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► **To cite this version:**

Olivier Geneste, Luis Paris. Coxeter groups, symmetries, and rooted representations. *Communications in Algebra*, 2018, 46 (5), pp.1996-2002. 10.1080/00927872.2017.1363221 . hal-01404050

**HAL Id: hal-01404050**

**<https://hal.science/hal-01404050v1>**

Submitted on 28 Nov 2016

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# Coxeter groups, symmetries, and rooted representations

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## 1 Introduction

Let  $\Gamma$  be a Coxeter graph, and let  $(W, S)$  be the Coxeter system of  $\Gamma$ . A *symmetry* of  $\Gamma$  is a permutation  $g$  of  $S$  such that  $m_{g(s),g(t)} = m_{s,t}$  for all  $s, t \in S$ , where  $(m_{s,t})_{s,t \in S}$  is the Coxeter matrix of  $\Gamma$ . Let  $G$  be a group of symmetries of  $\Gamma$ . Then  $G$  is necessarily finite, and it can be viewed as a group of automorphisms of  $W$ . We denote by  $W^G$  the subgroup of  $W$  fixed under the action of  $G$ . Mühlherr [7] and Hée [4], independently of one another, proved that  $W^G$  is a Coxeter group. None of them gave explicitly the Coxeter graph  $\tilde{\Gamma}$  which defines  $W^G$ . However, a third proof, different from the other two, with an explicit description of  $\tilde{\Gamma}$ , is given in Crisp [2, 3].

Let  $\Pi = \{\epsilon_s \mid s \in S\}$  be a set in one-to-one correspondence with  $S$ , let  $V$  be the real vector space having  $\Pi$  as a basis, and let  $\langle \cdot, \cdot \rangle$  be the symmetric bilinear form on  $V$  defined by  $\langle \epsilon_s, \epsilon_t \rangle = -\cos(\pi/m_{s,t})$  if  $m_{s,t} \neq \infty$  and  $\langle \epsilon_s, \epsilon_t \rangle = -1$  if  $m_{s,t} = \infty$ . For every  $s \in S$  we define the linear transformation  $f_s : V \rightarrow V$  by  $f_s(x) = x - 2\langle x, \epsilon_s \rangle \epsilon_s$ . Then the map  $S \rightarrow \text{GL}(V)$ ,  $s \mapsto f_s$ , induces a celebrated faithful linear representation  $f : W \rightarrow \text{GL}(V)$ , called the *canonical representation* of  $W$  (see Bourbaki [1]). In our context, the triple  $(V, \langle \cdot, \cdot \rangle, \Pi)$  will be called the *canonical root basis* of  $\Gamma$ .

Let  $G$  be a group of symmetries of  $\Gamma$ . Then  $G$  acts on  $V$  sending  $\epsilon_s$  to  $\epsilon_{g(s)}$  for all  $s \in S$ , and this action leaves invariant the canonical form  $\langle \cdot, \cdot \rangle$ . Hence, the canonical representation  $f : W \rightarrow \text{GL}(V)$  is equivariant, in the sense that  $f(g(w)) = g \circ f(w) \circ g^{-1}$  for all  $g \in G$  and all  $w \in W$ , and therefore  $f$  induces a linear representation  $f^G : W^G \rightarrow \text{GL}(V^G)$ , where  $V^G = \{x \in V \mid g(x) = x \text{ for all } g \in G\}$ .

A naive question would be: Is  $f^G : W^G \rightarrow \text{GL}(V^G)$  the canonical representation of  $W^G$ ? A positive answer would provide a way to (re)prove that  $W^G$  is a Coxeter group and to determine the Coxeter graph of  $W^G$ . Unfortunately, simple calculations show that  $f^G$  is not the canonical representation in general. Nevertheless, one can transpose

this question to a larger family of linear representations, the rooted representations introduced by Krammer [5, 6], and, in this context, the answer is yes. Our purpose is to show that.

A *root basis* of  $\Gamma$  is a triple  $(V, \langle \cdot, \cdot \rangle, \Pi)$ , where  $V$  is a finite dimensional real vector space,  $\langle \cdot, \cdot \rangle$  is a symmetric bilinear form on  $V$ , and  $\Pi = \{\epsilon_s \mid s \in S\}$  is a collection of vectors in  $V$  in one-to-one correspondence with  $S$ , that satisfies the following properties:

- (a)  $\langle \epsilon_s, \epsilon_s \rangle = 1$  for all  $s \in S$ ;
- (b) for all  $s, t \in S$ ,  $s \neq t$ , we have
 
$$\begin{aligned} \langle \epsilon_s, \epsilon_t \rangle &= -\cos(\pi/m_{s,t}) && \text{if } m_{s,t} \neq \infty, \\ \langle \epsilon_s, \epsilon_t \rangle &\in (-\infty, -1] && \text{if } m_{s,t} = \infty; \end{aligned}$$
- (c) there exists  $\chi \in V^*$  such that  $\chi(\epsilon_s) > 0$  for all  $s \in S$ .

As mentioned above, if  $\Pi$  is a basis of  $V$  and  $\langle \epsilon_s, \epsilon_t \rangle = -1$  whenever  $m_{s,t} = \infty$ , then  $(V, \langle \cdot, \cdot \rangle, \Pi)$  is called the *canonical root basis* of  $\Gamma$ .

This definition is taken from Krammer's thesis [5, 6]. It is both, a generalization of the canonical spaces and canonical forms defined by Bourbaki [1], and a new point of view on the theory of reflection groups developed by Vinberg [8]. Note also that Condition (c) in the above definition often follows from Conditions (a) and (b), but not always (see Krammer [6, Proposition 6.1.2]).

Let  $(V, \langle \cdot, \cdot \rangle, \Pi)$  be a root basis of  $\Gamma$ . For every  $s \in S$  we define the linear transformation  $f_s : V \rightarrow V$  by  $f_s(x) = x - 2\langle x, \epsilon_s \rangle \epsilon_s$ . The following theorem can be proved for any root basis in the same way as it is proved in Bourbaki [1] for the canonical root basis.

**Theorem 1.1** (Krammer [5, 6]) *The map  $S \rightarrow \text{GL}(V)$ ,  $s \mapsto f_s$ , induces a faithful linear representation  $f : W \rightarrow \text{GL}(V)$ .*

The representation  $f : W \rightarrow \text{GL}(V)$  of Theorem 1.1 is called the *rooted representation* of  $W$  associated with  $(V, \langle \cdot, \cdot \rangle, \Pi)$ .

Let  $G$  be a group of symmetries of  $\Gamma$ , and let  $(V, \langle \cdot, \cdot \rangle, \Pi)$  be a root basis. As for the canonical root basis, we assume that  $G$  embeds in  $\text{GL}(V)$ , satisfies  $g(\epsilon_s) = \epsilon_{g(s)}$  for all  $g \in G$  and all  $s \in S$ , and leaves invariant the form  $\langle \cdot, \cdot \rangle$ . Then the representation  $f : W \rightarrow \text{GL}(V)$  is equivariant in the sense that  $f(g(w)) = g \circ f(w) \circ g^{-1}$  for all  $g \in G$  and all  $w \in W$ , and therefore  $f$  induces a linear representation  $f^G : W^G \rightarrow \text{GL}(V^G)$ , where  $V^G = \{x \in V \mid g(x) = x \text{ for all } g \in G\}$ . The goal of this paper is to prove the following.

**Theorem 1.2** (1) *The group  $W^G$  is a Coxeter group.*

(2) *Let  $\tilde{\Gamma}$  denote the Coxeter graph of  $W^G$ . There exists a subset  $\tilde{\Pi}$  of  $V^G$  such that  $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$  is a root basis of  $\tilde{\Gamma}$ , and the induced representation  $f^G : W^G \rightarrow \text{GL}(V^G)$  is the rooted representation associated with  $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$ . In particular,  $f^G$  is faithful.*

A similar approach is adopted in Hée [4, Section 3], with different definitions. One can easily show that the root system obtained from a root basis is a root system in Hée sense [4], and that part of the results of the paper, such as the fact that  $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$  is a root basis, can be deduced from Hée [4]. However, to get the explicit expression of the Coxeter graph  $\tilde{\Gamma}$  of  $W^G$ , one would need extra arguments that can be either a rewrite of Lemma 3.3, or some arguments similar to that given in Crisp [2, 3]. More generally, the whole theorem is more or less in the literature. In particular, as mentioned before, Part (1) is explicit in Mühlherr [7], Hée [4] and Crisp [2, 3]. But, our aim is to provide a new point of view on the question with unified, short and self-contained proofs.

A more precise statement of Theorem 1.2 is given in Section 2. In particular, the Coxeter graph  $\tilde{\Gamma}$  and the set  $\tilde{\Pi}$  are explicitly described. Section 3 is dedicated to the proofs.

## 2 Statement

The *length* of an element  $w \in W$ , denoted by  $\text{lg}(w)$ , is the shortest length of an expression of  $w$  over the elements of  $S$ . An expression  $w = s_1 \cdots s_\ell$  is called *reduced* if  $\ell = \text{lg}(w)$ . It is known that, if  $W$  is finite, then  $W$  has a unique *longest element*, that is, an element  $w_0 \in W$  such that  $\text{lg}(w) \leq \text{lg}(w_0)$  for all  $w \in W$ , and this element is an involution (see Bourbaki [1]).

For  $X \subset S$ , we denote by  $\Gamma_X$  the full subgraph of  $\Gamma$  spanned by  $X$ , and by  $W_X$  the subgroup of  $W$  generated by  $X$ . The subgroup  $W_X$  is called a *standard parabolic subgroup* of  $W$ . By Bourbaki [1],  $(W_X, X)$  is a Coxeter system of  $\Gamma_X$ . If  $W_X$  is finite, then we denote by  $w_X$  the longest element of  $W_X$ .

Let  $G$  be a group of symmetries of  $\Gamma$ . Now, we define a Coxeter matrix  $\tilde{M} = \tilde{M}^G = (\tilde{m}_{X,Y})_{X,Y \in \mathcal{S}}$  (and its associated Coxeter graph,  $\tilde{\Gamma}$ ). This will be the Coxeter matrix (and the Coxeter graph) of  $W^G$  (see Theorem 1.2 and Theorem 2.2).

We denote by  $\mathcal{O}$  the set of orbits of  $G$  in  $S$ , and we set  $\mathcal{S} = \{X \in \mathcal{O} \mid W_X \text{ is finite}\}$ . Then  $\mathcal{S}$  is the set of indices of  $\tilde{M}$  (which is the set of vertices of  $\tilde{\Gamma}$ ). Let  $X, Y \in \mathcal{S}$ .

- If  $m_{s,t} = 2$  for all  $s \in X$  and all  $t \in Y$ , then we set  $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = 2$ .
- Let  $k \in \{1, 2, 3, 4, 5\}$ . If  $\Gamma_{X \cup Y}$  is a disjoint union of copies of the Coxeter graph depicted in Figure 2.1 (k), where the vertices corresponding to  $x_1, x_2, \dots$  belong to  $X$  and the vertices corresponding to  $y_1, y_2, \dots$  belong to  $Y$ , then we set  $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = m$  if  $k = 1$ ,  $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = 4$  if  $k \in \{2, 3\}$ ,  $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = 8$  if  $k = 4$ , and  $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = 6$  if  $k = 5$ . In this case we say that  $(X, Y)$  is a *bi-orbit of type  $k$* .
- We set  $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = \infty$  in the remaining cases.

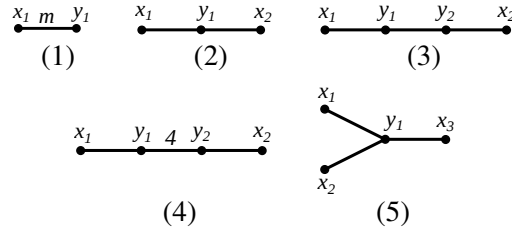


Figure 2.1: Bi-orbits.

The next lemma will be used in the definition of the set  $\tilde{\Pi}$ . It is well-known and can be easily proved using [1, Chapter V, Section 4, Subsection 8].

**Lemma 2.1** *Let  $(V, \langle \cdot, \cdot \rangle, \Pi)$  be a root basis of  $\Gamma$ . Suppose that  $W$  is finite and that  $\Pi$  spans  $V$ . Then  $\langle \cdot, \cdot \rangle$  is a scalar product, and  $(V, \langle \cdot, \cdot \rangle, \Pi)$  is the canonical root basis of  $\Gamma$ . In particular,  $\Pi$  is a basis of  $V$ .*

We turn back to the hypothesis of Theorem 1.2, that is,  $\Gamma$  is any Coxeter graph,  $G$  is a group of symmetries of  $\Gamma$ , and  $(V, \langle \cdot, \cdot \rangle, \Pi)$  is a root basis of  $\Gamma$ . We assume that  $G$  embeds in  $\text{GL}(V)$  so that the form  $\langle \cdot, \cdot \rangle$  is invariant under the action of  $G$ , and  $g(\epsilon_s) = \epsilon_{g(s)}$  for all  $s \in S$  and all  $g \in G$ .

Let  $X$  be an element of  $\mathcal{S}$ , that is, an orbit of  $G$  in  $S$  such that  $W_X$  is finite. Set  $\Pi_X = \{\epsilon_s \mid s \in X\}$ , and denote by  $V_X$  the linear subspace of  $V$  spanned by  $\Pi_X$ , and by  $\langle \cdot, \cdot \rangle_X$  the restriction of  $\langle \cdot, \cdot \rangle$  to  $V_X \times V_X$ . By Lemma 2.1,  $\Pi_X$  is a basis of  $V_X$  and  $\langle \cdot, \cdot \rangle_X$  is a scalar product. Let  $a_X = \sum_{s \in X} \epsilon_s$ . Note that  $a_X \in V^G$ , hence, by the above,  $a_X \neq 0$  and  $\|a_X\| > 0$ . We set  $\tilde{\epsilon}_X = \frac{a_X}{\|a_X\|}$  for all  $X \in \mathcal{S}$ , and  $\tilde{\Pi} = \tilde{\Pi}^G = \{\tilde{\epsilon}_X \mid X \in \mathcal{S}\}$ . The main result of the paper, with a precise statement, is the following.

**Theorem 2.2** (1) *The set  $\mathcal{S}_W = \{w_X \mid X \in \mathcal{S}\}$  generates  $W^G$ , and  $(W^G, \mathcal{S}_W)$  is a Coxeter system of  $\tilde{\Gamma}$ .*

- (2) The triple  $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$  is a root basis of  $\tilde{\Gamma}$ , and the induced representation  $f^G : W^G \rightarrow \text{GL}(V^G)$  is the rooted representation associated with  $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$ . In particular,  $f^G$  is faithful.

**Remark** The proof of Part (1) of Theorem 2.2 uses the induced representation  $f^G : W^G \rightarrow \text{GL}(V^G)$ . Nevertheless, the conclusion of Part (1) is always true because there is always a root basis which satisfies the hypothesis of the theorem: the canonical root basis.

### 3 Proof

We assume given a Coxeter graph  $\Gamma$ , a root basis  $(V, \langle \cdot, \cdot \rangle, \Pi)$  of  $\Gamma$ , and a group  $G$  of symmetries of  $\Gamma$ . We assume that  $G$  embeds in  $\text{GL}(V)$ , satisfies  $g(\epsilon_s) = \epsilon_{g(s)}$  for all  $g \in G$  and all  $s \in S$ , and leaves invariant the form  $\langle \cdot, \cdot \rangle$ .

Let  $f : W \rightarrow \text{GL}(V)$  be the rooted representation of  $W$  associated with  $(V, \langle \cdot, \cdot \rangle, \Pi)$ . From now on, in order to simplify the notations, we will assume that  $W$  acts on  $V$  via  $f$ , and we will write  $w(x)$  in place of  $f(w)(x)$  for  $w \in W$  and  $x \in V$ . Lemmas 3.1 to 3.4 are preliminaries to the proof of Theorem 2.2. Lemma 3.1 (1) is well-known. It is a direct consequence of Mühlherr [7, Lemma 2.8], and its proof can be found in the beginning of the proof of Mühlherr [7, Theorem 1.3]. Lemma 3.1 (2) is also known. Its proof is implicit in Crisp [2], but, as far as we know, it is not explicitly given anywhere else.

**Lemma 3.1** (1) The group  $W^G$  is generated by  $S_W$ .

- (2) We have  $(w_X w_Y)^{\tilde{m}_{X,Y}} = 1$  for all  $X, Y \in \mathcal{S}$  such that  $\tilde{m}_{X,Y} \neq \infty$ .

**Proof** As mentioned above, the proof of Part (1) can be found in Mühlherr [7]. So, we only need to prove Part (2). Let  $X \subset S$  be such that  $\Gamma_X$  is a disjoint union of vertices (*i.e.*  $\Gamma_X$  has no edge). Then  $W_X$  is finite and  $w_X = \prod_{s \in X} s$ . Let  $X = \{s, t\}$  be a pair included in  $S$  such that  $m_{s,t} = m < \infty$ . Then  $W_X$  is finite,  $w_X = (st)^{\frac{m}{2}}$  if  $m$  is even, and  $w_X = (st)^{\frac{m-1}{2}} s$  if  $m$  is odd. Now, let  $X, Y \in \mathcal{S}$ . If  $m_{s,t} = 2$  for all  $s \in X$  and  $t \in Y$ , then  $w_X$  and  $w_Y$  commute, hence  $(w_X w_Y)^2 = 1$ , as  $w_X$  and  $w_Y$  are both involutions. Suppose that  $(X, Y)$  is a bi-orbit of type  $j$ , where  $j \in \{1, 2, 3, 4, 5\}$ . Let  $\Gamma_1, \dots, \Gamma_\ell$  be the connected components of  $\Gamma_{X \cup Y}$ . For  $i \in \{1, \dots, \ell\}$ , we denote by  $Z_i$  the set of vertices of  $\Gamma_i$ , and we set  $X_i = X \cap Z_i$  and  $Y_i = Y \cap Z_i$ . We have  $w_X = \prod_{i=1}^{\ell} w_{X_i}$  and  $w_Y = \prod_{i=1}^{\ell} w_{Y_i}$ . Moreover, using the above observation together

with Theorem 1.1, it is easily checked that  $(w_{X_i} w_{Y_i})^{\tilde{m}_{X,Y}} = 1$  for all  $i$ . It follows that  $(w_X w_Y)^{\tilde{m}_{X,Y}} = \prod_{i=1}^{\ell} (w_{X_i} w_{Y_i})^{\tilde{m}_{X,Y}} = 1$ .  $\square$

**Lemma 3.2** *Let  $X \in \mathcal{S}$ . Then one of the following two alternatives holds.*

- (I)  $\Gamma_X$  is a disjoint union of vertices (i.e.  $\Gamma_X$  has no edge).
- (II) There exists  $m \in \mathbb{N}$ ,  $m \geq 3$ , such that  $\Gamma_X$  is a disjoint union of copies of the Coxeter graph depicted in Figure 2.1 (1).

**Proof** For  $s \in X$  we set  $v_s(X) = |\{t \in X \mid m_{s,t} \geq 3\}|$ . Since  $W_X$  is finite, the connected components of  $\Gamma_X$  are trees (see Bourbaki [1]), hence there exists  $s \in X$  such that  $v_s(X) \leq 1$ . On the other hand, since  $G$  acts transitively on  $X$ , we have  $v_s(X) = v_t(X)$  for all  $s, t \in X$ . So, either  $v_s(X) = 0$  for all  $s \in X$ , or  $v_s(X) = 1$  for all  $s \in X$ . If  $v_s(X) = 0$  for all  $s \in X$ , then we are in Alternative (I). If  $v_s(X) = 1$  for all  $s \in X$ , then we are in Alternative (II).  $\square$

Let  $X \in \mathcal{S}$ . We say that  $X$  is of *type I* if  $\Gamma_X$  satisfies Condition (I) of Lemma 3.2, and that  $X$  is of *type II<sub>m</sub>* if  $\Gamma_X$  satisfies Condition (II).

**Lemma 3.3** *Let  $X, Y \in \mathcal{S}$ ,  $X \neq Y$ . Then*

$$\begin{aligned} \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle &= -\cos(\pi/\tilde{m}_{X,Y}) && \text{if } \tilde{m}_{X,Y} \neq \infty, \\ \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle &\in (-\infty, -1] && \text{if } \tilde{m}_{X,Y} = \infty. \end{aligned}$$

**Proof** Observe that, if  $m_{s,t} = 2$  for all  $s \in X$  and all  $t \in Y$ , then  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = 0$  and  $\tilde{m}_{X,Y} = 2$ . Hence, we can assume that there exist  $s \in X$  and  $t \in Y$  such that  $m_{s,t} \geq 3$ . Since  $G$  acts transitively on  $X$  and leaves invariant  $Y$ , it follows that, for all  $s \in X$ , there exists  $t \in Y$  such that  $m_{s,t} \geq 3$ . Similarly, for all  $t \in Y$ , there exists  $s \in X$  such that  $m_{s,t} \geq 3$ .

Recall that  $a_X = \sum_{s \in X} \epsilon_s$ ,  $a_Y = \sum_{t \in Y} \epsilon_t$ ,  $\tilde{\epsilon}_X = \frac{a_X}{\|a_X\|}$ ,  $\tilde{\epsilon}_Y = \frac{a_Y}{\|a_Y\|}$ . Choose  $s \in X$  and set  $v_X = |\{t \in Y \mid m_{s,t} \geq 3\}|$  and  $p_X = \sum_{t \in Y} \langle \epsilon_s, \epsilon_t \rangle = \langle \epsilon_s, a_Y \rangle$ . Since  $G$  acts transitively on  $X$  and leaves invariant  $Y$ , these definitions do not depend on the choice of  $s$ . Similarly, choose  $t \in Y$  and set  $v_Y = |\{s \in X \mid m_{s,t} \geq 3\}|$  and  $p_Y = \sum_{s \in X} \langle \epsilon_t, \epsilon_s \rangle = \langle \epsilon_t, a_X \rangle$ . The hypothesis that there exist  $s \in X$  and  $t \in Y$  such that  $m_{s,t} \geq 3$  implies that  $v_X \geq 1$  and  $v_Y \geq 1$ .

Let  $s \in X$  and  $t \in Y$ . If  $m_{s,t} \geq 3$ , then  $\langle \epsilon_s, \epsilon_t \rangle \leq -\frac{1}{2}$ , and if  $m_{s,t} = 2$ , then  $\langle \epsilon_s, \epsilon_t \rangle = 0$ . It follows that

$$(3-1) \quad p_X \leq -\frac{v_X}{2}.$$

On the other hand, we have

$$(3-2) \quad |X| v_X = |Y| v_Y.$$

This is the number of edges in  $\Gamma$  connecting an element of  $X$  with an element of  $Y$ . A direct calculation shows that

$$(3-3) \quad \|a_X\| = \begin{cases} \sqrt{|X|} & \text{if } X \text{ is of type } I, \\ \sqrt{|X|(1 - \cos(\pi/m))} & \text{if } X \text{ is of type } II_m. \end{cases}$$

Finally, by definition of  $p_X$ ,

$$(3-4) \quad \langle a_X, a_Y \rangle = |X| p_X.$$

*Case 1:  $X$  and  $Y$  are of type  $I$ .* Applying Equations (3-2), (3-3), and (3-4) we get

$$(3-5) \quad \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = \frac{p_X \sqrt{v_Y}}{\sqrt{v_X}}.$$

Applying Equation (3-1) to this equality we get  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -\frac{\sqrt{v_X v_Y}}{2}$ . It follows that, if either  $v_X \geq 4$ , or  $v_Y \geq 4$ , or  $v_X, v_Y \geq 2$ , then  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -1$ . If  $v_X = 1$ ,  $v_Y \geq 2$  and  $p_X \leq -\cos(\pi/4) = -\frac{1}{\sqrt{2}}$ , then, by Equation (3-5),  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -1$ . If  $v_X = 1$ ,  $v_Y = 3$  and  $p_X = -\cos(\pi/3) = -\frac{1}{2}$ , then, by Equation (3-5),  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = -\frac{\sqrt{3}}{2} = -\cos(\pi/6)$ . In this case  $(Y, X)$  is a bi-orbit of type 5 and  $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = 6$ . If  $v_X = 1$ ,  $v_Y = 2$  and  $p_X = -\cos(\pi/3) = -\frac{1}{2}$ , then, by Equation (3-5),  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = -\frac{\sqrt{2}}{2} = -\cos(\pi/4)$ . In this case  $(Y, X)$  is a bi-orbit of type 2 and  $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = 4$ . If  $v_X = 1$ ,  $v_Y = 1$  and  $p_X = -\cos(\pi/m)$  with  $m \neq \infty$ , then, by Equation (3-5),  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = -\cos(\pi/m)$ . In this case  $(Y, X)$  is a bi-orbit of type 1 and  $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = m$ . Finally, if  $v_X = v_Y = 1$  and  $p_X \leq -1$ , then, by Equation (3-5),  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = p_X \leq -1$ . In this case  $(Y, X)$  is a bi-orbit of type 1 and  $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = \infty$ .

*Case 2:  $X$  is of type  $II_m$  and  $Y$  is of type  $I$ .* Applying Equations (3-2), (3-3), and (3-4) we get

$$(3-6) \quad \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = \frac{p_X \sqrt{v_Y}}{\sqrt{v_X(1 - \cos(\pi/m))}}.$$

Applying Equation (3-1) to this equality, we get

$$(3-7) \quad \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -\frac{\sqrt{v_X v_Y}}{2\sqrt{(1 - \cos(\pi/m))}}.$$

If  $m \geq 5$ , then  $\sqrt{1 - \cos(\pi/m)} < \frac{1}{2}$ , hence, by Equation (3-7),  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -\sqrt{v_X v_Y} \leq -1$ . So, we can assume that  $m \in \{3, 4\}$ . Then we have  $\sqrt{1 - \cos(\pi/m)} \leq \frac{1}{\sqrt{2}}$  and, by Equation (3-7),  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -\frac{\sqrt{v_X v_Y}}{\sqrt{2}}$ . It follows that, if either  $v_X \geq 2$ , or  $v_Y \geq 2$ , then



$\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -1$ . If  $v_X = 1$ ,  $v_Y = 1$  and  $p_X \leq -\cos(\pi/4) = -\frac{1}{\sqrt{2}}$ , then, by Equation (3-6),  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -1$ . If  $v_X = 1$ ,  $v_Y = 1$ ,  $p_X = -\cos(\pi/3) = -\frac{1}{2}$  and  $m = 4$ , then, by Equation (3-6),  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = -\frac{\sqrt{2+\sqrt{2}}}{2} = -\cos(\pi/8)$ . In this case  $(Y, X)$  is a bi-orbit of type 4 and  $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = 8$ . If  $v_X = 1$ ,  $v_Y = 1$ ,  $p_X = -\cos(\pi/3) = -\frac{1}{2}$  and  $m = 3$ , then, by Equation (3-6),  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = -\frac{1}{\sqrt{2}} = -\cos(\pi/4)$ . In this case  $(Y, X)$  is a bi-orbit of type 3 and  $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = 4$ .

*Case 3:  $X$  is of type  $II_m$  and  $Y$  is of type  $II_{m'}$ .* Applying Equations (3-2), (3-3) and (3-4) we get

$$\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = \frac{p_X \sqrt{v_Y}}{\sqrt{v_X(1 - \cos(\pi/m))(1 - \cos(\pi/m'))}}.$$

Applying Equation (3-1) to this equality we get

$$\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -\frac{\sqrt{v_X v_Y}}{2\sqrt{(1 - \cos(\pi/m))(1 - \cos(\pi/m'))}}.$$

Since  $\sqrt{(1 - \cos(\pi/m))} \leq \frac{1}{\sqrt{2}}$  and  $\sqrt{(1 - \cos(\pi/m'))} \leq \frac{1}{\sqrt{2}}$ , it follows that  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -\sqrt{v_X v_Y} \leq -1$ .  $\square$

**Lemma 3.4** *Let  $X \in \mathcal{S}$ , and let  $x \in V^G$ . Then  $w_X(x) = x - 2\langle x, \tilde{\epsilon}_X \rangle \tilde{\epsilon}_X$ .*

**Proof** Let  $\Gamma'$  be a Coxeter graph, and let  $(W', S')$  be its associated Coxeter system, such that  $W'$  is finite. Let  $w'_0$  be the longest element of  $W'$ , and let  $(V', \langle \cdot, \cdot \rangle', \Pi')$  be the canonical root basis of  $\Gamma'$ . Then, by Bourbaki [1],  $w'_0(\Pi') = -\Pi'$ .

Let  $X \in \mathcal{S}$ . Recall that  $\Pi_X = \{\epsilon_s \mid s \in X\}$ . By Lemma 2.1 and the above, we have  $w_X(\Pi_X) = -\Pi_X$ , hence  $w_X(a_X) = -a_X$ , therefore  $w_X(\tilde{\epsilon}_X) = -\tilde{\epsilon}_X$ .

Recall that  $V_X$  denotes the linear subspace of  $V$  spanned by  $\Pi_X$ . For all  $x \in V$  and all  $u \in W_X$  there exists  $y \in V_X$  such that  $u(x) = x + y$ . This is true by definition for all  $s \in X$ , hence it is true for all  $u \in W_X$ . Let  $x \in V^G$ . Let  $y \in V_X$  be such that  $w_X(x) = x + y$ . Let  $y = \sum_{s \in X} \lambda_s \epsilon_s$  be the expression of  $y$  in the basis  $\Pi_X$ . For  $g \in G$  we have

$$\begin{aligned} x + \sum_{s \in X} \lambda_s \epsilon_s &= w_X(x) = g(w_X)(g(x)) = g(w_X(x)) \\ &= g(x) + \sum_{s \in X} \lambda_s g(\epsilon_s) = x + \sum_{s \in X} \lambda_s \epsilon_{g(s)}, \end{aligned}$$

hence  $\lambda_s = \lambda_{g^{-1}(s)}$  for all  $s \in X$ . Since  $G$  acts transitively on  $X$ , it follows that  $\lambda_s = \lambda_t$  for all  $s, t \in X$ . So, there exists  $\lambda \in \mathbb{R}$  such that  $w_X(x) = x + \lambda a_X = x + \lambda \|a_X\| \tilde{\epsilon}_X$ .

We have

$$\langle x, \tilde{\epsilon}_X \rangle = \langle w_X(x), w_X(\tilde{\epsilon}_X) \rangle = \langle x + \lambda \|a_X\| \tilde{\epsilon}_X, -\tilde{\epsilon}_X \rangle = -\langle x, \tilde{\epsilon}_X \rangle - \lambda \|a_X\|,$$

hence  $\lambda \|a_X\| = -2\langle x, \tilde{\epsilon}_X \rangle$ . So,  $w_X(x) = x - 2\langle x, \tilde{\epsilon}_X \rangle \tilde{\epsilon}_X$ .  $\square$

**Proof of Theorem 2.2** We have  $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_X \rangle = 1$  for all  $X \in \mathcal{S}$  by definition. We have

$$\begin{aligned} \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle &= -\cos(\pi/\tilde{m}_{X,Y}) && \text{if } \tilde{m}_{X,Y} \neq \infty, \\ \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle &\in (-\infty, -1] && \text{if } \tilde{m}_{X,Y} = \infty, \end{aligned}$$

by Lemma 3.3. Let  $\chi \in V^*$  be such that  $\chi(\epsilon_s) > 0$  for all  $s \in \mathcal{S}$ . Let  $\tilde{\chi} : V^G \rightarrow \mathbb{R}$  be the restriction of  $\chi$  to  $V^G$ . Then, for  $X \in \mathcal{S}$ ,  $\tilde{\chi}(\tilde{\epsilon}_X) = \frac{1}{\|a_X\|} \sum_{s \in \mathcal{S}} \chi(\epsilon_s) > 0$ . So,  $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi}^G)$  is a root basis of  $\tilde{\Gamma}$ .

Let  $(\tilde{W}, \tilde{\mathcal{S}})$  be a Coxeter system of  $\tilde{\Gamma}$ , where  $\tilde{\mathcal{S}} = \{\tilde{s}_X \mid X \in \mathcal{S}\}$  is a set in one-to-one correspondence with  $\mathcal{S}$ . By Lemma 3.1, the map  $\tilde{\mathcal{S}} \rightarrow \mathcal{S}_W$ ,  $\tilde{s}_X \mapsto w_X$ , induces a surjective homomorphism  $\gamma : \tilde{W} \rightarrow W^G$ . By Lemma 3.4, the composition  $f^G \circ \gamma : \tilde{W} \rightarrow \text{GL}(V^G)$  is the rooted representation associated with  $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$ . By Theorem 1.1, it follows that  $f^G \circ \gamma$  is injective, hence  $\gamma$  is an isomorphism. So,  $(W, \mathcal{S}_W)$  is a Coxeter system of  $\tilde{\Gamma}$ , and  $f^G : W^G \rightarrow \text{GL}(V^G)$  is the rooted representation associated with  $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$ .  $\square$

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