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EXCEPTIONAL ZEROS OF L -SERIES AND BERNOULLI-CARLITZ NUMBERS

BRUNO ANGLÈS, TUAN NGO DAC, AND FLORIC TAVARES RIBEIRO

ABSTRACT. Bernoulli-Carlitz numbers were introduced by L. Carlitz in 1935, they are the analogues in positive characteristic of Bernoulli numbers. We prove a conjecture formulated by F. Pellarin and the first author on the non-vanishing modulo a given prime of families of Bernoulli-Carlitz numbers. We then show that the "exceptional zeros" of certain L -series are intimately connected to the Bernoulli-Carlitz numbers.

With an appendix by B. Anglès, D. Goss, F. Pellarin, F. Tavares Ribeiro

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1. INTRODUCTION

Recently, M. Kaneko and D. Zagier have introduced the \mathbb{Q} -algebra of finite multiple zeta values which is a sub- \mathbb{Q} -algebra of $\mathcal{A} := \frac{\prod_p \mathbb{F}_p}{\oplus_p \mathbb{F}_p}$ (p runs through the prime numbers). This algebra of finite multiple zeta values contains the following elements:

$$\forall k \geq 2, \mathcal{Z}(k) = \left(\left(\frac{B_{p-k}}{k} \right)_p \right) \in \mathcal{A},$$

where B_n denotes the n th Bernoulli number. It is not known that the algebra of finite multiple zeta values is non-trivial. In particular, it is an open problem to prove that $\mathcal{Z}(k) \neq 0$ for $k \geq 3, k \equiv 1 \pmod{2}$ (observe that $\mathcal{Z}(k) = 0$ if $k \geq 2, k \equiv 0 \pmod{2}$). This latter problem is equivalent to the following:

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Conjecture 1. *Let $k \geq 3$ be an odd integer. Then, there exist infinitely many primes p such that $B_{p-k} \not\equiv 0 \pmod{p}$.*

Let $k \geq 3$ be an odd integer. M. Kaneko ([14]) remarked that, viewing the B_{p-k} 's as being random modulo p when p varies through the prime numbers, taking into account that $\sum_p \frac{1}{p}$ diverges, then it is reasonable to expect that there exist infinitely many prime numbers p such that $B_{p-k} \equiv 0 \pmod{p}$.

Let \mathbb{F}_q be a finite field having q elements, q being a power of a prime number p , and let θ be an indeterminate over \mathbb{F}_q . In 1935, L. Carlitz has introduced the analogues of Bernoulli numbers for $A := \mathbb{F}_q[\theta]$ ([9]). The Bernoulli-Carlitz numbers, $BC_n \in K := \mathbb{F}_q(\theta)$, $n \in \mathbb{N}$, are defined as follows:

- $BC_n = 0$ if $n \not\equiv 0 \pmod{q-1}$,
- for $n \equiv 0 \pmod{q-1}$, we have:

$$\frac{BC_n}{\Pi(n)} = \frac{\zeta_A(n)}{\tilde{\pi}^n},$$

where $\Pi(n) \in A$ is the Carlitz factorial ([13], chapter 9, paragraph 9.1), $\tilde{\pi}$ is the Carlitz period ([13], chapter 3, paragraph 3.2), and $\zeta_A(n) := \sum_{a \in A, a \text{ monic}} \frac{1}{a^n} \in K_\infty := \mathbb{F}_p((\frac{1}{\theta}))$ is the value at n of the Carlitz-Goss zeta function. The Bernoulli-Carlitz numbers are connected to Taelman's class modules introduced in [19] (see for example [21] and [6]). L. Carlitz established a von-Staudt result for these numbers ([13], chapter 9, paragraph 9.2), and as an easy consequence, we get that if P is a monic irreducible polynomial in A , then BC_n is P -integral for $0 \leq n \leq q^{\deg_\theta P} - 2$. It is natural to ask if Conjecture 1 is valid in the Carlitzian context. In this paper, we prove a stronger result which answers positively to a Conjecture formulated in [3]:

Theorem 1.1. *Let $N \geq 2$ be an integer, $N \equiv 1 \pmod{q-1}$. Let $\ell_q(N)$ be the sum of the digits in base q of N . Let $P \in A$ be a monic irreducible polynomial of degree d such that $q^d > N$. If $d \geq \frac{\ell_q(N)-1}{q-1}N$, then:*

$$BC_{q^d-N} \not\equiv 0 \pmod{P}.$$

The above Theorem is linked with the study of exceptional zeros of certain L -series introduced in 2012 by F. Pellarin ([15]), but from a slightly different point of view. More precisely, let N be as above and for simplicity we assume that $\ell_q(N) \geq q$, let t be an indeterminate over K_∞ , let's consider:

$$\mathcal{L}_N(t) := \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a^N}{a(t)} \in A\left[\left[\frac{1}{t}\right]\right]^\times,$$

where $A_{+,d}$ is the set of monic elements in A of degree d . It was already noticed by F. Pellarin ([16]) that such L -series can be related with Anderson's solitons and should play an important role in the arithmetic theory of function fields. Let \mathbb{C}_∞ be the completion of a fixed algebraic closure of K_∞ . Then, one can show that $\mathcal{L}_N(t)$ converges on $\{x \in \mathbb{C}_\infty, v_\infty(x) < 0\}$, where v_∞ is the valuation on \mathbb{C}_∞ normalized such that $v_\infty(\theta) = -1$. Furthermore, one can easily see that the elements of $S := \{\theta^{q^j}, j \in \mathbb{Z}, q^j \leq N\}$ are zeros of the function $\mathcal{L}_N(t)$. We call the zeros of $\mathcal{L}_N(t)$ which belong to $\{x \in \mathbb{C}_\infty, v_\infty(x) < 0\} \setminus S$ the *exceptional zeros* of $\mathcal{L}_N(t)$. Let's briefly describe the case $q = p$. In this case, the exceptional zeros of $\mathcal{L}_N(t)$ are simple, belong to $\mathbb{F}_p((\frac{1}{\theta}))$ and are the eigenvalues of a certain K -linear

endomorphism $\phi_t^{(N)}$ of a finite dimensional K -vector space $H(\phi^{(N)})$ connected to the generalization of Taelman's class modules introduced in [5]. The proof of the fact that the exceptional zeros are simple and "real" uses combinatorial techniques introduced by F. Diaz-Vargas ([12]) and J. Sheats ([18]). Furthermore, if $p^d > N$, then:

$$BC_{p^d-N} \frac{(-1)^{\frac{\ell_p(N)-p}{p-1}} \prod_{l=0}^k \prod_{n=0, n \neq l}^{d-1} (\theta^{p^l} - \theta^{p^n})^{n_l}}{\Pi(N)\Pi(p^d - N)} = \det_K (\theta^{p^d} \text{Id} - \phi_t^{(N)} |_{H(\phi^{(N)})}),$$

where $\Pi(\cdot)$ is the Carlitz factorial, and $N = \sum_{l=0}^k n_l p^l$, $n_l \in \{0, \dots, p-1\}$. Since the eigenvalues of $\phi_t^{(N)}$ are exactly in this situation the exceptional zeros of $\mathcal{L}_N(t)$, we also obtain another proof of Theorem 1.1 as a consequence of the fact that:

$$\det_{K[Z]} (Z\text{Id} - \phi_t^{(N)} |_{H(\phi^{(N)})}) \in \mathbb{F}_p[Z, \theta],$$

and therefore (P is a monic irreducible polynomial of degree d):

$$\det_K (\theta^{p^d} \text{Id} - \phi_t^{(N)} |_{H(\phi^{(N)})}) \equiv \det_K (\theta \text{Id} - \phi_t^{(N)} |_{H(\phi^{(N)})}) \pmod{P}.$$

Let's observe that Theorem 1.1 implies the following (see [3], page 248):

$$\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{(a')^N}{a} \neq 0,$$

where a' denotes the derivative of a and $N \equiv 1 \pmod{q-1}$. In the appendix of this paper, we discuss a digit principle for such Euler type sums.

We mention that the construction of Kaneko-Zagier's objects in the positive characteristic world is the subject of a forthcoming work of F. Pellarin and R. Perkins ([17]), they prove, in this context, that the algebra of finite multiple zeta values is non-trivial. In this situation, it would be very interesting to examine the validity of Conjecture 1 for Bernoulli-Goss numbers (see [2] for a special case).

2. PROOF OF THEOREM 1.1

2.1. Notation. Let \mathbb{F}_q be a finite field having q elements and let p be the characteristic of \mathbb{F}_q . Let θ be an indeterminate over \mathbb{F}_q and let $A = \mathbb{F}_q[\theta]$, $K = \mathbb{F}_q(\theta)$, $K_\infty = \mathbb{F}_q((\frac{1}{\theta}))$. Let \mathbb{C}_∞ be the completion of a fixed algebraic closure of K_∞ . Let $v_\infty : \mathbb{C}_\infty \rightarrow \mathbb{Q} \cup \{+\infty\}$ be the valuation on \mathbb{C}_∞ normalized such that $v_\infty(\theta) = -1$. For $d \in \mathbb{N}$, let $A_{+,d}$ be the set of monic elements in A of degree d .

2.2. The L -series $L_N(t)$.

Let $N \geq 1$ be an integer. Let t be an indeterminate over \mathbb{C}_∞ . Let \mathbb{T}_t be the Tate algebra in the variable t with coefficients in \mathbb{C}_∞ . Let's set:

$$L_N(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} \in \mathbb{T}_t^\times.$$

Then, we can write:

$$L_N(t) = \sum_{i \geq 0} \alpha_{i,N}(t) \theta^{-i}, \quad \alpha_{i,N}(t) \in \mathbb{F}_q[t].$$

Observe that $\alpha_{0,N}(t) = 1$.

Lemma 2.1. *We have:*

$$\forall i \geq 0, \deg_t \alpha_{i,N}(t) \leq N(\text{Max}\{\frac{\log(i)}{\log(q)}, 0\} + \lfloor \frac{\ell_q(N)}{q-1} \rfloor + 1).$$

In particular $L_N(t)$ is an entire function.

Proof. Let $u = \lfloor \frac{\ell_q(N)}{q-1} \rfloor \in \mathbb{N}$. This Lemma is a consequence of the proof of [3], Lemma 7. We give a proof for the convenience of the reader. We will use the following elementary fact ([3], Lemma 4):

Let $s \geq 1$ be an integer and let t_1, \dots, t_s be s indeterminate over \mathbb{F}_p . If $d(q-1) > s$ then $\sum_{a \in A_{+,d}} a(t_1) \cdots a(t_s) = 0$.

If a is a monic polynomial in A , we will set:

$$\langle a \rangle_\infty = \frac{a}{\theta^{\deg_\theta a}} \in 1 + \frac{1}{\theta} \mathbb{F}_q \left[\left[\frac{1}{\theta} \right] \right].$$

Let $S_d := \sum_{a \in A_{+,d}} \frac{a(t)^N}{a}$. Observe that:

$$\deg_t S_d = dN.$$

We have :

$$S_d = \frac{1}{\theta^d} \sum_{a \in A_{+,d}} a(t)^N \langle a \rangle_\infty^{-1}.$$

Observe that (p -adically) $-1 = \sum_{n \geq 0} (q-1)q^n$. For $m \geq 0$, set:

$$y_m = \sum_{n=0}^m (q-1)q^n.$$

Then:

$$y_m \equiv -1 \pmod{q^{m+1}}, \quad \ell_q(y_m) = (m+1)(q-1).$$

Therefore:

$$v_\infty \left(\sum_{a \in A_{+,d}} a(t)^N \langle a \rangle_\infty^{-1} - \sum_{a \in A_{+,d}} a(t)^N \langle a \rangle_\infty^{y_m} \right) \geq q^{m+1},$$

where v_∞ is the ∞ -adic valuation on \mathbb{T}_t such that $v_\infty(\theta) = -1$. Thus, if $\ell_q(N) + (m+1)(q-1) < d(q-1)$, we get:

$$\sum_{a \in A_{+,d}} a(t)^N a^{y_m} = 0.$$

We therefore get, if $d \geq u+2$:

$$v_\infty(S_d) \geq d + q^{d-u-1}.$$

This implies that $L_N(t)$ is an entire function. Let j such that t^j appears in $\alpha_{i,N}(t)$. Let $x = \lfloor \frac{j}{N} \rfloor$. Let d be minimal such that t^j comes from S_d . We must have $d \geq x$ and $j \leq dN$. Furthermore, if $d \geq u+2$, we have:

$$i \geq d + q^{d-u-1}.$$

In particular:

$$i \geq d + q^{d-u-1} \geq q^{d-u-1}.$$

Therefore:

$$d \leq \text{Max}\left\{ \frac{\log(i)}{\log(q)}, 0 \right\} + u + 1.$$

□

2.3. The two variable polynomial $B_N(t, \theta)$.

Let $N \geq 2$, $N \equiv 1 \pmod{q-1}$. If $\ell_q(N) = 1$, we set $B_N(t, \theta) = 1$. Let's assume that $\ell_q(N) \neq 1$ and let's set $s = \ell_q(N) \geq 2$. Let t_1, \dots, t_s be s indeterminates over \mathbb{C}_∞ . Let \mathbb{T}_s be the Tate algebra in the indeterminates t_1, \dots, t_s with coefficients in \mathbb{C}_∞ . Let $\tau : \mathbb{T}_s \rightarrow \mathbb{T}_s$ be the continuous morphism of $\mathbb{F}_q[t_1, \dots, t_s]$ -algebras such that $\forall c \in \mathbb{C}_\infty, \tau(c) = c^q$. For $i = 1, \dots, s$, we set:

$$\omega(t_i) = \lambda_\theta \prod_{j \geq 0} \left(1 - \frac{t_i}{\theta^{q^j}}\right)^{-1} \in \mathbb{T}_s,$$

where λ_θ is a fixed $(q-1)$ th-root of $-\theta$ in \mathbb{C}_∞ . Set:

$$\tilde{\pi} = \lambda_\theta \theta \prod_{j \geq 1} (1 - \theta^{1-q^j})^{-1}.$$

Set:

$$L_s = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t_1) \cdots a(t_s)}{a} \in \mathbb{T}_s^\times.$$

We also set:

$$\mathbb{B}_s = (-1)^{\frac{s-1}{q-1}} \frac{L_s \omega(t_1) \cdots \omega(t_s)}{\tilde{\pi}} \in \mathbb{T}_s.$$

Then, by [5], Lemma 7.6 (see also [3], Corollary 21), $\mathbb{B}_s \in \mathbb{F}_q[t_1, \dots, t_s, \theta]$ is a monic polynomial in θ of degree $r = \frac{s-q}{q-1}$. Write $N = \sum_{n=1}^{\ell_q(N)} q^{e_n}$, $e_1 \leq e_2 \leq \dots \leq e_{\ell_q(N)}$. We set:

$$B_N(t, \theta) = \mathbb{B}_s |_{t_i = t^{q^{e_i}}} \in \mathbb{F}_q[t, \theta].$$

We observe that $B_N(t, \theta)$ is a monic polynomial in θ such that $\deg_\theta B_N(t, \theta) = r$.

Lemma 2.2. *Let $N \geq 2$, $N \equiv 1 \pmod{q-1}$. Then:*

- 1) $B_N(t^p, \theta^p) = B_N(t, \theta)^p$.
- 2) $B_{qN}(t, \theta) = B_N(t^q, \theta)$.
- 3) *We have:*

$$B_N(t, \theta) \equiv (\theta - t)^r - r(t^q - t)(\theta - t)^{r-1} \pmod{(t^q - t)^2 \mathbb{F}_p[t, \theta]}.$$

- 4) *If $N \equiv 0 \pmod{p}$, then $B_N(t, \theta) \in \mathbb{F}_p[t^p, \theta]$.*

Proof. Recall that:

$$B_N(t, \theta) = \frac{(-1)^{\frac{\ell_q(N)-1}{q-1}}}{\tilde{\pi}} \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} \prod_{l=0}^k \omega(t^{q^l})^{n_l},$$

where $N = \sum_{l=0}^k n_l q^l$, $n_0, \dots, n_k \in \{0, \dots, q-1\}$. Observe that:

$$\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} \in \mathbb{F}_p[t][[\frac{1}{\theta}]].$$

Thus:

$$B_N(t, \theta) \in \mathbb{F}_p[t, \theta].$$

Thus we get assertion 1). Assertion 2) is a consequence of the definition of $B_N(t, \theta)$. Let $\zeta \in \mathbb{F}_q$. By [4], theorem 2.9, we have:

$$\omega(t) |_{t=\zeta} = \exp_C\left(\frac{\tilde{\pi}}{\theta - \zeta}\right),$$

where $\exp_C : \mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ is the Carlitz exponential ([13], chapter 3, paragraph 3.2). Now, by [15] Theorem 1, we get:

$$\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} \Big|_{t=\zeta} = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(\zeta)}{a} = \frac{\tilde{\pi}}{(\theta - \zeta) \exp_C\left(\frac{\tilde{\pi}}{\theta - \zeta}\right)}.$$

But observe that:

$$\exp_C\left(\frac{\tilde{\pi}}{\theta - \zeta}\right)^{q-1} = -(\theta - \zeta).$$

Thus:

$$B_N(t, \theta) \Big|_{t=\zeta} = (\theta - \zeta)^r.$$

Observe that $\frac{d}{dt} B_N(t, \theta)$ is equal to:

$$\frac{n_0(-1)^{r+1}}{\tilde{\pi}} \left(\prod_{l=0}^k \omega(t^{q^l})^{-n_l} \right) \sum_{d \geq 0} \sum_{a \in A_{+,d}} \left(\frac{\frac{d}{dt}(a(t))a(t)^{N-1}}{a} - \frac{a(t)^N}{a} \frac{\frac{d}{dt}(\omega(t))}{\omega(t)} \right).$$

Thus we get assertion 4). Since for $\ell_q(N) = q$, we have $B_N(t) = 1$. We get:

$$\forall \zeta \in \mathbb{F}_q, \frac{d}{dt} B_N(t, \theta) \Big|_{t=\zeta} = 0.$$

This concludes the proof of the Lemma. \square

Lemma 2.3. *Let $N \geq 2, N \equiv 1 \pmod{q-1}$. Then $\deg_t B_N(t, \theta) \geq p$ if $r \geq 1$ and the total degree in t, θ of $B_N(t, \theta)$ is less than or equal to $\text{Max}\{rN + r - 2, 0\}$. Furthermore $B_N(t, \theta)$ (as a polynomial in t) is a primitive polynomial.*

Proof. Recall that if $r = 0$ then $B_N(t, \theta) = 1$. Let's assume that $r \geq 1$. Observe that by Lemma 2.2, we have:

$$B_N(t, 0) \equiv -(-t)^{r-1}(t + r(t^q - t)) \pmod{(t^q - t)^2 \mathbb{F}_p[t]}.$$

In particular $\deg_t B_N(t, \theta) \geq p$. Let $x \in \mathbb{C}_\infty$ such that $v_\infty(x) > \frac{-1}{N}$. Then:

$$\sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(x)^N}{a} = \prod_{P \text{ monic irreducible in } A} \left(1 - \frac{P(x)^N}{P}\right)^{-1} \in \mathbb{C}_\infty^\times.$$

Write $N = \sum_{l=0}^k n_l q^l, n_l \in \{0, \dots, q-1\}, n_k \neq 0$. For $l = 0, \dots, k$, we have:

$$\omega(t^{q^l}) \Big|_{t=x} \in \mathbb{C}_\infty^\times.$$

Therefore:

$$B_N(t, \theta) \Big|_{t=x} \neq 0.$$

This implies that, if $x \in \mathbb{C}_\infty$ is a root of $B_N(t, \theta)$ then $v_\infty(x) \leq \frac{-1}{N} < 0$. Write in $\mathbb{C}_\infty[t]$:

$$B_N(t, \theta) = \lambda \prod_{j=1}^m (t - x_j), \lambda \in \mathbb{F}_p[\theta] \setminus \{0\}, \deg_\theta \lambda \leq r-1, x_1, \dots, x_m \in \mathbb{C}_\infty,$$

where $m = \deg_t B_N(t, \theta)$. Then:

$$\theta^r = (-1)^m \lambda \prod_{j=1}^m x_j.$$

Therefore:

$$\deg_{\theta} \lambda - r \leq \frac{-m}{N}.$$

We finally get:

$$\deg_t B_N(t, \theta) \leq (r - \deg_{\theta} \lambda)N \leq rN.$$

Since $B_N(t, \theta)$ is a monic polynomial in θ , the total degree in t, θ of $B_N(t, \theta)$ is less than or equal to $\deg_t B_N(t, \theta) + r - 2$.

Write:

$$B_N(t, \theta) = \alpha F, \alpha \in \mathbb{F}_p[\theta] \setminus \{0\},$$

where F is a primitive polynomial (as a polynomial in t). In particular α must divide θ^r and $B_N(1, \theta)$ in $\mathbb{F}_p[\theta]$. By Lemma 2.2, this implies that $\alpha \in \mathbb{F}_p^{\times}$. \square

Remark 2.4. Let $f(\theta)$ be a monic irreducible polynomial in $\mathbb{F}_p[\theta]$. Let P_1, \dots, P_m be the monic irreducible polynomials in A such that $f(\theta) = P_1 \cdots P_m$. We can order them such that if $d = \deg_{\theta} f$, then for $i = 1, \dots, m$, we have:

$$P_i = \sigma^{i-1}(P_1),$$

where $\sigma : A \rightarrow A$ is the morphism of $\mathbb{F}_p[\theta]$ -algebras such that $\forall x \in \mathbb{F}_q$, $\sigma(x) = x^p$, and $m = [\mathbb{F}_{p^d} \cap \mathbb{F}_q : \mathbb{F}_p]$. Let $N \geq 1$, and let's set:

$$\chi_N(f) = \prod_{i=1}^m (P_i(\theta) - P_i(t)^N) - f(\theta) \in \mathbb{F}_p[t, \theta].$$

We have:

$$\deg_t \chi_N(f) = N \deg_{\theta} f,$$

$$\deg_{\theta} \chi_N(f) \leq \deg_{\theta} f - \deg_{\theta} P_1 \leq \deg_{\theta}(f) \left(1 - \frac{\log(p)}{\log(q)}\right).$$

If f_1, \dots, f_n are n irreducible monic polynomials in $\mathbb{F}_p[\theta]$, we set:

$$\chi_N(f_1 \cdots f_n) = \prod_{i=1}^n \chi_N(f_i) \in \mathbb{F}_p[t, \theta].$$

Thus in $\mathbb{F}_p[t][\left(\frac{1}{\theta}\right)]$:

$$L_N(t) = \sum_{d \geq 0} \sum_{a \in \mathbb{F}_p[\theta]_{+, d}} \frac{\chi_N(a)}{a}.$$

Observe that $\sum_{d \geq 0} \sum_{a \in \mathbb{F}_p[\theta]_{+, d}} \frac{\chi_N(a)}{a}$ converges on $\{x \in \mathbb{C}_{\infty}, v_{\infty}(x) > \frac{-1}{N}\}$ and does not vanish. Therefore on $\{x \in \mathbb{C}_{\infty}, v_{\infty}(x) > \frac{-1}{N}\}$:

$$\sum_{d \geq 0} \sum_{a \in \mathbb{F}_p[\theta]_{+, d}} \frac{\chi_N(a)}{a} = \frac{1}{\theta^{\frac{\ell_q(N)-q}{q-1}}} B_N(t, \theta) \prod_{l=0}^k \prod_{j \geq 0} \left(1 - \frac{t^q}{\theta^{q^j}}\right)^{n_l},$$

where $N \equiv 1 \pmod{q-1}$, $\ell_q(N) \geq q$, $N = \sum_{l=0}^k n_l q^l$, $n_l \in \{0, \dots, q-1\}$, $n_k \neq 0$.

Let $s \geq 2$, $s \equiv 1 \pmod{q-1}$. Recall that we have set:

$$\mathbb{B}_s = (-1)^{\frac{s-1}{q-1}} \frac{L_s \omega(t_1) \cdots \omega(t_s)}{\tilde{\pi}} \in \mathbb{T}_s,$$

where \mathbb{T}_s is the Tate algebra in the indeterminates t_1, \dots, t_s with coefficients in \mathbb{C}_{∞} , and for $i = 1, \dots, s$, $\omega(t_i) = \lambda_{\theta} \prod_{j \geq 0} \left(1 - \frac{t_i}{\theta^{q^j}}\right)^{-1}$. For $m \in \mathbb{N}$, we denote by $BC_m \in K$ the m th Bernoulli-Carlitz number ([13], chapter 9, paragraph 9.2).

Proposition 2.5.

1) Let $N \geq 1, N \equiv 1 \pmod{q-1}, \ell_q(N) \geq q$. Recall that $r = \frac{\ell_q(N)-q}{q-1}$. Let $d \geq 1$ such that $q^d > N$, then we have the following equality in \mathbb{C}_∞ :

$$\frac{B_N(\theta, \theta^{q^d})}{\prod_{l=0}^k \prod_{n=0, n \neq l}^{d-1} (\theta^{q^l} - \theta^{q^n})^{n_l}} = (-1)^r \frac{BC_{q^d-N}}{\Pi(N)\Pi(q^d-N)},$$

where $\Pi(\cdot)$ is the Carlitz factorial, and $N = \sum_{l=0}^k n_l q^l, n_l \in \{0, \dots, q-1\}$.

2) Let $N \geq 2, N \equiv 1 \pmod{q-1}$. Let P be a monic irreducible polynomial in A of degree $d \geq 1$ such that $q^d > N$. Then $BC_{q^d-N} \equiv 0 \pmod{P}$ if and only if $B_N(\theta, \theta) \equiv 0 \pmod{P}$.

Proof.

1) The first assertion of the Proposition is a consequence of the proof of Theorem 2 in [3]. For the convenience of the reader, we give the proof of this result. Let $s \geq q, s \equiv 1 \pmod{q-1}$. Then ([5], Lemma 7.6), we have that $\mathbb{B}_s \in \mathbb{F}_q[t_1, \dots, t_s, \theta]$ is a monic polynomial in θ of degree $\frac{s-q}{q-1}$.

Let $\tau : \mathbb{T}_s \rightarrow \mathbb{T}_s$ be the continuous morphism of $\mathbb{F}_q[t_1, \dots, t_s]$ -algebras such that $\forall x \in \mathbb{C}_\infty, \tau(x) = x^q$. Since $\lambda_\theta^q = -\theta\lambda_\theta$, we have:

$$\tau(\omega(t_i)) = (t_i - \theta)\omega(t_i), i = 1, \dots, s.$$

Let $d \geq 1$, we get:

$$(-1)^{\frac{s-1}{q-1}} \frac{\tau^d(L_s)}{\tilde{\pi}^{q^d}} \omega(t_1) \dots \omega(t_s) = \frac{\tau^d(\mathbb{B}_s)}{\prod_{l=0}^{d-1} (t_1 - \theta^{q^l}) \dots (t_s - \theta^{q^l})}.$$

Recall that, by formula (24) in [15], we have:

$$(t_i - \theta^{q^l})\omega(t_i) \Big|_{t=\theta^{q^l}} = -\frac{\tilde{\pi}^{q^l}}{D_l}.$$

Let $l_1, \dots, l_s \in \mathbb{N}$, we get:

$$(-1)^{\frac{s-1}{q-1}} \frac{\tau^d(L_s)}{\tilde{\pi}^{q^d}} (t_1 - \theta^{q^{l_1}})\omega(t_1) \dots (t_s - \theta^{q^{l_s}})\omega(t_s) = \frac{\tau^d(\mathbb{B}_s)(t_1 - \theta^{q^{l_1}}) \dots (t_s - \theta^{q^{l_s}})}{\prod_{l=0}^{d-1} (t_1 - \theta^{q^l}) \dots (t_s - \theta^{q^l})}.$$

Now, let $N \geq 1$ such that $\ell_q(N) = s$ (observe that $N \equiv 1 \pmod{q-1}$ and $\ell_q(N) \geq q$). Write $N = \sum_{l=0}^k n_l q^l, n_0, \dots, n_k \in \{0, \dots, q-1\}, n_k \neq 0$. Let $d \geq k+1$. We get:

$$(-1)^{\frac{s-1}{q-1}} \frac{\sum_{u \geq 0} \sum_{a \in A_+, u} \frac{a(t)^N}{a^{q^d}}}{\tilde{\pi}^{q^d}} \left(\prod_{l=0}^k ((t^{q^l} - \theta^{q^l})\omega(t^{q^l}))^{n_l} \right) = \frac{B_N(t, \theta^{q^d})}{\prod_{l=0}^k \prod_{n=0, n \neq l}^{d-1} (t^{q^l} - \theta^{q^n})^{n_l}}.$$

We get:

$$\frac{B_N(t, \theta^{q^d})}{\prod_{l=0}^k \prod_{n=0, n \neq l}^{d-1} (t^{q^l} - \theta^{q^n})^{n_l}} \Big|_{t=\theta} = (-1)^{\frac{\ell_q(N)-q}{q-1}} \frac{BC_{q^d-N}}{\Pi(N)\Pi(q^d-N)}.$$

2) The result is well-known for $\ell_q(N) = 1$ (this is a consequence of the definition of the Bernoulli-Carlitz numbers and [13], Lemma 8.22.4). Thus, we will assume $\ell_q(N) \geq q$. The assertion is then a consequence of the fact that:

$$B_N(\theta, \theta) \equiv B_N(\theta, \theta^{q^d}) \pmod{P}.$$

□

We have already mentioned that $\mathbb{B}_s \in \mathbb{F}_q[t_1, \dots, t_s, \theta]$ is a monic polynomial in θ of degree $\frac{s-q}{q-1}$ ([5], Lemma 7.6). Let's observe that we have:

Lemma 2.6. *For $s \geq 2q - 1$, $s \equiv 1 \pmod{q-1}$, we have:*

$$\mathbb{B}_s(t_1, \dots, t_{s-(q-1)}, 0, \dots, 0) = (\theta - t_1 \cdots t_{s-(q-1)}) \mathbb{B}_{s-(q-1)}(t_1, \dots, t_{s-(q-1)}).$$

More generally, if ζ is in the algebraic closure of \mathbb{F}_q in \mathbb{C}_∞ , let P be the monic irreducible polynomial in A such that $P(\zeta) = 0$. Let $s \equiv 1 \pmod{q-1}$, $s \geq q + q^d - 1$, where d is the degree of P . Write $s' = s - (q^d - 1)$. We have:

$$\mathbb{B}_s(t_1, \dots, t_{s'}, \zeta, \dots, \zeta) = (P - P(t_1) \cdots P(t_{s'})) \mathbb{B}_{s'}(t_1, \dots, t_{s'}).$$

Proof. The polynomial $\mathbb{B}_s(t_1, \dots, t_{s-(q-1)}, 0, \dots, 0)$ is equal to:

$$(-\theta)(-1)^{\frac{s-1}{q-1}} \frac{\omega(t_1) \cdots \omega(t_{s-(q-1)})}{\tilde{\pi}} \prod_{\substack{P \text{ monic irreducible in } A, \\ P \neq \theta}} \left(1 - \frac{P(t_1) \cdots P(t_{s-(q-1)})}{P}\right)^{-1}.$$

The proof of the second assertion of the Lemma is similar, using [4], Theorem 2.9, and the properties of Gauss-Thakur sums ([22]). \square

Lemma 2.7.

$$\mathbb{B}_{2q-1} = \theta - \sum_{1 \leq i_1 < \dots < i_q \leq 2q-1} t_{i_1} \cdots t_{i_q}.$$

Proof. Let $\mathbb{T}_{2q-1}(K_\infty)$ be the Tate algebra in the variable t_1, \dots, t_{2q-1} with coefficients in K_∞ . Then:

$$\mathbb{T}_{2q-1} = \frac{1}{\theta} A[t_1, \dots, t_{2q-1}] \oplus N,$$

where $N = \{f \in \mathbb{T}_{2q-1}, v_\infty(f) \geq 2\}$. Let $\phi : A \rightarrow A[t_1, \dots, t_{2q-1}]\{\tau\}$ be the morphism of \mathbb{F}_q -algebras give by $\phi_\theta = (t_1 - \theta) \cdots (t_{2q-1} - \theta)\tau + \theta$. Then by the results in [5], we have:

$$N = \exp_\phi(\mathbb{T}_{2q-1}),$$

and $H_\phi := \frac{\mathbb{T}_{2q-1}}{A[t_1, \dots, t_{2q-1}] \oplus N}$ is a free $\mathbb{F}_q[t_1, \dots, t_{2q-1}]$ -module of rank one generated by $\frac{1}{\theta}$. Furthermore:

$$\mathbb{B}_s = \det_{\mathbb{F}_q[t_1, \dots, t_{2q-1}][Z]} (Z \text{Id} - \phi_\theta |_{H_\phi \otimes_{\mathbb{F}_q[t_1, \dots, t_{2q-1}]} \mathbb{F}_q[t_1, \dots, t_{2q-1}][Z]}) |_{Z=\theta}.$$

Now, observe that:

$$\phi_\theta\left(\frac{1}{\theta}\right) \equiv \frac{\sum_{1 \leq i_1 < \dots < i_q \leq 2q-1} t_{i_1} \cdots t_{i_q}}{\theta} \pmod{A[t_1, \dots, t_{2q-1}] \oplus N}.$$

The Lemma follows. \square

For $\underline{m} = (m_1, \dots, m_d) \in \mathbb{N}^d$, we set $m_0 := s - (m_1 + \dots + m_d)$, and:

$$\sigma_s(\underline{m}) = \sum_{u=1}^d \prod_{i \in J_u} t_i^u,$$

where the sum runs through the disjoint unions $J_1 \sqcup \dots \sqcup J_k \subset \{1, \dots, s\}$ such that $|J_u| = m_u$, $u = 1, \dots, d$. Notice in particular that $\sigma_s(\underline{m}) = 0$ if $m_1 + \dots + m_d > s$, that is, if $m_0 < 0$. To give an example, the above lemma shows that $\mathbb{B}_{2q-1} = \theta - \sigma((q))$.

Lemma 2.8. *Let $\underline{m} \in \mathbb{N}^d$. We have :*

$$\sigma_s(\underline{m}) \mid_{t_s=0, \dots, t_{s-(q-2)}=0} = \sigma_{s-(q-1)}(\underline{m}).$$

In particular, if $m_0 < q - 1$, we have:

$$\sigma_s(\underline{m}) \mid_{t_s=0, \dots, t_{s-(q-2)}=0} = 0.$$

Proof. This is a straight computation. \square

Let $\rho : \mathbb{F}_p[t_1, \dots, t_s] \rightarrow \mathbb{N} \cup \{+\infty\}$ be the function given by:
- if $f = 0$, $\rho(f) = +\infty$,
- if $f \neq 0$, $f = \sum \alpha_{j_1, \dots, j_s} t_1^{j_1} \cdots t_s^{j_s}$, $\alpha_{j_1, \dots, j_s} \in \mathbb{F}_p$, then $\rho(f) = \text{Inf}\{j_1 + \dots + j_s, \alpha_{j_1, \dots, j_s} \neq 0\}$.

Let's write:

$$\mathbb{B}_s = \sum_{i=0}^r B_{i,s} \theta^{r-i},$$

where $B_{i,s} \in \mathbb{F}_p[t_1, \dots, t_s]$ is a symmetric polynomial, and $B_{0,s} = 1$.

Proposition 2.9. *For $i = 1, \dots, r$, we have:*

$$\rho(B_{i,s}) \geq i(q-1) + 1.$$

Proof. By Lemma 2.7, this is true for $r = 1$, thus we can assume that $r \geq 2$. The proof is by induction on r . Recall that by Lemma 2.2, we have:

$$\mathbb{B}_s \mid_{t_1=\dots=t_s=0} = \theta^r.$$

Thus, for $i = 1, \dots, r$, we can write:

$$B_{i,s} = \sum_{\underline{m} \in S} x_{i,\underline{m}} \sigma_s(\underline{m}), \quad x_{i,\underline{m}} \in \mathbb{F}_p,$$

where $S = \{(m_1, \dots, m_s) \in \mathbb{N}^s, 1 \leq m_1 + \dots + m_d \leq s\}$. Set:

$$\widetilde{B}_{i,s} = B_{i,s} - \sum_{\underline{m} \in S, m_0 < q-1} x_{i,\underline{m}} \sigma_s(\underline{m}).$$

Then:

$$\rho(B_{i,s} - \widetilde{B}_{i,s}) \geq r(q-1) + 2.$$

Therefore we have to prove:

$$\rho(\widetilde{B}_{i,s}) \geq i(q-1) + 1.$$

Observe that, by Lemma 2.8, we have:

$$B_{i,s} \mid_{t_s=\dots=t_{s-(q-2)}=0} = \widetilde{B}_{i,s} \mid_{t_s=\dots=t_{s-(q-2)}=0} = \sum_{\underline{m} \in S, m_0 \geq q-1} x_{i,\underline{m}} \sigma_{s-(q-1)}(\underline{m}).$$

By Lemma 2.6, we have:

$$\mathbb{B}_s \mid_{t_s=\dots=t_{s-(q-2)}=0} = (\theta - t_1 \cdots t_{s-(q-1)}) \mathbb{B}_{s-(q-1)}.$$

We therefore get, for $i = 1, \dots, r$:

$$\widetilde{B}_{i,s} \mid_{t_s=\dots=t_{s-(q-2)}=0} = B_{i,s-(q-1)} - t_1 \cdots t_{s-(q-1)} B_{i-1,s-(q-1)},$$

where we have set $B_{r,s-(q-1)} = 0$. Now, by the induction hypothesis:

$$\rho(B_{i,s-(q-1)} - t_1 \cdots t_{s-(q-1)} B_{i-1,s-(q-1)}) \geq i(q-1) + 1.$$

Thus:

$$\rho(B_{i,s}) \geq i(q-1) + 1.$$

□

Corollary 2.10. *Let $N \equiv 1 \pmod{q-1}$, $\ell_q(N) \geq 2q-1$. Then $\forall a \in \overline{\mathbb{F}}_q[\theta]$, we have:*

$$B_N(t, \theta) |_{t=a} \neq 0.$$

Proof. By Proposition 2.9, we have:

$$B_N(t, \theta) - \theta^r \in t(t, \theta)^r.$$

The Corollary follows easily. □

Proof of Theorem 1.1

We can assume that $\ell_q(N) \geq q$. By Lemma 2.3, the total degree in t, θ of $B_N(t, \theta)$ is strictly less than $(r+1)N$, where $r = \frac{\ell_q(N)-q}{q-1}$. Now, by Corollary 2.10:

$$B_N(\theta, \theta) \neq 0.$$

Furthermore, $\deg_\theta B_N(\theta, \theta) < (r+1)N$. Thus if P is a monic irreducible polynomial in A such that $\deg_\theta P \geq (r+1)d$, we have:

$$B_N(\theta, \theta) \not\equiv 0 \pmod{P}.$$

We conclude the proof of the Theorem by Proposition 2.5.

3. EXCEPTIONAL ZEROS AND EIGENVALUES OF CERTAIN K -ENDOMORPHISMS

3.1. The L -series $L_N(t)$.

Let $N \geq 1$ be an integer. Recall that

$$L_N(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a} = \sum_{i \geq 0} \alpha_{i,N}(t) \theta^{-i}, \quad \alpha_{i,N}(t) \in \mathbb{T}_t^\times.$$

Let $d \geq 1$ be an integer, we set for $\mathbf{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$:

- $\ell(\mathbf{k}) = d$,
- $|\mathbf{k}| = k_0 + \dots + k_{d-1}$,
- if $a = a_0 + a_1\theta + \dots + a_{d-1}\theta^{d-1} + \theta^d$, $a_i \in \mathbb{F}_q$, $i = 0, \dots, d-1$, $a^{\mathbf{k}} = \prod_{i=0}^{d-1} a_i^{k_i}$.

Let's begin by a simple observation. Let $d \geq 1$ and let $\mathbf{k} = (k_0, \dots, k_{d-1}) \in \mathbb{N}^d$. Let $N \geq 1$ be an integer, we get:

$$\sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = \sum_{a \in A_{+,d}} \sum_{\mathbf{m} \in \mathbb{N}^{d+1}, |\mathbf{m}|=N} C(N, \mathbf{m}) a^{\mathbf{k}} a^{\mathbf{m}} t^{m_1+2m_2+\dots+dm_d},$$

where:

$$C(N, \mathbf{m}) = \frac{N!}{m_0! \dots m_d!} \in \mathbb{F}_p.$$

Recall that by Luca's Theorem $C(N, \mathbf{m}) \neq 0$ if and only if there is no carryover p -digits in the sum $N = m_0 + \dots + m_d$. Furthermore, recall that, for $n \in \mathbb{N}$, $\sum_{\lambda \in \mathbb{F}_q} \lambda^n \neq 0$ if and only if $n \equiv 0 \pmod{q-1}$ and $n \geq 1$. Thus, for $\mathbf{m} \in \mathbb{N}^{d+1}$, $\sum_{a \in A_{+,d}} a^{\mathbf{m}} = 0$ unless $(m_0, \dots, m_{d-1}) \in ((q-1)(\mathbb{N} \setminus \{0\}))^d$ and in this latter case $\sum_{a \in A_{+,d}} a^{\mathbf{m}} = (-1)^d$.

Thus for $d, N \geq 1$, $\mathbf{k} \in \{0, \dots, q-1\}^d$, we denote by $U_d(N, \mathbf{k})$ the set of elements $\mathbf{m} \in \mathbb{N}^{d+1}$ such that:

- there is no carryover p -digits in the sum $N = m_0 + \dots + m_d$,
- for $n = 0, \dots, d-1$, $m_n - k_n \in (q-1)\mathbb{N}$.

For $\mathbf{m} \in U_d(N, \mathbf{k})$, we set:

$$\deg \mathbf{m} = m_1 + 2m_2 + \cdots + dm_d.$$

An element $\mathbf{m} \in U_d(N, \mathbf{k})$ is called optimal if $\deg \mathbf{m} = \text{Max}\{\deg \mathbf{n}, \mathbf{n} \in U_d(N, \mathbf{k})\}$. If $U_d(N, \mathbf{k}) \neq \emptyset$, the greedy element of $U_d(N, \mathbf{k})$ is the element $\mathbf{m} = (m_0, \dots, m_d) \in U_d(N, \mathbf{k})$ such that (m_d, \dots, m_1) is largest lexicographically.

Let $\mathbf{k} \in \mathbb{N}^d, d \geq 1$. For $n = 0, \dots, d-1$, let $\bar{k}_n \in \{0, \dots, q-1\}$ be the least integer such that $k_n + \bar{k}_n \in (q-1)(\mathbb{N} \setminus \{0\})$. We set:

$$\bar{\mathbf{k}} = (\bar{k}_0, \dots, \bar{k}_{d-1}).$$

We get:

$$\sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = (-1)^d \sum_{\mathbf{m} \in U_d(N, \bar{\mathbf{k}})} C(N, \mathbf{m}) t^{\deg \mathbf{m}}.$$

Let $N \geq 1$ be an integer and let $\ell_q(N)$ be the sum of digits of N in base q . Then we can write in a unique way:

$$N = \sum_{n=1}^{\ell_q(N)} q^{e_n}, \quad e_1 \leq e_2 \leq \cdots \leq e_{\ell_q(N)}.$$

We set:

$$r = \text{Max}\{0, \lfloor \frac{\ell_q(N) - q}{q-1} \rfloor\} \in \mathbb{N}.$$

Lemma 3.1. *We have:*

$$\forall i \geq 0, \alpha_{i,N}(t) = \sum_{\ell(\mathbf{k})+w(\mathbf{k})=i} (-1)^{\ell(\mathbf{k})} C_{\mathbf{k}} \sum_{a \in A_{+, \ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}} \in \mathbb{F}_p[t],$$

where $C_{\mathbf{k}} = (-1)^{|\mathbf{k}|} \frac{|\mathbf{k}|!}{k_0! \cdots k_{d-1}!} \in \mathbb{F}_p$, $w(\mathbf{k}) = k_{d-1} + \cdots + (d-1)k_1 + dk_0$, for $\mathbf{k} = (k_0, \dots, k_{d-1})$.

Proof. Let $a \in A_{+,d}$. We have:

$$\frac{1}{a} = \frac{1}{\theta^d} \sum_{\mathbf{k} \in \mathbb{N}^d} C_{\mathbf{k}} a^{\mathbf{k}} \frac{1}{\theta^{w(\mathbf{k})}},$$

where $C_{\mathbf{k}} = (-1)^{|\mathbf{k}|} \frac{|\mathbf{k}|!}{k_0! \cdots k_{d-1}!} \in \mathbb{F}_p$, $w(\mathbf{k}) = k_{d-1} + \cdots + (d-1)k_1 + dk_0$. Thus:

$$\sum_{a \in A_{+,d}} \frac{a(t)^N}{a} = \frac{(-1)^d}{\theta^d} \sum_{\mathbf{k} \in \mathbb{N}^d} C_{\mathbf{k}} \frac{1}{\theta^{w(\mathbf{k})}} \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}}.$$

Therefore:

$$\alpha_{i,N}(t) = \sum_{\ell(\mathbf{k})+w(\mathbf{k})=i} (-1)^{\ell(\mathbf{k})} C_{\mathbf{k}} \sum_{a \in A_{+, \ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}} \in \mathbb{F}_p[t].$$

□

Lemma 3.2. *Let $j \in \mathbb{Z}$. Then $L_N(t) \big|_{t=\theta^j} = 0$ if and only if $N \equiv 1 \pmod{q-1}$, and $q^j N > 1$.*

Proof. This comes from the following facts:

- $\forall n \geq 1, \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{1}{a^n} \neq 0$,

- for $n \geq 0, \sum_{d \geq 0} \sum_{a \in A_{+,d}} a^n = 0$ if and only if $n \geq 1, n \equiv 0 \pmod{q-1}$. □

We will need the following Lemma in the sequel:

Lemma 3.3. *Let $F(t) = \sum_{n \geq 0} \beta_n(t) \frac{1}{\theta^n} \in \mathbb{F}_q[t][[\frac{1}{\theta}]]$, $\beta_n(t) \in \mathbb{F}_q[t]$, and let $\rho \in \mathbb{R}$ such that $F(t)$ converges on $\{x \in \mathbb{C}_\infty, v_\infty(x) \geq \rho\}$. Let $M \geq 1$ and set $F_M(t) = \sum_{n=0}^M \beta_n(t) \frac{1}{\theta^n} \in K_\infty[t]$. Let $\varepsilon \in \mathbb{R}, \varepsilon \geq \rho$. Suppose that $F_M(t)$ has exactly $k \geq 1$ zeros in \mathbb{C}_∞ with valuation ε . Then either $F(t)$ has k zeros with valuation ε or $F(t)$ has at least $\deg_t F_M(t) + 1$ zeros with valuation $> \varepsilon$.*

Proof. Let's assume that the side of the Newton polygon of $F_M(t)$ corresponding to the k zeros of valuation ε is not a portion of a side of the Newton polygon of $F(t)$, then $F(t)$ has a side of slope $-\varepsilon' < -\varepsilon$ with end point of abscissa $k' > \deg_t F_M(t)$. Thus the Newton polygon of $F(t)$ delimited by the vertical axis of abscissas 0 and k' has only sides of slope $\leq -\varepsilon'$. Thus $F(t)$ has k' zeros of valuation $\geq \varepsilon'$. \square

3.2. An example.

For the convenience of the reader, we treat a basic example: $N = 1$. We set $\ell_0 = 1$ and for $d \geq 1$, $\ell_d = (\theta - \theta^q) \cdots (\theta - \theta^{q^d})$.

Lemma 3.4. *Let $d \geq 0$. Then:*

$$\sum_{a \in A_{+,d}} \frac{1}{a} = \frac{1}{\ell_d}.$$

Proof. This is a well-known consequence of a result of Carlitz ([13], Theorem 3.1.5). Let's give a proof for the convenience of the reader. We can assume that $d \geq 1$. Set:

$$e_d(X) = \prod_{a \in A, \deg_\theta a < d} (X - a) \in A[X].$$

Then ([13], Theorem 3.1.5):

$$e_d(X) = \sum_{i=0}^d \frac{D_d}{D_i \ell_d^{q^i}} X^{q^i},$$

where $D_0 = 1$, and for $i \geq 1$, $D_i = (\theta^{q^i} - \theta) D_{i-1}^q$. Now observe that:

$$\frac{\frac{d}{dX}(e_d(X - \theta^d))}{e_d(X - \theta^d)} \Big|_{X=0} = - \sum_{a \in A_{+,d}} \frac{1}{a}.$$

Since $e_d(\theta^d) = D_d$ ([13], Corollary 3.1.7), we get the desired result. \square

Lemma 3.5. *Let $d \geq 0$. then:*

$$\sum_{a \in A_{+,d}} \frac{a(t)}{a} = \frac{(t - \theta) \cdots (t - \theta^{q^{d-1}})}{\ell_d}.$$

Proof. We can assume that $d \geq 1$. Set:

$$S = \sum_{a \in A_{+,d}} \frac{a(t)}{a}.$$

Then for $i = 0, \dots, d-1$, we have:

$$S \Big|_{t=\theta^{q^i}} = 0.$$

Furthermore, by Lemma 3.4, S has degree d in t and the coefficient of t^d is $\frac{1}{\ell_d}$. The Lemma follows. \square

Lemma 3.6. *The edge points of the Newton polygon of $L_1(t)$ are $(d, q^{\frac{d-q}{q-1}})$, $d \in \mathbb{N}$.*

Proof. Let's write:

$$L_1(t) = \sum_{d \geq 0} S_d(t),$$

where:

$$S_d(t) = \sum_{a \in A_{+,d}} \frac{a(t)}{a}.$$

Let $d \in \mathbb{N}$ and let $d' > d$. Let $x \in K$ be the coefficient of t^d in $S_{d'}(t)$. Then, by Lemma 3.5, we get:

$$v_\infty(x) \geq -v_\infty(\ell_{d'}) - q^d - \dots - q^{d'-1} > q^{\frac{d-1}{q-1}}.$$

Thus, if we write:

$$L_1(t) = \sum_{d \geq 0} \alpha_d t^d \in K_\infty[[t]], \alpha_d \in K_\infty, d \in \mathbb{N},$$

by the above observation and again by Lemma 3.5, we get:

$$\forall d \geq 0, v_\infty(\alpha_d) = q^{\frac{d-1}{q-1}}.$$

□

This latter Lemma implies the following formula due to F. Pellarin ([15], Theorem 1):

Proposition 3.7. *Let $\lambda_\theta \in \mathbb{C}_\infty^\times$ be a fixed $(q-1)$ th root of $-\theta$. Set:*

$$\tilde{\pi} = \lambda_\theta \theta \prod_{j \geq 1} (1 - \theta^{1-q^j})^{-1} \in \mathbb{C}_\infty^\times.$$

Then:

$$(\theta - t)L_1(t) = \frac{\tilde{\pi}}{\lambda_\theta} \prod_{j \geq 0} (1 - \frac{t}{\theta^{q^j}}).$$

Proof. We observe that:

$$\begin{aligned} \forall n \geq 1, L_1(t) |_{t=\theta^{q^n}} &= 0, \\ L_1(\theta) &= 1. \end{aligned}$$

By Lemma 3.6, the entire function $(t - \theta)L_1(t)$ has simple zeros in K_∞ and if $x \in K_\infty$ is such a zero, $v_\infty(x) \in \{-q^i, i \in \mathbb{N}\}$. Thus, there exists $\alpha \in \mathbb{C}_\infty^\times$ such that:

$$(t - \theta)L_1(t) = \alpha \prod_{j \geq 0} (1 - \frac{t}{\theta^{q^j}}).$$

But, observe that:

$$\frac{\tilde{\pi}}{\lambda_\theta} \prod_{j \geq 1} (1 - \frac{t}{\theta^{q^j}}) |_{t=\theta} = \theta.$$

Therefore:

$$\alpha = \frac{-\tilde{\pi}}{\lambda_\theta}.$$

□

3.3. Eigenvalues and Bernoulli-Carlitz numbers.

In this paragraph, we slightly change our point of view. Let t be an indeterminate over \mathbb{C}_∞ and let $\varphi : \mathbb{C}_\infty[[\frac{1}{t}]] \rightarrow \mathbb{C}_\infty[[\frac{1}{t}]]$ be the continuous (for the $\frac{1}{t}$ -adic topology) morphism of \mathbb{C}_∞ -algebras such that $\varphi(t) = t^q$. We first recall some consequences of the work of F. Demeslay's (see the appendix of [5] or [11]) generalizing the work of L. Taelman ([20]).

Let $N \geq 1$, $N \equiv 1 \pmod{q-1}$, $\ell_q(N) \geq q$. Write $N = \sum_{l=0}^k n_l q^l$, $n_l \in \{0, \dots, q-1\}$, $l = 0, \dots, k$, and $n_k \neq 0$. We set $B = K[t]$. Let $\phi^{(N)} : B \rightarrow B\{\varphi\}$ be the morphism of K -algebras given by:

$$\phi_t^{(N)} = \left(\prod_{l=0}^k (\theta^{q^l} - t)^{n_l} \right) \varphi + t.$$

Since t is transcendental over \mathbb{F}_q , there exists a unique "power series" $\exp_{\phi^{(N)}} \in K(t)\{\{\varphi\}\}$ such that:

$$\begin{aligned} \exp_{\phi^{(N)}} &\equiv 1 \pmod{\varphi}, \\ \exp_{\phi^{(N)}} t &= \phi_t^{(N)} \exp_{\phi^{(N)}}. \end{aligned}$$

One can easily see that:

$$\exp_{\phi^{(N)}} = \sum_{j \geq 0} \frac{\prod_{l=0}^k (\prod_{n=0}^{j-1} (\theta^{q^l} - t^{q^n}))^{n_l}}{\prod_{n=0}^{j-1} (t^{q^n} - t^{q^j})} \varphi^j.$$

In particular $\exp_{\phi^{(N)}}$ induces a continuous K -linear endomorphism of $K((\frac{1}{t}))$ which is an isometry on a sufficiently small neighborhood of zero (for the $\frac{1}{t}$ -adic topology). Let's set:

$$H(\phi^{(N)}) = \frac{K((\frac{1}{t}))}{(B + \exp_{\phi^{(N)}}(K((\frac{1}{t})))}.$$

Then $H(\phi^{(N)})$ is a finite K -vector space and a B -module via ϕ . Let's denote by $[H(\phi^{(N)})]_B$ the monic generator (as a polynomial in t) of the Fitting ideal of the B -module $H(\phi^{(N)})$, i.e.:

$$[H(\phi^{(N)})]_B = \det_{K[Z]} (Z \text{Id} - \phi_t^{(N)} |_{H(\phi^{(N)})}) |_{Z=t}.$$

As in [5], Proposition 7.2, one can prove that:

$$\begin{aligned} \dim_K H(\phi^{(N)}) &= \frac{\ell_q(N) - q}{q - 1}, \\ \{x \in K((\frac{1}{t})), \exp_{\phi^{(N)}}(x) \in B\} &= \frac{\bar{\pi}}{t^{\frac{\ell_q(N)-q}{q-1}} \prod_{l=0}^k \bar{\omega}(\theta^{q^l})^{n_l}} B, \end{aligned}$$

where:

$$\begin{aligned} \bar{\pi} &= \prod_{j \geq 1} (1 - t^{1-q^j})^{-1} \in \mathbb{F}_p[[\frac{1}{t}]]^\times, \\ \bar{\omega}(\theta^{q^l}) &= \prod_{j \geq 0} (1 - \frac{\theta^{q^l}}{t^{q^j}})^{-1} \in A[[\frac{1}{t}]]^\times. \end{aligned}$$

Furthermore, if we set:

$$\mathcal{L}_N(t) = \prod_{P(t) \text{ monic irreducible polynomial of } \mathbb{F}_q[t]} \frac{[\frac{B}{P(t)B}]_B}{[\phi^{(N)}(\frac{B}{P(t)B})]_B},$$

then, by the appendix of [5], $\mathcal{L}_N(t)$ converges in $K((\frac{1}{t}))$, and:

$$[H(\phi^{(N)})]_B \frac{\bar{\pi}}{t^{\frac{\ell_q(N)-q}{q-1}} \prod_{l=0}^k \bar{\omega}(\theta^{q^l})^{n_l}} = \mathcal{L}_N(t).$$

Now, one can compute $\mathcal{L}_N(t)$ as in [5], paragraph 5.3, and we get:

$$\mathcal{L}_N(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a^N}{a(t)} \in A[[\frac{1}{t}]]^\times.$$

Therefore:

$$[H(\phi^{(N)})]_B = B_N(\theta, t).$$

We warn the reader not to confuse $B_N(\theta, t)$ and $B_N(t, \theta)$, here and in the sequel of the paper, since we will be interested in those two polynomials. Recall that $r = \frac{\ell_q(N)-q}{q-1}$. Let $\alpha_1(N), \dots, \alpha_r(N) \in \mathbb{C}_\infty$ be the eigenvalues (counted with multiplicity) of the K -endomorphism of $H(\phi^{(N)})$: $\phi_t^{(N)}$. We get:

$$B_N(\theta, \theta) = \prod_{j=1}^r (\theta - \alpha_j(N)).$$

Recall that $\mathcal{L}_N(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a^N}{a(t)} \in A[[\frac{1}{t}]]$. By Lemma 2.1, $\mathcal{L}_N(t)$ converges on $\{x \in \mathbb{C}_\infty, v_\infty(x) < 0\}$. Let's write $N = \sum_{l=0}^k n_l q^l$, $n_l \in \{0, \dots, q-1\}$, $n_k \neq 0$. Set $S = \{\theta^{q^i}, i \leq k\}$. Then, by Lemma 3.2, the elements of S are zeros of $\mathcal{L}_N(t)$. The elements of S are called the trivial zeros of $\mathcal{L}_N(t)$. A zero of $\mathcal{L}_N(t)$ which does belong to $\{x \in \mathbb{C}_\infty, v_\infty(x) < 0\} \setminus S$ will be called an exceptional zero of $\mathcal{L}_N(t)$. It is clear that the exceptional zeros of $\mathcal{L}_N(t)$ are roots of $B_N(\theta, t)$ with the same multiplicity. Our aim in the remaining of the article is to study the following problem:

Problem 1.

Let $N \geq 2$, $N \equiv 1 \pmod{q-1}$, $\ell_q(N) \geq q$. Then all the eigenvalues of $\phi_t^{(N)}$ (viewed as a K -endomorphism of $H(\phi^{(N)})$) are simple and belong to $\mathbb{F}_p((\frac{1}{\theta}))$.

Theorem 1.1 implies that θ is not an eigenvalue of $\phi_t^{(N)}$. We presently do not know whether another trivial zero of $\mathcal{L}_N(t)$ can be an eigenvalue of $\phi_t^{(N)}$. On the other side, the above problem implies that the exceptional zeros of $\mathcal{L}_N(t)$ are simple. Observe that, by Lemma 2.2, the above Problem has an affirmative answer for $q \leq \ell_q(N) \leq 2q-1$.

4. ANSWER TO PROBLEM 1 FOR $q = p$

In this section we give an affirmative answer to Problem 1 in the case $q = p$. By Proposition 2.5 and Proposition 4.6 below, this implies Theorem 1.1. For the convenience of the reader, we have tried to keep the text of this section as self-contained as possible.

In this section $q = p$.

4.1. Preliminaries.

Lemma 4.1 and Proposition 4.2 below are slight generalizations of the arguments used in the proof of Theorem 1 in [12].

Lemma 4.1. *Let $d, N \geq 1$ and $\mathbf{k} = (k_0, \dots, k_{d-1}) \in \{0, \dots, p-1\}^d$. We assume that $|\mathbf{k}| \leq \ell_p(N)$. For $1 \leq i \leq d$, we set $\sigma_i = \sum_{n=0}^{i-1} k_n$. We also set $\sigma_0 = 0$ and $\sigma_{d+1} = \ell_p(N)$. Let $\mathbf{m} = (m_0, \dots, m_d) \in \mathbb{N}^{d+1}$ be the element defined as follows:*

$$n = 0, \dots, d, \quad m_n = \sum_{l=\sigma_n+1}^{\sigma_{n+1}} p^{el}.$$

Then:

$$\mathbf{m} \in U_d(N, \mathbf{k}).$$

Furthermore \mathbf{m} is the greedy element of $U_d(N, \mathbf{k})$. In particular $U_d(N, \mathbf{k}) \neq \emptyset$ if and only if $|\mathbf{k}| \leq \ell_p(N)$.

Proof. Observe that $\sigma_d = |\mathbf{k}| \leq \ell_p(N)$. Thus \mathbf{m} is well-defined. It is then straightforward to verify that $\mathbf{m} \in U_d(N, \mathbf{k})$ and that \mathbf{m} is the greedy element of $U_d(N, \mathbf{k})$. Now assume that $U_d(N, \mathbf{k}) \neq \emptyset$. Let $\mathbf{m}' \in U_d(N, \mathbf{k})$. Then:

$$n = 0, \dots, d-1, \ell_p(m'_n) \equiv k_n \pmod{p-1}.$$

This implies:

$$n = 0, \dots, d-1, \ell_p(m'_n) \geq k_n.$$

Thus:

$$\ell_p(N) \geq \sum_{n=0}^{d-1} \ell_p(m'_n) \geq |\mathbf{k}|.$$

□

Proposition 4.2. *Let $d, N \geq 1$ and $\mathbf{k} \in \mathbb{N}^d$. We assume that $|\bar{\mathbf{k}}| \leq \ell_p(N)$. Then $U_d(N, \bar{\mathbf{k}})$ contains a unique optimal element which is equal to the greedy element of $U_d(N, \bar{\mathbf{k}})$. In particular $\sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} \neq 0$ if and only if $|\bar{\mathbf{k}}| \leq \ell_p(N)$.*

Proof. Let $\mathbf{u} = (u_0, \dots, u_d)$ be the greedy element of $U_d(N, \bar{\mathbf{k}})$. Let $\mathbf{m} \in U_d(N, \bar{\mathbf{k}})$ such that $\mathbf{m} \neq \mathbf{u}$. We will show that \mathbf{m} is not optimal.

Write $c_n = \ell_p(m_n)$, $n = 0, \dots, d-1$. Then :

$$n = 0, \dots, d-1, c_n \geq \bar{k}_n, c_n \equiv \bar{k}_n \pmod{p-1}.$$

For $n = 0, \dots, d-1$, there exist $f_{n,1} \leq \dots \leq f_{n,c_n}$ such that we can write in a unique way:

$$m_n = \sum_{l=1}^{c_n} p^{f_{n,l}}.$$

Case 1) There exists an integer j , $0 \leq j \leq d-1$, such that $c_j > \bar{k}_j$.

Let $\mathbf{m}' \in \mathbb{N}^{d+1}$ be defined as follows:

- $m'_n = m_n$ for $0 \leq n \leq d-1$, $n \neq j$,

- $m'_j = \sum_{l=1}^{\bar{k}_j} p^{f_{j,l}}$.

- $m'_d = N - m'_0 - \dots - m'_{d-1} = m_d + m_j - m'_j$.

Then $\mathbf{m}' \in U_d(N, \mathbf{k})$ and:

$$\deg \mathbf{m}' = \deg \mathbf{m} + (d-j)(m_j - m'_j) > \deg \mathbf{m}.$$

Thus \mathbf{m} is not optimal.

Case 2) For $n = 0, \dots, d-1$, $c_n = \bar{k}_n$.

Let $j \in \{0, \dots, d-1\}$ be the smallest integer such that $m_j \neq u_j$. Then, by the construction of \mathbf{u} , we have:

$$m_j > u_j.$$

Thus there exists an integer l such that the number of times p^l appears in the sum of m_j as \bar{k}_j powers of p is strictly greater than the number of times it appears in the sum of u_j as \bar{k}_j powers of p . Also, there exists an integer v such that the number of times p^v appears in the sum of u_j as \bar{k}_j powers of p is strictly greater than the number of times it appears in the sum of m_j as \bar{k}_j powers of p . Thus there exists an integer $t > j$ such that p^v appears in the sum of m_t as $\ell_p(m_t)$ powers of p . We observe that, by the construction of \mathbf{u} , we can choose v and l such that $v < l$. Now set:

- for $n = 0, \dots, d$, $n \neq j, n \neq t$, $m'_n = m_n$,
- $m'_j = m_j - p^l + p^v$,
- $m'_t = m_t - p^v + p^l$.

Let $\mathbf{m}' = (m'_0, \dots, m'_{d-1}) \in \mathbb{N}^d$. Then $\mathbf{m}' \in U_d(N, \bar{\mathbf{k}})$ and:

$$\deg \mathbf{m}' = \sum_{l=0}^d l m'_l = \deg \mathbf{m} + (t-j)(p^l - p^v) > \deg \mathbf{m}.$$

Thus \mathbf{m} is not optimal. □

We have the following key result:

Proposition 4.3. *Let $d, N \geq 1$ and $\mathbf{k} \in \mathbb{N}^d$. We assume that $\ell_p(N) \geq p$ and that $d(p-1) \leq \ell_p(N) - p$. Then:*

$$N(d-1) < \deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} \leq Nd.$$

Proof. It is clear that $\deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} \leq Nd$. Observe that $|\bar{\mathbf{k}}| \leq d(p-1)$. Let \mathbf{m} be the greedy element of $U_d(N, \bar{\mathbf{k}})$. By Proposition 4.2, we have:

$$\deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = \sum_{n=0}^d n m_n = dN - \sum_{n=1}^d n m_{d-n}.$$

By Lemma 4.1, we have:

$$n = 0, \dots, d-1, m_n \leq \bar{k}_n p^{e_{\bar{k}_0 + \dots + \bar{k}_n}},$$

where we recall that:

$$N = \sum_{n=1}^{\ell_p(N)} p^{e_n}.$$

Let $l = \ell_p(N) - 1$. Observe that:

$$e_{\ell_p(N) - p - (p-1)t} \leq e_l - 1 - t.$$

Since $d(p-1) \leq \ell_p(N) - p$, we get:

$$n = 1, \dots, d, \bar{k}_0 + \dots + \bar{k}_{d-n} \leq (p-1)(d-n+1) \leq \ell_p(N) - p - (p-1)(n-1).$$

Thus:

$$n = 1, \dots, d, m_{d-n} \leq (p-1)p^{e_l - n},$$

Therefore:

$$\sum_{n=1}^d nm_{d-n} \leq (p-1)p^{e_l} \sum_{n=1}^d np^{-n}.$$

Recall that if $x \in \mathbb{R} \setminus \{1\}$, we have:

$$\sum_{n=1}^d nx^{n-1} = \frac{1-x^{d+1} + (d+1)(x-1)x^d}{(x-1)^2}.$$

Thus:

$$(p-1) \sum_{n=1}^d np^{-n} = \frac{p-p^{-d} - (d+1)(p-1)p^{-d}}{p-1} < \frac{p}{p-1}.$$

Now:

$$\sum_{n=1}^d nm_{d-n} < \frac{p}{p-1} p^{e_l}.$$

Thus:

$$\sum_{n=1}^d nm_{d-n} < 2p^{e_l}.$$

But $2p^{e_l} \leq N$ since $l = \ell_p(N) - 1$. Thus:

$$\deg_t \sum_{a \in A_{+,d}} a(t)^N a^{\mathbf{k}} = dN - \sum_{n=1}^d nm_{d-n} > dN - N.$$

□

4.2. Newton polygons of truncated L -series.

For $i, j \geq 0$, we set:

$$S_j(i) = \sum_{a \in A_{+,j}} a(t)^i \in \mathbb{F}_p[t].$$

Note that by Proposition 4.2, we have $S_i(N) \neq 0$ for $i = 0, \dots, r$.

Proposition 4.4. *We have:*

$$i = 0, \dots, r, \deg_t \alpha_{i,N}(t) = \deg_t S_i(N).$$

Proof. We recall that:

$$r = \text{Max}\left\{\left\lfloor \frac{\ell_p(N) - p}{p-1} \right\rfloor, 0\right\} \in \mathbb{N}.$$

We can assume that $r \geq 1$. By Proposition 4.3, for $i = 0, \dots, r$,

$$\text{Max}\left\{\deg_t \sum_{a \in A_{+, \ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}}, w(\mathbf{k}) + \ell(\mathbf{k}) = i\right\}$$

is attained for a unique \mathbf{k} which is $(0, \dots, 0) \in \mathbb{N}^i$. It remains to apply Lemma 3.1. □

We set:

$$\Lambda_r(N) = \sum_{i=0}^r \alpha_i(N) \theta^{-i} \in K_\infty[t].$$

Let's write:

$$N = \sum_{l=0}^k n_l p^l, n_0, \dots, n_k \in \{0, \dots, p-1\}, n_k \neq 0.$$

Proposition 4.5. *We have $\deg_t \Lambda_r(N) = \deg_t S_r(N)$. Furthermore the edge points of the Newton polygon of $\Lambda_r(N)$ are:*

$$(\deg_t S_j(N), j), j = 0, \dots, r.$$

Proof. We can assume that $r \geq 1$. By Proposition 4.2, we have $U_r(N, (p-1, \dots, p-1)) \neq \emptyset$. Let $\mathbf{m} \in \mathbb{N}^{r+1}$ be the optimal element of $U_r(N, (p-1, \dots, p-1))$ given by Proposition 4.2 and Lemma 4.1. For $j = 0, \dots, r$, let $\mathbf{m}(j) = (m_0, \dots, m_{j-1}, N - \sum_{n=0}^{j-1} m_n) \in \mathbb{N}^{j+1}$. Then, again by Proposition 4.2 and Lemma 4.1, $\mathbf{m}(j)$ is the optimal element of $U_j(N, (p-1, \dots, p-1))$. Therefore, $\deg_t S_j(N) = \deg \mathbf{m}(j)$, $j = 0, \dots, r$. For $j = 0, \dots, r$, we have:

$$p^{k+1} > N - \sum_{n=0}^{j-1} m_n > n_k p^k.$$

Now let $j \in \{0, \dots, r-1\}$, we have:

$$p^{k+1} > \deg \mathbf{m}(j+1) - \deg \mathbf{m}(j) = N - \sum_{n=0}^j m_n > n_k p^k.$$

Thus, by Proposition 4.4, we get $\deg_t \Lambda_r(N) = \deg_t S_r(N)$. Furthermore, we observe that for $j \in \{1, \dots, r-1\}$, we have:

$$\deg \mathbf{m}(j) - \deg \mathbf{m}(j-1) > \deg \mathbf{m}(j+1) - \deg \mathbf{m}(j).$$

Thus, one easily sees that the edge points of the Newton polygon of $\Lambda_r(N)$ are $(\deg \mathbf{m}(j), j)$, $j = 0, \dots, r$. \square

4.3. A positive answer to Problem 1.

Let $N \geq 2$ be an integer, $N \equiv 1 \pmod{p-1}$. Recall that:

$$N = \sum_{l=0}^k n_l p^l, n_0, \dots, n_k \in \{0, \dots, p-1\}, n_k \neq 0.$$

Recall that $\ell_p(N) = n_0 + \dots + n_k$ and $r = \text{Max}\{\frac{\ell_p(N)-p}{p-1}, 0\}$. Let $b_N \in \mathbb{N}$ be the total degree in t, θ of the polynomial $B_N(t)$.

Proposition 4.6. *The polynomial $B_N(t, \theta)$ has only one monomial of total degree b_N , which is of the form t^{b_N} . Furthermore:*

$$b_N = \deg_t S_r(N).$$

Proof. We can assume that $r \geq 1$. First let's observe that:

$$\prod_{j \geq 1} (1 - \theta^{1-p^j})^{-1} B_N(t, \theta) \equiv (-1)^{\frac{\ell_p(N)-1}{p-1}} \delta_N \prod_{l=0}^k \prod_{j \geq 0} (1 - \frac{t^{p^l}}{\theta^{p^j}})^{-n_l} \pmod{\frac{1}{\theta} \mathbb{F}_p[t][[\frac{1}{\theta}]]},$$

where:

$$\delta_N = \sum_{i=0}^r \alpha_{r-i}(N) \theta^i.$$

Let $\varepsilon_N \in \mathbb{F}_p[t, \theta]$ be uniquely determined by the congruence:

$$\varepsilon_N \equiv \delta_N \prod_{l=0}^k \prod_{j \geq 0} (1 - \frac{t^{p^l}}{\theta^{p^j}})^{-n_l} \pmod{\frac{1}{\theta} \mathbb{F}_p[t][[\frac{1}{\theta}]]}.$$

Let $i \in \{0, \dots, r\}$. A monomial in the product

$$\alpha_i(N) \theta^i \prod_{l=0}^k \prod_{j \geq 0} (1 - \frac{t^{p^l}}{\theta^{p^j}})^{-n_l} \pmod{\frac{1}{\theta} \mathbb{F}_p[t][[\frac{1}{\theta}]]}$$

is of the form:

$$\theta^i t^j \frac{t^\alpha}{\theta^\beta}, \quad j \leq \deg_t \alpha_i(t), \beta \leq i, \alpha \leq p^k \beta.$$

Thus the total degree of a monomial in

$$\alpha_i(N) \theta^i \prod_{l=0}^k \prod_{j \geq 0} (1 - \frac{t^{p^l}}{\theta^{p^j}})^{-n_l} \pmod{\frac{1}{\theta} \mathbb{F}_p[t][[\frac{1}{\theta}]]}$$

is less than or equal to $p^k i + \deg_t \alpha_i(t)$. To conclude the proof of the Proposition, we use the same arguments as that used in the proof of Proposition 4.5. By Proposition 4.2, we have $U_r(N, (p-1, \dots, p-1)) \neq \emptyset$. Let $\mathbf{m} \in \mathbb{N}^{r+1}$ be the optimal element of $U_r(N, (p-1, \dots, p-1))$ given by Proposition 4.2 and Lemma 4.1. For $j = 0, \dots, r$, let $\mathbf{m}(j) = (m_0, \dots, m_{r-j-1}, N - \sum_{n=0}^{r-j-1} m_n) \in \mathbb{N}^{r-j+1}$. Then, again by Proposition 4.2 and Lemma 4.1, $\mathbf{m}(j)$ is the optimal element of $U_{r-j}(N, (p-1, \dots, p-1))$. For $j = 0, \dots, r$, we have:

$$N - \sum_{n=0}^{r-j-1} m_n > p^k.$$

Now let $j \in \{0, \dots, r-1\}$, we have:

$$\deg \mathbf{m}(j) - \deg \mathbf{m}(j+1) = N - \sum_{n=0}^{r-j-1} m_n > p^k.$$

Thus, by Proposition 4.3 and Proposition 4.4, $\text{Max}\{p^k i + \deg_t \alpha_i(t), i = 0, \dots, r\}$ is attained exactly at $i = 0$. Again by Proposition 4.4, this implies that the total degree in t, θ of ε_N is equal to $\deg_t S_r(N)$ and that $\varepsilon_N(t)$ has only one monomial of total degree $\deg_t S_r(N)$ which is of the form $t^{\deg_t S_r(N)}$. The Proposition follows. \square

Theorem 4.7. *Let $N \geq 2, N \equiv 1 \pmod{p-1}$. The polynomial $B_N(\theta, t)$ (viewed as a polynomial in t) has r simple roots and all its roots are contained in $\mathbb{F}_p((\frac{1}{\theta})) \setminus \{\theta^{p^i}, i \in \mathbb{Z}\}$.*

Proof.

Recall that $B_N(t, \theta)$ is a monic polynomial in θ such that $\deg_\theta B_N(t, \theta) = r$. We can assume that $r \geq 1$. By Proposition 4.6, the leading coefficient of $B_N(t, \theta)$ as a polynomial in t is in \mathbb{F}_p^\times and:

$$b_N = \deg_t B_N(t, \theta) > r.$$

Let $S = \{\theta^{p^i}, i \geq -k\}$. Then if $\alpha \in \mathbb{C}_\infty$ is a zero of $L_N(t)$ and $\alpha \notin S$, α must be a zero of $B_N(t, \theta)$. Observe that by Proposition 4.5 and Proposition 4.6, we have:

$$\deg_t \Lambda_r(N) = \deg_t B_N(t, \theta).$$

By Proposition 4.5, the zeros in \mathbb{C}_∞ of $\Lambda_r(N)$ are not in S . By Lemma 2.1 and Lemma 3.3, $\theta^{-r}B_N(t, \theta)$ and $\Lambda_r(t)$ have the same Newton polygon. Thus, by the proof of Proposition 4.5 and the properties of Newton polygons ([13], chapter 2), we get in $K_\infty[t]$:

$$B_N(t, \theta) = \lambda \prod_{j=1}^r P_j(t),$$

where $\lambda \in \mathbb{F}_p^\times$, $P_j(t)$ is an irreducible monic element in $K_\infty[t]$, $P_j(t) \neq P_{j'}(t)$ for $j \neq j'$. Furthermore each root of $P_j(t)$ generates a totally ramified extension of K_∞ and $p^{k+1} > \deg_t P_j(t) > n_k p^k$. Also note that $\deg_t P_j(t) \not\equiv 0 \pmod{p^k}$ and $\deg_t P_j(t) \equiv 1 \pmod{p-1}$. Observe that if $x \in \cup_{i \in \mathbb{Z}} (\mathbb{F}_p((\frac{1}{\theta p^i}))^{ab})^{perf}$, then there exist $l \geq 0, m \in \mathbb{Z}, d \geq 1, p \equiv 1 \pmod{d}$, such that $v_\infty(x) = \frac{m}{dp^l}$. Thus, $P_j(t)$ has no roots in $\cup_{i \in \mathbb{Z}} (\mathbb{F}_p((\frac{1}{\theta p^i}))^{ab})^{perf}$.

Write:

$$\theta^{-r}B_N(t, \theta) = \sum_{j=0}^r \beta_j(t) \theta^{-j}, \beta_j(t) \in \mathbb{F}_p[t].$$

Observe that $\beta_0(t) = 1$ and by the above discussion, $\theta^{-r}B_N(t, \theta)$ and $\Lambda_r(t)$ have the same Newton polygon (as polynomials in t). Now, by Proposition 4.5 and Proposition 4.6, we get:

$$\deg_t \beta_r(t) = \deg_t \alpha_r(t).$$

We deduce that:

$$i = 0, \dots, r, \deg_t \beta_i(t) = \deg_t \alpha_i(t).$$

By the proof of Proposition 4.5, for $i \in \{1, \dots, r-1\}$, $\deg_t \beta_{i+1}(t) - \deg_t \beta_i(t) < \deg_t \beta_i(t) - \deg_t \beta_{i-1}(t)$. Thus the edges of the Newton polygon of $\theta^{-r}B_N(t, \theta)$ viewed as polynomial in $\frac{1}{\theta}$ are $(i, -\deg_t(\beta_i(t))), i = 0, \dots, r$. \square

5. SOME HINTS FOR PROBLEM 1 FOR GENERAL q .

In this section q is no longer assumed to be equal to p .

5.1. The work of J. Sheats.

For $N, d \geq 1$, we set $U_d(N) = U_d(N, (q-1, \dots, q-1))$. Thus:

$$S_d(N) := \sum_{a \in A_{+,d}} a(t)^N = (-1)^d \sum_{\mathbf{m} \in U_d(N)} C(N, \mathbf{m}) t^{\deg \mathbf{m}}.$$

J. Sheats proved ([18], Lemma 1.3) that if $U_d(N) \neq \emptyset$, $U_d(N)$ has a unique optimal element and it is the greedy element of $U_d(N)$. In particular $U_d(N) \neq \emptyset \Leftrightarrow S_d(N) \neq 0$. Observe that if $\mathbf{m} = (m_0, \dots, m_d) \in U_d(N)$, then $(m_0, \dots, m_{d-2}, m_{d-1} + m_d) \in U_{d-1}(N)$. In particular $U_d(N) \neq \emptyset \Rightarrow U_{d-1}(N) \neq \emptyset$.

Proposition 5.1.

1) Let $d \geq 1$ such that $U_d(N) \neq \emptyset$. Then, for $j \in \{1, \dots, d-1\}$, we have:

$$\deg_t S_j(N) - \deg_t S_{j-1}(N) > \deg_t S_{j+1}(N) - \deg_t S_j(N).$$

2) Let $d \geq 1$ such that $U_{d+1}(N) \neq \emptyset$. Let \mathbf{m} be the greedy element of $U_{d+1}(N)$. Then:

$$\deg_t S_d(N) > N(d-1),$$

and:

$$\deg_t S_d(N) - \deg_t S_{d-1}(N) > m_{d+1}.$$

Proof.

1) Observe that this assertion is a consequence of the proof of [18], Theorem 1.1 (see pages 127 and 128 of [18]).

2) Let $\mathbf{m} = (m_0, \dots, m_{d+1})$ be the greedy element of $U_{d+1}(N)$. Define $\mathbf{m}' = (m_0, \dots, m_d) \in U_d(N - m_{d+1})$. Then:

$$m_d \equiv 0 \pmod{q-1}, m_d \geq q-1.$$

Furhermore, observe that \mathbf{m}' is the greedy element of $U_d(N - m_{d+1})$. By [18], Lemma 1.3 and Proposition 4.6, we get:

$$\deg_t S_d(N - m_{d+1}) > (N - m_{d+1})(d-1).$$

Let $\mathbf{m}'' = (m_0, \dots, m_{d-1}, m_d + m_{d+1}) \in U_d(N)$. We have:

$$\deg_t S_d(N) \geq m_1 + \dots + (d-1)(m_{d-1}) + d(m_d + m_{d+1}).$$

Thus:

$$\begin{aligned} \deg_t S_d(N) &\geq \deg_t S_d(N - m_{d+1}) + dm_{d+1} \\ &> (N - m_{d+1})(d-1) + dm_{d+1} = (d-1)N + m_{d+1}. \end{aligned}$$

Thus:

$$\begin{aligned} \deg_t S_d(N) &> N(d-1), \\ \deg_t S_d(N) - \deg_t S_{d-1}(N) &\geq \deg_t S_d(N) - (d-1)N > m_{d+1}. \end{aligned}$$

□

To conclude this paragraph, we recall the following crucial result due to G. Böckle ([8], Theorem 1.2):

$$S_d(N) \neq 0 \Leftrightarrow d(q-1) \leq \text{Min}\{\ell_q(p^i N), i \in \mathbb{N}\}.$$

An integer $N \geq 1$ will be called q -minimal if:

$$\left\lfloor \frac{\ell_q(N)}{q-1} \right\rfloor = \text{Min}\left\{\left\lfloor \frac{\ell_q(p^i N)}{q-1} \right\rfloor, i \in \mathbb{N}\right\}.$$

5.2. Consequences of Sheats results.

Let $N \geq 1$, and write:

$$L_N(t) = \sum_{i \geq 0} \alpha_{i,N}(t) \theta^{-i}, \alpha_{i,N}(t) \in \mathbb{F}_q[t].$$

Proposition 5.2. *Let $d \geq 1$ such that $U_{d+1}(N) \neq \emptyset$. Set:*

$$\Lambda_d(t) = \sum_{i=0}^d \alpha_{i,N}(t) \theta^{-i} \in K_\infty[t].$$

Then $\deg_t \Lambda_d(t) = \deg_t S_d(N)$ and the edge points of the Newton polygon of $\Lambda_d(t)$ are:

$$(\deg_t S_i(N), i), i = 0, \dots, d.$$

Proof. The proof uses similar arguments as that used in the proof of Proposition 4.5. Let $j \geq 0$, then (see Lemma 3.1), we have:

$$\alpha_{j,N}(t) = \sum_{\ell(\mathbf{k})+w(\mathbf{k})=j} (-1)^{\ell(\mathbf{k})} C_{\mathbf{k}} \sum_{a \in A_{+, \ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}}.$$

Observe that:

$$\deg_t \sum_{a \in A_{+, \ell(\mathbf{k})}} a(t)^N a^{\mathbf{k}} \leq \ell(\mathbf{k})N.$$

Thus, for $j = 0, \dots, d$, by Proposition 5.1, assertion 2), we get:

$$\deg_t \alpha_{j,N}(t) = \deg_t S_j(N).$$

In particular, again by Proposition 5.1, assertion 2), we have:

$$\deg_t \Lambda_d(t) = \deg_t S_d(N).$$

Finally, by Proposition 5.1, assertion 1), $(\deg_t S_i(N), i), i = 0, \dots, d$, are the edge points of the Newton polygon of $\Lambda_d(t)$. □

Lemma 5.3. *We assume that N is q -minimal, $N \equiv 1 \pmod{q-1}$. We also assume that $r \geq 1$ (recall that $r = \frac{\ell_q(N)-q}{q-1}$).*

1) *Write $N = \sum_{l=0}^k n_l q^l, n_0, \dots, n_k \in \{0, \dots, q-1\}, n_k \neq 0$. Then there exists an integer $0 \leq m \leq k$ such that $n_m \not\equiv 0 \pmod{p}$.*

2) *Let $n = \text{Max}\{l, 0 \leq l \leq k, n_l \not\equiv 0 \pmod{p}\}$. Let $\mathbf{m} = (m_0, \dots, m_{r+1}) \in U_{r+1}(N)$ be the greedy element, then:*

$$m_{r+1} = q^n.$$

3) *Assume that $n < k$. Then, for $i \in \{1, \dots, r-1\}$, we have:*

$$\deg_t S_i(N) - \deg_t S_{i-1}(N) > pq^k.$$

Proof.

1) Let's assume that the assertion is false. Then:

$$\ell_q\left(\frac{q}{p}N\right) = \frac{\ell_q(N)}{p}.$$

This contradicts the q -minimality of N .

2) Observe that:

$$\forall i \geq 0, \ell_q(p^i N) = \ell_q(p^i N - p^i q^n) + \ell_q(p^i).$$

Therefore, $N - q^n$ is q -minimal. Thus, by Böckle's result: $U_{r+1}(N - q^n) \neq \emptyset$. This easily implies that there exists $\mathbf{n} = (n_0, \dots, n_{r+1}) \in U_{r+1}(N)$ such that:

$$n_{r+1} = q^n.$$

We have:

$$1 + (q-1)(r+1) = \ell_q(N) = \sum_{i=0}^{r+1} \ell_q(m_i).$$

Furthermore:

$$\sum_{i=0}^r \ell_q(m_i) \equiv 0 \pmod{q-1},$$

and

$$\sum_{i=0}^r \ell_q(m_i) \geq (q-1)(r+1).$$

Thus:

$$\ell_q(m_{r+1}) = 1.$$

This implies that m_{r+1} is a power of q and since \mathbf{m} is the greedy element of $U_{r+1}(N)$, we also have: $m_{r+1} \geq q^n$. Since there is no carryover p -digits in the sum $m_0 + \dots + m_{r+1}$, by the definition of n , we deduce that $m_{r+1} = q^n$.

3) Let $\mathbf{m}' = (m_0, \dots, m_{r-1}, m_r + m_{r+1}) \in U_r(N)$. If \mathbf{n} is the greedy element of $U_r(N)$ then:

$$n_r \geq m_r + m_{r+1}.$$

Since \mathbf{m} is the greedy element of $U_{r+1}(N)$, we have:

$$m_0 \leq m_1 \leq \dots \leq m_r.$$

Since there is no carryover p -digits in the sum $m_0 + \dots + m_{r+1}$, and $n < k$, this implies that:

$$m_r = \sum_{l=0}^k j_l q^l, j_l \in \{0, \dots, q-1\}, j_k \neq 0, j_k \equiv 0 \pmod{p}.$$

Thus:

$$m_r > p q^k.$$

It remains to apply Proposition 5.1. □

5.3. Zeros of $B_N(\theta, t)$.

The following theorem implies in particular Theorem 1.1 in the case where N is q -minimal.

Theorem 5.4. *We assume that N is q -minimal, $N \equiv 1 \pmod{q-1}$. We also assume that $r \geq 1$.*

1) $B_N(\theta, \theta) \neq 0$ and the zeros of $B_N(t, \theta)$ are algebraic integers (i.e. they are integral over A). Furthermore: $(r-1)N < \deg_t B_N(t, \theta) < rN$.

2) $B_N(\theta, t)$ has only simple roots and its roots belong to $\mathbb{F}_p((\frac{1}{\theta})) \setminus \{\theta\}$.

Proof. The proof is in the same spirit as that of the proofs of Proposition 4.6 and Theorem 4.7.

Write $N = \sum_{l=0}^k n_l q^l$, where $n_0, \dots, n_k \in \{0, \dots, q-1\}$, and $n_k \neq 0$. Recall that $r+1 = \lceil \frac{\ell_q(N)}{q-1} \rceil$. Let $n \geq 0$ be the integer such that $n = \text{Max}\{l, n_l \not\equiv 0 \pmod{p}\}$ (see Lemma 5.3). Let $\varepsilon_N(t) \in \mathbb{F}_p[t, \frac{1}{\theta}]$ be the polynomial determined by the congruence:

$$\varepsilon_N(t) \equiv \Lambda_r(t) \prod_{j=0}^k \prod_{i \geq 0} (1 - \frac{t q^j}{\theta q^i})^{n_j} \pmod{\frac{1}{\theta^{r+1}} \mathbb{F}_p[t][[\frac{1}{\theta}]]},$$

where $\Lambda_r(t) = \sum_{l=0}^r \alpha_{l,N}(t) \theta^{-l}$. We can write:

$$\varepsilon_N(t) = \sum_{l=0}^r \eta_l(t) \theta^{-l}, \eta_l(t) \in \mathbb{F}_p[t].$$

Note that $\eta_l(t)$ is a \mathbb{F}_p -linear combination of terms of the form

$$x_{l,j,u} = \alpha_{l-u,N}(t) \theta^{-l+u} t^j \theta^{-u}, \quad j \leq q^k u.$$

By Proposition 5.2, we have:

$$l = 0, \dots, r, \deg_t \alpha_{l,N}(t) = \deg_t S_l(N).$$

1) Case $n = k$.

By Proposition 5.1 and Lemma 5.3 :

$$l = 0, \dots, h-1, \deg_t \alpha_{h-l,N}(t) < \deg_t \alpha_{h,N}(t) - q^k l.$$

Thus, if $l \neq h$ or $u \neq 0$, we get:

$$l = 0, \dots, h, \deg_t x_{l,j,u} < \deg_t \alpha_{l,N}(t).$$

Therefore:

$$l = 0, \dots, r, \deg_t \eta_l(t) = \deg_t S_l(N).$$

2) Case $n < k$.

As in the proof of the case $n = k$, we get by Proposition 5.1 and Lemma 5.3:

$$l = 0, \dots, r-1, \deg_t \eta_l(t) = \deg_t S_l(N).$$

Furthermore, by Proposition 5.1 and Lemma 5.3, we have for $l \geq 2$:

$$\begin{aligned} \deg_t S_{r-l}(N) &< \deg_t S_{r-1}(N) - pq^k(l-1), \\ \deg_t S_r(N) - \deg_t S_{r-1}(N) &> q^n. \end{aligned}$$

Thus, for $u \geq 2$:

$$\deg_t x_{r,j,u} \leq \deg_t S_{r-u}(N) + q^k u < \deg_t S_r(N).$$

Now, observe that:

$$\prod_{j=0}^k \prod_{i \geq 0} \left(1 - \frac{t^{q^j}}{\theta^{q^i}}\right)^{n_j} \equiv 1 - \frac{1}{\theta} \sum_{l=0}^k n_l t^{q^l} \pmod{\frac{1}{\theta^2} \mathbb{F}_p[t][[\frac{1}{\theta}]]}.$$

Thus:

$$\deg_t x_{r,j,1} \leq \deg_t S_{r-1}(N) + q^n < \deg_t S_r(N).$$

Thus we get:

$$\deg_t \eta_r(t) = \deg_t S_r(N).$$

Now, observe that:

$$\frac{(-1)^{\frac{\ell q(N)-1}{q-1}} \theta^{-r} B_N(t, \theta)}{\prod_{j \geq 1} (1 - \theta^{1-q^j})} \equiv \varepsilon_N \pmod{\frac{1}{\theta^{r+1}} \mathbb{F}_p[t][[\frac{1}{\theta}]]}.$$

We easily deduce that:

$$\theta^{-r} B_N(t, \theta) = \sum_{l=0}^r \beta_l(t) \theta^{-l}, \beta_l(t) \in \mathbb{F}_p[t], \deg_t \beta_l(t) = \deg_t S_l(N), l = 0, \dots, r.$$

Observe that, by Proposition 5.1, we have $\deg_t S_r(N) > N(r-1)$, and it is obvious that $\deg_t S_r(N) < rN$. Now, we get assertion 1) and 2) by the same reasoning as that used in the proof of Theorem 4.7. \square

Corollary 5.5. *We assume that N is q -minimal, $N \equiv 1 \pmod{q-1}$. We also assume that $r \geq 1$. Then, $B_N(t, \theta)$ has at most one zero in $\{\theta^{q^i}, i \in \mathbb{Z}\}$.*

Proof. Let's assume that $B_N(t, \theta)$ has a zero $\alpha \in \{\theta^{q^i}, i \in \mathbb{Z}\}$. Let n be the integer introduced in the proof of Theorem 5.4. Then:

$$\alpha = \theta^{q^{-i}}, k \geq i > n,$$

where $q^k \leq N < q^{k+1}$. Thus $n < k$, and therefore, by Lemma 5.3, we must have:

$$\deg_t S_r(N) - \deg_t S_{r-1}(N) = q^i.$$

By the proof of Theorem 5.4, we have:

$$\theta^{-r} B_N(t, \theta) = \left(\frac{t^{q^i}}{\theta} - 1 \right) F(t),$$

where $F(t) = \sum_{l=0}^{r-1} \nu_l(t)\theta^{-l}$, $\nu_l(t) \in \mathbb{F}_p[t]$, $\deg_t \nu_l(t) = \deg_t S_l(N)$, $l = 0, \dots, r-1$. Furthermore $\nu_0(t) = -1$. This implies that the zeros of $F(t)$ are not in $\{\theta^{q^i}, i \in \mathbb{Z}\}$. \square

Corollary 5.6.

1) The polynomial \mathbb{B}_s is square-free, i.e. \mathbb{B}_s is not divisible by the square of a non-trivial polynomial in $\mathbb{F}_q[t_1, \dots, t_s, \theta]$.

2) For all $l, n \in \mathbb{N}$, \mathbb{B}_s is relatively prime to $(t_1^{q^l} - \theta^{q^n})$ (observe that \mathbb{B}_s is a symmetric polynomial in t_1, \dots, t_s).

3) For all monic irreducible prime P of A , \mathbb{B}_s is relatively prime to $P(t_1) \cdots P(t_s) - P$.

Proof. Let $N = q^{e_1} + \dots + q^{e_s}$, $0 \leq e_1 < e_2 < \dots < e_s$. Then:

$$B_N(t, \theta) = \mathbb{B}_s \Big|_{t_i = t^{q^{e_i}}}.$$

We observe that N is q -minimal. Thus we can apply Theorem 5.4. This Theorem and its proof imply that $B_N(t, \theta)$ is square-free and has no roots in $\{\theta^{q^i}, i \in \mathbb{Z}\}$. This proves 1) and 2).

Let P be a monic irreducible polynomial in A . Suppose that $P(t_1) \cdots P(t_s) - P$ and \mathbb{B}_s are not relatively prime. Then $P(t)^N - P$ and $B_N(t, \theta)$ are not relatively prime. But, by the proof of Theorem 5.4, if $\alpha \in \mathbb{C}_\infty$ is a root of $B_N(t, \theta)$, then:

$$v_\infty(\alpha) > \frac{-1}{N}.$$

Now, observe that if $\beta \in \mathbb{C}_\infty$ is a root of $P(t)^N - P$, then $v_\infty(\beta) = \frac{-1}{N}$. This leads to a contradiction. \square

Note that assertion 1) of the above Corollary gives the cyclicity result implied by [7], Theorem 3.17, but by a completely different method.

6. AN EXAMPLE

We study here an example of an N which is not q -minimal, so that our method does not apply. We choose $q = 4$, and $N = 682 = 2 + 2 \times 4 + 2 \times 4^2 + 2 \times 4^3 + 2 \times 4^4$. We get $l_q(N) = 10 = 3q - 2$ so that $\deg_\theta(B_N(t, \theta)) = 2$. Moreover, $l_q(pN) = 5$ so that N is not q -minimal. By using the table of section B, we get :

$$B_N(t, \theta) = \theta^2 + \theta (t^{10} + t^{34} + t^{40} + t^{130} + t^{136} + t^{160} + t^{514} + t^{520} + t^{544} + t^{640}) \\ + (t^{170} + t^{554} + t^{650} + t^{674} + t^{680}).$$

The Newton polygon of $B_N(t, \theta)$ has then the end points $(0, -2)$, $(640, -1)$, $(680, 0)$. We deduce that $B_N(t, \theta)$ has 640 distinct zeroes of valuation $-\frac{1}{640}$ and 40 distinct zeros of valuation $-\frac{1}{40}$. Similarly, $B_N(\theta, t)$ has two zeros of respective valuations -40 and -640 . In particular, we still have that $B_N(t, \theta)$ has no zero of the form θ^{q^i} , $i \in \mathbb{Z}$, and an affirmative answer to Problem 1.

A. APPENDIX: THE DIGIT PRINCIPLE AND DERIVATIVES OF CERTAIN L -SERIES, BY B. ANGLÈS, D. GOSS, F. PELLARIN AND F. TAVARES RIBEIRO

We keep the notation of the article.

Let N be a positive integer. We consider its base- q expansion $N = \sum_{i=0}^r n_i q^i$, with $n_i \in \{0, \dots, q-1\}$. We recall that $\ell_q(N) = \sum_{i=0}^r n_i$ and the definition of the Carlitz factorial :

$$\Pi(N) = \prod_{i \geq 0} D_i^{n_i} \in A^+,$$

where $[k] = \theta^{q^k} - \theta$ if $k > 0$ and $D_j = [j][j-1]^q \cdots [1]^{q^{j-1}}$ for $j > 0$.

It is easy to see (the details are in §A.1, A.2 and A.3) that, if we denote by a' the derivative $\frac{d}{d\theta}a$ of $a \in A$ with respect to θ , the series

$$\sum_{k \geq 1} \sum_{a \in A_{+,k}} \frac{a'^N}{a}$$

converges in K_∞ to a limit that we denote by δ_N . In particular, if $n = q^j$ with $j > 0$, we will see (Proposition A.4) that

$$\delta_1 = -\sum_{k \geq 1} \frac{1}{[k]} \quad \text{and} \quad \delta_{q^j} = \frac{D_j}{[j]} \tilde{\pi}^{1-q^j}.$$

Our aim is to prove the following:

Theorem A.1. *If $N \geq q$ is such that $N \equiv 1 \pmod{q-1}$ and $\ell_q(N) \geq q$, then*

$$\frac{\delta_N}{\tilde{\pi}} = \beta_N \frac{\Pi(N)}{\Pi(\lfloor \frac{N}{q} \rfloor)^q} \prod_{k=1}^r \left(\frac{\delta_{q^k}}{\tilde{\pi}} \right)^{n_k},$$

where for $x \in \mathbb{R}$, $[x]$ denotes the integer part of x , and where $\beta_N = (-1)^{\frac{s-1}{q-1}} B_N(\theta, \theta)$.

Our Theorem A.1 can be viewed as a kind of *digit principle* for the values δ_j in the sense of [10].

The plan of this appendix is the following. In §A.1, we recall the first properties of Anderson and Thakur function ω . In §A.2 we discuss the one-digit case of our Theorem, while the general case is discussed in §A.3.

A.1. The Anderson-Thakur function. Recall that \mathbb{T}_t denotes the Tate algebra over \mathbb{C}_∞ in the variable t , $C : A \rightarrow A\{\tau\}$ is the Carlitz module ([13], chapter 3), in other words, C is the morphism of \mathbb{F}_q -algebras given by $C_\theta = \tau + \theta$, and

$$\exp_C = \sum_{i \geq 0} \frac{1}{D_i} \tau^i \in \mathbb{T}_t\{\{\tau\}\}$$

is the Carlitz exponential. In particular, we have the following equality in $\mathbb{T}_t\{\{\tau\}\}$:

$$\exp_C \theta = C_\theta \exp_C.$$

Let us choose a $(q-1)$ -th root ${}^{q-1}\sqrt{-\theta}$ of $-\theta$ in \mathbb{C}_∞ and set:

$$\tilde{\pi} = \theta {}^{q-1}\sqrt{-\theta} \prod_{j \geq 1} (1 - \theta^{1-q^j})^{-1} \in \mathbb{C}_\infty^\times.$$

We recall the Anderson-Thakur function ([1], proof of Lemma 2.5.4):

$$\omega(t) = {}^{q-1}\sqrt{-\theta} \prod_{j \geq 0} \left(1 - \frac{t}{\theta^{q^j}} \right)^{-1} \in \mathbb{T}_t^\times.$$

To give an idea of how to compute $\exp_C(f)$ for certain f in \mathbb{T}_t , we verify here that

$$\exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right) = \sum_{j \geq 0} \frac{\tilde{\pi}^{q^j}}{D_j(\theta^{q^j} - t)}$$

is a well defined element of \mathbb{T}_t . Indeed, for $j \geq 0$:

$$v_\infty\left(\frac{\tilde{\pi}^{q^j}}{D_j(\theta^{q^j} - t)}\right) = q^j \left(j + 1 - \frac{q}{q-1}\right).$$

Therefore $\sum_{j \geq 0} \frac{\tilde{\pi}^{q^j}}{D_j(\theta^{q^j} - t)}$ converges in \mathbb{T}_t .

We will need the following crucial result in the sequel:

Proposition A.2. *We have the following equality in \mathbb{T}_t :*

$$\omega(t) = \exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right).$$

Proof. It is a consequence of the formulas established in [15]. We give details for the convenience of the reader. Let us set

$$F(t) = \exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right).$$

We observe that:

$$\begin{aligned} C_\theta(F(t)) &= \exp_C\left(\frac{\theta\tilde{\pi}}{\theta-t}\right) = \exp_C\left(\frac{(\theta-t+t)\tilde{\pi}}{\theta-t}\right) \\ &= \exp_C(\tilde{\pi}) + \exp_C\left(\frac{t\tilde{\pi}}{\theta-t}\right) = t \exp_C\left(\frac{\tilde{\pi}}{\theta-t}\right) = tF(t). \end{aligned}$$

Therefore:

$$\tau(F(t)) = (t - \theta)F(t).$$

But we also have:

$$\tau(\omega(t)) = (t - \theta)\omega(t).$$

Finally, we get:

$$\tau\left(\frac{F(t)}{\omega(t)}\right) = \frac{F(t)}{\omega(t)}.$$

It is a simple and well-known consequence of a ultrametric variant of Weierstrass preparation Theorem that $\{f \in \mathbb{T}_t, \tau(f) = f\} = \mathbb{F}_q[t]$. Since $\omega \in \mathbb{T}_t^\times$, we have then:

$$\frac{F(t)}{\omega(t)} \in \mathbb{F}_q[t].$$

Now observe that

$$F(t) = \exp_C\left(\sum_{j \geq 0} \frac{\tilde{\pi}}{\theta^{j+1}} t^j\right) = \sum_{j \geq 0} \lambda_{\theta^{j+1}} t^j$$

and that, for all $j \geq 0$, $v_\infty(\lambda_{\theta^{j+1}}) = j + 1 - \frac{q}{q-1}$. This implies $v_\infty\left(\frac{F(t)}{\lambda_\theta} - 1\right) > 0$. By the definition of $\omega(t)$, we also have $v_\infty\left(\frac{\omega(t)}{\lambda_\theta} - 1\right) > 0$. Thus:

$$v_\infty\left(\frac{F(t)}{\omega(t)} - 1\right) > 0.$$

Since $\frac{F(t)}{\omega(t)} \in \mathbb{F}_q[t]$, we get $\omega(t) = F(t)$. □

Notice that $\omega(t)$ defines a meromorphic function on \mathbb{C}_∞ without zeroes. Its only poles, simple, are located at $t = \theta, \theta^q, \theta^{q^2}, \dots$. As a consequence of Proposition A.2, we get:

Corollary A.3. *Let $j \geq 0$ be an integer, then:*

$$(t - \theta^{q^j})\omega(t) \Big|_{t=\theta^{q^j}} = -\frac{\tilde{\pi}^{q^j}}{D_j}.$$

A.2. The one digit case. Let us consider the following L -series:

$$L(t) = L_1(t) = \sum_{k \geq 0} \sum_{a \in A_{+,k}} \frac{a(t)}{a} \in \mathbb{T}_t.$$

By Proposition 3.7, we have the following equality in \mathbb{T}_t (see [15], Theorem 1):

$$\frac{L(t)\omega(t)}{\tilde{\pi}} = \frac{1}{\theta - t}.$$

This implies that $L(t)$ extends to an entire function on \mathbb{C}_∞ (see also Lemma 2.1 or [3, Proposition 6]). We set:

$$L'(t) = \sum_{k \geq 0} \sum_{a \in A_{+,k}} \frac{a'(t)}{a} \in \mathbb{T}_t,$$

where $a'(t)$ denotes the derivative $\frac{d}{dt}a(t)$ of $a(t)$ with respect to t . The derivative $\frac{d}{dt}$ inducing a continuous endomorphism of the algebra of entire functions $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$, $L'(t)$ extends to an entire function on \mathbb{C}_∞ . Thus, for $j \geq 0$ an integer, $\sum_{k \geq 1} \sum_{a \in A_{+,k}} \frac{a'^{q^j}}{a}$ converges in K_∞ and we have:

$$\delta_{q^j} = \sum_{k \geq 1} \sum_{a \in A_{+,k}} \frac{a'^{q^j}}{a} = L'(t) \Big|_{t=\theta^{q^j}}.$$

Proposition A.4. *The following properties hold:*

(1) *We have:*

$$\delta_1 = -\sum_{k \geq 1} \frac{1}{[k]}.$$

(2) *Let $j \geq 1$ be an integer, then:*

$$\delta_{q^j} = \frac{\Pi(q^j)}{[j]} \tilde{\pi}^{1-q^j}.$$

Proof. (1) It is well known that, for $n > 0$, $D_n = \prod_{a \in A_{+,n}} a$. Therefore,

$$\sum_{a \in A_{+,n}} \frac{a'}{a} = -\frac{1}{[n]}$$
 from which the first formula follows.

(2) By [13, Remark 8.13.10], we have:

$$L(t) \Big|_{t=\theta^{q^j}} = 0.$$

Thus:

$$\delta_{q^j} = L'(t) \Big|_{t=\theta^{q^j}} = \frac{L(t)}{t - \theta^{q^j}} \Big|_{t=\theta^{q^j}}.$$

But,

$$\frac{\frac{L(t)}{t - \theta^{q^j}}(t - \theta^{q^j})\omega(t)}{\tilde{\pi}} = \frac{1}{\theta - t}.$$

It remains to apply Corollary A.3. □

Remark A.5. The transcendence over K of the “bracket series” $\delta_1 = \sum_{i \geq 1} \frac{1}{[i]}$ was first obtained by Wade [23]. The transcendence of δ_1 directly implies the transcendence of $\tilde{\pi}$.

A.3. The several digits case. As a consequence of [5], Lemma 7.6 (see also [3], Corollary 21), the series $L_N(t) = \sum_{d \geq 0} \sum_{a \in A_{+,d}} \frac{a(t)^N}{a}$ has a zero of order at least N at $t = \theta$. Thus,

$$\tilde{L}_N(t) = \sum_{d \geq 1} \sum_{a \in A_{+,d}} \frac{a'(t)^N}{a}$$

defines an entire function on \mathbb{C}_∞ such that

$$\delta_N = \tilde{L}_N(\theta).$$

Proof of Theorem A.1. Recall that $N = \sum_{i=0}^r n_i q^i$, is the q -expansion of N . We set $s = \ell_q(N)$. We can assume that $s \geq q$ by Proposition A.4. From the definition of $B_N(t, \theta)$ in §2.3, we have :

$$(-1)^{\frac{s-1}{q-1}} B_N(t, \theta) = L_N(t) \left(\prod_{i=0}^r \omega(t^{q^i})^{n_i} \right) \tilde{\pi}^{-1} \in A[t].$$

Since

$$\delta_N = \tilde{L}_N(\theta) = \left(\frac{L_N(t)}{\prod_{i=0}^r (t - \theta^{q^i})^{n_i}} \right)_{|t=\theta},$$

we obtain, by Corollary A.3 and our previous discussions:

$$\beta_N = \frac{\delta_n \prod_{i=0}^r \left(\frac{-\tilde{\pi}^{q^i}}{D_i} \right)^{n_i}}{\tilde{\pi}}.$$

Now, by Proposition A.4, we have, for all $k \geq 1$, $D_k = [k] \delta_{q^k} \tilde{\pi}^{q^k - 1}$. We obtain the Theorem by using the fact that:

$$\frac{\Pi(N)}{\Pi\left(\left[\frac{N}{q}\right]^q\right)} = \prod_{k \geq 1} [k]^{n_k}.$$

□

B. TABLE

We give an explicit expression of the polynomials \mathbb{B}_s for $s \in \{q, 2q - 1, 3q - 2\}$. We recall that \mathbb{B}_s is monic of degree $r = \frac{s-q}{q-1}$. One obtains the corresponding

expressions for $B_N(t, \theta)$ if $\ell_q(N) = s$ by evaluating the variables t_i 's as in §2.3.

$$\begin{aligned} \mathbb{B}_q &= 1, \\ \mathbb{B}_{2q-1} &= \theta - \sum_{i_1 < \dots < i_q} t_{i_1} \cdots t_{i_q}, \\ \mathbb{B}_{3q-2} &= \theta^2 - \theta \left(\sum_{i_1 < \dots < i_{2q-1}} \prod_{j=1}^{2q-1} t_{i_j} + \sum_{i_1 < \dots < i_q} \prod_{j=1}^q t_{i_j} \right) + \\ &\quad + \left(\sum_{i_1 < \dots < i_q} \prod_{j=1}^q t_{i_j}^2 + \sum_{m_1 < \dots < m_{q-1}, m_j \neq i_j, j=1} \prod_{j=1}^{q-1} t_{i_j} + \sum_{i_1 < \dots < i_{2q}} \prod_{j=1}^{2q} t_{i_j} \right). \end{aligned}$$

One easily computes the discriminant of \mathbb{B}_{3q-2} from this table. It is then an easy computation to prove that $B_N(\theta, t)$ has only simple roots for all N such that $\ell_q(N) = 3q - 2$.

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