



**HAL**  
open science

## Twisted characteristic $p$ zeta functions

Bruno Anglès, Tuan Ngo Dac, Floric Tavares Ribeiro

► **To cite this version:**

Bruno Anglès, Tuan Ngo Dac, Floric Tavares Ribeiro. Twisted characteristic  $p$  zeta functions. *Journal of Number Theory*, 2016, 168, pp.180-214. 10.1016/j.jnt.2016.05.002 . hal-01404884

**HAL Id: hal-01404884**

**<https://hal.science/hal-01404884>**

Submitted on 29 Nov 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# TWISTED CHARACTERISTIC $p$ ZETA FUNCTIONS

BRUNO ANGLÈS, TUAN NGO DAC, AND FLORIC TAVARES RIBEIRO

ABSTRACT. We propose a “twisted” variation of zeta functions introduced by David Goss in 1979.

## CONTENTS

1. Introduction	1
2. Notation and preliminaries	3
3. A Key Lemma	7
4. Several variable twisted zeta functions	12
4.1. The $\infty$ -adic case	12
4.2. The case of finite places	16
5. Examples	17
5.1. The case $A = \mathbb{F}_q[\theta]$	17
5.2. Twisted Goss zeta functions	20
5.3. $A$ -harmonic series attached to some admissible maps	22
6. Multiple several variable twisted zeta functions	24
References	27

## 1. INTRODUCTION

Let  $\mathbb{F}_q$  be a finite field of characteristic  $p$ , having  $q$  elements, and let  $\theta$  be an indeterminate over  $\mathbb{F}_q$ . We set:  $A := \mathbb{F}_q[\theta]$ ,  $A_{+,d} := \{a \in A, a \text{ monic, } \deg_\theta a = d\}$ , and  $\mathbb{C}_\infty$  is the completion of an algebraic closure of  $\mathbb{F}_q((\frac{1}{\theta}))$  which is equipped with the canonical topology. We consider the following zeta function ([21], chapter 8): for  $s = (x; y) \in \mathbb{S}_\infty := \mathbb{C}_\infty^\times \times \mathbb{Z}_p$ , put:

$$\zeta_A(s) := \sum_{d \geq 0} \left( \sum_{a \in A_{+,d}} \frac{1}{(\frac{a}{\theta^d})^y} \right) x^{-d} \in \mathbb{C}_\infty^\times.$$

The facts that  $\zeta_A(s)$  converges on  $\mathbb{S}_\infty$  and is “essentially algebraic” (i.e. for  $y \in \mathbb{N}$ ,  $\zeta_A((\theta^y x; -y))$  is a polynomial in  $x^{-1}$  with coefficients in  $A$ ) can be proved by using the following vanishing result: for  $n \in \mathbb{N}$ , for  $d(q-1) > \ell_q(n)$ ,  $\sum_{a \in A_{+,d}} a^n = 0$ , where  $\ell_q(n)$  is the sum of digits of  $n$  in base  $q$  (this is a consequence of [21], Lemma 8.8.1). The zeta function  $\zeta_A(s)$  is an example of a new kind of  $L$ -series introduced by D. Goss in [19] (see also [20]). The “special values” of these type of  $L$ -functions are at the heart of function field arithmetic and we refer the interested reader to (this list is clearly not exhaustive): [3], [13], [24], [36].

---

*Date:* May 3, 2016.

Let  $m, n \in \mathbb{N}$ ,  $n \geq 1$ , let  $s = (x, y_1, \dots, y_n) \in \mathbb{S}_{\infty, n} := \mathbb{C}_{\infty}^{\times} \times \mathbb{Z}_p^n$  (or  $s = (x, \underline{y})$  for short), let's consider the following "twisted" zeta function:

$$\zeta_A(\underline{t}; \underline{\theta}; s) := \sum_{d \geq 0} \left( \sum_{a \in A_{+, d}} \frac{a(t_1) \cdots a(t_m)}{\left(\frac{a(\theta_1)}{\theta_1^d}\right)^{y_1} \cdots \left(\frac{a(\theta_n)}{\theta_n^d}\right)^{y_n}} \right) x^{-d},$$

where  $\underline{t} = (t_1, \dots, t_m) \in \mathbb{C}_{\infty}^m$ ,  $\underline{\theta} = (\theta_1, \dots, \theta_n) \in (\mathbb{C}_{\infty}^{\times})^n$  and are such that  $\frac{1}{\theta_i}$ ,  $i = 1, \dots, n$ , is in the maximal ideal of the valuation ring of  $\mathbb{C}_{\infty}$ . Using the fact that for  $\underline{t} \in \mathbb{C}_{\infty}^m$ , and for  $d(q-1) > m$ ,  $\sum_{a \in A_{+, d}} a(t_1) \cdots a(t_m) = 0$  (this is again a consequence of [21], Lemma 8.8.1), one can prove that the twisted zeta function  $\zeta_A(\underline{t}; \underline{\theta}; s)$  converges on  $\mathbb{S}_{\infty, n}$  and that this function is essentially algebraic, i.e. for  $\underline{y} = (y_1, \dots, y_n) \in \mathbb{N}^n$ :

$$\zeta_A(\underline{t}; \underline{\theta}; \left(\prod_{j=1}^n \theta_j^{y_j} x; -\underline{y}\right)) \in \mathbb{F}_q[\underline{t}, \underline{\theta}, x^{-1}].$$

Observe that if  $m = 0$ ,  $n = 1$  and  $\theta_1 = \theta$ , we recover the zeta function  $\zeta_A(s)$ , and if  $m \geq 1$ ,  $n = 1$ ,  $\theta_1 = \theta$ , we recover the  $L$ -series introduced by F. Pellarin in [28].

Our aim in this article is to extend the above construction to the case where  $K/\mathbb{F}_q$  is a global function field ( $\mathbb{F}_q$  is algebraically closed in  $K$ ),  $\infty$  is a place of  $K$ , and  $A$  is the ring of elements of  $K$  which are regular outside  $\infty$ . Although the twisted zeta functions proposed in this paper are in the spirit of the twisted zeta functions  $\zeta_A(\underline{t}; \underline{\theta}; s)$  defined above, the proof of the convergence of these functions, and their  $v$ -adic interpolation at finite places  $v$  of  $K$ , are more subtle and based on a technical key Lemma which generalizes the vanishing result mentioned above (Lemma 3.2). The zeta functions introduced by D. Goss as well as their deformations over affinoid algebras are examples of such twisted zeta functions (see Example 5.2).

The main ingredient to our construction is what we call *admissible maps* (Definition 2.3). Let  $K_{\infty}$  be the  $\infty$ -adic completion of  $K$ , let  $\mathbb{F}_{\infty}$  be the residue field of  $K_{\infty}$ , and let  $\text{sgn} : K_{\infty}^{\times} \rightarrow \mathbb{F}_{\infty}^{\times}$  be a group homomorphism such that  $\text{sgn}|_{\mathbb{F}_{\infty}^{\times}} = \text{Id}_{\mathbb{F}_{\infty}^{\times}}$ . Let  $\pi$  be a uniformizer of  $K_{\infty}$  such that  $\pi \in \text{Ker } \text{sgn}$ . Let  $\overline{K}_{\infty}$  be a fixed algebraic closure of  $K_{\infty}$  equipped with the canonical topology, let  $\overline{\mathbb{F}}_q$  be the algebraic closure of  $\mathbb{F}_q$  in  $\overline{K}_{\infty}$ , then  $\text{sgn}$  can be naturally extended to a sign function  $\text{sgn} : \overline{K}_{\infty}^{\times} \rightarrow \overline{\mathbb{F}}_q^{\times}$  (Definition 2.1) according to our choice of  $\pi$ . Let  $v_{\infty} : \overline{K}_{\infty} \rightarrow \mathbb{Q} \cup \{+\infty\}$  be the valuation on  $\overline{K}_{\infty}$  such that  $v_{\infty}(\pi) = 1$ . For  $x \in \overline{K}_{\infty}^{\times}$ , let's set:

$$\langle x \rangle = \frac{x}{\text{sgn}(x)\pi^{v_{\infty}(x)}}.$$

Let  $\mathcal{I}(A)$  be the group of non-zero fractional ideals of  $A$ . By definition, an admissible map  $\eta : \mathcal{I}(A) \rightarrow \overline{K}_{\infty}^{\times}$  is a map such that there exist an open subgroup of finite index  $N(\eta) \subset K_{\infty}^{\times}$ , an element  $n(\eta) \in \mathbb{Z}_p$ , and an element  $\gamma_{\eta} \in \overline{K}_{\infty}^{\times}$ , such that:

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^{\times} \cap N(\eta), \quad \eta(I\alpha) = \eta(I)\langle \alpha \rangle^{n(\eta)} \gamma_{\eta}^{v_{\infty}(\alpha)}.$$

The twisted zeta functions considered in this work are constructed with the help of such admissible maps (Section 4). Let's give a basic example. Let  $\eta : \mathcal{I}(A) \rightarrow \overline{K}_{\infty}^{\times}$  be an admissible map such that  $n(\eta) = 1$  and  $\eta$  is algebraic (i.e.  $\eta(\mathcal{I}(A)) \subset \overline{K}^{\times}$ , where  $\overline{K}$  is the algebraic closure of  $K$  in  $\overline{K}_{\infty}$ ). Let's set:

$$K(\eta) = K(\mathbb{F}_{\infty}, \gamma_{\eta}\pi^{-1}, \eta_i(I), I \in \mathcal{I}(A)),$$

$$K_\infty(\langle \eta \rangle) = K_\infty(\langle \gamma_\eta \rangle, \langle \eta_i(I) \rangle, I \in \mathcal{I}(A)).$$

Observe that  $K(\eta)/K$  is a finite extension, and  $K_\infty(\langle \eta \rangle)/K_\infty$  is also a finite extension. Let  $m, n \in \mathbb{N}$ ,  $n \geq 1$ . Let  $\mathbb{C}_\infty$  be the completion of  $\overline{K}_\infty$ . Let  $\rho_1, \dots, \rho_m : K(\eta) \hookrightarrow \mathbb{C}_\infty$  be  $m$  continuous  $\mathbb{F}_p$ -algebra homomorphisms, and let  $\sigma_1, \dots, \sigma_n : K_\infty(\langle \eta \rangle) \hookrightarrow \mathbb{C}_\infty$  be  $n$  continuous  $\mathbb{F}_p$ -algebra homomorphisms. For  $s = (x; \underline{y}) \in \mathbb{S}_{\infty, n} = \mathbb{C}_\infty^\times \times \mathbb{Z}_p^n$ , we set:

$$\zeta_{\eta, A}(\underline{\rho}; \underline{\sigma}; s) = \sum_{d \geq 0} \left( \sum_{\substack{I \in \mathcal{I}(A), I \subset A \\ \deg I = d}} \frac{\prod_{i=1}^m \rho_i(\eta(I))}{\prod_{j=1}^n \sigma_j(\langle \eta(I) \rangle^{-y_j})} \right) x^{-d}.$$

Then, as a consequence of Theorem 4.1,  $\zeta_{\eta, A}(\underline{\rho}; \underline{\sigma}; s)$  converges on  $\mathbb{S}_{\infty, n}$ . One can easily notice that the above function, and its twists by some characters, generalize the twisted zeta function  $\zeta_A(t; \underline{\theta}; s)$ . We refer the reader to Section 5 for more detailed examples.

For the sake of completeness, in the last section of this article, we also introduce multiple twisted zeta functions, as well as their  $v$ -adic interpolation at finite places  $v$  of  $K$ , in the spirit of multiple zeta values introduced by D. Thakur ([4],[39]).

This article grew out of discussions between David Goss, Federico Pellarin and the first author, and the authors warmly thank David Goss and Federico Pellarin. The authors thank the referee for fruitful suggestions that enabled them to improve the content of this paper. The second author was partially supported by ANR Grant PerCoLaTor ANR-14-CE25-0002.

## 2. NOTATION AND PRELIMINARIES

Let  $\mathbb{F}_q$  be a finite field having  $q$  elements and of characteristic  $p$ . Let  $K/\mathbb{F}_q$  be a global function field ( $\mathbb{F}_q$  is algebraically closed in  $K$ ). Let  $\infty$  be a place of  $K$ . We denote by  $K_\infty$  the  $\infty$ -adic completion of  $K$ , and by  $\mathbb{F}_\infty$  the residue field of  $K_\infty$ . We fix  $\overline{K}_\infty$  an algebraic closure of  $K_\infty$  equipped with the canonical topology. Let  $\overline{K}$  be the algebraic closure of  $K$  in  $\overline{K}_\infty$ , and let  $\overline{\mathbb{F}}_q \subset \overline{K}$  be the algebraic closure of  $\mathbb{F}_q$  in  $\overline{K}_\infty$ . Let  $\mathbb{C}_\infty$  be the completion of  $\overline{K}_\infty$ . Let  $A$  be the ring of elements of  $K$  which are regular outside of  $\infty$ . Let  $v_\infty : K_\infty \rightarrow \mathbb{Z} \cup \{+\infty\}$  be the normalized discrete valuation associated to the local field  $K_\infty$ . We still denote by  $v_\infty : \mathbb{C}_\infty \rightarrow \mathbb{R} \cup \{+\infty\}$  the extension of  $v_\infty$  to  $\mathbb{C}_\infty$ .

Let  $\mathcal{I}(A)$  be the group of fractional ideals of  $A$ , and  $\mathcal{P}(A) = \{\alpha A, \alpha \in K^\times\}$ . Recall that  $\text{Pic}(A) = \frac{\mathcal{I}(A)}{\mathcal{P}(A)}$  is a finite abelian group. For any ideal  $I \subset A, I \neq \{0\}$ , we set:

$$\deg I = \dim_{\mathbb{F}_q} A/I.$$

Note that this function on non-zero ideals of  $A$  extends naturally to a morphism  $\deg : \mathcal{I}(A) \rightarrow \mathbb{Z}$ . In particular, we have:

$$\forall \alpha \in K^\times, \quad \deg \alpha := \deg \alpha A = -d_\infty v_\infty(\alpha),$$

where  $d_\infty = \dim_{\mathbb{F}_q} \mathbb{F}_\infty$ .

**Definition 2.1.** A sign function is a group homomorphism  $\text{sgn} : K_\infty^\times \rightarrow \overline{\mathbb{F}}_q^\times$  such that  $\text{sgn}(\zeta) = \zeta$  for all  $\zeta \in \mathbb{F}_q^\times$ .

We will fix a uniformizer  $\pi$  in  $K_\infty$ . Let  $(\pi_n)_{n \geq 1}$  be a sequence of elements in  $\overline{K}_\infty$  such that  $\pi_1 = \pi$ , and for  $n \geq 1$ ,  $\pi_{n+1}^{n+1} = \pi_n$ . Let  $y \in \mathbb{Q}$ , write  $y = \frac{m}{n!}$ ,  $m \in \mathbb{Z}$ ,  $n \geq 1$ , we set:

$$\pi^y := \pi_n^m.$$

This is well-defined. We set:

$$U_\infty = \{u \in \overline{K}_\infty, v_\infty(u-1) > 0\}.$$

Observe that  $U_\infty$  is a  $\mathbb{Q}_p$ -vector space. We have with the above definition:

$$\overline{K}_\infty^\times = \pi^\mathbb{Q} \times \overline{\mathbb{F}}_q^\times \times U_\infty.$$

Let  $x \in \overline{K}_\infty^\times$ , then  $x$  can be written in a unique way:

$$x = \pi^{v_\infty(x)} \text{sgn}(x) \langle x \rangle, \quad \text{with} \quad \text{sgn}(x) \in \overline{\mathbb{F}}_q^\times, \langle x \rangle \in U_\infty.$$

Observe that the map  $\langle \cdot \rangle : \overline{K}_\infty^\times \rightarrow U_\infty$  is a group homomorphism with  $\langle \cdot \rangle|_{U_\infty} = \text{Id}_{U_\infty}$ , and that  $\text{sgn}|_{\overline{K}_\infty^\times} : \overline{K}_\infty^\times \rightarrow \overline{\mathbb{F}}_q^\times$  is a sign function (see Definition 2.1).

We will need the following Lemma in the sequel (see also [22], Lemma 2):

**Lemma 2.2.**

1) Let  $\langle \cdot \rangle' : \overline{K}_\infty^\times \rightarrow U_\infty$  be a group homomorphism with  $\langle \cdot \rangle'|_{U_\infty} = \text{Id}_{U_\infty}$ . Then there exists a unique element  $u \in U_\infty$  such that:

$$\forall x \in \overline{K}_\infty^\times, \quad \langle x \rangle' = \langle x \rangle u^{v_\infty(x)}.$$

2) Let  $\text{sgn}' : \overline{K}_\infty^\times \rightarrow \overline{\mathbb{F}}_q^\times$  be a sign function. Then there exist  $e \in \mathbb{N}$ ,  $e \equiv 1 \pmod{q-1}$ , and an element  $\zeta \in \overline{\mathbb{F}}_q^\times$ , such that:

$$\forall x \in \overline{K}_\infty^\times, \quad \text{sgn}'(x) = \text{sgn}(x)^e \zeta^{v_\infty(x)}.$$

3) For  $i = 1, 2$ , let  $\langle \cdot \rangle'_i : \overline{K}_\infty^\times \rightarrow U_\infty$  be a group homomorphism such that  $\langle \cdot \rangle'_i|_{U_\infty} = \text{Id}_{U_\infty}$ , and let  $\gamma_i \in \overline{K}_\infty^\times$ . Suppose that there exist an open subgroup of finite index  $N \subset \overline{K}_\infty^\times$ , and  $n_1, n_2 \in \mathbb{Z}_p$ , such that:

$$\forall \alpha \in N \cap \overline{K}_\infty^\times, \quad \langle \alpha \rangle_1^{n_1} \gamma_1^{v_\infty(\alpha)} = \langle \alpha \rangle_2^{n_2} \gamma_2^{v_\infty(\alpha)}.$$

Then  $n_1 = n_2$ .

*Proof.*

1) We put:  $u = \langle \pi \rangle'$ .

2) There exists  $e \in \mathbb{N}$ ,  $e \equiv 1 \pmod{q-1}$  such that:

$$\forall \zeta \in \overline{\mathbb{F}}_q^\times, \quad \text{sgn}'(\zeta) = \zeta^e.$$

Let  $U = \overline{K}_\infty^\times \cap U_\infty$ . We get:

$$\frac{\text{sgn}'(\cdot)}{\text{sgn}(\cdot)^e} \Big|_{\overline{\mathbb{F}}_q^\times \times U} = 1.$$

We set:  $\zeta = \text{sgn}'(\pi) \in \overline{\mathbb{F}}_q^\times$ .

3) Let  $\alpha \in U_\infty \cap N \cap \overline{K}_\infty^\times$ ,  $\alpha \neq 1$ . Then:

$$\alpha^{n_1} = \alpha^{n_2}.$$

Thus  $n_1 = n_2$ . □

Let  $\eta : \mathcal{I}(A) \rightarrow \overline{K}_\infty$  be a map such that there exist an open subgroup of finite index  $N(\eta) \subset K_\infty^\times$ , an element  $n(\eta) \in \mathbb{Z}_p$ , an element  $\gamma'_\eta \in \overline{K}_\infty^\times$ , and a group homomorphism  $\langle \cdot \rangle' : \overline{K}_\infty \rightarrow U_\infty$ , with  $\langle \cdot \rangle' |_{U_\infty} = \text{Id}_{U_\infty}$ , such that:

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^\times \cap N(\eta), \quad \eta(I\alpha) = \eta(I)(\langle \alpha \rangle')^{n(\eta)} (\gamma'_\eta)^{v_\infty(\alpha)}.$$

By Lemma 2.2, there exists another element  $\gamma_\eta \in \overline{K}_\infty^\times$  such that:

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^\times \cap N(\eta), \quad \eta(I\alpha) = \eta(I)\langle \alpha \rangle^{n(\eta)} \gamma_\eta^{v_\infty(\alpha)}.$$

**Definition 2.3.**

1) A map  $\eta : \mathcal{I}(A) \rightarrow \overline{K}_\infty$  will be called *admissible* if there exist an open subgroup of finite index  $N(\eta) \subset K_\infty^\times$ , an element  $n(\eta) \in \mathbb{Z}_p$ , and an element  $\gamma_\eta \in \overline{K}_\infty^\times$ , such that:

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^\times \cap N(\eta), \quad \eta(I\alpha) = \eta(I)\langle \alpha \rangle^{n(\eta)} \gamma_\eta^{v_\infty(\alpha)}.$$

2) An admissible map  $\eta : \mathcal{I}(A) \rightarrow \overline{K}_\infty$  will be called *algebraic* if  $\eta(\mathcal{I}(A)) \subset \overline{K}$ .

Again, by Lemma 2.2,  $n(\eta)$  is well-defined, and one sees that  $\langle \gamma_\eta \rangle$  is well-defined. Note also that the product of admissible maps is an admissible map and any constant map from  $\mathcal{I}(A)$  to  $\overline{K}_\infty$  is admissible.

**Example 2.4.** Let's give a fundamental example, due to D. Goss, of such an admissible map ([21], section 8.2). Write  $d_\infty = p^k d'_\infty$  with  $d'_\infty \not\equiv 0 \pmod{p}$ . Let  $p^m$  be the biggest power of  $p$  that divides  $|\text{Pic}(A)|$ . Set  $l = \text{Max}\{m, k\}$ , and  $e_\infty = p^l d'_\infty$ . We fix  $\pi_* \in \overline{K}_\infty$  an  $e_\infty$ -th root of  $\pi$ . We set:

$$T = \mathbb{F}_\infty((\pi_*)).$$

If  $I \in \mathcal{I}(A)$ , then there exists an integer  $h$  dividing  $|\text{Pic}(A)|$ , such that:

$$I^h = \alpha A, \quad \alpha \in K^\times.$$

We set:

$$[I] = \langle \alpha \rangle^{\frac{1}{h}} \pi^{-\frac{\deg I}{d_\infty}} \in T.$$

Then,  $K([I], I \in \mathcal{I}(A))/K$  is a finite extension. The group homomorphism

$$[\cdot] : \mathcal{I}(A) \rightarrow \overline{K}_\infty^\times$$

satisfies:

$$\forall \alpha \in K^\times, \quad [\alpha A] = \frac{\alpha}{\text{sgn}(\alpha)}.$$

Hence,  $[\cdot]$  is an algebraic admissible map.

We refer the reader to [21, section 8.2] or [23, section 2, Remark 1] to see how the admissible map  $[\cdot]$  varies with different choices of uniformizer.

**Definition 2.5.**

1) Let  $\eta : \mathcal{I}(A) \rightarrow \overline{K}_\infty^\times$  be an admissible map. We set:

$$K_\infty(\langle \eta \rangle) := K_\infty(\langle \gamma_\eta \rangle, \langle \eta(I) \rangle, I \in \mathcal{I}(A)).$$

It is a finite extension of  $K_\infty$ .

2) Let  $\eta : \mathcal{I}(A) \rightarrow \overline{K}_\infty$  be an admissible map with  $n(\eta) \in \mathbb{Z}$ . We set:

$$K(\eta) := K(\mathbb{F}_\infty, \langle \gamma_\eta \rangle, \eta(I), I \in \mathcal{I}(A)).$$

If  $\eta$  is algebraic (Definition 2.3), then  $K(\eta)$  is a finite extension of  $K$ .

**Remark 2.6.** Let  $N \subset K_\infty^\times$  be an open subgroup of finite index, and let's set:

$$\mathcal{P}(N) = \{xA, x \in K^\times \cap N\}.$$

Then  $\mathcal{I}(A)/\mathcal{P}(N)$  is a finite abelian group. Let  $\chi : \mathcal{I}(A) \rightarrow \overline{K}_\infty$  be a map such that:

$$\forall I \in \mathcal{I}(A), \forall J \in \mathcal{P}(N), \quad \chi(IJ) = \chi(I).$$

Observe that  $\chi$  is an admissible map. Let  $\gamma \in \overline{K}_\infty^\times$ , and let  $y \in \mathbb{Z}_p$ . Then

$$\forall I \in \mathcal{I}(A), \quad \eta(I) = \chi(I) \langle I \rangle^y \gamma^{\deg(I)}$$

is an admissible map with  $N(\eta) = N$ ,  $n(\eta) = y$ , and  $\gamma_\eta = \gamma^{-d_\infty}$ .

Reciprocally, let  $\eta$  be an admissible map. Choose  $\gamma \in \overline{K}_\infty^\times$  such that  $\gamma^{-d_\infty} = \gamma_\eta$ . We define  $\chi : \mathcal{I}(A) \rightarrow \overline{K}_\infty$  as follows:

$$\forall I \in \mathcal{I}(A), \quad \chi(I) = \frac{\eta(I)}{\langle I \rangle^{n(\eta)} \gamma^{\deg(I)}}.$$

Then, we have:

$$\forall I \in \mathcal{I}(A), \forall J \in \mathcal{P}(N(\eta)), \quad \chi(IJ) = \chi(I).$$

**Remark 2.7.** Let  $F$  be a complete field with respect to a non-trivial valuation  $v_F : F \rightarrow \mathbb{R} \cup \{+\infty\}$  and such that  $F$  is a  $\overline{\mathbb{F}}_q$ -algebra. Let  $T/K_\infty$  be a finite extension, let  $\mathbb{F}_T$  be the residue field of  $T$ , and let  $\pi_T$  be a uniformizer of  $T$ . Let  $\sigma : T \hookrightarrow F$  be a continuous  $\mathbb{F}_p$ -algebra homomorphism. Let  $\varphi \in \text{Gal}(T/\mathbb{F}_p((\pi_T))) \simeq \text{Gal}(\mathbb{F}_T/\mathbb{F}_p)$ , such that:

$$\forall \zeta \in \mathbb{F}_T, \quad \sigma(\zeta) = \varphi(\zeta).$$

Let  $y = \sigma(\pi_T)$ . Then  $v_F(y) > 0$ , and:

$$\forall a \in T, \quad \sigma(a) = \varphi(a) |_{\pi_T=y}.$$

Thus the choice of a continuous  $\mathbb{F}_p$ -algebra homomorphism  $\sigma : T \hookrightarrow F$  is equivalent to the choice of an element  $y \in F^\times$ ,  $v_F(y) > 0$ , and  $\varphi \in \text{Gal}(\mathbb{F}_T/\mathbb{F}_p)$ .

Let  $F$  be as in the above remark. For any  $n \geq 1$ , we set:

$$\mathbb{S}_{F,n} = F^\times \times \mathbb{Z}_p^n.$$

We view  $\mathbb{S}_{F,n}$  as an abelian group with group action written additively. If  $F = \mathbb{C}_\infty$  and  $n = 1$ , then  $\mathbb{S}_{F,n}$  is called the ‘‘complex plane’’  $\mathbb{S}_\infty$  ([21], section 8.1). Following D. Goss, an element  $s \in \mathbb{S}_{F,n}$  is always written as  $s = (x; y_1, \dots, y_n)$  (or  $s = (x, \underline{y})$  for short) with  $x \in F^\times$  and  $y_1, \dots, y_n \in \mathbb{Z}_p$ .

We end this section with some notation for the case of finite places. Let  $v$  be a finite place of  $K$ , i.e.  $v : K \rightarrow \mathbb{Z} \cup \{+\infty\}$  is a discrete valuation on  $K$  such that there exists  $a \in A$  with  $v(a) = 1$ . Let  $P_v = \{a \in A, v(a) > 0\}$  (or  $P$  for short) be the maximal ideal of  $A$  corresponding to the finite place  $v$ . We denote by  $K_v$  the  $v$ -adic completion of  $K$ . We fix  $\overline{K}_v$  an algebraic closure of  $K_v$  equipped with the canonical topology. Then  $v$  extends to a valuation  $v : \overline{K}_v \rightarrow \mathbb{Q} \cup \{+\infty\}$ . We fix a  $K$ -morphism  $\overline{K} \hookrightarrow \overline{K}_v$ , and, by abuse of notation, we will identify the elements in  $\overline{K}$  with their images in  $\overline{K}_v$ . We will fix a uniformizer  $\pi_v$  of  $K_v$ . Then, every  $x \in \overline{K}_v^\times$ , can be written in a unique way:

$$x = \pi_v^{v(x)} \text{sgn}_v(x) \langle x \rangle_v \quad \text{with} \quad \text{sgn}_v(x) \in \overline{\mathbb{F}}_q^\times, v(\langle x \rangle_v - 1) > 0.$$

Let  $n \geq 1$  be an integer and let's set:

$$\mathbb{S}_{F,v,n} = F^\times \times \mathbb{Z}_p^n \times \mathbb{Z}^n.$$

We view  $\mathbb{S}_{F,v,n}$  as an abelian group with group action written additively. An element  $s \in \mathbb{S}_{F,v,n}$  is always written as  $s = (x; \underline{y}; \underline{\delta})$  with  $x \in F^\times$ ,  $y_1, \dots, y_n \in \mathbb{Z}_p$ , and  $\delta_1, \dots, \delta_n \in \mathbb{Z}$ .

### 3. A KEY LEMMA

Let  $E \subset \bar{K}$  be a finite extension of  $K$  of genus  $g_E$ . Let  $O_E$  be the integral closure of  $A$  in  $E$  and  $\mathbb{F}_E$  be the algebraic closure of  $\mathbb{F}_q$  in  $E$ . We set  $\mathcal{I}(O_E)$  to be the group of fractional ideals of  $O_E$ , and  $\mathcal{P}(O_E) = \{xO_E, x \in E^\times\}$ . Let  $N_{E/K} : \mathcal{I}(O_E) \rightarrow \mathcal{I}(A)$  be the group homomorphism such that, if  $\mathfrak{P}$  is a maximal ideal of  $O_E$  and if  $P = \mathfrak{P} \cap A$  is the corresponding maximal ideal of  $A$ , then:

$$N_{E/K}(\mathfrak{P}) = P^{[\frac{O_E}{\mathfrak{P}} : \frac{A}{P}]}$$

Note that, if  $\mathfrak{J} = xO_E, x \in E^\times$ , then:

$$N_{E/K}(\mathfrak{J}) = N_{E/K}(x)A,$$

where  $N_{E/K} : E \rightarrow K$  denotes also the usual norm map. Let  $\infty_1, \dots, \infty_r$  be the places of  $E$  above  $\infty$ . For  $i = 1, \dots, r$ , let  $f_i$  be the residual degree of  $\infty_i/\infty$ ,  $v_i$  be the associated valuation on  $E$  and  $E_{v_i}$  be the  $v_i$ -adic completion of  $E$ .

Let  $\mathcal{B} \subset O_E$ ,  $\mathcal{B} \in \mathcal{I}(O_E)$ . We denote by  $\mathcal{I}(\mathcal{B})$  the group of fractional ideals of  $O_E$  which are relatively prime to  $\mathcal{B}$ . For  $i = 1, \dots, r$ , let  $N_i$  be an open subgroup of finite index of  $E_{v_i}^\times$ , and let  $d_i \geq 0$  be the least integer such that:

$$\{x \in E_{v_i}^\times, v_i(x-1) \geq d_i\} \subset N_i.$$

We set  $N = \prod_{i=1}^r N_i$ , and:

$$\mathcal{P}(\mathcal{B}, N) = \{xO_E \in \mathcal{I}(\mathcal{B}), x \in E^\times, x \equiv 1 \pmod{\mathcal{B}}, x \in N_i, i = 1, \dots, r\}.$$

Observe that  $\mathcal{I}(\mathcal{B})/\mathcal{P}(\mathcal{B}, N)$  is a finite abelian group. Let:

$$d(\mathcal{B}) = \dim_{\mathbb{F}_q} O_E/\mathcal{B},$$

$$d_\infty(N) = \sum_{i=1}^r d_\infty f_i (d_i + 1).$$

More generally, let  $\mathcal{S}$  be a finite set, possibly empty, of maximal ideals of  $O_E$  which are relatively prime to  $\mathcal{B}$ . We denote by  $\mathcal{I}_{\mathcal{S}}(\mathcal{B})$  the group of fractional ideals of  $\mathcal{I}(O_E)$  which are relatively prime to  $\mathcal{B} \prod_{\mathfrak{P} \in \mathcal{S}} \mathfrak{P}$ . We also set  $\mathcal{P}_{\mathcal{S}}(\mathcal{B}, N) = \mathcal{P}(\mathcal{B}, N) \cap \mathcal{I}_{\mathcal{S}}(\mathcal{B})$ . Observe that we have a natural group isomorphism:

$$\mathcal{I}_{\mathcal{S}}(\mathcal{B})/\mathcal{P}_{\mathcal{S}}(\mathcal{B}, N) \simeq \mathcal{I}(\mathcal{B})/\mathcal{P}(\mathcal{B}, N).$$

We set:

$$d_{\mathcal{S}}(\mathcal{B}) = \dim_{\mathbb{F}_q} O_E/(\mathcal{B} \prod_{\mathfrak{P} \in \mathcal{S}} \mathfrak{P}),$$

and

$$(3.1) \quad d_{\mathcal{S}}(\mathcal{B}, N, t) = 2g_E[\mathbb{F}_E : \mathbb{F}_q] + d_\infty(N) + d_{\mathcal{S}}(\mathcal{B}) + \frac{t[E : K]_{\text{sep}}}{(p-1)[\mathbb{F}_q : \mathbb{F}_p]}$$

where  $[E : K]_{\text{sep}}$  is the separable degree of the finite extension  $E/K$ .

We recall a basic result ([21], Lemma 8.8.1):

**Lemma 3.1.** *Let  $h \in \mathbb{N}$ , and let  $W$  be a finite dimensional  $\mathbb{F}_p$ -vector space. For  $i = 1, \dots, h$ , let  $f_i : W \rightarrow F$  be an  $\mathbb{F}_p$ -linear map. If  $\dim_{\mathbb{F}_p} W > \frac{h}{p-1}$ , then:*

$$\forall x_1, \dots, x_h \in F, \quad \sum_{w \in W} (x_1 + f_1(w)) \cdots (x_h + f_h(w)) = 0.$$

Let  $t \geq 0$  be an integer. For  $i = 1, \dots, t$ , let  $\text{sgn}_i : K^\times \rightarrow \overline{\mathbb{F}}_q^\times$  be a sign function and  $\rho_i : K(\text{sgn}_i(K^\times)) \hookrightarrow F$  be an  $\mathbb{F}_p$ -algebra homomorphism.

Lemma 3.1 combined with the Riemann-Roch Theorem yields the following result which will be crucial in the sequel.

**Lemma 3.2.** *We keep the previous notation. Let  $\mathfrak{J}$  be a non-zero ideal of  $O_E$  with  $\mathfrak{J} \in \mathcal{I}_{\mathcal{S}}(\mathcal{B})$  and  $d$  be an integer. We set:*

$$S_{d,\mathcal{S}}(\mathfrak{J}, \mathcal{B}, N) = \{aO_E \in \mathcal{P}_{\mathcal{S}}(\mathcal{B}, N), a\mathfrak{J} \subset O_E, \deg(N_{E/K}(a\mathfrak{J})) = d\}.$$

Then, if  $d \geq d_{\mathcal{S}}(\mathcal{B}, N, t)$  (3.1), we have:

$$\sum_{aO_E \in S_{d,\mathcal{S}}(\mathfrak{J}, \mathcal{B}, N)} \rho_1\left(\frac{N_{E/K}(a)}{\text{sgn}_1(N_{E/K}(a))}\right) \cdots \rho_t\left(\frac{N_{E/K}(a)}{\text{sgn}_t(N_{E/K}(a))}\right) = 0.$$

*Proof.* The proof of the Lemma is based on the arguments used in the proof of [21], Theorem 8.9.2. We give a detailed proof for the convenience of the reader. We choose a generator  $\eta$  of the cyclic group  $\mathbb{F}_q^\times$ . For any integer  $k$ , we set:

$$S_{d,\mathcal{S}}(\mathfrak{J}, \mathcal{B}, N, k) = \{aO_E \in \mathcal{P}_{\mathcal{S}}(\mathcal{B}, N), \text{sgn}(N_{E/K}(a)) \equiv \eta^k \pmod{\mathbb{F}_q^\times}; \\ a\mathfrak{J} \subset O_E, \deg(N_{E/K}(a\mathfrak{J})) = d\}.$$

Then:

$$S_{d,\mathcal{S}}(\mathfrak{J}, \mathcal{B}, N) = \bigsqcup_{k=1}^{\frac{q^{d_\infty}-1}{q-1}} S_{d,\mathcal{S}}(\mathfrak{J}, \mathcal{B}, N, k).$$

Hence, it is enough to show that for any integer  $k$  such that  $1 \leq k \leq \frac{q^{d_\infty}-1}{q-1}$  and any integer  $d \geq d_{\mathcal{S}}(\mathcal{B}, N, t)$ :

$$\sum_{aO_E \in S_{d,\mathcal{S}}(\mathfrak{J}, \mathcal{B}, N, k)} \rho_1\left(\frac{N_{E/K}(a)}{\text{sgn}_1(N_{E/K}(a))}\right) \cdots \rho_t\left(\frac{N_{E/K}(a)}{\text{sgn}_t(N_{E/K}(a))}\right) = 0.$$

From now on, we will fix an integer  $k$  such that  $1 \leq k \leq \frac{q^{d_\infty}-1}{q-1}$  and an integer  $d \geq d_{\mathcal{S}}(\mathcal{B}, N, t)$ . Since there is a finite number of ideals  $J$  of  $A$  such that  $\deg J = d$ , and, given such an ideal  $J$ , there is a finite number of ideals  $\mathfrak{J}$  in  $O_E$  such that  $N_{E/K}(\mathfrak{J}) = J$ , we deduce that  $S_{d,\mathcal{S}}(\mathfrak{J}, \mathcal{B}, N, k)$  is a finite set, possibly empty. Let's fix  $aO_E \in S_{d,\mathcal{S}}(\mathfrak{J}, \mathcal{B}, N, k)$ . Let  $V(a)$  be the set of elements  $b \in E$  such that

- i)  $b\mathfrak{J} \subset O_E$ ,
- ii)  $v_i(b) \geq v_i(a) + d_i + 1$ , for  $i = 1, \dots, r$ ,
- iii)  $\text{ord}_{\mathfrak{P}}(b) \geq \text{ord}_{\mathfrak{P}}(\mathcal{B})$ , for all  $\mathfrak{P}$  dividing  $\mathcal{B}$ ,
- iv)  $\text{ord}_{\mathfrak{P}}(b) \geq 1$ , for all  $\mathfrak{P} \in \mathcal{S}$ .

Observe that:

$$\deg N_{E/K}(a\mathfrak{J}) = \deg N_{E/K}(\mathfrak{J}) - d_\infty \sum_{i=1}^r f_i v_i(a).$$

But:

$$\deg N_{E/K}(\mathfrak{J}) = [\mathbb{F}_E : \mathbb{F}_q] \dim_{\mathbb{F}_E} \frac{O_E}{\mathfrak{J}},$$

therefore:

$$\frac{-d_\infty}{[\mathbb{F}_E : \mathbb{F}_q]} \sum_{i=1}^r f_i v_i(a) = \frac{d}{[\mathbb{F}_E : \mathbb{F}_q]} - \dim_{\mathbb{F}_E} \frac{O_E}{\mathfrak{J}}.$$

Since  $d \geq d_S(\mathcal{B}, N, t)$ , then we have:

$$d - d_\infty(N) - d_S(\mathcal{B}) \geq [\mathbb{F}_E : \mathbb{F}_q](2g_E - 1).$$

By the Riemann-Roch Theorem,  $V(a)$  is a finite  $\mathbb{F}_E$ -vector space of dimension

$$\dim_{\mathbb{F}_E} V(a) = \frac{d}{[\mathbb{F}_E : \mathbb{F}_q]} + 1 - g_E - \frac{d_\infty(N) + d_S(\mathcal{B})}{[\mathbb{F}_E : \mathbb{F}_q]}.$$

Again, the hypothesis  $d \geq d_S(\mathcal{B}, N, t)$  implies:

$$(3.2) \quad \dim_{\mathbb{F}_p} V(a) > \frac{t[E : K]_{\text{sep}}}{p-1}.$$

We have (see [21], Lemma 8.9.3):

$$(a + V(a))O_E \subset S_{d,S}(\mathfrak{J}, \mathcal{B}, N, k),$$

$$\forall b, b' \in V(a), \quad (a + b)O_E = (a + b')O_E \Leftrightarrow b = b'.$$

Furthermore, if  $\text{sgn}' : K_\infty^\times \rightarrow \overline{\mathbb{F}_q}^\times$  is a sign function, we have:

$$\forall b \in V(a), \quad \text{sgn}'(N_{E/K}(a + b)) = \text{sgn}'(N_{E/K}(a)).$$

Let  $cO_E \in S_{d,S}(\mathfrak{J}, \mathcal{B}, N, k)$ . Then:

$$(c + V(c))O_E \cap (a + V(a))O_E \neq \emptyset \Leftrightarrow cO_E \in (a + V(a))O_E.$$

Thus, if  $S_{d,S}(\mathfrak{J}, \mathcal{B}, N, k) \neq \emptyset$ , let's select  $a_1O_E, \dots, a_lO_E \in S_{d,S}(\mathfrak{J}, \mathcal{B}, N, k)$  such that  $S_{d,S}(\mathfrak{J}, \mathcal{B}, N, k)$  is the disjoint union of the  $(a_i + V(a_i))O_E$ . We have:

$$\begin{aligned} & \sum_{cO_E \in S_{d,S}(\mathfrak{J}, \mathcal{B}, N, k)} \rho_1\left(\frac{N_{E/K}(c)}{\text{sgn}_1(N_{E/K}(c))}\right) \cdots \rho_t\left(\frac{N_{E/K}(c)}{\text{sgn}_t(N_{E/K}(c))}\right) = \\ & \sum_{i=1}^l \sum_{b \in V(a_i)} \rho_1\left(\frac{N_{E/K}(a_i + b)}{\text{sgn}_1(N_{E/K}(a_i))}\right) \cdots \rho_t\left(\frac{N_{E/K}(a_i + b)}{\text{sgn}_t(N_{E/K}(a_i))}\right). \end{aligned}$$

Let's fix  $1 \leq i \leq l$ , and we set

$$S = \sum_{b \in V(a_i)} \rho_1(N_{E/K}(a_i + b)) \cdots \rho_t(N_{E/K}(a_i + b)).$$

Let  $E_1/K$  consist of the elements of  $E$  which are separable over  $K$ . Let  $p^{l_1} = [E : E_1]$ . Then:

$$S = \sum_{b \in V(a_i)} \rho_1(N_{E_1/K}(a_i^{p^{l_1}} + b^{p^{l_1}})) \cdots \rho_t(N_{E_1/K}(a_i^{p^{l_1}} + b^{p^{l_1}})).$$

Therefore, we can assume that  $E/K$  is a finite separable extension. Let  $\overline{F}$  be an algebraic closure of  $F$ . Then, for  $i = 1, \dots, t$ ,  $\rho_i$  extends to a morphism  $\rho_i : \overline{K} \rightarrow \overline{F}$ . Let  $\sigma_j : E \rightarrow \overline{K}$  be the distinct  $K$ -embeddings of  $E$  in  $\overline{K}$ ,  $j = 1, \dots, [E : K]$ . Then:

$$\forall i = 1, \dots, t, \quad \rho_i(N_{E/K}(a_i + b)) = \prod_{j=1}^{[E:K]} \rho_i(\sigma_j(a_i + b)).$$

Since  $\dim_{\mathbb{F}_p} V(a_i) > \frac{t[E:K]_{\text{sep}}}{p-1} = \frac{t[E:K]}{p-1}$  (3.2), Lemma 3.1 implies that  $S = 0$ . The proof is finished.  $\square$

For  $i = 1, \dots, t$ , let  $\psi_i : \mathcal{I}(A) \rightarrow \overline{K}_\infty$  be a map such that there exists  $N(\psi_i) \subset K_\infty^\times$  an open subgroup of finite index with the following property:

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^\times \cap N(\psi_i), \quad \psi_i(I\alpha) = \psi_i(I) \frac{\alpha}{\text{sgn}_i(\alpha)}.$$

We set  $K(\psi_i) = K(\text{sgn}_i(K^\times), \psi_i(I), I \in \mathcal{I}(A)) \subset \overline{K}_\infty$  and let  $\rho_i : K(\psi_i) \hookrightarrow F$  be an  $\mathbb{F}_p$ -algebra homomorphism.

Let  $\chi : \mathcal{I}(\mathcal{B}) \rightarrow F$  be a map such that:

$$\forall \mathfrak{J} \in \mathcal{I}(\mathcal{B}), \forall \mathfrak{J} \in \mathcal{P}(\mathcal{B}, N), \quad \chi(\mathfrak{J}\mathfrak{J}) = \chi(\mathfrak{J}).$$

For  $j = 1, \dots, r$ , let  $N'_j = N_j \cap N_{E_{v_j}/K_\infty}^{-1}(N(\psi_1) \cap \dots \cap N(\psi_t))$ , then  $N'_j$  is an open subgroup of finite index of  $E_{v_j}^\times$ . We set  $N' = \prod_{j=1}^r N'_j$ .

**Corollary 3.3.** *We keep the previous notation. Then, if  $d \geq d_S(\mathcal{B}, N', t)$  (3.1), we have:*

$$\sum_{\substack{\mathfrak{J} \subset O_E, \mathfrak{J} \in \mathcal{I}_S(\mathcal{B}) \\ \deg N_{E/K}(\mathfrak{J}) = d}} \chi(\mathfrak{J}) \rho_1(\psi_1(N_{E/K}(\mathfrak{J}))) \cdots \rho_t(\psi_t(N_{E/K}(\mathfrak{J}))) = 0.$$

*Proof.* Let's set:

$$U_d = \sum_{\substack{\mathfrak{J} \subset O_E, \mathfrak{J} \in \mathcal{I}_S(\mathcal{B}) \\ \deg N_{E/K}(\mathfrak{J}) = d}} \chi(\mathfrak{J}) \rho_1(\psi_1(N_{E/K}(\mathfrak{J}))) \cdots \rho_t(\psi_t(N_{E/K}(\mathfrak{J}))).$$

Let  $h = |\mathcal{I}(\mathcal{B})/\mathcal{P}(\mathcal{B}, N')|$ . Select  $\mathfrak{J}_1, \dots, \mathfrak{J}_h \in \mathcal{I}_S(\mathcal{B})$ ,  $\mathfrak{J}_1, \dots, \mathfrak{J}_h \subset O_E$ , a system of representatives of  $\frac{\mathcal{I}(\mathcal{B})}{\mathcal{P}(\mathcal{B}, N')}$ . Then  $U_d$  is equal to

$$\sum_{j=1}^h \chi(\mathfrak{J}_j) \prod_{i=1}^t \rho_i(\psi_i(N_{E/K}(\mathfrak{J}_j))) \sum_{a \in O_E \in S_{d,S}(\mathfrak{J}_j, \mathcal{B}, N')} \prod_{i=1}^t \rho_i\left(\frac{N_{E/K}(a)}{\text{sgn}_i(N_{E/K}(a))}\right).$$

Since  $d \geq d_S(\mathcal{B}, N', t)$ , by Lemma 3.2, we have:

$$U_d = 0.$$

$\square$

Let  $f : \mathbb{Z} \rightarrow 1 + (q-1)\mathbb{Z}$  be a map. For  $i = 1, \dots, t$ , let  $\psi_i : \mathcal{I}(A) \rightarrow \overline{K}_\infty$  be a map such that :

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^\times, \quad \psi_i(I\alpha) = \psi_i(I) \frac{\alpha}{\text{sgn}_i(\alpha)^{f(\deg I)}}.$$

By Lemma 2.2, there exist  $e_i \in \mathbb{Z}$ ,  $e_i \equiv 1 \pmod{q-1}$ , and  $\zeta_i \in \overline{\mathbb{F}}_q^\times$  such that:

$$\forall x \in K_\infty^\times, \quad \text{sgn}_i(x) = \text{sgn}(x)^{e_i} \zeta_i^{v_\infty(x)}.$$

We set  $K(\psi_i) = K(\text{sgn}_i(K^\times), \psi_i(I), I \in \mathcal{I}(A)) \subset \overline{K}_\infty$  and let  $\rho_i : K(\psi_i) \hookrightarrow F$  be an  $\mathbb{F}_p$ -algebra homomorphism. Observe that there exists  $l_i \in \mathbb{N}$  such that:

$$\forall \alpha \in K^\times, \quad \rho_i(\text{sgn}_i(\alpha)) = \text{sgn}_i(\alpha)^{p^{l_i}}.$$

Let  $\chi : \mathcal{I}(A) \rightarrow F$  be a map such that:

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^\times, \quad \chi(I\alpha) = \chi(I).$$

**Corollary 3.4.** *We keep the previous notation and suppose that  $t \geq 1$ . If*

$$p^{l_1}e_1 + \cdots + p^{l_t}e_t \equiv 0 \pmod{q^{d_\infty} - 1},$$

and

$$\prod_{i=1}^t \zeta_i^{p^{l_i}} = 1,$$

then :

$$\sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(A), I \subset A \\ \deg I = d}} \chi(I) \rho_1(\psi_1(I)) \cdots \rho_t(\psi_t(I)) = 0.$$

*Proof.* Here  $E = K$ ,  $\mathcal{S} = \emptyset$ ,  $\mathcal{B} = A$ ,  $N = K^\times$ . For  $d \geq 0$ , let's set:

$$U_d = \sum_{\substack{I \in \mathcal{I}(A), I \subset A \\ \deg I = d}} \chi(I) \rho_1(\psi_1(I)) \cdots \rho_t(\psi_t(I)).$$

Let  $h = |\text{Pic}(A)|$ . Select  $I_1, \dots, I_h \in \mathcal{I}(A)$ ,  $I_1, \dots, I_h \subset A$ , a system of representatives of  $\text{Pic}(A)$ . For simplicity, we will write  $S_d(I_j)$  instead of  $S_{d, \mathcal{S}}(I_j, A, N)$ . Then  $U_d$  is equal to:

$$\sum_{j=1}^h \chi(I_j) \prod_{i=1}^t \rho_i(\psi_i(I_j)) \sum_{aA \in S_d(I_j)} \prod_{i=1}^t \rho_i\left(\frac{a}{\text{sgn}_i(a)^{f(\deg I_j)}}\right).$$

Now, assume that  $p^{l_1}e_1 + \cdots + p^{l_t}e_t \equiv 0 \pmod{q^{d_\infty} - 1}$ , and  $\prod_{i=1}^t \zeta_i^{p^{l_i}} = 1$ . By Lemma 3.2,  $\sum_{d \geq 0} U_d$  is equal to the finite sum:

$$\sum_{j=1}^h \chi(I_j) \rho_1(\psi_1(I_j)) \cdots \rho_t(\psi_t(I_j)) \sum_{d \geq 0} \sum_{aA \in S_d(I_j)} \rho_1(a) \cdots \rho_t(a).$$

Let's select  $d \geq 0$  such that:

$$\sum_{d \geq 0} \sum_{aA \in S_d(I_j)} \rho_1(a) \cdots \rho_t(a) = \sum_{m=0}^d \sum_{aA \in S_m(I_j)} \rho_1(a) \cdots \rho_t(a).$$

Let  $V_d = \{a \in K^\times, aA \in S_m(I_j), m \leq d\} \cup \{0\}$ . Then  $V_d$  is a finite dimensional  $\mathbb{F}_q$ -vector space and:

$$\sum_{m=0}^d \sum_{aA \in S_m(I_j)} \rho_1(a) \cdots \rho_t(a) = \sum_{a \in V_d} \rho_1(a) \cdots \rho_t(a).$$

Now, by the Riemann-Roch Theorem, for  $d \geq 2g_K - 1$ , we have:

$$\dim_{\mathbb{F}_q} V_d = d + 1 - g_K.$$

Hence, we can choose  $d$  big enough such that  $\dim_{\mathbb{F}_p} V_d > \frac{t}{(p-1)[\mathbb{F}_q:\mathbb{F}_p]}$ . Then, by Lemma 3.1, we have:

$$\sum_{a \in V_d} \rho_1(a) \dots \rho_t(a) = 0.$$

The proof is finished.  $\square$

#### 4. SEVERAL VARIABLE TWISTED ZETA FUNCTIONS

##### 4.1. The $\infty$ -adic case.

Let  $E/K$  be a finite extension, and let  $\mathcal{B}$ ,  $N$  and  $\mathcal{S}$  as in Section 3. Recall that  $\mathcal{S}$  is a finite set, possibly empty, of maximal ideals of  $O_E$  which are relatively prime to  $\mathcal{B}$ , and that  $F$  is a complete field with respect to a non-trivial valuation  $v_F : F \rightarrow \mathbb{R} \cup \{+\infty\}$  and such that  $F$  is a  $\overline{\mathbb{F}_q}$ -algebra. Let  $\chi : \mathcal{I}(\mathcal{B}) \rightarrow F$  be a map such that:

$$\forall \mathfrak{J} \in \mathcal{I}(\mathcal{B}), \forall \mathfrak{J} \in \mathcal{P}(\mathcal{B}, N), \quad \chi(\mathfrak{J}\mathfrak{J}) = \chi(\mathfrak{J}).$$

Let  $m \geq 0$  be an integer. For  $i = 1, \dots, m$ , let  $\psi_i : \mathcal{I}(A) \rightarrow \overline{K}_\infty$  be an admissible map with  $n(\psi_i) \in \mathbb{N}$  and  $\rho_i : K(\psi_i) \hookrightarrow F$  be an  $\mathbb{F}_p$ -algebra homomorphism (see Definition 2.5 for the definition of  $K(\psi_i)$ ).

Let  $n \geq 1$  be another integer. For  $j = 1, \dots, n$ , let  $\eta_j : \mathcal{I}(A) \rightarrow \overline{K}_\infty^\times$  be an admissible map and  $\sigma_j : K_\infty(\langle \eta_j \rangle) \hookrightarrow F$  be a continuous  $\mathbb{F}_p$ -algebra homomorphism (see Definition 2.5 for the definition of  $K_\infty(\langle \eta_j \rangle)$ ).

For  $I \in \mathcal{I}(A)$ ,  $s = (x, \underline{y}) \in \mathbb{S}_{F,n}$ , and  $\alpha \in K^\times$ , we set:

$$\begin{aligned} I_{\underline{\rho}, \underline{\psi}} &= \prod_{i=1}^m \rho_i(\psi_i(I)) \in F, \\ I_{\underline{\sigma}, \underline{\eta}}^{\underline{y}} &= \prod_{j=1}^n \sigma_j(\langle \eta_j(I) \rangle^{y_j}) \in F^\times, \\ I_{\underline{\sigma}, \underline{\eta}}^s &= x^{\deg I} \prod_{j=1}^n \sigma_j(\langle \eta_j(I) \rangle^{y_j}) \in F^\times, \\ (\alpha A)_{\underline{\sigma}, \underline{\eta}}^s &= (x^{-d_\infty} \prod_{j=1}^n \sigma_j(\langle \eta_j \rangle^{y_j}))^{v_\infty(\alpha)} \prod_{j=1}^n \sigma_j(\langle \alpha \rangle^{n(\eta_j)y_j}) \in F^\times. \end{aligned}$$

Then, for  $I \in \mathcal{I}(A)$ ,  $s, s' \in \mathbb{S}_{F,n}$  and  $\alpha \in K^\times \cap N(\eta_1) \cap \dots \cap N(\eta_n)$ , we have:

$$\begin{aligned} I_{\underline{\sigma}, \underline{\eta}}^{s+s'} &= I_{\underline{\sigma}, \underline{\eta}}^s I_{\underline{\sigma}, \underline{\eta}}^{s'}, \\ (I\alpha)_{\underline{\sigma}, \underline{\eta}}^s &= I_{\underline{\sigma}, \underline{\eta}}^s (\alpha A)_{\underline{\sigma}, \underline{\eta}}^s. \end{aligned}$$

We define a zeta function

$$\zeta_{\mathcal{S}, O_E}(\underline{\rho}, \underline{\psi}; \underline{\sigma}, \underline{\eta}; \chi; \cdot) : \mathbb{S}_{F,n} \rightarrow F$$

which sends  $s \in \mathbb{S}_{F,n}$  to the following sum:

$$\sum_{d \geq 0} \sum_{\substack{\mathfrak{J} \in \mathcal{I}_{\mathcal{S}}(\mathcal{B}), \mathfrak{J} \subset O_E \\ \deg N_{E/K}(\mathfrak{J}) = d}} \chi(\mathfrak{J}) N_{E/K}(\mathfrak{J})_{\underline{\rho}, \underline{\psi}} N_{E/K}(\mathfrak{J})_{\underline{\sigma}, \underline{\eta}}^{-s}.$$

For each fixed  $\underline{y} \in \mathbb{Z}_p^n$ , the next theorem proves that  $\zeta_{\mathcal{S}, O_E}(\underline{\rho}, \underline{\psi}; \underline{\sigma}, \underline{\eta}; \chi; \cdot)$  is an entire power series in  $x^{-1}$  with the resulting function on  $\mathbb{S}_{F,n}$  having good continuity properties.

**Theorem 4.1.** *The zeta function  $\zeta_{\mathcal{S}, O_E}(\underline{\rho}, \underline{\psi}; \underline{\sigma}, \underline{\eta}; \chi; \cdot)$  continues analytically to an entire function on  $\mathbb{S}_{F,n}$ .*

*Proof.* The proof is in the spirit of the proof of [5], Proposition 6. Recall that  $\infty_1, \dots, \infty_r$  are the places of  $E$  above  $\infty$ . For  $i = 1, \dots, r$ ,  $v_i$  is the corresponding valuation on  $E$  associated to  $\infty_i$  and  $E_{v_i}$  is the  $v_i$ -adic completion of  $E$ . Moreover,  $N = \prod_{i=1}^r N_i$  where  $N_i \subset E_{v_i}^\times$  is an open subgroup of finite index. For  $i = 1, \dots, r$ , we put:

$$N'_i = N_i \cap N_{E_{v_i}/K_\infty}^{-1}(N(\psi_1) \cap \dots \cap N(\psi_m) \cap N(\eta_1) \dots \cap N(\eta_n)).$$

Then  $N'_i \subset E_{v_i}^\times$  is also an open subgroup of finite index. Let's set:

$$N' = \prod_{i=1}^r N'_i.$$

Let  $\mathfrak{J}_1, \dots, \mathfrak{J}_h \in \mathcal{I}_{\mathcal{S}}(\mathcal{B})$ ,  $\mathfrak{J}_1, \dots, \mathfrak{J}_h \subset O_E$  be a system of representatives of  $\mathcal{I}_{\mathcal{S}}(\mathcal{B})/\mathcal{P}_{\mathcal{S}}(\mathcal{B}, N')$ . Recall that

$$S_{d,\mathcal{S}}(\mathfrak{J}_j, \mathcal{B}, N') = \{aO_E \in \mathcal{P}_{\mathcal{S}}(\mathcal{B}, N'), a\mathfrak{J}_j \subset O_E, \deg(N_{E/K}(a\mathfrak{J}_j)) = d\}.$$

For  $j = 1, \dots, h$ , we define  $V_{d,j}$  to be the following sum:

$$\begin{aligned} \sum_{aO_E \in S_{d,\mathcal{S}}(\mathfrak{J}_j, \mathcal{B}, N')} & \left( \prod_{i=1}^m \rho_i(\gamma_{\psi_i} \pi^{-n(\psi_i)}) \right)^{v_\infty(N_{E/K}(a))} \prod_{i=1}^m \rho_i \left( \frac{N_{E/K}(a)}{\text{sgn}(N_{E/K}(a))} \right)^{n(\psi_i)} \times \\ & \times \left( \prod_{j=1}^n \sigma_j(\langle \gamma_{\eta_j} \rangle^{-y_j}) \right)^{v_\infty(N_{E/K}(a))} \prod_{j=1}^n \sigma_j(\langle N_{E/K}(a) \rangle)^{-y_j n(\eta_j)}. \end{aligned}$$

Then:

$$\begin{aligned} x^d & \sum_{\substack{\mathfrak{J} \in \mathcal{I}_{\mathcal{S}}(\mathcal{B}), \mathfrak{J} \subset O_E \\ \deg N_{E/K}(\mathfrak{J}) = d}} \chi(\mathfrak{J}) N_{E/K}(\mathfrak{J})_{\underline{\rho}, \underline{\psi}} N_{E/K}(\mathfrak{J})_{\underline{\sigma}, \underline{\eta}}^{-s} \\ & = \sum_{j=1}^h \chi(\mathfrak{J}_j) N_{E/K}(\mathfrak{J}_j)_{\underline{\rho}, \underline{\psi}} N_{E/K}(\mathfrak{J}_j)_{\underline{\sigma}, \underline{\eta}}^{-y} V_{d,j}. \end{aligned}$$

Note that  $v_F \circ \rho_i \mid_M$  is a valuation on  $K$ , thus, it is equivalent to the trivial valuation or to the valuation attached to a place of  $K$ . This implies that there exist  $D, D' \in \mathbb{R}$ , such that for all  $d \geq 0$ , for all  $j \in \{1, \dots, h\}$  and for all  $aO_E \in S_{d,\mathcal{S}}(\mathfrak{J}_j, \mathcal{B}, N')$ , we have:

$$v_F \left( \prod_{i=1}^m \rho_i(N_{E/K}(a))^{n(\psi_i)} \right) \geq -Dd - D'.$$

If  $k \in \mathbb{N}$ , we denote by  $\ell_p(k)$  the sum of the digits of  $k$  in base  $p$ . By Lemma 3.2, if  $k_1, \dots, k_n \in \mathbb{N}$  are such that  $k_1 + \dots + k_n \geq 1$  and if

$$d \geq 2g_E[\mathbb{F}_E : \mathbb{F}_q] + d_\infty(N') + d_{\mathcal{S}}(\mathcal{B}) + \frac{(\sum_{i=1}^m n(\psi_i) + \sum_{j=1}^n \ell_p(k_j)) [E : K]_{\text{sep}}}{(p-1)[\mathbb{F}_q : \mathbb{F}_p]},$$

then:

$$\sum_{aO_E \in S_{d,\mathcal{S}}(\mathfrak{J}_j, \mathcal{B}, N')} \prod_{i=1}^m \rho_i \left( \frac{N_{E/K}(a)}{\text{sgn}(N_{E/K}(a))} \right)^{n(\psi_i)} \prod_{j=1}^n \sigma_j(\langle N_{E/K}(a) \rangle)^{k_j} = 0.$$

Now, let  $l \in \mathbb{N}$ , and select  $k_1, \dots, k_n \in \mathbb{N} \setminus \{0\}$  such that:

- i)  $\ell_p(k_j) \leq (l+1)(p-1)$ ,
- ii)  $y_j n(\eta_j) + k_j \equiv 0 \pmod{p^{l+1}\mathbb{Z}_p}$ .

For example, write  $-y_j = \sum_{i \geq 0} u_{i,j} p^i$ ,  $u_{i,j} \in \{0, \dots, p-1\}$ , and set  $k_j = \sum_{i=0}^l u_{i,j} p^i + p^{l+1}$ . Let

$$C = 2g_E[\mathbb{F}_E : \mathbb{F}_q] + d_\infty(N') + d_S(\mathcal{B}) + \frac{(\sum_{i=1}^m n(\psi_i) + p - 1)[E : K]_{\text{sep}}}{(p-1)[\mathbb{F}_q : \mathbb{F}_p]}.$$

Then, if  $d \geq Cn(l+1)$ , we have:

$$\sum_{a \in O_E \in S_{d,S}(\mathcal{J}, \mathcal{B}, N')} \prod_{i=1}^m \rho_i \left( \frac{N_{E/K}(a)}{\text{sgn}(N_{E/K}(a))} \right)^{n(\psi_i)} \prod_{j=1}^n \sigma_j \left( \frac{N_{E/K}(a)}{\text{sgn}(N_{E/K}(a))} \right)^{k_j} = 0.$$

Thus, there exists  $D'' \in \mathbb{R}$  such that:

$$v_F(V_{d,j}) \geq p^{l+1} \text{Inf}\{v_F(\sigma_i(\pi)), i = 1, \dots, n\} - D''d - D'.$$

Note also that  $v_F(\sigma_i(\pi)) > 0$ ,  $i = 1, \dots, n$  (see Remark 2.7). Therefore, if  $d \geq Cn$ , we get:

$$v_F(V_{d,j}) \geq p^{\lfloor \frac{d}{Cn} \rfloor} \text{Inf}\{v_F(\sigma_i(\pi)), i = 1, \dots, n\} - D''d - D'.$$

Therefore, for  $d \geq Cn$ , there exists  $D''' \in \mathbb{R}$  such that we have:

$$\begin{aligned} v_F \left( x^d \sum_{\substack{\mathcal{J} \in \mathcal{I}_S(\mathcal{B}), \mathcal{J} \subset O_E \\ \deg N_{E/K}(\mathcal{J})=d}} \chi(\mathcal{J}) N_{E/K}(\mathcal{J})_{\underline{\rho}, \underline{\psi}} N_{E/K}(\mathcal{J})_{\underline{\sigma}, \underline{\eta}}^{-s} \right) \\ \geq p^{\lfloor \frac{d}{Cn} \rfloor} \text{Inf}\{v_F(\sigma_i(\pi)), i = 1, \dots, n\} - D''d - D'''. \end{aligned}$$

In other words, the valuation  $v_F$  of the coefficient of  $x^{-d}$  in  $\zeta_{S, O_E}(\underline{\rho}, \underline{\psi}; \underline{\sigma}, \underline{\eta}; \chi; s)$  grows exponentially in  $d$  for large  $d$ . Hence, the zeta function  $\zeta_{S, O_E}(\underline{\rho}, \underline{\psi}; \underline{\sigma}, \underline{\eta}; \chi; s)$  continues analytically to an entire function on  $s \in \mathbb{S}_{F,n}$ .  $\square$

**Remark 4.2.** Assume that  $\psi_1, \dots, \psi_m, \eta_1, \dots, \eta_n$ , and  $\chi$  are group homomorphisms, then for  $s = (x; \underline{y}) \in \mathbb{S}_{F,n}$  with  $v_F(x) \ll 0$ , we have:

$$\zeta_{S, O_E}(\underline{\rho}, \underline{\psi}; \underline{\sigma}, \underline{\eta}; \chi; s) = \prod_{\mathfrak{P}} (1 - \chi(\mathfrak{P}) N_{E/K}(\mathfrak{P})_{\underline{\rho}, \underline{\psi}} N_{E/K}(\mathfrak{P})_{\underline{\sigma}, \underline{\eta}}^{-s})^{-1},$$

where  $\mathfrak{P}$  runs through the set of maximal ideals of  $O_E$  that are contained in  $\mathcal{I}_S(\mathcal{B})$ .

Let  $f : \mathbb{Z} \rightarrow 1 + (q-1)\mathbb{Z}$  be a map. For  $i = 1, \dots, m$ , let  $\text{sgn}_i : K_\infty^\times \rightarrow \overline{\mathbb{F}_q}^\times$  be a sign function,  $\psi_i : \mathcal{I}(A) \rightarrow \overline{K}_\infty$  be a map such that:

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^\times, \quad \psi_i(I\alpha) = \psi_i(I) \frac{\alpha}{\text{sgn}_i(\alpha)^{f(\deg I)}},$$

and  $\rho_i : K(\psi_i) \hookrightarrow F$  be an  $\mathbb{F}_p$ -algebra homomorphism (see Definition 2.5 for the definition of  $K(\psi_i)$ ).

For  $j = 1, \dots, n$ , let  $f_j : \mathbb{Z} \rightarrow 1 + (q-1)\mathbb{Z}$  be a map, and let  $\eta_j : \mathcal{I}(A) \rightarrow \overline{K}_\infty^\times$  be a map such that:

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^\times, \quad \eta_j(I\alpha) = \eta_j(I) \frac{\alpha}{\text{sgn}_j'(\alpha)^{f_j(\deg I)}},$$

where  $\text{sgn}'_j : K_\infty^\times \rightarrow \overline{\mathbb{F}}_q^\times$  is a sign function. Finally, for  $j = 1, \dots, n$ , let

$$\sigma_j : K_\infty(\text{sgn}'_j(K^\times), \pi^{\frac{1}{d_\infty}}, \langle \eta_j(I) \rangle, I \in \mathcal{I}(A)) \rightarrow F$$

be a continuous  $\mathbb{F}_p$ -algebra homomorphism.

Let's define  $l_1, \dots, l_m, l'_1, \dots, l'_n \in \mathbb{N}$  as follows:

$$\begin{aligned} \forall i = 1, \dots, m, \forall \alpha \in K^\times, \quad \rho_i(\text{sgn}_i(\alpha)) &= \text{sgn}_i(\alpha)^{p^{l_i}}, \\ \forall j = 1, \dots, n, \forall \zeta \in \mathbb{F}_q^\times, \quad \sigma_j(\zeta) &= \zeta^{p^{l'_j}}. \end{aligned}$$

For  $i = 1, \dots, m$ , there exist  $e_i \in \mathbb{Z}$ ,  $e_i \equiv 1 \pmod{q-1}$ , and  $\zeta_i \in \overline{\mathbb{F}}_q^\times$  such that:

$$\forall a \in K_\infty^\times, \quad \text{sgn}_i(a) = \text{sgn}(a)^{e_i} \zeta_i^{v_\infty(a)}.$$

We assume that:

$$\forall a \in K^\times, \quad \prod_{i=1}^m \left( \frac{\text{sgn}_i(a)}{\text{sgn}(a)} \right)^{e_i p^{l_i}} = 1, \quad \text{and} \quad \prod_{i=1}^m \zeta_i^{p^{l_i}} = 1.$$

Let  $\chi : \mathcal{I}(A) \rightarrow F$  be a map such that:

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^\times, \quad \chi(I\alpha) = \chi(I).$$

Note that  $\psi_1, \dots, \psi_m, \eta_1, \dots, \eta_n$  are admissible maps.

**Corollary 4.3.** *We assume that  $E = K$ ,  $\mathcal{B} = A$ ,  $N = K_\infty^\times$ ,  $\mathcal{S} = \emptyset$ . Let  $m_1, \dots, m_n \in \mathbb{N}$  such that  $p^{l_1} e_1 + \dots + p^{l_m} e_m + m_1 p^{l'_1} + \dots + m_n p^{l'_n} \equiv 0 \pmod{q^{d_\infty} - 1}$ , and  $p^{l_1} e_1 + \dots + p^{l_m} e_m + m_1 p^{l'_1} + \dots + m_n p^{l'_n} \geq 1$ . Then:*

$$\zeta_{\mathcal{S}, A}(\underline{\rho}, \underline{\psi}; \underline{\sigma}, \underline{\eta}; \chi; (\sigma_1(\pi^{\frac{1}{d_\infty}})^{m_1} \dots \sigma_n(\pi^{\frac{1}{d_\infty}})^{m_n}; -m_1, \dots, -m_n)) = 0.$$

*Proof.* For  $i = 1, \dots, n$ , let  $x_i = \sigma_i(\pi^{\frac{1}{d_\infty}})$ . Set  $s = (x_1^{m_1} \dots x_n^{m_n}; -m_1, \dots, -m_n) \in \mathbb{S}_{F, n}$ . Let  $h = |\text{Pic}(A)|$ . Select  $I_1, \dots, I_h \in \mathcal{I}(A)$ ,  $I_1, \dots, I_h \subset A$ , a system of representatives of  $\text{Pic}(A)$ . Let's set:

$$V_{d,j} = \sum_{aA \in S_{d,\mathcal{S}}(I_j, A, N)} \prod_{i=1}^m \rho_i \left( \frac{a}{\text{sgn}_i(a)^{f(\deg I_j)}} \right) (aA)_{\underline{\sigma}, \underline{\eta}}^{(m_1, \dots, m_n)}.$$

Then, we have:

$$V_{d,j} = \sum_{aA \in S_{d,\mathcal{S}}(I_j, A, N)} \prod_{i=1}^m \frac{\rho_i(a)}{\text{sgn}(a)^{p^{l_i} e_i}} \prod_{j=1}^n \sigma_j \left( \frac{a}{\text{sgn}(a) \pi^{v_\infty(a)}} \right)^{m_j}.$$

Since  $p^{l_1} e_1 + \dots + p^{l_m} e_m + m_1 p^{l'_1} + \dots + m_n p^{l'_n} \equiv 0 \pmod{q^{d_\infty} - 1}$  and  $x = x_1^{m_1} \dots x_n^{m_n}$ ,

$$V_{d,j} x^{-d} = - \left( \prod_{j=1}^n x_j^{-m_j \deg I_j} \right) \sum_{\substack{a \in K^\times \\ aA \in S_{d,\mathcal{S}}(I_j, A, N)}} \rho_1(a) \dots \rho_m(a) \sigma_1(a)^{m_1} \dots \sigma_n(a)^{m_n}.$$

The assertion is then a consequence of the proof of Corollary 3.4.  $\square$

#### 4.2. The case of finite places.

Let  $v$  be a finite place of  $K$ . We use freely the notation introduced in Section 2. Let  $\psi_1, \dots, \psi_m, \eta_1, \dots, \eta_n$  be  $m+n$  admissible maps. We assume that

- 1) All the  $m+n$  admissible maps  $\psi_1, \dots, \psi_m, \eta_1, \dots, \eta_n$  are algebraic.
- 2) For  $i = 1, \dots, m$ ,  $n(\psi_i) \in \mathbb{N}$ .
- 3) For  $j = 1, \dots, n$ ,  $n(\eta_j) \in \mathbb{Z}$  and  $\eta_j(\mathcal{I}(A)) \subset \overline{K}_v^\times$ .

For  $i = 1, \dots, m$ , we recall that  $K(\psi_i) = K(\mathbb{F}_\infty, \gamma_{\psi_i} \pi^{-n(\psi_i)}, \psi_i(I), I \in \mathcal{I}(A)) \subset \overline{K}_v$ , and let  $\rho_i : K(\psi_i) \hookrightarrow F$  be an  $\mathbb{F}_p$ -algebra homomorphism. For  $j = 1, \dots, n$ , we set  $K_v(\langle \eta_j \rangle_v) := K_v(\langle \eta_j(I) \rangle_v, I \in \mathcal{I}(A))$  which is a finite extension of  $K_v$  and let  $\sigma_j : K_v(\langle \eta_j \rangle_v) \hookrightarrow F$  be a continuous  $\mathbb{F}_p$ -algebra homomorphism.

Let  $n \geq 1$ , be an integer and recall that:

$$\mathbb{S}_{F,v,n} = F^\times \times \mathbb{Z}_p^n \times \mathbb{Z}^n.$$

For  $s = (x; \underline{y}; \underline{\delta}) \in \mathbb{S}_{F,v,n}$ ,  $I \in \mathcal{I}(A)$ , we set:

$$I_{\underline{\rho}, \underline{\psi}} = \prod_{i=1}^m \rho_i(\psi_i(I)) \in F,$$

$$I_{\underline{\sigma}, \underline{\eta}}^s = x^{\deg I} \prod_{j=1}^n \text{sgn}_v(\eta_j(I))^{\delta_j} \prod_{j=1}^n \sigma_j(\langle \eta_j(I) \rangle_v^{y_j}) \in F^\times.$$

Let  $E/K$  be a finite extension, and let  $\mathcal{B}$  and  $N$  as in Section 3. Let  $\mathcal{S}$  be a finite set, possibly empty, of maximal ideals of  $O_E$  which are relatively prime to  $\mathcal{B}$ . Let  $\mathcal{S}_v$  be the union of  $\mathcal{S}$  and the maximal ideals of  $O_E$  above  $P_v$  and that do not divide  $\mathcal{B}$ . Let  $\chi : \mathcal{I}(\mathcal{B}) \rightarrow F$  be a map such that:

$$\forall \mathfrak{J} \in \mathcal{I}(\mathcal{B}), \forall \mathfrak{J} \in \mathcal{P}(\mathcal{B}, N), \quad \chi(\mathfrak{J}\mathfrak{J}) = \chi(\mathfrak{J}).$$

**Theorem 4.4.** *We consider the function from  $\mathbb{S}_{F,v,n}$  to  $F$  which sends  $s = (x; \underline{y}; \underline{\delta}) \in \mathbb{S}_{F,v,n}$  to the following sum:*

$$\sum_{d \geq 0} \sum_{\substack{\mathfrak{J} \in \mathcal{I}_{\mathcal{S}_v}(\mathcal{B}), \mathfrak{J} \subset O_E \\ \deg(N_{E/K}(\mathfrak{J})) = d}} \chi(\mathfrak{J}) N_{E/K}(\mathfrak{J})_{\underline{\rho}, \underline{\psi}} N_{E/K}(\mathfrak{J})_{\underline{\sigma}, \underline{\eta}}^{-s}.$$

Then, it continues analytically to an entire function on  $s \in \mathbb{S}_{F,v,n}$ .

*Proof.* The proof of the Theorem is similar to the proof of Theorem 4.1. We only give a sketch of the proof. Let  $N'$  be as in the proof of Theorem 4.1.

Let  $\mathfrak{J}_1, \dots, \mathfrak{J}_h \in \mathcal{I}_{\mathcal{S}_v}(\mathcal{B})$ ,  $\mathfrak{J}_1, \dots, \mathfrak{J}_h \subset O_E$  be a system of representatives of  $\mathcal{I}(\mathcal{B})/\mathcal{P}(\mathcal{B}, N')$ . Recall that

$$\mathcal{S}_{d, \mathcal{S}_v}(\mathfrak{J}_j, \mathcal{B}, N') = \{aO_E \in \mathcal{P}_{\mathcal{S}_v}(\mathcal{B}, N'), a\mathfrak{J}_j \subset O_E, \deg(N_{E/K}(a\mathfrak{J}_j)) = d\}.$$

Let  $V_{d,j}$  be the following sum:

$$\sum_{aO_E \in \mathcal{S}_{d, \mathcal{S}_v}(\mathfrak{J}_j, \mathcal{B}, N')} \left( \prod_{i=1}^m \rho_i(\gamma_{\psi_i} \pi^{-n(\psi_i)}) \right)^{v_\infty(N_{E/K}(a))} \text{sgn}_v \left( \prod_{j=1}^n \sigma_j(\gamma_{\psi_j} \pi^{-n(\eta_j)})^{-\delta_j} \right)^{v_\infty(N_{E/K}(a))}$$

$$\times \left( \prod_{j=1}^n \langle \sigma_j(\gamma_{\psi_j} \pi^{-n(\eta_j)}) \rangle_v^{-y_j} \right)^{v_\infty(N_{E/K}(a))} \text{sgn}_v \left( \frac{N_{E/K}(a)}{\text{sgn}(N_{E/K}(a))} \right)^{-\delta_1 n(\eta_1) - \dots - \delta_n n(\eta_n)}$$

$$\times \prod_{i=1}^m \rho_i \left( \frac{N_{E/K}(a)}{\text{sgn}(N_{E/K}(a))} \right)^{n(\psi_i)} \prod_{j=1}^n \langle \sigma_j \left( \frac{N_{E/K}(a)}{\text{sgn}(N_{E/K}(a))} \right) \rangle_v^{-y_j n(\eta_j)}.$$

Note that there exist integers  $D, D' \in \mathbb{R}$ , such that, for  $d \geq 0$ ,  $j \in \{1, \dots, h\}$  and  $aO_E \in S_{d, S_v}(\mathcal{J}_j, \mathcal{B}, N')$ , we have:

$$v_F\left(\prod_{i=1}^m \rho_i(N_{E/K}(a))^{n(\psi_i)}\right) \geq -Dd - D'.$$

By the proof of Lemma 3.2, there exists an integer  $C \geq 1$  such that, for all  $k_1, \dots, k_n \in \mathbb{N}$ , with  $k_1 + \dots + k_n \geq 1$ , for all  $d \geq \frac{C(\sum_{i=1}^n \ell_p(k_i))}{p-1}$  and for all  $\delta \in \mathbb{Z}$ , we have:

$$\begin{aligned} \sum_{aO_E \in S_{d, S_v}(\mathcal{J}_j, \mathcal{B}, N')} \operatorname{sgn}_v\left(\frac{N_{E/K}(a)}{\operatorname{sgn}(N_{E/K}(a))}\right)^{-\delta} \prod_{i=1}^m \rho_i\left(\frac{N_{E/K}(a)}{\operatorname{sgn}(N_{E/K}(a))}\right)^{n(\psi_i)} \times \\ \times \prod_{j=1}^n \sigma_j\left(\frac{N_{E/K}(a)}{\operatorname{sgn}(N_{E/K}(a)) \operatorname{sgn}_v\left(\frac{N_{E/K}(a)}{\operatorname{sgn}(N_{E/K}(a))}\right)}\right)^{k_j} = 0. \end{aligned}$$

Now, let  $l \in \mathbb{N}$ , and select  $k_1, \dots, k_n \in \mathbb{N} \setminus \{0\}$  such that:

- i)  $\ell_p(k_j) \leq (l+1)(p-1)$ ,
- ii)  $y_j n(\eta_j) + k_j \equiv 0 \pmod{p^{l+1}\mathbb{Z}_p}$ .

Then, there exists  $D'' \in \mathbb{R}$ , such that if  $d \geq Cn(l+1)$ , we have:

$$v_F(V_{d,j}) \geq p^{l+1} \operatorname{Inf}\{v_F(\sigma_i(\pi_v)), i = 1, \dots, n\} - D''d - D'.$$

We conclude as in the proof of Theorem 4.1.  $\square$

## 5. EXAMPLES

### 5.1. The case $A = \mathbb{F}_q[\theta]$ .

We take  $\pi = 1/\theta$ . In that case  $d_\infty = 1$ ,  $\operatorname{Pic}(A) = \{1\}$ , and we set

$$A_+ = \{a \in A, a \text{ monic}\} = \{a \in A, \operatorname{sgn}(a) = 1\}.$$

Let  $\varphi : \mathbb{F}_q \rightarrow \mathbb{F}_q, x \mapsto x^p$ . Let  $a = \sum_{i=0}^m \lambda_i \theta^i$ ,  $\lambda_i \in \mathbb{F}_q$ , we set:

$$\forall j \geq 0, \quad \varphi^j(a) = \sum_{i=0}^m \lambda_i^{p^j} \theta^i.$$

Let  $m \geq 0$ ,  $n \geq 1$ , and  $e_1, \dots, e_m, l_1, \dots, l_n \in \mathbb{N}$ . Let  $E/K$  be a finite extension. Let  $E_1/E$  be a finite abelian extension and let  $(\cdot, E_1/E)$  be the Artin symbol. Let  $\chi : \operatorname{Gal}(E_1/E) \rightarrow F^\times$  be a group homomorphism. If  $\mathfrak{P}$  is a maximal ideal of  $O_E$ , we set:

$$\chi(\mathfrak{P}) = \begin{cases} 0 & \text{if } \mathfrak{P} \text{ is ramified in } E_1/E, \\ \chi((\mathfrak{P}, E_1/E)) & \text{otherwise.} \end{cases}$$

If we apply Theorem 4.1, we get:

**Corollary 5.1.** *Let  $x_1, \dots, x_m \in F$ , and let  $z_1, \dots, z_n \in F$  such that  $v_F(z_i) < 0, i = 1, \dots, n$ . Then, for  $s = (x, y) \in \mathbb{S}_{F,n}$ , the following sum converges in  $F$  :*

$$\sum_{d \geq 0} \left( \sum_{\substack{\mathcal{J} \in \mathcal{I}(O_E), \mathcal{J} \subset O_E, \\ N_{E/K}(\mathcal{J}) = aA, \\ a \in A_+, \deg_\theta a = d}} \chi(\mathcal{J}) \prod_{i=1}^m \varphi^{e_i}(a) \Big|_{\theta=x_i} \prod_{j=1}^n \langle \varphi^{l_j}(a) \Big|_{\theta=z_j} \rangle^{y_j} \right) x^d,$$

where for  $y \in F, v_\infty(y) < 0$ , and for  $a \in A_+$ ,  $\langle a(y) \rangle = \frac{a(y)}{y^{\deg_\theta a}}$ .

Let  $X$  be an indeterminate over  $K$ , and write:

$$\forall a \in A, \quad a(\theta + X) = \sum_{k \geq 0} a^{(k)} X^k, a^{(k)} \in A.$$

Then, for all  $k \geq 0$ ,  $\cdot^{(k)} : A \rightarrow A$  is an  $\mathbb{F}_q$ -linear map and we have:

$$\forall l \geq 0, \quad (\theta^l)^{(k)} = \binom{l}{k} \theta^{l-k},$$

where

$$\binom{l}{k} = \begin{cases} 0 & \text{if } l < k, \\ \frac{l!}{k!(l-k)!} \pmod{p} & \text{if } l \geq k. \end{cases}$$

Observe that:

$$\forall i \geq 0, \forall k \geq 0, \forall a \in A, \quad \varphi^i(a^{(k)}) = (\varphi^i(a))^{(k)}.$$

**Proposition 5.2.** *Let  $x_1, \dots, x_m \in F$ ,  $k_1, \dots, k_m \in \mathbb{N}$ , and let  $z_1, \dots, z_n \in F$  such that  $v_F(z_i) < 0, i = 1, \dots, n$ . Then, for  $s = (x, \underline{y}) \in \mathbb{S}_{F,n}$ , the following sum converges in  $F$  :*

$$S(k_1, \dots, k_m) := \sum_{d \geq 0} \left( \sum_{\substack{\mathfrak{J} \in \mathcal{I}(O_E), \mathfrak{J} \subset O_E, \\ N_{E/K}(\mathfrak{J}) = aA, \\ a \in A_+, \deg_\theta a = d}} \chi(\mathfrak{J}) \prod_{i=1}^m \varphi^{e_i}(a^{(k_i)}) \Big|_{\theta=x_i} \prod_{j=1}^n \langle \varphi^{l_j}(a) \Big|_{\theta=z_j} \rangle^{y_j} \right) x^d.$$

Furthermore:

$$\lim_{k_1 + \dots + k_m \rightarrow +\infty} S(k_1, \dots, k_m) = 0.$$

*Proof.* Let  $t_1, \dots, t_m$  be  $m$  indeterminates over  $F$  and let  $\mathbb{T}_m(F)$  be the Tate algebra in the variables  $t_1, \dots, t_m$  with coefficients in  $F$ . Let  $F'$  be the completion of the field of fraction of  $\mathbb{T}_m(F)$ . By Corollary 5.1, the following sum converges in  $F'$  :

$$S := \sum_{d \geq 0} \left( \sum_{\substack{\mathfrak{J} \in \mathcal{I}(O_E), \mathfrak{J} \subset O_E, \\ N_{E/K}(\mathfrak{J}) = aA, \\ a \in A_+, \deg_\theta a = d}} \chi(\mathfrak{J}) \prod_{i=1}^m \varphi^{e_i}(a) \Big|_{\theta=t_i+x_i} \prod_{j=1}^n \langle \varphi^{l_j}(a) \Big|_{\theta=z_j} \rangle^{y_j} \right) x^d.$$

Since  $\mathbb{T}_m(F)$  is closed in  $F'$ , we get  $S \in \mathbb{T}_m(F)$ . For  $i = 1, \dots, m$ , we have:

$$\varphi^{e_i}(a) \Big|_{\theta=t_i+x_i} = \sum_{k \geq 0} \varphi^{e_i}(a^{(k)}) \Big|_{\theta=x_i} t_i^k.$$

But, we have:

$$S = \sum_{k_1, \dots, k_m \in \mathbb{N}} S(k_1, \dots, k_m) t_1^{k_1} \cdots t_m^{k_m},$$

where  $S(k_1, \dots, k_m) \in F$ , and  $\lim_{k_1 + \dots + k_m \rightarrow +\infty} S(k_1, \dots, k_m) = 0$ . Therefore, we get:

$$S(k_1, \dots, k_m) = \sum_{d \geq 0} \left( \sum_{\substack{\mathfrak{J} \in \mathcal{I}(O_E), \mathfrak{J} \subset O_E, \\ N_{E/K}(\mathfrak{J}) = aA, \\ a \in A_+, \deg_\theta a = d}} \chi(\mathfrak{J}) \prod_{i=1}^m \varphi^{e_i}(a^{(k_i)}) \Big|_{\theta=x_i} \prod_{j=1}^n \langle \varphi^{l_j}(a) \Big|_{\theta=z_j} \rangle^{y_j} \right) x^d.$$

□

We refer the interested reader to [5] and [11] for the arithmetic properties of a special case of the above sums.

Recall that  $\mathbb{C}_\infty$  is the completion of  $\overline{K}_\infty$ . Let  $t_1, \dots, t_m, z$  be  $m+1$  indeterminates over  $\mathbb{C}_\infty$ , and let  $\mathbb{T}$  be the Tate algebra in the variables  $t_1, \dots, t_m, z$  with coefficients in  $\mathbb{C}_\infty$ . Let  $F$  be the completion of the field of fractions of  $\mathbb{T}$ . Take in Corollary 5.1  $n = 1$ ,  $E_1 = E = K$ ,  $e_1 = \dots = e_m = l_1 = 0$ ,  $x_i = t_i$ ,  $z_1 = \theta$ ,  $s = (z, y) \in \mathbb{S}_\infty$ , we get that the following infinite sum converges in  $\mathbb{T}$  and is in fact an entire function on  $\mathbb{C}_\infty^{m+1}$ :

$$\sum_{d \geq 0} \left( \sum_{a \in A_+, \deg_\theta a = d} \frac{a(t_1) \cdots a(t_m)}{\langle a \rangle^y} \right) z^d.$$

The above sums were introduced in [28]. In particular, for all  $n \in \mathbb{Z}$ , the sum

$$L(n; \underline{t}; z) := \sum_{d \geq 0} \left( \sum_{a \in A_+, \deg_\theta a = d} \frac{a(t_1) \cdots a(t_m)}{a^n} \right) z^d \in \mathbb{T}$$

defines an entire function on  $\mathbb{C}_\infty^{m+1}$ .

Observe that, by Corollary 3.3 and Corollary 3.4, if  $n \leq 0$ , this sum is finite and furthermore it vanishes at  $z = 1$  if  $m-n \equiv 0 \pmod{q-1}$ ,  $m-n \geq 1$ . Using a special case of Anderson's log-algebraicity Theorem for  $\mathbb{F}_q[\theta]$  ([1], [2], [38] paragraphs 8.9 and 8.10, see also [9], [8], [10] and the forthcoming work of M. Papanikolas [27]), F. Pellarin proved ([28]), for  $m = 1$ , a formula connecting  $L(1; t_1; 1)$  to a special function introduced by G. Anderson and D. Thakur ([3]). This formula reflects an analytic class number formula à la Taelman ([35], [36], [17], [18]) for  $L(1; t; 1)$  (see [8]). Such an analytic class number formula has been generalized in [16] to a larger class of  $L$ -series (in particular for  $L(n; \underline{t}; z)$ , for  $n \geq 1$ ). We also refer the reader to [5], [6], [23], [30], [31], [32], [33], [34], [37] for various arithmetic and analytic properties of the series  $L(n; \underline{t}; z)$ ,  $n \in \mathbb{Z}$ .

Now, we look at the case of finite places. Let  $v$  be a finite place of  $A$  and let  $P$  be the corresponding monic irreducible polynomial in  $A$  of degree  $d$ . Let  $\mathbb{C}_v$  be the completion of an algebraic closure  $K_v$  of the  $v$ -adic completion of  $K$ . Let  $A_v$  be the valuation ring of  $K_v$ . Then:

$$\forall a \in A_v^\times, \quad a = \text{sgn}_v(a) \langle a \rangle_v, \quad \text{with } v(\langle a \rangle_v - 1) \geq 1, \text{sgn}_v(a) \in \mathbb{F}_{q^d}^\times.$$

Note that  $\frac{1}{q^d - 1} \in \mathbb{Z}_p^\times$ , thus:

$$\forall a \in A_v^\times, \quad \langle a \rangle_v = (a^{q^d - 1})^{\frac{1}{q^d - 1}}.$$

Let  $\sigma : K_v \hookrightarrow F$  be a continuous  $\mathbb{F}_p$ -algebra homomorphism. Then:

$$\forall a \in A_+, \quad \sigma(\langle a \rangle_v) = (\sigma(a)^{q^d - 1})^{\frac{1}{q^d - 1}}.$$

Let  $z = \sigma(\theta)$ , then  $v_F(z) \geq 0$ , and  $z \notin \mathbb{F}_{q^d}$ . Furthermore, there exists  $i \geq 0$ , such that:

$$\forall a \in A, \quad \sigma(a) = \varphi^i(a) |_{\theta=z}.$$

Thus:

$$v_F(\varphi^i(P) |_{\theta=z}) > 0.$$

Furthermore:

$$\sigma(\langle a \rangle_v) = ((\varphi^i(a) |_{\theta=z})^{q^d - 1})^{\frac{1}{q^d - 1}} =: \langle \varphi^i(a) |_{\theta=z} \rangle_v.$$

Let  $E/K$  be a finite extension and let  $E_1/E$  be a finite abelian extension. Let  $\chi : \text{Gal}(E_1/E) \rightarrow F^\times$  be a group homomorphism. Let  $\mathcal{S}_P$  be the set of maximal ideals of  $O_E$  above  $P$ . If we apply Theorem 4.4, then, by the proof of Proposition 5.2, we get:

**Corollary 5.3.** *Let  $e_1, \dots, e_m, k_1, \dots, k_m, l_1, \dots, l_n \in \mathbb{N}$ . Let  $x_1, \dots, x_m \in F$ . Let  $z_1, \dots, z_n \in F \setminus \mathbb{F}_{q^d}$  such that  $v_F(\varphi^{l_i}(P)|_{\theta=z_i}) > 0$ . Then for  $\delta \in \mathbb{Z}$  and for  $s = (x, y) \in \mathbb{S}_{F,n}$ , the following sum converges in  $F$ :*

$$\sum_{d \geq 0} \left( \sum_{\substack{\mathfrak{J} \in \mathcal{I}_{\mathcal{S}_P}(O_E), \mathfrak{J} \subset O_E, \\ N_{E/K}(\mathfrak{J}) = aA, \\ a \in A_+, \deg_\theta a = d}} \text{sgn}_v(a)^\delta \chi(\mathfrak{J}) \prod_{i=1}^m \varphi^{e_i}(a^{(k_i)})|_{\theta=x_i} \prod_{j=1}^n \langle \varphi^{l_j}(a)|_{\theta=z_j} \rangle_v^{y_j} \right) x^d.$$

Let  $t_1, \dots, t_m, z$  be  $m+1$  indeterminates over  $\mathbb{C}_v$ , and let  $\mathbb{T}_v$  be the Tate algebra in the variables  $t_1, \dots, t_m, z$  with coefficients in  $\mathbb{C}_v$ . Let  $F_v$  be the completion of the field of fractions of  $\mathbb{T}_v$ . Take in the above Corollary  $n = 1$ ,  $E_1 = E = K$ ,  $e_1 = \dots = e_m = k_1 = \dots = k_m = l_1 = 0$ ,  $x_i = t_i$ ,  $z_1 = \theta$ ,  $s = (z, y) \in \mathbb{S}_\infty$ ,  $\delta \in \mathbb{Z}$ , we get that the following sum converges in  $\mathbb{T}_v$  and is in fact an entire function on  $\mathbb{C}_v^{m+1}$ :

$$\sum_{d \geq 0} \left( \sum_{\substack{a \in A_+, \deg_\theta a = m, \\ a \not\equiv 0 \pmod{P}}} \text{sgn}_v(a)^\delta \frac{a(t_1) \cdots a(t_m)}{\langle a \rangle_v^y} \right) z^d.$$

In particular,  $\forall n \in \mathbb{Z}$ , the sum

$$L_v(n; \underline{t}; z) := \sum_{d \geq 0} \left( \sum_{\substack{a \in A_+, \deg_\theta a = m, \\ a \not\equiv 0 \pmod{P}}} \frac{a(t_1) \cdots a(t_m)}{a^n} \right) z^d \in \mathbb{T}_v$$

defines an entire function on  $\mathbb{C}_v^{m+1}$ . We refer the reader to [7], [10], for various arithmetic properties of “special values” of the series  $L_v(n; \underline{t}; z)$ ,  $n \in \mathbb{Z}$ .

## 5.2. Twisted Goss zeta functions.

In this example,  $A$  is general. Recall that  $\mathbb{C}_\infty$  is the completion of  $\overline{K}_\infty$ . Let  $m \geq 1$  be an integer. Let  $\mathbb{F} \subset \overline{\mathbb{F}}_q$  be a field containing  $\mathbb{F}_\infty$ . Let  $k_m(\mathbb{F})$  be defined as follows:

- if  $m = 1$ ,  $k_1(\mathbb{F}) = \mathbb{F}((y_1))$  where  $y_1$  is an indeterminate over  $\mathbb{F}$ ,
  - if  $m \geq 2$ , let  $y_m$  be an indeterminate over  $k_{m-1}(\mathbb{F})$ , and set  $k_m(\mathbb{F}) = k_{m-1}(\mathbb{F})((y_m))$ .
- Observe that  $\mathbb{F}$  is algebraically closed in  $k_m(\mathbb{F})$ .

Let  $L/K_\infty$  be a finite extension and let  $O_L$  be the valuation ring of  $L$ . Then:

$$O_L = \mathbb{F}_L[[\pi_L]],$$

where  $\mathbb{F}_L$  is the residue field of  $L$ , and  $\pi_L$  is a uniformizer of  $O_L$ . Let's consider the following tensor product:

$$k_m(\mathbb{F}_\infty) \otimes_{\mathbb{F}_\infty} O_L.$$

This ring can be identified naturally with  $k_m(\mathbb{F}_L) \otimes_{\mathbb{F}_L} O_L$ . Any element  $f \in k_m(\mathbb{F}_L) \otimes_{\mathbb{F}_L} O_L$  can be written in a unique way:

$$\sum_{i \geq 0} \alpha_i \otimes \pi_L^i, \quad \alpha_i \in k_m(\mathbb{F}_L).$$

We set:

$$v_\infty(f) = \text{Inf}\{v_\infty(\pi_L)^i, i \in \mathbb{N}, \alpha_i \neq 0\}.$$

Then  $v_\infty$  is a valuation on  $k_m(\mathbb{F}_\infty) \otimes_{\mathbb{F}_\infty} O_L$  which does not depend on the choice of  $\pi_L$ . We observe that  $k_m(\mathbb{F}_\infty) \otimes_{\mathbb{F}_\infty} O_L$  is complete with respect to  $v_\infty$ . Finally denote by  $L_m$  the completion (for  $v_\infty$ ) of the field of fractions of  $k_m(\mathbb{F}_\infty) \widehat{\otimes}_{\mathbb{F}_\infty} O_L$ . If we identify  $1 \otimes \mathbb{F}_L$  with  $\mathbb{F}_L$ , and  $1 \otimes \pi_L$  with  $\pi_L$  which is thus an indeterminate over  $k_m(\mathbb{F}_L)$ , we have:

$$L_m = k_m(\mathbb{F}_L)((\pi_L)).$$

If  $L \subset L'$ , then we have a natural injective map compatible with  $v_\infty$ :

$$L_m \hookrightarrow L'_m.$$

We denote by  $\mathbb{C}_{\infty, m}$  the completion (for  $v_\infty$ ) of the inductive limit  $\varinjlim_{L/K_\infty \text{ finite}} L_m$ . Note that  $\mathbb{C}_\infty, k_m(\overline{\mathbb{F}}_q) \subset \mathbb{C}_{\infty, m}$ , and the residue field of  $\mathbb{C}_{\infty, m}$  is  $k_m(\overline{\mathbb{F}}_q)$ .

Let  $z$  be an indeterminate over  $\mathbb{C}_{\infty, m}$ , we denote by  $\mathbb{T}_z(\mathbb{C}_{\infty, m})$  the Tate algebra in the variable  $z$  with coefficients in  $\mathbb{C}_{\infty, m}$ .

We fix a  $K$ -embedding of  $\overline{K}$  in  $\mathbb{C}_\infty$ . As in Section 2, we consider the Goss admissible map  $[\cdot]$  (Example 2.4). Note that  $V = K([I], I \in \mathcal{I}(A))$  is a finite extension of  $K$ . Recall that  $V$  is viewed as a subfield of  $K_\infty([I], I \in \mathcal{I}(A))$ . Let  $O_V$  be the integral closure of  $A$  in  $V$ . Since, for  $a \in K^\times$ , we have  $[aA] = a/\text{sgn}(a)$ , we deduce that if  $I$  is a non-zero ideal of  $A$  then  $[I] \in O_V$ . Also recall that  $\mathbb{F}_\infty \subset O_V$  is algebraically closed in  $V$ .

Let  $m \in \mathbb{N}$ . For  $i = 1, \dots, m$ , let  $\rho_i : K_\infty([I], I \in \mathcal{I}(A)) \rightarrow \mathbb{F}_\infty((y_i))$  be a continuous  $\mathbb{F}_p$ -algebra homomorphism. Let's observe that  $K_\infty([I], I \in \mathcal{I}(A))/K_\infty$  is a totally ramified extension of degree dividing the l.c.m. of  $p^k$  and  $d_\infty$ , where  $p^k$  is the exact power of  $p$  dividing  $|\text{Pic}(A)|$ , thus the morphisms  $\rho_i$  are described as in Remark 2.7 where  $F$  is replaced by  $\mathbb{F}_\infty((y_i))$ .

Let  $\mathbb{T}_{\rho, z}(\mathbb{C}_\infty)$  be the closure of  $\mathbb{C}_\infty[z][\rho_i(O_V), i = 1, \dots, m]$  in  $\mathbb{T}_z(\mathbb{C}_{\infty, m})$  with respect to the Gauss norm. Let also  $\mathbb{T}_\rho(\mathbb{C}_\infty)$  be the closure of  $\mathbb{C}_\infty[\rho_i(O_V), i = 1, \dots, m]$  in  $\mathbb{C}_{\infty, m}$  with respect to the Gauss norm.

Let's observe that  $\mathbb{T}_{\rho, z}(\mathbb{C}_\infty)$  is an affinoid algebra over  $\mathbb{C}_\infty$  (this is of also the case for  $\mathbb{T}_\rho(\mathbb{C}_\infty)$ ). In fact, select  $\theta \in A \setminus \mathbb{F}_q$ . Then there exist  $v_1, \dots, v_r \in O_V$  such that:

$$O_V = \bigoplus_{j=1}^r \mathbb{F}_\infty[\theta]v_j.$$

For  $i = 1, \dots, m$ , let  $t_i = \rho_i(\theta)$ . Then  $t_1, \dots, t_m, z$  are  $m+1$  indeterminates over  $\mathbb{C}_\infty$ . Let  $\mathbb{T}_{m, z}(\mathbb{C}_\infty)$  be the Tate algebra in the variables  $t_1, \dots, t_m, z$  with coefficients in  $\mathbb{C}_\infty$ . Clearly:

$$\begin{aligned} \mathbb{T}_{m, z}(\mathbb{C}_\infty) &\subset \mathbb{T}_z(\mathbb{C}_{\infty, m}), \\ \mathbb{T}_{\rho, z}(\mathbb{C}_\infty) &= \sum_{1 \leq i_1, \dots, i_m \leq r} \mathbb{T}_{m, z}(\mathbb{C}_\infty) \rho_1(v_{i_1}) \cdots \rho_m(v_{i_m}). \end{aligned}$$

If we apply Theorem 4.1, we get as a special case:

**Corollary 5.4.** *Let  $E/K$  be a finite extension. Let  $y \in \mathbb{Z}_p$ , the following sum converges in  $\mathbb{T}_{\rho, z}(\mathbb{C}_\infty)$ :*

$$\sum_{d \geq 0} \sum_{\substack{\mathfrak{J} \in \mathcal{I}(O_E), \mathfrak{J} \subset O_E \\ \deg N_{E/K}(\mathfrak{J}) = d}} \rho_1([N_{E/K}(\mathfrak{J})]) \cdots \rho_m([N_{E/K}(\mathfrak{J})]) \langle N_{E/K}(\mathfrak{J}) \rangle^y z^d.$$

Furthermore, as a function in  $z$ , it defines an entire function on  $\mathbb{C}_\infty$  with values in  $\mathbb{T}_\rho(\mathbb{C}_\infty)$ .

**Remark 5.5.** Let's suppose that  $A = \mathbb{F}_q[\theta]$ ,  $\pi = \frac{1}{\theta}$ ,  $E = K$ . For  $i = 1, \dots, m$ , let  $\rho_i : K_\infty \rightarrow \mathbb{F}_\infty((y_i))$  be the continuous morphism of  $\mathbb{F}_q$ -algebras such that  $\rho_i(\frac{1}{\theta}) = y_i$ , and let  $t_i = \rho_i(\theta) = \frac{1}{y_i}$ . Then  $\mathbb{T}_{\rho, z}(\mathbb{C}_\infty)$  is equal to the Tate algebra in the variable  $t_1, \dots, t_m, z$  with coefficients in  $\mathbb{C}_\infty$ . Let  $y \in \mathbb{Z}_p$ , then the sum in the above Corollary is in this case:

$$\sum_{d \geq 0} \sum_{a \in A_{+, d}} a(t_1) \cdots a(t_m) \left(\frac{a}{\theta^d}\right)^y z^d.$$

Thus, we recover the  $L$ -series treated in example 5.1.

### 5.3. $A$ -harmonic series attached to some admissible maps.

In this example, we work in the case  $A$  general. Let  $\eta : \mathcal{I}(A) \rightarrow \overline{K}_\infty^\times$  be an admissible map such that  $n(\eta) \in \mathbb{Z}$ . Recall that there exist an open subgroup of finite index  $N(\eta) \subset K_\infty^\times$ , and an element  $\gamma_\eta \in \overline{K}_\infty^\times$ , such that:

$$\forall I \in \mathcal{I}(A), \forall \alpha \in K^\times \cap N(\eta), \quad \eta(I\alpha) = \eta(I) \left(\frac{\alpha}{\text{sgn}(\alpha)}\right)^{n(\eta)} \gamma_\eta^{v_\infty(\alpha)}.$$

Note that there exists an open subgroup of finite index  $N \subset K_\infty^\times$ ,  $N \subset N(\eta)$ , such that:

$$\forall \alpha \in K^\times \cap N, \quad \text{sgn}(\gamma_\eta)^{v_\infty(\alpha)} = 1.$$

Let  $\chi : \mathcal{I}(A) \rightarrow \overline{K}_\infty^\times$  be the map defined by:

$$\forall I \in \mathcal{I}(A), \quad \chi(I) = \pi^{v_\infty(\eta(I)) + \frac{\text{deg } I}{d_\infty} (n(\eta) + v_\infty(\gamma_\eta))} \text{sgn}(\eta(I)).$$

If we set:

$$\mathcal{P} = \{xA, x \in K^\times \cap N\}.$$

Then  $\mathcal{I}(A)/\mathcal{P}$  is a finite abelian group and:

$$\forall I \in \mathcal{I}(A), \forall J \in \mathcal{P}, \quad \chi(IJ) = \chi(I).$$

Let  $n \in \mathbb{Z}$ . For  $s = (x, y) \in \mathbb{S}_\infty$ , let's set:

$$L_{\eta, A}(\chi^n; s) = \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(A), I \subset A \\ \text{deg } I = d}} \chi^n(I) \langle \eta(I) \rangle^{-y} x^{-d}.$$

Then, by Theorem 4.1,  $L_{\eta, A}(\chi^n; \cdot)$  converges on  $\mathbb{S}_\infty$ . Now, let  $x \in \mathbb{C}_\infty^\times$ , and observe that:

$$L_{\eta, A}(\chi^{-n}; (\pi^{\frac{-n}{d_\infty} (n(\eta) + v_\infty(\gamma_\eta))} x, n)) = \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(A), I \subset A \\ \text{deg } I = d}} \eta(I)^{-n} x^{-d}.$$

Thus, for all  $n \in \mathbb{Z}$ , the following function in the variable  $z$  is entire on  $\mathbb{C}_\infty$ :

$$Z_{\eta, A}(n; z) = \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(A), I \subset A \\ \text{deg } I = d}} \eta(I)^{-n} z^d.$$

Observe that, by Lemma 3.2, if  $n n(\eta) \leq 0$ , we have:

$$Z_{\eta, A}(n; z) \in K(\eta(I), I \in \mathcal{I}(A))[z].$$

Let's furthermore assume that  $\eta$  is an algebraic admissible map. Let  $E = K(\eta(I), I \in \mathcal{I}(A))$  which is a finite extension of  $K$ , and let  $E_\infty = E \otimes_M K_\infty$ . Let  $z$  be an indeterminate over  $K_\infty$ , following [10], we denote by  $\mathbb{T}_z(E_\infty)$  the closure of  $E_\infty[z]$  in  $E \otimes_M \widehat{K}_\infty$ , where  $\widehat{K}_\infty$  denotes the completion of the field of

fractions of the Tate algebra in the variable  $z$  with coefficients in  $K_\infty$ . From the above discussion, we get:

**Corollary 5.6.** *Let  $n \in \mathbb{Z}$ . The following sum converges in  $\mathbb{T}_z(E_\infty)$ :*

$$\zeta_{\eta,A}(n; z) := \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(A), I \subset A \\ \deg I = d}} \eta(I)^{-n} z^d.$$

Furthermore, if  $nn(\eta) \leq 0$ , we have:

$$\zeta_{\eta,A}(n; z) \in E[z].$$

Let  $v$  be a finite place of  $K$ , and let  $P_v$  be the maximal ideal of  $A$  associated to  $v$ . Let  $\mathcal{I}(P_v)$  be the group of fractional ideals of  $A$  which are relatively prime to  $P_v$ . Let  $\mathbb{C}_v$  be the  $v$ -adic completion of  $\bar{K}_v$ . Let  $\eta$  be our admissible map introduced in the beginning of the paragraph, and we assume that  $\eta$  is algebraic. Let  $\chi : \mathcal{I}(P_v) \rightarrow \bar{K}_v$  such that:

$$\forall I \in \mathcal{I}(P_v), \quad \chi(I) = \text{sgn}_v(\eta(I)) \pi_v^{v(\eta(I)) + \frac{\deg I}{d_\infty} v(\gamma_\eta)}.$$

Note that there exists an open subgroup of finite index  $N' \subset K_\infty^\times$ ,  $N' \subset N(\eta)$ , such that:

$$\forall \alpha \in K^\times \cap N', \quad \text{sgn}_v(\gamma_\eta)^{v_\infty(\alpha)} = 1.$$

Then:

$$\forall I \in \mathcal{I}(P_v), \forall J \in \mathcal{P}', \quad \chi(IJ) = \chi(I),$$

where

$$\mathcal{P}' = \{xA \in \mathcal{I}(P_v), x \equiv 1 \pmod{P_v}, x \in K^\times \cap N' \cap \text{Ker sgn}\}.$$

Let  $n \in \mathbb{Z}$ . For  $s = (x, y) \in \mathbb{C}_v^\times \times \mathbb{Z}_p$ , let's set:

$$L_{v,\eta,A}(\chi^n; s) = \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(P_v), I \subset A \\ \deg I = d}} \chi^n(I) \langle \eta(I) \rangle_v^{-y} x^{-d}.$$

Then, by Theorem 4.4,  $L_{v,\eta,A}(\chi^n; \cdot)$  converges on  $\mathbb{C}_v^\times \times \mathbb{Z}_p$ . Now, let  $x \in \mathbb{C}_v^\times$ , and observe that:

$$L_{v,\eta,A}(\chi^{-n}; (\pi_v^{\frac{-n}{d_\infty} v(\gamma_\eta)} x, n)) = \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(P_v), I \subset A \\ \deg I = d}} \eta(I)^{-n} x^{-d}.$$

Thus, for all  $n \in \mathbb{Z}$ , the following function in the variable  $z$  is entire on  $\mathbb{C}_v$ :

$$Z_{v,\eta,A}(n; z) = \sum_{d \geq 0} \sum_{\substack{I \in \mathcal{I}(P_v), I \subset A \\ \deg I = d}} \eta(I)^{-n} z^d.$$

By Lemma 3.2, if  $nn(\eta) \leq 0$ , then:

$$Z_{v,\eta,A}(n; z) \in K(\eta(I), I \in \mathcal{I}(A))[z].$$

In particular, if  $P_v = \alpha A$ ,  $\alpha \in N(\eta)$  (there exist infinitely many such maximal ideals by Chebotarev's density Theorem), then, for  $n \in \mathbb{Z}$  we get:

$$Z_{v,\eta,A}(n; z) = \left(1 - \left(\frac{\alpha}{\text{sgn}(\alpha)}\right)^{-nn(\eta)} \gamma_\eta^n \frac{\deg P_v}{d_\infty} z^{\deg P_v}\right) Z_{\eta,A}(n; z).$$

We refer the interested reader to a forthcoming work of the authors dedicated to the arithmetic of such  $A$ -harmonic series ([12]).

## 6. MULTIPLE SEVERAL VARIABLE TWISTED ZETA FUNCTIONS

We briefly explain how the constructions in Section 4 can easily be generalized in the spirit of Thakur's construction of positive characteristic multiple zeta values; the reader interested by the arithmetic of multiple zeta values for  $K = \mathbb{F}_q(\theta)$  is referred to (this list is not exhaustive): [4], [39],[40], [25], [26], [14], [15], [29]. We keep the notation of Sections 3 and 4.

Let  $m \in \mathbb{N}$ ,  $m \geq 1$ . Let  $k_1, \dots, k_m \in \mathbb{N}$  and  $n_1, \dots, n_m \in \mathbb{N} \setminus \{0\}$ . We set:

$$\mathbb{S}_{F, \underline{n}} = \mathbb{S}_{F, n_1} \times \cdots \times \mathbb{S}_{F, n_m}.$$

For  $i = 1, \dots, m$ , let  $\psi_{1,i}, \dots, \psi_{k_i,i}$  be  $k_i$  admissible maps such that  $n(\psi_{j,i}) \in \mathbb{N}$ . For  $i = 1, \dots, m$ ,  $j = 1, \dots, k_i$ , let  $\rho_{j,i} : K(\psi_{j,i}) \rightarrow F$  be an  $\mathbb{F}_p$ -algebra homomorphism. For  $i = 1, \dots, m$ , we set:

$$\forall I \in \mathcal{I}(A), \quad I_{\underline{\rho}_i, \underline{\psi}_i} = \prod_{j=1}^{k_i} \rho_{j,i}(\psi_{j,i}(I)) \in F.$$

We recall the notation in Section 3. Let  $E/K$  be a finite extension. Let  $\mathcal{B} \subset O_E$  be a non-zero ideal. Let  $v_1, \dots, v_r$  be the places of  $E$  above  $\infty$ , and for  $j = 1, \dots, r$ , let  $N_j$  be an open subgroup of finite index of  $E_{v_j}^\times$  ( $E_{v_j}$  is the  $v_j$ -adic completion of  $E$ ), and we set  $N = \prod_{j=1}^r N_j$ . Let  $\mathcal{S}$  be a finite set, possibly empty, of maximal ideals of  $O_E$  which are relatively prime to  $\mathcal{B}$ . For  $i = 1, \dots, m$ , let  $\chi_i : \mathcal{I}(\mathcal{B}) \rightarrow F$  be a map such that:

$$\forall \mathfrak{J} \in \mathcal{I}(\mathcal{B}), \forall \mathfrak{J} \in \mathcal{P}(\mathcal{B}, N), \quad \chi_i(\mathfrak{J}\mathfrak{J}) = \chi_i(\mathfrak{J}).$$

For  $i = 1, \dots, m$ , let  $\eta_{1,i}, \dots, \eta_{n_i,i}$  be  $n_i$  admissible maps with values in  $\overline{K}_\infty^\times$ . For  $i = 1, \dots, m$ ,  $j = 1, \dots, n_i$ , let  $\sigma_{j,i} : K(\langle \eta_{j,i} \rangle) \rightarrow F$  be a continuous  $\mathbb{F}_p$ -algebra homomorphism. For  $i = 1, \dots, m$ , let  $s_i = (x_i; \underline{y}_i) \in \mathbb{S}_{F, n_i}$ , for  $I \in \mathcal{I}(A)$ , we set:

$$I_{\underline{\sigma}_i, \underline{\eta}_i}^{s_i} = x_i^{\deg I} \prod_{j=1}^{n_i} \sigma_{j,i}(\langle \eta_{j,i}(I) \rangle)^{y_{j,i}} \in F^\times.$$

We fix  $i$  such that  $1 \leq i \leq m$ . Let  $d \geq 0$  be an integer, for  $s_i = (x_i; \underline{y}_i) \in \mathbb{S}_{F, n_i}$ , we denote by  $S_{d; \mathcal{S}, O_E}(\underline{\rho}_i, \underline{\psi}_i; \underline{\sigma}_i, \underline{\eta}_i; \chi_i; s_i)$  (or  $S_d(\underline{\rho}_i, \underline{\psi}_i; \underline{\sigma}_i, \underline{\eta}_i; \chi_i; s_i)$  for short) the following sum:

$$\sum_{\substack{\mathfrak{J} \in \mathcal{I}_{\mathcal{S}}(\mathcal{B}), \mathfrak{J} \subset O_E \\ \deg N_{E/K}(\mathfrak{J}) = d}} \chi_i(\mathfrak{J}) N_{E/K}(\mathfrak{J})_{\underline{\rho}_i, \underline{\psi}_i} N_{E/K}(\mathfrak{J})_{\underline{\sigma}_i, \underline{\eta}_i}^{-s_i}.$$

By the proof of Theorem 4.1, we get:

**Corollary 6.1.** *For any  $\underline{s} = (s_1, \dots, s_m) \in \mathbb{S}_{F, \underline{n}}$ , we associate the following sums with values in  $F$ :*

$$\sum_{d \geq 0} S_d(\underline{\rho}_1, \underline{\psi}_1; \underline{\sigma}_1, \underline{\eta}_1; \chi_1; s_1) \sum_{d > d_2 > \dots > d_m \geq 0} \prod_{j=2}^m S_{d_j}(\underline{\rho}_j, \underline{\psi}_j; \underline{\sigma}_j, \underline{\eta}_j; \chi_j; s_j),$$

$$\sum_{d \geq 0} S_d(\underline{\rho}_1, \underline{\psi}_1; \underline{\sigma}_1, \underline{\eta}_1; \chi_1; s_1) \sum_{d \geq d_2 \geq \dots \geq d_m \geq 0} \prod_{j=2}^m S_{d_j}(\underline{\rho}_j, \underline{\psi}_j; \underline{\sigma}_j, \underline{\eta}_j; \chi_j; s_j).$$

Then, these functions continue analytically to an entire function on  $\mathbb{S}_{F, \underline{n}}$ .

We furthermore assume that  $n(\eta_{j,i}) \in \mathbb{Z}$ ,  $j = 1, \dots, n_i$ ,  $i = 1, \dots, m$ , and that all our admissible maps are algebraic. Let  $v$  be a finite place of  $K$ , and let  $P_v$  be the maximal ideal of  $A$  corresponding to the finite place  $v$ .

Let's set:

$$\mathbb{S}_{F, v, \underline{n}} = \mathbb{S}_{F, v, n_1} \times \dots \times \mathbb{S}_{F, v, n_m}.$$

For  $i = 1, \dots, m$ ,  $j = 1, \dots, k_i$ , let  $\rho_{j,i} : K(\psi_i) \hookrightarrow F$  be an  $\mathbb{F}_p$ -algebra homomorphism. For  $i = 1, \dots, m$ , we set as above:

$$\forall I \in \mathcal{I}(A), \quad I_{\underline{\rho}_i, \underline{\psi}_i} = \prod_{j=1}^{k_i} \rho_{j,i}(\psi_{j,i}(I)) \in F.$$

For  $i = 1, \dots, m$ ,  $j = 1, \dots, n_i$ , let  $\sigma_{j,i} : K_v(\langle \eta_i \rangle_v) \hookrightarrow F$  be a continuous  $\mathbb{F}_p$ -algebra homomorphism. For  $i = 1, \dots, m$ , for  $s_i = (x; \underline{y}_i; \underline{\delta}_i) \in S_{F, v, n_i}$ , for  $I \in \mathcal{I}(A)$ , we set:

$$I_{\underline{\sigma}_i, \underline{\eta}_i}^{s_i} = x^{\deg I} \prod_{j=1}^{n_i} \text{sgn}_v(\eta_{j,i}(I))^{\delta_{j,i}} \prod_{j=1}^{n_i} \sigma_{j,i}(\langle \eta_{j,i}(I) \rangle_v^{y_j}) \in F^\times.$$

Let  $E/K$  be a finite extension, and let  $\mathcal{B}, N, \chi_1, \dots, \chi_m$ , and  $\mathcal{S}$  as above. Let  $\mathcal{S}_v$  be the union of  $\mathcal{S}$  and the maximal ideals of  $O_E$  above  $P_v$  and that do not divide  $\mathcal{B}$ .

We fix  $i \in \{1, \dots, m\}$ . Let  $d \geq 0$  be an integer, for  $s_i = (x; \underline{y}_i; \underline{\delta}_i) \in S_{F, v, n_i}$ , we set:

$$S_{v; d}(\underline{\rho}_i, \underline{\psi}_i; \underline{\sigma}_i, \underline{\eta}_i; \chi_i; s_i) = \sum_{\substack{\mathfrak{J} \in \mathcal{I}_{\mathcal{S}}(\mathcal{B}), \mathfrak{J} \subset O_E \\ \deg N_{E/K}(\mathfrak{J}) = d}} \chi_i(\mathfrak{J}) N_{E/K}(\mathfrak{J})_{\underline{\rho}_i, \underline{\psi}_i} N_{E/K}(\mathfrak{J})_{\underline{\sigma}_i, \underline{\eta}_i}^{-s_i}.$$

By the proof of Theorem 4.4, we get:

**Corollary 6.2.** *For any  $\underline{s} = (s_1, \dots, s_m) \in \mathbb{S}_{F, v, \underline{n}}$ , we associate the following sums with values in  $F$ :*

$$\sum_{d \geq 0} S_{v; d}(\underline{\rho}_1, \underline{\psi}_1; \underline{\sigma}_1, \underline{\eta}_1; \chi_1; s_1) \sum_{d \geq d_2 \geq \dots \geq d_m \geq 0} \prod_{j=2}^m S_{v; d_j}(\underline{\rho}_j, \underline{\psi}_j; \underline{\sigma}_j, \underline{\eta}_j; \chi_j; s_j),$$

$$\sum_{d \geq 0} S_{v; d}(\underline{\rho}_1, \underline{\psi}_1; \underline{\sigma}_1, \underline{\eta}_1; \chi_1; s_1) \sum_{d \geq d_2 \geq \dots \geq d_m \geq 0} \prod_{j=2}^m S_{v; d_j}(\underline{\rho}_j, \underline{\psi}_j; \underline{\sigma}_j, \underline{\eta}_j; \chi_j; s_j).$$

Then, these functions continue analytically to an entire function on  $\mathbb{S}_{F, v, \underline{n}}$ .

Let's pursue our basic example 5.3. Let  $\eta : \mathcal{I}(A) \rightarrow \overline{K}_\infty^\times$  be an admissible map such that  $n(\eta) \in \mathbb{Z}$ . Let  $m \geq 1$  be an integer and let  $z_1, \dots, z_m$  be  $m$  indeterminates over  $\mathbb{C}_\infty$ . Let  $\underline{n} = (n_1, \dots, n_m) \in \mathbb{Z}^m$ . Let's define  $Z_{\eta, A}(\underline{n}; \underline{z}) \in K(\eta_i(I), I \in \mathcal{I}(A), i = 1, \dots, m)[z_2, \dots, z_m][[z_1]]$  to be the following sum:

$$\sum_{d \geq 0} \sum_{\substack{I_1, \dots, I_m \in \mathcal{I}(A) \\ I_1, \dots, I_m \subset A \\ d = \deg I_1 > \deg I_2 > \dots > \deg I_m \geq 0}} \frac{1}{\eta_1(I_1)^{n_1} \dots \eta_m(I_m)^{n_m}} z_1^{\deg I_1} \dots z_m^{\deg I_m};$$

let's also define  $Z_{\eta,A}^*(\underline{n}; \underline{z}) \in K(\eta_i(I), I \in \mathcal{I}(A), i = 1, \dots, m)[z_2, \dots, z_m][[z_1]]$  as the following sum:

$$\sum_{d \geq 0} \sum_{\substack{I_1, \dots, I_m \in \mathcal{I}(A) \\ I_1, \dots, I_m \subset A \\ d = \deg I_1 \geq \deg I_2 \geq \dots \geq \deg I_m \geq 0}} \frac{1}{\eta_1(I_1)^{n_1} \dots \eta_m(I_m)^{n_m}} z_1^{\deg I_1} \dots z_m^{\deg I_m}.$$

Then, as in Example 5.3, by Corollary 6.1, we deduce that  $Z_{\eta,A}(\underline{n}; \underline{z})$  and  $Z_{\eta,A}^*(\underline{n}; \underline{z})$  define entire functions on  $\mathbb{C}_\infty^m$ . Furthermore, by Lemma 3.2, if  $n_1 n(\eta) \leq 0$ , then:

$$Z_{\eta,A}(\underline{n}; \underline{z}), Z_{\eta,A}^*(\underline{n}; \underline{z}) \in K(\eta_i(I), I \in \mathcal{I}(A), i = 1, \dots, r)[z_1, z_2, \dots, z_m].$$

Let's observe that, if  $\eta = [\cdot]$ ,  $A = \mathbb{F}_q[\theta]$ , and  $\pi = 1/\theta$ , we recover Thakur's multiple zeta values (as in Example 5.1, one can also recover their deformations in Tate algebras treated in [29]):

$$\sum_{d \geq 0} \sum_{\substack{a_1, \dots, a_m \in A_+ \\ d = \deg a_1 > \dots > \deg a_m \geq 0}} \frac{1}{a_1^{n_1} \dots a_m^{n_m}} \prod_{i=1}^m z_i^{\deg_\theta a_i} \in K[z_2, \dots, z_m][[z_1]].$$

Let's assume that  $\eta$  is algebraic. Let  $v$  be a finite place of  $K$ , and let  $P_v$  be the maximal ideal of  $A$  associated to  $v$ . Let  $\mathcal{I}(P_v)$  be the group of fractional ideals of  $A$  which are relatively prime to  $P_v$ . Let  $\mathbb{C}_v$  be the  $v$ -adic completion of  $\bar{K}_v$ . We define  $Z_{v,\eta,A}(\underline{n}; \underline{z}) \in K(\eta_i(I), I \in \mathcal{I}(A), i = 1, \dots, r)[z_2, \dots, z_m][[z_1]]$  to be the following sum:

$$\sum_{d \geq 0} \sum_{\substack{I_1 \in \mathcal{I}(P_v), I_2, \dots, I_m \in \mathcal{I}(A) \\ I_1, \dots, I_m \subset A \\ d = \deg I_1 > \dots > \deg I_m \geq 0}} \frac{1}{\eta_1(I_1)^{n_1} \dots \eta_m(I_m)^{n_m}} z_1^{\deg I_1} \dots z_m^{\deg I_m};$$

we also define  $Z_{v,\eta,A}^*(\underline{n}; \underline{z}) \in K(\eta_i(I), I \in \mathcal{I}(A), i = 1, \dots, r)[z_2, \dots, z_m][[z_1]]$  to be:

$$\sum_{d \geq 0} \sum_{\substack{I_1 \in \mathcal{I}(P_v), I_2, \dots, I_m \in \mathcal{I}(A) \\ I_1, \dots, I_m \subset A \\ d = \deg I_1 \geq \dots \geq \deg I_m \geq 0}} \frac{1}{\eta_1(I_1)^{n_1} \dots \eta_m(I_m)^{n_m}} z_1^{\deg I_1} \dots z_m^{\deg I_m}.$$

Again, as in Example 5.3, by Corollary 6.2, we deduce that  $Z_{v,\eta,A}(\underline{n}; \underline{z})$  and  $Z_{v,\eta,A}^*(\underline{n}; \underline{z})$  define entire functions on  $\mathbb{C}_v^m$ .

If  $\eta = [\cdot]$ ,  $A = \mathbb{F}_q[\theta]$ , and  $\pi = 1/\theta$ , we have:

$$Z_{v,\eta,A}(\underline{n}; \underline{z}) = \sum_{d \geq 0} \sum_{\substack{a_1, \dots, a_m \in A_+, a_1 \notin P_v \\ d = \deg a_1 > \deg a_2 > \dots > \deg a_m \geq 0}} \frac{1}{a_1^{n_1} \dots a_m^{n_m}} \prod_{i=1}^m z_i^{\deg_\theta a_i}.$$

Note that, if  $n_1, \dots, n_m \leq 0$ , then:

$$Z_{v,\eta,A}(\underline{n}; \underline{z}) \equiv Z_{\eta,A}(\underline{n}; \underline{z}) \pmod{P^{n_1} A[z_1, \dots, z_m]}.$$

Furthermore, for  $\underline{n} \in \mathbb{Z}^m$ , we observe that  $Z_{v,\eta,A}(\underline{n}; \underline{z})$  is the  $P_v$ -adic limit of certain sequences  $Z_{\eta,A}((m_k, n_2, \dots, n_m); \underline{z}) \in K[z_1, \dots, z_m]$ ,  $m_k \leq 0$ , where  $m_k$  is suitably chosen and  $m_k$  converges  $p$ -adically to  $n_1$ . Indeed, let  $\mathbb{T}_{\underline{z}}(K_v)$  be the Tate algebra in the variables  $z_1, \dots, z_m$ , with coefficients in  $K_v$  equipped with the Gauss valuation associated to  $v$  and still denoted by  $v$ . Now, as in the proof of Theorem 4.4, for  $k \geq 0$ , let's select  $-m_k \in \mathbb{N}$  such that:

$$\text{i) } -m_k \equiv -n_1 \pmod{q^{k+1} \mathbb{Z}_p},$$

- ii)  $-m_k \equiv -n_1 \pmod{q^{\deg P_v} - 1}$ ,
- iii)  $\ell_q(-m_k) \leq (k + \deg P_v)(q - 1)$ ,
- iv)  $-m_k \geq q^{k+1}$ .

For example, if  $-n_1 = \sum_{i \geq 0} a_i q^i$ ,  $a_i \in \{0, \dots, q-1\}$ , select  $\delta_k \in \{1, \dots, q^{\deg P_v} - 1\}$  such that  $-n_1 - \sum_{i=0}^k a_i q^i \equiv \delta_k \pmod{q^{\deg P_v} - 1}$ , and set:

$$-m_k = \sum_{i=0}^k a_i q^i + \delta_k q^{(k+1) \deg P_v}.$$

Then:

$$\begin{aligned} v(Z_{\eta,A}((m_k, n_2, \dots, n_m); \underline{z})) - \sum_{d=0}^{k+\deg P_v} \sum_{\substack{a_1, \dots, a_m \in A_+, a_1 \notin P_v \\ d = \deg a_1 > \dots > \deg a_m \geq 0}} \frac{1}{a_1^{n_1} \dots a_m^{n_m}} \prod_{i=1}^m z_i^{\deg_{\theta} a_i}) \\ \geq q^{k+1} - (|n_2| + \dots + |n_m|)(\deg P_v + k). \end{aligned}$$

Thus in  $\mathbb{T}_{\underline{z}}(K_v)$ :

$$Z_{v,\eta,A}(\underline{n}; \underline{z}) = \lim_m Z_{\eta,A}((m_m, n_2, \dots, n_m); \underline{z}).$$

To our knowledge, and in the case  $A = \mathbb{F}_q[\theta]$ , these type of elements  $Z_{v,\eta,A}(\underline{n}; \underline{1}) \in K_v$  have not been studied so far, and it would be very interesting to obtain informations on such type of objects, especially in the spirit of [4].

#### REFERENCES

- [1] G. Anderson, Rank one elliptic  $A$ -modules and  $A$ -harmonic series, *Duke Mathematical Journal* **73** (1994), 491-542.
- [2] G. Anderson, Log-Algebraicity of Twisted  $A$ -Harmonic Series and Special Values of  $L$ -series in Characteristic  $p$ , *Journal of Number Theory* **60** (1996), 165-209.
- [3] G. Anderson, D. Thakur, Tensor powers of the Carlitz module and zeta values, *Annals of Mathematics* **132** (1990), 159-191.
- [4] G. Anderson, D. Thakur, Multizeta values for  $\mathbb{F}_q[t]$ , their period interpretation, and relations between them, *International Mathematics Research Notices* no. 11 (2009), 2038-2055.
- [5] B. Anglès, F. Pellarin, Functional identities for  $L$ -series values in positive characteristic, *Journal of Number Theory* **142** (2014), 223-251.
- [6] B. Anglès, F. Pellarin, Universal Gauss-Thakur sums and  $L$ -series, *Inventiones mathematicae* **200** (2015), 653-669.
- [7] B. Anglès, L. Taelman, with an appendix by V. Bosser, Arithmetic of characteristic  $p$  special  $L$ -values, *Proceedings of the London Mathematical Society* **110** (2015), 1000-1032.
- [8] B. Anglès, F. Pellarin, F. Tavares Ribeiro, with an appendix by F. Demeslay, Arithmetic of positive characteristic  $L$ -series values in Tate algebras, *Compositio Mathematica* **152** (2016), 1-61.
- [9] B. Anglès, F. Pellarin, F. Tavares Ribeiro, Anderson-Stark Units for  $\mathbb{F}_q[\theta]$ , arXiv: 1501.06804 (2015).
- [10] B. Anglès, F. Tavares-Ribeiro, Arithmetic of function fields units, to appear in *Mathematische Annalen*, arXiv:1506.06286 (2015).
- [11] B. Anglès, T. Ngo Dac, F. Tavares Ribeiro, Exceptional Zeros of  $L$ -series and Bernoulli-Carlitz Numbers (with an appendix by B. Anglès, D. Goss, F. Pellarin, F. Tavares Ribeiro), arXiv: 1511.06209 (2015).
- [12] B. Anglès, T. Ngo Dac, F. Tavares Ribeiro, Regulator of Stark units in positive characteristic, in preparation.
- [13] G. Böckle, Global  $L$ -functions over functions fields, *Mathematische Annalen* **323** (2002), 737-795.
- [14] C.-Y. Chang, Linear independence of monomials of multizeta values in positive characteristic, *Compositio Mathematica* **150** (2014), 1789-1808.

- [15] C.-Y. Chang, M. Papanikolas, J. Yu, An effective criterion for Eulerian multizeta values in positive characteristic, arXiv: 1411.0124 (2014).
- [16] F. Demeslay, A class formula for  $L$ -series in positive characteristic, arXiv:1412.3704 (2014).
- [17] J. Fang, Special  $L$ -values of abelian  $t$ -modules, *Journal of Number Theory* **147** (2015), 300-325.
- [18] J. Fang, Equivariant Special  $L$ -values of abelian  $t$ -modules, arXiv: 1503.07243 (2015) .
- [19] D. Goss,  $v$ -adic zeta functions,  $L$ -series and measures for function fields, *Inventiones mathematicae* **55** (1979), 107-116.
- [20] D. Goss, On a new type of  $L$  function for algebraic curves over finite fields, *Pacific Journal of Mathematics* **105** (1983), 143-181.
- [21] D. Goss, Basic Structures of Function Field Arithmetic, Springer, 1996.
- [22] D. Goss, Zeta phenomenology, Noncommutative geometry, arithmetic, and related topics, Johns Hopkins University Press, Baltimore, 2011, 159-182.
- [23] D. Goss, On the  $L$ -series of F. Pellarin, *Journal of Number Theory* **133** (2013), 955-962.
- [24] V. Lafforgue, Valeurs spéciales de fonctions  $L$  en caractéristique  $p$ , *Journal of Number Theory* **129** (2009), 2600-2634.
- [25] J. Lara Rodríguez, D. Thakur, Zeta-like multizeta values for  $\mathbb{F}_q[t]$ , *Indian Journal of Pure and Applied Mathematics* **45** (2014), 787-801.
- [26] J. Lara Rodríguez, D. Thakur, Multizeta shuffle relations for function fields with non rational infinite place, *Finite Fields and their Applications* **37** (2016), 344-356.
- [27] M. Papanikolas, Log-Algebraicity on Tensor Powers of the Carlitz Module and Special Values of Goss  $L$ -Functions, work in progress.
- [28] F. Pellarin, Values of certain  $L$ -series in positive characteristic, *Annals of Mathematics* **176** (2012), 2055-2093 .
- [29] F. Pellarin, A note on multiple zeta values in Tate algebras, arXiv:1601.07348 (2016).
- [30] F. Pellarin, R. Perkins, On certain generating functions in positive characteristic, arXiv: 1409.6006 (2014).
- [31] F. Pellarin, R. Perkins, On twisted  $A$ -harmonic sums and Carlitz finite zeta values, arXiv:1512.05953 (2015).
- [32] R. Perkins, On Pellarin's  $L$ -series, *Proceedings of the American Mathematical Society* **142** (2014), 3355-3368.
- [33] R. Perkins, An exact degree for multivariate special polynomials, *Journal of Number Theory* **142** (2014), 252-263.
- [34] R. Perkins, Explicit formulae for  $L$ -values in positive characteristic, *Mathematische Zeitschrift* **278** (2014), 279-299.
- [35] L. Taelman, A Dirichlet unit theorem for Drinfeld modules, *Mathematische Annalen* **348** (2010), 899-907.
- [36] L. Taelman, Special  $L$ -values of Drinfeld modules, *Annals of Mathematics* **175** (2012), 369-391.
- [37] L. Taelman, A Herbrand-Ribet theorem for function fields, *Inventiones mathematicae* **188** (2012), 253-275.
- [38] D. Thakur, Function Field Arithmetic, World Scientific, 2004.
- [39] D. Thakur, Relations between multizeta values for  $\mathbb{F}_q[t]$ , *International Mathematics Research Notices* no. 12 (2009), 2318-2346.
- [40] D. Thakur, Shuffle relations for function field multizeta values, *International Mathematics Research Notices* no. 11 (2010), 1973-1980.

UNIVERSITÉ DE CAEN NORMANDIE, LABORATOIRE DE MATHÉMATIQUES NICOLAS ORESME, CNRS  
 UMR 6139, CAMPUS II, BOULEVARD MARÉCHAL JUIN, B.P. 5186, 14032 CAEN CEDEX, FRANCE.  
*E-mail address:* bruno.angles@unicaen.fr, tuan.ngodac@unicaen.fr, floric.tavares-ribeiro@unicaen.fr