proof. Let $n \geq 1$. First let's observe that for any place w of H above ∞ :

$$\lim_{I \subset A, \deg I \rightarrow +\infty} w(\psi(I)) = +\infty.$$

Let P be a maximal ideal of A . Note that:

$$\forall k \geq 0, \quad \psi(P^{k+1}) = \sigma_P(\psi(P^k))\psi(P) = \sigma_P^k(\psi(P))\psi(P^k).$$

Thus:

$$\sum_{m \geq 0} \frac{z^{m \deg P}}{\psi(P^m)^n} \sigma_P^m \in \mathbb{T}_z(H_\infty)[G],$$

and we have:

$$\left(1 - \frac{z^{\deg P}}{\psi(P)^n} \sigma_P\right) \left(\sum_{m \geq 0} \frac{z^{m \deg P}}{\psi(P^m)^n} \sigma_P^m\right) = \left(\sum_{m \geq 0} \frac{z^{m \deg P}}{\psi(P^m)^n} \sigma_P^m\right) \left(1 - \frac{z^{\deg P}}{\psi(P)^n} \sigma_P\right) = 1.$$

Thus, we have:

$$\left(1 - \frac{z^{\deg P}}{\psi(P)^n} \sigma_P\right)^{-1} := \sum_{m \geq 0} \frac{z^{m \deg P}}{\psi(P^m)^n} \sigma_P^m \in (\mathbb{T}_z(H_\infty)[G])^\times.$$

Let P, Q be two distinct maximal ideals of A . We have:

$$\left(1 - \frac{z^{\deg P}}{\psi(P)^n} \sigma_P\right) \left(1 - \frac{z^{\deg Q}}{\psi(Q)^n} \sigma_Q\right) = \left(1 - \frac{z^{\deg Q}}{\psi(Q)^n} \sigma_Q\right) \left(1 - \frac{z^{\deg P}}{\psi(P)^n} \sigma_P\right) = \left(1 - \frac{z^{\deg(PQ)}}{\psi(PQ)^n} \sigma_{PQ}\right).$$

Therefore:

$$\mathcal{L}(\phi/B; n; z) = \prod_P \left(1 - \frac{z^{\deg P}}{\psi(P)^n} \sigma_P\right)^{-1} = \sum_{I \in \mathcal{I}(A), I \subset A} \frac{z^{\deg I}}{\psi(I)^n} \sigma_I \in (\mathbb{T}_z(H_\infty)[G])^\times.$$

Let $n \in \mathbb{Z}$. For $d \in \mathbb{N}$, we set:

$$S_{\psi,d}(B; n) = \sum_{\substack{I \in \mathcal{I}(A), I \subset A, \\ \deg I = d}} \psi(I)^{-n} \sigma_I \in H[G].$$

Let h be the order of $\frac{\mathcal{I}(A)}{\mathcal{P}_+(A)}$. Let $I_1, \dots, I_h \in \mathcal{I}(A) \cap A$ be a system of representatives of $\frac{\mathcal{I}(A)}{\mathcal{P}_+(A)}$. Then:

$$S_{\psi,d}(B; n) = \sum_{j=1}^h \psi(I_j)^{-n} \sigma_{I_j} \sum_{\substack{a \in K^\times, \text{sgn}(a)=1, \\ aI_j \subset A, \\ \deg(aI_j)=d}} a^{-n}.$$

Now, let's assume that $n \leq 0$. Then, by [5], Lemma 3.2, there exists an integer $d_0(n, \psi, H) \in \mathbb{N}$ such that, for all $d \geq d_0(n, \psi, H)$, for all $j \in \{1, \dots, h\}$, we have:

$$\sum_{\substack{a \in K^\times, \text{sgn}(a)=1, \\ aI_j \subset A, \\ \deg(aI_j)=d}} a^{-n} = 0.$$

Therefore, for $d \geq d_0(n, \psi, H)$, we have:

$$S_{\psi,d}(B; n) = 0.$$

Thus:

$$\forall n \in \mathbb{N}, \quad \mathcal{L}(\phi/B; -n; z) \in B[z][G].$$

□

The element $\mathcal{L}(\phi/B) := \mathcal{L}(\phi/B; 1; 1) \in (H_\infty[G])^\times$ will be called *the equivariant A -harmonic series* attached to ϕ/B .

Note that $\mathcal{L}(\phi/B; 1; z)$ induces a $\mathbb{T}_z(K_\infty)$ -linear map $\mathcal{L}(\phi/B; 1; z) : \mathbb{T}_z(H_\infty) \rightarrow \mathbb{T}_z(H_\infty)$. Since $\mathbb{T}_z(H_\infty)$ is a free $\mathbb{T}_z(K_\infty)$ -module of rank $[H : K]$ (recall that $\mathbb{T}_z(K_\infty)$ is a principal ideal domain), $\det_{\mathbb{T}_z(K_\infty)} \mathcal{L}(\phi/B; 1; z)$ is well-defined. We also observe that $\mathcal{L}(\phi/B)$ induces a K_∞ -linear map $\mathcal{L}(\phi/B) : H_\infty \rightarrow H_\infty$, and we denote by $\det_{K_\infty} \mathcal{L}(\phi/B)$ its determinant. Recall that $\text{ev} : \mathbb{T}_z(H_\infty) \rightarrow H_\infty$ is the H_∞ -linear map given by:

$$\forall f \in \mathbb{T}_z(H_\infty), \quad \text{ev}(f) = f|_{z=1}.$$

Observe that, if $\{e_1, \dots, e_n\}$ is a K -basis of H/K (recall that $n = [H : K]$), then:

$$H_\infty = \bigoplus_{i=1}^n K_\infty e_i,$$

$$\mathbb{T}_z(H_\infty) = \bigoplus_{i=1}^n \mathbb{T}_z(K_\infty) e_i.$$

We deduce that:

$$\det_{K_\infty} \mathcal{L}(\phi/B) = \text{ev}(\det_{\mathbb{T}_z(K_\infty)} \mathcal{L}(\phi/B; 1; z)).$$

Theorem 3.6. *We have:*

$$\det_{\mathbb{T}_z(K_\infty)} \mathcal{L}(\phi/B; 1; z) = Z_B(1; z).$$

In particular:

$$\det_{K_\infty} \mathcal{L}(\phi/B) = \zeta_B(1).$$

Proof. First, we recall that, by Lemma 3.5, we have the following equality in $\mathbb{T}_z(H_\infty)[G]$:

$$\prod_P \left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_P\right)^{-1} = \mathcal{L}(\phi/B; 1; z),$$

where P runs through the maximal ideals of A , and:

$$\left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_P\right)^{-1} = \sum_{n \geq 0} \frac{z^{n \deg P}}{\psi(P^n)} \sigma_{P^n}.$$

By the properties of ψ (Lemma 3.2), we have:

$$\lim_{N \rightarrow +\infty} \prod_{\deg P \geq N} \left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_P\right)^{-1} = 1.$$

Thus:

$$\det_{\mathbb{T}_z(K_\infty)} \mathcal{L}(\phi/B; 1; z) = \prod_P \det_{\mathbb{T}_z(K_\infty)} \left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_P\right)^{-1}.$$

Thus, we are led to compute:

$$\det_{\mathbb{T}_z(K_\infty)} \left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_P\right).$$

But $1 - \frac{z^{\deg P}}{\psi(P)} \sigma_P$ induces a $K[z]$ -linear map on $H[z]$. Thus:

$$\det_{\mathbb{T}_z(K_\infty)}\left(1 - \frac{z^{\deg P}}{\psi(P)}\sigma_P\right) = \det_{K[z]}\left(1 - \frac{z^{\deg P}}{\psi(P)}\sigma_P\right) \Big|_{H[z]}.$$

Let $e \geq 1$ be the order of P in $\frac{\mathcal{I}(A)}{\mathcal{P}_+(A)}$. Write $\xi = \frac{z^{\deg P}}{\psi(P)}\sigma_P \Big|_{H[z]}$. We have $\xi^e = \frac{z^e \deg P}{\psi(P^e)} \in K[z]$. Since e is the order of σ_P in G , by Dedekind's Theorem $\sigma_P^0, \sigma_P, \dots, \sigma_P^{e-1}$ are linearly independent over $H(z)$. We deduce that $X^e - \frac{z^e \deg P}{\psi(P^e)}$ is the minimal polynomial of ξ over $K(z)$ and also over $H^{(\sigma_P)}(z)$, and that:

$$\det_{K[z]}\left(1 - \frac{z^{\deg P}}{\psi(P)}\sigma_P\right) \Big|_{H[z]} = \left(1 - \frac{z^e \deg P}{\psi(P^e)}\right)^{\frac{[H:K]}{e}}.$$

Now, let \mathfrak{P} be a maximal ideal of B above P . Then, by class field theory, we have:

$$\left[\frac{B}{\mathfrak{P}} : \frac{A}{P}\right] = e.$$

Therefore:

$$\left[\frac{B}{\mathfrak{P}}\right]_A = \psi(P^e).$$

Thus:

$$\det_{K[z]}\left(1 - \frac{z^{\deg P}}{\psi(P)}\sigma_P\right) \Big|_{H[z]} = \prod_{\mathfrak{P}|P} \left(1 - \frac{z^{\deg(N_{H/K}(\mathfrak{P}))}}{\left[\frac{B}{\mathfrak{P}}\right]_A}\right).$$

Finally, we get:

$$\det_{\mathbb{T}_z(K_\infty)}\mathcal{L}(\phi/B; 1; z) = \prod_{\mathfrak{P}} \left(1 - \frac{z^{\deg(N_{H/K}(\mathfrak{P}))}}{\left[\frac{B}{\mathfrak{P}}\right]_A}\right)^{-1},$$

where \mathfrak{P} runs through the maximal ideals of B . Thus:

$$\det_{\mathbb{T}_z(K_\infty)}\mathcal{L}(\phi/B; 1; z) = Z_B(1; z).$$

Now:

$$\det_{K_\infty}\mathcal{L}(\phi/B) = \text{ev}(\det_{\mathbb{T}_z(K_\infty)}\mathcal{L}(\phi/B; 1; z)) = \text{ev}(Z_B(1; z)) = \zeta_B(1).$$

□

Although this is not evident, the above theorem reflects a class formula à la Taelman which will be proved in Section 3.5.

3.4. Stark units.

We keep the notation of the previous sections. We will need the following basic result:

Lemma 3.7. *Let L/K be a finite extension, and let O_L be the integral closure of A in L . Let $\rho : A \rightarrow O_L\{\tau\}$ be a Drinfeld module of rank $r \geq 1$. Let $\exp_\rho, \log_\rho \in 1 + L\{\{\tau\}\}\tau$ be such that:*

$$\begin{aligned} \forall a \in A, \quad \exp_\rho a &= \rho_a \exp_\rho, \\ \exp_\rho \log_\rho &= \log_\rho \exp_\rho = 1. \end{aligned}$$

Write:

$$\begin{aligned} \exp_\rho &= \sum_{i \geq 0} e_i(\rho)\tau^i, \\ \log_\rho &= \sum_{i \geq 0} l_i(\rho)\tau^i, \end{aligned}$$

with $e_i(\rho), l_i(\rho) \in L$.

1) Let P be a maximal ideal of A . Let A_P be the P -adic completion of A . Then:

$$\begin{aligned} \forall n \geq 0, \quad P^{q^n} e_n(\rho) O_L &\subset P O_L \otimes_A A_P, \\ \forall n \geq 0, \quad P^{\lfloor \frac{n}{\deg P} \rfloor} l_n(\rho) O_L &\subset O_L \otimes_A A_P. \end{aligned}$$

2) Let $\sigma : L \hookrightarrow \overline{K}_\infty$ be a field homomorphism such that $\sigma|_K = \text{Id}_K$. Then, there exist $n(\rho, \sigma) \in \mathbb{N}, C(\rho, \sigma) \in]0; +\infty[$, such that:

$$\forall n \geq n(\rho, \sigma), \quad v_\infty(\sigma(e_n(\rho))) \geq C(\rho, \sigma) n q^n.$$

Proof.

1) Let $\theta \in A \setminus \mathbb{F}_q$ such that $\theta A_P = P A_P$. Let $d = r \deg(\theta)$, and let's write:

$$\rho_\theta = \sum_{j=0}^d \rho_{\theta,j} \tau^j.$$

From $\exp_\rho \theta = \rho_\theta \exp_\rho$, we get:

$$\forall n \geq 0, \quad (\theta^{q^n} - \theta) e_n(\rho) = \sum_{l=1}^d \rho_{\theta,l} e_{n-l}(\rho)^{q^l}$$

where $e_i = 0$ if $i < 0$. Since $e_0(\rho) = 1$, one proves by induction on $n \geq 0$ that

$$e_n(\rho) \theta^{q^n} \in \theta^{\inf\{q-1, q^n\}} O_L \otimes_A A_P.$$

Observe that:

$$\forall a \in A, \quad a \log_\rho = \log_\rho \rho_a.$$

Thus:

$$\forall a \in A, \forall n \geq 0, \quad (a - a^{q^n}) l_n(\rho) = \sum_{l=1}^{r \deg a} l_{n-l}(\rho) \rho_{a,l}^{q^{n-l}}.$$

Thus, if $n \not\equiv 0 \pmod{\deg P}$, we get:

$$l_n(\rho) O_E \otimes_A A_P \subset \sum_{l=1}^n l_{n-l}(\rho) O_L \otimes_A A_P.$$

If $n \equiv 0 \pmod{\deg P}$, we have:

$$(\theta - \theta^{q^n}) l_n(\rho) = \sum_{l=1}^d l_{n-l}(\rho) \rho_{\theta,l}^{q^{n-l}}.$$

In any case, we get:

$$\theta^{\lfloor \frac{n}{\deg P} \rfloor} l_n(\rho) \in \sum_{l=1}^n \theta^{\lfloor \frac{n-l}{\deg P} \rfloor} l_{n-l}(\rho) O_L \otimes_A A_P.$$

Since $l_0(\rho) = 1$, we get the desired second assertion by induction on $n \geq 0$.

2) This is a consequence of the proof of [19], Theorem 4.6.9. We give a proof for the convenience of the reader. We keep the previous notation. In particular, let $\theta \in A \setminus \mathbb{F}_q$, and write:

$$\rho_\theta = \sum_{j=0}^{r \deg(\theta)} \rho_{\theta,j} \tau^j, \quad \text{with } \rho_{\theta,j} \in \overline{K}_\infty.$$

Recall that $\rho_{\theta,0} = \theta$. Set $d = r \deg(\theta)$. Then:

$$\forall n \geq 0, \quad (\theta^{q^n} - \theta)e_n(\rho) = \sum_{l=1}^d \rho_{\theta,l} e_{n-l}(\rho)^{q^l}.$$

Set $u = \frac{\deg(\theta)}{d_\infty} = -v_\infty(\theta) \geq 1$. We get :

$$\frac{v_\infty(e_n(\rho))}{q^n} \geq u + \inf\left\{\frac{v_\infty(e_{n-j}(\rho))}{q^{n-j}} + \frac{v_\infty(\rho_{\theta,j})}{q^n}, j = 1, \dots, d\right\}.$$

Let $\beta \in]0; u[$. There exists an integer n_0 such that:

$$\forall n \geq n_0, \quad \inf\left\{\frac{v_\infty(\rho_{\theta,j})}{q^n}, j = 1, \dots, d\right\} \geq \beta - u.$$

Therefore:

$$\forall n \geq n_0, \quad \frac{v_\infty(e_n(\rho))}{q^n} \geq \beta + \inf\left\{\frac{v_\infty(e_{n-j}(\rho))}{q^{n-j}}, j = 1, \dots, d\right\}.$$

Thus, for $n \in [n_0; n_0 + d - 1]$, we get:

$$\frac{v_\infty(e_n(\rho))}{q^n} \geq \beta + \inf\left\{\frac{v_\infty(e_{n_0-j}(\rho))}{q^{n_0-j}}, j = 1, \dots, d\right\}.$$

Set:

$$C = \inf\left\{\frac{v_\infty(e_{n_0-j}(\rho))}{q^{n_0-j}}, j = 1, \dots, d\right\}.$$

By induction, we show that if $n \geq n_0 + md, m \in \mathbb{N}$, then:

$$\frac{v_\infty(e_n(\rho))}{q^n} \geq \beta(m+1) + C.$$

Therefore there exist $n_1 \geq n_0, C', C \in \mathbb{Q}$, with $C' > 0$, such that:

$$\forall n \geq n_1, \quad v_\infty(e_n(\rho)) \geq C' n q^n + C.$$

□

Let E/K be a finite abelian extension $H \subset E$. Let $G = \text{Gal}(E/K)$. We denote by S_E the set of maximal ideals P of A which are wildly ramified in E/K (note that we can have $S_E = \emptyset$). Let P be a maximal ideal of A such that $P \notin S_E$. We fix a maximal ideal \mathfrak{P} of O_E above P . Let $D_P \subset G$ be the decomposition group associated to P , i.e. $D_P = \{g \in G, g(\mathfrak{P}) = \mathfrak{P}\}$. We have a natural surjective homomorphism $D_P \twoheadrightarrow \text{Gal}(\frac{O_E}{\mathfrak{P}}/\frac{A}{P}), g \mapsto \bar{g}$. Let I_P be the inertia group at P , i.e. $I_P = \ker(D_P \rightarrow \text{Gal}(\frac{O_E}{\mathfrak{P}}/\frac{A}{P}))$. Then, since $P \notin S_E$, we have:

$$|I_P| \not\equiv 0 \pmod{p}.$$

Let $\text{Frob}_P \in \text{Gal}(\frac{O_E}{\mathfrak{P}}/\frac{A}{P})$ be the Frobenius at P , i.e.

$$\forall x \in \frac{O_E}{\mathfrak{P}}, \quad \text{Frob}_P(x) = x^{q^{\deg P}}.$$

We set:

$$\sigma_{P, O_E} := \frac{1}{|I_P|} \sum_{g \in D_P, \bar{g} = \text{Frob}_P} g \in \mathbb{F}_p[G].$$

If $P \in S_E$, we set:

$$\sigma_{P, O_E} = 0.$$

Note that, if L/K is a finite abelian extension, $L \subset E$, and if P is unramified in L with $P \notin S_E$, then:

$$\sigma_{P, O_E} |_{L=} = (P, L/K).$$

If $I \in \mathcal{I}(A)$, $I \subset A$, $I = \prod_P P^{m_P}$, we set:

$$\sigma_{I, O_E} = \prod_P \sigma_{P, O_E}^{m_P} \in \mathbb{F}_p[G].$$

For all $n \in \mathbb{Z}$, we set:

$$\mathcal{L}(\phi/O_E; n; z) = \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(A), I \subset A, \\ \deg I = d}} \frac{z^d}{\psi(I)^n} \sigma_{I, O_E} \in H[G][[z]].$$

By the proof of Lemma 3.5, we have:

$$\forall n \in \mathbb{Z}, \quad \mathcal{L}(\phi/O_E; n; z) \in \mathbb{T}_z(H_\infty)[G],$$

and:

$$\mathcal{L}(\phi/O_E; 1; z) = \prod_P \left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_{P, O_E}\right)^{-1} \in (\mathbb{T}_z(H_\infty)[G])^\times.$$

Note that, if L/K is a finite abelian extension, $H \subset L \subset E$, we have:

$$\mathcal{L}(\phi/O_E; 1; z) |_{\mathbb{T}_z(L_\infty)} = \left(\prod_{P \in S_E \setminus S_L} \left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_{P, O_L}\right) \mathcal{L}(\phi/O_L; 1; z) \right) |_{\mathbb{T}_z(L_\infty)}.$$

We set:

$$I(O_E) = \prod_{P \in S_E} P.$$

Recall that

$$U(\tilde{\phi}/O_E[z]) = \{f \in \mathbb{T}_z(E_\infty), \exp_{\tilde{\phi}}(f) \in O_E[z]\}.$$

Theorem 3.8. *We always have:*

$$\psi(I(O_E)) \mathcal{L}(\phi/O_E; 1; z) O_E[z] \subset U(\tilde{\phi}/O_E[z]).$$

Furthermore, if $S_E = \emptyset$, we have an equality:

$$\mathcal{L}(\phi/O_E; 1; z) O_E[z] = U(\tilde{\phi}/O_E[z]).$$

Proof. We divide the proof into several steps.

1) We will first work in $E[[z]]$. Observe that $\exp_{\tilde{\phi}} : E[[z]] \rightarrow \tilde{\phi}(E[[z]])$ is an isomorphism of A -modules. In fact, if we write: $\log_{\tilde{\phi}} = \sum_{i \geq 0} l_i(\phi) \tau^i$, then we set:

$$\log_{\tilde{\phi}} = \sum_{i \geq 0} l_i(\phi) z^i \tau^i.$$

Thus, $\log_{\tilde{\phi}}$ converges on $E[[z]]$, and $\log_{\tilde{\phi}} \exp_{\tilde{\phi}} = \exp_{\tilde{\phi}} \log_{\tilde{\phi}} = 1$.

2) Let P be a maximal ideal of A . Let $R_P = S^{-1}O_E \subset E$, where $S = A \setminus P$. Then:

$$PR_P = \psi(P)R_P.$$

By Lemma 3.7, we have:

$$\exp_{\tilde{\phi}}(PR_P[[z]]) \subset PR_P[[z]],$$

$$\log_{\tilde{\phi}}(PR_P[[z]]) \subset PR_P[[z]].$$

Thus:

$$(3.2) \quad \exp_{\tilde{\phi}}(PR_P[[z]]) = PR_P[[z]].$$

3) Recall that there exists a sign-normalized rank one Drinfeld module $\varphi := P * \phi : A \hookrightarrow B\{\tau\}$ such that:

$$\forall a \in A, \quad \phi_P \phi_a = \varphi_a \phi_P.$$

Furthermore ([19], Theorem 7.4.8):

$$\forall a \in A, \quad \varphi_a = \sigma_P(\phi_a) := \sum_{i=0}^{r \deg a} \sigma_P(\phi_{a,i}) \tau^i.$$

Thus:

$$\begin{aligned} \exp_{\varphi} &= \sigma_P(\exp_{\phi}) := \sum_{i \geq 0} \sigma_P(e_i(\phi)) \tau^i, \\ \log_{\varphi} &= \sigma_P(\log_{\phi}) := \sum_{i \geq 0} \sigma_P(l_i(\phi)) \tau^i. \end{aligned}$$

In particular:

$$\begin{aligned} \phi_P \exp_{\phi} &= \sigma_P(\exp_{\phi}) \psi(P), \\ \psi(P) \log_{\phi} &= \sigma_P(\log_{\phi}) \phi_P. \end{aligned}$$

The same properties hold for $\tilde{\phi}$.

4) Let's set:

$$U(\tilde{\phi}/R_P[[z]]) = \{x \in E[[z]]; \exp_{\tilde{\phi}}(x) \in R_P[[z]]\}.$$

Let's assume that $P \notin S_E$. Then, by 1) and 2), $\exp_{\tilde{\phi}}$ induces an isomorphism of A -modules:

$$\frac{E[[z]]}{PR_P[[z]]} \simeq \tilde{\phi}\left(\frac{E[[z]]}{PR_P[[z]]}\right).$$

Therefore, we get an isomorphism of A -modules:

$$\frac{U(\tilde{\phi}/R_P[[z]])}{PR_P[[z]]} \simeq \tilde{\phi}\left(\frac{R_P[[z]]}{PR_P[[z]]}\right).$$

Now observe that:

$$(\tilde{\phi}_P - z^{\deg P} \sigma_{P,O_E}) \tilde{\phi}\left(\frac{R_P[[z]]}{PR_P[[z]]}\right) = \{0\}.$$

Furthermore, if $x \in E[[z]] \setminus R_P[[z]]$, then one can easily verify that:

$$(\tilde{\phi}_P - z^{\deg P} \sigma_{P,O_E})(x) \notin PR_P[[z]].$$

Thus:

$$\tilde{\phi}\left(\frac{R_P[[z]]}{PR_P[[z]]}\right) = \{x \in \tilde{\phi}\left(\frac{E[[z]]}{PR_P[[z]]}\right), (\tilde{\phi}_P - z^{\deg P} \sigma_{P,O_E})(x) = 0\}.$$

Let $x \in E[[z]]$, we deduce that:

$$x \in U(\tilde{\phi}/R_P[[z]]) \Leftrightarrow (\tilde{\phi}_P - z^{\deg P} \sigma_{P,O_E})(\exp_{\tilde{\phi}}(x)) \in PR_P[[z]].$$

Observe that, by 3), we have:

$$\sigma_{P,O_E}(\exp_{\tilde{\phi}}) = \exp_{\tilde{\varphi}},$$

and also:

$$\tilde{\phi}_P \exp_{\tilde{\phi}} = \sigma_{P,O_E}(\exp_{\tilde{\varphi}}) \psi(P).$$

Thus:

$$x \in U(\tilde{\phi}/R_P[[z]]) \Leftrightarrow \exp_{\tilde{\phi}}(\psi(P)x - z^{\deg P} \sigma_{P, O_E}(x)) \in PR_P[[z]].$$

Applying (3.2) for φ , we have:

$$x \in U(\tilde{\phi}/R_P[[z]]) \Leftrightarrow \psi(P)x - z^{\deg P} \sigma_{P, O_E}(x) \in PR_P[[z]].$$

Thus:

$$U(\tilde{\phi}/R_P[[z]]) = \left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_{P, O_E}\right)^{-1} R_P[[z]].$$

5) Let P be a maximal ideal of A . If $P \notin S_E$, by 4), we have:

$$U(\tilde{\phi}/R_P[[z]]) = \psi(I(O_E))\mathcal{L}(\phi/O_E; 1; z)R_P[[z]] = \left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_{P, O_E}\right)^{-1} R_P[[z]].$$

If $P \in S_E$, then:

$$\psi(I(O_E))\mathcal{L}(\phi/O_E; 1; z)R_P[[z]] = PR_P[[z]] \subset U(\tilde{\phi}/R_P[[z]]).$$

Since $\psi(I(O_E))\mathcal{L}(\phi/O_E; 1; z) \in \mathbb{T}_z(H_\infty)[G]$, we get:

$$\psi(I(O_E))\mathcal{L}(\phi/O_E; 1; z)R[z] \subset \mathbb{T}_z(E_\infty).$$

Observe that $O_E[[z]] = \bigcap_P R_P[[z]]$. Therefore, we get:

$$\exp_{\tilde{\phi}}(\psi(I(O_E))\mathcal{L}(\phi/O_E; 1; z)O_E[z]) \subset O_E[[z]] \cap \mathbb{T}_z(E_\infty) = O_E[z].$$

Thus, we get the first assertion.

Now, let's assume that $S_E = \emptyset$. We have:

$$\bigcap_P U(\tilde{\phi}/R_P[[z]]) = \{x \in E_\infty[[z]], \exp_{\tilde{\phi}}(x) \in O_E[[z]]\}.$$

By 4), we get:

$$\prod_P \left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_{P, O_E}\right) \{x \in E_\infty[[z]], \exp_{\tilde{\phi}}(x) \in O_E[[z]]\} = O_E[[z]].$$

Thus:

$$\{x \in E_\infty[[z]], \exp_{\tilde{\phi}}(x) \in O_E[[z]]\} = \mathcal{L}(\phi/O_E; 1; z)O_E[[z]].$$

Hence:

$$U(\tilde{\phi}/R[[z]]) = \mathcal{L}(\phi/O_E; 1; z)O_E[[z]] \cap \mathbb{T}_z(E_\infty).$$

Since $\mathcal{L}(\phi/O_E; 1; z) \in (\mathbb{T}_z(H_\infty)[G])^\times$, we have:

$$\mathcal{L}(\phi/O_E; 1; z)O_E[[z]] \cap \mathbb{T}_z(E_\infty) = \mathcal{L}(\phi/O_E; 1; z)O_E[z].$$

This concludes the proof of the theorem. \square

3.5. A class formula à la Taelman.

Recall that $\text{ev} : \mathbb{T}_z(E_\infty) \rightarrow E_\infty$ is the evaluation at $z = 1$.

Definition 3.9. We define the *equivariant A -harmonic series* $\mathcal{L}(\phi/O_E)$ attached to ϕ/O_E by:

$$\mathcal{L}(\phi/O_E) = \text{ev}(\mathcal{L}(\phi/O_E; 1; z)) \in (H_\infty[G])^\times.$$

Note that:

$$\mathcal{L}(\phi/O_E) = \prod_P \left(1 - \frac{1}{\psi(P)} \sigma_{P, O_E}\right)^{-1} = \sum_{I \in \mathcal{I}(A), I \subset A} \frac{1}{\psi(I)} \sigma_{I, O_E}.$$

Theorem 3.10. *We have:*

$$\alpha_A(\phi/O_E) = 1,$$

i.e.

$$\zeta_{O_E}(1) = [O_E : U(\phi/O_E)]_A [H(\phi/O_E)]_A.$$

Furthermore:

$$\psi(I(O_E)) \mathcal{L}(\phi/O_E) O_E \subset U_{St}(\phi/O_E),$$

and

$$\left[\frac{U_{St}(\phi/O_E)}{\psi(I(O_E)) \mathcal{L}(\phi/O_E) O_E} \right]_A = \left[\phi \left(\frac{O_E}{I(O_E) O_E} \right) \right]_A.$$

Proof.

1) Let $J \subset I(O_E)$ be a finite product of maximal ideals of A . Set:

$$\mathcal{L}_J(\phi/O_E) := \prod_P \left(1 - \frac{1}{\psi(P)} \sigma_{P, O_E}\right)^{-1} \in (H_\infty[G])^\times,$$

$$\mathcal{L}_J(\phi/O_E; 1; z) := \prod_P \left(1 - \frac{z^{\deg P}}{\psi(P)} \sigma_{P, O_E}\right)^{-1} \in (\mathbb{T}_z(H_\infty)[G])^\times,$$

where P runs through the maximal ideals of A that do not divide J .

By Lemma 3.7 and the proof of Theorem 3.8, we have:

$$\{x \in E[[z]], \exp_{\tilde{\phi}}(x) \in \psi(J) O_E[[z]]\} = \psi(J) \mathcal{L}_J(\phi/O_E; 1; z) O_E[[z]].$$

We can conclude as in the proof of Theorem 3.8 that:

$$\psi(J) \mathcal{L}_J(\phi/O_E; 1; z) O_E[z] = \{x \in \mathbb{T}_z(E_\infty), \exp_{\tilde{\phi}}(x) \in \psi(J) O_E[z]\}.$$

Therefore, we have a short exact sequence of A -modules:

$$(3.3) \quad 0 \rightarrow \frac{U(\tilde{\phi}/O_E[z])}{\psi(J) \mathcal{L}_J(\phi/O_E; 1; z) O_E[z]} \rightarrow \tilde{\phi} \left(\frac{O_E[z]}{\psi(J) O_E[z]} \right) \rightarrow \\ \rightarrow \frac{\mathbb{T}_z(E_\infty)}{\psi(J) O_E[z] + \exp_{\tilde{\phi}}(\mathbb{T}_z(E_\infty))} \rightarrow H(\tilde{\phi}/O_E[z]) \rightarrow 0.$$

Note that $\tilde{\phi} \left(\frac{O_E[z]}{\psi(J) O_E[z]} \right)$ is a finitely generated and free $\mathbb{F}_q[z]$ -module. Let ρ be the Drinfeld module defined over O_E such that:

$$\exp_\rho = \psi(J)^{-1} \exp_{\tilde{\phi}} \psi(J).$$

Then, the map $x \mapsto \psi(J)^{-1} x$ induces an isomorphism of A -modules (the left module is an A -module via ϕ and the right module is an A -module via ρ):

$$\frac{\mathbb{T}_z(E_\infty)}{\psi(J) O_E[z] + \exp_{\tilde{\phi}}(\mathbb{T}_z(E_\infty))} \simeq H(\tilde{\rho}/O_E[z]).$$

Observe that $\ker \text{ev} = (z-1)\mathbb{T}_z(E_\infty)$. Furthermore, since $O_E[z] \cap (z-1)\mathbb{T}_z(E_\infty) = (z-1)O_E[z]$, we have :

$$U(\tilde{\phi}/O_E[z]) \cap \ker \text{ev} = (z-1)U(\tilde{\phi}/O_E[z]),$$

$$\psi(J)\mathcal{L}_J(\phi/O_E; 1; z)O_E[z] \cap \ker \text{ev} = (z-1)\psi(J)\mathcal{L}_J(\phi/O_E; 1; z)O_E[z].$$

Thus, the evaluation at $z = 1$ induces the following exact sequence of A -modules:

$$(3.4) \quad 0 \rightarrow (z-1) \frac{U(\tilde{\phi}/O_E[z])}{\psi(J)\mathcal{L}_J(\phi/O_E; 1; z)O_E[z]} \rightarrow \frac{U(\tilde{\phi}/O_E[z])}{\psi(J)\mathcal{L}_J(\phi/O_E; 1; z)O_E[z]} \rightarrow \frac{U_{St}(\phi/O_E)}{\psi(J)\mathcal{L}_J(\phi/O_E)O_E} \rightarrow 0.$$

Note also that the evaluation at $z = 1$ induces a sequence of A -modules:

$$(3.5) \quad 0 \rightarrow (z-1)\tilde{\phi}\left(\frac{O_E[z]}{\psi(J)O_E[z]}\right) \rightarrow \tilde{\phi}\left(\frac{O_E[z]}{\psi(J)O_E[z]}\right) \rightarrow \phi\left(\frac{O_E}{\psi(J)O_E}\right) \rightarrow 0.$$

For an $\mathbb{F}_q[z]$ -module M , we denote by $M[z-1]$ the $(z-1)$ -torsion. By (3.3), (3.4), (3.5) and the Snake Lemma, we get the following exact sequence of finite A -modules:

$$\begin{aligned} 0 \rightarrow H(\tilde{\rho}/O_E[z])[z-1] &\rightarrow H(\tilde{\phi}/O_E[z])[z-1] \rightarrow \frac{U_{St}(\phi/O_E)}{\psi(J)\mathcal{L}_J(\phi/O_E)O_E} \rightarrow \\ &\rightarrow \phi\left(\frac{O_E}{\psi(J)O_E}\right) \rightarrow H(\rho/O_E) \rightarrow H(\phi/O_E) \rightarrow 0. \end{aligned}$$

By the proof of Theorem 2.7, we have:

$$[H(\tilde{\rho}/O_E[z])[z-1]]_A = [H(\rho/O_E)]_A,$$

$$[H(\tilde{\phi}/O_E[z])[z-1]]_A = [H(\phi/O_E)]_A.$$

Thus:

$$\left[\frac{U_{St}(\phi/O_E)}{\psi(J)\mathcal{L}_J(\phi/O_E)O_E}\right]_A = \left[\phi\left(\frac{O_E}{JO_E}\right)\right]_A.$$

2) Now, we have:

$$[O_E : \mathcal{L}_J(\phi/O_E)O_E]_A = \frac{\det_{K_\infty} \mathcal{L}_J(\phi/O_E)}{\text{sgn}(\det_{K_\infty} \mathcal{L}_J(\phi/O_E))}.$$

Thus:

$$[O_E : \psi(J)\mathcal{L}_J(\phi/O_E)O_E]_A = \left[\frac{O_E}{JO_E}\right]_A \frac{\det_{K_\infty} \mathcal{L}_J(\phi/O_E)}{\text{sgn}(\det_{K_\infty} \mathcal{L}_J(\phi/O_E))}.$$

And finally, we get:

$$[O_E : U_{St}(\phi/O_E)]_A = \frac{\left[\frac{O_E}{JO_E}\right]_A}{\left[\phi\left(\frac{O_E}{JO_E}\right)\right]_A} \frac{\det_{K_\infty} \mathcal{L}_J(\phi/O_E)}{\text{sgn}(\det_{K_\infty} \mathcal{L}_J(\phi/O_E))}.$$

Set:

$$L_J = \prod_{\mathfrak{p}|J} \frac{\left[\frac{O_E}{\mathfrak{p}}\right]_A}{\left[\phi\left(\frac{O_E}{\mathfrak{p}}\right)\right]_A}.$$

Then:

$$[O_E : U_{St}(\phi/O_E)]_A = L_J \frac{\det_{K_\infty} \mathcal{L}_J(\phi/O_E)}{\text{sgn}(\det_{K_\infty} \mathcal{L}_J(\phi/O_E))}.$$

3) Let $N \geq 1$, and we define J_N to be the l.c.m. of the product of all maximal ideals of degree $\leq N$ and $I(O_E)$. We have:

$$\begin{aligned} \lim_{N \rightarrow +\infty} L_{J_N} &= L_A(\phi/O_E), \\ \lim_{N \rightarrow +\infty} \mathcal{L}_{J_N}(\phi/O_E) &= 1. \end{aligned}$$

In particular:

$$\lim_{N \rightarrow +\infty} \det_{K_\infty} \mathcal{L}_{J_N}(\phi/O_E) = 1.$$

Thus:

$$[O_E : U_{St}(\phi/O_E)]_A = L_A(\phi/O_E).$$

If we apply Theorem 2.7 and Proposition 3.4, we get:

$$\zeta_{O_E}(1) = [O_E : U(\phi/O_E)]_A [H(\phi/O_E)]_A.$$

□

4. LOG-ALGEBRAICITY THEOREM

4.1. A refinement of Anderson's log-algebraicity theorem.

We keep the notation of the previous sections.

Lemma 4.1. *Let E/K be a finite separable extension, $H \subset E$. Let P be a maximal ideal of A which is unramified in E . Let $\lambda_P \in \overline{K} \setminus \{0\}$ be a root of ϕ_P . Then:*

$$O_{E(\lambda_P)} = O_E[\lambda_P].$$

Proof. Let $F = E(\lambda_P)$. Recall that F/E is a finite abelian extension unramified outside P, ∞ , and totally ramified at P ([19], Proposition 7.5.18). We also have:

$$[F : E] = q^{\deg P} - 1.$$

Write: $\phi_P = \sum_{k=0}^{\deg P} \phi_{P,k} T^k$, $\phi_{P,k} \in B \subset O_E$. Recall that $\phi_{P,0} = \psi(P)$ and $\phi_{P,\deg P} = 1$. Furthermore, P is unramified in E/K and:

$$\psi(P)O_E = PO_E.$$

Let:

$$G(X) = \sum_{k=0}^{\deg P} \phi_{P,k} X^{q^k - 1} \in O_E[X].$$

Then, for any maximal ideal \mathfrak{P} of O_E above P :

$$G(X) \equiv X^{q^{\deg P} - 1} \pmod{\mathfrak{P}}.$$

This implies that $G(X)$ is an Eisenstein polynomial at \mathfrak{P} for every maximal ideal of O_E above P . Furthermore:

$$XG'(X) + G(X) = \psi(P).$$

Therefore:

$$N_{F/E}(G'(\lambda_P))O_E = P^{q^{\deg P} - 2}O_E.$$

But $P^{q^{\deg P} - 2}O_E$ is the discriminant of O_F/O_E . Thus $O_F = O_E[\lambda_P]$. □

Let E/K be a finite abelian extension, $H \subset E$. Let $n \geq 0$ be an integer, let X_1, \dots, X_n be n indeterminates over K . Let $\tau : E[X_1, \dots, X_n][[z]] \rightarrow E[X_1, \dots, X_n][[z]]$ be the $\mathbb{F}_q[[z]]$ -homomorphism continuous for the z -adic topology such that:

$$\forall f \in E[X_1, \dots, X_n], \quad \tau(f) = f^q.$$

For a non-zero ideal I of A and for $f = \sum_{i_1, \dots, i_n \in \mathbb{N}} f_{i_1, \dots, i_n} X_1^{i_1} \cdots X_n^{i_n} \in E[X_1, \dots, X_n]$, with $f_{i_1, \dots, i_n} \in E$, we set:

$$I *_{E} f = \sum_{i_1, \dots, i_n \in \mathbb{N}} \sigma_{I, O_E}(f_{i_1, \dots, i_n}) \phi_I(X_1)^{i_1} \cdots \phi_I(X_n)^{i_n},$$

where σ_{I, O_E} is defined in Section 3.4. Recall that $I(O_E)$ is the product of maximal ideals of A that are wildly ramified in E/K .

Theorem 4.2. *For all $f \in O_E[X_1, \dots, X_n]$, we have:*

$$\exp_{\tilde{\phi}}(\psi(I(O_E))) \sum_{I \in \mathcal{I}(A), I \subset A} \frac{I *_{E} f}{\psi(I)} z^{\deg I} \in O_E[X_1, \dots, X_n, z].$$

In particular, for all $f \in B[X_1, \dots, X_n]$, we have:

$$\exp_{\tilde{\phi}}\left(\sum_{I \in \mathcal{I}(A), I \subset A} \frac{I *_{H} f}{\psi(I)} z^{\deg I}\right) \in B[X_1, \dots, X_n, z].$$

Remark 4.3. This result is a generalization of the Log-Algebraicity Theorems established in [1], [2] (in these papers the theorem is proved for $E = H$, $d_{\infty} = 1$ and $n \leq 1$). Furthermore, the result in the case $E = H$ can be proved along the same lines as that used to prove [2], Theorem 3. Following [9], Section 2.6, we will show below how Theorem 3.8 implies the Log-Algebraicity Theorem. Observe also that the case $n = 0$ is a direct consequence of Theorem 3.8.

Proof. Let's write:

$$\exp_{\tilde{\phi}}(\psi(I(O_E))) \sum_I \frac{I *_{E} f}{\psi(I)} z^{\deg I} = \sum_{m \geq 0} g_m(X_1, \dots, X_n) z^m,$$

with $g_m(X_1, \dots, X_n) \in E[X_1, \dots, X_n]$.

1) Let P_1, \dots, P_n be n distinct maximal ideals of A which are unramified in E , with $q^{\deg P_i} \geq 3$, $i = 1, \dots, n$, and for $i = 1, \dots, n$, let $\lambda_i \neq 0$ be a root of ϕ_{P_i} . Set:

$$F = E(\lambda_1, \dots, \lambda_n).$$

Then F/E is unramified outside P_1, \dots, P_n, ∞ , F/K a finite abelian extension of K which is tamely ramified at P_1, \dots, P_n . Let O_F be the integral closure of A in F . Let Q be any maximal ideal of A , if Q is not wildly ramified in E , we have ([19], Proposition 7.5.4):

$$\sigma_{Q, O_F}(\lambda_i) = \phi_Q(\lambda_i), \quad \text{if } Q \neq P_i,$$

and:

$$\sigma_{P_i, O_F}(\lambda_i) = 0.$$

We deduce that:

$$\psi(I(O_E)) \sum_I \frac{I *_{E} f}{\psi(I)} z^{\deg I} \Big|_{X_i = \lambda_i} = \psi(I(O_F)) \mathcal{L}(\phi/O_F; 1; z) f(\lambda_1, \dots, \lambda_n).$$

Therefore, by Theorem 3.8, we get:

$$\forall m \geq 0, \quad g_m(\lambda_1, \dots, \lambda_n) \in O_F.$$

Let $i \in \{1, \dots, n\}$. Then:

$$E(\lambda_i) \cap E(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n) = E.$$

Furthermore, the discriminant of $O_{E(\lambda_i)}/O_E$ and $O_{E(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n)}/O_E$ are relatively prime, thus, by Lemma 4.1, we have:

$$O_F = O_E[\lambda_1, \dots, \lambda_n].$$

Finally, for $m \geq 0$, for n distinct maximal ideals P_1, \dots, P_n of A that are unramified in E/K , with $q^{\deg P_i} \geq 3, i = 1, \dots, n$, and for $i = 1, \dots, n$, if $\lambda_i \neq 0$ be a root of ϕ_{P_i} , then we have:

$$g_m(\lambda_1, \dots, \lambda_n) \in O_E[\lambda_1, \dots, \lambda_n].$$

This implies:

$$\forall m \geq 0, \quad g_m(X_1, \dots, X_n) \in O_E[X_1, \dots, X_n].$$

2) We fix a K -embedding of \overline{K} in \mathbb{C}_∞ . For $\sigma \in \text{Gal}(H/K)$, let $\Lambda(\phi^\sigma) \subset \mathbb{C}_\infty$ be the A -module of periods of ϕ^σ , and let $\Lambda(\phi^\sigma)K_\infty$ be the K_∞ -vector space generated by $\Lambda(\phi^\sigma)$. Then $\frac{\Lambda(\phi^\sigma)K_\infty}{\Lambda(\phi^\sigma)}$ is compact, thus there exists a constant $C \in \mathbb{R}$ such that:

$$\forall \sigma \in \text{Gal}(H/K), \forall x \in \Lambda(\phi^\sigma)K_\infty, \quad v_\infty(\exp_{\phi^\sigma}(x)) \geq C.$$

Recall that, if $\sigma \in \text{Gal}(H/K)$, then there exists a non-zero ideal J of A such that $\sigma = (J, H/K) = \sigma_J$, and we have ([19], Theorem 7.4.8):

$$\phi_J \phi_a = \phi_a^\sigma \phi_J.$$

Thus:

$$\exp_{\phi^\sigma} \psi(J) = \phi_J \exp_\phi.$$

In particular:

$$\begin{aligned} \Lambda(\phi^\sigma) &= \psi(J)J^{-1}\Lambda(\phi), \\ \Lambda(\phi^\sigma)K_\infty &= \psi(J)\Lambda(\phi)K_\infty. \end{aligned}$$

Therefore, there exists a constant $C' \in \mathbb{R}$, such that:

$$\forall \sigma \in \text{Gal}(H/K), \forall x_1, \dots, x_n \in \Lambda(\phi^\sigma)K_\infty, \forall I \in \mathcal{I}(A), \quad v_\infty(I *_{E}^{\sigma} f^\sigma |_{X_i = \exp_{\phi^\sigma}(x_i)}) \geq C',$$

where $*_{E}^{\sigma}$ is the map $*$ attached to ϕ^σ . Now, recall that $\exp_\phi = \sum_{j \geq 0} e_j(\phi)\tau^j$, then there exists a constant $C'' > 0$ such that (Lemma 3.7):

$$\forall \sigma \in \text{Gal}(H/K), \forall j \gg 0, \quad v_\infty(e_j(\phi^\sigma)) \geq C'' j q^j.$$

Note also that there exists $C''' \in \mathbb{R}$ such that:

$$\forall \sigma \in \text{Gal}(H/K), \forall I \in \mathcal{I}(A), \deg I = m \gg 0, \quad v_\infty\left(\frac{1}{\sigma(\psi(I))}\right) \geq \frac{m}{d_\infty} + C'''.$$

This implies that there exists an integer $m_0 \in \mathbb{N}$, such that:

$$\forall m \geq m_0, \forall \sigma \in \text{Gal}(E/K), \forall \lambda_1, \dots, \lambda_n \in \exp_{\phi^\sigma}(\Lambda(\phi^\sigma)K_\infty), \quad v_\infty(g_m^\sigma(\lambda_1, \dots, \lambda_n)) > 0.$$

3) Let $m_0 \in \mathbb{N}$ be as in 2). Let $\lambda_1, \dots, \lambda_n$ be n torsion points for ϕ . Let $F = E(\lambda_1, \dots, \lambda_n)$. Then F/K is a finite abelian extension. Let w be a place of F above ∞ . Let $i_w : E \rightarrow \mathbb{C}_\infty$ be the K -embedding of F in \mathbb{C}_∞ corresponding to w . Then there exists $\sigma \in \text{Gal}(F/K)$ such that:

$$\forall m \geq 0, \quad i_w(g_m(\lambda_1, \dots, \lambda_n)) = \sigma(g_m(\lambda_1, \dots, \lambda_n)) = g_m^\sigma(\sigma(\lambda_1), \dots, \sigma(\lambda_n)).$$

Observe that $\sigma(\lambda_i) \in \exp_{\phi^\sigma}(\Lambda(\phi^\sigma)K_\infty)$, $i = 1, \dots, n$ ([19], Proposition 7.5.16). Therefore:

$$\forall m \geq m_0, \quad w(g_m(\lambda_1, \dots, \lambda_n)) > 0.$$

Thus, we get that for any place w of F above ∞ :

$$\forall m \geq m_0, \quad w(g_m(\lambda_1, \dots, \lambda_n)) > 0.$$

But by 1), $\forall m \geq 0$, $g_m(\lambda_1, \dots, \lambda_n) \in O_F$. Since O_F is the set of elements of F which are regular outside the places of F above ∞ , we deduce that:

$$\forall m \geq m_0, \quad g_m(\lambda_1, \dots, \lambda_n) = 0.$$

And the above property is true for any n torsion points of ϕ , thus:

$$\forall m \geq m_0, \quad g_m(X_1, \dots, X_n) = 0.$$

□

M. Papanikolas informed us that, together with N. Green, they obtained explicit formulas for Anderson's Log-Algebraicity Theorem ([1], Theorem 5.1.1) when the genus g of K is one and $d_\infty = 1$.

4.2. Several variable L -series and shtukas.

In this section, we present an alternative approach to the several variable Log-Algebraicity Theorem (Theorem 4.2) by using the seminal works of Drinfeld [12], [13], [14] (see also [1], [27], and [19], Chapter 6).

We recall some notation for the convenience of the reader. Let X/\mathbb{F}_q be a smooth projective geometrically irreducible curve of genus g whose function field is K . We will consider ∞ as a closed point of X of degree d_∞ . Recall that K_∞ is the completion of K at ∞ , \bar{K}_∞ is a fixed algebraic closure of K_∞ , and \mathbb{C}_∞ is the completion of \bar{K}_∞ . Let $\text{sgn} : K_\infty^\times \rightarrow \mathbb{F}_\infty^\times$ be a sign function (\mathbb{F}_∞ is the residue field of K_∞ and $d_\infty = [\mathbb{F}_\infty : \mathbb{F}_q]$), i.e. sgn is a group homomorphism such that $\text{sgn}|_{\mathbb{F}_\infty^\times} = \text{Id}|_{\mathbb{F}_\infty^\times}$. We fix $\pi \in K \cap \text{Ker}(\text{sgn})$ and such that $K_\infty = \mathbb{F}_\infty((\pi))$.

We set $\bar{X} = X \otimes_{\mathbb{F}_q} \mathbb{C}_\infty$, and $\bar{A} := A \otimes_{\mathbb{F}_q} \mathbb{C}_\infty$. Then $F := \text{Frac}(\bar{A})$ is the function field of \bar{X} . We identify \mathbb{C}_∞ with its image $1 \otimes \mathbb{C}_\infty$ in F . Note that \bar{A} is the set of elements of F/\mathbb{C}_∞ which are "regular outside ∞ ". We denote by $\tau : F \rightarrow F$ the K -algebra homomorphism such that:

$$\tau|_{\bar{A}} = \text{Id}_{\bar{A}} \otimes \text{Frob}_{\mathbb{C}_\infty}.$$

For $m \geq 0$, we also set:

$$\forall x \in F, \quad x^{(m)} = \tau^m(x).$$

Let P be a point of $\bar{X}(\mathbb{C}_\infty)$. We denote by $P^{(i)}$ the point of $\bar{X}(\mathbb{C}_\infty)$ obtained by applying τ^i to the coordinates of P . If $D \in \text{Div}(\bar{X})$, $D = \sum_{j=1}^n n_{P_j}(P_j)$, $P_j \in \bar{X}(\mathbb{C}_\infty)$, $n_{P_j} \in \mathbb{Z}$, we set:

$$D^{(i)} = \sum_{j=1}^n n_{P_j}(P_j^{(i)}).$$

If $D = (x)$, $x \in F^\times$, then:

$$D^{(i)} = (x^{(i)}).$$

We fix a point $\bar{\infty}$ of $X(\mathbb{C}_\infty)$ above ∞ . Let ξ be the point of $\bar{X}(\mathbb{C}_\infty)$ corresponding to the kernel of the map $\bar{A} \rightarrow \mathbb{C}_\infty, \sum x_i \otimes a_i \mapsto \sum x_i a_i$. Let $\rho : K \rightarrow K \otimes 1, x \mapsto x \otimes 1$. Then:

$$F = \mathbb{C}_\infty(\rho(K)).$$

By [27] (see also [19], section 7.11), there exists a function $f \in F^\times$, such that:

$$V^{(1)} - V + (\xi) - (\bar{\infty}) = (f),$$

for some effective divisor V of $\bar{X}/\mathbb{C}_\infty$ of degree g . The points ξ and $\bar{\infty}^{(-1)}$ do not belong to the support of V ([27], Corollary 0.3.3). We identify the completion of F at $\bar{\infty}$ with:

$$\mathbb{C}_\infty\left(\left(\frac{1}{t}\right)\right),$$

where $t = \rho(\pi^{-1})$. We have a natural sign function $\overline{\text{sgn}} : \mathbb{C}_\infty\left(\left(\frac{1}{t}\right)\right)^\times \rightarrow \mathbb{C}_\infty^\times$ attached to $\frac{1}{t}$. We normalize f such that $\overline{\text{sgn}}(f) = 1$.

We set:

$$(\infty) = \sum_{i=0}^{d_\infty-1} (\bar{\infty}^{(i)}),$$

$$W(\mathbb{C}_\infty) = \bigcup_{m \geq 0} L(V + m(\infty)),$$

where:

$$L(V + m(\infty)) = \{x \in F^\times, (x) + V + m(\infty) \geq 0\} \cup \{0\}.$$

Observe that for $i > 0$:

$$(4.1) \quad (ff^{(1)} \dots f^{(i-1)}) = V^{(i)} - V + (\xi) + \dots + (\xi^{(i-1)}) - \sum_{k=0}^{i-1} (\bar{\infty}^{(k)}).$$

We have (see for example [27], paragraph 0.3.5):

$$W(\mathbb{C}_\infty) = \bigoplus_{i \geq 0} \mathbb{C}_\infty f \dots f^{(i-1)}.$$

If L is a sub- \mathbb{F}_q -algebra of \mathbb{C}_∞ , we set:

$$W(L) = \bigoplus_{i \geq 0} L f \dots f^{(i-1)}.$$

Let $a \in A$, then we can write:

$$\rho(a) = a \otimes 1 = \sum_{i=0}^{\deg a} \phi_{a,i} f \dots f^{(i-1)},$$

where $\phi_{a,i} \in \mathbb{C}_\infty$, and:

$$\begin{aligned} \phi_{a, \deg a} &= \overline{\text{sgn}}(a), \\ \phi_{a,0} &= a. \end{aligned}$$

In particular, note that $\bar{\infty}$ does not belong to the support of V . The map $\phi : A \rightarrow \mathbb{C}_\infty\{\tau\}$ such that:

$$\forall a \in A, \quad \phi_a = \sum \phi_{a,i} \tau^i,$$

is a sign-normalized rank one Drinfeld module by the Drinfeld correspondence attached to f ([27], paragraph 0.3.5, see also [19], section 7.11). Let's write:

$$\exp_\phi = \sum e_i(\phi) \tau^i, \quad e_i(\phi) \in \mathbb{C}_\infty.$$

We have ([27], Proposition 0.3.6):

$$\forall i \geq 0, \quad e_i(\phi) = \frac{1}{f \cdots f^{(i-1)} \big|_{\xi^{(i)}}}.$$

Let $\mathbb{H} = \text{Frac}(A \otimes B) \subset F$. By Drinfeld's correspondence (see [19], Chapter 6), $f \in \mathbb{H}$. Thus:

$$f = t + \sum_{i \geq 0} f_i \frac{1}{t^i} \in H\left(\left(\frac{1}{t}\right)\right) \subset \mathbb{C}_\infty\left(\left(\frac{1}{t}\right)\right),$$

where $f_i \in H, \forall i \geq 0$.

We view \mathbb{H} as a function field over $\rho(K) = K \otimes 1$. Let $\mathbb{K} = \text{Frac}(A \otimes A)$. Let ∞ be the unique place of $\mathbb{K}/\rho(K)$ which is above the place ∞ of K/\mathbb{F}_q . Then the completion of \mathbb{K} above ∞ is:

$$\mathbb{K}_\infty = \rho(K)(\mathbb{F}_\infty)((1 \otimes \pi)).$$

Observe that the set of elements of $\mathbb{K}/\rho(K)$ which are regular outside ∞ is:

$$\mathbb{A} := A[\rho(K)] = K \otimes A.$$

We set $\mathbb{B} := B[\rho(K)] = K \otimes B$, then \mathbb{B} is the integral closure of \mathbb{A} in \mathbb{H} . Let $G = \text{Gal}(H/K) \simeq \text{Gal}(\mathbb{H}/\mathbb{K})$. Let $\varphi : \mathbb{A} \rightarrow \mathbb{H}\{\tau\}$ be the $\rho(K)$ -algebra homomorphism such that:

$$\forall a \in A, \quad \varphi_a = \sum_{i=0}^{\deg a} \phi_{a,i} f \cdots f^{(i-1)} \tau^i \in \mathbb{H}\{\tau\}.$$

Let $\exp_\varphi \in \mathbb{H}\{\{\tau\}\}$ be the following element:

$$\exp_\varphi = \sum_{i \geq 0} f \cdots f^{(i-1)} e_i(\phi) \tau^i = \sum_{i \geq 0} \frac{f \cdots f^{(i-1)}}{f \cdots f^{(i-1)} \big|_{\xi^{(i)}}} \tau^i.$$

Then:

$$\forall a \in \mathbb{A}, \quad \exp_\varphi a = \varphi_a \exp_\varphi.$$

Let $\mathbb{H}_\infty = \mathbb{H} \otimes_{\mathbb{K}} \mathbb{K}_\infty$, then \exp_φ converges on \mathbb{H}_∞ .

Let \mathfrak{P} be a maximal ideal of B . Then $\mathfrak{P}\mathbb{B}$ is a maximal ideal of \mathbb{B} . Let $v_{\mathfrak{P}} : \mathbb{H} \rightarrow \mathbb{Z} \cup \{+\infty\}$ be the valuation on \mathbb{H} attached to $\mathfrak{P}\mathbb{B}$. Since for all $a \in A$, $\rho(a) = \sum_{j=0}^{\deg a} \phi_{a,j} f \cdots f^{(j-1)}$, we deduce that:

$$\forall i \geq 0, \quad v_{\mathfrak{P}}(f^{(i)}) = q^i v_{\mathfrak{P}}(f) = 0.$$

However, we warn the reader that, if $g > 0$, we have:

$$f \notin \mathbb{B}.$$

We set:

$$W(B) = \bigoplus_{i \geq 0} B f \cdots f^{(i-1)}.$$

Lemma 4.4.

- 1) $W(B)$ is a $A \otimes B$ -module containing $A \otimes B$, furthermore $W(B)$ is a $A \otimes A$ -module via φ .
- 2) Let $W(B)\mathbb{B}$ be the \mathbb{B} -module generated by $W(B)$. Let \mathfrak{P} be a maximal ideal of B . The inclusion $\mathbb{B} \subset W(B)\mathbb{B}$ induces an equality:

$$\frac{\mathbb{B}}{\mathfrak{P}\mathbb{B}} = \frac{W(B)\mathbb{B}}{\mathfrak{P}W(B)\mathbb{B}}.$$

- 3) $W(B)\mathbb{B}$ is a fractional ideal of \mathbb{B} . In particular, it is discrete in \mathbb{H}_∞ .

Proof. We have:

$$\forall i \geq 0, \forall a \in A, \quad \rho(a)f \cdots f^{(i-1)} = \sum_{j=0}^{\deg a} \phi_{a,j}^{q^i} f \cdots f^{(i+j-1)} \in W(B).$$

Observe that:

$$\forall i, j \geq 0, \quad f \cdots f^{(j-1)} \tau^j (f \cdots f^{(i-1)}) = f \cdots f^{(i+j-1)}.$$

The assertion 1) follows.

We set: $O_{\mathfrak{P}} = \{x \in \mathbb{H}, v_{\mathfrak{P}}(x) \geq 0\}$. Since $\frac{O_{\mathfrak{P}}}{\mathfrak{P}O_{\mathfrak{P}}} \simeq \frac{\mathbb{B}}{\mathfrak{P}\mathbb{B}}$ and $\mathbb{B} \subset W(B)\mathbb{B} \subset O_{\mathfrak{P}}$, the assertion 2) holds.

Let's prove the assertion 3). Note that $A \otimes H$ is the set of elements of \mathbb{H} which are regular outside ∞ . By the expression (4.1) of the divisor of $f \cdots f^{(i-1)}$, $i \geq 0$, there exists $a \in A \otimes B \setminus \{0\}$ such that:

$$\forall i \geq 0, \quad af \cdots f^{(i-1)} \in A \otimes H.$$

Since for every maximal ideal \mathfrak{P} of B , and for all $i \geq 0$, $v_{\mathfrak{P}}(f \cdots f^{(i-1)}) = 0$, we deduce that:

$$\forall i \geq 0, af \cdots f^{(i-1)} \in A \otimes B.$$

Thus, there exists $a \in \mathbb{B} \setminus \{0\}$ such that $aW(B) \subset \mathbb{B}$. Since \mathbb{B} is discrete in \mathbb{H}_{∞} , we get the desired result. \square

Let's observe that, by Lemma 4.4, $W(B)\mathbb{B}$ is an \mathbb{A} -module via φ . Let \mathfrak{P} be a maximal ideal of B , then, again by Lemma 4.4, $\frac{\mathbb{B}}{\mathfrak{P}\mathbb{B}}$ is an \mathbb{A} -module via φ , and we denote this latter \mathbb{A} -module by $\varphi(\frac{\mathbb{B}}{\mathfrak{P}\mathbb{B}})$.

Lemma 4.5. *Let \mathfrak{P} be a maximal ideal of B . Then:*

$$\text{Fitt}_{\mathbb{A}}\varphi\left(\frac{\mathbb{B}}{\mathfrak{P}\mathbb{B}}\right) = \left([\frac{B}{\mathfrak{P}B}]_A - \rho([\frac{B}{\mathfrak{P}B}]_A)\right)\mathbb{A}.$$

Proof. Recall that:

$$[\frac{B}{\mathfrak{P}B}]_A = \psi(P^e),$$

where $e = \dim_{\frac{A}{\mathfrak{P}}} \frac{B}{\mathfrak{P}}$. Set $aA = P^e$ where $\text{sgn}(a) = 1$. Then:

$$\rho(a) = \sum \phi_{a,i} f \cdots f^{(i-1)}.$$

Therefore:

$$\forall x \in \frac{\mathbb{B}}{\mathfrak{P}\mathbb{B}}, \quad \varphi_{a-\rho(a)}(x) = 0.$$

Thus, by similar arguments to those of [6], Lemma 5.8, we have an \mathbb{A} -module isomorphism:

$$\varphi\left(\frac{\mathbb{B}}{\mathfrak{P}\mathbb{B}}\right) \simeq \frac{\mathbb{A}}{(a - \rho(a))\mathbb{A}}.$$

\square

If M is an \mathbb{A} -module such that M is a finite dimensional $\rho(K)$ -vector space and its Fitting ideal is principal, $\text{Fitt}_{\mathbb{A}}(M) = x\mathbb{A}$, then we set:

$$[M]_{\mathbb{A}} = \frac{x}{\text{sgn}(x)}.$$

By the above Lemma, we can form the L -series attached to $\varphi/W(B)$:

$$L(\varphi/W(B)) = \prod_{\mathfrak{p}} \frac{[\frac{B}{\mathfrak{p}B}]_A}{[\varphi(\frac{B}{\mathfrak{p}B})]_A} = \prod_{\mathfrak{p}} (1 - \frac{\rho([\frac{B}{\mathfrak{p}B}]_A)}{[\frac{B}{\mathfrak{p}B}]_A})^{-1} \in \mathbb{K}_\infty^\times.$$

Note that $L(\varphi/W(B))$ is in fact an element in the ∞ -adic completion of $K_\infty[\rho(A)] = A \otimes K_\infty$ which is an affinoid algebra over K_∞ , and $L(\varphi/W(B))$ is a special value of a twisted zeta function (see [5], Section 5.2).

We denote by $\tau : \mathbb{H}_\infty \rightarrow \mathbb{H}_\infty$ the continuous homomorphism of $\rho(K)$ -algebras such that $\forall x \in H_\infty, \tau(x) = x^q$. Let z be an indeterminate. The map $\tau : \mathbb{H}_\infty \rightarrow \mathbb{H}_\infty$ extends uniquely into a continuous homomorphism (for the z -adic topology) of $\mathbb{F}_q[[z]]$ -algebras $\tau : \mathbb{H}_\infty[[z]] \rightarrow \mathbb{H}_\infty[[z]]$. Let $\mathbb{T}_z(\mathbb{H}_\infty) \subset \mathbb{H}_\infty[[z]]$ be the ∞ -adic completion of $\mathbb{H}_\infty[[z]]$, i.e. an element $g \in \mathbb{T}_z(\mathbb{H}_\infty)$ can be uniquely written $g = \sum_{i \geq 0} g_i z^i, g_i \in \mathbb{H}_\infty$, such that $\lim_{i \rightarrow +\infty} g_i = 0$. We also denote by $\mathbb{T}_z(\mathbb{K}_\infty)$ the ∞ -adic completion of $\mathbb{K}_\infty[[z]]$. Note that $\mathbb{T}_z(\mathbb{H}_\infty)$ is a free $\mathbb{T}_z(\mathbb{K}_\infty)$ -module of rank $[H : K]$, and if (e_1, \dots, e_n) is a K -basis of H ($n = [H : K]$), then:

$$\mathbb{T}_z(\mathbb{H}_\infty) = \bigoplus_{i=1}^n e_i \mathbb{T}_z(\mathbb{K}_\infty).$$

Observe also that G acts on $\mathbb{T}_z(\mathbb{H}_\infty)$ and $\mathbb{T}_z(\mathbb{H}_\infty)$ is a free $\mathbb{T}_z(\mathbb{K}_\infty)[G]$ -module of rank one by the normal basis Theorem. We denote by $\mathbb{T}_z(\mathbb{H}_\infty)[G]$ the ring:

$$\mathbb{T}_z(\mathbb{H}_\infty)[G] := \bigoplus_{\sigma \in G} \mathbb{T}_z(\mathbb{H}_\infty)\sigma,$$

where the product rule is given by:

$$\forall \sigma_1, \sigma_2 \in G, \forall g_1, g_2 \in \mathbb{T}_z(\mathbb{H}_\infty), \quad (g_1 \sigma_1)(g_2 \sigma_2) = g_1 \sigma_1(g_2) \sigma_1 \sigma_2.$$

Let's set:

$$\exp_{\bar{\varphi}} = \sum_{i \geq 0} \frac{f \dots f^{(i-1)}}{f \dots f^{(i-1)} |_{\xi^{(i)}}} z^i \tau^i \in \mathbb{H}[[z]]\{\{\tau\}\}.$$

Let I be a non-zero ideal of A . We set:

$$u_I = \sum_{i=0}^{\deg I} \phi_{I,i} f \dots f^{(i-1)} \in W(B),$$

where $\phi_I = \sum_{i=0}^{\deg I} \phi_{I,i} \tau^i, \phi_{I,i} \in B$. Note that if $I = aA$, we have:

$$u_I = \frac{\rho(a)}{\text{sgn}(a)}.$$

Furthermore, we prove (see [1], Section 3.7 for the case $d_\infty = 1$):

Lemma 4.6. *Let I, J be two non-zero ideals of A . We have:*

$$\begin{aligned} u_I |_{\xi} &= \psi(I), \\ \sigma_I(f)u_I &= f\tau(u_I), \\ u_{IJ} &= \sigma_I(u_J)u_I. \end{aligned}$$

Proof. The fact that $u_I |_{\xi} = \psi(I)$ comes from the definition of u_I . Note that we have a natural isomorphism of B -modules:

$$\gamma_\phi : W(B) \simeq B\{\tau\}, \quad f \dots f^{(i-1)} \mapsto \tau^i.$$

In particular:

$$\begin{aligned} \forall x \in W(B), \quad \gamma_\phi(fx^{(1)}) &= \tau\gamma_\phi(x), \\ \forall x \in W(B), \forall a \in A, \quad \gamma_\phi(\rho(a)x) &= \gamma_\phi(x)\phi_a. \end{aligned}$$

By explicit reciprocity law (see [19], Theorem 7.4.8), we have:

$$\forall a \in A, \quad \phi_I \phi_a = \sigma_I(\phi)_a \phi_I.$$

By direct calculations, we deduce from this:

$$\sigma_I(f)u_I = f\tau(u_I),$$

Now, let J be a non-zero ideal of A . We have:

$$\gamma_\phi(u_{IJ}) = \phi_{IJ} = \sigma_I(\phi_J)\phi_I.$$

But, since $\forall i \geq 0$, $\sigma_I(f \cdots f^{(i-1)})u_I = f \cdots f^{(i-1)}u_I^{(i)}$, we have :

$$\gamma_\phi(\sigma_I(u_J)u_I) = \sigma_I(\phi_J)\phi_I.$$

Thus:

$$u_{IJ} = \sigma_I(u_J)u_I.$$

□

We deduce that if P, Q are maximal ideals of A :

$$\left(1 - \frac{u_P}{\psi(P)} z^{\deg P} \sigma_P\right) \left(1 - \frac{u_Q}{\psi(Q)} z^{\deg Q} \sigma_Q\right) = \left(1 - \frac{u_Q}{\psi(Q)} z^{\deg(Q)} \sigma_Q\right) \left(1 - \frac{u_P}{\psi(P)} z^{\deg P} \sigma_P\right).$$

For every integer $n \geq 1$, we set:

$$\left(1 - \frac{u_P}{\psi(P)^n} z^{\deg P} \sigma_P\right)^{-1} := \sum_{k \geq 0} \frac{u_{P^k}}{\psi(P^k)^n} z^{k \deg P} \sigma_{P^k} \in \mathbb{T}_z(\mathbb{H}_\infty)[G].$$

We define:

$$\forall n \geq 1, \quad \mathcal{L}(\varphi; n; z) = \prod_P \left(1 - \frac{u_P}{\psi(P)^n} z^{\deg P} \sigma_P\right)^{-1} \in (\mathbb{T}_z(\mathbb{H}_\infty)[G])^\times,$$

where P runs through the maximal ideals of A . Note that, for any $n \geq 1$, $\mathcal{L}(\varphi; n; z)$ induces a $\mathbb{T}_z(\mathbb{K}_\infty)$ -linear endomorphism of $\mathbb{T}_z(\mathbb{H}_\infty)$, and we denote by $\det_{\mathbb{T}_z(\mathbb{K}_\infty)} \mathcal{L}(\varphi; n; z)$ its determinant. Let's set:

$$W(B[z]) = \bigoplus_{i \geq 0} B[z] f \cdots f^{(i-1)} \subset \mathbb{H}[z].$$

Proposition 4.7. *We have:*

$$\forall n \geq 1, \quad \det_{\mathbb{T}_z(\mathbb{K}_\infty)} \mathcal{L}(\varphi; n; z) = \prod_{\mathfrak{P}} \left(1 - \frac{\rho\left(\left[\frac{B}{\mathfrak{P}B}\right]_A\right) z^{\deg N_{H/K}(\mathfrak{P})}}{\left[\frac{B}{\mathfrak{P}B}\right]_A^n}\right)^{-1} \in \mathbb{T}_z(\mathbb{K}_\infty)^\times,$$

where \mathfrak{P} runs through the maximal ideals of B .

Proof. The proof is similar to that of Theorem 3.6 . We give a sketch of the proof for the convenience of the reader.

Let $n \geq 1$. We have:

$$\det_{\mathbb{T}_z(\mathbb{K}_\infty)} \mathcal{L}(\varphi; n; z) = \prod_P \det_{\mathbb{K}[z]} \left(1 - \frac{u_P}{\psi(P)^n} z^{\deg P} \sigma_P \big|_{\mathbb{H}[z]}\right)^{-1}.$$

Let P be a maximal ideal of A . Let $e \geq 1$ be the order of P in $\text{Pic}(A)$. Then $1, \sigma_P, \dots, \sigma_P^{e-1}$ are linearly independent over $\mathbb{H}(z)$. We have :

$$\left(\frac{u_P}{\psi(P)^n} z^{\deg P} \sigma_P\right)^e = \frac{\rho(\psi(P^e)) z^{e \deg P}}{\psi(P^e)^n} \in \mathbb{K}[z].$$

Thus the minimal polynomial of $\frac{u_P}{\psi(P)^n} z^{\deg P} \sigma_P |_{\mathbb{H}(z)}$ over $\mathbb{K}(z)$ (and also over $\mathbb{H}^{(\sigma_P)}(z)$) is equal to:

$$X^e - \frac{\rho(\psi(P^e)) z^e \deg P}{\psi(P^e)^n} \in \mathbb{K}[z][X].$$

Therefore the characteristic polynomial of $\frac{u_P}{\psi(P)^n} z^{\deg P} \sigma_P |_{\mathbb{H}(z)}$ over $\mathbb{K}(z)$ is equal to:

$$\left(X^e - \frac{\rho(\psi(P^e)) z^e \deg P}{\psi(P^e)^n} \right)^{\frac{[H:K]}{e}}.$$

One obtains the desired result by the same arguments as that used in the proof of Theorem 3.6. \square

Remark 4.8. Let $L = \rho(K)(\mathbb{F}_\infty)((q^{d_\infty-1}\sqrt{-\pi}))$, and let $\tau : L \rightarrow L$ be the continuous morphism of $\rho(K)$ -algebras such that $\forall x \in \mathbb{F}_\infty((q^{d_\infty-1}\sqrt{-\pi}))$, $\tau(x) = x^q$. Then there exists an element $\omega \in L^\times$ (unique up to the multiplication of an element in $\rho(K)^\times$) such that:

$$\tau(\omega) = f\omega.$$

This element is a generalization of the special function introduced by G. Anderson and D. Thakur in [3]. The existence of this element (combined with the log-algebraicity theorem) gives new arithmetic informations on special values of L -series. We refer the interested reader to a forthcoming work of the authors.

4.3. Stark units and several variable log-algebraicity theorem.

We set:

$$U(\tilde{\varphi}/W(B[z])) = \{x \in \mathbb{T}_z(\mathbb{H}_\infty), \exp_{\tilde{\varphi}}(x) \in W(B[z])\}.$$

The following result is a twisted (by the shtuka function f) version of [1], Theorem 5.1.1. :

Theorem 4.9. *We have:*

$$U(\tilde{\varphi}/W(B[z])) = \mathcal{L}(\varphi; 1; z)W(B[z]).$$

In particular,

$$\exp_{\tilde{\varphi}}(\mathcal{L}(\varphi; 1; z)W(B[z])) \subset W(B[z]),$$

Proof. The proof is similar to that of Theorem 3.8. We give a sketch of the proof for the convenience of the reader.

Observe that $\exp_{\tilde{\varphi}} : \mathbb{H}[[z]] \rightarrow \mathbb{H}[[z]]$ is an isomorphism of $\mathbb{A}[[z]]$ -modules. Furthermore, if we set:

$$W(H[[z]]) = \bigoplus_{i \geq 0} H[[z]] f \cdots f^{(i-1)},$$

we get:

$$\exp_{\tilde{\varphi}}(W(H[[z]])) = W(H[[z]]).$$

Let:

$$W(B[[z]]) = \bigoplus_{i \geq 0} B[[z]] f \cdots f^{(i-1)} \subset \mathbb{H}[[z]].$$

Let P be a maximal ideal of A . Let $W_P = S^{-1}W(B[[z]])$, where $S = A \setminus P$. Then:

$$PW_P = \psi(P)W_P.$$

By Lemma 3.7, we have:

$$\exp_{\tilde{\varphi}}(PW_P) = PW_P.$$

If

$$\phi_P = \sum_{i=0}^{\deg P} \phi_{P,i} \tau^i,$$

we set:

$$\tilde{\varphi}_P = \sum_{i=0}^{\deg P} \phi_{P,i} f \cdots f^{(i-1)} z^i \tau^i.$$

We have:

$$\tilde{\varphi}_P \exp_{\tilde{\varphi}} = \exp_{\sigma_P} \psi(P),$$

where:

$$\exp_{\sigma_P} = \sum_{i \geq 0} \sigma_P(e_i(\phi)) f \cdots f^{(i-1)} z^i \tau^i.$$

Let's set:

$$U(\tilde{\varphi}/W_P) = \{x \in \mathbb{H}[[z]], \exp_{\tilde{\varphi}}(x) \in W_P\} \subset W(H[[z]]).$$

We have an isomorphism of $A[[z]]$ -modules induced by $\exp_{\tilde{\varphi}}$:

$$\frac{U(\tilde{\varphi}/W_P)}{PW_P} \simeq \tilde{\varphi} \left(\frac{W_P}{PW_P} \right).$$

Note that:

$$\forall i \geq 0, \quad \sigma_P(f \cdots f^{(i-1)}) u_P = f \cdots f^{(i-1)} \tau^i(u_P) \in W(B).$$

Therefore:

$$(\tilde{\varphi}_P - z^{\deg P} u_P \sigma_P) \tilde{\varphi} \left(\frac{W_P}{PW_P} \right) = \{0\}.$$

Since u_P is a “ P -unit”, for $x \in W(H[[z]]) \setminus W_P$, $(\tilde{\varphi}_P - z^{\deg P} u_P \sigma_P)(x)$ is not P -integral as an element of $\mathbb{H}[[z]]$. Thus:

$$\tilde{\varphi} \left(\frac{W_P}{PW_P[[z]]} \right) = \{x \in \tilde{\varphi} \left(\frac{W(H[[z]])}{PW_P} \right), (\tilde{\varphi}_P - z^{\deg P} u_P \sigma_P)(x) = 0\}.$$

Let $x \in W(H[[z]])$, we deduce that:

$$x \in U(\tilde{\varphi}/W_P) \Leftrightarrow (\tilde{\varphi}_P - z^{\deg P} u_P \sigma_P)(\exp_{\tilde{\varphi}}(x)) \in PW_P.$$

Thus:

$$x \in U(\tilde{\varphi}/W_P) \Leftrightarrow \exp_{\sigma_P}(\psi(P)x - z^{\deg P} u_P \sigma_P(x)) \in PW_P.$$

Lemma 3.7 implies:

$$x \in U(\tilde{\varphi}/W_P) \Leftrightarrow \psi(P)x - z^{\deg P} u_P \sigma_P(x) \in PW_P.$$

Thus:

$$U(\tilde{\varphi}/W_P) = \left(1 - \frac{z^{\deg P} u_P}{\psi(P)} \sigma_P\right)^{-1} W_P.$$

Observe that $W(B[[z]]) = \bigcap_P W_P$. We conclude that:

$$W(B[[z]]) = \exp_{\tilde{\varphi}}(\mathcal{L}(\varphi; 1; z)W(B[[z]])).$$

By Lemma 4.4, we get:

$$\exp_{\tilde{\varphi}}(\mathcal{L}(\varphi; 1; z)W(B[z])) \subset \mathbb{T}_z(\mathbb{H}_\infty) \cap W(B[[z]]) = W(B[z]).$$

Recall that:

$$U(\tilde{\varphi}/W(B[z])) = \{x \in \mathbb{T}_z(\mathbb{H}_\infty), \exp_{\tilde{\varphi}}(x) \in W(B[z])\}.$$

Then:

$$U(\tilde{\varphi}/W(B[z])) = \mathcal{L}(\varphi; 1; z)W(B[[z]]) \cap \mathbb{T}_z(\mathbb{H}_\infty).$$

But recall that:

$$\mathcal{L}(\varphi; 1; z) \in (\mathbb{T}_z(\mathbb{H}_\infty)[G])^\times.$$

Thus:

$$U(\tilde{\varphi}/W(B[z])) = \mathcal{L}(\varphi; 1; z)W(B[z]).$$

□

Let $\text{ev} : \mathbb{T}_z(\mathbb{H}_\infty) \rightarrow \mathbb{H}_\infty$ be the evaluation map at $z = 1$. Then by Proposition 4.7, we get:

$$L(\varphi/W(B)) = \det_{\mathbb{K}_\infty} \text{ev}(\mathcal{L}(\varphi; 1; z)),$$

where:

$$\text{ev}(\mathcal{L}(\varphi; 1; z)) = \prod_P \left(1 - \frac{u_P}{\psi(P)} \sigma_P\right)^{-1} = \sum_I \frac{u_I}{\psi(I)} \sigma_I \in (\mathbb{H}_\infty[G])^\times,$$

where I runs through the non-zero ideals of A . Furthermore, by the above Theorem:

$$\exp_\varphi(\text{ev}(\mathcal{L}(\varphi; 1; z))W(B)) \subset W(B).$$

And also:

$$\exp_\varphi(\text{ev}(\mathcal{L}(\varphi; 1; z))W(B)\mathbb{B}) \subset W(B)\mathbb{B}.$$

If we define the regulator of Stark units $\text{ev}(\mathcal{L}(\varphi; 1; z))W(B)\mathbb{B}$ as follows:

$$[W(B)\mathbb{B} : \text{ev}(\mathcal{L}(\varphi; 1; z))W(B)\mathbb{B}]_\mathbb{A} := \det_{\mathbb{K}_\infty} \text{ev}(\mathcal{L}(\varphi; 1; z)),$$

then:

$$L(\varphi/W(B)) = [W(B)\mathbb{B} : \text{ev}(\mathcal{L}(\varphi; 1; z))W(B)\mathbb{B}]_\mathbb{A}.$$

We now briefly discuss the several variable version of Theorem 4.9. Let $s \geq 0$ be an integer. Let:

$$K_s = \text{Frac}(A^{\otimes s}),$$

where:

$$A^{\otimes s} = A \otimes_{\mathbb{F}_q} \cdots \otimes_{\mathbb{F}_q} A.$$

If $s = 0$, then $K_0 = \mathbb{F}_q$. Let:

$$\mathbb{H}_s = \text{Frac}(A^{\otimes s} \otimes_{\mathbb{F}_q} B),$$

$$\mathbb{K}_s = \text{Frac}(A^{\otimes s} \otimes_{\mathbb{F}_q} A).$$

For $i = 1, \dots, s$, let:

$$\rho_i : A \rightarrow \mathbb{H}_s, \quad a \mapsto (1 \otimes \cdots \otimes 1 \otimes a \otimes \cdots \otimes 1) \otimes 1,$$

where a appears at the i -th position. We still denote by $\rho_i : \mathbb{H} \rightarrow \mathbb{H}_s$ the homomorphism of H -algebras such that:

$$\forall a \in A, \quad \rho_i(\rho(a)) = \rho_i(a).$$

We view \mathbb{H}_s and \mathbb{K}_s as functions fields over $K_s \otimes 1$. Let ∞ be the unique place of $\mathbb{K}_s/K_s \otimes 1$ above the place ∞ of K/\mathbb{F}_q . Then:

$$\mathbb{K}_{s,\infty} = (K_s \otimes 1)(\mathbb{F}_\infty)((1^{\otimes s} \otimes \pi)),$$

and we set:

$$\mathbb{H}_{s,\infty} = \mathbb{H}_s \otimes_{\mathbb{K}_s} \mathbb{K}_{s,\infty}.$$

Let $\mathbb{T}_z(\mathbb{H}_{s,\infty})$ be the Tate algebra in the variable z with coefficients in $\mathbb{H}_{s,\infty}$. Let $\tau : \mathbb{T}_z(\mathbb{H}_{s,\infty}) \rightarrow \mathbb{T}_z(\mathbb{H}_{s,\infty})$ be the continuous homomorphism of $(K_s \otimes 1)[z]$ -algebras such that:

$$\forall x \in H_\infty, \quad \tau(x) = x^q.$$

Let's set:

$$W_s(B[z]) = \bigoplus_{i_1, \dots, i_s \geq 0} B[z] \prod_{j=1}^s \rho_j(f) \cdots \tau^{(i_j-1)}(\rho_j(f)) \subset \mathbb{H}_s[z].$$

In particular $W_0(B[z]) = B[z]$. By similar arguments as those of the proof of Lemma 4.4, we show that $W_s(B[z])$ is discrete in $\mathbb{T}_z(\mathbb{H}_{s,\infty})$. For $n \geq 1$, we set:

$$\mathcal{L}(\varphi_s; n; z) := \prod_P \left(1 - \frac{\prod_{j=1}^s \rho_j(u_P)}{\psi(P)^n} z^{\deg P} \sigma_P\right)^{-1} \in (\mathbb{T}_z(\mathbb{H}_{s,\infty})[G])^\times,$$

where P runs through the maximal ideals of A . Then, by the same proof as that of Proposition 4.7, for all $n \geq 1$, we get:

$$\det_{\mathbb{T}_z(\mathbb{K}_{s,\infty})} \mathcal{L}(\varphi_s; n; z) = \prod_{\mathfrak{P}} \left(1 - \frac{(\prod_{j=1}^s \rho_j([\frac{B}{\mathfrak{P}B}]_A)) z^{\deg N_{H/K}(\mathfrak{P})}}{[\frac{B}{\mathfrak{P}B}]_A^n}\right)^{-1} \in \mathbb{T}_z(\mathbb{K}_{s,\infty})^\times,$$

where \mathfrak{P} runs through the maximal ideals of B .

We define:

$$\exp_{\tilde{\varphi}_s} = \sum_{i \geq 0} e_i(\phi) \left(\prod_{j=1}^s \rho_j(f) \cdots \tau^{i-1}(\rho_j(f))\right) z^i \tau^i \in \mathbb{H}_s\{\{\tau\}\}.$$

Then $\exp_{\tilde{\varphi}_s}$ converges on $\mathbb{T}_s(\mathbb{H}_{s,\infty})$, and we set:

$$U(\tilde{\varphi}_s/W_s(B[z])) = \{x \in \mathbb{T}_z(\mathbb{H}_{s,\infty}), \exp_{\tilde{\varphi}_s}(x) \in W_s(B[z])\}.$$

By similar arguments as those of the proof of Theorem 4.9, we get:

Corollary 4.10. *We have:*

$$U(\tilde{\varphi}_s/W_s(B[z])) = \mathcal{L}(\varphi_s; 1; z) W_s(B[z]).$$

Example 4.11. We consider the Carlitz example, where $g = 0$ and $d_\infty = 1$. Observe that there exists $\theta \in K$ such that $\text{sgn}(\theta) = 1$, and $A = \mathbb{F}_q[\theta]$. Thus, $K = \mathbb{F}_q(\theta)$, and $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$.

Let $\phi : A \rightarrow \overline{K}_\infty\{\tau\}$ be the Carlitz module defined by

$$\phi_\theta = \theta + \tau.$$

Then the Carlitz exponential is given by:

$$\exp_\phi = \sum_{i \geq 0} \frac{1}{D_i} \tau^i,$$

where for $i \geq 0$, $D_i = \prod_{k=0}^{i-1} (\theta^{q^i} - \theta^{q^k})$.

The Hilbert class field H of K is K , and then $B = A$. Then, the shtuka function $f \in K \otimes H$ associated to the Carlitz module via the Drinfeld correspondence is given by:

$$f = \theta \otimes 1 - 1 \otimes \theta.$$

Let $s \geq 0$ be an integer. For $i = 1, \dots, s$, let $t_i = \rho_i(\theta)$. We have:

$$\rho_i(f) = t_i - \theta,$$

$$\begin{aligned}\mathbb{H}_s &= \mathbb{K}_s = \mathbb{F}_q(t_1, \dots, t_s, \theta), \\ \mathbb{H}_{s, \infty} &= \mathbb{K}_{s, \infty} = \mathbb{F}_q(t_1, \dots, t_s) \left(\left(\frac{1}{\theta} \right) \right).\end{aligned}$$

For $i \geq 0, j = 1, \dots, s$, set:

$$b_i(t_j) = \prod_{k=0}^{i-1} (t_j - \theta^{q^k}).$$

We get:

$$W_s(B[z]) = A[t_1, \dots, t_s][z].$$

Observe that:

$$\exp_{\tilde{\varphi}_s} = \sum_{i \geq 0} \frac{\prod_{j=1}^s b_i(t_j)}{D_i} \tau^i.$$

We have:

$$\mathcal{L}(\varphi_s; 1; z) = \sum_{a \in A_+} \frac{a(t_1) \cdots a(t_s)}{a} z^{\deg_{\theta} a},$$

where A_+ denotes the set of monic polynomials in $A = \mathbb{F}_q[\theta]$. In particular, for $s = 1$, we recover the zeta function introduced by Pellarin [21].

Corollary 4.10 implies:

$$\exp_{\tilde{\varphi}_s}(\mathcal{L}(\varphi_s; 1; z)A[t_1, \dots, t_s, z]) \subset A[t_1, \dots, t_s, z].$$

We refer the interested reader to [4], [6], [7], [9], for arithmetic applications of this latter result.

4.4. Another proof of Anderson's log-algebraicity theorem.

Corollary 4.12. *Let $n \geq 0$ and let X_1, \dots, X_n, z be $n+1$ indeterminates over K . Let $\tau : K[X_1, \dots, X_n][[z]] \rightarrow K[X_1, \dots, X_n][[z]]$ be the continuous $\mathbb{F}_q[[z]]$ -algebra homomorphism for the z -adic topology such that $\forall x \in K[X_1, \dots, X_n], \tau(x) = x^q$. Then:*

$$\forall b \in B, \quad \exp_{\tilde{\varphi}} \left(\sum_I \frac{\sigma_I(b)}{\psi(I)} \phi_I(X_1) \cdots \phi_I(X_n) z^{\deg I} \right) \in B[X_1, \dots, X_n, z],$$

where I runs through the non-zero ideals of A , and:

$$\exp_{\tilde{\varphi}} = \sum_{i \geq 0} e_i(\phi) z^i \tau^i.$$

Proof. We first treat the case $n = 0$. Let $b \in B$. By Theorem 4.9, we get:

$$\forall k \geq 0, \quad \sum_{\deg I + i = k} e_i(\phi) f \cdots f^{(i-1)} \frac{\tau^i(u_I \sigma_I(b))}{\psi(I)^{q^i}} \in W(B),$$

and:

$$\forall k \gg 0, \quad \sum_{\deg I + i = k} e_i(\phi) f \cdots f^{(i-1)} \frac{\tau^i(u_I \sigma_I(b))}{\psi(I)^{q^i}} = 0.$$

The coefficient of $f \cdots f^{(k-1)}$ in $\sum_{\deg I + i = k} e_i(\phi) f \cdots f^{(i-1)} \frac{\tau^i(u_I \sigma_I(b))}{\psi(I)^{q^i}}$ is:

$$\sum_{\deg I + i = k} e_i(\phi) \frac{\sigma_I(b)^{q^i}}{\psi(I)^{q^i}}.$$

Therefore:

$$\forall k \geq 0, \quad \sum_{\deg I+i=k} e_i(\phi) \frac{\sigma_I(b)^{q^i}}{\psi(I)^{q^i}} \in B.$$

$$\forall k \gg 0, \quad \sum_{\deg I+i=k} e_i(\phi) \frac{\sigma_I(b)^{q^i}}{\psi(I)^{q^i}} = 0.$$

Thus:

$$\exp_{\tilde{\varphi}}\left(\sum_I \frac{\sigma_I(b)}{\psi(I)} z^{\deg I}\right) \in B[z].$$

We now assume that $n \geq 1$. We have an isomorphism of $B[z]$ -modules

$$\gamma : W(B[z]) \rightarrow \bigoplus_{i_1, \dots, i_n \geq 0} B[z] X_1^{q^{i_1}} \cdots X_n^{q^{i_n}}$$

such that:

$$\forall i_1, \dots, i_n \in \mathbb{N}, \quad \gamma\left(\prod_{j=1}^n \rho_j(f \cdots f^{(i_j-1)})\right) = \prod_{j=1}^n X_j^{q^{i_j}}.$$

Observe that:

$$\gamma \circ \prod_{j=1}^n \rho_j(f) \tau = \tau \circ \gamma.$$

Furthermore:

$$\gamma\left(\prod_{j=1}^n \rho_j(u_I)\right) = \phi_I(X_1) \cdots \phi_I(X_n).$$

Thus, we get by Corollary 4.10:

$$\exp_{\tilde{\varphi}_n}(\mathcal{L}(\varphi_n; 1; z)b) \in W_n(B[z]),$$

and thus:

$$\exp_{\tilde{\varphi}}\left(\sum_I \frac{\sigma_I(b)}{\psi(I)} \phi_I(X_1) \cdots \phi_I(X_n) z^{\deg I}\right) \in \bigoplus_{i_1, \dots, i_n \geq 0} B[z] X_1^{q^{i_1}} \cdots X_n^{q^{i_n}}.$$

□

Remark 4.13. Let $s \geq 1$ be an integer and let $B\{\tau_1, \dots, \tau_s\}$ be the non-commutative polynomial ring in the variables τ_1, \dots, τ_s , such that:

$$\tau_i \tau_j = \tau_j \tau_i,$$

$$\forall b \in B, \forall n \geq 0, \quad \tau_i^n b = b^{q^n} \tau_i.$$

For $i = 1, \dots, s$, we set:

$$\forall a \in A, \quad \varphi_{i,a} = \sum_{j=0}^{\deg a} \phi_{a,j} \tau_i^j \in B\{\tau_1, \dots, \tau_s\},$$

and:

$$\forall a \in A, \quad \varphi_a = \sum_{j=0}^{\deg a} \phi_{a,j} \tau^j \in B\{\tau_1, \dots, \tau_s\},$$

where $\tau = \tau_1 \cdots \tau_s$.

Let $W_s(B) = \bigoplus_{i_1, \dots, i_s} B \prod_{j=1}^s \rho_j(f) \cdots \tau_j^{i_j-1}(\rho_j(f)) \subset \mathbb{H}_s$. Then $W_s(B)$ is an $A^{\otimes s} \otimes B$ -module. Let $j \in \{1, \dots, s\}$. Let $a \in A$, we have a natural B -module homomorphism:

$$\tilde{\rho}_j(a) : B\{\tau_1, \dots, \tau_s\} \rightarrow B\{\tau_1, \dots, \tau_s\},$$

such that:

$$\forall i_1, \dots, i_s \in \mathbb{N}, \quad \tilde{\rho}_j(a) \cdot (\tau_1^{i_1} \cdots \tau_s^{i_s}) = (\tau_j^{i_j} \varphi_{j,a}) \prod_{k=1, k \neq j}^s \tau_k^{i_k}.$$

Observe that:

$$\forall i, j \in \{1, \dots, s\}, \forall a, b \in A, \quad \tilde{\rho}_j(a) \circ \tilde{\rho}_i(b) = \tilde{\rho}_i(b) \circ \tilde{\rho}_j(a).$$

Thus $B\{\tau_1, \dots, \tau_s\}$ becomes an $A^{\otimes s} \otimes B$ -module via:

$$\forall x \in B\{\tau_1, \dots, \tau_s\}, \quad \left(\sum_i b_i \prod_{j=1}^s \rho_j(a_{i,j}) \right) \cdot x = \sum_i b_i \left(\prod_{j=1}^s \tilde{\rho}_j(a_{i,j}) \right) (x).$$

Then, by the proof of Corollary 4.12, we have an $A^{\otimes s} \otimes B$ -module isomorphism:

$$B\{\tau_1, \dots, \tau_s\} \simeq W_s(B).$$

In particular, $B\{\tau_1, \dots, \tau_s\}$ is a finitely generated $A^{\otimes s} \otimes B$ -module of rank one. The case $s = 1$ was already observed by G. Anderson ([19], page 230, line 21 - there is a misprint in line 24, since in general $f \notin A \otimes_{\mathbb{F}_q} \mathbb{C}_\infty$). If I is a non-zero ideal of A , we define $I * \cdot : B\{\tau_1, \dots, \tau_s\} \rightarrow B\{\tau_1, \dots, \tau_s\}$ to be the B -module homomorphism such that:

$$I * (\tau_1^{i_1} \cdots \tau_s^{i_s}) = \sum_{j_1, \dots, j_s \in \{0, \dots, \deg I\}} \phi_{I, j_1}^{q^{i_1}} \cdots \phi_{I, j_s}^{q^{i_s}} \tau_1^{i_1 + j_1} \cdots \tau_s^{i_s + j_s},$$

where $\phi_I = \sum_{k=0}^{\deg I} \phi_{I,k} \tau^k$.

Let $\mathcal{L} : B\{\tau_1, \dots, \tau_s\} \rightarrow H\{\{\tau_1, \dots, \tau_s\}\}$ be defined as follows:

$$\mathcal{L} \left(\sum_{i_1, \dots, i_s} b_{i_1, \dots, i_s} \tau_1^{i_1} \cdots \tau_s^{i_s} \right) = \sum_{i_1, \dots, i_s} \sum_I \frac{\sigma_I(b_{i_1, \dots, i_s})}{\psi(I)} I * (\tau_1^{i_1} \cdots \tau_s^{i_s}).$$

Then by Corollary 4.10, we get that the multiplication by $\exp_\phi \in H\{\{\tau\}\}$ on $H\{\{\tau_1, \dots, \tau_s\}\}$ yields to the following property:

$$\forall x \in B\{\tau_1, \dots, \tau_s\}, \quad \exp_\phi(\mathcal{L}(x)) \in B\{\tau_1, \dots, \tau_s\}.$$

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