Persistent Random Walks. II. Functional Scaling Limits

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Abstract We give a complete and unified description – under some stability assumptions – of the functional scaling limits associated with some persistent random walks for which the recurrent or transient type is studied in [1]. As a result, we highlight a phase transition phenomenon with respect to the memory. It turns out that the limit process is either Markovian or not according to – to put it in a nutshell – the rate of decrease of the distribution tails corresponding to the persistent times. In the memoryless situation, the limits are classical strictly stable Lévy processes of infinite variations. However, we point out that the description of the critical Cauchy case fills some lacuna even in the closely related context of Directionally Reinforced Random Walks (DRRWs) for which it has not been considered yet. Besides, we need to introduced some relevant generalized drift – extended the classical one – in order to study the critical case but also the situation when the limit is no longer Markovian. It appears to be in full generality a drift in mean for the Persistent Random Walk (PRW). The limit processes keeping some memory – given by some variable length Markov chain – of the underlying PRW are called arcsine Lamperti anomalous diffusions due to their marginal distribution which are computed explicitly here. To this end, we make the connection with the governing equations for Lévy walks, the occupation times of skew Bessel processes and a more general class modelled on Lamperti processes. We also stress that we clarify some misunderstanding regarding this marginal distribution in the framework of DRRWs. Finally, we stress that the latter situation is more flexible – as in the first paper – in the sense that the results can be easily generalized to a wider class of PRWs without renewal pattern.

Key words Persistent random walk . Functional scaling limit . Arcsine Lamperti marginal distributions . Directionally reinforced random walk . Lévy walk . Anomalous diffusion

Mathematics Subject Classification (2000) 60F17 . 60G50 . 60J15 . 60G17 . 60J05 . 60G22 . 60K20

Contents

1 Introduction

This paper is a continuation of [1] in which recurrence versus transience features of some PRWs are described. More specifically, we still consider a walker $\{S_n\}_{n\geqslant 0}$ on $\mathbb{Z}$, whose jumps are of unit size, and such that at each step it keeps the same direction (or switches) with a probability directly depending on the time already spent in the direction the walker is currently moving. Here we aim at investigating functional scaling limits of the form

$$\left\{\frac{S_{\lfloor ut\rfloor}-\mathbf{m}_s\,ut}{\lambda(u)}\right\}_{t\geqslant 0} \quad \text{or} \quad \left\{\frac{S_{ut}-\mathbf{m}_s\,ut}{\lambda(u)}\right\}_{t\geqslant 0} \xrightarrow[u\to\infty]{\mathscr{L}} \{Z(t)\}_{t\geqslant 0}, \tag{1.1}$$

for which some functional convergence in distribution toward a stochastic process Z holds. The continuous time stochastic process $\{S_t\}_{t\geqslant 0}$ above denotes the piecewise linear interpolation of the discrete time one $\{S_n\}_{n\geqslant 0}$. Due to the sizes of its jumps, the latter is obviously ballistic or sub-ballistic. In particular, the drift parameter $\mathbf{m}_s$ belongs to $[-1,1]$ and that the growth rate of the normalizing positive function $\lambda(u)$ is at most linear. In full generality, we aim at investigating PRWs given by

$$S_0=0 \quad \text{and} \quad S_n:=\sum_{k=1}^{n}X_k, \quad \text{for all } n\geqslant 1, \tag{1.2}$$

where a two-sided process of jumps $\{X_k\}_{n\in\mathbb{Z}}$ in an additive group G is considered. In order to take into account possibly infinite reinforcements, the increment process is supposed to have a finite but possibly unbounded variable memory. More precisely, we assume that it is built from a Variable Length Markov Chain (VLMC) given by some probabilized context tree. To be more explicit, let us give the general construction that fits to our model.

1.1 VLMC structure of increments

Let $\mathscr{L} = \mathscr{A}^{-\mathbb{N}}$ be the set of left-infinite words on the alphabet $\mathscr{A} := \{\mathtt{d},\mathtt{u}\} \simeq \{-1,1\}$ and consider a complete tree on this alphabet, *i.e.* such that each node has 0 or 2 children, whose leaves $\mathscr{C}$ are words (possibly infinite) on $\mathscr{A}$. To each leaf $c \in \mathscr{C}$, called a context, is attached a probability distribution q_c on $\mathscr{A}$. Endowed with this probabilistic structure, such a tree is named a probabilized context tree. The related VLMC is the Markov Chain $\{U_n\}_{n\geqslant 0}$ on $\mathscr{L}$ whose transitions are given by

$$\mathbb{P}(U_{n+1} = U_n\ell | U_n) = q_{\overleftarrow{\text{pref}}\,(U_n)}(\ell), \tag{1.3}$$

where $\overleftarrow{\text{pref}}\,(w) \in \mathscr{C}$ is defined as the shortest prefix of $w = \cdots w_{-1}w_0$ – read from right to left – appearing as a leaf of the context tree. The kth increment X_k of the corresponding PRW is given as the rightmost letter of $U_k := \cdots X_{k-1}X_k$ with the one-to-one correspondence $\mathtt{d} = -1$ (for a descent) and $\mathtt{u} = 1$ (for a rise). Supposing the context tree is infinite, the resulting PRW is no longer Markovian and somehow very persistent. The associated two-sided process of jumps has a finite but possibly unbounded variable memory whose successive lengths are given by the so-called age time process defined for any $n \geqslant 0$ by

$$A_n := \inf\{k \geqslant 1 : X_n \cdots X_{n-k} \in \mathscr{C}\}. \tag{1.4}$$

1.2 Outline of the article

The paper is organized as follows. In the next section, we recall the elementary assumption on S – the double-infinite comb PRW – and we introduce our main Assumption 2.2 together with the quantity $\mathbf{m}_s$ in (1.1) – named the mean drift – and the normalizing function $\lambda(u)$. Thereafter – in Section 3 – we state our main results, namely Theorems 3.1, 3.2 and 3.3. Together, the two latter are refinements and complements of the first one. In Section 4 we make some remarks to step back on these results. The two following Sections 5 and 6 are focused on the proofs in the two fundamental cases. Finally, in the last Section 7 we state two useful and straightforward lemmas allowing several interpretations of the Assumption 2.2 and often implicitly used in the proofs.

2 Framework and assumptions

The model we consider corresponds to the double-infinite comb. Roughly speaking, the leaves – coding the memory – are words on $\{\mathtt{d},\mathtt{u}\} \simeq \{-1,1\}$ of the form $\mathtt{d}^n\mathtt{u}$ and $\mathtt{u}^n\mathtt{d}$. It follows that the probability to invert the current direction depends only on the direction itself and of its present length. In the sequel, we refer to Figure 2.1 that illustrates our notations and assumptions on the so-called double-infinite comb PRW.

2.1 Elementary settings and hypothesis

We recall that this process is characterized by the transition probabilities

$$\alpha_k^{\mathtt{d}} := \mathbb{P}(X_{k+1} = \mathtt{u} | X_k \cdots X_0 = \mathtt{d}^k\mathtt{u}) \quad \text{and} \quad \alpha_k^{\mathtt{u}} := \mathbb{P}(X_{k+1} = \mathtt{d} | X_k \cdots X_0 = \mathtt{u}^k\mathtt{d}). \tag{2.1}$$

where X_k is the kth jump in $\{\mathtt{d},\mathtt{u}\} \simeq \{-1,1\}$ given as the rightmost letter of the left-infinite word U_k – the kth term of underlying double-infinite comb VLMC defined in [2]. The latter conditional probabilities

are invariant by shifting the sequence of increments and thus $\alpha_k^{\mathtt{u}}$ and $\alpha_k^{\mathtt{d}}$ stand respectively for the probabilities of changing direction after k rises and k descents. Therefore, we can write

$$\alpha_k^{\mathtt{u}} = \mathbb{P}\left(S_{n+1} - S_n = 1 | X_n = \mathtt{u}, A_n = k\right) = 1 - \mathbb{P}\left(S_{n+1} - S_n = -1 | X_n = \mathtt{u}, A_n = k\right), \tag{2.2}$$

and vice versa replacing the direction $\mathtt{u}$ by $\mathtt{d}$ and inverting the jumps 1 and -1.

Besides, in order to avoid trivial cases, we assume that S can not be frozen in one of the two directions with a positive probability so that it makes infinitely many U-turns almost surely. We deal throughout this paper with the conditional probability with respect to the event $(X_0, X_1) = (\mathtt{u}, \mathtt{d})$. In other words, the initial time is suppose to be an up-to-down turn. Obviously, there is no loss of generality supposing this and the long time behaviour of S is not affected as well. Therefore, we assume the following

Assumption 2.1 (finiteness of the length of runs)**.** *For any* $\ell \in \{\mathtt{u}, \mathtt{d}\}$,

$$\prod_{k=1}^{\infty}(1 - \alpha_k^{\ell}) = 0 \iff \left(\exists k \geqslant 1 \ \ s.t. \ \ \alpha_k^{\ell} = 1 \quad or \quad \sum_{k=1}^{\infty} \alpha_k^{\ell} = \infty\right). \tag{2.3}$$

In addition, we exclude implicitly the situation when both of the length of runs are almost surely constant. We denote by $\tau_n^{\mathtt{u}}$ and $\tau_n^{\mathtt{d}}$ the length of the nth rise and descent respectively. The two sequences of *i.i.d.* random variables $\{\tau_n^{\mathtt{d}}\}_{n\geqslant 1}$ and $\{\tau_n^{\mathtt{u}}\}_{n\geqslant 1}$ are independent – a renewal property somehow – and it is clear that their distribution tails and truncated means are given for any $\ell \in \{\mathtt{u}, \mathtt{d}\}$ and $t \geqslant 0$ by

$$\mathscr{T}_{\ell}(t) := \mathbb{P}(\tau_1^{\ell} > t) = \prod_{k=1}^{\lfloor t \rfloor}(1 - \alpha_k^{\ell}) \quad \text{and} \quad \Theta_{\ell}(t) := \mathbb{E}[\tau_1^{\ell} \wedge t] = \sum_{n=1}^{\lfloor t \rfloor} \prod_{k=1}^{n-1}(1 - \alpha_k^{\ell}). \tag{2.4}$$

We will also need to consider their truncated second moments defined by

$$V_{\ell}(t) := \mathbb{E}[(\tau_1^{\ell})^2 \mathbb{1}_{\{|\tau_1^{\ell}| \leqslant t\}}]. \tag{2.5}$$

Furthermore, in order to deal with a more tractable random walk with *i.i.d.* increments, we introduce the underlying skeleton random walk $\{M_n\}_{n\geqslant 0}$ associated with the even U-turns – the original walk observed at the random times of up-to-down turns. These observation times form also a random walk (increasing) which is denoted by $\{T_n\}_{n\geqslant 0}$. Note that the expectation $\mathbf{d}_M$ of an increment $Y_k := \tau_k^{\mathtt{u}} - \tau_k^{\mathtt{d}}$ of M is meaningful whenever (at least) one of the persistence times is integrable. By contrast, the mean $\mathbf{d}_T$ of a jump $\tau_k := \tau_k^{\mathtt{u}} + \tau_k^{\mathtt{d}}$ related to T is always defined. Finally, we set when it makes sense,

$$\mathbf{d}_S := \frac{\mathbf{d}_M}{\mathbf{d}_T} = \frac{\mathbb{E}[\tau_1^{\mathtt{u}}] - \mathbb{E}[\tau_1^{\mathtt{d}}]}{\mathbb{E}[\tau_1^{\mathtt{u}}] + \mathbb{E}[\tau_1^{\mathtt{d}}]}, \tag{2.6}$$

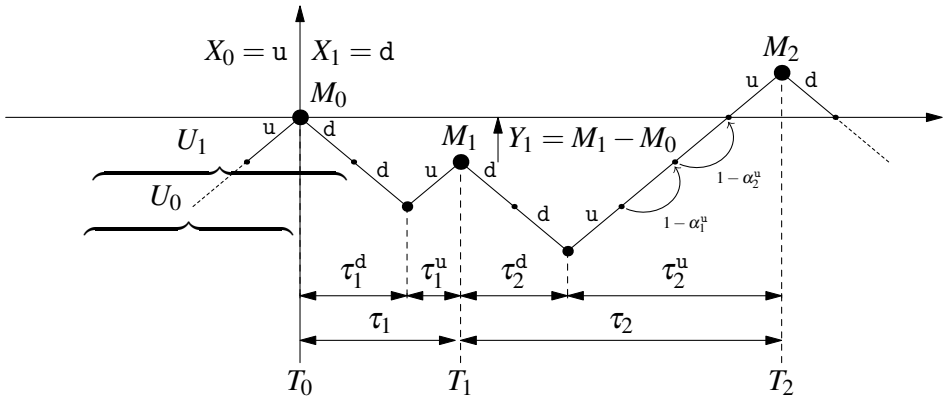

Figure 2.1: A trajectory of S

extended by continuity to ± 1 if only one of the persistence times has a finite expectation. This characteristic naturally arises in the recurrence and transience features and as an almost sure drift of S as it is shown in [1].

2.2 Mean drift and stability assumption

To go further and introduce the suitable centering term $\mathbf{m}_S$ in the scaling limits (1.1) we need to consider, when it exists, the tail balance parameter defined by

$$\mathbf{b}_S := \lim_{t\to\infty} \frac{\mathscr{T}_{\mathtt{u}}(t) - \mathscr{T}_{\mathtt{d}}(t)}{\mathscr{T}_{\mathtt{u}}(t) + \mathscr{T}_{\mathtt{d}}(t)}, \tag{2.7}$$

and set

$$\mathbf{m}_S := \begin{cases} \mathbf{b}_S, & \text{when } \tau_1^{\mathtt{u}} \text{ and } \tau_1^{\mathtt{d}} \text{ are both not integrable,} \\ \mathbf{d}_S, & \text{otherwise.} \end{cases} \tag{2.8}$$

In the light of the L^1-convergence in (4.4) below, this term is naturally called the mean drift of S. Note also that it generalizes $\mathbf{d}_S$ since, when it is well defined,

$$\mathbf{m}_S = \lim_{t\to\infty} \frac{\Theta_{\mathtt{u}}(t) - \Theta_{\mathtt{d}}(t)}{\Theta_{\mathtt{u}}(t) + \Theta_{\mathtt{d}}(t)}. \tag{2.9}$$

A Strong Law of Large Number (SLLN) is established in [2] as well as a (non-functional) CLT under some strong moment conditions on the running times τ_1^{ℓ}. Here the assumptions are drastically weakened. More precisely, our main hypothesis to get functional invariance principles – assumed throughout the article unless otherwise stated – is the following

Assumption 2.2 (α-stability)**.** *The mean drift $\mathbf{m}_S$ is well defined and not extremal, that is $\mathbf{m}_S \in (-1,1)$. Moreover, there exists $\alpha \in (0,2]$ such that*

$$\tau_1^{\mathtt{c}} := (1-\mathbf{m}_S)\tau_1^{\mathtt{u}} - (1+\mathbf{m}_S)\tau_1^{\mathtt{d}} \in \mathrm{D}(\alpha), \tag{2.10}$$

i.e. $\tau_1^{\mathtt{c}}$ belongs to the domain of attraction of an α-stable distribution.

Note that $\tau_1^{\mathtt{c}}$ is a centered random variable when $\mathbf{m}_S = \mathbf{d}_S$. When $\mathbf{m}_S = \mathbf{b}_S$, we will see that it is always – in some sense – well balanced. Obviously, the α-stable distribution in the latter hypothesis is supposed to be non-degenerate. Closely related to stable distributions and their domains of attraction are the notions of regularly varying functions, infinitely divisible distributions and Lévy processes. We refer, for instance, to [3–8] for a general panorama.

2.3 Normalizing functions

For the statement of the functional invariance principle, it is necessary to define the appropriate sub-linear normalizing function $\lambda(u)$ in the functional convergences (1.1). To this end, we introduce the non-negative and non-decreasing functions $\Sigma(t)$ and $\Theta(t)$ respectively given by

$$\Sigma(t)^2 := (1-\mathbf{m}_S)V_{\mathtt{u}}\left(\frac{t}{1-\mathbf{m}_S}\right) + (1+\mathbf{m}_S)V_{\mathtt{d}}\left(\frac{t}{1+\mathbf{m}_S}\right) \text{ and } \quad \Theta(t) := \Theta_{\mathtt{u}}(t) + \Theta_{\mathtt{d}}(t). \tag{2.11}$$

Then we shall see that a relevant choice can be given by setting $\lambda(u) := a \circ s(u)$ with

$$a(u) := \inf\left\{t > 0 : \frac{t^2}{\Sigma(t)^2} \geqslant u\right\} \quad \text{and} \quad s(u) := \inf\left\{t > 0 : \Theta \circ a(t)\, t \geqslant u\right\}. \tag{2.12}$$

3 Statements of the results

We establish – under the α-stability Assumption 2.2 – a general functional invariance principle, stated in its compact form in Theorem 3.1 below. In this one, the Skorohod space of all right-continuous functions $\omega : [0,\infty) \longrightarrow \mathbb{R}$ having left limits (*càdlàg*) is endowed with the M_1-topology (making it a Polish space) and the convergence in distribution with respect to the related Borel σ-fields is denoted by M_1. For more details, we refer especially to [8].

Theorem 3.1. *There exists a non-trivial càdlàg process Z_α such that*

$$\left\{\frac{S_{\lfloor ut\rfloor}-\mathbf{m}_S ut}{\lambda(u)}\right\}_{t\geqslant 0} \quad \text{and} \quad \left\{\frac{S_{ut}-\mathbf{m}_S ut}{\lambda(u)}\right\}_{t\geqslant 0} \xrightarrow[u\to\infty]{\mathrm{M}_1} \{Z_\alpha(t)\}_{t\geqslant 0}. \tag{3.1}$$

Remark 3.1. *We point out that we make the effort to give an unified functional convergence. To enforce the scaling limit, we do not have to know the index of stability α since computing the mean drift $\mathbf{m}_s$ and the normalization function $\lambda(u)$ only involve to observe the truncated mean and second moment of the running times. This could have some statistical interests.*

Furthermore, it turns out that there are mainly four situations according to the position of the index of stability $\alpha \in (0,2]$ with respect to the partition $(0,1) \sqcup \{1\} \sqcup (1,2) \sqcup \{2\}$. More precisely, reading the four latter intervals from the right to the left, the limit stochastic process Z_α can be equal to

1) B – a brownian motion;

2) $S_{\alpha,\beta}$ – a strictly α-stable Lévy process with skewness parameter β;

3) $\mathfrak{C}$ – a symmetric Cauchy process;

4) $\mathscr{S}_\alpha$ – an arcsine Lamperti anomalous diffusion.

The first three situations cover all classical strictly stable Lévy processes having infinite variations and we will specify later what is meant by anomalous diffusion and the arcsine Lamperti terminology, which is undoubtedly the most fruitful situation. Note that the authors in [2] show the convergence of some rescaling toward a kind of non-symmetric generalized telegraph process. In this paper the limit processes are no longer generalizations of the telegraph process since the probabilities of changing directions are fixed. Here actual scaling limits are investigated. Theorem 3.1 is divided and completed below according to the four latter possible situations. In particular, we consider stronger convergence in distribution: the J_1-convergence and its restriction $\mathscr{C}$ to the Wiener space of all continuous functions (see [9] for instance).

3.1 Classical Lévy situation

Here we deepen the case when the limit process is a strictly stable Lévy motion of infinite variation. We shall prove – in the generic situation $\alpha \in (1,2)$ – that up to some scale parameter it is an α-stable Lévy process $S_{\alpha,\beta}$ with skewness parameter

$$\beta = \frac{(1-\mathbf{m}_S)^\alpha(1+\mathbf{b}_S)-(1+\mathbf{m}_S)^\alpha(1-\mathbf{b}_S)}{(1-\mathbf{m}_S)^\alpha(1+\mathbf{b}_S)+(1+\mathbf{m}_S)^\alpha(1-\mathbf{b}_S)}. \tag{3.2}$$

More precisely, we shall see that its symbol $\log \mathbb{E}[e^{iuS_{\alpha,\beta}(1)}]$ is equal to

$$-\frac{(2-\alpha)\Gamma(2-\alpha)}{\alpha}\frac{\sin\left(\frac{\pi}{2}(\alpha-1)\right)}{\alpha-1}\left[1-i\beta\tan\left(\frac{\pi\alpha}{2}\right)\right]|u|^\alpha, \tag{3.3}$$

where Γ is the Gamma function. In particular, its Lévy jump measure is given by

$$\left[\left(\frac{1-\beta}{2}\right)\mathbb{1}_{\{x<0\}}+\left(\frac{1+\beta}{2}\right)\mathbb{1}_{\{x>0\}}\right]\frac{(2-\alpha)}{|x|^{\alpha+1}}\,dx. \tag{3.4}$$

Regarding the case $\alpha=2$, the log-characteristic function of the limit process is still given by the right hand side of (3.3) – whatever the value of β is – and it is nothing but a standard Brownian motion denoted by B. Concerning the last situation $\alpha=1$, the symbol is given by the right hand side of (3.3) with a null skewness parameter $\beta=0$. In other words, it is a symmetric Cauchy process denoted by $\mathfrak{C}$ and characterized by its marginal distribution

$$\mathfrak{C}(1)\sim\frac{1}{2}\frac{1}{(\pi/2)^2+x^2}\,dx. \tag{3.5}$$

As a matter of fact, the skewness parameter (3.2) is associated with the random variable τ_1^{c} introduced in (2.10). This expression will be explained in the last section where equivalent formulations of Assumption 2.2 can be deduced from Lemma 7.1. Besides, it also comes from this lemma – together with Lemma 7.2 and classical results on regular variations in [3] for instance – that $\mathbf{d}_T<\infty$ and $\mathbf{m}_S=\mathbf{d}_S$ when $\alpha\in(1,2)$ but also that there exists a slowly varying function $\Xi_\alpha(u)$ such that

$$a(u)\underset{u\to\infty}{\sim}\Xi_\alpha(u)\,u^{1/\alpha}\quad\text{and}\quad s(u)\underset{u\to\infty}{\sim}\frac{u}{\mathbf{d}_T}, \tag{3.6}$$

In particular, the random variable τ_1^{c} is centered. As far as $\alpha=2$, the latter considerations also hold but with an ultimately non-decreasing slowly varying function $\Xi_2(u)$ since it can be chosen as $\Sigma\circ a(u)$.

When $\alpha=1$, it is possible for both of the running times τ_1^{u} and τ_1^{d} to be integrable or not. In the first situation, the latter estimations are still effective. In the second one, the random variable τ_1^{c} is no longer centered but it is well balanced – the skewness parameter β is equal to zero – in the sight of equality (3.2) since $\mathbf{m}_S=\mathbf{b}_S$. Anyway, when $\alpha=1$, we can formulate the following

Lemma 3.1. *When $\alpha=1$ there exist two slowly varying functions $\Xi_1(u)$ and $D(u)$, the latter being ultimately non-decreasing, such that*

$$a(u)\underset{u\to\infty}{\sim}\Xi_1(u)\,u\quad\textit{and}\quad s(u)\underset{u\to\infty}{\sim}\frac{u}{D(u)},\quad\textit{with}\quad\lim_{u\to\infty}\frac{D(u)}{\Xi_1\circ s(u)}=\infty. \tag{3.7}$$

The proof of these estimates – especially the right hand side of (3.7) which is not a direct consequence of regular variations – is postponed to the end of Section 5.

Theorem 3.2 (classical Lévy situation)**.** *Suppose $\alpha\in[1,2]$, then the scaling limits in (3.1) can be rewritten – in a stronger way – as follows.*

1) If $\alpha=2$ – the Gaussian case – then

$$\left\{\frac{\sqrt{\mathbf{d}_T\,u}}{\Xi_2(u)}\left(\frac{S_{\lfloor ut\rfloor}}{u}-\mathbf{d}_S t\right)\right\}_{t\geqslant0}\underset{u\to\infty}{\overset{\mathrm{J}_1}{\Longrightarrow}}\{B(t)\}_{t\geqslant0},$$

$$\textit{and}\quad\left\{\frac{\sqrt{\mathbf{d}_T}\sqrt{u}}{\Xi_2(u)}\left(\frac{S_{ut}}{u}-\mathbf{d}_S t\right)\right\}_{t\geqslant0}\underset{u\to\infty}{\overset{\mathscr{C}}{\Longrightarrow}}\{B(t)\}_{t\geqslant0}. \tag{3.8}$$

2) If $\alpha\in(1,2)$ – the generic case – then

$$\left\{\frac{\mathbf{d}_T^{1/\alpha}\,u^{1-1/\alpha}}{\Xi_\alpha(u)}\left(\frac{S_{\lfloor ut\rfloor}}{u}-\mathbf{d}_S t\right)\right\}_{t\geqslant0}\underset{u\to\infty}{\overset{\mathrm{J}_1}{\Longrightarrow}}\{S_{\alpha,\beta}(t)\}_{t\geqslant0}. \tag{3.9}$$

3) If $\alpha=1$ – the Cauchy case – then

$$\left\{\frac{D(u)}{\Xi_1\circ s(u)}\left(\frac{S_{\lfloor ut\rfloor}}{u}-\mathbf{m}_S t\right)\right\}_{t\geqslant0}\underset{u\to\infty}{\overset{\mathrm{J}_1}{\Longrightarrow}}\{\mathfrak{C}(t)\}_{t\geqslant0}. \tag{3.10}$$

3.2 Anomalous situation

In this section, we detail the functional convergence (3.1) when $\alpha \in (0,1)$. It turns out that the limit process is no longer a stable Lévy process nor even a Markov process. Roughly speaking, it is – up to a linear and multiplicative term – a random continuous piecewise linear function built from an α-stable subordinator as follows.

1) We attach *i.i.d.* Bernoulli distributions of parameter $(1+\mathbf{b}_S)/2$ to every maximal open intervals in the complementary the range of the α-stable subordinator.

2) According to their success or their failure, the slope on the corresponding jump interval is chosen to be equal to 1 or -1 accordingly.

We refer carefully to Figure 3.2 illustrating some notations and the construction of this process. Again, classical results on regularly varying functions together with Lemma 7.1 and 7.2 apply and we can check that $\mathbf{d}_T = \infty$ and $\mathbf{m}_S = \mathbf{b}_S$ in this section. The reason we used $\mathbf{m}_S$ rather than $\mathbf{b}_S$ or vice versa is motivated by the will to focus on different meanings – a mean drift or a balance term – according to the situations. In the following, we mention [10] concerning classical results about Lévy subordinators and their local times and [11, Chap. 6.] for general Markov processes. Finally, we allude to [12] for the general theory of excursions and Poisson point processes which is strongly connected to these notions.

3.2.1 Preliminaries

Let T_α be an α-stable Lévy subordinator with no drift and recall that it is a *càdlàg* non-decreasing pure jump process. Let $\mathscr{J}$ be its random set of jumps – the random set of discontinuity points. It follows that the closure of the image $\mathscr{R}_\alpha := \{T_\alpha(t) : t \geqslant 0\}$ – the so-called regenerative set or range – is a perfect set of zero Lebesgue measure satisfying

$$\overline{\mathscr{R}_\alpha} = \mathscr{R}_\alpha \sqcup \{T_\alpha(u-) : u \in \mathscr{J}\} \quad \text{and} \quad [0,\infty)\backslash\overline{\mathscr{R}_\alpha} = \bigsqcup_{u\in\mathscr{J}} (T_\alpha(u-), T_\alpha(u)). \tag{3.11}$$

The backward recurrence time (the current life time) and the forward renewal time (the residual life time) processes are respectively defined as

$$G_\alpha(t) := \sup\{s \leqslant t : s \in \mathscr{R}_\alpha\} \quad \text{and} \quad H_\alpha(t) = \inf\{s \geqslant t : s \in \mathscr{R}_\alpha\}. \tag{3.12}$$

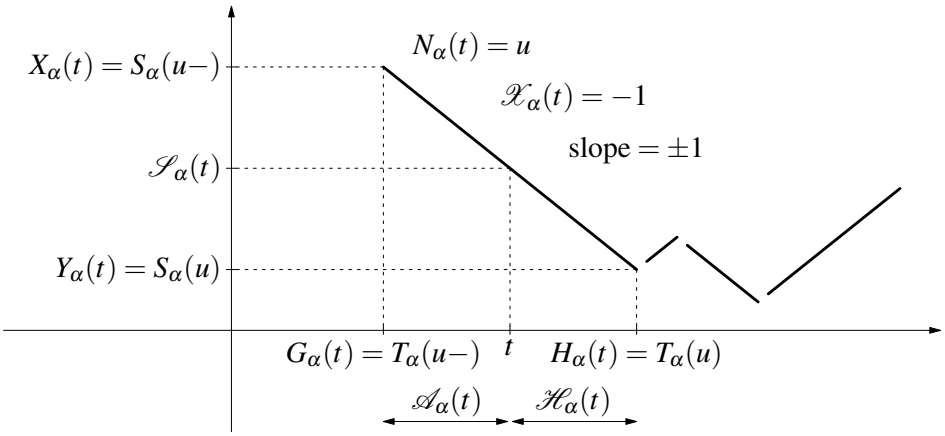

Figure 3.2: Construction of the anomalous diffusion

Those are *càdlàg* stochastic processes and we can check that $G_\alpha(t) = T_\alpha(u-)$ and $H_\alpha(t) = T_\alpha(u)$ when $T_\alpha(u-) \leqslant t < T_\alpha(u)$ whereas $G_\alpha(t) = H_\alpha(t) = T_\alpha(u)$ when $t = T_\alpha(u)$. To go further, we consider the local time of (the inverse of) the stable subordinator T_α defined by

$$N_\alpha(t) := \inf\{s > 0 : T_\alpha(s) > t\} = \sup\{s > 0 : T_\alpha(s) \leqslant t\}. \tag{3.13}$$

Note the usefull switching identity $\{T_\alpha(s) \leqslant t\} = \{s \leqslant N_\alpha(t)\}$ and the fact that N_α is a continuous stochastic process. When $\alpha = 1/2$ it is nothing but the classical local time of a Brownian motion. In full generality, it is is a non-decreasing continuous stochastic process such that the support of the random Stieljes measure dN_α is equal to the range of the subordinator. Then, one has $N_\alpha(t) = u$ when $T_\alpha(u-) \leqslant t \leqslant T_\alpha(u)$ and the first and last renewal time processes can be rewritten as

$$G_\alpha(t) = [T_\alpha^- \circ N_\alpha]^+(t) \quad \text{and} \quad H_\alpha(t) = T_\alpha \circ N_\alpha(t), \tag{3.14}$$

where $F^\pm(t) := F(t\pm)$ is the right or the left continuous version of a function $F(t)$.

Furthermore, it is well-known that stable subordinator can be decomposed as

$$T_\alpha(t) = T_\alpha^{\mathrm{u}}(t) + T_\alpha^{\mathrm{d}}(t) := \left(\frac{1+\mathbf{b}_s}{2}\right)^{1/\alpha} T'_\alpha(t) + \left(\frac{1-\mathbf{b}_s}{2}\right)^{1/\alpha} T''_\alpha(t), \tag{3.15}$$

where T'_α and T''_α are *i.i.d.* with the same distribution as T_α. This decomposition can be obtained intrinsically from T_α by labelling each jump interval (an excursion) $I = (T_\alpha(u-), T_\alpha(u))$ as in [13, pp. 342-343] by *i.i.d.* Rademacher random variables $\mathscr{X}_I$ independent of T_α and of parameter $(1+\mathbf{b}_s)/2$. It follows that $T_\alpha^{\mathrm{u}}(t)$ and $T_\alpha^{\mathrm{d}}(t)$ can be viewed as the sums up to the time t of the jumps $\Delta T_\alpha(u) := T_\alpha(u) - T_\alpha(u-)$ for which the corresponding labels are respectively equal to one and minus one. They are both thinning of the initial subordinator. Coupled with the latter decomposition, we can consider the α-stable Lévy process

$$S_\alpha(t) := T_\alpha^{\mathrm{u}}(t) - T_\alpha^{\mathrm{d}}(t) = \left(\frac{1+\mathbf{b}_s}{2}\right)^{1/\alpha} T'_\alpha(t) - \left(\frac{1-\mathbf{b}_s}{2}\right)^{1/\alpha} T''_\alpha(t). \tag{3.16}$$

This one can be viewed as a pure jump process whose increment $\Delta S_\alpha(u)$ is equal to $\Delta T_\alpha(u)$ or $-\Delta T_\alpha(u)$ according to the label of the corresponding excursion. Following the terminology used in [14] we can introduce the so-called lagging and leading *càdlàg* stochastic processes respectively defined by

$$X_\alpha(t) := [S_\alpha^- \circ N_\alpha]^+(t) \quad \text{and} \quad Y_\alpha(t) := S_\alpha \circ N_\alpha(t). \tag{3.17}$$

It is not difficult to check that $X_\alpha(t) = S_\alpha(u-)$ and $Y_\alpha(t) = S_\alpha(u)$ when $T_\alpha(u-) \leqslant t < T_\alpha(u)$ whereas $X_\alpha(t) = Y_\alpha(t) = S_\alpha(u)$ when $t = T_\alpha(u)$.

3.2.2 Arcsine Lamperti anomalous diffusions

We can now define the so-called Arcsine Lamperti anomalous diffusion by

$$\mathscr{S}_\alpha(t) := X_\alpha(t) + \frac{t - G_\alpha(t)}{H_\alpha(t) - G_\alpha(t)}(Y_\alpha(t) - X_\alpha(t)), \tag{3.18}$$

with $\mathscr{S}_\alpha(t) = X_\alpha(t)$ when $t \in \mathscr{R}_\alpha$. In other words, the so-called anomalous diffusion $\mathscr{S}_\alpha(t)$ is the center of mass of the lagging and leading processes $X_\alpha(t)$ and $Y_\alpha(t)$ with respective weights given by the so-called remaining time and age time processes

$$\mathscr{H}_\alpha(t) := H_\alpha(t) - t \quad \text{and} \quad \mathscr{A}_\alpha(t) = t - G_\alpha(t). \tag{3.19}$$

This kind of construction as already been exposed in [15]. From the self-similarity of T_α we get that the law of $\mathscr{S}_\alpha$ is self-similar of index 1: for any $\lambda > 0$,

$$\{\mathscr{S}_\alpha(\lambda t)\}_{t\geqslant 0} \overset{\mathscr{L}}{=} \{\lambda \mathscr{S}_\alpha(t)\}_{t\geqslant 0}. \tag{3.20}$$

Besides, its law does not depend on the scale parameter chosen to define T_α. To go further, it is continuous and of bounded variation and it can be written as

$$\mathscr{S}_\alpha(t) = \int_0^t \mathscr{X}_\alpha(s)\, ds, \tag{3.21}$$

where $\{\mathscr{X}_\alpha(t)\}_{t\geqslant 0}$ denote the so-called label process taking its values in $\{-1,1\}$ and such that $\mathscr{X}_\alpha(t) := \mathscr{X}_I$ when $t \in I$ – no matter its values on the regenerative set is. We recall that the $\mathscr{X}_I$ are *i.i.d.* Rademacher random variables of parameter $(1+\mathbf{b}_s)/2$ attached on the excursion intervals I. To get the latter integral representation one can see that $\mathscr{S}_\alpha$ is linear on each excursion interval I with a slope given by $\mathscr{X}_I$. Then, noting that there is clearly equality on the random set $\{H_\alpha(t) : t > 0\}$ with Y_α – the so-called overshoot continuous time random walk – we obtain the equality on $[0,\infty)$.

Furthermore, one can see that $\lambda(u) = a \circ s(u)$ is equivalent to $(2-\alpha)(1-\alpha)u/\alpha$ as u goes to infinity and thus we only need to study the asymptotic of normalized processes $t \mapsto S_{ut}/u$.

Theorem 3.3 (anomalous situation)**.** *Suppose $\alpha \in (0,1)$, then the functional convergence (3.1) can be rewritten – in a stronger form – as*

$$\left\{\frac{S_{ut}}{u}\right\}_{t\geqslant 0} \underset{u\to\infty}{\overset{\mathscr{C}}{\Longrightarrow}} \{\mathscr{S}_\alpha(t)\}_{t\geqslant 0}. \tag{3.22}$$

Moreover, the marginal $\mathscr{S}_\alpha(t)$ has a density function $f_t(x)$ on $(-t,t)$ satisfying – in a weak sense – the fractional partial differential equation

$$\left[\frac{1+\mathbf{m}_s}{2}\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)^\alpha + \frac{1-\mathbf{m}_s}{2}\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial t}\right)^\alpha\right] f_t(x) = \frac{1}{\Gamma(1-\alpha)t^\alpha}\left[\frac{1+\mathbf{m}_s}{2}\delta_t(dx) + \frac{1-\mathbf{m}_s}{2}\delta_{-t}(dx)\right], \tag{3.23}$$

In addition, the latter density admits the integral representation

$$f_t(x) = \int_0^t \left[\frac{1+\mathbf{m}_s}{2}\frac{u_\alpha(x-(t-s),s)}{\Gamma(1-\alpha)} + \frac{1-\mathbf{m}_s}{2}\frac{u_\alpha(x+(t-s),s)}{\Gamma(1-\alpha)}\right]\frac{ds}{(1-s)^\alpha}, \tag{3.24}$$

where $u_\alpha(x,t)$ denotes the 0-potential density of (S_α, T_α). Furthermore, this it is – up to an affine transformation – an arsine Lamperti distribution given by

$$f_t(x) = \frac{2\sin(\pi\alpha)}{\pi t}\frac{(t-x)^{\alpha-1}(t+x)^{\alpha-1}}{\mathbf{r}_s(t-x)^{2\alpha} + 2\cos(\pi\alpha)(t+x)^\alpha(t-x)^\alpha + \mathbf{r}_s^{-1}(t+x)^{2\alpha}}, \tag{3.25}$$

with $\mathbf{r}_s := (1+\mathbf{m}_s)/(1-\mathbf{m}_s)$. Also, the stochastic process $(\mathscr{X}_\alpha, \mathscr{A}_\alpha)$ on the state space $\{-1,1\}\times[0,\infty)$ is Markovian

3.2.3 On general PRWs

In [1, Section 4.] the recurrence and transience criteria are extended to a wider class of PRWs – without any renewal assumption – suitable perturbations of the double-infinite comb model when the persistence

times are non-integrable. In the same way, it would be possible – adding some other natural stability assumptions – to deduce the same theorem for these walks as it is stated below.

Consider a double-infinite comb and attach to each finite leaf c another context tree $\mathbb{T}_c$ (possibly trivial) as in Figure 3.3. The leaves of the related graft are denoted by $\mathscr{C}_c$ and this one is endowed with Bernoulli distributions $\{q_l : l \in \mathscr{C}_c\}$ on $\{\mathtt{u},\mathtt{d}\}$. Note that any probabilized context tree on $\{\mathtt{u},\mathtt{d}\}$ can be constructed in this way. We denote by $S^{\mathtt{g}}$ the corresponding PRW. In this case, the random walk is particularly persistent in the sense that the rises and descents are no longer independent. A renewal property may still hold but it is more tedious to expect in general. Let $\underline{S}$ and $\overline{S}$ be the double-infinite comb PRWs with respective transitions

$$\underline{\alpha}_n^{\mathtt{u}} := \sup\{q_c(\mathtt{d}) : c \in \mathscr{C}_{\mathtt{u}^n\mathtt{d}}\}, \quad \underline{\alpha}_n^{\mathtt{d}} = \inf\{q_c(\mathtt{u}) : c \in \mathscr{C}_{\mathtt{d}^n\mathtt{u}}\},$$
$$\text{and} \quad \overline{\alpha}_n^{\mathtt{u}} := \inf\{q_c(\mathtt{d}) : c \in \mathscr{C}_{\mathtt{u}^n\mathtt{d}}\}, \quad \overline{\alpha}_n^{\mathtt{d}} := \sup\{q_c(\mathtt{u}) : c \in \mathscr{C}_{\mathtt{d}^n\mathtt{u}}\}. \quad (3.26)$$

The proof of the following proposition is omitted and follows from comparison results exposed in [1].

Proposition 3.1. *Assume that $\underline{S}$ and $\overline{S}$ satisfy the same Assumption 2.2 with $\alpha \in (0,1)$. To be more precise, we suppose that the associated mean-drifts are equal and non-extremal but also that the corresponding random variables $\underline{\tau}_1^{\mathtt{c}}$ and $\overline{\tau}_1^{\mathtt{c}}$ are both in $\mathrm{D}(\alpha)$, the stable limit distribution being the same by taking the same normalization and centering terms. Then Theorem 3.3 holds for $S^{\mathtt{g}}$.*

For instance, Theorem 3.3 holds when the (non-trivial) grafts are some finite trees in finite number such that the attached Bernoulli distributions are non-degenerated and induce running times in $\mathrm{D}(\alpha)$.

4 Few comments and contributions

After some clarifications on the choice of the normalizing functions – see Section 4.1 – and the connections between the scaling limits associated with the piecewise constant or the linear interpolations in Theorem 3.1, we give some interpretations on various parameters and limits involved in Theorems 3.2 and 3.3 – see Section 4.2. Thereafter – see Section 4.3 – we briefly explain how to extend our results to the extremal mean drift situations. Then – see Section 4.4 – we interpret the essential difference between the limit processes in Theorems 3.2 and 3.3 as a phase transition phenomenon with respect to the memory. Finally, the last Section 4.5 is devoted to the relation with a part of the literature on CTRWs and their connections with the Lévy walks or the DRRWs.

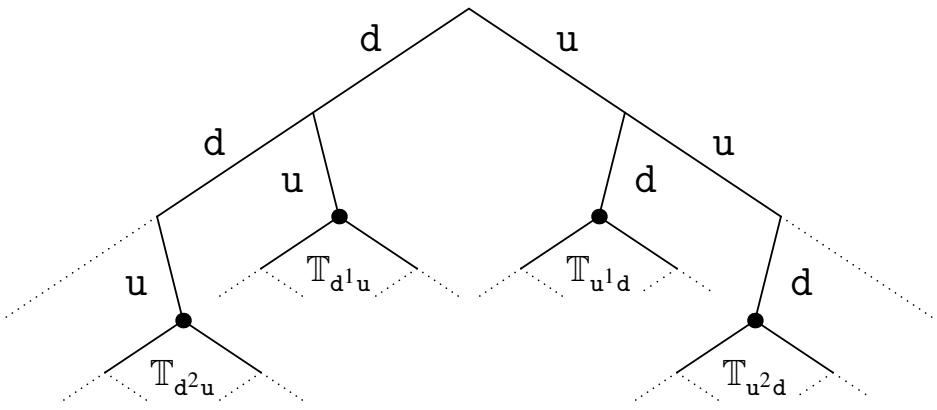

Figure 3.3: Grafting of the double-infinite comb

4.1 Regular variations and Slutsky type arguments

First of all, we shall see that the normalization function and those involved in its construction are regularly varying. More precisely, we can show that they are characterized by the relations

$$\frac{a(u)}{(\Sigma\circ a(u))^2} \underset{u\to\infty}{\sim} u \quad \text{and} \quad \Theta\circ a\circ s(u)\, s(u) \underset{u\to\infty}{\sim} u. \tag{4.1}$$

Then, in virtue of Slutsky type arguments – see [16, Theorem 3.4, p. 55] and [16, Proposition 3.1., p. 57] or [9, Theorem 3.1., p. 27] for instance – together with classical properties about regularly varying functions – see the reference [3] – it comes that in the definition of $\lambda(u)$, along with $\Sigma(t)$ and $\Theta(t)$, the functions $a(u)$ and $s(u)$ can replaced by any other equivalent function in a neighbourhood of infinity to get the same functional convergences.

In this spirit, when the persistence times are both square integrable, one can replace $\Xi_2(u)$ in (3.8) by the the standard deviation of τ_1^{c} to retrieve the CLT shown in [2]. Also, when $\mathbf{d}_T < \infty$, one can replace $D(u)$ by $\mathbf{d}_T$ and thus $\mathbf{m}_s$ by $\mathbf{d}_s$ in (3.10).

4.2 About the three theorems

First, we stress that each of the functional convergences in (3.1) implies the other one. Indeed, due to the tightness criterion [8, Theorem 12.12.3., p. 426] and the form of the oscillation function for the M_1-metric, the tightness of one induces the tightness of the other. Then, the identification of the finite-dimensional distribution follows from $S_t - 1 \leqslant S_{\lfloor t\rfloor} \leqslant S_t + 1$ and the Slutsky lemma. As a consequence, the general Theorem 3.1 is a direct consequence of Theorems 3.2 and 3.3.

Furthermore, we deduce from (3.10) and (3.7) the following interpretation of the so-called mean drift $\mathbf{m}_s$ – appearing here as a drift in probability – since for $\alpha \in [1,2]$, the following holds

$$\lim_{n\to\infty} \frac{S_n}{n} \overset{\mathbb{P}}{=} \mathbf{m}_s. \tag{4.2}$$

Even if the latter convergence is a straightforward consequence of the almost sure convergence exposed in [1] under the assumption that the running times are both integrable, it is completely new otherwise.

To go further, if $\alpha = 1/2$ and $\mathbf{m}_s = 0$ in Theorem 3.3 then one can note that the distribution $f_t(x)$ given in (3.25) is nothing but the push forward image by $x \longmapsto 2tx - 1$ of the classical arcsine distribution, the law of the occupation time of the half-line, up to the time t, of a one-dimensional standard Brownian motion. In full generality, it also appears as the limit of the mean sojourn time of some classical random walks, including a wide class of discrete time processes described by Lamperti in [17]. It is also explained in [18, 19] that it is – up to an affine transformation – the law of the occupation time of the half-line of a skew Bessel process of dimension $2-2\alpha$ with a skewness parameter $(1+\mathbf{m}_s)/2$. Moreover, it is shown in [13, Corollary 4.2., p. 343] that $f_1(x)$ can be represented as the distribution of the random variable

$$\mathbf{D}_\alpha := \frac{T_\alpha^{\mathrm{u}}(1) - T_\alpha^{\mathrm{d}}(1)}{T_\alpha^{\mathrm{u}}(1) + T_\alpha^{\mathrm{d}}(1)}. \tag{4.3}$$

Hence, the marginal convergence at time $t = 1$ in (3.22) can be interpreted as a kind of law of large number in distribution which extends the classical one in [1] – the means in (2.6) being replaced by some canonical positive α-stable random variables.

In addition, the terminology used for the so-called mean drift $\mathbf{m}_s$ is a direct consequence of the latter considerations since in any case $\alpha \in]0,2]$ we get that

$$\lim_{n\to\infty} \frac{S_n}{n} \overset{\mathrm{L}^1}{=} \mathbf{m}_s. \tag{4.4}$$

Besides – among other considerations – it turns out that $(\mathscr{S}_\alpha, \mathscr{X}_\alpha, \mathscr{A}_\alpha)$ is somehow the continuous time counterpart of (S, X, A) and the label process $\mathscr{X}_\alpha$ is in some sense a continuous time VLMC. We recall that the discrete age time process A is introduced in (1.4) as S the corresponding PRW and the two-sided sequence of jumps X. Indeed – as it is explained in [2] – the stochastic process $\{(X_n, A_n)\}_{n\geqslant 0}$ on $\{\mathtt{u},\mathtt{d}\}\times\mathbb{Z}^+$ is also a Markov process.

4.3 On extremal situations

We have exclude in Assumption 2.2 the extremal mean drifts – say $\mathbf{m}_s = 1$ for the example – situation that may arise when $\alpha = 1$ and $\mathbf{d}_\tau = \infty$ or when $\alpha \in (0,1)$. In some sense, this mean – in a first approximation – that the weight of the persistence time $\tau_1^{\mathtt{d}}$ in the scaling limits can be omitted.

As a matter of facts, we can obtain similar functional convergences as (3.10) or (3.22) but there toward the null or the identity process respectively. To this end, it suffices to replace $\Sigma(t)^2$ by $V(t)$ defined in (7.4) in the settings of $a(u)$ and the proofs follows the same lines.

The additional conditions to get non trivial limits are more subtle and related to the negligible persistence time. Roughly speaking, when $\tau_1^{\mathtt{d}}$ belongs to $\mathrm{D}(\gamma)$ with $0<\gamma\leqslant\alpha<1$ – hypothesis (#) – or when it is a relatively stable distribution in the sense of [3, Chap. 8.8, p. 372] – hypothesis $(\star)$ – then the generic form of the functional convergence seems to be

$$\left\{-\frac{u^{1-\frac{\alpha}{\gamma}}}{\Xi(u)}\left(\frac{S_{\lfloor ut\rfloor}}{u}-t\right)\right\}_{t\geqslant 0}\xRightarrow[u\to\infty]{\mathrm{J}_1}\left\{T_\gamma^{\mathtt{d}}\circ N_\alpha^{\mathtt{u}}(t))\right\}_{t\geqslant 0}\quad(\#),$$

$$\text{or}\quad\left\{-\frac{u^{1-\alpha}}{\Xi(u)}\left(\frac{S_{ut}}{u}-t\right)\right\}_{t\geqslant 0}\xRightarrow[u\to\infty]{\mathscr{C}}\left\{N_\alpha^{\mathtt{u}}(t))\right\}_{t\geqslant 0}\quad(\star),\qquad(4.5)$$

where $\Xi(u)$ is slowly varying and $T_\gamma^{\mathtt{d}}$ is a γ-stable subordinator independent – contrary to the non-extremal situation – of an α-local time $N_\alpha^{\mathtt{u}}$. In other words the limit process is an uncoupled CTRW under the assumption (#).

Furthermore, assuming in place of the relative stability that $\tau_1^{\mathtt{d}}\in\mathrm{D}(\gamma)$ with $\gamma\in[1,2]$, the next term in the asymptotic expansion of $(\star)$ is – heuristically – of the order of $S_\gamma^{\mathtt{d}}\circ N_\alpha^{\mathtt{u}}(t)/u^{\alpha/\gamma}$ where $S_\gamma^{\mathtt{d}}$ is a γ-stable Lévy process independent of $N_\alpha^{\mathtt{u}}$.

4.4 Phase transition phenomenon on the memory

It follows from our results and Assumption 2.2 that the limit process and the proper normalizations of the walk given in (3.1) strongly depend on the tail distributions of the length of runs. In fact, we bring out phase transition phenomena concerning the speed of diffusivity and the memory of the limit process.

When the tail distributions are sufficiently light – that is $\alpha\in[1,2]$ in (2.10) – the limit process is Markovian – we lose the long term memory. In addition, the sub-linear normalizing function $\lambda(u)$ will be $(1/\alpha)$-regularly varying as the index of self-similarity of the limit process.

When the tails are sufficiently heavy – that is $\alpha\in(0,1)$ – the limit process is no-longer Markovian. We keep an unbounded memory encoded – in addition to its position – by the "current" label $\mathscr{X}_\alpha(t)$ and the age time $\mathscr{A}_\alpha(t)$. Moreover, the limit process is somehow ballistic and it is self-similar of index 1.

4.5 Around and beyond CTRWs

In full generality, anomalous diffusions – as the lagging process $X_\alpha(t)$ in (3.17) – appear as scaling limits of (possibly coupled) CTRWs named also renewal reward processes or Lévy walks following the context. Those are nothing but classical RW subordinated to a counting process. Their scaling limits and their finite-dimensional distributions has been widely investigated in [14, 20–23] for example. Besides,

closely related to our model, DRRW has been introduced in [24] to model ocean surface wave fields. Those are nearest neighbourhood random walks on $\mathbb{Z}^d$ keeping their directions during random times τ, independently and identically drawn after every change of directions. We refer to [25–27] where are revealed diffusive and super-diffusive behaviours. These walks are intrinsically continuous and can be seen as a linear interpolation of a CTRW – as the "true" Lévy walks studied in [15].

Our model generalizes one dimensional DRRW since asymmetrical transition probabilities α_n^{u} and α_n^{d} lead in general to running times τ_1^{u} and τ_1^{d} with distinct distributions. Even if the mean drift $\mathbf{m}_s$ represents the classical almost sure drift $\mathbf{d}_s$ of the PRW when it is well defined, it appears to be a drift in mean when the persistence times are not integrable and thus also in full generality as it is pointed out by (4.4). This relevant characteristic has not yet been revealed to our knowledge.

Furthermore, it seems that some results in [25, 27] are not entirely true when the persistence times are not integrable, and so it must have some misunderstanding in their proofs. We think more precisely to [25, Theorem 1.3., p. 370] and equation (3.24), p. 3281 relative to the proof of [27, Theorem 2.6., p. 3272]. In fact, with our settings, those results would imply that

$$\frac{S_n}{n} \xRightarrow[n\to\infty]{\mathscr{L}} Y_\alpha(1). \tag{4.6}$$

However, the latter convergence can not be true since $Y_\alpha(1)$ is the overshoot limit of the underlying CTRW and thus it is not compactly supported – contrary to $X_\alpha(1)$. Moreover, in their results it misses the residual term insuring the continuity of the limit process.

As a matter of fact, this kind of subtle mistake – corrected in [22] – has already been made in [21]. In that paper the authors prove some powerful limit theorems concerning coupled CTRW. The outcomes of our interest are [21, Theorem 3.1., p. 738] together with [21, Theorem 3.4., p.744] upgraded with the slight but significant modification in [22].

Finally, we also fill some gaps in the literature. In addition to only investigating symmetric DRRWs in [25, 27], the authors have omitted to treat the Cauchy situation $\alpha = 1$. This case is more delicate as it is explained in Section 5. To be more precise, the choice of the suitable centering term is not clear and the counting process $N(t)$ associated with the time $\tau_1^{\mathrm{d}} + \tau_1^{\mathrm{u}}$ no longer satisfies a SLLN so that we need to revise the proof when $\alpha \in (1,2]$.

Moreover, in the case when $\alpha \in (0,1)$, there is no available proof of a functional convergence (at best a marginal one) and the limit distribution is not explicit to our knowledge. In this paper, we give the explicit density of such marginal limit in (3.25) and we adopt the following complementary approaches, according to the desired aims, in Section 6:

1) We adapt the results in [15, Section 5.] representing the marginal densities of some Lévy walks as the solution of pseudo partial differential equations.

2) We make the link with the excursions theory of α-stable subordinators presented for instance in [13, section 4., pp. 341-343] and [18, Theorem 1].

3) We make the connection with the occupation time of wide class of stochastic processes modelled on those defined by Lamperti in [17] and including $\{X_n X_{n+1}\}_{n\geqslant 0}$.

5 Proof of Theorem 3.2

We begin with the case of a Lévy motion as a limit process. The proofs follow more or less the same lines depending on the index of stability and therefore we only insist on the most difficult – and never exposed – situation when $\alpha = 1$. The proof in the other cases will be only outline.

5.1 Sketch of the proof and preliminaries

Let N be the continuous time counting process associated with T – the increasing discrete time RW. The latter is introduced and represented – as M the skeleton RW – in Section 2.1. More precisely, this renewal process is defined by

$$N(t) := \max\{n \geqslant 0 : T_n \leqslant t\} = \inf\{n \geqslant 0 : T_{n+1} > t\}. \tag{5.1}$$

The main idea of the proofs when $\alpha \in (1,2]$ consists of taking advantage of the decomposition

$$S_{\lfloor ut \rfloor} - \mathbf{m}_s ut =: \left(M_{N(ut)} - \mathbf{m}_s T_{N(ut)}\right) + R(ut) =: C_{N(ut)} + R(ut), \tag{5.2}$$

where $\{C_n\}_{n\geqslant 0}$ denotes the RW whose jumps are distributed as the random variable $\tau_1^{\mathtt{c}}$ given in (2.10) and $\{R(v)\}_{v\geqslant 0}$ is the corresponding residual continuous time process. Note that

$$|R(v)| \leqslant (1-\mathbf{m}_s)\tau^{\mathtt{u}}_{N(v)+1} + (1+\mathbf{m}_s)\tau^{\mathtt{d}}_{N(v)+1}. \tag{5.3}$$

Actually, the proofs of (3.8) and (3.9) is organized as follows.

Step 1. Direct consequence of Assumption 2.2 and estimations (7.8) together with classical results on α-stable distributions we observe that C belongs to the domain of attraction of the α-stable Lévy process $S_{\alpha,\beta} - B$ when $\alpha = 2$ – with the normalization function given by $a(u)$.

Step 2. Applying the weak law of large numbers to T, we show that $\{N(ut)/s(u)\}_{t\geqslant 0}$ converge in probability towards the identity process $t \mapsto t$ as u goes to infinity.

Step 3. We prove that the normalization of the residual process $\{R(ut)/a(u)\}_{t\geqslant 0}$ converges in probability towards the null process $t \mapsto 0$ for u large enough.

Applying a classical continuous mapping argument as in [9, Sect. 17., Chap. 3., pp. 143-150] for instance – or to be more explicit by using [8, Theorem 13.2.2., p. 430, Chap. 13.] coupled with the continuous mapping [8, Theorem 3.4.3., p. 86, Chap. 3.] – we can deduce the required scaling limits with the normalizing function $\lambda(u) = a \circ s(u)$ and this ends the proof when $\alpha \in (1,2]$ assuming the three latter steps.

5.2 Focus on the Cauchy situation

Unfortunately, in the case when $\alpha = 1$, the Step 1. is not true – with the corresponding limit $\mathfrak{C}$ – if the persistence times are both integrable $\mathbf{d}_T < \infty$. Besides, we also need to be careful about the Step 2. if $\mathbf{d}_T = \infty$ since the weak law of large numbers does not hold in that case. Therefore, we need to adapt the reasoning above according the two latter situations.

First, when the persistence times are both of infinite mean, we can see that the proof follows exactly the same steps as previously by noting that Step 2. remains valid. Indeed, in any case T is relatively stable in the sense of [3, Chap. 8.8, p. 372] with the inverse of $u \mapsto s(u)$ as normalizing function.

Lemma 5.1 (Step 2.)**.** *The following convergences hold in probability: for any $v \geqslant 0$,*

$$\sup_{0\leqslant t\leqslant v} \left| \frac{N(ut)}{s(u)} - t \right| \xrightarrow[u\to\infty]{\mathbb{P}} 0. \tag{5.4}$$

Proof. Using the switching identity we first remark that it suffices to show that

$$\sup_{0\leqslant t\leqslant v}\left|\frac{T_{\lfloor s(u)t\rfloor}}{u}-t\right|\xrightarrow[u\to\infty]{\mathbb{P}}0. \tag{5.5}$$

When $\mathbf{d}_T<\infty$ the convergence in (5.5) at any fixed time t but without the supremum follows immediately from the weak law of large numbers since $s(u)$ can be chosen to be equal to $u/\mathbf{d}_T$.

When $\mathbf{d}_T=\infty$, the same marginal convergence is a direct consequence of the considerations on the tail distribution $\mathscr{T}(t)$ of T in Lemma 7.1 and classical results on stable distributions – especially [28, p. 174] – which imply the convergence in distribution as u tends to infinity of

$$\frac{T_{\lfloor ut\rfloor}}{a(u)}-\frac{\Theta\circ a(u)\,u}{a(u)}t. \tag{5.6}$$

Thus, in both cases, it only remains to prove the local uniform convergence in probability. To this end, it is well-known for stable distributions that the convergence of one marginal distribution implies its functional counterpart. Here the same is true. One way to prove this is to use the characteristics of semi-martingales which are in addition processes with independent increments. More precisely, by using [29, Theorem 2.52, Chap. VII.2, p. 409] and [29, Theorem 3.4, Chap. VII.3, p. 414] we only need to prove that the characteristics $(b_u,\tilde{c}_u,\nu_u)$ relative to a truncation function h of $t\mapsto T_{\lfloor s(u)t\rfloor}/u$ tends to $(t\mapsto t,0,0)$ as u tends to infinity uniformly on compact subset. Following [29, Theorem 3.11, Chap. II.3, p. 94] and [29, Equation 3.18, Chap. II.3, p. 96] it turns out that

$$b_u(t)=\mathbb{E}\left[h\left(\frac{\tau_1}{u}\right)\right]\lfloor ut\rfloor,\quad \tilde{c}_u(t)=\mathbb{V}\left[h\left(\frac{\tau_1}{u}\right)\right]\lfloor ut\rfloor\quad\text{and}\quad \nu_u([0,t]\times g):=\mathbb{E}\left[g\left(\frac{\tau_1}{u}\right)\right]\lfloor ut\rfloor, \tag{5.7}$$

where τ_1 is a jump of T and g is any bounded continuous function bounded by $x\mapsto x^2$ in a neighbourhood of the origin. Then the punctual convergence of this triplet follows from the convergence for $t=1$. Furthermore, their uniform convergence on compact sets are obvious and we deduce the convergence in distribution of the associated processes in the Skorokhod space endowed with the J_1-Borel σ-field. Since the convergence in distribution to a constant in metric spaces implies the convergence in probability, the lemma is proved. □

By contrast, when the persistence times are both integrable, we will see that we need to adjust the first step and more precisely the decomposition (5.2) involving the jump distribution $\tau_1^{\mathfrak{c}}$ which is no longer well balanced. This dichotomy appears in the choice of the centering term in the classical convergence towards a symmetric Cauchy process.

Lemma 5.2 (Step 1. when $\alpha=1$)**.** *Let W be a random walk whose jump w_1 belongs to the domain of attraction of a symmetric Cauchy distribution. Then the centering term can be chosen to be equal to zero, when w_1 is not integrable, and to the drift otherwise, in such way that*

$$\left\{\frac{W_{\lfloor ut\rfloor}}{k(u)}\right\}_{t\geqslant 0}\quad or\quad \left\{\frac{W_{\lfloor ut\rfloor}-\mathbb{E}[w_1]\,ut}{k(u)}\right\}_{n\geqslant 0}\xrightarrow[u\to\infty]{\mathrm{J}_1}\{\mathfrak{C}(t)\}_{t\geqslant 0}, \tag{5.8}$$

according to the situations. Here we denote by $k(u)$ the suitable normalizing function so that the limit process is the symmetric Cauchy process defined previously.

Proof. First, we can show by using [28, pp. 170-174] that a suitable centering term (in the numerator above) to get the marginal convergence at time $t=1$ above is

$$b(u):=u\left(\int_0^{k(u)}\mathbb{P}(+w_1>s)ds-\int_0^{k(u)}\mathbb{P}(-w_1>s)ds\right), \tag{5.9}$$

which can be rewritten when w_1 is integrable as

$$b(u) := u\left[\mathbb{E}[w_1] - \left(\int_{k(u)}^{\infty} \mathbb{P}(w_1 > s)ds - \int_{k(u)}^{\infty} \mathbb{P}(-w_1 > s)ds\right)\right]. \tag{5.10}$$

In fact, the convergence holds when u runs through the integers in most of the papers but since $b(u)$ and $k(u)$ are regularly varying (indexes 0 and 1 respectively) a Slutsky type argument implies the convergence for u along the real numbers. Besides, it is well known that the functional convergence is a consequence of the marginal at $t = 1$.

To go further, we can see that for t sufficiently large there exist $\lambda^{\pm}(t)$ such that

$$\int_t^{\infty} \mathbb{P}(\pm w_1 > s)ds = \frac{1}{2}\int_{\lambda^{\pm}(t)}^{\infty} \mathbb{P}(|w_1| > s)ds \quad \text{or} \quad \int_0^{t} \mathbb{P}(\pm w_1 > s)ds = \frac{1}{2}\int_0^{\lambda^{\pm}(t)} \mathbb{P}(|w_1| > s)ds, \tag{5.11}$$

depending on whether w_1 is integrable or not. Moreover, since the right tail and the left tail of w_1 are well balanced, standard results on regularly functions (in particular those of slowly variations) and their inverse allow us to see that $\lambda^{\pm}(t)/t$ tend to 1 as t goes to infinity. Furthermore, noting that the two-sided tail of w_1 is regularly varying of index 1, the de Haan theory applies – especially [3, Theorem 3.7.3, pp. 162-163] coupled with [3, Theorem 3.1.16, p. 139] – and with (5.11) it implies that

$$\int_u^{\infty} \mathbb{P}(w_1 > s)ds - \int_u^{\infty} \mathbb{P}(-w_1 > s)ds \quad \text{or} \quad \int_0^{u} \mathbb{P}(w_1 > s)ds - \int_0^{u} \mathbb{P}(-w_1 > s)ds, \tag{5.12}$$

depending on whether w_1 is integrable or not, is negligible with respect to $u\mathbb{P}(|w_1| > u)$ as u goes to infinity. To conclude, it suffices to note that $u\mathbb{P}(|w_1| > k(u))$ is equivalent to 1 in a neighbourhood of infinity and thus by a Slutsky argument we deduce convergences (5.8). □

Therefore, when the persistence times are both of infinite mean, one can see the first step above holds with a symmetric Cauchy process for limit since τ_1^{c} is well balanced and the centering term is null. This is not the case when the running times are both integrable. To overcome those difficulties, we write in place of the decomposition (5.2) the more adapted relation

$$S_{\lfloor ut\rfloor} - \mathbf{m}_s ut =: \left(S_{\lfloor ut\rfloor} - \mathbf{b}_s ut\right) - (\mathbf{d}_s - \mathbf{b}_s)ut =: C^{\circ}_{N(ut)} - (\mathbf{d}_s - \mathbf{b}_s)ut + R^{\circ}(ut). \tag{5.13}$$

We recall that $\mathbf{m}_s = \mathbf{d}_s$ in this situation. Here C° denotes the RW associated with the (well balanced) jump $\tau_1^{\circ} := (1 - \mathbf{b}_s)\tau_1^{\mathrm{u}} - (1 + \mathbf{b}_s)\tau_1^{\mathrm{d}}$ and R° is a residual process satisfying the same upper bound as the previous R replacing $\mathbf{m}_s$ by $\mathbf{b}_s$. Note that τ_1° is in the domain of attraction of a symmetric Cauchy distribution – a direct consequence of Lemma 7.1 – and has for expectation

$$\mathbb{E}[\tau_1^{\circ}] = \mathbf{d}_T(\mathbf{d}_s - \mathbf{b}_s). \tag{5.14}$$

As a consequence – since we can take $s(u) := u/\mathbf{d}_T$ – we deduce from Lemmas 5.1 and 5.2 and that

$$\left\{\frac{C^{\circ}_{N(ut)} - (\mathbf{d}_s - \mathbf{b}_s)ut}{a \circ s(u)}\right\}_{n\geqslant 0} \xrightarrow[u\to\infty]{\mathrm{J}_1} \{\mathfrak{C}(t)\}_{t\geqslant 0} \tag{5.15}$$

In any cases, to conclude, we need to show that the normalization of the residual term is negligible and it is done in the proof of the following

Lemma 5.3 (Step 3.)**.** *The following convergences hold in probability: for any $v \geqslant 0$,*

$$\sup_{0\leqslant t\leqslant v}\left|\frac{R(ut)}{a \circ s(u)}\right| \xrightarrow[u\to\infty]{\mathbb{P}} 0 \quad \textit{and} \quad \sup_{0\leqslant t\leqslant v}\left|\frac{R^{\circ}(ut)}{a \circ s(u)}\right| \xrightarrow[u\to\infty]{\mathbb{P}} 0. \tag{5.16}$$

Proof. First note that whatever $\alpha \in [1,2]$ it suffices to prove that for any $\ell \in \{\mathtt{u},\mathtt{d}\}$,

$$\sup_{0\leqslant t\leqslant v} \frac{\tau^{\ell}_{N(ut)+1}}{a\circ s(u)} \xrightarrow[u\to\infty]{\mathbb{P}} 0. \tag{5.17}$$

It follows from Lemma 5.1 above that for any $\delta > 0$, there exists $s \geqslant 0$ such that for all $u \geqslant s$, $\mathbb{P}(\Omega_{u,\delta}) \geqslant 1-\delta$, with

$$\Omega_{u,\delta} := \left\{ \sup_{0\leqslant t\leqslant v} \left| \frac{N(ut)}{s(u)} - t \right| \leqslant \delta \right\}. \tag{5.18}$$

Then, one can see for any $\varepsilon > 0$ that

$$\mathbb{P}\left(\sup_{0\leqslant t\leqslant v} \frac{\tau^{\ell}_{N(ut)+1}}{a\circ s(u)} > \varepsilon \ , \ \Omega_{u,\delta} \right) \leqslant 1 - \left[1 - \mathbb{P}(\tau^{\ell}_1 > \varepsilon\, a\circ s(u))\right]^{2\delta s(u)}. \tag{5.19}$$

Furthermore, we obtain from the regular variations of the involved functions and Lemma 7.1 the existence of a positive constant k depending only on $\mathbf{m}_s$, $\mathbf{b}_s$ and ℓ such that

$$r\,\mathbb{P}(\tau^{\ell}_1 > \varepsilon a(r)) \underset{r\to\infty}{\sim} k \left(\frac{2-\alpha}{\alpha}\right) \varepsilon^{-\alpha}. \tag{5.20}$$

Therefore, we deduce from (5.19) and (5.20) that

$$\limsup_{u\to\infty} \mathbb{P}\left(\sup_{0\leqslant t\leqslant v} \frac{\tau^{\ell}_{N(ut)+1}}{a\circ s(u)} > \varepsilon \right) \leqslant \delta + \left[1 - \exp\left(-2\delta k \left(\frac{2-\alpha}{\alpha}\right) \varepsilon^{-\alpha}\right)\right], \tag{5.21}$$

for any $\delta > 0$, which achieves the proof of this lemma. □

Since the first step is obvious when $\alpha \in (1,2]$, this completes the proof of the Lévy situations, excepted Lemma 3.1 whose proof is given below. □

Proof of Lemma 3.1. First, one can see from Lemma 7.1 and standard results on regularly varying functions that $\Xi_1(u) := a(u)\mathscr{T}\circ a(u)$. Besides, the second asymptotic holds with $D(u) := \Theta\circ a\circ s(u)$. Then Karamata's Theorem [3, Proposition 1.5.9a., p. 26] implies that $u\mathscr{T}(u)$ is negligible with respect to $\theta(u)$ as u goes to infinity and we deduce the lemma. □

6 Proof of Theorem 3.3

We treat in this section the anomalous case, *i.e.* $\alpha \in (0,1)$. The proof is divided into four parts. First, we show the functional invariance principle – Section 6.1. Then – Section 6.2 – we describe the marginal distributions of the process before we investigate – Section 6.3 – the Markov property given at the end of the theorem. Finally, we identify the marginal distributions in Section 6.4.1 by the mean of the excursion theory but also – Section 6.4.2 – by extending the structure of Lamperti processes introduced in [17].

6.1 Functional scaling limits

First note that the tightness in (3.22) is obvious since the modulus of continuity of $\{S_{ut}/u\}_{t\geqslant 0}$ is equal to 1 almost surely. Hence, we only need to show the convergence of the finite-dimensional marginal distributions. To this end, we shall adapt the results in [15] for Lévy walks. In that paper, the authors consider Lévy walks of the form

$$\mathfrak{S}(t) := \sum_{k=1}^{N(t)} \Lambda_k J_k + (t - T_{N(t)})\Lambda_{N(t)+1}, \tag{6.1}$$

with

$$T_n := \sum_{k=1}^{n} J_k \quad \text{and} \quad N(t) := \max\{k \geqslant 1 : T_k \leqslant t\}. \tag{6.2}$$

Here the random moving times J_k and the jumps $\Lambda_k J_k$ are *i.i.d.* – the Λ_k being *i.i.d.* and independent of the jumping times and of unit size. Here, one can interpret S – the PRW – as such Lévy walk for which the random moving times are given by τ^{d}_{2n-1} or τ^{u}_{2n} alternatively and corresponding random jumps $-\tau^{\mathrm{d}}_{2n-1}$ or τ^{u}_{2n}. Hence, the directions are deterministic and equal to -1 or 1 alternatively.

Even if their assumptions do not fit to this model, we shall see that their results extend easily to our situation. Indeed, the central convergence [15, Theorem 3.4., pp. 4023-4024] becomes in our situation

$$\left(\frac{M_n}{a(n)}, \frac{T_n}{a(n)}\right) \underset{n\to\infty}{\overset{\mathscr{L}}{\Longrightarrow}} (S_\alpha(1), T_\alpha(1)), \tag{6.3}$$

where T_α and S_α are the coupled α-stable Lévy processes introduced to define the arcsine Lamperti anomalous diffusion. Thereafter, it is not difficult to check that the same continuous mapping and topological arguments used in the proof of [15, Theorem 4.11., p. 4032] and its Corollary 4.14., p. 4033, hold to obtain the functional convergence (3.22).

6.2 About the governing equation

We observe that $\mathscr{S}_\alpha$ is the same scaling limit of a "true" Lévy walk – in the sense of [15] – whose random moving times and jumps are respectively given by $\xi_n\tau^{\mathrm{u}}_n + (1-\xi_n)\tau^{\mathrm{d}}_n$ and $\xi_n\tau^{\mathrm{u}}_n - (1-\xi_n)\tau^{\mathrm{d}}_n$ with *i.i.d.* random directions $2\xi_n - 1$ independent of the running times and distributed as a Rademacher distribution of parameter $(1+\mathbf{b}_s)/2$ denoted in the following by $\mathscr{X}(d\ell)$.

Remark 6.1. *One can see the latter Lévy walk as randomized version of the PRW which is somehow more convenient to study as it is already appeared for the recurrence and transience criteria in [1].*

It is also shown in [15, Theorem 5.6., pp. 4036-4037] that the density distribution $f_t(x)$ satisfies – in a weak sense – the fractional partial differential equation (3.23) since it can be write as

$$\int_{\mathscr{A}} \left(\frac{\partial}{\partial t} + \ell\,\frac{\partial}{\partial t}\right)^{\alpha} f_t(x)\,\mathscr{X}(d\ell) = \frac{1}{\Gamma(1-\alpha)}\frac{1}{t^\alpha}\int_{\mathscr{A}} \delta_{\ell t}(dx)\,\mathscr{X}(d\ell). \tag{6.4}$$

Besides, it follows from [15, Section 5., p. 4035] that $f_t(x)$ can be expressed as

$$f_t(x) = \frac{1}{\Gamma(1-\alpha)}\int_{\mathscr{A}}\int_0^t \frac{u_\alpha(x-\ell(t-s),s)}{(1-s)^\alpha}\,ds\,\mathscr{X}(d\ell), \tag{6.5}$$

where $u_\alpha(x,t)$ denotes the 0-potential density of (S_α, T_α) – the occupation time density – and it comes (3.24). Furthermore, the Fourier-Laplace Transform (FLT) of $u_\alpha(x,t)$ can be computed explicitly and it leads to the FLT of $\mathscr{S}_\alpha(1)$ given in equation (5.5) of [15] – equal here to

$$\frac{(s-iy)^{\alpha-1} + (s+iy)^{\alpha-1}}{\left(\frac{1+\mathbf{b}_S}{2}\right)(s-iy)^\alpha + \left(\frac{1-\mathbf{b}_S}{2}\right)(s+iy)^\alpha}. \tag{6.6}$$

Taking the derivative at the origin with respect to the spacial variable we can see that the mean distribution of $f_1(x)$ is equal to $\mathbf{m}_s$. However, we have not been able to invert directly this FLT to get the expression (3.25).

Remark 6.2. *By using the results in [23] it may be possible to provide the same representations of the finite-dimensional marginal distributions.*

6.3 Markov property and continuous time VLMC

Here we show that $(\mathscr{X}_\alpha, \mathscr{A}_\alpha)$ is Markovian. Let $\mathbb{P}_0^\alpha$ be the distribution of such stochastic process. First, from the regenerative property of a stable subordinator we get that $\{(\mathscr{X}_\alpha(t+s), \mathscr{A}_\alpha(t+s))\}_{t\geqslant 0}$ is distributed as $\mathbb{P}_0^\alpha$ for any $s \in \mathscr{R}_\alpha$. Furthermore – by using [10, Lemma 1.10, p. 15] – one can see that

$$\mathbb{P}(\mathscr{H}_\alpha(t) \in dh \mid \mathscr{A}_\alpha(t) = a) = \frac{\alpha\, a^\alpha}{(a+h)^{\alpha+1}}\, dh, \tag{6.7}$$

for any $t > 0$ and we denote this homogeneous kernel by $N(a; dh)$. Then we introduce for any $\ell \in \{-1, 1\}$ and $a \geqslant 0$ the distribution $\mathbb{H}^\alpha_{(\ell,a)}$ of the process on $\{-1,1\} \times [0,\infty)$ equal to $t \mapsto (\ell, a+t)$ up the random time H_a distributed as $N(a; dh)$ and whatever elsewhere. Therefore, by construction and the regenerative property, it turns out that the distribution of the stochastic process $\{(\mathscr{X}_\alpha(t+s), \mathscr{A}_\alpha(t+s))\}_{t\geqslant 0}$ is – almost surely – the push-forward image of $\mathbb{H}^\alpha_{(\mathscr{X}_\alpha(s), \mathscr{A}_\alpha(s))} \otimes \mathbb{P}_0^\alpha$ under the gluing map G defined by

$$G(w_1, w_2) := w_1(t)\mathbb{1}_{\{t<H_a\}} + w_2(t)\mathbb{1}_{\{t>H_a\}}, \tag{6.8}$$

The Markov property is then a simple consequence of this representation.

As a consequence, one can see $\mathscr{X}_\alpha$ as a continuous time VLMC. Besides, it is well known – see [3, Chap. 8.6] for instance – that the distribution of the couple formed with the age time and the remaining time $(\mathscr{A}_\alpha(t), \mathscr{H}_\alpha(t))$ can be computed explicitly and some scaling limits of this couple leads to other generalized arcsine distributions. Moreover, one can easily be convinced using [30, Theorem (3.2), p. 506] or the results in [31, 32]) that this process admit the invariant (infinite) measure given by

$$\int_{\mathscr{A}} \int_0^\infty G_*(\mathbb{H}^\alpha_{(\ell,a)} \otimes \mathbb{P}_0^\alpha) \frac{1}{a^{\alpha+1}}\, da\, \mathscr{X}(d\ell), \tag{6.9}$$

where $G_*\mu$ is the push-forward image of a distribution μ by the map G. One can note that this measure is not a probability and that to be compared with the result [2] stating that a the double-comb PRW admits an invariant probability measure if and only if the persistence times are both integrable.

6.4 Arcsine Lamperti density

To conclude the proof of Theorem 3.3 it remains to we compute explicitly the density expression. The first proof uses the theory of excursion whereas the second one generalizes the concept of Lamperti and is a more general result.

6.4.1 Via the excursion theory

We deduce from the integral representation (3.21) that [13, section 4., pp. 341-343] holds and more precisely that [18, Theorem 1] apply to our situation. Hence, we get the density expression, making in addition the link with the distribution of the occupation time in the positive half-line of a skew Bessel process of dimension $2-2\alpha$ and skewness parameter $(1+\mathbf{b}_s)/2$.

6.4.2 A second class of Lamperti processes

Lamperti investigate in [17] the general question of the distribution of the occupation time of a set A for some stochastic process $\{\mathbf{X}_n\}_{n\geqslant 0}$ on a state space E. Regarding the dynamic, it is surprisingly only assumed that E can be divided into two sets – the aforementioned A and an other one B – plus a recurrent state σ in such a way that for any $n \geqslant 1$, if $\mathbf{X}_{n-1} \in A$ and $\mathbf{X}_{n+1} \in B$ or vice versa, then necessarily $\mathbf{X}_n = \sigma$.

Thereafter, given such process starting from σ, Lamperti consider N_n the occupation time of A up to the time n – the state σ being counted when the process comes from A – and he introduces $F(x)$ the

generating function associated with the recurrence time of σ. This decomposition of the state space and how to count the occupation time is resumed by the weighted diagram in the left-hand side of Figure 6.4. The following Theorem is stated and proved in its paper.

Theorem 6.1. *The mean occupation time $\{N_n/n\}_{n\geqslant 1}$ converges in distribution as n goes to infinity to a non-degenerate limit if and only if there exist α and p in $(0,1)$ such that*

$$\lim_{n\to\infty} \mathbb{E}[N_n/n] = p \quad \textit{and} \quad \lim_{x\uparrow 1}(1-x)F'(x)/(1-F(x)) = \alpha. \tag{6.10}$$

Besides –in that case– the limit distribution has for density on $(0,1)$ the function

$$\frac{\sin(\pi\alpha)}{\pi}\,\frac{t^{\alpha-1}(1-t)^{\alpha-1}}{rt^{2\alpha}+2\cos(\pi\alpha)t^{\alpha}(1-t)^{\alpha}+r^{-1}(1-t)^{2\alpha}}, \tag{6.11}$$

with $r := (1-p)/p$.

The proof is based on a precise asymptotic estimate of the moments associated with N_n as n goes to infinity – via suitable generating functions and their regular variations – in order to identify the limit of the Stieljes transform of N_n/n.

In our case, it would seem natural to see A as the rises of S so that it can be write more or less as $S_n = 2N_n - n$. Unfortunately, we have not been able to fit this situation to the latter Lamperti dynamic. As a matter of fact, the communication diagram and the way to compute the suitable occupation time N_n can be given as in the right hand side of Figure 6.4.

To be more precise, if we consider the 2-letters process $\mathbf{X}_n := X_nX_{n+1}$ with the decomposition of $\{\mathtt{u},\mathtt{d}\}\times\{\mathtt{u},\mathtt{d}\}$ given by $A := \{\mathtt{uu}\}$, $B := \{\mathtt{dd}\}$, $\sigma := \mathtt{ud}$ and $\eta := \mathtt{du}$ and if we denote by N_n the occupation time of A up to the time n – the state η being counted or not according to well chosen weights – the PRW satisfies for any $n \geqslant 1$ the relation $S_n = 2N_{n-1} - n$. Note also that σ and η are recurrent states and since the random times spend in A and B are distributed as $\tau_1^{\mathtt{u}}$ and $\tau_1^{\mathtt{d}}$ respectively the recurrence times of σ and η are both equal in law to $\tau_1 = \tau_1^{\mathtt{u}} + \tau_1^{\mathtt{d}}$.

To get the limit distribution we adapt and follow a part of the proof in [17]. To this end, we introduce $p_{n,k}$ the probability – starting from σ – that $N_n = k$ and we denote by u_n and d_n the probability that $\tau_1^{\mathtt{u}} = n$ and $\tau_1^{\mathtt{d}} = n$ respectively. Thereafter, by conditioning with respect to the complete three events $\{\tau_1 \leqslant n\} \sqcup \{\tau_1 \geqslant n+1, \tau_1^{\mathtt{d}} \leqslant n\} \sqcup \{\tau_1^{\mathtt{d}} \geqslant n+1\}$ we obtain

$$p_{n,k} = \sum_{\substack{m+l\leqslant n\\ m\leqslant k}} d_l\, u_m\, p_{n-(m+l),k-m} + \sum_{l\leqslant n} d_l\, \mathscr{T}_{\mathtt{u}}(n-l)\,\delta_{k,n-l+1} + \mathscr{T}_{\mathtt{d}}(n)\,\delta_{k,0}, \tag{6.12}$$

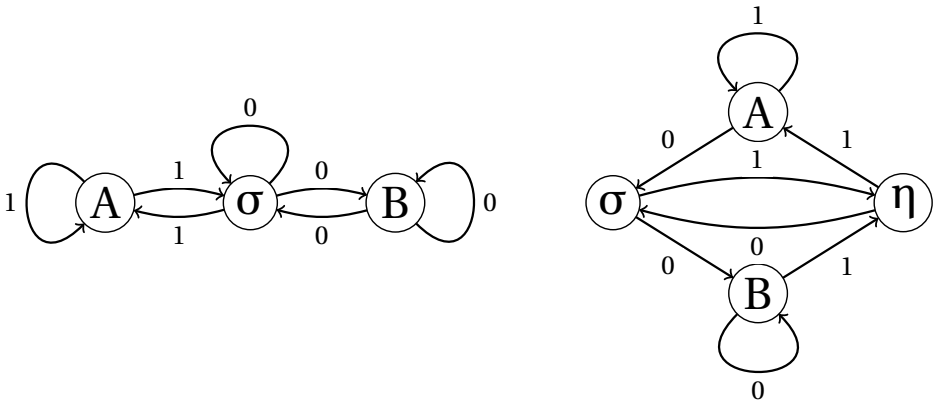

Figure 6.4: Original and current Lamperti decomposition of the state space

with $\delta_{i,j}$ is 1 or 0 according to $i=j$ or not. Then, let $F_\ell(z)$ be the generating function of τ_1^ℓ and $T_\ell(z)$ be the one associated with its tail distribution. it follows that the double generating function associated with the latter difference equation satisfies

$$P(x,y) := \sum_{n,k\geqslant 0} p_{n,k}\, x^n y^k = \frac{F_{\mathtt{d}}(y)T_{\mathtt{u}}(xy)y + T_{\mathtt{d}}(x)}{1-F_{\mathtt{d}}(x)F_{\mathtt{u}}(xy)}, \tag{6.13}$$

From our stability assumption and classical results on regular variations we get that there exists a slowly varying function $L(z)$ such that

$$1-F_{\mathtt{u}}(z) = (1-z)T_{\mathtt{u}}(z) \underset{z\uparrow 1}{\sim} \left(\frac{1+\mathbf{b}_s}{2}\right) L\left(\frac{1}{1-z}\right)(1-z)^\alpha, \tag{6.14}$$

the same asymptotic and equality being also true replacing $\mathtt{u}$ by $\mathtt{d}$ and $(1+\mathbf{b}_s)/2$ by $(1-\mathbf{b}_s)/2$. Then we can check that for any $\lambda>0$,

$$\lim_{x\uparrow 1}(1-x)P\left(x, e^{-\lambda(1-x)}\right) = \frac{(1+\lambda)^{\alpha-1}+\mathbf{r}_s}{(1+\lambda)^\alpha+\mathbf{r}_s}, \tag{6.15}$$

and thereafter the proof follows exactly the same lines as [17]. We obtain the arcsine Lamperti density. Note that it may be possible to state and prove a general theorem for such Lamperti processes.

This completes the proof of the anomalous situation. □

7 Equivalent characterizations of Assumption 2.2

In the continuity of (2.4) and (2.5) we can consider $\mathscr{T}_{\mathtt{c}}(t)$ and $V_{\mathtt{c}}(t)$ the two-sided distribution tail and the truncated second moment of the central random variable $\tau_1^{\mathtt{c}}$ given in (2.10). In the following, we denote by $\mathrm{RV}(\lambda)$ the set of regularly varying functions of index $\lambda\in\mathbb{R}$, also named the set of slowly varying functions SV if $\lambda=0$. It is well known that hypothesis (2.10) is equivalent to

$$V_{\mathtt{c}}(t)\in \mathrm{RV}(2-\alpha), \quad \text{or equivalently when } \alpha\neq 2, \quad \mathscr{T}_{\mathtt{c}}(t)\in \mathrm{RV}(-\alpha), \tag{7.1}$$

with an additional tail balance criterion when $\alpha\neq 2$: there exists $\beta\in[-1,1]$ with

$$\lim_{t\to\infty}\frac{\mathbb{P}(\tau_1^{\mathtt{c}}>t)-\mathbb{P}(\tau_1^{\mathtt{c}}<-t)}{\mathbb{P}(\tau_1^{\mathtt{c}}>t)+\mathbb{P}(\tau_1^{\mathtt{c}}<-t)} = \beta. \tag{7.2}$$

Remark from [7, Theorem 2, Chap. VIII.9, p. 283] that

$$\lim_{t\to\infty}\frac{t^2\mathscr{T}_{\mathtt{c}}(t)}{V_{\mathtt{c}}(t)} = \frac{2-\alpha}{\alpha}, \tag{7.3}$$

To go further, introduce

$$\mathscr{T}(t) := \mathscr{T}_{\mathtt{u}}(t)+\mathscr{T}_{\mathtt{d}}(t) \quad \text{and} \quad V(t) := V_{\mathtt{u}}(t)+V_{\mathtt{d}}(t). \tag{7.4}$$

The two following lemmas gives another possible interpretations and choices of condition (2.10) and of the normalizing functions. Their proofs are straightforward computations using only classical results on regularly varying functions which can be found in [3] for instance, they are omitted.

Lemma 7.1. *Assuming the existence of the mean drift $\mathbf{m}_s \in (-1,1)$, the hypothesis (2.10) is equivalent when $\alpha \neq 2$ to $Y_1 = \tau_1^{\mathsf{u}} - \tau_1^{\mathsf{d}} \in \mathrm{D}(\alpha)$ but also to*

$$\Big[V(t) \in \mathrm{RV}(2-\alpha) \quad or \quad \mathscr{T}(t) \in \mathrm{RV}(-\alpha)\Big] \quad and \quad \lim_{t\to\infty} \frac{\mathscr{T}_{\mathsf{u}}(t) - \mathscr{T}_{\mathsf{d}}(t)}{\mathscr{T}_{\mathsf{u}}(t) + \mathscr{T}_{\mathsf{d}}(t)} = \mathbf{b}_s. \tag{7.5}$$

In that case, the functions $\mathscr{T}(t)$ and $V(t)$ are respectively the tail distribution and the truncated second moment of Y_1 but also of τ_1. Besides, one has

$$\mathscr{T}_{\mathsf{c}}(t) \underset{t\to\infty}{\sim} \left[(1-\mathbf{m}_s)^{\alpha}\left(\frac{1+\mathbf{b}_s}{2}\right) + (1+\mathbf{m}_s)^{\alpha}\left(\frac{1-\mathbf{b}_s}{2}\right)\right]\mathscr{T}(t), \tag{7.6}$$

and thus the same asymptotic between $V_{\mathsf{c}}(t)$ and $V(t)$. Moreover, the balance term β in (7.2) satisfies

$$\beta = \frac{(1-\mathbf{m}_s)^{\alpha}(1+\mathbf{b}_s) - (1+\mathbf{m}_s)^{\alpha}(1-\mathbf{b}_s)}{(1-\mathbf{m}_s)^{\alpha}(1+\mathbf{b}_s) + (1+\mathbf{m}_s)^{\alpha}(1-\mathbf{b}_s)}. \tag{7.7}$$

Finally, when $\alpha = 2$,

$$\mathbb{V}[\tau_1^{\mathsf{c}} \mathbb{1}_{\{|\tau_1^{\mathsf{c}}| \leqslant t\}}] \underset{t\to\infty}{\sim} V_{\mathsf{c}}(t) \underset{t\to\infty}{\sim} \Sigma(t)^2. \tag{7.8}$$

Hence, when $\alpha \neq 2$, Assumption 2.2 means that at least one of the persistence times belongs to the domain of attraction of a stable distribution, the tail distribution of the other one being in some sense comparable with the first one. However, in view of [33, Theorem 4.5., p. 790], we can state

Remark 7.1. *When $\alpha = 2$, it is possible for all linear combination of τ_1^{u} and τ_1^{d} distinct of τ_1^{c} to not belong to the domain of attraction of a standard normal distribution.*

As a consequence, it turns out that $a(u)$, $s(u)$ and thus $\lambda(u)$ – together with $\Sigma(t)^2$ and $\Theta(t)$ inside their definitions – are regularly varying functions. In particular, we can see that the sub-linear normalizing function $\lambda(u)$ is asymptotically linear if and only if $\alpha \in (0,1)$. in any case, there exists a slowly varying function $\ell(u)$ such that

$$\lambda(u) \underset{u\to\infty}{\sim} \ell(u)\, u^{\frac{1}{\alpha \vee 1}}.$$

The second lemma, coupled with the first one, will be useful – among other considerations – to discriminate whether the persistence times are integrable or not and thus to identify $\mathbf{m}_s$. Again, the proof is omitted. Given two real functions $f(t)$ and $g(t)$ defined on a neighbourhood of infinity, we set $f(t) \asymp g(t)$ when there exists $c > 0$ such that for t sufficiently large $c^{-1} g(t) \leqslant f(t) \leqslant c g(t)$.

Lemma 7.2. *Let μ be a positive measure on $[0,\infty)$ and for any $p \geqslant 0$ and $t \geqslant 0$,*

$$M_p(t) := \int_{[0,t]} x^p \mu(dx) \quad and \quad T_p(t) := \int_{(t,\infty)} x^p \mu(dx).$$

Then for any $q > p \geqslant 0$,

$$M_p(t) \asymp \int_{[1,t]} \frac{M_q(u)}{u^{q-p+1}} du + \frac{M_q(t)}{t^{q-p}} \quad and \quad T_p(t) \asymp \int_{(t,\infty)} \frac{M_q(u)}{u^{q-p+1}} du + \frac{M_q(t)}{t^{q-p}}.$$

Therefore, the persistence times are both integrable $\mathbf{d}_T < \infty$ when $\alpha \in (1,2]$ and in that case $\mathbf{m}_s = \mathbf{d}_s$. Besides, they are both of infinite mean when $\alpha \in (0,1)$ and then $\mathbf{m}_s = \mathbf{b}_s$, the two latter situations being possible when $\alpha = 1$.

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