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Convergence of a finite volume scheme for a stochastic conservation law involving a Q -Brownian motion

Tadahisa Funaki · Yueyuan Gao · Danielle Hilhorst

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Abstract We study a time explicit finite volume method for a first order conservation law with a multiplicative source term involving a Q -Wiener process. After having presented the definition of a measure-valued weak entropy solution of the stochastic conservation law, we apply a finite volume method together with Godunov scheme for the space discretization, and we denote by $\{u_{\mathcal{T},k}\}$ its discrete solution. We present some a priori estimates including a weak BV estimate. After performing a time interpolation, we prove two entropy inequalities for the discrete solution. We show that the discrete solution $\{u_{\mathcal{T},k}\}$ converges along a subsequence to a measure-valued entropy solution of the conservation law in the sense of Young measures as the maximum diameter of the volume elements and the time step tend to zero. Some numerical simulations are presented in the case of the stochastic Burgers equation.

Keywords Finite volume methods · Stochastic partial differential equations · Convergence of numerical methods

Mathematics Subject Classification (2010) 65M08 · 60H15 · 65M20

1 Introduction

The convergence of numerical methods for the discretization of stochastic conservation laws is a topic of high interest. In this article we study the convergence of a finite volume scheme for the problem

$$\begin{cases} du + \operatorname{div}(\mathbf{v}f(u))dt = g(u)dW(x, t) & \text{in } \Omega \times \mathbb{T}^d \times (0, T), \\ u(\omega, x, 0) = u_0(x) & \text{for all } \omega \in \Omega, x \in \mathbb{T}^d, \end{cases}$$

where \mathbb{T}^d is the d -dimensional torus, $W(x, t)$ is a Q -Brownian motion, the function f is Lipschitz continuous and the function g is Lipschitz continuous and bounded. We suppose that $\mathbf{v} = \mathbf{v}(x, t)$ is a given vector function and that u_0 is a given square integrable function on \mathbb{T}^d .

T. Funaki
Graduate School of Mathematical Sciences, University of Tokyo, 3-8-1, Komaba, Meguro-ku, Tokyo, 153-8914, Japan
Tel.: +81 (0)3 5465 7033
Fax : +81 (0)3 5465 7011
E-mail: funaki@ms.u-tokyo.ac.jp

Y. Gao
Laboratoire de Mathématiques, University Paris-Sud Paris-Saclay, 91405 Orsay Cedex, France
Tel.: +33 (0)1 69 15 57 60
E-mail: yueyuan.gao@u-psud.fr

D. Hilhorst
Laboratoire de Mathématiques, University Paris-Sud Paris-Saclay and CNRS, 91405 Orsay Cedex, France
Tel.: +33 (0)1 69 15 60 21
Fax: +33 (0)1 69 15 67 18
E-mail: danielle.hilhorst@math.u-psud.fr

A number of articles have been devoted to the study of scalar conservation laws with a multiplicative stochastic forcing term involving a white noise in time.

Let us mention the one-dimensional study of Feng-Nualart [15], where the authors introduced a notion of entropy solution in order to prove the existence and uniqueness of an entropy solution. Chen-Ding-Karlsen [9] extended the work of Feng-Nualart to the multi-dimensional case. They proved a uniform spatial BV bound by means of vanishing viscosity approximations. Moreover they proved the temporal equicontinuity of approximations in $L^1(\Omega \times D \times (0, T))$, uniformly in the viscosity coefficient.

Debussche-Vovelle [13] proved the existence and uniqueness of a kinetic solution for multi-dimensional scalar conservation laws in a d -dimensional torus driven by a general multiplicative space-time noise. Hofmanová [22] then presented a Bhatnagar-Gross-Krook-like approximation of this problem. Applying the stochastic characteristics method, the author established the existence of an approximate solution and proved its convergence to the kinetic solution introduced by [13].

Bauzet-Vallet-Wittbold [4] proved the existence and uniqueness of a weak stochastic entropy solution of the multi-dimensional Cauchy problem in $L^2(\Omega \times \mathbb{R}^d \times (0, T))$ in the case of a multiplicative one-dimensional white noise in time;

In Bauzet-Vallet-Wittbold [5] the authors investigated a corresponding Dirichlet Problem in a bounded domain of \mathbb{R}^d .

Concerning the study of numerical schemes for stochastic conservation laws, Bauzet-Charrier-Gallouët [6] studied explicit flux-splitting finite volume discretizations of multi-dimensional nonlinear scalar conservation laws with monotone flux perturbed by a multiplicative one-dimensional white noise in time with a given initial function in $L^2(\mathbb{R}^d)$. Under a stability condition on the time step, they proved the convergence of the finite volume approximation towards the unique stochastic entropy solution of the corresponding initial value problem. Then Bauzet-Charrier-Gallouët [7] studied the case of a more general flux and in [8], Bauzet-Charrier-Gallouët studied the convergence of the scheme when the stochastic conservation law is defined on a bounded domain with inhomogeneous Dirichlet boundary conditions. Let us also mention the convergence results of time-discretization of Holden-Risebro [23] and Bauzet [3] on a bounded domain of \mathbb{R}^d , as well as an article of Kröker-Rohde [25] of a finite volume schemes in a one-dimensional context.

In a recent study, Audusse-Boyaval-Gao-Hilhorst [1] performed numerical simulations in the one-dimensional torus for the first order Burgers equation forced by a stochastic source term. The source term is a white noise in time while various regularities in space are considered. The authors applied the Monte-Carlo method, and observed that the empirical mean introduces a small diffusion effect to the deterministic numerical solution and converges to the space average of the initial condition as the time t tends to infinity and that the empirical variance stabilizes for large time.

The present article extends the article by Bauzet-Charrier-Gallouët [7] mentioned above. Its organisation is as follows: In section 2 we define a weak stochastic entropy solution and a measure-valued stochastic entropy solution of Problem (1) and recall basic results from probability theory. In section 3, we apply a finite volume method together with a Godunov scheme to Problem (1) and define the discrete solution $\{u_{\mathcal{T},k}\}$. In section 4, we present an estimate of the discrete noise term as well as a priori estimates on the discrete solution $\{u_{\mathcal{T},k}\}$. The a priori estimates imply that $\{u_{\mathcal{T},k}\}$ converges up to a subsequence in the sense of Young measures to an entropy process denoted by \mathbf{u} in $L^2(\Omega \times \mathbb{T}^d \times (0, T) \times (0, 1))$. We then prove a weak BV estimate which is essential in the sequel in order to ensure that the difference between the piece-wise constant solution in time and the solution linearly interpolated in time can be controlled by the maximum diameter of the volume elements and the time step. Meanwhile, in order to prove the discrete entropy inequality, we need the weak BV estimate for showing that a certain residue tends to zero as the maximum diameter of the volume elements and the time step tend to zero. In section 5, we introduce a time interpolation and prove two inequalities, a discrete entropy inequality and a continuous entropy inequality on the discrete solution which are fundamental for the convergence proof. Then in section 6, using the two entropy inequalities, we show that the discrete solution $\{u_{\mathcal{T},k}\}$ converges along a subsequence to a limit \mathbf{u} in the sense of Young measures as the maximum diameter of the volume elements and the time step tend to zero; moreover \mathbf{u} is a measure-valued entropy solution of Problem (1). In section 7, some numerical simulations for stochastic Burgers equation involving a Brownian motion and a Q -Brownian motion are presented. It turns out that the variance increases more as a function of time in the case of a unidimensional Brownian motion than in the case of the Q -Brownian motion.

In a forthcoming work [16], we will show that the measured-value entropy solution \mathbf{u} is unique and coincides with the unique weak stochastic entropy solution; this will ensure that the whole approximate sequence $\{u_{\mathcal{T},k}\}$ converges to the entropy solution u .

2 A stochastic conservation law involving a Q -Brownian motion

We study the convergence of a finite volume scheme for the discretization of the stochastic scalar conservation law

$$\begin{cases} du + \operatorname{div}(\mathbf{v}f(u))dt = g(u)dW(x, t) & \text{in } \Omega \times \mathbb{T}^d \times [0, T], \\ u(\omega, x, 0) = u_0(x) & \omega \in \Omega, x \in \mathbb{T}^d, \end{cases} \quad (2.1)$$

where \mathbb{T}^d is the d -dimensional torus and $W(x, t)$ is a Q -Brownian motion [10]. More precisely, let Q be a nonnegative definite symmetric operator on $L^2(\mathbb{T}^d)$ such that Q is of trace-class, namely

$$\operatorname{Tr} Q = \sum_{m=1}^{\infty} (Qe_m, e_m)_{L^2(\mathbb{T}^d)} = \sum_{m=1}^{\infty} \lambda_m \leq \Lambda_0, \quad (2.2)$$

for some positive constant Λ_0 . Let $\{e_m\}_{m \geq 1}$ be an orthonormal basis in $L^2(\mathbb{T}^d)$ diagonalizing Q , and let $\{\lambda_m\}_{m \geq 1}$ be the corresponding eigenvalues, such that

$$Qe_m = \lambda_m e_m$$

for all $m \geq 1$. Actually, Q is an integral operator with the kernel

$$Q(x, y) = \sum_{m=1}^{\infty} \lambda_m e_m(x) e_m(y). \quad (2.3)$$

We suppose furthermore that $e_m \in L^\infty(\mathbb{T}^d)$ for $m = 1, 2, \dots$ and that there exists a positive constant Λ_1 such that

$$\sum_{m=1}^{\infty} \lambda_m \|e_m\|_{L^\infty(\mathbb{T}^d)}^2 \leq \Lambda_1. \quad (2.4)$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration (\mathcal{F}_t) [24] and $\{\beta_m(t)\}_{m \geq 1}$ be a sequence of independent (\mathcal{F}_t) -Brownian motions defined on $(\Omega, \mathcal{F}, \mathbb{P})$; the process W defined by

$$W(x, t) = \sum_{m=1}^{\infty} \beta_m(t) Q^{\frac{1}{2}} e_m(x) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t) e_m(x) \quad (2.5)$$

is a Q -Brownian motion in $L^2(\mathbb{T}^d)$ [cf. [18], Definition 2.6, page 20], and the series defined by (2.5) is convergent in $C([0, T], L^2(\Omega \times \mathbb{T}^d))$ [cf. [18], page 20]. We recall that a Brownian motion $\beta(t)$ is called an (\mathcal{F}_t) -Brownian motion if it is (\mathcal{F}_t) -adapted and the increment $\beta(t) - \beta(s)$ is independent of \mathcal{F}_s for every $0 \leq s < t$.

Moreover we assume that the following hypotheses (H) hold:

- $u_0 \in L^2(\mathbb{T}^d)$,
- $f : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz continuous function with Lipschitz constant C_f such that $f(0) = 0$,
- $g : \mathbb{R} \rightarrow \mathbb{R}$ is a bounded Lipschitz continuous function with Lipschitz constant C_g such that $|g(u)| \leq M_g$ for some positive constant M_g ,
- There exists $V < \infty$ such that $|\mathbf{v}(x, t)| \leq V$ for all $(x, t) \in \mathbb{T}^d \times [0, T]$,

and we introduce some further notations,

- Let $\mathbb{E}[\cdot]$ denote the expectation, namely the integral over Ω with respect to the probability measure \mathbb{P} . In general for a random variable Y defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, if Y is integrable with respect to the measure \mathbb{P} , then $\mathbb{E}[Y] = \int_{\Omega} Y(\omega) \mathbb{P}(d\omega)$.

- We denote by $\mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d))$ the subclass of $L^2(\Omega \times \mathbb{T}^d \times (0, T))$ consisting of predictable $L^2(\mathbb{T}^d)$ -valued processes [cf. [11], page 98].

Next we define the notions of stochastic entropy solution and of measure-valued entropy solutions of Problem (2.1):

Definition 2.1 (Entropy solution of Problem (2.1))

A function $u \in \mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d)) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d))$ is a weak entropy solution of the stochastic scalar conservation law (2.1) with the initial condition $u_0 \in L^2(\mathbb{T}^d)$, if P-a.s in Ω ,

$$\begin{aligned} & \int_{\mathbb{T}^d} \eta(u_0(x))\varphi(x, 0)dx + \int_0^T \int_{\mathbb{T}^d} \{\eta(u)\partial_t\varphi(x, t) + F^\eta(u)\mathbf{v} \cdot \nabla_x\varphi(x, t)\}dxdt \\ & + \int_{\mathbb{T}^d} \int_0^T \eta'(u)g(u)\varphi(x, t)dW(x, t)dx + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \eta''(u)g^2(u)\varphi(x, t)Q(x, x)dxdt \\ & \geq 0 \end{aligned}$$

with

$$F^\eta(\tau) = \int_0^\tau \eta'(\sigma)f'(\sigma)d\sigma \quad (2.6)$$

for all $\varphi \in \mathcal{C} := \{\varphi \in C_0^\infty(\mathbb{T}^d \times [0, T]), \varphi \geq 0\}$ and for all $\eta \in \mathcal{A}$ where \mathcal{A} is the set of C^3 convex functions such that the support of η'' is compact.

Definition 2.2 (Measure-valued entropy solution of Problem (2.1))

A function \mathbf{u} of $\mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$ is a measure-valued entropy solution of the stochastic scalar conservation law (2.1) with the initial condition $u_0 \in L^2(\mathbb{T}^d)$, if P-a.s in Ω , for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$

$$\begin{aligned} & \int_{\mathbb{T}^d} \eta(u_0(x))\varphi(x, 0)dx + \int_0^T \int_{\mathbb{T}^d} \int_0^1 \{\eta(\mathbf{u}(\cdot, \alpha))\partial_t\varphi(x, t) + F^\eta(\mathbf{u}(\cdot, \alpha))\mathbf{v} \cdot \nabla_x\varphi(x, t)\}d\alpha dxdt \\ & + \int_{\mathbb{T}^d} \int_0^T \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))\varphi(x, t)d\alpha dW(x, t)dx \\ & + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(\cdot, \alpha))g^2(\mathbf{u}(\cdot, \alpha))\varphi(x, t)Q(x, x)d\alpha dxdt \\ & \geq 0 \end{aligned}$$

3 The finite volume discretization

3.1 The numerical scheme

Definition 3.1 (Admissible mesh) An admissible mesh \mathcal{T} of \mathbb{T}^d for the discretization is given by a family of disjoint polygonal connected subsets of \mathbb{T}^d such that \mathbb{T}^d is the union of the closure of the elements of \mathcal{T} and the common interface of any two control volumes is included in a hyperplane of \mathbb{T}^d . We assume that

$$h = \text{size}(\mathcal{T}) = \sup\{\text{diam}(K), K \in \mathcal{T}\} < \infty,$$

and that, for some $\alpha_{\mathcal{T}} \in \mathbb{R}^+$,

$$\alpha_{\mathcal{T}}h^d \leq |K| \text{ and } |\partial K| \leq \frac{1}{\alpha_{\mathcal{T}}}h^{d-1}, \quad \text{for all } K \in \mathcal{T},$$

which implies

$$\frac{|\partial K|}{|K|} \leq \frac{1}{\alpha_{\mathcal{T}}^2 h}, \quad (3.1)$$

where

- x_K is a point in the control volume K ,
- $|K|$ is the d -dimensional Lebesgue measure of K ,
- ∂K is the boundary of the control volume K ,
- $|\partial K|$ is the $(d-1)$ -dimensional Lebesgue measure of ∂K ,
- $\mathcal{N}(K)$ is the set of control volumes neighbors of the control volume K ,
- $\sigma_{K,L}$ is the common interface between K and L for all $L \in \mathcal{N}(K)$,
- $\mathbf{n}_{K,L}$ is the unit normal vector which is perpendicular to the interface $\sigma_{K,L}$, outward to the control volume K , for all $L \in \mathcal{N}(K)$.

Consider an admissible mesh \mathcal{T} in the sense of Definition 3.1. In order to compute an approximation of u on $(0, T)$, we take $N \in \mathbb{N}^+$ and define the time step $k = \frac{T}{N}$. In this way $(0, T) = \bigcup_{n=0}^{N-1} (nk, (n+1)k)$. We set $t^n = nk$ for all $n = 0, 1, 2, \dots, N$ and assume that k and h satisfy a Courant-Friedrichs-Lewy (CFL) condition: $k \leq Ch$ for a certain constant C . We recall the definition of Godunov scheme.

Definition 3.2 (Godunov flux) A function $F^G \in C^2(\mathbb{R}^2, \mathbb{R})$ is called a Godunov flux if it satisfies

$$F^G(a, b) = \begin{cases} \min_{s \in [a, b]} f(s) & \text{if } a \leq b \\ \max_{s \in [b, a]} f(s) & \text{if } a > b. \end{cases} \quad (3.2)$$

For all $(a, b) \in \mathbb{R}^2$ we denote by $s(a, b) \in [\min(a, b), \max(a, b)]$ a real number such that $F^G(a, b) = f(s(a, b))$.

Remark 3.1 There hold $|F^G(b, a) - F^G(a, a)| \leq C_f |a - b|$ and $|F^G(a, b) - F^G(a, a)| \leq C_f |a - b|$.

The discrete initial condition $\{u_K^0, K \in \mathcal{T}\}$ is given by:

$$u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx \quad \text{for all } K \in \mathcal{T}, \quad (3.3)$$

and $\{u_K^n\}$ satisfies the following explicit scheme:

For all $K \in \mathcal{T}$ and all $n \in \{0, 1, \dots, N-1\}$

$$\frac{|K|}{k} (u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} = \frac{|K|}{k} g(u_K^n) (W_K^{n+1} - W_K^n), \quad (3.4)$$

where denoting by $d\gamma$ the $(d-1)$ -dimensional Lebesgue measure,

$$v_{K,L}^n = \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\mathbf{v}(x, t) \cdot \mathbf{n}_{K,L})^+ d\gamma(x) dt, \quad (3.5)$$

$$\begin{aligned} v_{L,K}^n &= \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\mathbf{v}(x, t) \cdot \mathbf{n}_{L,K})^+ d\gamma(x) dt \\ &= \frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} (\mathbf{v}(x, t) \cdot \mathbf{n}_{K,L})^- d\gamma(x) dt. \end{aligned} \quad (3.6)$$

Moreover, since $\operatorname{div} \mathbf{v} = 0$ for all $(x, t) \in \mathbb{T} \times (0, T)$, we have

$$\begin{aligned} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n - v_{L,K}^n) &= \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left(\frac{1}{k|\sigma_{K,L}|} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} v(x, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \right) \\ &= \frac{1}{k} \int_{nk}^{(n+1)k} \int_K \operatorname{div} \mathbf{v} dx dt = 0. \end{aligned} \quad (3.7)$$

Remark 3.2 We remark that, if the flux function f is monotone, and $F^G(a, b)$ is the Godunov scheme, then the flux term in (3.4) coincides with the upwind scheme. Indeed, suppose that f is increasing, we use the definition of the Godunov flux (3.2) to deduce that $F^G(a, b) = f(a)$ for all $a, b \in \mathbb{R}$. Thus the flux term in the scheme (3.4) satisfies

$$v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n) = \begin{cases} v_{K,L}^n f(u_K^n) & \text{if } \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} \geq 0 \\ -v_{L,K}^n f(u_L^n) & \text{if } \mathbf{v}(x, t) \cdot \mathbf{n}_{K,L} < 0, \end{cases}$$

which coincides with the upwind scheme.

3.2 The main result of this article

We define the approximate finite volume solution $u_{\mathcal{T},k}$ on $\mathbb{T}^d \times (0, T)$ from the discrete unknowns u_K^n , for all $n \in \{0, 1, \dots, N-1\}$ and for all $K \in \mathcal{T}$, that:

$$u_{\mathcal{T},k}(x, t) = u_K^n \quad \text{for } x \in K \quad \text{and } t \in [nk, (n+1)k), \quad (3.8)$$

where the set $\{u_K^0, K \in \mathcal{T}\}$ is defined by (3.3).

The main result of this article is the following Theorem:

Theorem 3.1 (Convergence of the finite volume scheme and the existence of a measure-valued entropy solution of Problem (2.1)) *Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh, $T > 0$, $N \in \mathbb{N}^+$ and let $k = \frac{T}{N}$ satisfy that*

$$\frac{k}{h} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Then there exist a function $\mathbf{u} \in \mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$ and a subsequence of $\{u_{\mathcal{T},k}\}$ which we denote again by $\{u_{\mathcal{T},k}\}$ such that $\{u_{\mathcal{T},k}\}$ converges to \mathbf{u} in the sense of Young measures. Moreover \mathbf{u} is measure-valued entropy solution of Problem (2.1) in the sense of Definition 2.2.

3.3 The study of the discrete noise term

The discrete noise terms are given by

$$\begin{aligned} W_{M,K}(t) &= \sum_{m=1}^M \sqrt{\lambda_m} \beta_m(t) e_K^m, \\ W_K(t) &= \sum_{m=1}^{\infty} \sqrt{\lambda_m} \beta_m(t) e_K^m, \end{aligned} \quad (3.9)$$

where $e_K^m = \frac{1}{|K|} \int_K e_m(x) dx$ for all $K \in \mathcal{T}$. Moreover we denote by $W_{M,K}^n$ and W_K^n the values of $W_{M,K}$ and W_K at the time $t = nk$ respectively. We define for later use

$$\begin{aligned} W_{\mathcal{T},k}(x, t) &= W_K^n \quad \text{for } x \in K \quad \text{and } t \in [nk, (n+1)k), \\ W_{M,\mathcal{T},k}(x, t) &= W_{M,K}^n \quad \text{for } x \in K \quad \text{and } t \in [nk, (n+1)k). \end{aligned}$$

$$W_{\mathcal{T}}(x, t) = W_K(t) \quad \text{for } x \in K \quad \text{and } t \in [0, T].$$

Lemma 3.1 $W_{M,K}(t)$ and $W_K(t)$ being defined as in (3.9), we have that

$$\mathbb{E}[W_{M,K}^{n+1} - W_{M,K}^n] = 0 \quad (3.10)$$

and

$$\mathbb{E}[W_K^{n+1} - W_K^n] = 0 \quad (3.11)$$

Proof. Remark that for each $m \geq 1$ and each $n \in \{0, 1, \dots, N-1\}$

$$\beta_m(t^{n+1}) - \beta_m(t^n) \sim \mathcal{N}(0, t^{n+1} - t^n),$$

where $\mathcal{N}(0, t^{n+1} - t^n)$ denotes the Gaussian law $\mathcal{N}(\mu, \sigma^2)$ with density $\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{\sigma^2}}$ in the case $\mu = 0$ and $\sigma^2 = t^{n+1} - t^n$, so that,

$$\mathbb{E}[\beta_m(t^{n+1}) - \beta_m(t^n)] = 0. \quad (3.12)$$

and

$$\text{Var}[\beta_m(t^{n+1}) - \beta_m(t^n)] = \mathbb{E}[(\beta_m(t^{n+1}) - \beta_m(t^n))^2] - (\mathbb{E}[(\beta_m(t^{n+1}) - \beta_m(t^n))])^2,$$

so that by (3.12)

$$\mathbb{E}[(\beta_m(t^{n+1}) - \beta_m(t^n))^2] = \text{Var}[\beta_m(t^{n+1}) - \beta_m(t^n)] = t^{n+1} - t^n, \quad (3.13)$$

Let M and K be given. Then

$$\mathbb{E}[W_{M,K}^{n+1} - W_{M,K}^n] = \sum_{m=1}^M \sqrt{\lambda_m} \mathbb{E}[\beta_m(t^{n+1}) - \beta_m(t^n)] e_K^m = 0.$$

which completes the proof of (3.10). Next we show (3.11). Since

$$(e_K^m)^2 = \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2 \leq \frac{1}{|K|} \int_K e_m^2(x) dx \leq \frac{1}{|K|},$$

we have that

$$\mathbb{E}[(W_K^n)^2] = \sum_{m=1}^{\infty} \lambda_m t^n (e_K^m)^2 \leq \frac{t^n}{|K|} \sum_{m=1}^{\infty} \lambda_m < \infty,$$

where we have used that $\text{Var}[\beta_m(t_n)] = \text{Var}[\beta_m(t_n) - \beta_m(0)] = t_n$. Thus, $W_K^n \in L^2(\Omega) \subset L^1(\Omega)$, so that $\mathbb{E}[|W_K^n|] < \infty$. Therefore

$$\begin{aligned} \mathbb{E}[W_K^{n+1} - W_K^n] &= \mathbb{E}\left[\sum_{m=1}^{\infty} \sqrt{\lambda_m} (\beta_m(t^{n+1}) - \beta_m(t^n)) e_K^m \right] \\ &= \sum_{m=1}^{\infty} \sqrt{\lambda_m} \mathbb{E}[\beta_m(t^{n+1}) - \beta_m(t^n)] e_K^m \\ &= 0. \end{aligned} \quad (3.14)$$

Lemma 3.2 Suppose that the coefficients $\{\lambda_m\}_{m \geq 1}$ satisfy (2.2), then

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0, \quad (3.15)$$

holds for all $M \in \mathbb{N}^+$, $n \in \{0, 1, \dots, N-1\}$ and $K \in \mathcal{T}$.

Proof: We have that for fixed n , M and K ,

$$(W_{M,K}^{n+1} - W_{M,K}^n)^2 = \left(\sum_{m=1}^M (\beta_m(t^{n+1}) - \beta_m(t^n)) \frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2.$$

We take the expectation of both sides to obtain:

$$\begin{aligned} & \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] \\ &= \mathbb{E} \left[\left(\sum_{m=1}^M (\beta_m(t^{n+1}) - \beta_m(t^n)) \frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2 \right] \\ &= \mathbb{E} \left[\sum_{m=1}^M (\beta_m(t^{n+1}) - \beta_m(t^n))^2 \lambda_m \right] \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2 \\ &\quad + \mathbb{E} \left[2 \sum_{m_1 \neq m_2} (\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)) (\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n)) \sqrt{\lambda_{m_1}} e_K^{m_1} \sqrt{\lambda_{m_2}} e_K^{m_2} \right] \\ &= \sum_{m=1}^M \mathbb{E} [(\beta_m(t^{n+1}) - \beta_m(t^n))^2] \lambda_m \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2 \\ &\quad + 2 \sum_{m_1 \neq m_2} \mathbb{E} [(\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)) (\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n))] \sqrt{\lambda_{m_1}} e_K^{m_1} \sqrt{\lambda_{m_2}} e_K^{m_2}. \end{aligned}$$

Since $\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)$ and $\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n)$ are independent, we have

$$\begin{aligned} & \mathbb{E}[(\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)) (\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n))] \\ &= \mathbb{E}[\beta_{m_1}(t^{n+1}) - \beta_{m_1}(t^n)] \mathbb{E}[\beta_{m_2}(t^{n+1}) - \beta_{m_2}(t^n)] \\ &= 0. \end{aligned}$$

Therefore

$$\mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] = \sum_{m=1}^M \mathbb{E}[(\beta_m(t^{n+1}) - \beta_m(t^n))^2] \lambda_m \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2.$$

Next we deduce from (3.13) that

$$\begin{aligned} \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] &= \sum_{m=1}^M (t^{n+1} - t^n) \lambda_m \sum_{K \in \mathcal{T}} |K| \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2 \\ &= (t^{n+1} - t^n) \sum_{m=1}^M \lambda_m \sum_{K \in \mathcal{T}} |K| \frac{1}{|K|^2} \left(\int_K e_m(x) dx \right)^2 \\ &\leq (t^{n+1} - t^n) \sum_{m=1}^{\infty} \lambda_m \sum_{K \in \mathcal{T}} \int_K e_m^2(x) dx \\ &\leq (t^{n+1} - t^n) \sum_{m=1}^{\infty} \lambda_m \leq (t^{n+1} - t^n) \Lambda_0. \end{aligned}$$

Corollary 3.1 *There holds:*

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_K^{n+1} - W_K^n \right)^2 \right] \leq (t^{n+1} - t^n) \Lambda_0. \quad (3.16)$$

Proof: We first show the limiting property

$$\|W_{M,\mathcal{T},k}(x,t) - W_{\mathcal{T},k}(x,t)\|_{L^2(\Omega \times \mathbb{T}^d)} \rightarrow 0 \text{ as } M \rightarrow \infty,$$

for all $t \in (0, T)$. Indeed

$$\begin{aligned} & \|W_{M,\mathcal{T},k}(x,t) - W_{\mathcal{T},k}(x,t)\|_{L^2(\Omega \times \mathbb{T}^d)}^2 \\ &= \mathbb{E} \left[\int_{\mathbb{T}^d} (W_{M,\mathcal{T},k}(x,t) - W_{\mathcal{T},k}(x,t))^2 dx \right] \\ &= \mathbb{E} \left[\sum_{K \in \mathcal{T}} |K| \left(\frac{1}{|K|} \int_K (W_M(x,t) - W(x,t)) dx \right)^2 \right] \\ &\leq \mathbb{E} \left[\sum_{K \in \mathcal{T}} |K| \frac{1}{|K|^2} \int_K 1 dx \int_K (W_M(x,t) - W(x,t))^2 dx \right] \\ &\leq \mathbb{E} \left[\int_{\mathbb{T}^d} dx (W_M(x,t) - W(x,t))^2 \right] \\ &\rightarrow 0 \text{ as } M \rightarrow \infty, \end{aligned}$$

since the series defined by (2.5) is convergent in $C([0, T], L^2(\Omega \times \mathbb{T}^d))$. We have that,

$$\begin{aligned} & \left| \|W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} - \|W_{\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} \right| \\ &\leq \| (W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)) - (W_{\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^n)) \|_{L^2(\Omega \times \mathbb{T}^d)} \\ &= \| (W_{M,\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^{n+1})) - (W_{M,\mathcal{T},k}(t^n) - W_{\mathcal{T},k}(t^n)) \|_{L^2(\Omega \times \mathbb{T}^d)} \\ &\leq \|W_{M,\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^{n+1})\|_{L^2(\Omega \times \mathbb{T}^d)} + \|W_{M,\mathcal{T},k}(t^n) - W_{\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} \\ &\rightarrow 0 \text{ as } M \rightarrow \infty, \end{aligned}$$

that is to say

$$\lim_{M \rightarrow \infty} \|W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} = \|W_{\mathcal{T},k}(t^{n+1}) - W_{\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)}.$$

In view of (3.15), we obtain that

$$\|W_{M,\mathcal{T},k}(t^{n+1}) - W_{M,\mathcal{T},k}(t^n)\|_{L^2(\Omega \times \mathbb{T}^d)} = \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] \leq (t^{n+1} - t^n) A_0,$$

where we take the limit $M \rightarrow \infty$ to obtain:

$$\lim_{M \rightarrow \infty} \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_{M,K}^{n+1} - W_{M,K}^n \right)^2 \right] = \sum_{K \in \mathcal{T}} |K| \mathbb{E} \left[\left(W_K^{n+1} - W_K^n \right)^2 \right] \leq (t^{n+1} - t^n) A_0,$$

as the upper bound $(t^{n+1} - t^n) A_0$ does not depend on M .

4 A priori estimates

Lemma 4.1 *Assume that hypotheses (H) hold. Let $T > 0$, \mathcal{T} be an admissible mesh in the sense of Definition 3.1 and h and k satisfy the Courant-Friedrichs-Lewy (CFL) condition:*

$$k \leq \frac{\alpha_{\mathcal{T}}^2 h}{2VC_f}.$$

There hold

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{T}^d))}^2 \leq \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T A_0 M_g^2 |\mathbb{T}^d|$$

and

$$\|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T^2 A_0 M_g^2 |\mathbb{T}^d|,$$

where $Q_T = \mathbb{T}^d \times (0, T)$.

Proof. We recall the numerical scheme

$$\frac{|K|}{k}(u_K^{n+1} - u_K^n) + \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} = \frac{|K|}{k} g(u_K^n)(W_K^{n+1} - W_K^n). \quad (4.1)$$

We multiply both sides of (4.1) by ku_K^n :

$$\begin{aligned} |K|(u_K^{n+1} - u_K^n)u_K^n &= -k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} u_K^n \\ &\quad + |K|g(u_K^n)(W_K^{n+1} - W_K^n)u_K^n. \end{aligned}$$

Applying the formula $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$ with $a = u_K^{n+1} - u_K^n$ and $b = u_K^n$, we obtain that

$$\begin{aligned} \frac{|K|}{2} [(u_K^{n+1})^2 - (u_K^n)^2 - (u_K^{n+1} - u_K^n)^2] &= -k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} u_K^n \\ &\quad + |K|g(u_K^n)(W_K^{n+1} - W_K^n)u_K^n. \end{aligned}$$

Thus

$$\begin{aligned} \frac{|K|}{2} [(u_K^{n+1})^2 - (u_K^n)^2] &= \frac{|K|}{2} (u_K^{n+1} - u_K^n)^2 \\ &\quad - k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} u_K^n \\ &\quad + |K|g(u_K^n)(W_K^{n+1} - W_K^n)u_K^n. \end{aligned} \quad (4.2)$$

We substitute (4.1) into (4.2) and take the expectation of both sides to deduce that

$$\begin{aligned} &\frac{|K|}{2} \mathbb{E}[(u_K^{n+1})^2 - (u_K^n)^2] \\ &= \frac{|K|}{2} \mathbb{E} \left[\left(\frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \right)^2 + g^2(u_K^n)(W_K^{n+1} - W_K^n)^2 \right] \\ &\quad - 2 \frac{|K|}{2} \mathbb{E} \left[\frac{kg(u_K^n)}{|K|} (W_K^{n+1} - W_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \right] \\ &\quad - \mathbb{E} \left[k \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} u_K^n \right] \\ &\quad + \mathbb{E} [g(u_K^n)|K|(W_K^{n+1} - W_K^n)u_K^n]. \end{aligned}$$

We remark that two terms in the equality above vanish. Indeed since $W_K^{n+1} - W_K^n$ and u_K^n are independent variables, we have that, in view of (3.14)

$$\mathbb{E} \left[g(u_K^n)(W_K^{n+1} - W_K^n) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \right] = 0,$$

and similarly

$$\mathbb{E} \left[g(u_K^n)|K|(W_K^{n+1} - W_K^n)u_K^n \right] = \mathbb{E} \left[W_K^{n+1} - W_K^n \right] \mathbb{E} \left[g(u_K^n)u_K^n|K| \right] = 0.$$

Therefore,

$$\begin{aligned}
& \frac{|K|}{2} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] \\
&= \frac{k^2}{2|K|} \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \right)^2 \right] \\
&\quad - k \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} u_K^n \right] \\
&\quad + \frac{|K|}{2} \mathbb{E} [(W_K^{n+1} - W_K^n)^2] \mathbb{E} [g^2(u_K^n)].
\end{aligned} \tag{4.3}$$

We recall (3.7), which states that $\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n - v_{L,K}^n) = 0$, so that also

$$\sum_{L \in \mathcal{N}(K)} \{|\sigma_{K,L}| (v_{K,L}^n - v_{L,K}^n) f(u_K^n)\} = 0. \tag{4.4}$$

The equality (4.3) can be then rewritten as

$$\frac{|K|}{2} \mathbb{E} [(u_K^{n+1})^2 - (u_K^n)^2] = B_1 - B_2 + D, \tag{4.5}$$

where

$$\begin{aligned}
B_1 &= \frac{k^2}{2|K|} \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] \\
B_2 &= k \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} u_K^n \right] \\
D &= \frac{|K|}{2} \mathbb{E} [(W_K^{n+1} - W_K^n)^2] \mathbb{E} [g^2(u_K^n)]
\end{aligned} \tag{4.6}$$

Using a similar method as in the Part **I.2** of Proposition 4 in [7] we deduce that

$$\sum_{K \in \mathcal{T}} (B_1 - B_2) \leq 0,$$

which we substitute in (4.5); this together with the Definition of D in (4.6) and the inequality (3.16) yields:

$$\begin{aligned}
\sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^{n+1})^2] &\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^n)^2] + \sum_{K \in \mathcal{T}} |K| \mathbb{E} [(W_K^{n+1} - W_K^n)^2] \mathbb{E} [g^2(u_K^n)] \\
&\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^n)^2] + M_g^2 \sum_{K \in \mathcal{T}} |K| \mathbb{E} [(W_K^{n+1} - W_K^n)^2] \\
&\leq \sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^n)^2] + k \Lambda_0 M_g^2,
\end{aligned}$$

which implies that

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^n)^2] \leq \sum_{K \in \mathcal{T}} |K| \mathbb{E} [(u_K^0)^2] + nk \Lambda_0 M_g^2 \quad \text{for all } n \in \{1, 2, \dots, N\},$$

and that

$$\|u_{\mathcal{T},k}\|_{L^\infty(0,T;L^2(\Omega \times \mathbb{T}^d))}^2 \leq \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T \Lambda_0 M_g^2.$$

As a consequence, there holds

$$\|u_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T \|u_0\|_{L^2(\mathbb{T}^d)}^2 + T^2 \Lambda_0 M_g^2.$$

4.1 Weak BV estimate

Lemma 4.2 *Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 3.1, $T > 0$, $N \in \mathbb{N}^+$ and let $k = \frac{T}{N}$ satisfy the CFL condition*

$$k \leq \frac{(1-\xi)\alpha_{\mathcal{T}}^2 h}{2VC_f} \quad (4.7)$$

for some $\xi \in (0, 1)$. Then the following estimates hold

1. There exists a positive constant C_1 , depending on Λ_0 , T , M_g , ξ , C_f and $\|u_0\|_{L^2(\mathbb{T}^d)}$ such that

$$\begin{aligned} & \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \{F^G(u_K^n, u_L^n) - f(u_K^n)\}^2 \right. \\ & \left. + v_{L,K}^n \{F^G(u_L^n, u_K^n) - f(u_K^n)\}^2 \right] \leq C_1; \end{aligned} \quad (4.8)$$

2. There exists a positive constant C_2 depending on α , Λ_0 , T , M_g , ξ , C_f and $\|u_0\|_{L^2(\mathbb{T}^d)}$ such that

$$\begin{aligned} & \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c)) \right\} \right. \\ & \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(d) - F^G(c,d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(c) - F^G(c,d)) \right\} \right] \leq C_2 h^{-\frac{1}{2}}, \end{aligned} \quad (4.9)$$

where we define

$$\mathcal{I}_n := \{(K, L) \in \mathcal{T}^2 : L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n\}$$

and

$$\mathcal{C}(a, b) := \{(c, d) \in [\min(a, b), \max(a, b)]^2 : (d - c)(b - a) \geq 0\}.$$

Proof: 1, Multiplying the equation (4.1) by ku_K^n , inserting (4.4), taking the expectation and summing over $K \in \mathcal{T}$ and $n = 0, 1, \dots, N-1$ implies

$$\begin{aligned} & \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)u_K^n] \\ & + \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} u_K^n \right] \\ & = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)u_K^n g(u_K^n)], \end{aligned}$$

which we denote as $A + B = D$ with

$$\begin{aligned} A &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)u_K^n] \\ B &= \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E}[\{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} u_K^n] \\ D &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(W_K^{n+1} - W_K^n)u_K^n g(u_K^n)]. \end{aligned}$$

Note that the term $D = 0$ since the increment $(W_K^{n+1} - W_K^n)$ is independent of u_K^n and since by (3.14), $\mathbb{E}[W_K^{n+1} - W_K^n] = 0$. Applying the formula $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$ with $a = u_K^{n+1} - u_K^n$ and $b = u_K^n$ we obtain:

$$\begin{aligned} A &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2] + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1})^2 - (u_K^n)^2] \\ &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2] + \frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^N)^2 - (u_K^0)^2], \end{aligned}$$

where we set

$$A_1 = -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^{n+1} - u_K^n)^2],$$

and

$$A_2 = \frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^N)^2 - (u_K^0)^2] \geq -\frac{1}{2} \sum_{K \in \mathcal{T}} |K| \mathbb{E}[(u_K^0)^2]. \quad (4.10)$$

Next, we substitute (3.4) into A_1 . Also using (4.4) and the fact that $W_K^{n+1} - W_K^n$ and any function of $u_K^n, u_L^n, v_{K,L}^n$ and $v_{L,K}^n$ are independent, we deduce that

$$\begin{aligned} A_1 &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \left\{ \mathbb{E} [(W_K^{n+1} - W_K^n)^2 g^2(u_K^n)] \right. \\ &\quad \left. + \frac{k^2}{|K|} \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] \right\}. \end{aligned}$$

Using a similar idea in the proof of Proposition 2 in [7], we deduce that:

$$\begin{aligned} &\sum_{K \in \mathcal{T}} \frac{k^2}{2|K|} \mathbb{E} \left[\left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] \\ &\leq \frac{1-\xi}{2} k \sum_{(K,L) \in \mathcal{I}_n} \frac{|\sigma_{K,L}|}{2C_f} \\ &\quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))^2 \right\} \right. \\ &\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))^2 \right\} \right]. \end{aligned}$$

Therefore

$$\begin{aligned} A_1 &\geq -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \mathbb{E} [(W_K^{n+1} - W_K^n)^2 g^2(u_K^n)] - \frac{1-\xi}{2} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \frac{|\sigma_{K,L}|}{2C_f} \\ &\quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))^2 \right\} \right. \\ &\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))^2 \right\} \right], \end{aligned}$$

where we recall that $\mathcal{I}_n = \{(K, L) \in \mathcal{T}^2 : L \in \mathcal{N}(K) \text{ and } u_K^n > u_L^n\}$. Next we use (3.16) and (4.10) to deduce that

$$\begin{aligned} A &= A_1 + A_2 \\ &\geq -\frac{1}{2}(\|u_0\|_{L^2(\mathbb{T}^d)}^2 + T\Lambda_0 M_g^2) - \frac{1-\xi}{2} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \frac{|\sigma_{K,L}|}{2C_f} \\ &\quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \right. \\ &\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \right] \end{aligned}$$

The term B is estimated by using the same idea as in the proof of Proposition 2 in [7] such that,

$$\begin{aligned} B &\geq \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} \frac{|\sigma_{K,L}|}{2 \times 2C_f} \\ &\quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \right. \\ &\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \right]. \end{aligned}$$

Therefore, since $A + B = 0$,

$$\begin{aligned} &\|u_0\|_{L^2(\mathbb{T}^d)}^2 + T\Lambda_0 M_g^2 \\ &\geq \frac{\xi}{(2C_f)} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \\ &\quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \right. \\ &\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \right] \end{aligned}$$

which, in turn, implies the bound

$$\begin{aligned} &\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \\ &\quad \times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \right. \\ &\quad \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \right] \\ &\leq C_1, \end{aligned} \tag{4.11}$$

where the positive constant C_1 depends on Λ_0 , T , M_g , ξ , C_f and $\|u_0\|_{L^2(\mathbb{T}^d)}$. We then use a reordering of the summation to deduce that

$$\sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \left\{ (F^G(u_K^n, u_L^n) - f(u_K^n))^2 + (F^G(u_K^n, u_L^n) - f(u_L^n))^2 \right\} \right]$$

$$\begin{aligned}
& + v_{L,K}^n \left\{ (F^G(u_L^n, u_K^n) - f(u_K^n))^2 + (F^G(u_L^n, u_K^n) - f(u_L^n))^2 \right\} \Big] \\
& = \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \{F^G(u_K^n, u_L^n) - f(u_K^n)\}^2 + v_{L,K}^n \{F^G(u_L^n, u_K^n) - f(u_K^n)\}^2 \right]
\end{aligned}$$

which together with (4.11) implies

$$\sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \{F^G(u_K^n, u_L^n) - f(u_K^n)\}^2 + v_{L,K}^n \{F^G(u_L^n, u_K^n) - f(u_K^n)\}^2 \right] \leq C_1.$$

This completes the proof of the inequality (4.8).

Proof of the inequality (4.9). We estimate the term

$$\begin{aligned}
\bar{T}^2 = & \left\{ \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d)) \right\} \right. \right. \\
& \left. \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(c) - F^G(c,d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(d) - F^G(c,d)) \right\} \right] \right\}^2 \quad (4.12)
\end{aligned}$$

We define

$$T_1 := \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(c)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c,d) - f(d))$$

and

$$T_2 := \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(c) - F^G(c,d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(d) - F^G(c,d))$$

We deduce from the Cauchy-Schwarz inequality that:

$$\begin{aligned}
\bar{T}^2 & = \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} [v_{K,L}^n T_1 + v_{L,K}^n T_2] \right)^2 \\
& \leq \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \times \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[\frac{(v_{K,L}^n T_1 + v_{L,K}^n T_2)^2}{v_{K,L}^n + v_{L,K}^n} \right] \right) \quad (4.13) \\
& \leq \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \times \left(\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} [v_{K,L}^n T_1^2 + v_{L,K}^n T_2^2] \right),
\end{aligned}$$

where we have applied Jensen's inequality to deduce that

$$\left(\frac{v_{K,L}^n T_1 + v_{L,K}^n T_2}{v_{K,L}^n + v_{L,K}^n} \right)^2 \leq \frac{v_{K,L}^n}{v_{K,L}^n + v_{L,K}^n} T_1^2 + \frac{v_{L,K}^n}{v_{K,L}^n + v_{L,K}^n} T_2^2.$$

It follows from (3.1), that

$$\begin{aligned}
\sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) & = \sum_{n=0}^{N-1} k \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| v_{K,L}^n \\
& \leq T \sum_{K \in \mathcal{T}} |\partial K| V \\
& \leq \frac{TV |\mathbb{T}^d|}{\alpha_{\mathcal{T}}^2 h}.
\end{aligned}$$

Moreover, we remark that

$$\begin{aligned} T_1^2 &\leq 2 \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \\ T_2^2 &\leq 2 \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\}, \end{aligned}$$

which we substitute into (4.13) to deduce that

$$\begin{aligned} \bar{T}^2 &\leq \frac{2TV|\mathbb{T}^d|}{\alpha_{\mathcal{T}}^2 h} \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \\ &\times \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \right. \\ &\left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c))^2 + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d))^2 \right\} \right]. \end{aligned} \quad (4.14)$$

Substituting (4.11) into (4.14), we deduce that

$$\bar{T}^2 \leq \frac{2TV|\mathbb{T}^d|}{\alpha_{\mathcal{T}}^2 h} C_1$$

which combined with the inequality (4.12) implies

$$\begin{aligned} &\left\{ \sum_{n=0}^{N-1} k \sum_{(K,L) \in \mathcal{I}_n} |\sigma_{K,L}| \mathbb{E} \left[v_{K,L}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(c)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (F^G(c, d) - f(d)) \right\} \right. \right. \\ &\quad \left. \left. + v_{L,K}^n \left\{ \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(c) - F^G(c, d)) + \max_{(c,d) \in \mathcal{C}(u_K^n, u_L^n)} (f(d) - F^G(c, d)) \right\} \right] \right\}^2 \\ &\leq \frac{2TC_1|\mathbb{T}^d|V}{\alpha_{\mathcal{T}}^2 h}. \end{aligned}$$

We choose $C_2 = \frac{2TC_1|\mathbb{T}^d|V}{\alpha_{\mathcal{T}}^2 h}$ to deduce (4.9).

5 Convergence of the scheme

5.1 A time-continuous approximation

We define \bar{u}_K as the continuous in time stochastic process

$$\begin{aligned} \bar{u}_K(t) &= u_K^n - \frac{t - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)) \\ &\quad + g(u_K^n)(W_K(t) - W_K^n) \\ &= u_K^n - \int_{nk}^t \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \frac{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)}{|K|} ds \\ &\quad + \int_{nk}^t g(u_K^n) dW_K(t). \end{aligned} \quad (5.1)$$

on the domain $\Omega \times (nk, (n+1)k)$. In this way, $\bar{u}_K(nk) = u_K^n$ and $\bar{u}_K((n+1)k) = u_K^{n+1}$. We define the time-continuous approximate solution $\bar{u}_{\mathcal{T},k}$ on $\Omega \times \mathbb{T}^d \times (0, T)$ by

$$\bar{u}_{\mathcal{T},k}(x, t) = \bar{u}_K(t^n) \quad \text{for } x \in K \quad \text{and } t \in [nk, (n+1)k).$$

Next we estimate the difference between the time-continuous approximation $\bar{u}_{\mathcal{T},k}$ and the finite volume solution $u_{\mathcal{T},k}$ which is defined in (3.8).

Lemma 5.1 *Let $u_0 \in L^2(\mathbb{T}^d)$ and \mathcal{T} be an admissible mesh in the sense of Definition 3.1. Let $k = \frac{T}{N}$ satisfy the CFL condition (4.7). There exists a positive constant C depending on T, M_g, C_f, α, V and u_0 such that*

$$\|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq C(h+k).$$

Proof: There holds

$$\begin{aligned} & \|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \\ &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[g(u_K^n) (W_K(s) - W_K^n) + \right. \\ & \quad \left. \left(\frac{s-nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] dx ds \\ &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} [g^2(u_K^n) (W_K(s) - W_K^n)^2] dx ds \\ & \quad + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\frac{s-nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] dx ds \\ & \quad + 2 \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\frac{s-nk}{|K|} g(u_K^n) (W_K(s) - W_K^n) \right. \\ & \quad \left. \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right], \\ &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} [g^2(u_K^n) (W_K(s) - W_K^n)^2] dx ds \\ & \quad + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\frac{s-nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] dx ds \end{aligned}$$

where we used the fact that

$$\mathbb{E} [W_K(s) - W_K^n] = 0. \quad (5.2)$$

We remark that (5.2) can be proved in the same way as (3.14). Applying the counter part of (3.16)

$$\sum_{K \in \mathcal{T}} |K| \mathbb{E} [(W_K(s) - W_K^n)^2] \leq (s-t^n) \Lambda_0,$$

for all $s > nk$, we deduce that

$$\begin{aligned} & \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} [g^2(u_K^n)(W_K(s) - W_K^n)^2] dx ds \\ & \leq \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} (s - t^n) \Lambda_0 M_g^2 ds \\ & \leq T \Lambda_0 M_g^2 k. \end{aligned}$$

Moreover, using the CFL condition (4.7) and the inequality (4.8), we deduce that:

$$\begin{aligned} & \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \mathbb{E} \left[\left(\frac{s - nk}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) \right. \right. \\ & \quad \left. \left. - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n)) \} \right)^2 \right] dx ds \\ & \leq C_1 \frac{\alpha_{\mathcal{T}}^2 h}{V(2C_f)^2} \end{aligned}$$

Therefore

$$\|u_{\mathcal{T},k} - \bar{u}_{\mathcal{T},k}\|_{L^2(\Omega \times Q_T)}^2 \leq T \Lambda_0 M_g^2 k + C_1 \frac{\alpha_{\mathcal{T}}^2 h}{V(2C_f)^2}.$$

Finally we set $C = \max(T \Lambda_0 M_g^2, \frac{C_1 \alpha_{\mathcal{T}}^2}{V(2C_f)^2})$ to deduce the result of Lemma 5.1.

5.2 Entropy inequalities for the approximate solution

In this section, we show entropy inequalities satisfied by the approximate solution and use them in the proof of convergence of the numerical scheme.

Lemma 5.2 (Discrete entropy inequality) *Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 3.1, $N \in \mathbb{N}^+$, let $k = \frac{T}{N}$ be the time step and assume that*

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0. \quad (5.3)$$

Then P -a.s in Ω , for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$:

$$\begin{aligned} & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_K (\eta(u_K^{n+1}) - \eta(u_K^n)) \varphi(x, nk) dx \\ & + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K F^n(u_K^n) \mathbf{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt \\ & + \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx \\ & + \frac{1}{2} \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) q_K dx dt \\ & \geq R^{k,h} \end{aligned} \quad (5.4)$$

where $q_K = \sum_{m=1}^{\infty} \lambda_m \left(\frac{1}{|K|} \int_K e_m^2(x) dx \right)^2$, $F^\eta(a) = \int_0^a \eta'(s) f'(s) ds$ and for all sets $A \in \mathcal{F}$, $\mathbb{E}[\mathbf{1}_A R^{k,h}] \rightarrow 0$ as $h \rightarrow 0$.

Before proving Lemma 5.2, we recall Itô's formula (cf. [24], p. 153).

Lemma 5.3 (Itô's formula for an (\mathcal{F}_t) -Brownian motion)

Assume that the stochastic process $W(\tau)$ is an (\mathcal{F}_t) -Brownian motion multiplied by a positive constant $q^{\frac{1}{2}}$ and $X(\tau)$ is an \mathbb{R} -valued stochastic process given by

$$X(\tau) = X(0) + \int_0^\tau \psi(t) dt + \int_0^\tau \theta(t) dW(t), \quad 0 \leq \tau \leq T,$$

where $X(0)$ is \mathcal{F}_0 -measurable, ψ is (\mathcal{F}_t) -adapted and measurable in (ω, t) such that

$$\int_0^T \|\psi(t)\|_{\mathbb{R}} dt < \infty \quad \text{P-a.s.}$$

and θ is an (\mathcal{F}_t) -adapted and measurable process such that

$$\text{P} \left(\int_0^T \theta^2(t) dt < \infty \right) = 1.$$

Suppose that the function $\mathcal{G}(x_1, x_2) : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$ is such that its partial derivatives $\frac{\partial \mathcal{G}}{\partial x_2}$, $\frac{\partial \mathcal{G}}{\partial x_1}$ and $\frac{\partial^2 \mathcal{G}}{\partial x_1^2}$ are continuous on $\mathbb{R} \times [0, T]$. Then P-a.s. for all $\tau \in [0, T]$,

$$\begin{aligned} \mathcal{G}(X(\tau), \tau) &= \mathcal{G}(X(0), 0) + \int_0^\tau \left\{ \frac{\partial \mathcal{G}}{\partial x_2}(X(t), t) + \frac{\partial \mathcal{G}}{\partial x_1}(X(t), t) \psi(t) \right\} dt \\ &\quad + \int_0^\tau \frac{\partial \mathcal{G}}{\partial x_1}(X(t), t) \cdot \theta(t) dW(t) \\ &\quad + \frac{q}{2} \int_0^\tau \frac{\partial^2 \mathcal{G}}{\partial x_1^2}(X(t), t) \cdot \theta^2(t) dt. \end{aligned}$$

Lemma 5.4 (Itô isometry) (cf. [24], Page 144)

Assume that $\{\beta_i(t)\}_{i=1,2}$ and $\beta(t)$ are independent (\mathcal{F}_t) -Brownian motions, and let $X = (X_t(\omega))_{t \in (0, T)}$ and $Y = (Y_t(\omega))_{t \in (0, T)}$ be (\mathcal{F}_t) -adapted and measurable stochastic processes such that $X, Y \in L^2(\Omega \times (0, T))$. Then,

$$\mathbb{E} \left[\left(\int_0^T X_t d\beta_i(t) \right) \left(\int_0^T Y_t d\beta_j(t) \right) \right] = \mathbb{E} \left[\int_0^T X_t Y_t dt \right] \cdot \delta_{i,j}, \quad (5.5)$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise, which in turn implies that

$$\mathbb{E} \left[\left(\int_0^T X_t d\beta(t) \right)^2 \right] = \mathbb{E} \left[\int_0^T X_t^2 dt \right]. \quad (5.6)$$

We then prove an equality based upon Itô's formula.

Lemma 5.5 For all $\eta \in \mathcal{A}$, there holds:

$$\begin{aligned} &\eta(\bar{u}_K((n+1)k)) - \eta(\bar{u}_K(nk)) \\ &= - \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \frac{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)}{|K|} dt \\ &\quad + \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) g(u_K^n) dW_K(t) \\ &\quad + \frac{1}{2} q_K \int_{nk}^{(n+1)k} \eta''(\bar{u}_{\mathcal{T},k}(t)) g^2(u_K^n) dt \end{aligned} \quad (5.7)$$

P-a.s in Ω .

Proof: We apply the Itô's formula stated in Lemma 5.3 to the case that

- $\mathcal{G}(x_1, x_2) = \eta(x_1)$,
- $X(t) = \bar{u}_K(t)$ which is defined in (5.1) on the time interval $(nk, (n+1)k)$,
- $\psi = - \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \frac{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)}{|K|}$,
- $\theta = g(u_K^n)$
- The stochastic process $W(t)$ is given by $W_K(t) = \sum_{m=1}^{\infty} \frac{\sqrt{\lambda_m}}{|K|} \left(\int_K e_m(x) dx \right) \beta_m(t)$.

Thus:

$$\frac{\partial \mathcal{G}}{\partial x_2} = 0, \quad \frac{\partial \mathcal{G}}{\partial x_1} = \eta'(x_1) \quad \text{and} \quad \frac{\partial^2 \mathcal{G}}{\partial x_1^2} = \eta''(x_1)$$

with $x_1 = X(t) = \bar{u}_K(t)$, $t \in [nk, (n+1)k)$.

Using the formula of $W_K(t)$, we deduce that

$$W_K(t^{n+1}) - W_K(t^n) = \sum_{m=1}^{\infty} \left\{ \left(\frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right) (\beta_m(t^{n+1}) - \beta_m(t^n)) \right\}$$

where

$$\beta_m(t^{n+1}) - \beta_m(t^n) \sim \mathcal{N}(0, t^{n+1} - t^n),$$

Using the property that if a random variable $Y \sim \mathcal{N}(\mu, \sigma^2)$, then the random variable $aY + b \sim \mathcal{N}(a\mu + b, b^2 \sigma^2)$ [19], we deduce that

$$\left(\frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right) (\beta_m(t^{n+1}) - \beta_m(t^n)) \sim \mathcal{N} \left(0, \left(\frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2 (t^{n+1} - t^n) \right)$$

for all $m \in \{1, 2, \dots\}$. Because the Brownian motions $\{\beta_m(t)\}_{m \geq 1}$ are independent, and using the fact that if two independent random variables $Y_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $Y_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ then $Y_1 + Y_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ [19], we deduce that

$$W_K(t^{n+1}) - W_K(t^n) \sim \mathcal{N} \left(0, \sum_{m=1}^{\infty} \left(\frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2 (t^{n+1} - t^n) \right).$$

We recall

$$\begin{aligned} q_K &:= \sum_{m=1}^{\infty} \left(\frac{\sqrt{\lambda_m}}{|K|} \int_K e_m(x) dx \right)^2 \\ &= \frac{1}{|K|^2} \int_K \int_K Q(x, y) dx dy, \end{aligned}$$

and we rewrite the stochastic process $W_K(t)$ in the form $W_K(t) = q_K^{\frac{1}{2}} \beta(t)$ where β is a standard Brownian motion and apply Itô's formula (cf. Lemma 5.3) to deduce that

$$\begin{aligned} &\eta(\bar{u}_K((n+1)k)) - \eta(\bar{u}_K(nk)) \\ &= - \int_{nk}^{(n+1)k} \eta'(\bar{u}_K(t)) \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \frac{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)}{|K|} dt \\ &\quad + \int_{nk}^{(n+1)k} \eta'(\bar{u}_K(t)) g(u_K^n) dW_K(t) \\ &\quad + \frac{1}{2} q_K \int_{nk}^{(n+1)k} \eta''(\bar{u}_K(t)) g^2(u_K^n) dt \end{aligned}$$

P -a.s in Ω . This completes the proof of Lemma 5.5.

Proof of Lemma 5.2. Step 1: Recalling that $\bar{u}_K(nk) = u_K^n$, $\bar{u}_K((n+1)k) = u_K^{n+1}$, and using (3.7), after multiplying (5.7) by $|K|\varphi_K^n$, one deduces that P -a.s. in Ω ,

$$\begin{aligned}
& \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n [\eta(u_K^{n+1}) - \eta(u_K^n)] \\
= & - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \varphi_K^n \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) \sum_{L \in \mathcal{N}(K)} \left[|\sigma_{K,L}| \right. \\
& \quad \left. \times \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right] dt \\
& + \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n \int_{nk}^{(n+1)k} \eta'(\bar{u}_{\mathcal{T},k}(t)) g(u_K^n) dW_K(t) \\
& + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| q_K \varphi_K^n \int_{nk}^{(n+1)k} \eta''(\bar{u}_{\mathcal{T},k}(t)) g^2(u_K^n) dt,
\end{aligned} \tag{5.8}$$

with

$$\varphi_K^n = \frac{1}{|K|} \int_K \varphi(x, nk) dx.$$

The equation (5.8) can be written in the form $A^{k,h} = -B^{k,h} + C^{k,h} + D^{k,h}$, with

$$\begin{aligned}
A^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \varphi_K^n [\eta(u_K^{n+1}) - \eta(u_K^n)] \\
B^{k,h} &= \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \varphi_K^n \eta'(\bar{u}_{\mathcal{T},k}(t)) \sum_{L \in \mathcal{N}(K)} \left[|\sigma_{K,L}| \right. \\
& \quad \left. \times \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right] dt \\
C^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| \int_{nk}^{(n+1)k} \varphi_K^n \eta'(\bar{u}_{\mathcal{T},k}(t)) g(u_K^n) dW_K(t) \\
D^{k,h} &= \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} |K| q_K \varphi_K^n \int_{nk}^{(n+1)k} \eta''(\bar{u}_{\mathcal{T},k}(t)) g^2(u_K^n) dt
\end{aligned}$$

Since by the assumption (5.3), $\frac{k}{h} \rightarrow 0$ as $h \rightarrow 0$ in the theorem, we may suppose that the CFL condition

$$k \leq \frac{(1-\xi)\alpha_{\mathcal{T}}^2 h}{V(2C_f)}$$

holds for some $\xi \in (0, 1)$.

Study of $B^{k,h}$: we decompose $B^{k,h}$ as

$$B^{k,h} = (B^{k,h} - \tilde{B}^{k,h}) + (\tilde{B}^{k,h} - \bar{B}^{k,h}) + (\bar{B}^{k,h} - B_1^{k,h}) + B_1^{k,h}$$

where

$$\begin{aligned}\tilde{B}^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta'(u_K^n) \\ &\quad \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \\ \bar{B}^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n G^G(u_K^n, u_L^n) - v_{L,K}^n G^G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \\ B_1^{k,h} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \int_K F^\eta(u_K^n) \mathbf{v}(x, t) \cdot \nabla_x \varphi(x, nk) dx dt.\end{aligned}$$

where

$$G^G(a, b) = F^\eta(s(a, b)). \quad (\text{cf. (2.6)})$$

In the following we prove that $\tilde{B}^{h,k} - \bar{B}^{h,k} \geq 0$ almost surely. Recalling that

$$\begin{aligned}\sum_{L \in \mathcal{N}(K)} (v_{K,L}^n - v_{L,K}^n) F^\eta(u_K^n) &= 0, \\ \sum_{L \in \mathcal{N}(K)} (v_{K,L}^n - v_{L,K}^n) f(u_K^n) &= 0,\end{aligned}$$

and that for all $K \in \mathcal{T}$, $F^G(u_K^n, u_K^n) = f(u_K^n)$ and $G^G(u_K^n, u_K^n) = F^\eta(u_K^n)$; we rewrite $\tilde{B}^{h,k} - \bar{B}^{h,k}$ as:

$$\begin{aligned}&\tilde{B}^{h,k} - \bar{B}^{h,k} \\ &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \left\{ v_{K,L}^n [\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n))] \right. \\ &\quad \left. - v_{L,K}^n [\eta'(u_K^n)(F^G(u_L^n, u_K^n) - f(u_K^n)) - (G^G(u_L^n, u_K^n) - F^\eta(u_K^n))] \right\} \int_K \varphi(x, nk) dx\end{aligned}$$

Let $K, L \in \mathcal{T}$, $L \in \mathcal{N}(K)$ and suppose that $u_K^n < u_L^n$.

We study the sign of $\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n))$. Using the fact that F^G is a Godunov numerical flux (cf. (3.2)), we deduce that there exists $s(u_K^n, u_L^n) \in [u_K^n, u_L^n]$ such that $F^G(u_K^n, u_L^n) = f(s(u_K^n, u_L^n)) = \min_{s \in [u_K^n, u_L^n]} f(s)$. Thus

$$\begin{aligned}&\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n)) \\ &= \eta'(u_K^n)(f(s(u_K^n, u_L^n)) - f(u_K^n)) - (F^\eta(s(u_K^n, u_L^n)) - F^\eta(u_K^n)) \\ &= \int_{u_K^n}^{s(u_K^n, u_L^n)} f'(s) \eta'(u_K^n) ds - \int_{u_K^n}^{s(u_K^n, u_L^n)} f'(s) \eta'(s) ds \\ &= \int_{u_K^n}^{s(u_K^n, u_L^n)} f'(s) (\eta'(u_K^n) - \eta'(s)) ds \\ &= f(s(u_K^n, u_L^n)) \{ \eta'(u_K^n) - \eta'(s(u_K^n, u_L^n)) \} + \int_{u_K^n}^{s(u_K^n, u_L^n)} f(s) \eta''(s) ds \\ &\geq f(s(u_K^n, u_L^n)) \{ \eta'(u_K^n) - \eta'(s(u_K^n, u_L^n)) \} + \int_{u_K^n}^{s(u_K^n, u_L^n)} f(s(u_K^n, u_L^n)) \eta''(s) ds \\ &= 0.\end{aligned}$$

Similarly

$$\begin{aligned}
& \eta'(u_K^n)(F^G(u_L^n, u_K^n) - f(u_K^n)) - (G^G(u_L^n, u_K^n) - F^\eta(u_K^n)) \\
&= \eta'(u_K^n)(f(s(u_L^n, u_K^n)) - f(u_K^n)) - (F^\eta(s(u_K^n, u_L^n)) - F^\eta(u_K^n)) \\
&= \int_{u_K^n}^{s(u_L^n, u_K^n)} f'(s)(\eta'(u_K^n) - \eta'(s))ds \\
&= f(s(u_L^n, u_K^n))\{\eta'(u_K^n) - \eta'(s(u_L^n, u_K^n))\} + \int_{u_K^n}^{s(u_L^n, u_K^n)} f(s)\eta''(s)ds \\
&\leq f(s(u_L^n, u_K^n))\{\eta'(u_K^n) - \eta'(s(u_L^n, u_K^n))\} + \int_{u_K^n}^{s(u_L^n, u_K^n)} f(s(u_L^n, u_K^n))\eta''(s)ds \\
&= 0.
\end{aligned}$$

Also using (3.5) and (3.6) we obtain

$$\begin{aligned}
& v_{K,L}^n[\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n))] \\
& - v_{L,K}^n[\eta'(u_K^n)(F^G(u_K^n, u_L^n) - f(u_K^n)) - (G^G(u_K^n, u_L^n) - F^\eta(u_K^n))] \\
& \geq 0.
\end{aligned}$$

We deduce that

$$\begin{aligned}
& v_{K,L}^n[\eta'(u_K^n)(F_D^{LF}(u_K^n, u_L^n) - f(u_K^n)) - (G_D^{LF}(u_K^n, u_L^n) - F^\eta(u_K^n))] \\
& - v_{L,K}^n[\eta'(u_K^n)(F_D^{LF}(u_L^n, u_K^n) - f(u_K^n)) - (G_D^{LF}(u_L^n, u_K^n) - F^\eta(u_K^n))] \\
& \geq 0,
\end{aligned}$$

from which we deduce that $\tilde{B}^{h,k} - \bar{B}^{h,k} \geq 0$ almost surely in Ω .

Study of $C^{k,h}$: we decompose $C^{k,h}$ as

$$C^{k,h} = C^{k,h} - \tilde{C}^{k,h} + \tilde{C}^{k,h}$$

with

$$\tilde{C}^{k,h} = \sum_{K \in \mathcal{T}} \int_K \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx.$$

Study of $D^{k,h}$: we decompose $D^{k,h}$ as

$$D^{k,h} = D^{k,h} - \tilde{D}^{k,h} + \tilde{D}^{k,h}$$

with

$$\tilde{D}^{k,h} = \frac{1}{2} \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \left[\int_K q_K \eta''(u_K^n) g^2(u_K^n) \varphi(x, nk) dx dt \right],$$

Since P-a.s. in Ω , $A^{k,h} = -B^{k,h} + C^{k,h} + D^{k,h}$, we obtain

$$\begin{aligned}
B^{k,h} &= -A^{k,h} + C^{k,h} + D^{k,h} \\
&\geq (B^{k,h} - \tilde{B}^{k,h}) + (\bar{B}^{k,h} - B_1) + B_1,
\end{aligned}$$

so that

$$-A^{k,h} - B_1 \geq (B^{k,h} - \tilde{B}^{k,h}) + (\bar{B}^{k,h} - B_1) - C^{k,h} - D^{k,h},$$

and that

$$-A^{k,h} - B_1 + \tilde{C}^{k,h} + \tilde{D}^{k,h} \geq (B^{k,h} - \tilde{B}^{k,h}) + (\bar{B}^{k,h} - B_1) + (\tilde{C}^{k,h} - C^{k,h}) + (\tilde{D}^{k,h} - D^{k,h}). \quad (5.9)$$

Substituting the definition of $A^{k,h}$, B_1 , $\tilde{C}^{k,h}$ and $\tilde{D}^{k,h}$ into (5.9) yields

$$\begin{aligned}
& - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_K (\eta(u_K^{n+1}) - \eta(u_K^n)) \varphi(x, nk) dx \\
& + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K F^\eta(u_K^n) \mathbf{v} \cdot \nabla_x \varphi(x, nk) dx dt \\
& + \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K \eta'(u_K^n) g(u_K^n) \varphi(x, nk) dW_K(t) dx \\
& + \frac{1}{2} \sum_{n=0}^{N-1} \int_{nk}^{(n+1)k} \sum_{K \in \mathcal{T}} \int_K q_K \eta''(u_K^n) g(u_K^n) \varphi(x, nk) dx dt \\
& \geq (B^{k,h} - \tilde{B}^{k,h}) + (\bar{B}^{k,h} - B_1) + (\tilde{C}^{k,h} - C^{k,h}) + (\tilde{D}^{k,h} - D^{k,h}).
\end{aligned}$$

We define

$$R^{k,h} = (B^{k,h} - \tilde{B}^{k,h}) + (\bar{B}^{k,h} - B_1) + (\tilde{C}^{k,h} - C^{k,h}) + (\tilde{D}^{k,h} - D^{k,h})$$

to deduce the inequality (5.4).

Step 2: Next we prove that for all sets $A \in \mathcal{F}$, $\mathbb{E}[\mathbf{1}_A R^{k,h}] \rightarrow 0$ as $h \rightarrow 0$. Let $A \in \mathcal{F}$; we analyze separately the convergence of $\mathbb{E}[\mathbf{1}_A (B^{k,h} - \tilde{B}^{k,h})]$, $\mathbb{E}[\mathbf{1}_A (\bar{B}^{k,h} - B_1)]$, $\mathbb{E}[\mathbf{1}_A (\tilde{C}^{k,h} - C^{k,h})]$, $\mathbb{E}[\mathbf{1}_A (\tilde{D}^{k,h} - D^{k,h})]$. Note that the assumption that $\frac{k}{h} \rightarrow 0$ as $h \rightarrow 0$ is crucial.

Convergence of $\mathbb{E}[\mathbf{1}_A (B^{k,h} - \tilde{B}^{k,h})]$

We prove that for all sets $A \in \mathcal{F}$, $\mathbb{E}[\mathbf{1}_A (B^{k,h} - \tilde{B}^{k,h})] \rightarrow 0$ as $h \rightarrow 0$. For almost all $\omega \in \Omega$, all $K \in \mathcal{T}$ and all $n \in \{0, 1, \dots, N-1\}$, there exists $\zeta_K^n(\omega) \in \mathbb{R}$ such that

$$\begin{aligned}
B^{k,h} - \tilde{B}^{k,h} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K [\eta'(\bar{u}_{\mathcal{T},k}(t)) - \eta'(u_K^n)] \\
&\quad \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \\
&= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K [\eta''(\zeta_K^n)(\bar{u}_{\mathcal{T},k}(t) - u_K^n)] \\
&\quad \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt \\
&= T_1^{h,k} + T_2^{h,k}
\end{aligned}$$

where we set, in view of (5.1),

$$\begin{aligned}
T_1^{h,k} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta''(\zeta_K^n) \varphi(x, nk) \frac{t - nk}{|K|} \\
&\quad \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 dx dt \\
T_2^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K [\eta''(\zeta_K^n) g(u_K^n) (W_K(t) - W_K(nk))] \\
&\quad \times \sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n F^G(u_K^n, u_L^n) - v_{L,K}^n F^G(u_L^n, u_K^n)\} \varphi(x, nk) dx dt.
\end{aligned}$$

Let $A \in \mathcal{F}$; we first study $\mathbb{E}[\mathbf{1}_A T_1^{h,k}]$. Using Cauchy-Schwarz inequality as in (4.13), the assumption (3.1) and the estimate (4.8), we deduce that

$$\begin{aligned}
& \left| \mathbb{E}[\mathbf{1}_A T_1^{h,k}] \right| \\
&= \left| \mathbb{E} \left[\mathbf{1}_A \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \int_{nk}^{(n+1)k} \frac{1}{|K|} \int_K \eta''(\zeta_K^n) \frac{t-nk}{|K|} \varphi(x, nk) dx dt \right. \right. \\
&\quad \left. \left. \times \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n)) - v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))\} \right)^2 \right] \right| \\
&\leq \|\eta''\|_\infty \|\varphi\|_\infty \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k^2}{|K|} \left(\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n + v_{L,K}^n) \right) \\
&\quad \times \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \\
&\leq \|\eta''\|_\infty \|\varphi\|_\infty \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k^2}{|K|} V |\partial K| \\
&\quad \times \mathbb{E} \left[\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| \{v_{K,L}^n (F^G(u_K^n, u_L^n) - f(u_K^n))^2 + v_{L,K}^n (F^G(u_L^n, u_K^n) - f(u_K^n))^2\} \right] \\
&\leq C_1 \|\eta''\|_\infty \|\varphi\|_\infty \frac{k}{\alpha_{\mathcal{T}}^2 h} V,
\end{aligned}$$

which tends to 0 as h tends to 0. Next we remark that $\mathbb{E}[T_2^{h,k}] = 0$. This completes the proof of $\mathbb{E}[\mathbf{1}_A (B^{k,h} - \tilde{B}^{k,h})] \rightarrow 0$ as $h \rightarrow 0$.

Convergence of $\mathbb{E}[\mathbf{1}_A (\bar{B}^{h,k} - B_1^{h,k})]$

We prove that for all sets $A \in \mathcal{F}$, $\mathbb{E}[\mathbf{1}_A (\bar{B}^{h,k} - B_1^{h,k})] \rightarrow 0$ as $h \rightarrow 0$. Using the fact that

$$\sum_{L \in \mathcal{N}(K)} |\sigma_{K,L}| (v_{K,L}^n - v_{L,K}^n) F^\eta(u_K^n) = 0,$$

we deduce that

$$\begin{aligned}
\bar{B}^{h,k} &= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \frac{k}{|K|} \sum_{L \in \mathcal{N}(K)} \left[\int_K \varphi(x, nk) dx \right. \\
&\quad \left. \times |\sigma_{K,L}| \{v_{K,L}^n (G^G(u_K^n, u_L^n) - F^\eta(u_K^n)) - v_{L,K}^n (G^G(u_L^n, u_K^n) - F^\eta(u_K^n))\} \right]
\end{aligned}$$

so that, $\bar{B}^{h,k} = \bar{T}_1^{h,k} + \bar{T}_2^{h,k}$, where

$$\begin{aligned}
\bar{T}_1^{h,k} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \frac{k}{|K|} |\sigma_{K,L}| \int_K \varphi(x, nk) dx \\
&\quad \times \{v_{K,L}^n (G^G(u_K^n, u_L^n) - F^\eta(u_K^n)) - v_{L,K}^n (G^G(u_L^n, u_K^n) - F^\eta(u_K^n))\}
\end{aligned}$$

and

$$\begin{aligned}
\bar{T}_2^{h,k} &= \sum_{n=0}^{N-1} \sum_{(L,K) \in \mathcal{I}_n} \frac{k}{|K|} |\sigma_{K,L}| \int_K \varphi(x, nk) dx \\
&\quad \times \{v_{K,L}^n(G^G(u_K^n, u_L^n) - F^\eta(u_K^n)) - v_{L,K}^n(G^G(u_L^n, u_K^n) - F^\eta(u_L^n))\} \\
&= - \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \frac{k}{|L|} |\sigma_{K,L}| \int_L \varphi(x, nk) dx \\
&\quad \times \{v_{K,L}^n(G^G(u_K^n, u_L^n) - F^\eta(u_L^n)) - v_{L,K}^n(G^G(u_L^n, u_K^n) - F^\eta(u_L^n))\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
B_1^{h,k} &= - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} F^\eta(u_K^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} \varphi(\gamma, nk) d\gamma(x) dt \\
&= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(\gamma, nk) F^\eta(u_K^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\
&\quad + \sum_{n=0}^{N-1} \sum_{(L,K) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(\gamma, nk) F^\eta(u_K^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\
&= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(\gamma, nk) F^\eta(u_K^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\
&\quad - \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(\gamma, nk) F^\eta(u_L^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\
&:= T_1^{h,k} + T_2^{h,k}
\end{aligned}$$

where

$$\begin{aligned}
T_1^{h,k} &= \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(\gamma, nk) F^\eta(u_K^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt \\
T_2^{h,k} &= - \sum_{n=0}^{N-1} \sum_{(K,L) \in \mathcal{I}_n} \int_{nk}^{(n+1)k} \int_{\sigma_{K,L}} -\varphi(\gamma, nk) F^\eta(u_L^n) \mathbf{v}(\gamma, t) \cdot \mathbf{n}_{K,L} d\gamma(x) dt
\end{aligned}$$

In order to estimate the terms $|\bar{T}_1^{h,k} - T_1^{h,k}|$ and $|\bar{T}_2^{h,k} - T_2^{h,k}|$, we refer to the arguments of estimating $|\bar{T}_1^{h,k} - T_1^{h,k}|$ and $|\bar{T}_2^{h,k} - T_2^{h,k}|$ in Proposition 4 of [7], which yields that for all sets $A \in \mathcal{F}$, $\mathbb{E}[\mathbf{1}_A(\bar{B}^{h,k} - B_1^{h,k})] \rightarrow 0$ as $h \rightarrow 0$.

The detailed proofs of the convergence of the convergence of $\mathbb{E}[\mathbf{1}_A(\tilde{C}^{k,h} - C^{k,h})]$ and of $\mathbb{E}[\mathbf{1}_A(\tilde{D}^{k,h} - D^{k,h})]$ are given in [17]. We thus deduce that $\mathbb{E}[\mathbf{1}_A R^{k,h}] \rightarrow 0$ as $h \rightarrow 0$, which concludes the proof of Lemma 5.2.

We refer to [6, 7] and [17] for the detailed proofs for the following lemma.

Lemma 5.6 (Continuous entropy inequality on the discrete solution) *Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh in the sense of Definition 3.1, $N \in \mathbb{N}^+$, let $k = \frac{T}{N}$ be the time step and assume that*

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0.$$

Then P -a.s. in Ω , for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$, there holds

$$\begin{aligned}
& \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx + \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt + \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \\
& + \int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) \\
& + \frac{1}{2} \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) Q(x, x) dx dt \\
& \geq \tilde{R}^{k,h},
\end{aligned} \tag{5.10}$$

where for all sets $A \in \mathcal{F}$, $\mathbb{E} \left[\mathbf{1}_A \tilde{R}^{k,h} \right] \rightarrow 0$ as $h \rightarrow 0$.

6 Convergence proof

Theorem 6.1 [Convergence to a measure-valued entropy solution of Problem (2.1)] Assume that hypotheses (H) hold. Let \mathcal{T} be an admissible mesh, $T > 0$, $N \in \mathbb{N}^+$ and let $k = \frac{T}{N}$ satisfying that

$$\frac{k}{h} \rightarrow 0 \text{ as } h \rightarrow 0.$$

Then there exist a function $\mathbf{u} \in \mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d \times (0, 1))) \cap L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$ and a subsequence of $\{u_{\mathcal{T},k}\}$ which we denote again by $\{u_{\mathcal{T},k}\}$ such that it converges to \mathbf{u} in the sense of Young measures. Moreover \mathbf{u} is measure-valued entropy solution of Problem (2.1) in the sense of Definition 2.2.

Proof. We multiply the inequality (5.10) by $\mathbf{1}_A$, namely the characteristic function of the set $A \in \mathcal{F}$. We take the expectation, which yields:

$$\begin{aligned}
& \mathbb{E} \left[\mathbf{1}_A \int_{\mathbb{T}^d} \eta(u_0) \varphi(x, 0) dx \right] \\
& + \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \right] + \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right] \\
& + \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) dx \right] \\
& + \frac{1}{2} \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x, t) Q(x, x) dx dt \right] \\
& \geq \mathbb{E}[\mathbf{1}_A \tilde{R}^{k,h}].
\end{aligned} \tag{6.1}$$

It follows from Lemma 4.1 that there exist an entropy process [2, 14] \mathbf{u} of $L^\infty(0, T; L^2(\Omega \times \mathbb{T}^d \times (0, 1)))$ and a subsequence of $\{u_{\mathcal{T},k}\}$ which we denote again by $\{u_{\mathcal{T},k}\}$ such that $u_{\mathcal{T},k}$ converges to \mathbf{u} in the sense of Young measures. Moreover it follows from [2] and [6] that $\mathbf{u} \in \mathcal{N}_\omega^2(0, T; L^2(\mathbb{T}^d \times (0, 1)))$. More precisely the convergence in the sense of Young measures means that given a Carathéodory function $\Psi : \Omega \times \mathbb{T}^d \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ such that $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$ is uniformly integrable, one has:

$$\mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} \Psi(\cdot, u_{\mathcal{T},k}) dx dt \right] \rightarrow \mathbb{E} \left[\int_0^T \int_{\mathbb{T}^d} \int_0^1 \Psi(\cdot, \mathbf{u}(\cdot, \alpha)) d\alpha dx dt \right].$$

Recall that a function $\Psi : \Omega \times \mathbb{T}^d \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function if for almost all $(\omega, x, t) \in \Omega \times \mathbb{T}^d \times (0, T)$ the function $\nu \mapsto \Psi(\omega, x, t, \nu)$ is continuous and for all $\nu \in \mathbb{R}$, the function $(\omega, x, t) \mapsto \Psi(\omega, x, t, \nu)$ is measurable. In order to prove the convergence of $\{u_{\mathcal{T},k}\}$ to a measure-valued stochastic entropy solution

of Problem (2.1), we aim to pass to the limit in the inequality (6.1) as $h \rightarrow 0$. We have proved that $\mathbb{E}[\mathbf{1}_A \tilde{R}^{k,h}] \rightarrow 0$ as $h \rightarrow 0$. It remains to study the convergence of the terms on the left-hand side of (6.1).

Study of the term $\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x,t) dx dt \right]$

Note that

$$\Psi : (\omega, x, t, \nu) \in \Omega \times L^2(\mathbb{T}^d) \times [0, T] \times \mathbb{R} \mapsto \mathbf{1}_A(\omega) \partial_t \varphi(x, t) \eta(\nu) \in \mathbb{R}$$

is a Carathéodory function such that $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$ is bounded in $L^2(\Omega \times Q_T)$. Thus

$$\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta(u_{\mathcal{T},k}) \partial_t \varphi(x, t) dx dt \right] \rightarrow \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x, t, \alpha)) d\alpha \partial_t \varphi(x, t) dx dt \right]$$

as $h \rightarrow 0$.

Study of the term $\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right]$

Since F^η is bounded in $L^2(\Omega \times \mathbb{T}^d \times (0, T))$, using the same arguments as previously, we deduce that

$$\mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} F^\eta(u_{\mathcal{T},k}) \mathbf{v} \cdot \nabla_x \varphi(x, t) dx dt \right] \rightarrow \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x, t, \alpha)) \mathbf{v} \cdot \nabla_x \varphi(x, t) d\alpha dx dt \right]$$

as $h \rightarrow 0$.

Study of the term $\mathbb{E} \left[\mathbf{1}_A \int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi(x, t) dW_{\mathcal{T}}(x, t) dx \right]$

We denote by Ψ the mapping $\Psi : (\omega, x, t, \nu) \mapsto \mathbf{1}_A(\omega) \eta'(\nu) g(\nu) \varphi(x, t) \in \mathbb{R}$; $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$ is bounded in $L^\infty(\Omega \times Q_T)$, so that up to a subsequence $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$ converges weakly in $L^2(\Omega \times Q_T)$ to a function $\chi \in L^2(\Omega \times Q_T)$. Thus

$$\lim_{h,k \rightarrow 0} \mathbb{E} \left[\int_{Q_T} \Psi \phi dx dt \right] = \lim_{h,k \rightarrow 0} \int_{\Omega \times Q_T} \Psi \phi dx dt \mathbb{P}(d\omega) = \int_{\Omega \times Q_T} \chi \phi dx dt \mathbb{P}(d\omega).$$

For any $\phi \in L^2(\Omega \times Q_T)$, $(\omega, x, t, \nu) \mapsto \phi(\omega, x, t) \Psi(\omega, x, t, \nu)$ is a Carathéodory function such that $\phi \Psi(\cdot, u_{\mathcal{T},k})$ is uniformly integrable. It is based on the fact that there exists a positive constant C_3 such that

$$\begin{aligned} \mathbb{E} \left[\int_{Q_T} |\phi \Psi(\cdot, u_{\mathcal{T},k})| dx dt \right] &= \int_{\Omega \times Q_T} |\phi \Psi(\cdot, u_{\mathcal{T},k})| dx dt \mathbb{P}(d\omega) \\ &\leq C_3 \|\Psi(\cdot, u_{\mathcal{T},k})\|_{L^2(\Omega \times Q_T)} \|\phi\|_{L^2(\Omega \times Q_T)}. \end{aligned}$$

Thus

$$\lim_{h,k \rightarrow 0} \int_{\Omega \times Q_T} \Psi(\cdot, u_{\mathcal{T},k}) \phi dx dt \mathbb{P}(d\omega) = \int_{\Omega \times Q_T} \int_0^1 \Psi(\cdot, \mathbf{u}(\cdot, \alpha)) d\alpha \phi dx dt \mathbb{P}(d\omega).$$

By identification,

$$\Psi(\cdot, u_{\mathcal{T},k}) \rightharpoonup \int_0^1 \Psi(\cdot, \mathbf{u}(\cdot, \alpha)) d\alpha \tag{6.2}$$

weakly in $L^2(\Omega \times Q_T)$. We have that

$$\begin{aligned} &\int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW_{\mathcal{T}}(x, t) \\ &= \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW_{\mathcal{T}}(x, t) - \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW(x, t) \\ &\quad + \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW(x, t) \\ &= \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi d(W_{\mathcal{T}}(x, t) - W(x, t)) + \int_0^T \eta'(u_{\mathcal{T},k}) g(u_{\mathcal{T},k}) \varphi dW(x, t) \end{aligned}$$

We first study the term $\int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x,t)$; for $\mathcal{U} \in L^2(\Omega \times Q_T)$, we define the operator $\tilde{\mathcal{T}}$ as

$$\tilde{\mathcal{T}}(\mathcal{U}) = \int_0^T \mathcal{U} dW(x,t) \quad (6.3)$$

where $W(x,t)$ is the Q -Brownian motion defined by (2.5). In view of the Ito isometry Lemma 5.4, and the hypothesis (2.4), we have that:

$$\begin{aligned} & \mathbb{E} \left[\int_{\mathbb{T}^d} (\tilde{\mathcal{T}}(\mathcal{U}))^2 dx \right] = \mathbb{E} \left[\int_{\mathbb{T}^d} \left(\int_0^T \mathcal{U} dW(x,t) \right)^2 dx \right] \\ &= \mathbb{E} \left[\int_{\mathbb{T}^d} \sum_{m=1}^{\infty} \lambda_m \left(\int_0^T \mathcal{U}(x,t) d\beta_m(t) \right)^2 e_m^2(x) dx \right. \\ & \quad \left. + \int_{\mathbb{T}^d} 2dx \sum_{m_1 \neq m_2}^{\infty} \sqrt{\lambda_{m_1} \lambda_{m_2}} e_{m_1}(x) e_{m_2}(x) \int_0^T \mathcal{U}(x,t) d\beta_{m_1}(t) \int_0^T \mathcal{U}(x,t) d\beta_{m_2}(t) \right] \\ &= \mathbb{E} \left[\int_{\mathbb{T}^d} \int_0^T Q(x,x) \mathcal{U}^2(x,t) dt dx \right] + 0 \\ &\leq A_1 \mathbb{E} \left[\int_{\mathbb{T}^d} \int_0^T \mathcal{U}^2(x,t) dt dx \right] \\ &\leq A_1 \|\mathcal{U}\|_{L^2(\Omega \times Q_T)}^2. \end{aligned}$$

This proves that $\tilde{\mathcal{T}}$ is a bounded operator from $L^2(\Omega \times Q_T)$ to $L^2(\Omega \times \mathbb{T}^d)$. Let $\tilde{\mathcal{T}}^* : L^2(\Omega \times \mathbb{T}^d) \rightarrow L^2(\Omega \times Q_T)$ be the adjoint operator of $\tilde{\mathcal{T}}$; then

$$\left(\psi, \tilde{\mathcal{T}}\mathcal{U} \right)_{L^2(\Omega \times \mathbb{T}^d)} = \left(\tilde{\mathcal{T}}^*\psi, \mathcal{U} \right)_{L^2(\Omega \times Q_T)}$$

for all $\psi \in L^2(\Omega \times \mathbb{T}^d)$. Next we set $\mathcal{U} = \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi$. We recall that by (6.2),

$$\eta(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi \rightharpoonup \left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi$$

weakly in $L^2(\Omega \times Q_T)$ along a subsequence as h and k tend to zero. Thus as h and $k \rightarrow 0$,

$$\begin{aligned} & \left(\psi, \tilde{\mathcal{T}}(\eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi) \right)_{L^2(\Omega \times \mathbb{T}^d)} \\ &= \left(\tilde{\mathcal{T}}^*\psi, \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi \right)_{L^2(\Omega \times Q_T)} \\ &\rightarrow \left(\tilde{\mathcal{T}}^*\psi, \left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi \right)_{L^2(\Omega \times Q_T)} \\ &= \left(\psi, \tilde{\mathcal{T}} \left(\left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi \right) \right)_{L^2(\Omega \times \mathbb{T}^d)}. \end{aligned}$$

We conclude that along a subsequence

$$\tilde{\mathcal{T}}(\eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi) \rightharpoonup \tilde{\mathcal{T}} \left(\left\{ \int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right\} \varphi \right) \quad \text{weakly in } L^2(\Omega \times \mathbb{T}^d),$$

as h and $k \rightarrow 0$, or in other words, in view of the definition (6.3) of $\tilde{\mathcal{T}}$,

$$\int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x,t) \rightarrow \int_0^T \left(\int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right) \varphi dW(x,t)$$

Therefore, one obtains

$$\mathbb{E} \left[\int_{\mathbb{T}^d} \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi dW(x,t) dx \right] \rightarrow \mathbb{E} \left[\int_{\mathbb{T}^d} \int_0^T \left(\int_0^1 \eta'(\mathbf{u}(\cdot, \alpha))g(\mathbf{u}(\cdot, \alpha))d\alpha \right) \varphi dW(x,t) dx \right]$$

weakly in $L^2(\Omega \times \mathbb{T}^d)$, as h and $k \rightarrow 0$.

Next, we consider the term

$$I_h := I_h(\omega, x) = \int_0^T \eta'(u_{\mathcal{T},k})g(u_{\mathcal{T},k})\varphi d(W_{\mathcal{T}}(x,t) - W(x,t)). \quad (6.4)$$

To begin with, we prove below the following result:

Lemma 6.1 *Suppose that $A_h \in L^2(\Omega \times \mathbb{T}^d \times (0, T))$ is (\mathcal{F}_t) -adapted [i.e., $A_h(\cdot, t, x)$ is \mathcal{F}_t -measurable in ω for every $x \in \mathbb{T}^d$ and for every $t > 0$], such that*

$$\|A_h\|_{L^\infty(\Omega \times \mathbb{T}^d \times (0, T))} \leq C, \quad (6.5)$$

for some positive constant C which does not depend on h . Then

$$\mathcal{I}_h := \mathcal{I}_h(\omega, x) = \int_0^T A_h d(W_{\mathcal{T}}(x,t) - W(x,t)) \rightarrow 0 \quad (6.6)$$

strongly in $L^2(\Omega \times \mathbb{T}^d)$ as $h \rightarrow 0$.

Proof: We first rewrite

$$\begin{aligned} W_{\mathcal{T}}(x,t) &= \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K W(y,t) dy \cdot 1_K(x) \\ &= \sum_{m=1}^{\infty} \sqrt{\lambda_m} \sum_{K \in \mathcal{T}} e_K^m 1_K(x) \cdot \beta_m(t), \end{aligned}$$

where

$$e_K^m = \frac{1}{|K|} \int_K e_m(y) dy.$$

Then, the difference of stochastic integrals in (6.6) can be rewritten as

$$\mathcal{I}_h(\omega, x) := \sum_{m=1}^{\infty} \sqrt{\lambda_m} \left(\sum_{K \in \mathcal{T}} e_K^m 1_K(x) - e_m(x) \right) \int_0^T A_h(x,t) d\beta_m(t).$$

We have that

$$\begin{aligned} \|\mathcal{I}_h\|_{L^2(\Omega \times \mathbb{T}^d)}^2 &= \int_{\mathbb{T}^d} dx \|\mathcal{I}_h(\cdot, x)\|_{L^2(\Omega)}^2 \\ &= \int_{\mathbb{T}^d} dx \mathbb{E} \left[\left(\sum_{m=1}^{\infty} \sqrt{\lambda_m} \left(\sum_{K \in \mathcal{T}} e_K^m 1_K(x) - e_m(x) \right) \int_0^T A_h(x,t) d\beta_m(t) \right)^2 \right] \\ &= \int_{\mathbb{T}^d} dx \sum_{m=1}^{\infty} \lambda_m \left(\sum_{K \in \mathcal{T}} e_K^m 1_K(x) - e_m(x) \right)^2 \mathbb{E} \left[\left(\int_0^T A_h(x,t) d\beta_m(t) \right)^2 \right] \\ &\quad + \int_{\mathbb{T}^d} 2dx \sum_{m_1 \neq m_2}^{\infty} \sqrt{\lambda_{m_1} \lambda_{m_2}} \left(\sum_{K \in \mathcal{T}} e_K^{m_1} 1_K(x) - e_{m_1}(x) \right) \left(\sum_{K \in \mathcal{T}} e_K^{m_2} 1_K(x) - e_{m_2}(x) \right) \\ &\quad \times \mathbb{E} \left[\int_0^T A_h(x,t) d\beta_{m_1}(t) \int_0^T A_h(x,t) d\beta_{m_2}(t) \right] \end{aligned}$$

Therefore, by the independence of the Brownian motions $\{\beta_m(t)\}$ and by the Itô isometry for the stochastic integrals (5.5) and (5.6), in view of (6.5)

$$\begin{aligned}
& \|\mathcal{I}_h\|_{L^2(\Omega \times \mathbb{T}^d)}^2 \\
&= \int_{\mathbb{T}^d} dx \sum_{m=1}^{\infty} \lambda_m \left(\sum_{K \in \mathcal{T}} e_K^m 1_K(x) - e_m(x) \right)^2 \mathbb{E} \left[\left(\int_0^T A_h^2(x, t) dt \right) \right] \\
&+ \int_{\mathbb{T}^d} dx \left[2 \sum_{m_1 \neq m_2}^{\infty} \sqrt{\lambda_{m_1} \lambda_{m_2}} \left(\sum_{K \in \mathcal{T}} e_K^{m_1} 1_K(x) - e_{m_1}(x) \right) \left(\sum_{K \in \mathcal{T}} e_K^{m_2} 1_K(x) - e_{m_2}(x) \right) \right. \\
&\quad \left. \times \mathbb{E} \left[\int_0^T A_h^2(x, t) dt \right] \delta_{m_1, m_2} \right] \\
&= \int_{\mathbb{T}^d} dx \sum_{m=1}^{\infty} \lambda_m \left(\sum_{K \in \mathcal{T}} e_K^m 1_K(x) - e_m(x) \right)^2 \int_0^T \mathbb{E}[A_h(x, t)^2] dt \\
&\leq C \sum_{m=1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2.
\end{aligned} \tag{6.7}$$

By Cauchy-Schwarz's inequality, we have that

$$\begin{aligned}
& \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 = \sum_{K \in \mathcal{T}} (e_K^m)^2 |K| \\
&= \sum_{K \in \mathcal{T}} \frac{1}{|K|} \left(\int_K e_m(y) dy \right)^2 \leq \sum_{K \in \mathcal{T}} \int_K e_m^2(y) dy = \|e_m\|_{L^2(\mathbb{T}^d)}^2 = 1,
\end{aligned}$$

for all $m \in \{1, 2, \dots\}$, which implies that

$$\left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq 4 \quad \text{for each } m \in \{1, 2, \dots\}.$$

Next we prove that

$$\sum_{m=1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2$$

tends to zero as $\text{size}(\mathcal{T})$ tends to zero.

Fix $\epsilon > 0$, since $\sum_{m=1}^{\infty} \lambda_m$ is a converging series, there exists a $M \in \mathbb{N}$, such that

$$\sum_{m=M+1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq 4 \sum_{m=M+1}^{\infty} \lambda_m \leq \frac{\epsilon}{2}. \tag{6.8}$$

Next we consider the term $\sum_{m=1}^M \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2$. Let $m \in \{1, 2, \dots\}$ be arbitrary; there holds

$$\lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 = \lambda_m \left\| \sum_{K \in \mathcal{T}} \left(\frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right) 1_K(x) \right\|_{L^2(\mathbb{T}^d)}^2$$

$$\begin{aligned}
&= \lambda_m \int_{\mathbb{T}^d} dx \left(\sum_{K \in \mathcal{T}} \left(\frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right) 1_K(x) \right)^2 \\
&= \lambda_m \int_{\mathbb{T}^d} dx \left(\sum_{K \in \mathcal{T}} \left(\frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right)^2 1_K(x) \right) \\
&\quad + \lambda_m \int_{\mathbb{T}^d} dx \sum_{K_1 \neq K_2} \left(\frac{1}{|K_1|} \int_{K_1} e_m(y) dy - e_m(x) \right) \left(\frac{1}{|K_2|} \int_{K_2} e_m(y) dy - e_m(x) \right) 1_{K_1}(x) 1_{K_2}(x) \\
&= \lambda_m \sum_{K \in \mathcal{T}} \int_K \left(\frac{1}{|K|} \int_K e_m(y) dy - e_m(x) \right)^2 dx = \lambda_m \sum_{K \in \mathcal{T}} \int_K \left(\frac{1}{|K|} \int_K e_m(y) dy - \frac{1}{|K|} \int_K e_m(x) dy \right)^2 dx \\
&= \lambda_m \sum_{K \in \mathcal{T}} \int_K \frac{1}{|K|^2} \left(\int_K (e_m(y) - e_m(x)) dy \right)^2 dx \leq \lambda_m \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K \int_K (e_m(y) - e_m(x))^2 dx dy.
\end{aligned}$$

We denote by $B(h)$ the ball with center 0 and radius h with $h = \text{size}(\mathcal{T})$. Then

$$\begin{aligned}
&\lambda_m \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K \int_K (e_m(y) - e_m(x))^2 dx dy \\
&\leq \lambda_m \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K dx \int_{B(h)} dz (e_m(x+z) - e_m(x))^2 \\
&= \lambda_m \int_{B(h)} dz \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K (e_m(x+z) - e_m(x))^2 dx \\
&\leq \lambda_m |B(h)| \sup_{z \in B(h)} \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K (e_m(x+z) - e_m(x))^2 dx.
\end{aligned}$$

Since the condition (3.1) holds, it follows that

$$|B(h)| = C_d h^d \leq \frac{C_d}{\alpha} |K|,$$

for some positive constant C_d . Let $\mathbb{T}^d \subset \subset \mathcal{O}$ where \mathcal{O} is a open set. We suppose that e_m is prolonged by periodicity on \mathbb{R}^d and that $\text{dist}(\mathbb{T}^d, \partial\mathcal{O}) \leq \text{size}(\mathcal{T})$. Therefore

$$\begin{aligned}
&\lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m 1_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \\
&\leq \lambda_m \frac{C_d}{\alpha} |K| \sup_{z \in B(h)} \sum_{K \in \mathcal{T}} \frac{1}{|K|} \int_K (e_m(x+z) - e_m(x))^2 dx \\
&= \lambda_m \frac{C_d}{\alpha} \sup_{z \in B(h)} \|e_m(\cdot + z) - e_m(\cdot)\|_{L^2(\mathbb{T}^d)}^2.
\end{aligned}$$

Let $\epsilon > 0$ be arbitrary. It follows from the density of $C(\mathcal{O})$ into $L^2(\mathcal{O})$ that there exists a function $g \in C(\mathcal{O})$ such that

$$\|e_m(\cdot + z) - g(\cdot + z)\|_{L^2(\mathbb{T}^d)} + \|e_m(x) - g(x)\|_{L^2(\mathbb{T}^d)} \leq \frac{2\epsilon}{3} \cdot \frac{\alpha}{2M\lambda_m C_d},$$

for all $z \in B(h)$. Thus there exists a positive constant δ such that for all $z \in B(h)$

$$\begin{aligned}
&\|e_m(\cdot + z) - e_m(\cdot)\|_{L^2(\mathbb{T}^d)} \\
&\leq \|e_m(\cdot + z) - g(\cdot + z)\|_{L^2(\mathbb{T}^d)} + \|g(\cdot + z) - g(\cdot)\|_{L^2(\mathbb{T}^d)} + \|g(\cdot) - e_m(\cdot)\|_{L^2(\mathbb{T}^d)} \\
&\leq \frac{\alpha}{2M\lambda_m C_d} \epsilon
\end{aligned}$$

for all $h < \delta$.

Using the fact that the sum is finite, we have

$$\sum_{m=1}^M \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq \frac{\epsilon}{2}. \quad (6.9)$$

Combining (6.8) and (6.9), we deduce that if $h < \delta$

$$\sum_{m=1}^{\infty} \lambda_m \left\| \sum_{K \in \mathcal{T}} e_K^m \mathbf{1}_K(\cdot) - e_m(\cdot) \right\|_{L^2(\mathbb{T}^d)}^2 \leq \epsilon.$$

which in view of (6.7) completes the proof of (6.6).

It follows from Lemma 6.1 that the integral I_h defined by (6.4) converges to zero as h and k tend to zero.

Study of the term $\frac{1}{2} \mathbb{E} \left[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x,t) Q(x,x) dx dt \right]$:

Since $\Psi : (\omega, x, t, \nu) \in \Omega \times \mathbb{T}^d \times (0, T) \times \mathbb{R} \mapsto \mathbf{1}_A(\omega) \eta''(\nu) g(\nu) \varphi(x, t) Q(x, x) \in \mathbb{R}$ is a Carathéodory function such that $\Psi(\cdot, \cdot, \cdot, u_{\mathcal{T},k})$ is bounded in $L^2(\Omega \times \mathbb{T}^d \times (0, T))$, we deduce that:

$$\begin{aligned} & \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \eta''(u_{\mathcal{T},k}) g^2(u_{\mathcal{T},k}) \varphi(x,t) Q(x,x) dx dt] \\ & \rightarrow \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x,t,\alpha)) g^2(\mathbf{u}(x,t,\alpha)) \varphi(x,t) Q(x,x) \alpha dx dt], \end{aligned}$$

as $h \rightarrow 0$, for all $m \geq 1$.

We deduce that for all sets $A \in \mathcal{F}$, for all $\eta \in \mathcal{A}$ and for all $\varphi \in \mathcal{C}$

$$\begin{aligned} & \mathbb{E}[\mathbf{1}_A \int_{\mathbb{T}^d} \eta(u_0) \varphi(x,0) dx] + \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x,t,\alpha)) d\alpha \partial_t \varphi(x,t) dx dt] \\ & + \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x,t,\alpha)) \mathbf{v} \cdot \nabla_x \varphi(x,t) d\alpha dx dt] \\ & + \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta'(\mathbf{u}(x,t,\alpha)) \varphi(x,t) d\alpha dW(x,t) dx] \\ & + \frac{1}{2} \mathbb{E}[\mathbf{1}_A \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x,t,\alpha)) g^2(\mathbf{u}(x,t,\alpha)) \varphi(x,t) Q(x,x) d\alpha dx dt] \\ & \geq 0. \end{aligned} \quad (6.10)$$

In turn (6.10) implies that for all $\chi \in L^2(\Omega)$

$$\begin{aligned} & \mathbb{E}[\chi \int_{\mathbb{T}^d} \eta(u_0) \varphi(x,0) dx] + \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta(\mathbf{u}(x,t,\alpha)) d\alpha \partial_t \varphi(x,t) dx dt] \\ & + \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 F^\eta(\mathbf{u}(x,t,\alpha)) \mathbf{v} \cdot \nabla_x \varphi(x,t) d\alpha dx dt] \\ & + \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta'(\mathbf{u}(x,t,\alpha)) \varphi(x,t) d\alpha dW(x,t) dx] \\ & + \frac{1}{2} \mathbb{E}[\chi \int_0^T \int_{\mathbb{T}^d} \int_0^1 \eta''(\mathbf{u}(x,t,\alpha)) g^2(\mathbf{u}(x,t,\alpha)) \varphi(x,t) Q(x,x) d\alpha dx dt] \\ & \geq 0. \end{aligned}$$

Hence \mathbf{u} is a measure-valued entropy solution of Problem (2.1) in the sense of Definition 2.2.

7 Numerical simulations

Taking inspiration from the numerical computations in [1], we propose to perform numerical simulations for a stochastic Burgers equation involving a Q -Brownian motion:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = \alpha_Q \dot{W}(x, t) \quad (7.1)$$

in the space interval $[0, 1]$ together with periodic boundary conditions and the initial condition $u(x, 0) = u_0(x)$. The constant α_Q is the amplitude of the noise term.

We consider the case that the Q -Brownian motion is given by

$$W(x, t) = \sum_{m=1}^{\infty} \sqrt{\lambda_m} \{ \beta_m(t) e_m(x) + \gamma_m(t) f_m(x) \},$$

with $\{e_m(x), f_m(x)\} = \{\sqrt{2} \sin(2m\pi x), \sqrt{2} \cos(2m\pi x)\}$, $m = 1, 2, \dots$ and $\{\beta_m(t), \gamma_m(t)\}$, $m = 1, 2, \dots$ is a sequence of independent (\mathcal{F}_t) -Brownian motions.

We suppose that Q is the Laplacian operator $u \rightarrow -u''$ with periodic boundary conditions, then the eigenvalue λ_m and eigenfunctions e_m and f_m of Q which satisfy

$$Qe_m = \lambda_m e_m, \quad Qf_m = \lambda_m f_m$$

for all $m = 1, 2, \dots$ and such that $\lambda_m = 4m^2\pi^2$. We remark that Burgers equation is not included in the class of problems studied in this article, since the flux function $f(u) = u^2/2$ is only locally Lipschitz continuous.

The following formula follows from [26]. For all u in the domain of $(-\Delta)^{-\beta}$, there holds, for the d -dimensional torus,

$$(-\Delta)^{-\beta} u = \sum_{m=1}^{\infty} \lambda_m^{-\beta} \left\{ (u, e_m)_{L^2(\mathbb{T}^d)} e_m + (u, f_m)_{L^2(\mathbb{T}^d)} f_m \right\},$$

which amounts to say that the eigenvalues of $(-\Delta)^{-\beta}$ are $\lambda_m^{-\beta}$, where $\{\lambda_m\}$ are the eigenvalues of the Laplacian operator $-\Delta$.

7.1 Numerical scheme

We propose the following numerical scheme for the discretization of the equation (7.1)

$$\frac{1}{\Delta t} (u_i^{n+1} - u_i^n) + \frac{1}{\Delta x} \left(F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n \right) = \frac{\alpha_Q}{\Delta t} (W_i^{n+1} - W_i^n), \quad (7.2)$$

for all $i \in \{1, 2, \dots, I\}$ and $n = \{0, 1, \dots, N-1\}$. If we choose $Q = (-\Delta)^{-\beta}$, the stochastic force terms are of the form

$$W_i^n = \sum_{m=1}^M \sqrt{\lambda_m^{-\beta}} \left\{ \beta_m(t^n) e_m(x_i) + \gamma_m(t^n) f_m(x_i) \right\}$$

with $\{e_m(x), f_m(x)\} = \{\sqrt{2} \sin(2m\pi x), \sqrt{2} \cos(2m\pi x)\}$ and $\lambda_m^{-\beta} = (4m^2\pi^2)^{-\beta}$ and M a truncation number. More precisely,

$$\begin{aligned} & \frac{\alpha_Q}{\Delta t} (W_i^{n+1} - W_i^n) \\ &= \frac{\alpha_Q}{\Delta t} \sum_{m=1}^M \sqrt{2} \sqrt{\lambda_m^{-\beta}} \left((\beta_m(t^{n+1}) - \beta_m(t^n)) \sin(2m\pi x_i) + (\gamma_m(t^{n+1}) - \gamma_m(t^n)) \cos(2m\pi x_i) \right) \\ &= \alpha_Q \sqrt{\frac{2}{\Delta t}} \sum_{m=1}^M \frac{1}{(2m\pi)^\beta} \left(\left(\frac{\beta_m(t^{n+1}) - \beta_m(t^n)}{\sqrt{\Delta t}} \right) \sin(2m\pi x_i) + \left(\frac{\gamma_m(t^{n+1}) - \gamma_m(t^n)}{\sqrt{\Delta t}} \right) \cos(2m\pi x_i) \right) \end{aligned} \quad (7.3)$$

where $\frac{\beta_m(t^{n+1}) - \beta_m(t^n)}{\sqrt{\Delta t}}$ and $\frac{\gamma_m(t^{n+1}) - \gamma_m(t^n)}{\sqrt{\Delta t}}$ follow the Gaussian law $\mathcal{N}(0, 1)$.

We recall that in the simulation of stochastic Burgers equation [1], the authors proposed the following scheme,

$$\frac{1}{\Delta t}(u_i^{n+1} - u_i^n) + \frac{1}{\Delta x} (F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) = \alpha \sqrt{\frac{1}{\Delta x \Delta t}} G_i^n \quad (7.4)$$

with

$$\begin{aligned} \alpha \sqrt{\frac{1}{\Delta x \Delta t}} G_i^n &= \alpha \sqrt{\frac{1}{\Delta x \Delta t}} \sqrt{\frac{2}{I}} \left(\sum_{m=1}^{\frac{I-1}{2}} \left(\frac{C_m^n}{m^\beta} \cos(2m\pi x_i) - \frac{S_m^n}{m^\beta} \sin(2m\pi x_i) \right) \right) \\ &= \alpha \sqrt{\frac{2}{\Delta t}} \left(\sum_{m=1}^{\frac{I-1}{2}} \left(\frac{C_m^n}{m^\beta} \cos(2m\pi x_i) - \frac{S_m^n}{m^\beta} \sin(2m\pi x_i) \right) \right) \end{aligned}$$

Comparing (7.4) with (7.3), we deduce that those two formulas are equivalent when $M = (I - 1)/2$ and $\alpha = \frac{\alpha_Q}{(2\pi)^\beta}$, where I is the number of volumes.

7.1.1 Burgers equation involving a Brownian motion

In order to compare the numerical results, we propose to perform simulations for a stochastic Burgers equation involving a Brownian motion:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(\frac{u^2}{2} \right) = \alpha \dot{\beta}(t), \quad (7.5)$$

where $\beta(t)$ is an (\mathcal{F}_t) -Brownian motion. We propose the following numerical scheme for the discretization of the equation (7.5)

$$\frac{1}{\Delta t}(u_i^{n+1} - u_i^n) + \frac{1}{\Delta x} (F_{i+\frac{1}{2}}^n - F_{i-\frac{1}{2}}^n) = \frac{\alpha}{\Delta t} (\beta^{n+1} - \beta^n),$$

where $\beta^{n+1} - \beta^n$ follows the Gaussian distribution $\mathcal{N}(0, t^{n+1} - t^n)$. We remark that for each time step n , we use the same random variable $\beta^{n+1} - \beta^n$ for all $i \in \{1, 2, \dots, I\}$.

7.2 Numerical results

In this section, we present some numerical results. The initial condition is given by $u_0(x) = \sin(2\pi x) + 0.5$. We subdivide the space interval $[0, 1]$ into 201 volumes and we perform 16 384 realizations for each fixed α_Q and β . The numerical results are presented in two periods in space.

First we present the results in the deterministic case, namely the case that the source term is equal to zero.

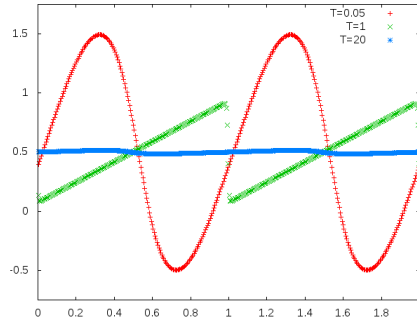


Figure 1: Solutions in the deterministic case

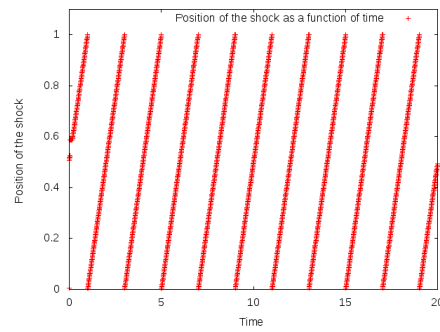


Figure 2: The positions of the shock

The figures show that after a certain time, a shock appears and moves with a constant speed.

7.3 Comparison between two different types of noise term

We consider the case that $\alpha_Q = 1$ and $\beta = 1$, so that $\alpha = 1/2\pi$ and we compare the numerical results of two different types of noise.

7.3.1 Empirical mean value and one single realization

For the Brownian motion case;

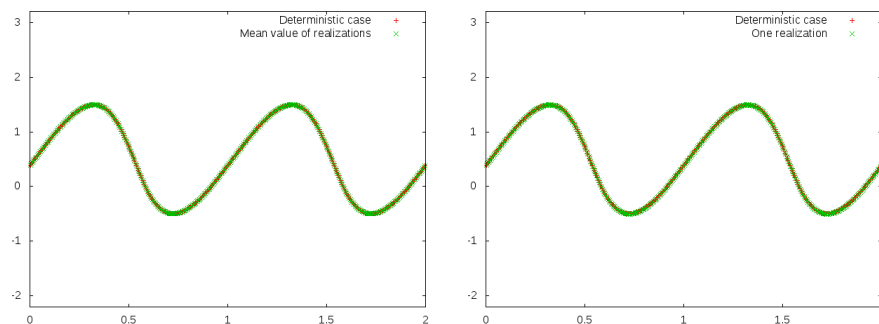


Figure 3: Empirical average (left) and one realization (right) comparing with the solution in the deterministic case at $t = 0.05$

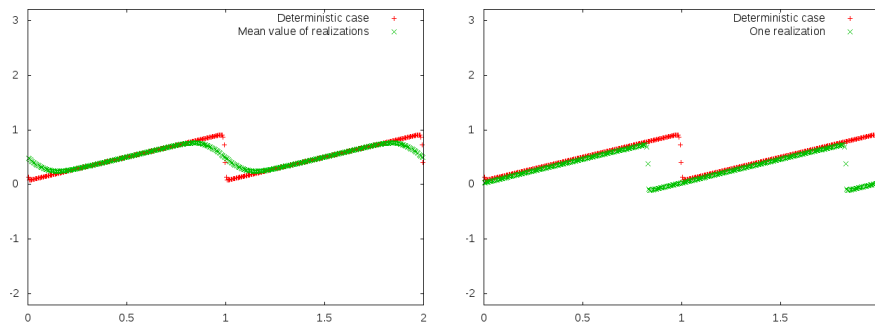


Figure 4: Empirical average (left) and one realization (right) comparing with the solution in the deterministic case at $t = 1$

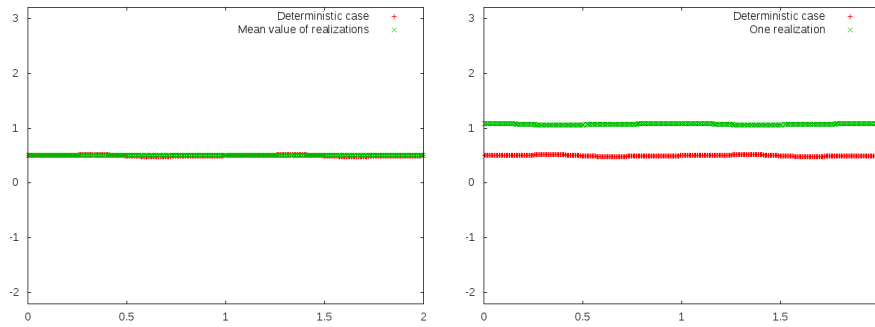


Figure 5: Empirical average (left) and one realization (right) comparing with the solution in the deterministic case at $t = 20$

For the Q -Brownian motion case;

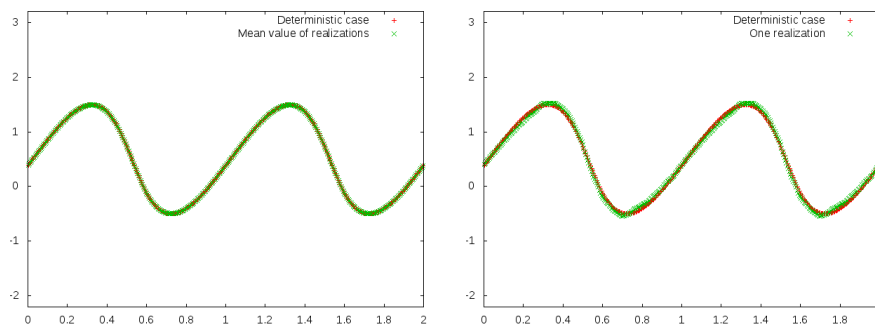


Figure 6: Empirical average (left) and one realization (right) comparing with the solution in the deterministic case at $t = 0.05$

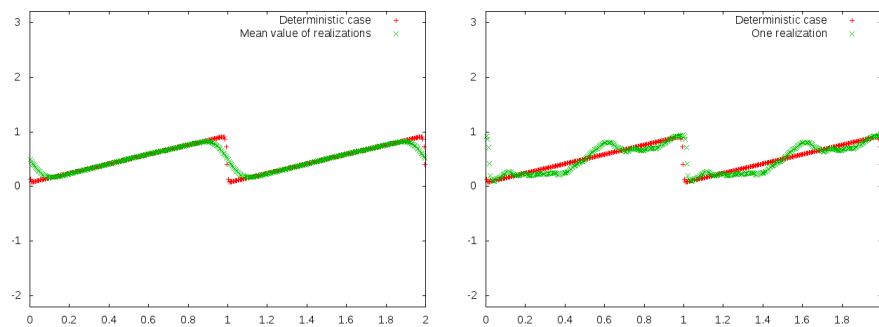


Figure 7: Empirical average (left) and one realization (right) comparing with the solution in the deterministic case at $t = 1$

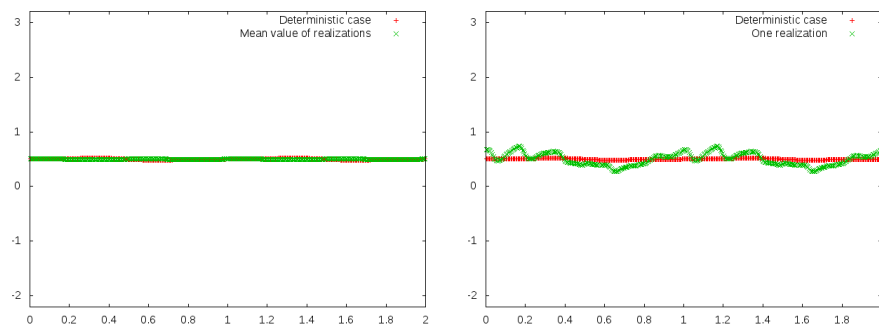


Figure 8: Empirical average (left) and one realization (right) comparing with the solution in the deterministic case at $t = 20$

The mean value of the realizations is close to the deterministic solution in both cases. For single realizations, at time $t = 1$ the Brownian motion shifts the position of the shock; and in the Q -Brownian motion case, the stochastic solution is perturbed around the deterministic solution.

7.3.2 Variance and covariance for two fixed points

We first present the variance $\text{Var}[u(x_i, t)]$ for all $i \in \{1, 2, \dots, I\}$ at different times $t = 0.05$, $t = 1$ and $t = 20$,

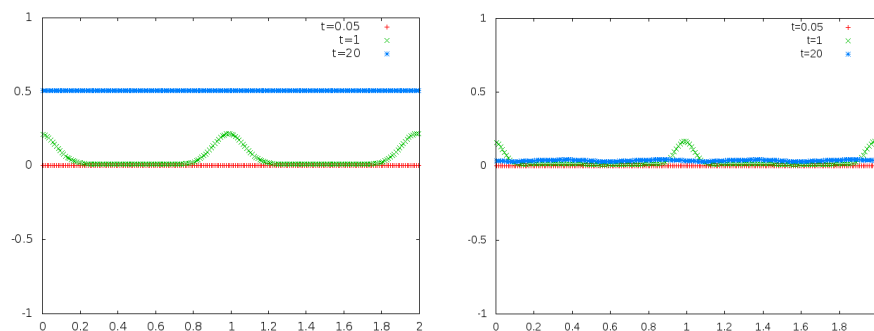


Figure 9: Variance in the case of Brownian motion (left) and Q -Brownian motion (right) for fixed time

We then present the discrete L^1 norm of the variance as a function of time, namely

$$\|\text{Var}[u(\cdot, t)]\|_{L^1([0,1])} = \sum_{i=1}^I |\text{Var}[u(x_i, t)]| dx.$$

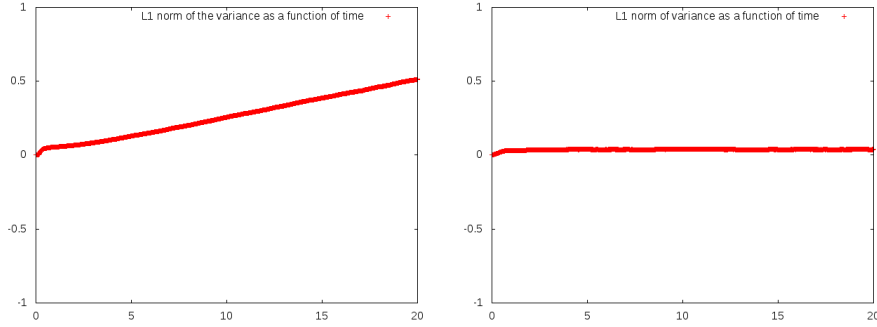


Figure 10: L^1 norm of the variance as a function of time in the case of Brownian motion (left) and Q -Brownian motion (right)

These results show that in the Brownian motion case, the L^1 norm is increasing as a function of time; meanwhile in the Q -Brownian motion case, the L^1 norm of the variance tends to a stable value as time increases. Since the L^1 norm of the variance is increasing in the Brownian motion case, we propose to compare the results to the case that the amplitude $\alpha = 1/\pi$, which is twice as the previous amplitude that we considered.

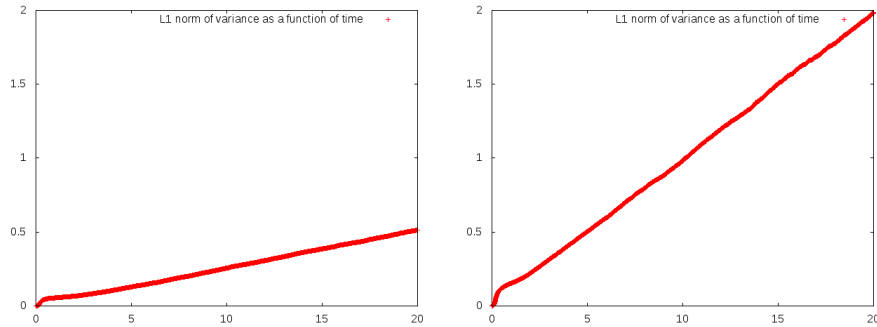


Figure 11: L^1 norm of the variance as a function of time in the case of the Brownian motion with $\alpha = 1/2\pi$ (left) and $\alpha = 1/\pi$ (right).

Remark 7.1 The phenomena that a richer randomness in noise implies less fluctuation in the solution can be observed in a different setting of SPDEs. Consider a stochastic heat equation $du = \Delta u dt + dW(x, t)$ on \mathbb{R}^d with an initial value $u(x, 0) = 0$ for simplicity and with an additive noise $W(x, t)$ which is a Q -Brownian motion. Then the solution u is given in a mild form $u(x, t) = \int_0^t \int_{\mathbb{R}^d} p(t-s, x, y) dW(y, s) dy$, and therefore its variance is easily computed as

$$\text{Var}(u(x, t)) = \int_0^t ds \int_{\mathbb{R}^d \times \mathbb{R}^d} p(s, x, y_1) p(s, x, y_2) \mathbf{Q}(y_1, y_2) dy_1 dy_2,$$

where p is the heat kernel. In particular, if W is the space-time white noise, $\mathbf{Q}(y_1, y_2) = \delta(y_1 - y_2)$, so that $\text{Var}(u(x, t)) = \int_0^t p(2s, x, x) ds = \sqrt{t/(2\pi)}$ when $d = 1$. While, if W is the Brownian motion $\beta(t)$ only

in time, $\mathbf{Q}(y_1, y_2) = 1$, so that $\text{Var}(u(x, t)) = t$ for every $d \geq 1$. Indeed, $u(x, t) = \beta(t)$ in this case. This simple example, though it is totally different from our equation, indicates that the fluctuation of the solution determined from the noise with richer randomness is asymptotically smaller as the time grows.

We fix $x_1 = 0.25$ and $x_2 = 0.75$ and then present the covariance $\text{Cov}[u(x_1, t), u(x_2, t)]$ as a function of time, where

$$\text{Cov}[u(x_1, t), u(x_2, t)] = \mathbb{E}[(u(x_1, t) - \mathbb{E}[u(x_1, t)])(u(x_2, t) - \mathbb{E}[u(x_2, t)])]$$

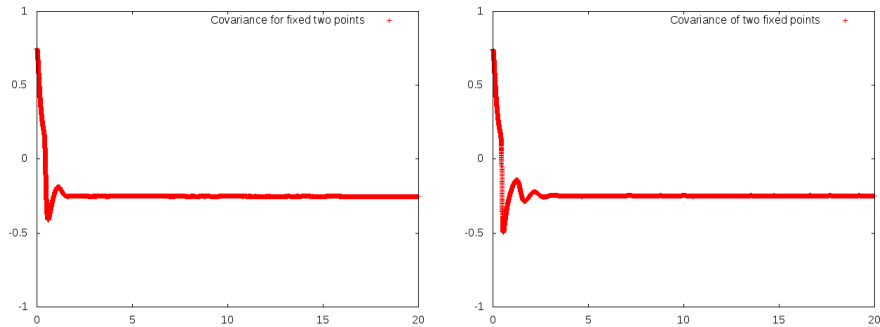


Figure 12: Covariance in the case of Brownian motion (left) and Q -Brownian motion (right) as a function of time

The covariance tends to a fixed value as time increases and the limit values are close in the two cases.

7.3.3 Comparison between different amplitudes in the Q -Brownian motion case

In order to study the influence of the amplitude of the noise in the Q -Brownian motion case, we propose to consider the case that $\alpha_Q = 2\pi$ and $\beta = 1$

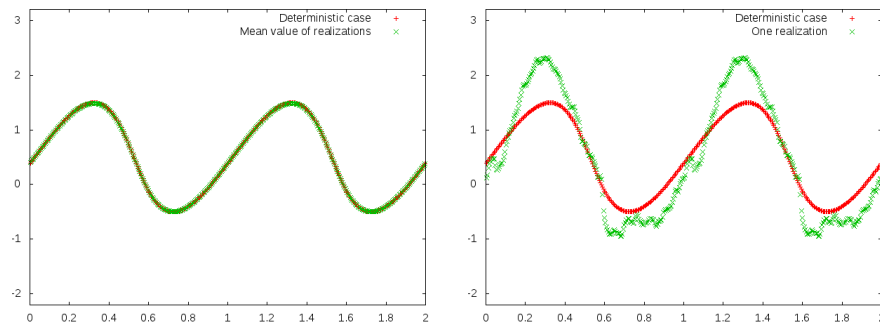


Figure 13: Empirical average (left) and one realization (right) comparing with the solution in the deterministic case at $t = 0.05$

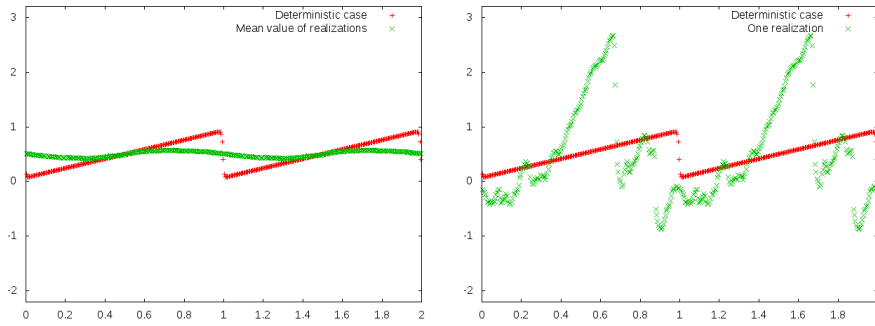


Figure 14: Empirical average (left) and one realization (right) comparing with the solution in the deterministic case at $t = 1$

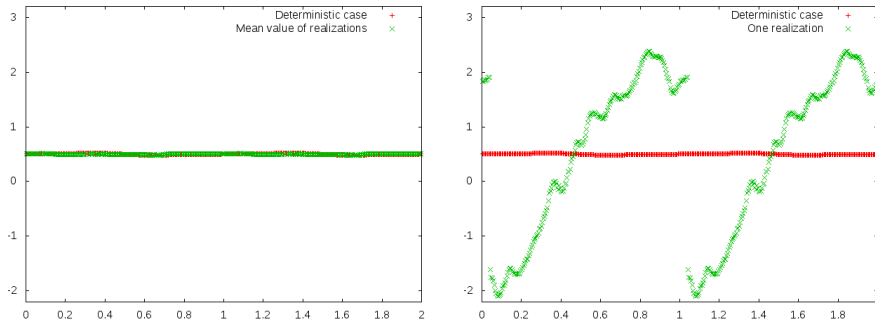


Figure 15: Empirical average (left) and one realization (right) comparing with the solution in the deterministic case at $t = 20$

We present the L^1 norm of the variance in this case:

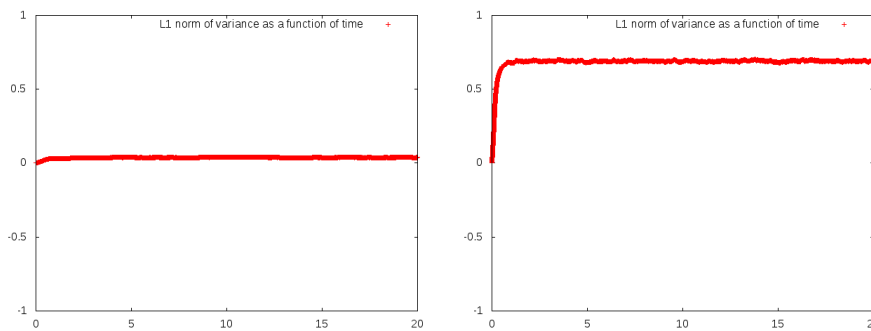


Figure 16: The L^1 norm of the variance in the cases that $\alpha_Q = 1$ (left) and $\alpha_Q = 2\pi$ (right) as a function of time

In these two cases, the L^1 norm of the variance tends to a constant as time increases, and as the amplitude of the noise is stronger, the limit value is larger, which is as it could be expected.

7.4 Some conclusions and discussions

In the Q -Brownian motion case, we have considered cases that $Q = -\Delta^{-1}$ with the series truncated by a fixed number $M = 100$ for the numerical simulations. We consider different noise amplitudes, namely $\alpha_Q = 1$, $\beta = 1$ and $\alpha_Q = 2\pi$, $\beta = 1$. The numerical results show that when α_Q is larger, one realization in stochastic case is more dispersed. The average of the realizations is a good approximation of the deterministic solution and as time tends to infinity it converges to the space-average of the initial function [12], which is constant $1/2$ in our case. While the deterministic solution is discontinuous at the shock, the average of the realizations of stochastic solutions has a smoothing effect. When the amplitude is larger, the smoothing effect is stronger, and the average goes faster to the space-average.

While comparing the results of the Brownian motion case and the Q -Brownian motion case, the averages of realizations approximate the solution in the deterministic case. In single realizations, the Brownian motion type noise shifts the shock position; meanwhile in the Q -Brownian motion case, one single realization is perturbed around the deterministic solution. If we consider the variance, the L^1 norm of the empirical variance increases as a function of time in the Brownian motion case while the L^1 norm of variance tends to a constant which depends on the amplitude of the noise term.

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