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1           **CONVERGENCE OF A FINITE VOLUME SCHEME FOR A**  
2           **STOCHASTIC CONSERVATION LAW INVOLVING A**  
3           **Q-BROWNIAN MOTION**

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ABSTRACT. We study a time explicit finite volume method for a first order conservation law with a multiplicative source term involving a  $Q$ -Wiener process. After having presented the definition of a measure-valued weak entropy solution of the stochastic conservation law, we apply a finite volume method together with Godunov scheme for the space discretization, and we denote by  $\{u_{\mathcal{T},k}\}$  its discrete solution. We present some a priori estimates including a weak BV estimate. After performing a time interpolation, we prove two entropy inequalities for the discrete solution. We show that the discrete solution  $\{u_{\mathcal{T},k}\}$  converges along a subsequence to a measure-valued entropy solution of the conservation law in the sense of Young measures as the maximum diameter of the volume elements and the time step tend to zero. Some numerical simulations are presented in the case of the stochastic Burgers equation. The empirical average turns out to be a regularization of the deterministic solution; moreover, the variance in the case of the  $Q$ -Brownian motion converges to a constant while that in the Brownian motion case keeps increasing as time tends to infinity.

4   **1. Introduction.** The convergence of numerical methods for the discretization of  
5 stochastic conservation laws is a topic of high interest. In this article we study the  
6 convergence of a finite volume scheme for the problem

$$\begin{cases} du + \operatorname{div}(\mathbf{v}f(u))dt = g(u)dW(x,t) & \text{in } \Omega \times \mathbb{T}^d \times [0, T], \\ u(\omega, x, 0) = u_0(x) & \text{for all } \omega \in \Omega, x \in \mathbb{T}^d, \end{cases} \quad (1)$$

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1 where  $\mathbb{T}^d$  is the  $d$ -dimensional torus,  $W(x, t)$  is a  $Q$ -Brownian motion, the function  
 2  $f$  is Lipschitz continuous and the function  $g$  is Lipschitz continuous and bounded.  
 3 We suppose that  $\mathbf{v} = \mathbf{v}(x, t)$  is a given vector function and that  $u_0$  is a given square  
 4 integrable function on  $\mathbb{T}^d$ .

5 A number of articles have been devoted to the study of scalar conservation laws  
 6 with a multiplicative stochastic forcing term involving a white noise in time.

7 Let us mention the one-dimensional study of Feng-Nualart [14], where the au-  
 8 thors introduced a notion of entropy solution in order to prove the existence and  
 9 uniqueness of an entropy solution. Chen-Ding-Karlsen [9] extended the work of  
 10 Feng-Nualart to the multi-dimensional case. They proved a uniform spatial BV  
 11 bound by means of vanishing viscosity approximations. Moreover they proved the  
 12 temporal equicontinuity of approximations in  $L^1(\Omega \times D \times [0, T])$ , uniformly in the  
 13 viscosity coefficient.

14 Debussche-Vovelle [12] proved the existence and uniqueness of a kinetic solu-  
 15 tion for multi-dimensional scalar conservation laws in a  $d$ -dimensional torus driven  
 16 by a general multiplicative space-time noise. Hofmanová [19] then presented a  
 17 Bhatnagar-Gross-Krook-like approximation of this problem. Applying the stochas-  
 18 tic characteristics method, the author established the existence of an approximate  
 19 solution and proved its convergence to the kinetic solution introduced by [12].

20 Bauzet-Vallet-Wittbold [4] proved the existence and uniqueness of a weak sto-  
 21 chastic entropy solution of the multi-dimensional Cauchy problem in  $L^2(\Omega \times \mathbb{R}^d \times$   
 22  $[0, T])$  in the case of a multiplicative one-dimensional white noise in time. In Bauzet-  
 23 Vallet-Wittbold [5] the authors investigated a corresponding Dirichlet Problem in  
 24 a bounded domain of  $\mathbb{R}^d$ .

25 Concerning the study of numerical schemes for stochastic conservation laws,  
 26 Bauzet-Charrier-Gallouët [6] studied explicit flux-splitting finite volume discretiza-  
 27 tions of multi-dimensional nonlinear scalar conservation laws with monotone flux  
 28 perturbed by a multiplicative one-dimensional white noise in time with a given ini-  
 29 tial function in  $L^2(\mathbb{R}^d)$ . Under a stability condition on the time step, they proved  
 30 the convergence of the finite volume approximation towards the unique stochastic  
 31 entropy solution of the corresponding initial value problem. Then Bauzet-Charrier-  
 32 Gallouët [7] studied the case of a more general flux and in [8], Bauzet-Charrier-  
 33 Gallouët studied the convergence of the scheme when the stochastic conservation  
 34 law is defined on a bounded domain with inhomogeneous Dirichlet boundary condi-  
 35 tions. Let us also mention the convergence results of time-discretization of Holden-  
 36 Risebro [20] and Bauzet [3] on a bounded domain of  $\mathbb{R}^d$ , as well as an article of  
 37 Kröker-Rohde [21] of a finite volume schemes in a one-dimensional context.

38 In a recent study, Audusse-Boyaval-Gao-Hilhorst [1] performed numerical simu-  
 39 lations in the one-dimensional torus for the first order Burgers equation forced by  
 40 a stochastic source term. The source term is a white noise in time while various  
 41 regularities in space are considered. The authors applied the Monte-Carlo method,  
 42 and observed that the empirical mean introduces a small diffusion effect to the  
 43 deterministic numerical solution and converges to the space average of the initial  
 44 condition as the time  $t$  tends to infinity and that the empirical variance stabilizes  
 45 for large time.

46 The present article extends the article by Bauzet-Charrier-Gallouët [7] mentioned  
 47 above. The organisation is as follows: In section 2 we define a weak stochastic  
 48 entropy solution and a measure-valued stochastic entropy solution of Problem (1)  
 49 and recall basic results from probability theory. In section 3, we apply a finite

1 volume method together with a Godunov scheme to Problem (1) and define the  
 2 discrete solution  $\{u_{\mathcal{T},k}\}$ . In section 4, we present an estimate of the discrete noise  
 3 term as well as a priori estimates on the discrete solution  $\{u_{\mathcal{T},k}\}$ . The a priori  
 4 estimates imply that  $\{u_{\mathcal{T},k}\}$  converges up to a subsequence in the sense of Young  
 5 measures to an entropy process denoted by  $\mathbf{u}$  in  $L^2(\Omega \times \mathbb{T}^d \times [0, T] \times (0, 1))$ . We then  
 6 prove a weak BV estimate which is essential in the sequel in order to ensure that the  
 7 difference between the piece-wise constant solution in time and the solution linearly  
 8 interpolated in time can be controlled by the maximum diameter of the volume  
 9 elements and the time step. Meanwhile, in order to prove the discrete entropy  
 10 inequality, we need the weak BV estimate for showing that a certain residue tends  
 11 to zero as the maximum diameter of the volume elements and the time step tend  
 12 to zero. In section 5, we introduce a time interpolation and prove two inequalities,  
 13 a discrete entropy inequality and a continuous entropy inequality on the discrete  
 14 solution which are fundamental for the convergence proof. Then in section 6, using  
 15 the two entropy inequalities, we show that the discrete solution  $\{u_{\mathcal{T},k}\}$  converges  
 16 along a subsequence to a limit  $\mathbf{u}$  in the sense of Young measures as the maximum  
 17 diameter of the volume elements and the time step tend to zero; moreover  $\mathbf{u}$  is  
 18 a measure-valued entropy solution of Problem (1). In section 7, some numerical  
 19 simulations for stochastic Burgers equation involving a Brownian motion and a  $Q$ -  
 20 Brownian motion are presented. It turns out that the variance increases more as a  
 21 function of time in the case of a unidimensional Brownian motion than in the case  
 22 of the  $Q$ -Brownian motion.

23 In a forthcoming work [15], we will show that the measured-value entropy solution  
 24  $\mathbf{u}$  is unique and coincides with the unique weak stochastic entropy solution; this  
 25 will ensure that the whole approximate sequence  $\{u_{\mathcal{T},k}\}$  converges to the entropy  
 26 solution  $u$ .

**2. A stochastic conservation law involving a  $Q$ -Brownian motion.** We study the convergence of a finite volume scheme for the discretization of the stochastic scalar conservation law

$$\begin{cases} du + \operatorname{div}(\mathbf{v}f(u))dt = g(u)dW(x, t) & \text{in } \Omega \times \mathbb{T}^d \times [0, T], \\ u(\omega, x, 0) = u_0(x) & \omega \in \Omega, x \in \mathbb{T}^d, \end{cases}$$

where  $\mathbb{T}^d$  is the  $d$ -dimensional torus and  $W(x, t)$  is a  $Q$ -Brownian motion [10]. More precisely, let  $Q$  be a trace class nonnegative definite symmetric operator on  $L^2(\mathbb{T}^d)$  and let  $\{e_m\}_{m \geq 1}$  be an orthonormal basis in  $L^2(\mathbb{T}^d)$  diagonalizing  $Q$ , and  $\{\lambda_m\}_{m \geq 1}$  be the corresponding eigenvalues, such that

$$Qe_m = \lambda_m e_m$$

27 for all  $m \geq 1$ .  $Q$  is of trace class, namely

$$\operatorname{Tr} Q = \sum_{m=1}^{\infty} (Qe_m, e_m)_{L^2(\mathbb{T}^d)} = \sum_{m=1}^{\infty} \lambda_m \leq \Lambda_0, \quad (2)$$

28 for some positive constant  $\Lambda_0$ . Actually,  $Q$  is an integral operator with the kernel

$$Q(x, y) = \sum_{m=1}^{\infty} \lambda_m e_m(x) e_m(y). \quad (3)$$

1 We suppose furthermore that  $e_m \in L^\infty(\mathbb{T}^d)$  for  $m = 1, 2, \dots$  and that there exists a  
 2 positive constant  $\Lambda_1$  such that

$$\sum_{m=1}^{\infty} \lambda_m \|e_m\|_{L^\infty(\mathbb{T}^d)}^2 \leq \Lambda_1. \quad (4)$$

3 Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space equipped with a filtration  $(\mathcal{F}_t)$  [22] and  $\{\beta_m\}_{m \geq 1}$   
 4 be a sequence of independent  $(\mathcal{F}_t)$ -Brownian motions defined on  $(\Omega, \mathcal{F}, \mathbb{P})$ ; the process  $W$  defined by  
 5

$$W(x, t) = \sum_{m=1}^{\infty} \beta_m(t) Q^{\frac{1}{2}} e_m(x) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t) e_m(x) \quad (5)$$

6 is a  $Q$ -Brownian motion in  $L^2(\mathbb{T}^d)$  [cf. [17], Definition 2.6, page 20], and the series  
 7 defined by (5) is convergent in  $L^2(\Omega, C([0, T], L^2(\mathbb{T}^d)))$  [cf. [17], page 20]. We recall  
 8 that a Brownian motion  $\beta(t)$  is called an  $(\mathcal{F}_t)$ -Brownian motion if it is  $(\mathcal{F}_t)$ -adapted  
 9 and the increment  $\beta(t) - \beta(s)$  is independent of  $\mathcal{F}_s$  for every  $0 \leq s < t$ .

10 Moreover we assume that the following hypotheses (H) hold:

- 11 •  $u_0 \in L^2(\mathbb{T}^d)$ ,
- 12 •  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a Lipschitz continuous function with Lipschitz constant  $C_f$  and  
 13  $f(0) = 0$ ,
- 14 •  $g : \mathbb{R} \rightarrow \mathbb{R}$  is a bounded Lipschitz continuous function with Lipschitz constant  
 15  $C_g$  such that  $|g(u)| \leq M_g$  for some positive constant  $M_g$ ,
- 16 •  $\mathbf{v} \in C^1(\mathbb{T}^d \times [0, T])$  with  $\operatorname{div} \mathbf{v} = 0$  for all  $(x, t) \in \mathbb{T} \times [0, T]$  so that there exists  
 17  $V < \infty$  such that  $|\mathbf{v}(x, t)| \leq V$  for all  $(x, t)$ .

18 We introduce some notations:

- 19 • Let  $\mathbb{E}[\cdot]$  denote the expectation, and  $\mathcal{N}(\mu, \sigma^2)$  the Gaussian law with mean  
 20 value  $\mu$  and variance  $\sigma^2$ .
- 21 • We denote by  $\mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d))$  the subclass of  $L^2(\Omega \times \mathbb{T}^d \times [0, T])$  consisting  
 22 of predictable  $L^2(\mathbb{T}^d)$ -valued processes [cf. [10], page 98].

23 Next we define the notions of stochastic entropy solution and of measure-valued  
 24 entropy solution of Problem (1):

**Definition 2.1** (Entropy solution of Problem (1)). A function  $u \in \mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d)) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d))$  is a weak entropy solution of the stochastic scalar conservation law (1) with the initial condition  $u_0 \in L^2(\mathbb{T}^d)$ , if P-a.s in  $\Omega$ ,

$$\begin{aligned} & \int_{\mathbb{T}^d} \eta(u_0(x)) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \{ \eta(u) \partial_t \varphi(x, t) + F^\eta(u) \mathbf{v} \cdot \nabla_x \varphi(x, t) \} dx dt \\ & + \int_{\mathbb{T}^d} \int_0^T \eta'(u) g(u) \varphi(x, t) dW(x, t) dx + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \eta''(u) g^2(u) \varphi(x, t) Q(x, x) dx dt \\ & \geq 0 \end{aligned}$$

25 with

$$F^\eta(\tau) = \int_0^\tau \eta'(\sigma) f'(\sigma) d\sigma \quad (6)$$

26 for all  $\varphi \in \mathcal{C} := \{ \varphi \in C_0^\infty(\mathbb{T}^d \times [0, T]), \varphi \geq 0 \}$  and for all  $\eta \in \mathcal{A}$  where  $\mathcal{A}$  is the set  
 27 of  $C^2$  convex functions such that the support of  $\eta''$  is compact.

**Definition 2.2** (Measure-valued entropy solution of Problem (1)). A function  $\mathbf{u}$  of  $\mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$  is a measure-valued entropy

solution of the stochastic scalar conservation law (1) with the initial condition  $u_0 \in L^2(\mathbb{T}^d)$ , if P-a.s. in  $\Omega$ , for all  $\eta \in \mathcal{A}$  and for all  $\varphi \in \mathcal{C}$

$$\begin{aligned} & \int_{\mathbb{T}^d} \eta(u_0(x))\varphi(x, 0)dx \\ & + \int_0^T \int_{\mathbb{T}^d} \int_0^1 \{\eta(\mathbf{u}(\cdot, \alpha))\partial_t\varphi(x, t) + F^\eta(\mathbf{u}(\cdot, \alpha))\mathbf{v} \cdot \nabla_x\varphi(x, t)\}d\alpha dx dt \\ & + \int_{\mathbb{T}^d} \int_0^T \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))\varphi(x, t)d\alpha dW(x, t)dx \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(\cdot, \alpha))g^2(\mathbf{u}(\cdot, \alpha))\varphi(x, t)Q(x, x)d\alpha dx dt \\ & \geq 0. \end{aligned}$$

### 1 3. The finite volume discretization.

#### 2 3.1. The numerical scheme.

**Definition 3.1** (Admissible mesh). An admissible mesh  $\mathcal{T}$  of  $\mathbb{T}^d$  for the discretization is given by a family of disjoint polygonal connected subsets of  $\mathbb{T}^d$  such that  $\mathbb{T}^d$  is the union of the closure of the elements of  $\mathcal{T}$  and the common interface of any two control volumes is included in a hyperplane of  $\mathbb{T}^d$ . We assume that

$$h = \text{size}(\mathcal{T}) = \sup\{\text{diam}(K), K \in \mathcal{T}\} < \infty,$$

3 and that, for some  $\alpha_{\mathcal{T}} \in \mathbb{R}^+$ ,

$$\alpha_{\mathcal{T}}h^d \leq |K| \text{ and } |\partial K| \leq \frac{1}{\alpha_{\mathcal{T}}}h^{d-1} \quad \text{for all } K \in \mathcal{T}, \quad (7)$$

4 which implies

$$\frac{|\partial K|}{|K|} \leq \frac{1}{\alpha_{\mathcal{T}}^2 h}. \quad (8)$$

5 We also introduce the following notations:

- 6 •  $x_K$  is a point in the control volume  $K$ ,
- 7 •  $|K|$  is the  $d$ -dimensional Lebesgue measure of  $K$ ,
- 8 •  $\partial K$  is the boundary of the control volume  $K$ ,
- 9 •  $|\partial K|$  is the  $(d-1)$ -dimensional Lebesgue measure of  $\partial K$ ,
- 10 •  $\mathcal{N}(K)$  is the set of control volumes neighbors of the control volume  $K$ ,
- 11 •  $\sigma_{K,L}$  is the common interface between  $K$  and  $L$  for all  $L \in \mathcal{N}(K)$ ,
- 12 •  $\mathbf{n}_{K,L}$  is the unit normal vector which is perpendicular to the interface  $\sigma_{K,L}$ ,
- 13 outward to the control volume  $K$ , for all  $L \in \mathcal{N}(K)$ .

14 Consider an admissible mesh  $\mathcal{T}$  in the sense of Definition 3.1. In order to compute  
15 an approximation of  $u$  on  $[0, T]$ , we take  $N \in \mathbb{N}^+$  and define the time step  $k = \frac{T}{N}$ .

16 In this way  $[0, T] = \bigcup_{n=0}^{N-1} [nk, (n+1)k]$ . We set  $t^n = nk$  for all  $n = 0, 1, 2, \dots, N$  and

17 assume that  $k$  and  $h$  satisfy a Courant-Friedrichs-Lewy (CFL) condition:  $k \leq Ch$   
18 for a certain constant  $C$ . We recall the definition of Godunov scheme.

1 **Definition 3.2** (Godunov flux). A function  $F^G \in C^2(\mathbb{R}^2, \mathbb{R})$  is called a Godunov  
 2 flux if it satisfies

$$F^G(a, b) = \begin{cases} \min_{s \in [a, b]} f(s) & \text{if } a \leq b \\ \max_{s \in [b, a]} f(s) & \text{if } a > b. \end{cases} \quad (9)$$

3 For all  $(a, b) \in \mathbb{R}^2$ , we denote by  $s(a, b) \in [\min(a, b), \max(a, b)]$  a real number such  
 4 that  $F^G(a, b) = f(s(a, b))$ .

5 **Remark 1.**  $F^G$  is a Lipschitz continuous function such that  $|F^G(b, a) - F^G(a, a)| \leq$   
 6  $C_f|a - b|$  and  $|F^G(a, b) - F^G(a, a)| \leq C_f|a - b|$ .

7 Denoting by  $d\gamma$  the  $(d - 1)$ -dimensional Lebesgue measure, we define

$$v_{K,L}^n = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\mathbf{v}(x, t) \cdot \mathbf{n}_{K,L})^+ d\gamma(x) dt, \quad (10)$$

8

$$\begin{aligned} v_{L,K}^n &= \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\mathbf{v}(x, t) \cdot \mathbf{n}_{L,K})^+ d\gamma(x) dt \\ &= \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\mathbf{v}(x, t) \cdot \mathbf{n}_{K,L})^- d\gamma(x) dt. \end{aligned} \quad (11)$$

9 Since  $\operatorname{div} \mathbf{v} = 0$  for all  $(x, t) \in \mathbb{T}^d \times [0, T]$ ,

$$\begin{aligned} &\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n - v_{L,K}^n) \\ &= \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left( \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} v(x, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \right) \\ &= \frac{1}{k} \int_{nk}^{(n+1)k} \int_K \operatorname{div} \mathbf{v} dx dt = 0. \end{aligned} \quad (12)$$

10 We define the discrete noise terms

$$\begin{aligned} W_{M,K}(t) &= \sum_{m=1}^M \sqrt{\lambda_m} \beta_m(t) e_K^m, \\ W_K(t) &= \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t) e_K^m, \end{aligned} \quad (13)$$

where  $e_K^m = \frac{1}{|K|} \int_K e_m(x) dx$  for all  $K \in \mathcal{T}$ . Moreover we denote by  $W_{M,K}^n$  and  $W_K^n$   
 the values of  $W_{M,K}$  and  $W_K$  at the time  $t = nk$  respectively. We define for later  
 use

$$\begin{aligned} W_{\mathcal{T},k}(x, t) &= W_K^n \quad \text{for } x \in K \quad \text{and } t \in [nk, (n+1)k], \\ W_{M,\mathcal{T},k}(x, t) &= W_{M,K}^n \quad \text{for } x \in K \quad \text{and } t \in [nk, (n+1)k], \\ W_{\mathcal{T}}(x, t) &= W_K(t) \quad \text{for } x \in K \quad \text{and } t \in [0, T]. \end{aligned}$$

11 We propose the following numerical scheme. The discrete initial condition  $\{u_K^0\}_{K \in \mathcal{T}}$   
 12 is given by:

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx \quad \text{for all } K \in \mathcal{T}, \quad (14)$$

13 and  $\{u_K^n\}$  satisfies the explicit scheme:

14

1 For all  $K \in \mathcal{T}$  and all  $n \in \{0, 1, \dots, N-1\}$

$$\begin{aligned} \frac{|K|}{k}(u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)) \\ = \frac{|K|}{k} g(u_K^n) (W_K^{n+1} - W_K^n). \end{aligned} \quad (15)$$

**Remark 2.** We remark that, if the flux function  $f$  is monotone, and  $F^G(a, b)$  is the Godunov scheme, then the flux term in (15) coincides with the upwind scheme. Indeed, suppose that  $f$  is increasing, we use the definition of the Godunov flux (9) to deduce that  $F^G(a, b) = f(a)$  for all  $a, b \in \mathbb{R}$ . Thus the flux term in the scheme (15) satisfies

$$v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n) = \begin{cases} v_{K,L}^n f(u_K^n) & \text{if } \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} \geq 0 \\ -v_{L,K}^n f(u_L^n) & \text{if } \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} < 0, \end{cases}$$

2 which coincides with the upwind scheme.

3 In the following, we denote by

$$F_{K,L}^{G,n} = v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n), \quad (16)$$

4 and we define for later use that

$$F_{K,L}^{G,n,f} = v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n)). \quad (17)$$

5 In view of (12),

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} = \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n,f}, \quad (18)$$

6 for all  $K \in \mathcal{T}$ .

7 **3.2. The main result of this article.** We define the approximate finite volume  
8 solution  $u_{\mathcal{T},k}$  on  $\mathbb{T}^d \times [0, T]$  from the discrete unknowns  $u_K^n$ , for all  $n \in \{0, 1, \dots, N-$   
9  $1\}$  and for all  $K \in \mathcal{T}$ , that:

$$u_{\mathcal{T},k}(x, t) = u_K^n \quad \text{for } x \in K \quad \text{and } t \in [nk, (n+1)k], \quad (19)$$

10 where the set  $\{u_K^0\}_{K \in \mathcal{T}}$  is defined by (14).

11

12 The main result of this article is the following Theorem:

13 **Theorem 3.3** (Convergence of the finite volume scheme and the existence of a  
14 measure-valued entropy solution of Problem (1)). *Assume that hypotheses (H) hold.*

15 *Let  $\mathcal{T}$  be an admissible mesh,  $T > 0$ ,  $N \in \mathbb{N}^+$  and let  $k = \frac{T}{N}$  satisfy that*

$$\frac{k}{h} \rightarrow 0 \quad \text{as } h \rightarrow 0. \quad (20)$$

16 *Then there exist a function  $\mathbf{u} \in \mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times$   
17  $(0, 1)))$  and a subsequence of  $\{u_{\mathcal{T},k}\}$  which we denote again by  $\{u_{\mathcal{T},k}\}$  such that  
18  $\{u_{\mathcal{T},k}\}$  converges to  $\mathbf{u}$  in the sense of Young measures as  $h, k \rightarrow 0$ . Moreover  $\mathbf{u}$  is  
19 measure-valued entropy solution of Problem (1) in the sense of Definition 2.2.*

The convergence in the sense of Young measures can be defined as follows: given a Carathéodory function  $\Psi : \Omega \times \mathbb{T}^d \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  such that  $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T}, k})$  is uniformly integrable, one has

$$\mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} \Psi(\cdot, u_{\mathcal{T}, k}) dx dt \right] \rightarrow \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^d} \int_0^1 \Psi(\cdot, \mathbf{u}(\cdot, \alpha)) d\alpha dx dt \right],$$

- 1 when  $h, k \rightarrow 0$ . We recall that a function  $\Psi : \Omega \times \mathbb{T}^d \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  is a  
 2 Carathéodory function if for almost all  $(\omega, x, t) \in \Omega \times \mathbb{T}^d \times [0, T]$  the function  
 3  $\nu \mapsto \Psi(\omega, x, t, \nu)$  is continuous and for all  $\nu \in \mathbb{R}$ , the function  $(\omega, x, t) \mapsto \Psi(\omega, x, t, \nu)$   
 4 is measurable.

### 5 3.3. The study of the discrete noise term.

6 **Lemma 3.4.**  $W_{M,K}(t)$  and  $W_K(t)$  being defined as in (13), we have the following  
 7 equalities

$$\mathbb{E}[W_{M,K}^{n+1} - W_{M,K}^n] = 0 \quad (21)$$

8 and

$$\mathbb{E}[W_K^{n+1} - W_K^n] = 0. \quad (22)$$

*Proof.* We show (22). Since

$$(e_K^m)^2 = \frac{1}{|K|^2} \left( \int_K e_m(x) dx \right)^2 \leq \frac{1}{|K|} \int_K e_m^2(x) dx \leq \frac{1}{|K|},$$

we have

$$\mathbb{E}[(W_K^n)^2] = \sum_{m=1}^{\infty} \lambda_m t^n (e_K^m)^2 \leq \frac{t^n}{|K|} \sum_{m=1}^{\infty} \lambda_m < \infty.$$

Thus,  $W_K^n \in L^2(\Omega) \subset L^1(\Omega)$ , so that  $\mathbb{E}[|W_K^n|] < \infty$ . Therefore

$$\begin{aligned} \mathbb{E}[W_K^{n+1} - W_K^n] &= \mathbb{E} \left[ \sum_{m=1}^{\infty} \sqrt{\lambda_m} (\beta_m(t^{n+1}) - \beta_m(t^n)) e_K^m \right] \\ &= \sum_{m=1}^{\infty} \sqrt{\lambda_m} \mathbb{E}[\beta_m(t^{n+1}) - \beta_m(t^n)] e_K^m \\ &= 0. \end{aligned}$$

9

□

10 **Lemma 3.5.** Suppose that the coefficients  $\{\lambda_m\}_{m \geq 1}$  satisfy (2), then

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[ \left( W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0, \quad (23)$$

11 holds for all  $M \in \mathbb{N}^+$ ,  $n \in \{0, 1, \dots, N-1\}$  and  $K \in \mathcal{T}$ .

*Proof.* We have that for fixed  $n$ ,  $M$  and  $K$ ,

$$(W_{M,K}^{n+1} - W_{M,K}^n)^2 = \left( \sum_{m=1}^M (\beta_m(t^{n+1}) - \beta_m(t^n)) \frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2.$$

We take the expectation of both sides to obtain:

$$\mathbb{E} \left[ \left( W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right]$$

$$\begin{aligned}
 &= \mathbb{E} \left[ \left( \sum_{m=1}^M (\beta_m(t^{n+1}) - \beta_m(t^n)) \frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2 \right] \\
 &= \mathbb{E} \left[ \sum_{m=1}^M (\beta_m(t^{n+1}) - \beta_m(t^n))^2 \lambda_m \right] \frac{1}{|K|^2} \left( \int_K e_m(x) dx \right)^2 \\
 &\quad + \mathbb{E} \left[ 2 \sum_{m_1 \neq m_2} (\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)) (\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n)) \sqrt{\lambda_{m_1}} e_K^{m_1} \sqrt{\lambda_{m_2}} e_K^{m_2} \right] \\
 &= \sum_{m=1}^M \mathbb{E} \left[ (\beta_m(t^{n+1}) - \beta_m(t^n))^2 \right] \lambda_m \frac{1}{|K|^2} \left( \int_K e_m(x) dx \right)^2
 \end{aligned}$$

we deduce that

$$\begin{aligned}
 \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[ \left( W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] &= \sum_{m=1}^M (t^{n+1} - t^n) \lambda_m \sum_{K \in \mathcal{T}} |K| \frac{1}{|K|^2} \left( \int_K e_m(x) dx \right)^2 \\
 &\leq (t^{n+1} - t^n) \sum_{m=1}^{\infty} \lambda_m \sum_{K \in \mathcal{T}} \int_K e_m^2(x) dx \\
 &\leq (t^{n+1} - t^n) \sum_{m=1}^{\infty} \lambda_m \leq (t^{n+1} - t^n) \Lambda_0.
 \end{aligned}$$

1

□

2 **Corollary 1.** *We deduce from Lemma 3.5 the following estimate*

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[ \left( W_K^{n+1} - W_K^n \right)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0. \quad (24)$$

*Proof.* We first show the limiting property

$$\|W_{M,\mathcal{T},k}(x, t) - W_{\mathcal{T},k}(x, t)\|_{L^2(\Omega \times \mathbb{T}^d)} \rightarrow 0 \text{ as } M \rightarrow \infty,$$

for all  $t \in [0, T]$ . Indeed

$$\begin{aligned}
 &\|W_{M,\mathcal{T},k}(x, t) - W_{\mathcal{T},k}(x, t)\|_{L^2(\Omega \times \mathbb{T}^d)}^2 \\
 &= \mathbb{E} \left[ \int_{\mathbb{T}^d} (W_{M,\mathcal{T},k}(x, t) - W_{\mathcal{T},k}(x, t))^2 dx \right] \\
 &= \mathbb{E} \left[ \sum_{K \in \mathcal{T}} |K| \left( \frac{1}{|K|} \int_K (W_M(x, t) - W(x, t)) dx \right)^2 \right] \\
 &\leq \mathbb{E} \left[ \sum_{K \in \mathcal{T}} |K| \frac{1}{|K|^2} \int_K 1 dx \int_K (W_M(x, t) - W(x, t))^2 dx \right] \\
 &\leq \mathbb{E} \left[ \int_{\mathbb{T}^d} dx (W_M(x, t) - W(x, t))^2 \right] \\
 &\rightarrow 0 \text{ as } M \rightarrow \infty,
 \end{aligned}$$

since the series defined by (5) is convergent in  $L^2(\Omega, C([0, T], L^2(\mathbb{T}^d)))$ . We deduce that

$$\begin{aligned} & \left| \|W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} - \|W_{\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} \right| \\ & \leq \| (W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)) - (W_{\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^n)) \|_{L^2(\Omega \times \mathbb{T}^d)} \\ & = \| (W_{M,\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^{n+1})) - (W_{M,\mathcal{T},k}(t^n) - W_{\mathcal{T},k}(t^n)) \|_{L^2(\Omega \times \mathbb{T}^d)} \\ & \leq \|W_{M,\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^{n+1})\|_{L^2(\Omega \times \mathbb{T}^d)} + \|W_{M,\mathcal{T},k}(t^n) - W_{\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} \\ & \rightarrow 0 \text{ as } M \rightarrow \infty, \end{aligned}$$

that is to say

$$\lim_{M \rightarrow \infty} \|W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} = \|W_{\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)}.$$

In view of (23), we obtain that

$$\begin{aligned} \|W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} &= \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[ \left( W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] \\ &\leq (t^{n+1} - t^n) \Lambda_0, \end{aligned}$$

where we take the limit  $M \rightarrow \infty$  to obtain:

$$\begin{aligned} \lim_{M \rightarrow \infty} \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[ \left( W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] &= \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[ \left( W_K^{n+1} - W_K^n \right)^2 \right] \\ &\leq (t^{n+1} - t^n) \Lambda_0, \end{aligned}$$

1 as the upper bound  $(t^{n+1} - t^n) \Lambda_0$  does not depend on  $M$ . □

#### 2 4. A priori estimates.

3 **Lemma 4.1.** *Assume that hypotheses (H) hold. Let  $T > 0$ ,  $\mathcal{T}$  be an admissible*  
4 *mesh in the sense of Definition 3.1 and  $h$  and  $k$  satisfy the Courant-Friedrichs-Lewy*  
5 *(CFL) condition:*

$$k \leq \frac{\alpha_{\mathcal{T}}^2 h}{2VC_f}. \quad (25)$$

Then, we have the following estimates

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{T}^d))}^2 \leq \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T \Lambda_0 M_g^2 |\mathbb{T}^d|$$

and

$$\|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T^2 \Lambda_0 M_g^2 |\mathbb{T}^d|,$$

6 where  $Q_T = \mathbb{T}^d \times [0, T]$ .

7 *Proof.* We recall the numerical scheme

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} = \frac{|K|}{k} g(u_K^n) (W_K^{n+1} - W_K^n). \quad (26)$$

We multiply both sides of (26) by  $ku_K^n$ :

$$|K| (u_K^{n+1} - u_K^n) u_K^n = -k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} u_K^n + |K| g(u_K^n) (W_K^{n+1} - W_K^n) u_K^n.$$

Applying the formula  $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$  with  $a = u_K^{n+1} - u_K^n$  and  $b = u_K^n$ , we obtain that

$$\begin{aligned} & \frac{|K|}{2} [(u_K^{n+1})^2 - (u_K^n)^2 - (u_K^{n+1} - u_K^n)^2] \\ &= -k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} u_K^n + |K| g(u_K^n) (W_K^{n+1} - W_K^n) u_K^n. \end{aligned}$$

1 Thus

$$\begin{aligned} \frac{|K|}{2} [(u_K^{n+1})^2 - (u_K^n)^2] &= \frac{|K|}{2} (u_K^{n+1} - u_K^n)^2 - k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} u_K^n \\ &\quad + |K| g(u_K^n) (W_K^{n+1} - W_K^n) u_K^n. \end{aligned} \quad (27)$$

We substitute (26) into (27) and take the expectation of both sides to deduce that

$$\begin{aligned} & \frac{|K|}{2} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] \\ &= \frac{|K|}{2} \mathbb{E} \left[ \left( \frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} \right)^2 \right] + \frac{|K|}{2} \mathbb{E} [g^2(u_K^n) (W_K^{n+1} - W_K^n)^2] \\ &\quad - 2 \frac{|K|}{2} \mathbb{E} \left[ \frac{k g(u_K^n)}{|K|} (W_K^{n+1} - W_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} \right] \\ &\quad + \mathbb{E} [g(u_K^n) |K| (W_K^{n+1} - W_K^n) u_K^n]. \end{aligned}$$

2 We remark that two terms in the equality above vanish. Indeed since  
3  $W_K^{n+1} - W_K^n$  and  $u_K^n$  are independent variables, we have that, in view of (22)

$$\mathbb{E} \left[ g(u_K^n) (W_K^{n+1} - W_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} \right] = 0,$$

and similarly

$$\mathbb{E} \left[ g(u_K^n) |K| (W_K^{n+1} - W_K^n) u_K^n \right] = \mathbb{E} [W_K^{n+1} - W_K^n] \mathbb{E} [g(u_K^n) u_K^n |K|] = 0.$$

4 Therefore,

$$\begin{aligned} \frac{|K|}{2} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] &= \frac{k^2}{2|K|} \mathbb{E} \left[ \left( \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} \right)^2 \right] \\ &\quad - k \mathbb{E} \left[ \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} u_K^n \right] \\ &\quad + \frac{|K|}{2} \mathbb{E} [(W_K^{n+1} - W_K^n)^2] \mathbb{E} [g^2(u_K^n)]. \end{aligned} \quad (28)$$

In view of (18), the equality (28) can be then rewritten as

$$\frac{|K|}{2} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] = B_1 - B_2 + D,$$

1 where

$$\begin{aligned}
B_1 &= \frac{k^2}{2|K|} \mathbb{E} \left[ \left( \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n,f} \right)^2 \right] \\
B_2 &= k \mathbb{E} \left[ \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| u_K^n F_{K,L}^{G,n,f} \right] \\
D &= \frac{|K|}{2} \mathbb{E} \left[ (W_K^{n+1} - W_K^n)^2 \right] \mathbb{E} [g^2(u_K^n)].
\end{aligned} \tag{29}$$

2 So that

$$\sum_{K \in \mathcal{T}} \frac{|K|}{2} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] = \sum_{K \in \mathcal{T}} (B_1 - B_2) + \sum_{K \in \mathcal{T}} D, \tag{30}$$

Using a similar method as in the Part **I.2** of Proposition 4 in [7] we deduce that

$$\sum_{K \in \mathcal{T}} (B_1 - B_2) \leq 0,$$

which we substitute in (30); this together with the definition of  $D$  in (29) and the inequality (24) yields:

$$\begin{aligned}
\sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^{n+1})^2] &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^n)^2] + \sum_{K \in \mathcal{T}} |K| \mathbb{E} [(W_K^{n+1} - W_K^n)^2] \mathbb{E} [g^2(u_K^n)] \\
&\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^n)^2] + k \Lambda_0 M_g^2,
\end{aligned}$$

which implies that

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^n)^2] \leq \sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^0)^2] + nk \Lambda_0 M_g^2 \quad \text{for all } n \in \{0, 1, 2, \dots, N\},$$

and that

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{T}^d))}^2 \leq \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T \Lambda_0 M_g^2.$$

As a consequence,

$$\|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T^2 \Lambda_0 M_g^2.$$

3

□

#### 4 4.1. Weak BV estimate.

5 **Lemma 4.2.** *Assume that hypotheses (H) hold. Let  $\mathcal{T}$  be an admissible mesh in the*  
6 *sense of Definition 3.1,  $T > 0$ ,  $N \in \mathbb{N}^+$  and let  $k = \frac{T}{N}$  satisfy the CFL condition*

$$k \leq \frac{(1-\xi)\alpha_{\mathcal{T}}^2 h}{2VC_f} \tag{31}$$

7 *for some  $\xi \in (0, 1)$ , which is stronger than the CFL condition (25). Then the*  
8 *following estimates hold*

9 1. *There exists a positive constant  $C_1$ , depending on  $\Lambda_0$ ,  $T$ ,  $M_g$ ,  $\xi$ ,  $C_f$  and*  
10  *$\|u_0\|_{L^2(\mathbb{T}^d)}$  such that*

$$\begin{aligned}
&\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[ v_{K,L}^n \{F^G(u_K^n, u_L^n) - f(u_K^n)\}^2 \right. \\
&\quad \left. + v_{L,K}^n \{F^G(u_L^n, u_K^n) - f(u_K^n)\}^2 \right] \leq C_1;
\end{aligned} \tag{32}$$

2. There exists a positive constant  $C_2$  depending on  $\alpha$ ,  $\Lambda_0$ ,  $T$ ,  $M_g$ ,  $\xi$ ,  $C_f$  and  $\|u_0\|_{L^2(\mathbb{T}^d)}$  such that

$$\begin{aligned} & \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \\ & \times \mathbb{E} \left[ v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c)) \right\} \right. \\ & \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(d) - F^G(c,d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(c) - F^G(c,d)) \right\} \right] \\ & \leq C_2 h^{-\frac{1}{2}}, \end{aligned}$$

where

$$\mathcal{I}_n := \{(K, L) \in \mathcal{T}^2 : L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n\}$$

and

$$\mathcal{C}(a, b) := \{(c, d) \in [\min(a, b), \max(a, b)]^2 : (d - c)(b - a) \geq 0\}.$$

*Proof.* 1, Multiplying the numerical scheme (15) by  $ku_K^n$ , inserting (18), taking the expectation and summing over  $K \in \mathcal{T}$  and  $n = 0, 1, \dots, N-1$  implies

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)u_K^n] + \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \mathbb{E} \left[ \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| u_K^n F_{K,L}^{G,n,f} \right] \\ & = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)u_K^n g(u_K^n)], \end{aligned}$$

which we denote as  $A + B = D$ . Note that the term  $D = 0$  since the increment  $(W_K^{n+1} - W_K^n)$  is independent of  $u_K^n$  and since by (22),  $\mathbb{E}[W_K^{n+1} - W_K^n] = 0$ . Applying the formula  $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$  with  $a = u_K^{n+1} - u_K^n$  and  $b = u_K^n$  we obtain:

$$\begin{aligned} A & = -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2] + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1})^2 - (u_K^n)^2] \\ & = -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2] + \frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^N)^2 - (u_K^0)^2]. \end{aligned}$$

We set

$$A_1 = -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2],$$

1 and

$$A_2 = \frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^N)^2 - (u_K^0)^2] \geq -\frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^0)^2]. \quad (33)$$

Next, substituting (15) into  $A_1$ , also using (18) and the fact that  $W_K^{n+1} - W_K^n$  and any function of  $u_K^n$ ,  $u_L^n$ ,  $v_{K,L}^n$  and  $v_{L,K}^n$  are independent, we deduce that

$$A_1 = -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \left\{ \mathbb{E}[(W_K^{n+1} - W_K^n)^2 g^2(u_K^n)] \right.$$

$$+ \frac{k^2}{|K|} \mathbb{E} \left[ \left( \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n,f} \right)^2 \right] \Bigg\}.$$

Using a similar idea in the proof of Proposition 2 in [7], we deduce that:

$$\begin{aligned} & \sum_{K \in \mathcal{T}} \frac{k^2}{2|K|} \mathbb{E} \left[ \left( \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n,f} \right)^2 \right] \\ & \leq \frac{1-\xi}{2} k \sum_{(K,L) \in \mathcal{I}_n} \frac{|\sigma_{K,L}|}{2C_f} \\ & \quad \times \mathbb{E} \left[ v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))^2 \right\} \right. \\ & \quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))^2 \right\} \right]. \end{aligned}$$

We denote by

$$\begin{aligned} M_1 & := v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))^2 \right\} \\ M_2 & := v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))^2 \right\}, \end{aligned}$$

therefore

$$\begin{aligned} A_1 & \geq -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \mathbb{E} [(W_K^{n+1} - W_K^n)^2 g^2(u_K^n)] \\ & \quad - \frac{1-\xi}{2} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \left( \frac{|\sigma_{K,L}|}{2C_f} \mathbb{E} [M_1 + M_2] \right). \end{aligned}$$

Next we deduce from (24) and (33) that

$$\begin{aligned} A & = A_1 + A_2 \\ & \geq -\frac{1}{2} (\|u_0\|_{L^2(\mathbb{T}^d)}^2 + T\Lambda_0 M_g^2) - \frac{1-\xi}{2} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \left( \frac{|\sigma_{K,L}|}{2C_f} \mathbb{E} [M_1 + M_2] \right). \end{aligned}$$

The term  $B$  is estimated by using the same idea as in the proof of Proposition 2 in [7]. Since

$$B \geq \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \left( \frac{|\sigma_{K,L}|}{4C_f} \mathbb{E} [M_1 + M_2] \right),$$

and since  $A + B = 0$ , it follows that

$$\|u_0\|_{L^2(\mathbb{T}^d)}^2 + T\Lambda_0 M_g^2 \geq \frac{\xi}{2C_f} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \left( |\sigma_{K,L}| \mathbb{E} [M_1 + M_2] \right),$$

1 which in turn implies the estimate

$$\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \left( |\sigma_{K,L}| \mathbb{E} [M_1 + M_2] \right) \leq C_1, \quad (34)$$

where the positive constant  $C_1$  depends on  $\Lambda_0$ ,  $T$ ,  $M_g$ ,  $\xi$ ,  $C_f$  and  $\|u_0\|_{L^2(\mathbb{T}^d)}$ . We then reorder the summation to deduce that

$$\begin{aligned} & \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[ v_{K,L}^n \left\{ (F^G(u_K^n, u_L^n) - f(u_K^n))^2 + (F^G(u_K^n, u_L^n) - f(u_L^n))^2 \right\} \right. \\ & \quad \left. + v_{L,K}^n \left\{ (F^G(u_L^n, u_K^n) - f(u_K^n))^2 + (F^G(u_L^n, u_K^n) - f(u_L^n))^2 \right\} \right] \\ &= \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[ v_{K,L}^n \left\{ F^G(u_K^n, u_L^n) - f(u_K^n) \right\}^2 \right. \\ & \quad \left. + v_{L,K}^n \left\{ F^G(u_L^n, u_K^n) - f(u_K^n) \right\}^2 \right] \end{aligned}$$

which together with (34) implies

$$\begin{aligned} & \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[ v_{K,L}^n \left\{ F^G(u_K^n, u_L^n) - f(u_K^n) \right\}^2 \right. \\ & \quad \left. + v_{L,K}^n \left\{ F^G(u_L^n, u_K^n) - f(u_K^n) \right\}^2 \right] \leq C_1. \end{aligned}$$

This completes the proof of the inequality (32). Next we present the proof of 2. We estimate the term

$$\begin{aligned} \bar{T}^2 &= \left( \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \right. \\ & \times \mathbb{E} \left[ v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d)) \right\} \right. \\ & \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(c) - F^G(c,d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(d) - F^G(c,d)) \right\} \right] \Big)^2. \end{aligned}$$

We define

$$\begin{aligned} T_1 &:= \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d)), \\ T_2 &:= \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(c) - F^G(c,d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(d) - F^G(c,d)). \end{aligned}$$

Then

$$\bar{T}^2 = \left( \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} [v_{K,L}^n T_1 + v_{L,K}^n T_2] \right)^2.$$

We apply the Cauchy-Schwarz inequality and Jensen's inequality

$$\left( \frac{v_{K,L}^n T_1 + v_{L,K}^n T_2}{v_{K,L}^n + v_{L,K}^n} \right)^2 \leq \frac{v_{K,L}^n}{v_{K,L}^n + v_{L,K}^n} T_1^2 + \frac{v_{L,K}^n}{v_{K,L}^n + v_{L,K}^n} T_2^2,$$

1 to deduce that:

$$\begin{aligned}
\bar{T}^2 &\leq \left( \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \\
&\quad \times \left( \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[ \frac{(v_{K,L}^n T_1 + v_{L,K}^n T_2)^2}{v_{K,L}^n + v_{L,K}^n} \right] \right) \\
&\leq \left( \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \\
&\quad \times \left( \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} [v_{K,L}^n T_1^2 + v_{L,K}^n T_2^2] \right). \tag{35}
\end{aligned}$$

It follows from (8) that

$$\begin{aligned}
\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| v_{K,L}^n \\
&\leq T \sum_{K \in \mathcal{T}} |\partial K| V \leq \frac{TV|\mathbb{T}^d|}{\alpha_7^2 h}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
T_1^2 &\leq 2 \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))^2 \right\}, \\
T_2^2 &\leq 2 \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))^2 \right\},
\end{aligned}$$

2 which we substitute into (35) to deduce that

$$\bar{T}^2 \leq \frac{2TV|\mathbb{T}^d|}{\alpha_7^2 h} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \left( |\sigma_{K,L}| \mathbb{E} [M_1 + M_2] \right). \tag{36}$$

Substituting (34) into (36) yields

$$\bar{T}^2 \leq \frac{2TV|\mathbb{T}^d|}{\alpha_7^2} C_1 h^{-1}$$

which combined with the definition of  $\bar{T}^2$  implies

$$\begin{aligned}
&\left( \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \right. \\
&\quad \times \mathbb{E} \left[ v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d)) \right\} \right. \\
&\quad \left. \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(c) - F^G(c,d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(d) - F^G(c,d)) \right\} \right] \right)^2 \\
&\leq \frac{2TV|\mathbb{T}^d|}{\alpha_7^2} C_1 h^{-1}.
\end{aligned}$$

1 We choose  $C_2 = \frac{2TV|\mathbb{T}^d|}{\alpha_T^2}C_1$ , which completes the proof of 2.  $\square$

## 2 5. Convergence of the scheme.

3 **5.1. A time-continuous approximation.** We define  $\bar{u}_K$  as the continuous in  
4 time stochastic process

$$\begin{aligned} \bar{u}_K(t) &= u_K^n - \frac{t - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} + g(u_K^n)(W_K(t) - W_K^n) \\ &= u_K^n - \int_{nk}^t \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \frac{F_{K,L}^{G,n}}{|K|} ds + \int_{nk}^t g(u_K^n) dW_K(t). \end{aligned} \quad (37)$$

on the domain  $\Omega \times [nk, (n+1)k]$ . In this way,  $\bar{u}_K(nk) = u_K^n$  and  $\bar{u}_K((n+1)k) = u_K^{n+1}$ . We define the time-continuous approximate solution  $\bar{u}_{\mathcal{T},k}$  on  $\Omega \times \mathbb{T}^d \times [0, T]$  by

$$\bar{u}_{\mathcal{T},k}(x, t) = \bar{u}_K(t^n) \quad \text{for } x \in K \quad \text{and } t \in [nk, (n+1)k].$$

5 Next we estimate the difference between the time-continuous approximation  $\bar{u}_{\mathcal{T},k}$   
6 and the finite volume solution  $u_{\mathcal{T},k}$  which is defined in (19).

**Lemma 5.1.** *Assume that the assumptions in Lemma 4.2 are satisfied. There exists a positive constant  $C$  depending on  $T, M_g, C_f, \alpha, V$  and  $u_0$  such that*

$$\|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq C(h+k).$$

*Proof.* In a same way as Lemma 3.4, one deduce that  $\mathbb{E}[W_K(s) - W_K^n] = 0$ . Also using the definitions of  $u_{\mathcal{T},k}$  and of  $\bar{u}_{\mathcal{T},k}$  and formula (18), we deduce

$$\begin{aligned} & \|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \\ &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[ \left( g(u_K^n)(W_K(s) - W_K^n) \right. \right. \\ & \quad \left. \left. + \frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n,f} \right)^2 \right] dx ds \\ &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} [g^2(u_K^n)(W_K(s) - W_K^n)^2] dx ds \\ & \quad + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[ \left( \frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n,f} \right)^2 \right] dx ds. \end{aligned}$$

Applying the counter part of (24)

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} [(W_K(s) - W_K^n)^2] \leq (s - t^n) \Lambda_0,$$

7 for all  $nk \leq s \leq (n+1)k$ , we deduce that

$$\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} [g^2(u_K^n)(W_K(s) - W_K^n)^2] dx ds$$

$$\begin{aligned} &\leq \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} (s - t^n) \Lambda_0 M_g^2 ds \\ &\leq T \Lambda_0 M_g^2 k. \end{aligned}$$

1 Moreover, using the CFL condition (31) and the inequality (32), we deduce that:

$$\begin{aligned} &\sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[ \left( \frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n,f} \right)^2 \right] dx ds \\ &\leq C_1 \frac{\alpha_{\mathcal{T}}^2 h}{V(2C_f)^2}. \end{aligned}$$

Therefore

$$\|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T \Lambda_0 M_g^2 k + \frac{C_1 \alpha_{\mathcal{T}}^2}{V(2C_f)^2} h.$$

2 Finally we set  $C = \max\left(T \Lambda_0 M_g^2, \frac{C_1 \alpha_{\mathcal{T}}^2}{V(2C_f)^2}\right)$  to deduce the result of Lemma 5.1.

3 □

4 **5.2. Entropy inequalities for the approximate solution.** In this section, we  
5 show entropy inequalities satisfied by the approximate solution and use them in the  
6 convergence proof of the numerical scheme.

7 **Lemma 5.2** (Discrete entropy inequality). *Assume the assumptions in Theorem*  
8 *3.3 are satisfied. Then P-a.s in  $\Omega$ , for all  $\eta \in \mathcal{A}$  and for all  $\varphi \in \mathcal{C}$ :*

$$\begin{aligned} & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_K (\eta(u_K^{n+1}) - \eta(u_K^n)) \varphi(x, nk) dx \\ & + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt \\ & + \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx \\ & + \frac{1}{2} \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) q_K dx dt \\ & \geq R^{k,h} \end{aligned} \tag{38}$$

9 where  $q_K = \sum_{m=1}^{\infty} \lambda_m \left( \frac{1}{|K|} \int_K e_m^2(x) dx \right)^2$ ,  $F^\eta(a) = \int_0^a \eta'(s) f'(s) ds$  and for all sets

10  $A \in \mathcal{F}$ ,  $\mathbb{E}[\mathbf{1}_A R^{k,h}] \rightarrow 0$  as  $h \rightarrow 0$ .

11 Before proving Lemma 5.2, we prove an equality based upon Itô's formula (cf. [22],  
12 Theorem 7.4.3).

1 **Lemma 5.3.** For all  $\eta \in \mathcal{A}$ , we deduce the following formula from Itô's formula

$$\begin{aligned}
 & \eta(\bar{u}_K((n+1)k)) - \eta(\bar{u}_K(nk)) \\
 &= - \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \frac{F_{K,L}^{G,n}}{|K|} dt \\
 & \quad + \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) g(u_K^n) dW_K(t) \\
 & \quad + \frac{1}{2} q_K \int_{nk}^{(n+1)k} \eta''(\bar{u}_{\mathcal{T},k}(t)) g^2(u_K^n) dt
 \end{aligned} \tag{39}$$

2 P-a.s in  $\Omega$ .

*Proof.* Using the formula of  $W_K(t)$ , we deduce that

$$W_K(t^{n+1}) - W_K(t^n) = \sum_{m=1}^{\infty} \left\{ \left( \frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right) (\beta_m(t^{n+1}) - \beta_m(t^n)) \right\}$$

where

$$\beta_m(t^{n+1}) - \beta_m(t^n) \sim \mathcal{N}(0, t^{n+1} - t^n).$$

Using the property that if a random variable  $Y \sim \mathcal{N}(\mu, \sigma^2)$ , then the random variable  $aY + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$  [18], we deduce that

$$\begin{aligned}
 & \left( \frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right) (\beta_m(t^{n+1}) - \beta_m(t^n)) \\
 & \sim \mathcal{N} \left( 0, \left( \frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2 (t^{n+1} - t^n) \right)
 \end{aligned}$$

for all  $m \in \{1, 2, \dots\}$ . Because the Brownian motions  $\{\beta_m(t)\}_{m \geq 1}$  are independent, and using the fact that if two independent random variables  $Y_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$  and  $Y_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$  then  $Y_1 + Y_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$  [18], we deduce that

$$W_K(t^{n+1}) - W_K(t^n) \sim \mathcal{N} \left( 0, \sum_{m=1}^{\infty} \left( \frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2 (t^{n+1} - t^n) \right),$$

so that we can rewrite the stochastic process  $W_K(t)$  in the form  $W_K(t) = q_K^{\frac{1}{2}} \beta(t)$  where  $\beta$  is a standard Brownian motion. We briefly recall Itô's formula [cf. [22], (7.4.3)] : Let  $W(\tau)$  be an  $(\mathcal{F}_t)$ -Brownian motion and let  $X(\tau)$  be an  $\mathbb{R}$ -valued stochastic process given by

$$X(\tau) = X(0) + \int_0^\tau \psi(t) dt + \int_0^\tau \theta(t) dW(t), \quad 0 \leq \tau \leq T,$$

where  $X(0)$  is  $\mathcal{F}_0$ -measurable,  $\psi$  is  $(\mathcal{F}_t)$ -adapted and measurable in  $(\omega, t)$  such that  $\int_0^T \|\psi(t)\|_{\mathbb{R}} dt < \infty$ , P-a.s. and  $\theta$  is an  $(\mathcal{F}_t)$ -adapted and measurable process such that  $\mathbb{P} \left( \int_0^T \theta^2(t) dt < \infty \right) = 1$ . Suppose that the function  $\mathcal{G}(x_1, x_2) : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$

is such that its partial derivatives  $\frac{\partial \mathcal{G}}{\partial x_2}$ ,  $\frac{\partial \mathcal{G}}{\partial x_1}$  and  $\frac{\partial^2 \mathcal{G}}{\partial x_1^2}$  are continuous on  $\mathbb{R} \times [0, T]$ .

Then P-a.s. for all  $\tau \in [0, T]$ ,

$$\mathcal{G}(X(\tau), \tau) = \mathcal{G}(X(0), 0) + \int_0^\tau \left\{ \frac{\partial \mathcal{G}}{\partial x_2}(X(t), t) + \frac{\partial \mathcal{G}}{\partial x_1}(X(t), t) \psi(t) \right\} dt$$

$$\begin{aligned}
& + \int_0^\tau \frac{\partial \mathcal{G}}{\partial x_1}(X(t), t) \cdot \theta(t) dW(t) \\
& + \frac{1}{2} \int_0^\tau \frac{\partial^2 \mathcal{G}}{\partial x_1^2}(X(t), t) \cdot \theta^2(t) dt.
\end{aligned}$$

1 We apply Itô's formula to the case that

- 2 •  $\mathcal{G}(X_t, t) = \eta(X_t)$ ,  
3 •  $X(t) = \bar{u}_K(t)$  which is defined in (37) on the time interval  $[nk, (n+1)k]$ ,  
4 •  $\psi = - \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \frac{F_{K,L}^{G,n}}{|K|}$ ,  
5 •  $\theta = q_K^{\frac{1}{2}} g(u_K^n)$ .

Thus:

$$\frac{\partial \mathcal{G}}{\partial x_2} = 0, \quad \frac{\partial \mathcal{G}}{\partial x_1} = \eta'(x_1) \quad \text{and} \quad \frac{\partial^2 \mathcal{G}}{\partial x_1^2} = \eta''(x_1)$$

with  $x_1 = X(t) = \bar{u}_K(t)$ , on  $t \in [nk, (n+1)k]$  we deduce that

$$\begin{aligned}
& \eta(\bar{u}_K((n+1)k)) - \eta(\bar{u}_K(nk)) \\
& = - \int_{nk}^{(n+1)k} \eta'(\bar{u}_K(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \frac{F_{K,L}^{G,n}}{|K|} dt \\
& \quad + \int_{nk}^{(n+1)k} \eta'(\bar{u}_K(t)) g(u_K^n) dW_K(t) \\
& \quad + \frac{q_K}{2} \int_{nk}^{(n+1)k} \eta''(\bar{u}_K(t)) g^2(u_K^n) dt
\end{aligned}$$

6 P *-a.s.* in  $\Omega$ . This completes the proof of Lemma 5.3. □

7 We then present the proof of Lemma 5.2.

8 *Proof. Step 1:* Recalling that  $\bar{u}_K(nk) = u_K^n$ ,  $\bar{u}_K((n+1)k) = u_K^{n+1}$ , and using (12),  
9 after multiplying (39) by  $|K| \varphi_K^n$ , one deduces that P *-a.s.* in  $\Omega$ ,

$$\begin{aligned}
& \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n [\eta(u_K^{n+1}) - \eta(u_K^n)] \\
& = - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \varphi_K^n \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) \sum_{L \in \mathcal{N}(K)} \left( |\sigma_{K,L}| F_{K,L}^{G,n,f} \right) dt \\
& \quad + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) g(u_K^n) dW_K(t) \\
& \quad + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| q_K \varphi_K^n \int_{nk}^{(n+1)k} \eta''(\bar{u}_{\mathcal{T},k}(t)) g^2(u_K^n) dt,
\end{aligned} \tag{40}$$

with

$$\varphi_K^n = \frac{1}{|K|} \int_K \varphi(x, nk) dx.$$

The equation (40) can be written in the form  $A^{k,h} = -B^{k,h} + C^{k,h} + D^{k,h}$ . Since by the assumption (20),  $\frac{k}{h} \rightarrow 0$  as  $h \rightarrow 0$  in the theorem, we may suppose that the

CFL condition

$$k \leq \frac{(1-\xi)\alpha_{\mathcal{T}}^2}{2VC_f}h$$

- 1 holds for some  $\xi \in (0, 1)$ . Thus the estimates in Lemma 4.2 hold.

**Study of  $B^{k,h}$ :** we decompose  $B^{k,h}$  as

$$B^{k,h} = (B^{k,h} - \tilde{B}^{k,h}) + (\tilde{B}^{k,h} - \bar{B}^{k,h}) + (\bar{B}^{k,h} - B_1^{k,h}) + B_1^{k,h}$$

where

$$\begin{aligned} \tilde{B}^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta'(u_K^n) \varphi(x, nk) dx dt \\ &\quad \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \\ \bar{B}^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \varphi(x, nk) dx dt \\ &\quad \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n G^G(u_K^n, u_L^n) - v_{L,K}^n G^G(u_L^n, u_K^n)\} \\ B_1^{k,h} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_K^n) \mathbf{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt. \end{aligned}$$

where

$$G^G(a, b) = F^\eta(s(a, b)). \quad (\text{cf. (6) and Definition 3.2})$$

In the following we prove that  $\tilde{B}^{h,k} - \bar{B}^{h,k} \geq 0$  almost surely. Recalling that

$$\begin{aligned} \sum_{L \in \mathcal{N}(K)} (v_{K,L}^n - v_{L,K}^n) F^\eta(u_K^n) &= 0, \\ \sum_{L \in \mathcal{N}(K)} (v_{K,L}^n - v_{L,K}^n) f(u_K^n) &= 0, \end{aligned}$$

and that for all  $K \in \mathcal{T}$ ,  $F^G(u_K^n, u_K^n) = f(u_K^n)$  and  $G^G(u_K^n, u_K^n) = F^\eta(u_K^n)$ ; we rewrite  $\tilde{B}^{h,k} - \bar{B}^{h,k}$  as:

$$\begin{aligned} &\tilde{B}^{h,k} - \bar{B}^{h,k} \\ &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k}{|K|} \int_K \varphi(x, nk) dx \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \times \\ &\quad \left\{ v_{K,L}^n [\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n))] \right. \\ &\quad \left. - v_{L,K}^n [\eta'(u_K^n)(F^G(u_L^n, u_K^n) - f(u_K^n)) - (G^G(u_L^n, u_K^n) - F^\eta(u_K^n))] \right\}. \end{aligned}$$

- 2 Let  $K, L \in \mathcal{T}$ ,  $L \in \mathcal{N}(K)$  and suppose that  $u_K^n < u_L^n$ .

We study the sign of  $\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n))$ . Using the fact that  $F^G$  is a Godunov numerical flux (cf. (9)), we deduce that there exists  $s(u_K^n, u_L^n) \in [u_K^n, u_L^n]$  such that  $F^G(u_K^n, u_L^n) = f(s(u_K^n, u_L^n)) = \min_{s \in [u_K^n, u_L^n]} f(s)$ .

Thus

$$\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n))$$

$$\begin{aligned}
&= \eta'(u_K^n)(f(s(u_K^n, u_L^n)) - f(u_K^n)) - (F^\eta(s(u_K^n, u_L^n)) - F^\eta(u_K^n)) \\
&= \int_{u_K^n}^{s(u_K^n, u_L^n)} f'(s)\eta'(u_K^n)ds - \int_{u_K^n}^{s(u_K^n, u_L^n)} f'(s)\eta'(s)ds \\
&= \int_{u_K^n}^{s(u_K^n, u_L^n)} f'(s)(\eta'(u_K^n) - \eta'(s))ds \\
&= f(s(u_K^n, u_L^n))\{\eta'(u_K^n) - \eta'(s(u_K^n, u_L^n))\} + \int_{u_K^n}^{s(u_K^n, u_L^n)} f(s)\eta''(s)ds \\
&\geq f(s(u_K^n, u_L^n))\{\eta'(u_K^n) - \eta'(s(u_K^n, u_L^n))\} + \int_{u_K^n}^{s(u_K^n, u_L^n)} f(s(u_K^n, u_L^n))\eta''(s)ds \\
&= 0.
\end{aligned}$$

Similarly

$$\begin{aligned}
&\eta'(u_K^n)(F^G(u_L^n, u_K^n) - f(u_K^n)) - (G^G(u_L^n, u_K^n) - F^\eta(u_K^n)) \\
&= \eta'(u_K^n)(f(s(u_L^n, u_K^n)) - f(u_K^n)) - (F^\eta(s(u_K^n, u_L^n)) - F^\eta(u_K^n)) \\
&= \int_{u_K^n}^{s(u_L^n, u_K^n)} f'(s)(\eta'(u_K^n) - \eta'(s))ds \\
&= f(s(u_L^n, u_K^n))\{\eta'(u_K^n) - \eta'(s(u_L^n, u_K^n))\} + \int_{u_K^n}^{s(u_L^n, u_K^n)} f(s)\eta''(s)ds \\
&\leq f(s(u_L^n, u_K^n))\{\eta'(u_K^n) - \eta'(s(u_L^n, u_K^n))\} + \int_{u_K^n}^{s(u_L^n, u_K^n)} f(s(u_L^n, u_K^n))\eta''(s)ds \\
&= 0.
\end{aligned}$$

Also using (10) and (11) we obtain

$$\begin{aligned}
&v_{K,L}^n[\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n))] \\
&- v_{L,K}^n[\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n))] \\
&\geq 0,
\end{aligned}$$

1 which implies  $\tilde{B}^{h,k} - \bar{B}^{h,k} \geq 0$  almost surely in  $\Omega$ .

**Study of  $C^{k,h}$ :** we decompose  $C^{k,h}$  as

$$C^{k,h} = C^{k,h} - \tilde{C}^{k,h} + \tilde{C}^{k,h}$$

with

$$\tilde{C}^{k,h} = \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \eta'(u_K^n)g(u_K^n)\varphi(x, nk)dW_K(t)dx.$$

**Study of  $D^{k,h}$ :** we decompose  $D^{k,h}$  as

$$D^{k,h} = D^{k,h} - \tilde{D}^{k,h} + \tilde{D}^{k,h}$$

with

$$\tilde{D}^{k,h} = \frac{1}{2} \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \left( \int_K q_K \eta''(u_K^n)g^2(u_K^n)\varphi(x, nk)dx \right) dt.$$

Since P-a.s. in  $\Omega$ ,  $A^{k,h} = -B^{k,h} + C^{k,h} + D^{k,h}$ , we obtain

$$\begin{aligned} B^{k,h} &= -A^{k,h} + C^{k,h} + D^{k,h} \\ &\geq (B^{k,h} - \tilde{B}^{k,h}) + (\bar{B}^{k,h} - B_1) + B_1, \end{aligned}$$

so that

$$-A^{k,h} - B_1 \geq (B^{k,h} - \tilde{B}^{k,h}) + (\bar{B}^{k,h} - B_1) - C^{k,h} - D^{k,h},$$

1 and that

$$\begin{aligned} &-A^{k,h} - B_1 + \tilde{C}^{k,h} + \tilde{D}^{k,h} \\ &\geq (B^{k,h} - \tilde{B}^{k,h}) + (\bar{B}^{k,h} - B_1) + (\tilde{C}^{k,h} - C^{k,h}) + (\tilde{D}^{k,h} - D^{k,h}). \end{aligned} \quad (41)$$

Substituting the definition of  $A^{k,h}$ ,  $B_1$ ,  $\tilde{C}^{k,h}$  and  $\tilde{D}^{k,h}$  into (41) yields

$$\begin{aligned} &-\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_K (\eta(u_K^{n+1}) - \eta(u_K^n)) \varphi(x, nk) dx \\ &+ \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx dt \\ &+ \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx \\ &+ \frac{1}{2} \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K q_K \eta''(u_K^n) g(u_K^n) \varphi(x, nk) dx dt \\ &\geq (B^{k,h} - \tilde{B}^{k,h}) + (\bar{B}^{k,h} - B_1) + (\tilde{C}^{k,h} - C^{k,h}) + (\tilde{D}^{k,h} - D^{k,h}). \end{aligned}$$

We define

$$R^{k,h} = (B^{k,h} - \tilde{B}^{k,h}) + (\bar{B}^{k,h} - B_1) + (\tilde{C}^{k,h} - C^{k,h}) + (\tilde{D}^{k,h} - D^{k,h})$$

2 to deduce the inequality (38).

3 **Step 2:** Next we prove that for all sets  $A \in \mathcal{F}$ ,  $\mathbb{E}[\mathbf{1}_A R^{k,h}] \rightarrow 0$  as  $h, k \rightarrow$   
 4 0. Let  $A \in \mathcal{F}$ ; we analyze separately the convergence of  $\mathbb{E}[\mathbf{1}_A (B^{k,h} - \tilde{B}^{k,h})]$ ,  
 5  $\mathbb{E}[\mathbf{1}_A (\bar{B}^{k,h} - B_1^{k,h})]$ ,  $\mathbb{E}[\mathbf{1}_A (\tilde{C}^{k,h} - C^{k,h})]$ ,  $\mathbb{E}[\mathbf{1}_A (\tilde{D}^{k,h} - D^{k,h})]$ . Note that the as-  
 6 sumption that  $\frac{k}{h} \rightarrow 0$  as  $h \rightarrow 0$  is crucial.

7 **Convergence of  $\mathbb{E}[\mathbf{1}_A (B^{k,h} - \tilde{B}^{k,h})]$**

We prove that for all sets  $A \in \mathcal{F}$ ,  $\mathbb{E}[\mathbf{1}_A (B^{k,h} - \tilde{B}^{k,h})] \rightarrow 0$  as  $h, k \rightarrow 0$ . For almost  
 all  $\omega \in \Omega$ , all  $K \in \mathcal{T}$  and all  $n \in \{0, 1, \dots, N-1\}$ , there exists  $\zeta_K^n(\omega) \in \mathbb{R}$  such that

$$\begin{aligned} &B^{k,h} - \tilde{B}^{k,h} \\ &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K [\eta'(\bar{u}_{\mathcal{T},k}(t)) - \eta'(u_K^n)] \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} \varphi(x, nk) dx dt \\ &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K [\eta''(\zeta_K^n)(\bar{u}_{\mathcal{T},k}(t) - u_K^n)] \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} \varphi(x, nk) dx dt \\ &=: T_1^{h,k} + T_2^{h,k}, \end{aligned}$$

where we set, in view of (37) and (18),

$$\begin{aligned}
T_1^{h,k} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta''(\zeta_K^n) \varphi(x, nk) \frac{t-nk}{|K|} dx dt \\
&\times \left( \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \\
T_2^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K [\eta''(\zeta_K^n) g(u_K^n) (W_K(t) - W_K(nk))] \\
&\times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| F_{K,L}^{G,n} \varphi(x, nk) dx dt.
\end{aligned}$$

We remark that  $\mathbb{E}[T_2^{h,k}] = 0$ . Let  $A \in \mathcal{F}$ ; we study  $\mathbb{E}[\mathbf{1}_A T_1^{h,k}]$ . Using Cauchy-Schwarz inequality as in (35), the assumption (8) and the estimate (32), we deduce that

$$\begin{aligned}
& \left| \mathbb{E}[\mathbf{1}_A T_1^{h,k}] \right| \\
&= \left| \mathbb{E} \left[ \mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta''(\zeta_K^n) \frac{t-nk}{|K|} \varphi(x, nk) dx dt \times \right. \right. \\
&\quad \left. \left. \left( \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] \right| \\
&\leq \|\eta''\|_\infty \|\varphi\|_\infty \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k^2}{|K|} \left( \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \times \\
&\quad \mathbb{E} \left[ \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \\
&\leq \|\eta''\|_\infty \|\varphi\|_\infty \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k^2}{|K|} V |\partial K| \times \\
&\quad \mathbb{E} \left[ \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \\
&\leq C_1 \|\eta''\|_\infty \|\varphi\|_\infty \frac{k}{\alpha_{\mathcal{T}}^2 h} V,
\end{aligned}$$

- 1 which tends to 0 as  $h, k \rightarrow 0$ . This completes the proof of  $\mathbb{E}[\mathbf{1}_A (B^{k,h} - \tilde{B}^{k,h})] \rightarrow 0$   
2 as  $h, k \rightarrow 0$ .

**Convergence of  $\mathbb{E}[\mathbf{1}_A (\bar{B}^{h,k} - B_1^{h,k})]$**

We prove that for all sets  $A \in \mathcal{F}$ ,  $\mathbb{E}[\mathbf{1}_A (\bar{B}^{h,k} - B_1^{h,k})] \rightarrow 0$  as  $h, k \rightarrow 0$ . Using the fact that

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n - v_{L,K}^n) F^\eta(u_K^n) = 0,$$

we deduce that

$$\begin{aligned} \bar{B}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} \left[ |\sigma_{K,L}| \int_K \varphi(x, nk) dx \right. \\ &\quad \left. \times \{v_{K,L}^n(G^G(u_K^n, u_L^n) - F^\eta(u_K^n)) - v_{L,K}^n(G^G(u_L^n, u_K^n) - F^\eta(u_K^n))\} \right] \end{aligned}$$

so that,  $\bar{B}^{h,k} = \bar{T}_1^{h,k} + \bar{T}_2^{h,k}$ , where

$$\begin{aligned} \bar{T}_1^{h,k} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \frac{k}{|K|} |\sigma_{K,L}| \int_K \varphi(x, nk) dx \\ &\quad \times \{v_{K,L}^n(G^G(u_K^n, u_L^n) - F^\eta(u_K^n)) - v_{L,K}^n(G^G(u_L^n, u_K^n) - F^\eta(u_K^n))\} \end{aligned}$$

and

$$\begin{aligned} \bar{T}_2^{h,k} &= \sum_{n=0}^{N-1} \sum_{(L,K) \in \mathcal{I}_n} \frac{k}{|K|} |\sigma_{K,L}| \int_K \varphi(x, nk) dx \\ &\quad \times \{v_{K,L}^n(G^G(u_K^n, u_L^n) - F^\eta(u_K^n)) - v_{L,K}^n(G^G(u_L^n, u_K^n) - F^\eta(u_K^n))\} \\ &= - \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \frac{k}{|L|} |\sigma_{K,L}| \int_L \varphi(x, nk) dx \\ &\quad \times \{v_{K,L}^n(G^G(u_K^n, u_L^n) - F^\eta(u_L^n)) - v_{L,K}^n(G^G(u_L^n, u_K^n) - F^\eta(u_L^n))\}. \end{aligned}$$

Similarly,

$$\begin{aligned} B_1^{h,k} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} F^\eta(u_K^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} \varphi(\gamma, nk) d\gamma(x) dt \\ &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(\gamma, nk) F^\eta(u_K^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\ &\quad + \sum_{n=0}^{N-1} \sum_{(L,K) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(\gamma, nk) F^\eta(u_K^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\ &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(\gamma, nk) F^\eta(u_K^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\ &\quad - \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(\gamma, nk) F^\eta(u_L^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\ &:= T_1^{h,k} + T_2^{h,k}. \end{aligned}$$

- 1 In order to estimate the terms  $|\bar{T}_1^{h,k} - T_1^{h,k}|$  and  $|\bar{T}_2^{h,k} - T_2^{h,k}|$ , we refer to the  
 2 arguments of estimating  $|\bar{T}_1^{h,k} - T_1^{h,k}|$  and  $|\bar{T}_2^{h,k} - T_2^{h,k}|$  in Proposition 4 of [7], which  
 3 yields that for all sets  $A \in \mathcal{F}$ ,  $\mathbb{E} \left[ \mathbf{1}_A (\bar{B}^{h,k} - B_1^{h,k}) \right] \rightarrow 0$  as  $h, k \rightarrow 0$ .  
 4

- 5 The detailed proofs of the convergence of the convergence of  $\mathbb{E}[\mathbf{1}_A(\tilde{C}^{k,h} - C^{k,h})]$   
 6 and of  $\mathbb{E}[\mathbf{1}_A(\tilde{D}^{k,h} - D^{k,h})]$  are given in [16]. We thus deduce that  $\mathbb{E}[\mathbf{1}_A R^{k,h}] \rightarrow 0$   
 7 as  $h, k \rightarrow 0$ , which concludes the proof of Lemma 5.2.  $\square$

1 We refer to [6, 7] and [16] for the detailed proofs for the following lemma.

2 **Lemma 5.4** (Continuous entropy inequality on the discrete solution). *Assume that*  
 3 *the assumptions in Theorem 3.3 are satisfied.*

4 *Then P-a.s. in  $\Omega$ , for all  $\eta \in \mathcal{A}$  and for all  $\varphi \in \mathcal{C}$ , the discrete solution satisfies*

$$\begin{aligned}
 & \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \\
 & + \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \\
 & + \int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) dx \\
 & + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) Q(x, x) dx dt \\
 & \geq \tilde{R}^{k,h},
 \end{aligned} \tag{42}$$

5 *where for all set  $A \in \mathcal{F}$ ,  $\mathbb{E}[\mathbf{1}_A \tilde{R}^{k,h}] \rightarrow 0$  as  $h, k \rightarrow 0$ .*

6 **6. Convergence proof.** The main result of this article is Theorem 3.3. In the  
 7 following, we proof this result.

8 *Proof.* The proof of Theorem 3.3. We multiply the inequality (42) by  $\mathbf{1}_A$ , namely  
 9 the characteristic function of the set  $A \in \mathcal{F}$ . We take the expectation, which yields:

$$\begin{aligned}
 & \mathbb{E} \left[ \mathbf{1}_A \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx \right] + \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \right] \\
 & + \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right] \\
 & + \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) dx \right] \\
 & + \frac{1}{2} \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) Q(x, x) dx dt \right] \\
 & \geq \mathbb{E}[\mathbf{1}_A \tilde{R}^{k,h}].
 \end{aligned} \tag{43}$$

10 It follows from Lemma 4.1 that there exist an entropy process [2, 13]  $\mathbf{u}$  of  $L^\infty(0, T;$   
 11  $L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$  and a subsequence of  $\{u_{\mathcal{T},k}\}$  which we denote again by  $\{u_{\mathcal{T},k}\}$   
 12 such that  $u_{\mathcal{T},k}$  converges to  $\mathbf{u}$  in the sense of Young measures. Moreover it follows  
 13 from [2] and [6] that  $\mathbf{u} \in \mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d \times (0, 1)))$ . In order to prove the convergence  
 14 of  $\{u_{\mathcal{T},k}\}$  to a measure-valued stochastic entropy solution of Problem (1), we aim  
 15 to pass to the limit in the inequality (43) as  $h, k \rightarrow 0$ . We have proved that  
 16  $\mathbb{E}[\mathbf{1}_A \tilde{R}^{k,h}] \rightarrow 0$  as  $h, k \rightarrow 0$ . It remains to study the convergence of the terms on  
 17 the left-hand side of (43).

**Study of the term**  $\mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \right]$

Note that

$$\Psi : (\omega, x, t, \nu) \in \Omega \times L^2(\mathbb{T}^d) \times [0, T] \times \mathbb{R} \mapsto \mathbf{1}_A(\omega) \partial_t \varphi(x, t) \eta(\nu) \in \mathbb{R}$$

is a Carathéodory function such that  $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$  is bounded in  $L^2(\Omega \times Q_T)$ . Thus

$$\begin{aligned} & \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \right] \\ & \rightarrow \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x, t, \alpha)) d\alpha \partial_t \varphi(x, t) dx dt \right] \end{aligned}$$

1 as  $h, k \rightarrow 0$ .

**Study of the term**  $\mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right]$

Since  $F^\eta$  is bounded in  $L^2(\Omega \times \mathbb{T}^d \times [0, T])$ , using the same arguments as previously, we deduce that

$$\begin{aligned} & \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right] \\ & \rightarrow \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x, t, \alpha)) \mathbf{v} \cdot \nabla_x \varphi(x, t) d\alpha dx dt \right] \end{aligned}$$

2 as  $h, k \rightarrow 0$ .

**Study of the term**  $\mathbb{E} \left[ \mathbf{1}_A \int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) dx \right]$

We denote by  $\Psi$  the mapping

$$\Psi : (\omega, x, t, \nu) \mapsto \mathbf{1}_A(\omega) \eta'(\nu) g(\nu) \varphi(x, t) \in \mathbb{R};$$

$\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$  is bounded in  $L^\infty(\Omega \times Q_T)$ , so that up to a subsequence  $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$  converges weakly in  $L^2(\Omega \times Q_T)$  to a function  $\chi \in L^2(\Omega \times Q_T)$ . Thus

$$\begin{aligned} \lim_{h,k \rightarrow 0} \mathbb{E} \left[ \int_{Q_T} \Psi \phi dx dt \right] &= \lim_{h,k \rightarrow 0} \int_{\Omega \times Q_T} \Psi \phi dx dt \mathbb{P}(d\omega) \\ &= \int_{\Omega \times Q_T} \chi \phi dx dt \mathbb{P}(d\omega). \end{aligned}$$

For any  $\phi \in L^2(\Omega \times Q_T)$ ,  $(\omega, x, t, \nu) \mapsto \phi(\omega, x, t) \Psi(\omega, x, t, \nu)$  is a Carathéodory function such that  $\phi \Psi(\cdot, u_{\mathcal{T},k})$  is uniformly integrable. It is based on the fact that there exists a positive constant  $C_3$  such that

$$\begin{aligned} \mathbb{E} \left[ \int_{Q_T} |\phi \Psi(\cdot, u_{\mathcal{T},k})| dx dt \right] &= \int_{\Omega \times Q_T} |\phi \Psi(\cdot, u_{\mathcal{T},k})| dx dt \mathbb{P}(d\omega) \\ &\leq C_3 \|\Psi(\cdot, u_{\mathcal{T},k})\|_{L^2(\Omega \times Q_T)} \|\phi\|_{L^2(\Omega \times Q_T)}. \end{aligned}$$

Thus

$$\lim_{h,k \rightarrow 0} \int_{\Omega \times Q_T} \Psi(\cdot, u_{\mathcal{T},k}) \phi dx dt \mathbb{P}(d\omega) = \int_{\Omega \times Q_T} \int_0^1 \Psi(\cdot, \mathbf{u}(\cdot, \alpha)) d\alpha \phi dx dt \mathbb{P}(d\omega).$$

3 By identification,

$$\Psi(\cdot, u_{\mathcal{T},k}) \rightharpoonup \int_0^1 \Psi(\cdot, \mathbf{u}(\cdot, \alpha)) d\alpha \tag{44}$$

weakly in  $L^2(\Omega \times Q_T)$ . We have that

$$\begin{aligned}
& \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW_{\mathcal{T}}(x,t) \\
&= \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW_{\mathcal{T}}(x,t) - \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x,t) \\
&\quad + \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x,t) \\
&= \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi d(W_{\mathcal{T}}(x,t) - W(x,t)) \\
&\quad + \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x,t)
\end{aligned}$$

- 1 We first study the term  $\int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x,t)$ ; for  $\mathcal{U} \in L^2(\Omega \times Q_T)$ , we  
2 define the operator  $\tilde{\mathcal{T}}$  as

$$\tilde{\mathcal{T}}(\mathcal{U}) = \int_0^T \mathcal{U} dW(x,t) \quad (45)$$

where  $W(x,t)$  is the  $Q$ -Brownian motion defined by (5). In view of the Itô isometry Lemma (cf. [22] Corollary 4.3.6 and Itô Table (2), page 107), and the hypothesis (4), we deduce that:

$$\begin{aligned}
& \mathbb{E} \left[ \int_{\mathbb{T}^d} (\tilde{\mathcal{T}}(\mathcal{U}))^2 dx \right] = \mathbb{E} \left[ \int_{\mathbb{T}^d} \left( \int_0^T \mathcal{U} dW(x,t) \right)^2 dx \right] \\
&= \mathbb{E} \left[ \int_{\mathbb{T}^d} dx \left\{ \sum_{m=1}^{\infty} \lambda_m \left( \int_0^T \mathcal{U}(x,t) d\beta_m(t) \right)^2 e_m^2(x) \right. \right. \\
&\quad \left. \left. + 2 \sum_{m_1 \neq m_2}^{\infty} \sqrt{\lambda_{m_1} \lambda_{m_2}} e_{m_1}(x) e_{m_2}(x) \int_0^T \mathcal{U}(x,t) d\beta_{m_1}(t) \int_0^T \mathcal{U}(x,t) d\beta_{m_2}(t) \right\} \right] \\
&= \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_0^T Q(x,x) \mathcal{U}^2(x,t) dt dx \right] + 0 \\
&\leq \Lambda_1 \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_0^T \mathcal{U}^2(x,t) dt dx \right] \\
&\leq \Lambda_1 \|\mathcal{U}\|_{L^2(\Omega \times Q_T)}^2.
\end{aligned}$$

This proves that  $\tilde{\mathcal{T}}$  is a bounded operator from  $L^2(\Omega \times Q_T)$  to  $L^2(\Omega \times \mathbb{T}^d)$ . Let  $\tilde{\mathcal{T}}^* : L^2(\Omega \times \mathbb{T}^d) \rightarrow L^2(\Omega \times Q_T)$  be the adjoint operator of  $\tilde{\mathcal{T}}$ ; then

$$(\psi, \tilde{\mathcal{T}}\mathcal{U})_{L^2(\Omega \times \mathbb{T}^d)} = (\tilde{\mathcal{T}}^*\psi, \mathcal{U})_{L^2(\Omega \times Q_T)}$$

for all  $\psi \in L^2(\Omega \times \mathbb{T}^d)$ . Next we set  $\mathcal{U} = \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi$ . We recall that by (44),

$$\eta(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi \rightharpoonup \left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi$$

- 3 weakly in  $L^2(\Omega \times Q_T)$  along a subsequence as  $h$  and  $k$  tend to zero. Thus as  $h, k \rightarrow 0$ ,

$$\begin{aligned}
 & \left( \psi, \tilde{\mathcal{T}}(\eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi) \right)_{L^2(\Omega \times \mathbb{T}^d)} \\
 &= \left( \tilde{\mathcal{T}}^* \psi, \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi \right)_{L^2(\Omega \times Q_T)} \\
 &\rightarrow \left( \tilde{\mathcal{T}}^* \psi, \left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi \right)_{L^2(\Omega \times Q_T)} \\
 &= \left( \psi, \tilde{\mathcal{T}} \left( \left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi \right) \right)_{L^2(\Omega \times \mathbb{T}^d)}.
 \end{aligned}$$

We conclude that along a subsequence

$$\tilde{\mathcal{T}}(\eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi) \rightharpoonup \tilde{\mathcal{T}} \left( \left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi \right)$$

weakly in  $L^2(\Omega \times \mathbb{T}^d)$ , as  $h, k \rightarrow 0$ , or in other words, in view of the definition (45) of  $\tilde{\mathcal{T}}$ ,

$$\int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x, t) \rightarrow \int_0^T \left( \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right) \varphi dW(x, t).$$

Therefore, one obtains

$$\begin{aligned}
 & \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x, t) dx \right] \\
 &\rightarrow \mathbb{E} \left[ \int_{\mathbb{T}^d} \int_0^T \left( \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right) \varphi dW(x, t) dx \right]
 \end{aligned}$$

1 weakly in  $L^2(\Omega \times \mathbb{T}^d)$ , as  $h, k \rightarrow 0$ . Next, we consider the term

$$I_h := I_h(\omega, x) = \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi d(W_{\mathcal{T}}(x, t) - W(x, t)). \quad (46)$$

2 To begin with, we prove the following result:

3 **Lemma 6.1.** *Suppose that  $A_h \in L^2(\Omega \times \mathbb{T}^d \times [0, T])$  is  $(\mathcal{F}_t)$ -adapted, i.e.  $A_h(\cdot, t, x)$*   
 4 *is  $\mathcal{F}_t$ -measurable in  $\omega$  for every  $x \in \mathbb{T}^d$  and for every  $t > 0$ , such that*

$$\|A_h\|_{L^\infty(\Omega \times \mathbb{T}^d \times [0, T])} \leq C, \quad (47)$$

5 *for some positive constant  $C$  which does not depend on  $h$ . Then*

$$\mathcal{I}_h := \mathcal{I}_h(\omega, x) = \int_0^T A_h d(W_{\mathcal{T}}(x, t) - W(x, t)) \rightarrow 0 \quad (48)$$

6 *strongly in  $L^2(\Omega \times \mathbb{T}^d)$  as  $h \rightarrow 0$ .*

*Proof.* We first rewrite

$$\begin{aligned}
 W_{\mathcal{T}}(x, t) &= \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K W(y, t) dy \cdot \mathbf{1}_K(x) \\
 &= \sum_{m=1}^{\infty} \sqrt{\lambda_m} \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(x) \cdot \beta_m(t),
 \end{aligned}$$

7 where  $e_K^m = \frac{1}{|K|} \int_K e_m(y) dy$ .

Then, the difference of stochastic integrals in (48) can be rewritten as

$$\mathcal{I}_h(\omega, x) := \sum_{m=1}^{\infty} \sqrt{\lambda_m} \left( \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(x) - e_m(x) \right) \int_0^T A_h(x, t) d\beta_m(t).$$

We calculate the  $L^2$  norm on  $\Omega \times \mathbb{T}^d$ , which yields

$$\begin{aligned} & \|\mathcal{I}_h\|_{L^2(\Omega \times \mathbb{T}^d)}^2 \\ &= \int_{\mathbb{T}^d} dx \mathbb{E} \left[ \left\{ \sum_{m=1}^{\infty} \sqrt{\lambda_m} \left( \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(x) - e_m(x) \right) \int_0^T A_h(x, t) d\beta_m(t) \right\}^2 \right] \\ &= \int_{\mathbb{T}^d} dx \sum_{m=1}^{\infty} \lambda_m \left( \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(x) - e_m(x) \right)^2 \mathbb{E} \left[ \left( \int_0^T A_h(x, t) d\beta_m(t) \right)^2 \right] \\ &+ \int_{\mathbb{T}^d} 2dx \mathbb{E} \left[ \int_0^T A_h(x, t) d\beta_{m_1}(t) \int_0^T A_h(x, t) d\beta_{m_2}(t) \right] \sum_{m_1 \neq m_2}^{\infty} \left[ \right. \\ & \quad \left. \sqrt{\lambda_{m_1} \lambda_{m_2}} \left( \sum_{K \in \mathcal{T}} e_K^{m_1} \mathbf{1}_K(x) - e_{m_1}(x) \right) \left( \sum_{K \in \mathcal{T}} e_K^{m_2} \mathbf{1}_K(x) - e_{m_2}(x) \right) \right]. \end{aligned}$$

- 1 Therefore, by the independence of the Brownian motions  $\{\beta_m(t)\}$  and by the Itô
- 2 isometry for the stochastic integrals (cf. [22] Corollary 4.3.6 and Itô Table (2), page
- 3 107), in view of (47)

$$\begin{aligned} & \|\mathcal{I}_h\|_{L^2(\Omega \times \mathbb{T}^d)}^2 \\ &= \int_{\mathbb{T}^d} dx \sum_{m=1}^{\infty} \lambda_m \left( \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(x) - e_m(x) \right)^2 \mathbb{E} \left[ \left( \int_0^T A_h^2(x, t) dt \right) \right] \\ &= \int_{\mathbb{T}^d} dx \sum_{m=1}^{\infty} \lambda_m \left( \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(x) - e_m(x) \right)^2 \int_0^T \mathbb{E}[A_h(x, t)^2] dt \quad (49) \\ &\leq C^2 T \sum_{m=1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2. \end{aligned}$$

By Cauchy-Schwarz inequality, we deduce that

$$\begin{aligned} \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 &= \sum_{K \in \mathcal{T}} (e_K^m)^2 |K| = \sum_{K \in \mathcal{T}} \frac{1}{|K|} \left( \int_K e_m(y) dy \right)^2 \\ &\leq \sum_{K \in \mathcal{T}} \int_K e_m^2(y) dy = \|e_m\|_{L^2(\mathbb{T}^d)}^2 = 1, \end{aligned}$$

for all  $m \in \{1, 2, \dots\}$ , which implies that

$$\left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq 4 \quad \text{for all } m \in \{1, 2, \dots\}.$$

Next we prove that

$$\sum_{m=1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2$$

1 tends to zero as  $h$  tends to zero.

2 Fix  $\epsilon > 0$ , since  $\sum_{m=1}^{\infty} \lambda_m$  is a converging series, there exists a  $M \in \mathbb{N}$ , such that

$$\sum_{m=M+1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq 4 \sum_{m=M+1}^{\infty} \lambda_m \leq \frac{\epsilon}{2}. \quad (50)$$

Next we consider the term  $\sum_{m=1}^M \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2$ . Let  $m \in \{1, 2, \dots\}$

be arbitrary; therefore

$$\begin{aligned} & \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \\ &= \lambda_m \left\| \sum_{K \in \mathcal{T}} \left( \frac{1}{|K|} \int_K e_m(y) dy - e_m(\cdot) \right) \mathbf{1}_K(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \\ &= \lambda_m \int_{\mathbb{T}^d} dx \left( \sum_{K \in \mathcal{T}} \left( \frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right) \mathbf{1}_K(x) \right)^2 \\ &= \lambda_m \int_{\mathbb{T}^d} dx \left( \sum_{K \in \mathcal{T}} \left( \frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right)^2 \mathbf{1}_K(x) \right) \\ & \quad + \lambda_m \int_{\mathbb{T}^d} dx \sum_{K_1 \neq K_2} \left[ \left( \frac{\int_{K_1} e_m(y) dy}{|K_1|} - e_m(x) \right) \right. \\ & \quad \quad \left. \times \left( \frac{\int_{K_2} e_m(y) dy}{|K_2|} - e_m(x) \right) \mathbf{1}_{K_1}(x) \mathbf{1}_{K_2}(x) \right] \\ &= \lambda_m \sum_{K \in \mathcal{T}} \int_K \left( \frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right)^2 dx \\ &= \lambda_m \sum_{K \in \mathcal{T}} \int_K \left( \frac{1}{|K|} \int_K e_m(y) dy - \frac{1}{|K|} \int_K e_m(x) dy \right)^2 dx \\ &= \lambda_m \sum_{K \in \mathcal{T}} \int_K \frac{1}{|K|^2} \left( \int_K (e_m(y) - e_m(x)) dy \right)^2 dx \\ &\leq \lambda_m \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K \int_K (e_m(y) - e_m(x))^2 dx dy. \end{aligned}$$

3 We denote by  $B(h)$  the ball with center 0 and radius  $h$  with  $h = \text{size}(\mathcal{T})$ . Then

$$\begin{aligned}
& \lambda_m \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K \int_K (e_m(y) - e_m(x))^2 dx dy \\
& \leq \lambda_m \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K dx \int_{B(h)} dz (e_m(x+z) - e_m(x))^2 \\
& = \lambda_m \int_{B(h)} dz \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K (e_m(x+z) - e_m(x))^2 dx \\
& \leq \lambda_m |B(h)| \sup_{z \in B(h)} \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K (e_m(x+z) - e_m(x))^2 dx.
\end{aligned}$$

Using (7), we deduce that

$$|B(h)| = C_d h^d \leq \frac{C_d}{\alpha_{\mathcal{T}}} |K|,$$

for some positive constant  $C_d$ . Let  $\mathbb{T}^d \subset \subset \mathcal{O}$  where  $\mathcal{O}$  is an open set. We suppose that  $e_m$  is prolonged by periodicity on  $\mathbb{R}^d$  and that  $\text{dist}(\mathbb{T}^d, \partial\mathcal{O}) \leq \text{size}(\mathcal{T})$ . Therefore

$$\begin{aligned}
& \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \\
& \leq \lambda_m \frac{C_d}{\alpha_{\mathcal{T}}} |K| \sup_{z \in B(h)} \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K (e_m(x+z) - e_m(x))^2 dx \\
& = \lambda_m \frac{C_d}{\alpha_{\mathcal{T}}} \sup_{z \in B(h)} \|e_m(\cdot+z) - e_m(\cdot)\|_{L^2(\mathbb{T}^d)}^2.
\end{aligned}$$

Let  $\epsilon > 0$  be arbitrary. It follows from the density of  $C(\mathcal{O})$  into  $L^2(\mathcal{O})$  that there exists a function  $g \in C(\mathcal{O})$  such that

$$\|e_m(\cdot+z) - g(\cdot+z)\|_{L^2(\mathbb{T}^d)} + \|e_m(\cdot) - g(\cdot)\|_{L^2(\mathbb{T}^d)} \leq \frac{2\epsilon}{3} \cdot \frac{\alpha_{\mathcal{T}}}{2M\lambda_m C_d},$$

for all  $z \in B(h)$ . Thus there exists a positive constant  $\delta$  such that for all  $z \in B(h)$

$$\begin{aligned}
& \|e_m(\cdot+z) - e_m(\cdot)\|_{L^2(\mathbb{T}^d)} \\
& \leq \|e_m(\cdot+z) - g(\cdot+z)\|_{L^2(\mathbb{T}^d)} + \|g(\cdot+z) - g(\cdot)\|_{L^2(\mathbb{T}^d)} \\
& \quad + \|g(\cdot) - e_m(\cdot)\|_{L^2(\mathbb{T}^d)} \\
& \leq \frac{\alpha_{\mathcal{T}}}{2M\lambda_m C_d} \epsilon
\end{aligned}$$

1 for all  $h < \delta$ .

2

3 Using the fact that the sum is finite, we have

$$\sum_{m=1}^M \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq \frac{\epsilon}{2}. \quad (51)$$

Combining (50) and (51), we deduce that if  $h < \delta$

$$\sum_{m=1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq \epsilon.$$

4 which in view of (49) completes the proof of (48).  $\square$

1 It follows from Lemma 6.1 that the integral  $I_h$  defined by (46) converges to zero  
 2 as  $h$  and  $k$  tend to zero.

**Study of the term**  $\frac{1}{2} \mathbb{E} \left[ \mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x,t) Q(x,x) dx dt \right]$

Since  $\Psi : (\omega, x, t, \nu) \in \Omega \times \mathbb{T}^d \times [0, T] \times \mathbb{R} \mapsto \mathbf{1}_A(\omega) \eta''(\nu) g(\nu) \varphi(x,t) Q(x,x) \in \mathbb{R}$  is a Carathéodory function such that  $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$  is bounded in  $L^2(\Omega \times \mathbb{T}^d \times [0, T])$ , we deduce that:

$$\begin{aligned} & \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x,t) Q(x,x) dx dt] \\ & \rightarrow \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x,t,\alpha)) g^2(\mathbf{u}(x,t,\alpha)) \varphi(x,t) Q(x,x) \alpha dx dt], \end{aligned}$$

3 as  $h, k \rightarrow 0$ , for all  $m \geq 1$ .

4 We deduce that for all sets  $A \in \mathcal{F}$ , for all  $\eta \in \mathcal{A}$  and for all  $\varphi \in \mathcal{C}$

$$\begin{aligned} & \mathbb{E}[\mathbf{1}_A \int_{\mathbb{T}^d} \eta(u_0) \varphi(x,0) dx] + \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x,t,\alpha)) d\alpha \partial_t \varphi(x,t) dx dt] \\ & + \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x,t,\alpha)) \mathbf{v} \cdot \nabla_x \varphi(x,t) d\alpha dx dt] \\ & + \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta'(\mathbf{u}(x,t,\alpha)) \varphi(x,t) d\alpha dW(x,t) dx] \\ & + \frac{1}{2} \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x,t,\alpha)) g^2(\mathbf{u}(x,t,\alpha)) \varphi(x,t) Q(x,x) d\alpha dx dt] \\ & \geq 0. \end{aligned} \tag{52}$$

In turn (52) implies that for all  $\chi \in L^2(\Omega)$

$$\begin{aligned} & \mathbb{E}[\chi \int_{\mathbb{T}^d} \eta(u_0) \varphi(x,0) dx] + \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x,t,\alpha)) d\alpha \partial_t \varphi(x,t) dx dt] \\ & + \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x,t,\alpha)) \mathbf{v} \cdot \nabla_x \varphi(x,t) d\alpha dx dt] \\ & + \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta'(\mathbf{u}(x,t,\alpha)) \varphi(x,t) d\alpha dW(x,t) dx] \\ & + \frac{1}{2} \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x,t,\alpha)) g^2(\mathbf{u}(x,t,\alpha)) \varphi(x,t) Q(x,x) d\alpha dx dt] \\ & \geq 0. \end{aligned}$$

5 Hence  $\mathbf{u}$  is a measure-valued entropy solution of Problem (1) in the sense of Defi-  
 6 nition 2.2.  $\square$

7 **7. Numerical simulations.** Taking inspiration from the numerical computations  
 8 in [1], we propose to perform numerical simulations for a stochastic Burgers equation  
 9 involving a  $Q$ -Brownian motion:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = \alpha_Q \dot{W}(x,t) \tag{53}$$

10 in the space interval  $[0, 1]$  together with periodic boundary conditions and the initial  
 11 condition  $u(x,0) = u_0(x)$ . The constant  $\alpha_Q$  is the amplitude of the noise term.

We consider the case that the  $Q$ -Brownian motion is given by

$$W(x, t) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \{ \beta_m(t) e_m(x) + \gamma_m(t) f_m(x) \},$$

- 1 with  $\{e_m(x), f_m(x)\} = \{\sqrt{2} \sin(2m\pi x), \sqrt{2} \cos(2m\pi x)\}$ ,  $m = 1, 2, \dots$  and  $\{\beta_m(t),$   
 2  $\gamma_m(t)\}$ ,  $m = 1, 2, \dots$  is a sequence of independent  $(\mathcal{F}_t)$ -Brownian motions.

We consider the Laplacian operator in space interval  $[0, 1]$ ,  $-\Delta : u \rightarrow -u''$  with periodic boundary conditions; then the eigenvalue  $\lambda_m$  and eigenfunctions  $e_m$  and  $f_m$  of  $-\Delta$  satisfy

$$-\Delta e_m = \lambda_m e_m, \quad -\Delta f_m = \lambda_m f_m$$

- 3 for all  $m = 1, 2, \dots$  and such that  $\lambda_m = 4m^2\pi^2$ . We remark that Burgers equation is  
 4 not included in the class of problems studied in this article, since the flux function  
 5  $f(u) = u^2/2$  is only locally Lipschitz continuous.

For all  $u$  in the domain of  $(-\Delta)^{-\zeta}$  with  $\zeta > 0$ , there holds, for the  $d$ -dimensional torus [23],

$$(-\Delta)^{-\zeta} u = \sum_{m=1}^{\infty} \lambda_m^{-\zeta} \left\{ (u, e_m)_{L^2(\mathbb{T}^d)} e_m + (u, f_m)_{L^2(\mathbb{T}^d)} f_m \right\},$$

- 6 which implies that the eigenvalues of  $(-\Delta)^{-\zeta}$  are  $\lambda_m^{-\zeta}$ , where  $\{\lambda_m\}$  are the eigen-  
 7 values of the Laplacian operator  $-\Delta$ .

- 8 **7.1. Numerical scheme.** We propose the following numerical scheme for the dis-  
 9 cretization of the equation (53)

$$\frac{1}{\Delta t} (u_i^{n+1} - u_i^n) + \frac{1}{\Delta x} \left( F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right) = \frac{\alpha Q}{\Delta t} (W_i^{n+1} - W_i^n), \quad (54)$$

for all  $i \in \{1, 2, \dots, I\}$  and  $n = \{0, 1, \dots, N-1\}$  where  $I$  is the number of volumes and  $N$  is the number of time step and where  $\Delta x = 1/I$  and  $\Delta t$  are respectively the length of a volume element and the time step. If  $Q$  is the operator  $(-\Delta)^{-\zeta}$  with  $\zeta > 0$ , then the stochastic force terms are of the form

$$W_i^n = \sum_{m=1}^M \sqrt{\lambda_m^{-\zeta}} \left\{ \beta_m(t^n) e_m(x_i) + \gamma_m(t^n) f_m(x_i) \right\}$$

- 10 with  $\{e_m(x), f_m(x)\} = \{\sqrt{2} \sin(2m\pi x), \sqrt{2} \cos(2m\pi x)\}$ ,  $\lambda_m^{-\zeta} = (4m^2\pi^2)^{-\zeta}$ ,  $M$  a  
 11 truncation number and  $x_i = i \cdot \Delta x$ . More precisely,

$$\begin{aligned} & \frac{\alpha Q}{\Delta t} (W_i^{n+1} - W_i^n) \\ &= \frac{\alpha Q}{\Delta t} \sum_{m=1}^M \sqrt{2} \sqrt{\lambda_m^{-\zeta}} \left( (\beta_m(t^{n+1}) - \beta_m(t^n)) \sin(2m\pi x_i) \right. \\ & \quad \left. + (\gamma_m(t^{n+1}) - \gamma_m(t^n)) \cos(2m\pi x_i) \right) \\ &= \alpha Q \sqrt{\frac{2}{\Delta t}} \sum_{m=1}^M \frac{1}{(2m\pi)^\zeta} \left( \left( \frac{\beta_m(t^{n+1}) - \beta_m(t^n)}{\sqrt{\Delta t}} \right) \sin(2m\pi x_i) \right. \\ & \quad \left. + \left( \frac{\gamma_m(t^{n+1}) - \gamma_m(t^n)}{\sqrt{\Delta t}} \right) \cos(2m\pi x_i) \right) \end{aligned} \quad (55)$$

1 where  $\frac{\beta_m(t^{n+1}) - \beta_m(t^n)}{\sqrt{\Delta t}}$  and  $\frac{\gamma_m(t^{n+1}) - \gamma_m(t^n)}{\sqrt{\Delta t}}$  follow the Gaussian law  $\mathcal{N}(0, 1)$ .

We recall that in the simulation of stochastic Burgers equation [1], the authors proposed the following scheme,

$$\frac{1}{\Delta t}(u_i^{n+1} - u_i^n) + \frac{1}{\Delta x} \left( F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right) = \alpha_1 \sqrt{\frac{1}{\Delta x \Delta t}} G_i^n$$

2 with

$$\begin{aligned} & \alpha_1 \sqrt{\frac{1}{\Delta x \Delta t}} G_i^n \\ &= \alpha_1 \sqrt{\frac{1}{\Delta x \Delta t}} \sqrt{\frac{2}{I}} \left( \sum_{m=1}^{\frac{I-1}{2}} \left( \frac{C_m^n}{m^\zeta} \cos(2m\pi x_i) - \frac{S_m^n}{m^\zeta} \sin(2m\pi x_i) \right) \right) \\ &= \alpha_1 \sqrt{\frac{2}{\Delta t}} \left( \sum_{m=1}^{\frac{I-1}{2}} \left( \frac{C_m^n}{m^\zeta} \cos(2m\pi x_i) - \frac{S_m^n}{m^\zeta} \sin(2m\pi x_i) \right) \right) \end{aligned} \quad (56)$$

3 where  $C_m^n$  and  $S_m^n$  follow the Gaussian law  $\mathcal{N}(0, 1)$ ,  $\alpha_1$  the amplitude and  $\zeta$  the  
4 space regularity.

5 **Remark 3.** Comparing (56) with (55), we deduce that those two formulas are  
6 equivalent when  $M = \frac{I-1}{2}$  and  $\alpha_1 = \frac{\alpha_Q}{(2\pi)^\zeta}$ .

7 7.1.1. *Burgers equation involving a Brownian motion.* In order to compare the nu-  
8 merical results, we propose to perform simulations for a stochastic Burgers equation  
9 involving a Brownian motion:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = \alpha_B \dot{\beta}(t), \quad (57)$$

10 where  $\beta(t)$  is an  $(\mathcal{F}_t)$ -Brownian motion. We propose the following numerical scheme  
11 for the discretization of the equation (57)

$$\frac{1}{\Delta t}(u_i^{n+1} - u_i^n) + \frac{1}{\Delta x} \left( F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right) = \frac{\alpha_B}{\Delta t} (\beta^{n+1} - \beta^n), \quad (58)$$

12 where  $\beta^n$  is the value of  $\beta(t^n)$  for all  $n$  and  $\beta^{n+1} - \beta^n$  has the Gaussian distribution  
13  $\mathcal{N}(0, t^{n+1} - t^n)$ . We remark that for each time step  $n$ , we use the same random  
14 variable  $\beta^{n+1} - \beta^n$  for all  $i \in \{1, 2, \dots, I\}$ .

15 7.2. **Numerical results.** In this section, we present some numerical results. The  
16 initial condition is given by  $u_0(x) = \sin(2\pi x) + 0.5$ . We subdivide the space interval  
17  $[0, 1]$  into 201 volumes so that  $\Delta x = 1/201$  and we fix  $\Delta t = 0.1\Delta x$  and obtain  
18  $N = T/\Delta t$ . We perform 16 384 realizations for each fixed  $\alpha_B$  and  $(\alpha_Q, \zeta)$ . The  
19 numerical results are presented in two periods in space.

20 First we present the results in the deterministic case in Figures 1 and 2, namely  
21 the case that the source term is equal to zero.

22 The figures show that after a certain time, a shock appears and moves with a  
23 constant speed.

24 7.3. **Comparison between two different types of noises.** We apply the scheme  
25 (58) for the Brownian motion case and the scheme (54) for the Q-Brownian motion  
26 case, and we compare the numerical results.

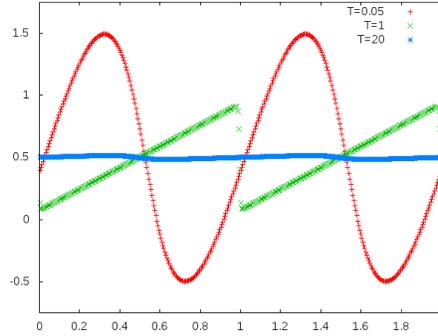


FIGURE 1. Solutions in the deterministic case

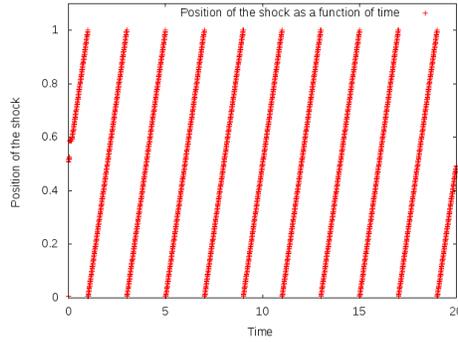
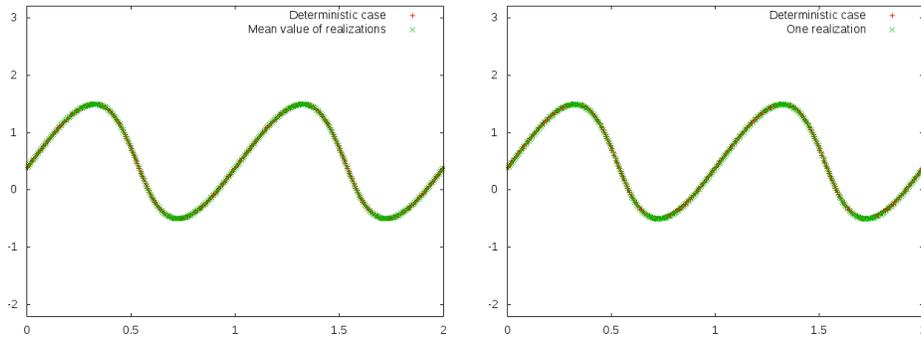


FIGURE 2. The positions of the shock

- 1 7.3.1. *Empirical mean value and one single realization.* In view of the Remark 3,  
 2 we choose  $\alpha_B = 1/(2\pi)$ ,  $\alpha_Q = 1$  and  $\zeta = 1$  and we present the results for the  
 Brownian motion case in Figures 3, 4 and 5; and the results for the  $Q$ -Brownian

FIGURE 3. Comparing the solution in the deterministic case with the empirical average (left) and one realization (right) in the Brownian motion case with  $\alpha_B = 1/(2\pi)$  at  $t = 0.05$ .

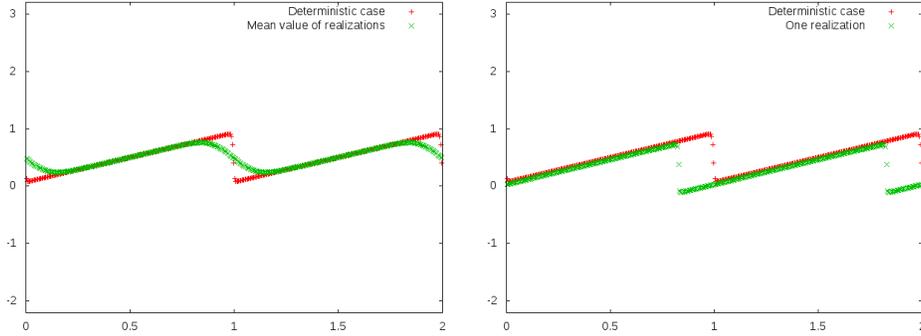


FIGURE 4. Comparing the solution in the deterministic case with the empirical average (left) and one realization (right) in the Brownian motion case with  $\alpha_B = 1/(2\pi)$  at  $t = 1$ .

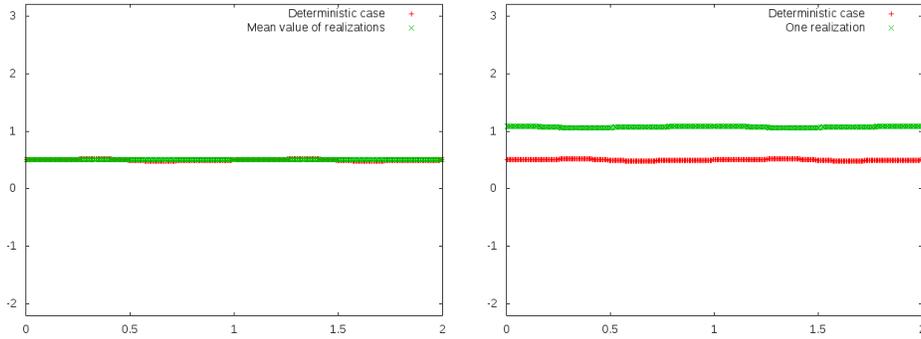


FIGURE 5. Comparing the solution in the deterministic case with the empirical average (left) and one realization (right) in the Brownian motion case with  $\alpha_B = 1/(2\pi)$  at  $t = 20$ .

1  
 2 motion case in Figures 6, 7 and 8.  
 3 The mean value of the realizations is close to the deterministic solution in both  
 4 cases. For single realizations, at time  $t = 1$  the Brownian motion shifts the posi-  
 5 tion of the shock; and in the  $Q$ -Brownian motion case, the stochastic solution is  
 6 perturbed around the deterministic solution.

7 **7.3.2. Variance and covariance for two fixed points.** We first present the variance  
 8  $\text{Var}[u(x_i, t)]$  for all  $i \in \{1, 2, \dots, I\}$  at different times  $t = 0.05$ ,  $t = 1$  and  $t = 20$   
 9 in the Figure 9. We recall that  $\alpha_B = 1/(2\pi)$ ,  $\alpha_Q = 1$  and  $\zeta = 1$ .

Next we present the discrete  $L^1$  norm of the variance as a function of time,  
 namely

$$\|\text{Var}[u(\cdot, t)]\|_{L^1([0,1])} = \sum_{i=1}^I \text{Var}[u(x_i, t)] \Delta x,$$

10 in Figures 10 and 11. These results show that in the Brownian motion case, the  
 11  $L^1$  norm is increasing as a function of time; meanwhile in the  $Q$ -Brownian motion

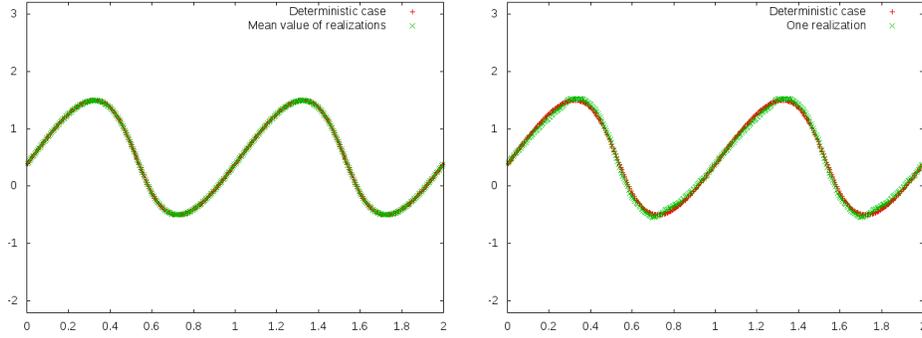


FIGURE 6. Comparing the solution in the deterministic case with the empirical average (left) and one realization (right) in the  $Q$ -Brownian motion case with  $\alpha_Q = 1$  and  $\zeta = 1$  at  $t = 0.05$ .

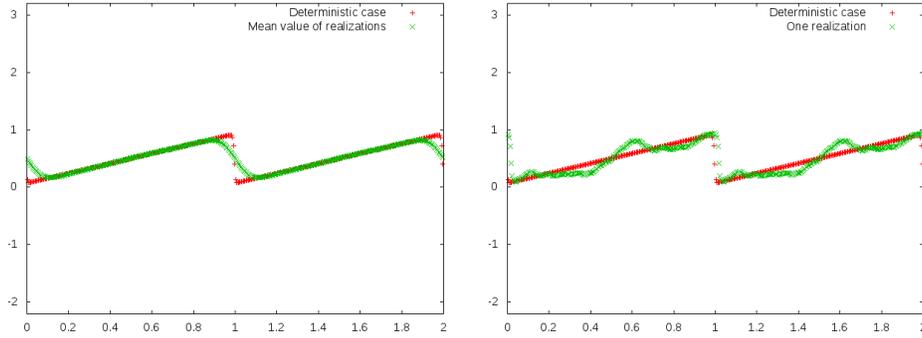


FIGURE 7. Comparing the solution in the deterministic case with the empirical average (left) and one realization (right) in the  $Q$ -Brownian motion case with  $\alpha_Q = 1$  and  $\zeta = 1$  at  $t = 1$ .

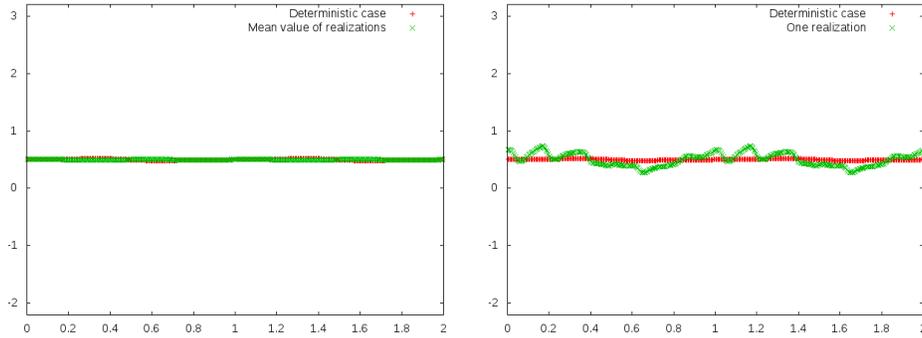


FIGURE 8. Comparing the solution in the deterministic case with the empirical average (left) and one realization (right) in the  $Q$ -Brownian motion case with  $\alpha_Q = 1$  and  $\zeta = 1$  at  $t = 20$ .

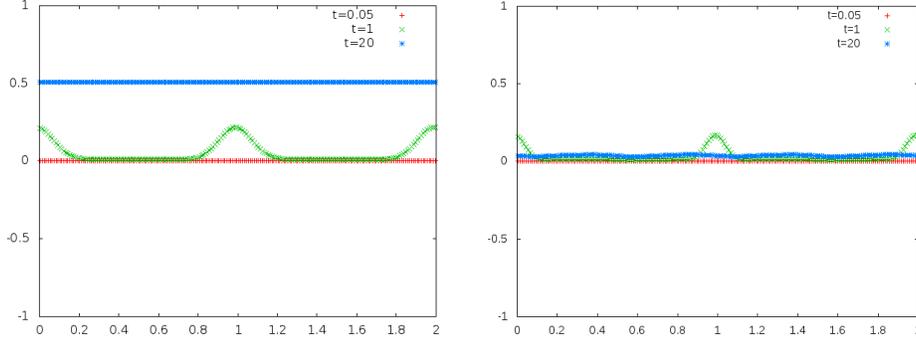


FIGURE 9. Variance in the Brownian motion case (left) and in the  $Q$ -Brownian motion case (right) for fixed time, with  $\alpha_B = 1/(2\pi)$ ,  $\alpha_Q = 1$  and  $\zeta = 1$ .

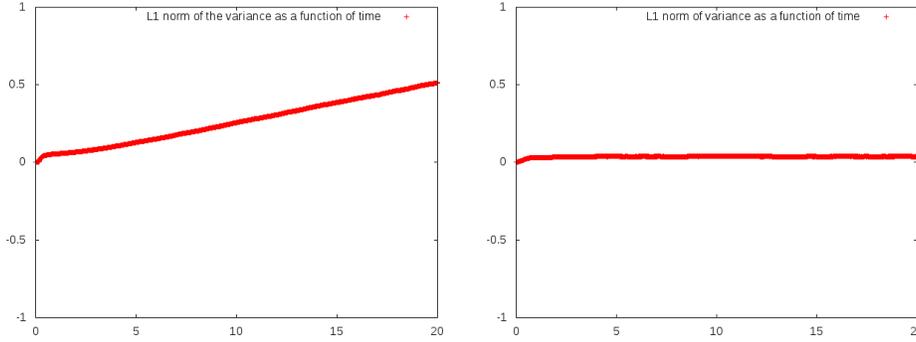


FIGURE 10.  $L^1$  norm of the variance as a function of time in the case of Brownian motion (left) and  $Q$ -Brownian motion (right) with  $\alpha_B = 1/(2\pi)$ ,  $\alpha_Q = 1$  and  $\zeta = 1$ .

1 case, the  $L^1$  norm of the variance tends to a stable value as time increases. Since  
 2 the  $L^1$  norm of the variance is increasing in the Brownian motion case, we propose  
 3 to compare the results to the case that the amplitude  $\alpha_B = 1/\pi$ , which is twice as  
 4 much as the previous amplitude that we considered.

5 **Remark 4.** The phenomenon that a richer randomness in noise implies less fluctu-  
 6 ation in the solution can be observed in a different setting of SPDEs. Consider a sto-  
 7 chastic heat equation  $du = \Delta u dt + dW(x, t)$  on  $\mathbb{R}^d$  with an initial value  $u(x, 0) = 0$   
 8 for simplicity and with an additive noise  $W(x, t)$  which is a  $Q$ -Brownian motion.  
 9 Then the solution  $u$  is given in a mild form

$$u(x, t) = \int_0^t \int_{\mathbb{R}^d} p(t-s, x, y) dW(y, s) dy, \quad (59)$$

where  $p$  is the heat kernel and therefore its variance is easily computed as

$$\text{Var}(u(x, t)) = \int_0^t ds \int_{\mathbb{R}^d \times \mathbb{R}^d} p(s, x, y_1) p(s, x, y_2) \mathbf{Q}(y_1, y_2) dy_1 dy_2.$$

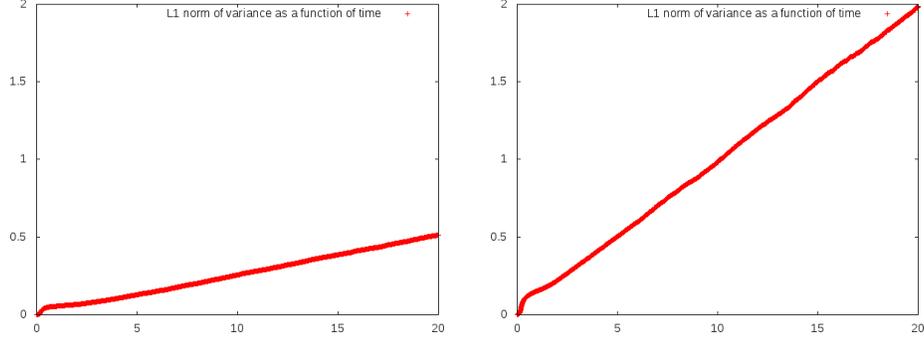


FIGURE 11.  $L^1$  norm of the variance as a function of time in the case of the Brownian motion with  $\alpha_B = 1/(2\pi)$  (left) and  $\alpha_B = 1/\pi$  (right).

- 1 In particular, if  $W$  is the space-time white noise,  $\mathbf{Q}(y_1, y_2) = \delta(y_1 - y_2)$ , so that  
 2  $\text{Var}(u(x, t)) = \int_0^t p(2s, x, x) ds = \sqrt{t/(2\pi)}$  when  $d = 1$ . While, if  $W$  is the Brownian  
 3 motion  $\beta(t)$  only in time,  $\mathbf{Q}(y_1, y_2) = 1$ , so that  $\text{Var}(u(x, t)) = t$  for every  $d \geq 1$ .  
 4 Indeed, using the mild form (59) and the initial value  $u(x, 0) = 0$ , we deduce that  
 5  $u(x, t) = \beta(t)$  in this case. This simple example, though it is totally different from  
 6 our equation, indicates that the fluctuation of the solution determined from the  
 7 noise with richer randomness is asymptotically smaller as the time grows.

We fix  $x_1 = 0.25$  and  $x_2 = 0.75$  and then present the corresponding covariance as a function of time, namely

$$\text{Cov}[u(x_1, t), u(x_2, t)] = \mathbb{E}[(u(x_1, t) - \mathbb{E}[u(x_1, t)])(u(x_2, t) - \mathbb{E}[u(x_2, t)])]$$

- 8 in the Figure 12.

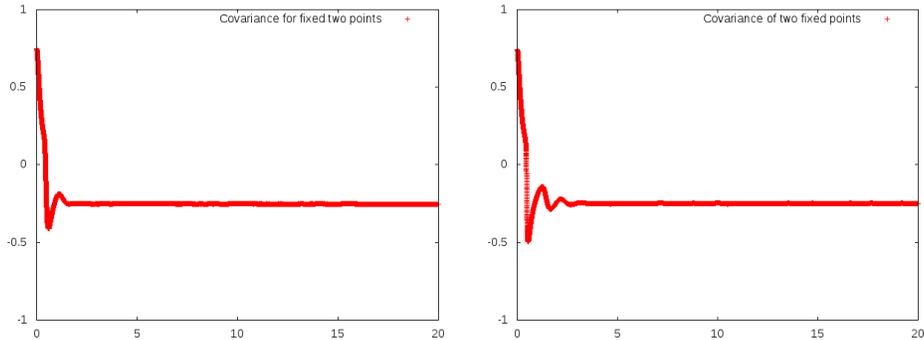


FIGURE 12. Covariance in the case of Brownian motion (left) and  $Q$ -Brownian motion (right) as a function of time with  $\alpha_Q = 1$  and  $\zeta = 1$

- 9 The covariance tends to a fixed value as time increases and the limit values are  
 10 close in the two cases.

1 7.3.3. *Comparison between different amplitudes in the  $Q$ -Brownian motion case.* In  
 2 order to study the influence of the amplitude of the noise in the  $Q$ -Brownian motion  
 3 case, we propose to consider the case that  $\alpha_Q = 2\pi$  and  $\zeta = 1$  in Figures 13, 14 and  
 15.

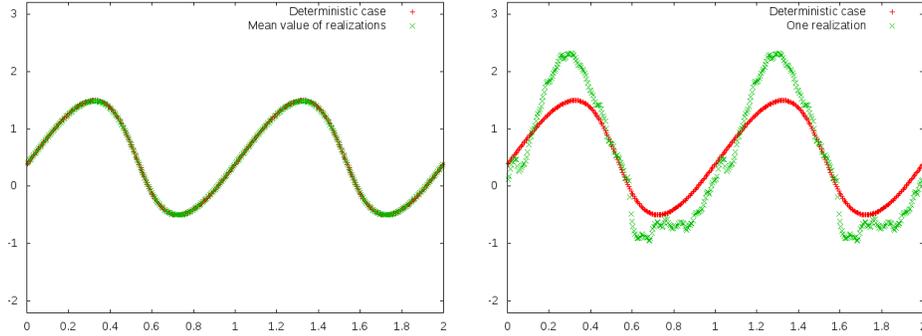


FIGURE 13. Comparing the solution in the deterministic case with the empirical average (left) and one realization (right) in the  $Q$ -Brownian motion case with  $\alpha_Q = 2\pi$  and  $\zeta = 1$  at  $t = 0.05$ .

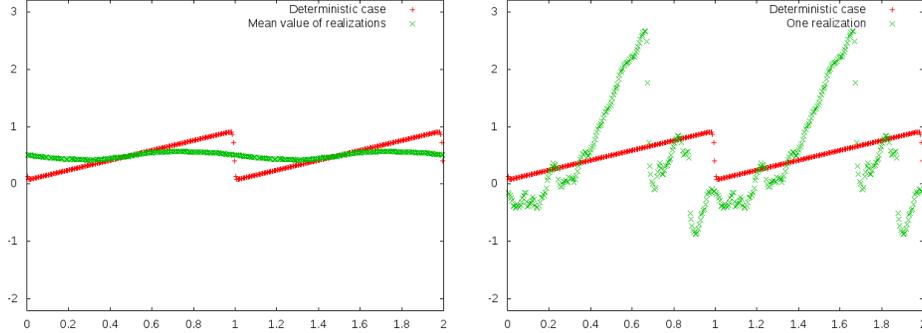


FIGURE 14. Comparing the solution in the deterministic case with the empirical average (left) and one realization (right) in the  $Q$ -Brownian motion case with  $\alpha_Q = 2\pi$  and  $\zeta = 1$  at  $t = 1$ .

4  
 5 We present the  $L^1$  norm of the variance in this case in Figure 16. In these two  
 6 cases, the  $L^1$  norm of the variance tends to a constant as time increases, and as the  
 7 amplitude of the noise is stronger, the limit value is larger, which is as it could be  
 8 expected.

9 **7.4. Some conclusions and discussions.**

10 In the  $Q$ -Brownian motion case, we have considered the case that  $Q = (-\Delta)^{-1}$   
 11 where the series was truncated at the fixed value  $M = 100$ . We have considered  
 12 different noise amplitudes, namely  $\alpha_Q = 1$ ,  $\zeta = 1$  and  $\alpha_Q = 2\pi$ ,  $\zeta = 1$ . The  
 13 numerical results show that when  $\alpha_Q$  is larger, one realization in the stochastic  
 14 case is more dispersed. The average of the realizations is a good approximation of

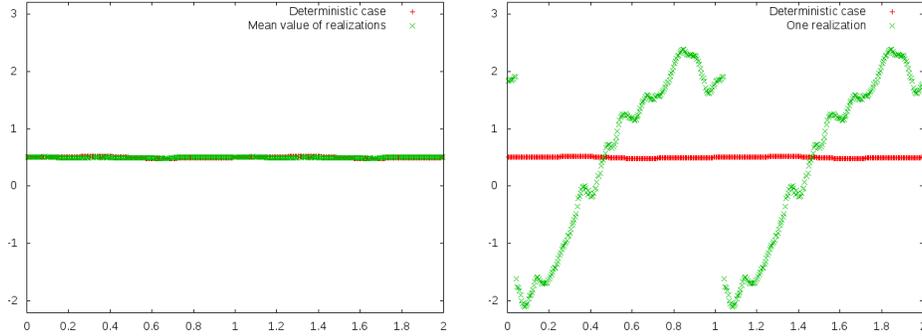


FIGURE 15. Comparing the solution in the deterministic case with the empirical average (left) and one realization (right) in the  $Q$ -Brownian motion case with  $\alpha_Q = 2\pi$  and  $\zeta = 1$  at  $t = 20$ .

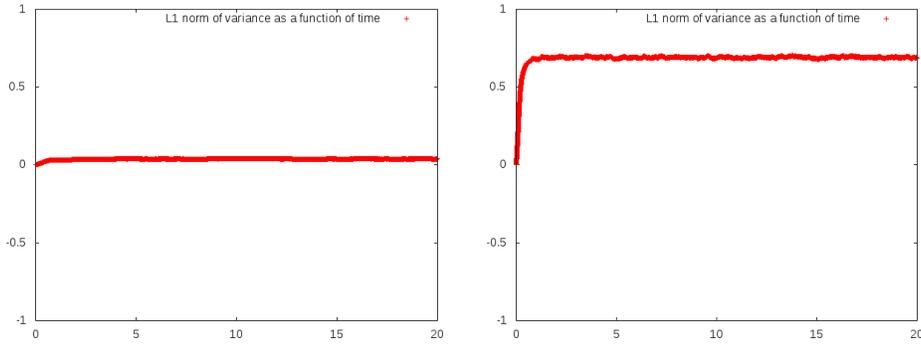


FIGURE 16. The  $L^1$  norm of the variance in the cases that  $\alpha_Q = 1$  (left) and  $\alpha_Q = 2\pi$  (right) as a function of time.

1 the deterministic solution and as time tends to infinity it converges to the space-  
 2 average of the initial function [11], which is the constant  $1/2$  in our case. While the  
 3 deterministic solution is discontinuous at the shock, the average of the realizations  
 4 of stochastic solutions has a smoothing effect. When the amplitude is larger, the  
 5 smoothing effect is stronger, and the average goes faster to the space-average.

6 Both in the Brownian motion case and in the  $Q$ -Brownian motion case, the  
 7 averages of realizations approximate the solution in the deterministic case. In single  
 8 realizations, the Brownian motion type noise shifts the shock position, but the  
 9 solution does not oscillate in space; however in the  $Q$ -Brownian motion case, one  
 10 single realization is perturbed around the deterministic solution. If we consider the  
 11 variance, the  $L^1$  norm of the empirical variance increases as a function of time  
 12 in the Brownian motion case while the  $L^1$  norm of the variance tends to a constant  
 13 which depends on the amplitude of the noise term in the  $Q$ -Brownian motion case.

14 **Acknowledgments.** The authors would like to thank Ludovic Goudenège, Max  
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1

## REFERENCES

- 2 [1] E. Audusse, S. Boyaval, Y. Gao and D. Hilhorst, Numerical simulations of the inviscid Burgers  
3 equation with periodic boundary conditions and stochastic forcing, *ESAIM: Proceedings and*  
4 *surveys*, **48** (2014), 308–320.
- 5 [2] E. J. Balder, Lectures on Young measure theory and its applications in economics, Work-  
6 shop on Measure Theory and Real Analysis, Grado, 1997, *Rend. Istit. Mat. Univ. Trieste*,  
7 **31(suppl. 1)** (2000), 1–69.
- 8 [3] C. Bauzet, On a time-splitting method for a scalar conservation law with a multiplicative sto-  
9 chastic perturbation and numerical experiments, *Journal of Evolution Equations*, **14** (2014),  
10 333–356.
- 11 [4] C. Bauzet, G. Vallet, and P. Wittbold, The Cauchy problem for a conservation law with a  
12 multiplicative stochastic perturbation, *Journal of Hyperbolic Differential Equations*, **09(4)**  
13 (2012), 661–709.
- 14 [5] C. Bauzet, G. Vallet and P. Wittbold, The Dirichlet problem for a conservation law with a  
15 multiplicative stochastic perturbation, *Journal of Functional Analysis*, **4(266)** (2014), 2503–  
16 2545.
- 17 [6] C. Bauzet, J. Charrier and T. Gallouët, Convergence of flux-splitting finite volume schemes  
18 for hyperbolic scalar conservation laws with a multiplicative stochastic perturbation, *Mathe-*  
19 *matics of Computation*, **85** (2016), 2777–2813.
- 20 [7] C. Bauzet, J. Charrier and T. Gallouët, Convergence of monotone finite volume schemes for  
21 hyperbolic scalar conservation laws with a multiplicative noise. *Stochastic Partial Differential*  
22 *Equations: Analysis and Computations*, **4(1)** (2016), 150–223.
- 23 [8] C. Bauzet, J. Charrier and T. Gallouët, Numerical approximation of stochastic conservation  
24 laws on bounded domains. *Mathematical Modelling and Numerical Analysis*, **51(1)** (2017),  
25 225–278.
- 26 [9] G.-Q. Chen, Q. Ding and K.H. Karlsen, On nonlinear stochastic balance laws, *Arch. Ration.*  
27 *Mech. Anal.*, **204(3)** (2012), 707–743.
- 28 [10] G. Da Prato and J. Zabczyk, Stochastic equations in infinite dimensions, *Cambridge Univer-*  
29 *sity Press*, Second edition, (2014).
- 30 [11] A. Debussche and J. Vovelle, Long-time behavior in scalar conservation laws, *Differential*  
31 *Integral Equations*, **22(3-4)** (2009), 225–238.
- 32 [12] A. Debussche and J. Vovelle, Scalar conservation laws with stochastic forcing, *Journal of*  
33 *Functional Analysis*, **259(4)** (2012), 1014–1042.
- 34 [13] R. Eymard, T. Gallouët and R. Herbin, Existence and Uniqueness of the entropy solution  
35 to a nonlinear hyperbolic equation, *Chinese Annals of Mathematics, Series B*, **16(1)** (1995),  
36 1–14.
- 37 [14] J. Feng and D. Nualart, Stochastic scalar conservation laws, *Journal of Functional Analysis*,  
38 **255(2)** (2008), 313–373.
- 39 [15] T. Funaki, Y. Gao and D. Hilhorst, Uniqueness results for a stochastic conservation law with  
40 a Q-Brownian motion, in preparation.
- 41 [16] Y. Gao, *Finite volume methods for deterministic and stochastic partial differential equations*,  
42 Ph.D thesis, Université Paris-Sud, 2015.
- 43 [17] L. Gawarecki and V. Mandrekar, Stochastic differential equations in infinite dimensions with  
44 Applications to Stochastic Partial Differential Equations, *Springer-Verlag, BerlinHeidelberg*,  
45 (2011).
- 46 [18] C.M. Grinstead, and J.L. Snell, Introduction to Probability, *American Mathematical Society*,  
47 (1997).
- 48 [19] M. Hofmanová, Bhatnagar-Gross-Krook approximation to stochastic scalar conservation laws,  
49 *Ann. Inst. H. Poincaré Probab. Statist.*, **51(4)** (2015), 1500–1528.
- 50 [20] H. Holden and N.H. Risebro, A stochastic approach to conservation laws, *In Third Interna-*  
51 *tional Conference on Hyperbolic Problems*, **I, II** (1990), 575–587.
- 52 [21] I. Kröker and C. Rohde, Finite volume schemes for hyperbolic balance laws with multiplicative  
53 noise, *Appl. Number. Math.*, **62(4)**, (2012), 441–456.
- 54 [22] H.H. Kuo, Introduction to Stochastic Integration, *Springer*, Springer Science + Business  
55 Media, Inc, (2006).
- 56 [23] R. Temam, Infinite-dimensional dynamical systems in mechanics and physics, *Applied Math-*  
57 *ematical Sciences*, Volume 68, Springer-Verlag, New York, Berlin, Heidelberg, (1988).

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