About a problem of definition of the geodesic erosion

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1. Introduction, context

This note can be seen as a complement of the course on geodesic transforms [2]. It aims at clarifying an issue in the definition of the numerical geodesic erosion and how this definition is inferred from the definition of the geodesic dilation. We will see that the definition of the numerical geodesic erosion (on functions) is not an extension of the set geodesic erosion. We explain the reason for this difference. This induces some difficulties when it comes to define operators built from the elementary operators as the numerical geodesic opening and closing. In fact, with the classical definition of the numerical geodesic erosion, this is simply not possible!

So we will explain the logical progression used to obtain a correct definition of the numerical geodesic operators from their definition on sets. This will allow both to understand where the problems lie but also to define a numerical geodesic erosion which is a true extension of the geodesic erosion given for sets.

Most definitions and proofs will be provided in a digital frame. These definitions could be introduced in a continuous context, however this would complicate them unnecessarily.

2. Reminder: geodesic erosion and dilation for sets

Let us recall the definition of the elementary geodesic dilation $\delta_X(Y)$ of a set Y included in the geodesic space X. We have:

$$\delta_X(Y) = (Y \oplus B) \cap X \tag{1}$$

where B is the elementary euclidean ball (size-1 hexagon or square according to the digital grid in use).

This operator is obviously extensive and increasing.

From the geodesic dilation, we can define the elementary geodesic erosion (of size 1) $\varepsilon_X(Y)$ of the set Y included in the geodesic space X by using the duality between these two operators with respect to the set complementation in X. We get the following definition:

$$\varepsilon_X(Y) = X \setminus \delta_X(X \setminus Y) = X \cap [((X \setminus Y) \oplus B) \cap X]^c = ((Y \cup X^c) \ominus B) \cap X \tag{2}$$

The larger operations are obtained by iteration:

$$\varepsilon_X^n(Y) = \underbrace{\varepsilon_X . \varepsilon_X ... \varepsilon_X(Y)}_{n} \tag{3}$$

Note that in the case of geodesic operators, dualities by complementation or by adjunction [3] lead to identical definitions (because, in a geodesic context, the only structuring elements in use are geodesic balls).

The set geodesic erosion is increasing (the geodesic space X being fixed) and anti-extensive.

Let us recall also the geometric interpretation of these two operators. The geodesic dilation of size n of Y corresponds to the space swept by the geodesic balls of size n whose center is included in Y. It is also the locus of the centers of the geodesic balls of size n in X that intersect Y. The geodesic eroded set corresponds to the locus of the centers of the geodesic balls of size n of X included in Y:

$$\varepsilon_X^n(Y) = \{x \in X : B_X(x,n) \subset Y\} \tag{4}$$

$$\delta_X^n(Y) = \{x \in X : B_X(x,n) \cap Y \neq \varnothing\} = \bigcup_{y \in Y} B_X(y,n) \tag{5}$$

3. Geodesic opening and closing for sets

In the set framework, it is easy to define the geodesic opening and closing of a set Y included in a geodesic space X. We have:

$$\gamma_X^n(Y) = \delta_X^n.\varepsilon_X^n(Y) \tag{6}$$

$$\varphi_X^n(Y) = \varepsilon_X^n.\delta_X^n(Y) \tag{7}$$

These two operations are increasing. The geodesic opening is anti-extensive (any point of $\gamma_X^n(Y)$ is included in a geodesic ball of size n itself included in Y). The open set $\gamma_X^n(Y)$ corresponds to the set of geodesic balls of size n included in Y. The geodesic closing corresponds to the complement in X of the geodesic opening of the complement of Y in X. This is an extensive operator.

This operator is also idempotent. Indeed:

$$\delta_X^n.\varepsilon_X^n.\delta_X^n.\varepsilon_X^n(Y) = \delta_X^n.\varphi_X^n.\varepsilon_X^n(Y) = \delta_X^n.[\varphi_X^n.\varepsilon_X^n(Y)] \tag{8}$$

The expression between brackets is the geodesic closing of the geodesic erosion of Y, $\varepsilon_X^n(Y)$. This closing, extensive, contains the geodesic eroded set. Therefore we have:

$\delta_X^n.\varepsilon_X^n.\delta_X^n.\varepsilon_X^n(Y) \supset \delta_X^n.\varepsilon_X^n(Y)$ (the geodesic dilation is increasing)

As $\delta_X^n.\varepsilon_X^n.\delta_X^n.\varepsilon_X^n(Y) \subset \delta_X^n.\varepsilon_X^n(Y)$ (increase and anti-extensivity of the geodesic opening), the property is proven. It is of course also true for the geodesic closing. Therefore, these operators can legitimately be called opening and closing.

4. Application example

The geodesic opening and closing are not much used. The use of the opening by reconstruction or of the closing by reconstrution is much more common.

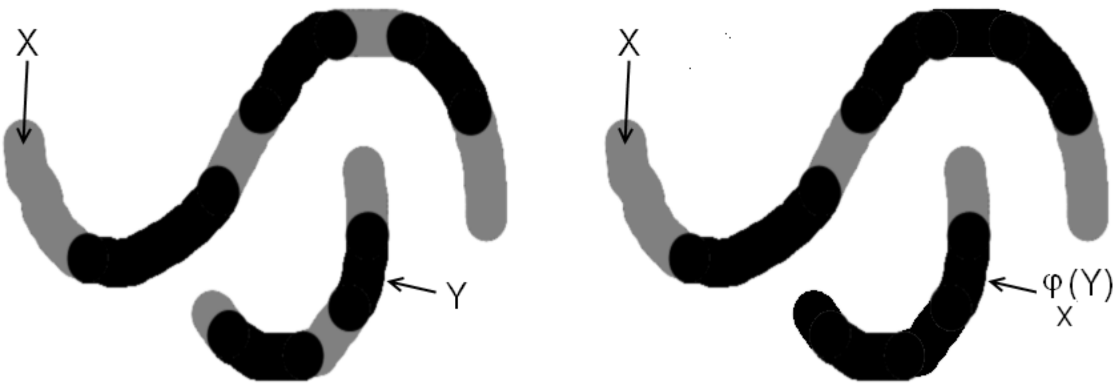

Figure 1 : Example of geodesic closing of a set Y in a geodesic space X. On the left, the initial sets, on the right, geodesic closing of set Y.

However, we can give a simple example of use of the geodesic closing (Figure 1). This example shows a set Y made of several connected components included in a set X. The geodesic closing allows to merge the close enough connected components of Y without deep modifications of the other components.

5. The geodesic dilation for functions

Extending the geodesic dilation for sets to the numerical domain is easy. Let f and g be two greyscale images (functions). Assume that $f \le g$. In order to define the elementary geodesic dilation $\delta_g(f)$ of f in the geodesic space defined by g, we just apply the classical procedure which consists in performing a geodesic dilation on all the sections (thresholds) of the subgraphs of f and g followed by a piling of the results to obtain the subgraph of $\delta_g(f)$. We know that this approach works only if the set operator is increasing, otherwise the re-stacking does not generate the subgraph of a function (Figure 2):

$X_i(f) = \{x : f(x) \ge i\}$
$i < j \quad X_j(f) \subset X_i(f)$
ψ increasing:
$\psi(X_j) \subset \psi(X_i)$
$X_j(\psi) \subset X_i(\psi)$

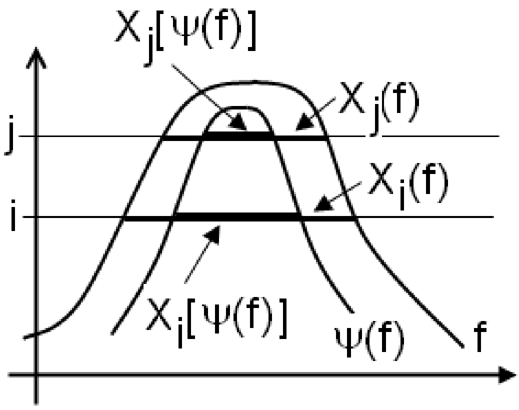

Figure 2 : Principle of the extension to the numerical domain of set increasing operators

In the case of the numerical geodesic dilation, this stacking generates a subgraph because the set geodesic dilation is increasing not only with respect to Y, the set to be dilated, but also with respect to X, the geodesic space. Indeed, by setting:

$$X_i(f) = \{x : f(x) \ge i\},\ X_j(f) = \{x : f(x) \ge j\},\ X_i(g) = \{x : g(x) \ge i\},\ X_j(g) = \{x : g(x) \ge j\} \quad (9)$$

With i < j, we have:

$$X_i(g) \supset X_j(g) \supset X_j(f) \quad (10)$$

and:

$$X_i(g) \supset X_i(f) \supset X_j(f) \quad (11)$$

Therefore this leads to:

$$X_i(\delta_g(f)) = (X_i(f) \oplus B) \cap X_i(g) \supset (X_i(f) \oplus B) \cap X_j(g) \supset (Xj(f) \oplus B) \cap X_j(g) = X_j(\delta_g(f)) \quad (12)$$

We can then easily give a specific algebraic formulation of the geodesic dilation of a function f in the geodesic space defined by g (we say that the geodesic dilation of f is carried out <u>under</u> g). We have:

$$:\delta_g(f) = (f \oplus B) \wedge g = \inf(f \oplus B, g) \quad (13)$$

(the transition from set operators to their numerical equivalents is straightforward).
The extension of the geodesic dilation to the numerical domain is immediate thanks to the good increasing property of the set operator with respect to the geodesic space.
We can give a geometric interpretation of this operator on the subgraphs of f and of g by introducing the concept of geodesic cylinder. A geodesic cylinder $C_g(x, n)$ defined in the subgraph G(g) of a function g, with origin x, point of the subgraph, and with a radius n, is made of points of G(g) at a geodesic distance less than or equal to n from every point y belonging to the vertical semi-axis $]-\infty, x]$ (Figure 3). The subgraph of the geodesic dilated function $\delta_g(f)$ is therefore made of all the geodesic cylinders of G(g) whose origin is in G(f).
The numerical geodesic dilation is obviously increasing and extensive.

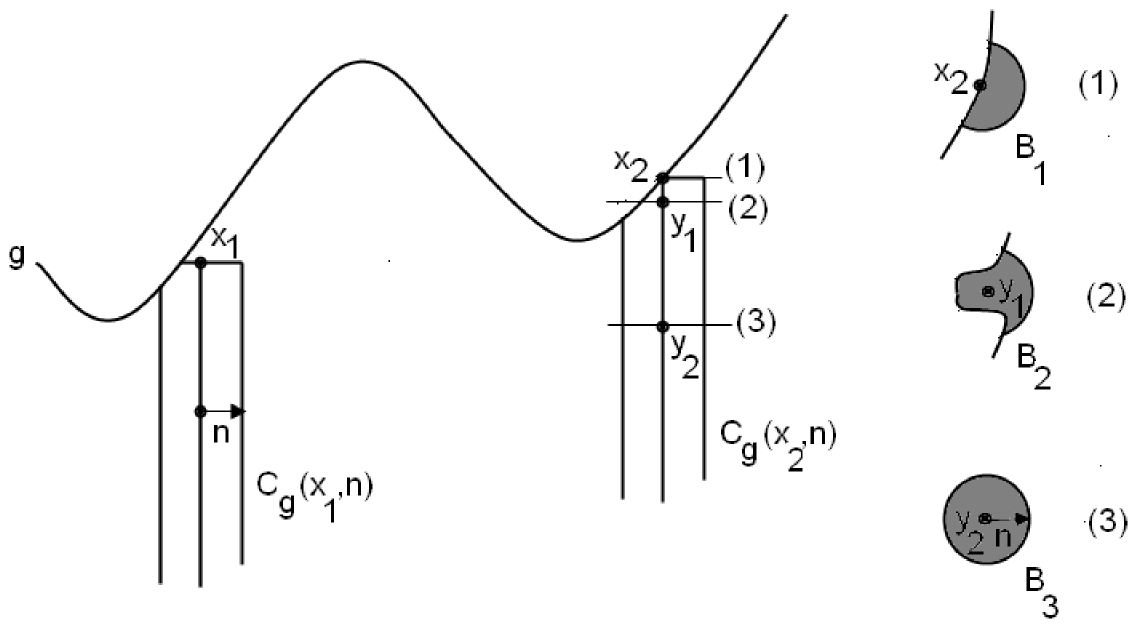

Figure 3 : Geodesic cylinders defined in the subgraph of a function g. On the right, sections of a bidimensional geodesic cylinder. Each section corresponds to a geodesic ball. We have obviously $B_1 \subset B_2 \subset B_3$.

6. The numerical geodesic erosion, analysis of a mix-up

The numerical geodesic erosion is generally defined from the numerical geodesic dilation by introducing a duality by inversion around a revolving value [2]. We define the elementary numerical geodesic erosion (size 1) of the function f above the function g (we indeed have $f \geq g$) as the operation:

$$\varepsilon_g(f) = p - \delta_{p-g}(p - f) = \sup(f \ominus B, g) \qquad (14)$$

where p is any revolving value (the result is independent of this value).

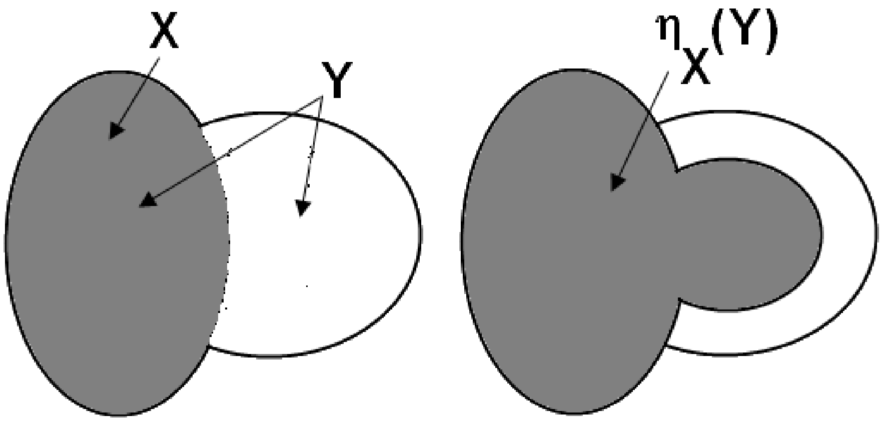

Figure 4 : Set equivalent of the geodesic erosion for functions. In this case, we have $Y \supset X$.

This transformation, however, has nothing to do with a possible extension of the set geodesic erosion to greyscale images! Indeed, in this definition, we have surreptitiously changed the geodesic space. We switched from the space G(g), subgraph of g, to its ongraph. It is as if, in the set case, we had defined the geodesic erosion in the geodesic space X^c (Figure 4). Y would then contain X and we would define the elemntary geodesic erosion $\eta_X(Y)$ as:

$$\eta_X(Y) = (Y \ominus B) \cup X \tag{15}$$

Note also that the sections of the elementary numerical geodesic erosion are defined with this formula.

This commonly used definition of the numerical geodesic erosion (formula (14)) is indeed interesting. In particular, it allows to introduce an operator such as the dual geodesic reconstruction which is widely used in Mathematical Morphology.

However, as erosion and dilation do not apply to the same geodesic space, defining a geodesic opening or closing by concatenating them makes no sense. Indeed, consider the transformation:

$$\varphi_g(f) = \varepsilon_g.\delta_g(f) \tag{16}$$

We see immediately that this operator is equal to (assuming that, when $f \leq g$, the geodesic erosion $\varepsilon_g(f)$ is, by convention, equal to the identity):

$$\varphi_g(f) = \varepsilon_g.\delta_g(f) = \delta_g(f) \tag{17}$$

This operator is the geodesic dilation (the geodesic erosion has no effect), so it does not verify the idempotence property and it is therefore difficult to argue that it is a closing. A solution consisting in using the result of the dilation as a geodesic space for the erosion of the function g (to avoid the above mentioned problem by ensuring that the function to be eroded is above) does not bring any improvement. Indeed, if we write:

$$\varphi'_g(f) = \varepsilon_{\delta_g(f)}(g) \tag{18}$$

The operator thus defined is always greater than or equal to $\delta_g(f)$.

Thus we see that, if we transpose in the numerical case the previous example given for sets (see Figure 1), by replacing the sets X and Y by two functions f and g defined by (valued indicator functions):

$$\begin{aligned}
&f(x) = m \text{ if } x \in Y\\
&f(x) = 0 \text{ otherwise}\\
&g(x) = M \text{ if } x \in Y\ (M \geqslant m)\\
&g(x) = 0 \text{ otherwise}
\end{aligned}$$

the operator $\varphi_g(f)$ does not produce the same result as in the set case since it stops after the dilation.

Can we avoid this problem and define a geodesic erosion in the same numerical geodesic space as the dilation? The answer is yes, just by using the duality by adjunction [3] and the geodesic cylinders.

Let us consider the subgraphs G(f) et G(g) of the functions f and g (with $f \leq g$). We can define the geodesic erosion of G(f) by geodesic cylinders of size n, $C_g(x, n)$, as the union of all sets Z included in G(g) such that their geodesic dilation is included in G(f):

$$\varepsilon_g[G(f)] = \cup\left\{Z : \delta^n_{G(g)}(Z) \subset G(f)\right\} \tag{19}$$

This formulation is equivalent to:

$$\varepsilon_g[G(f)] = \cup\{x \in G(g) : C_g(x, n) \subset G(f)\} \tag{20}$$

The geodesic erosion of f <u>under</u> g, $\underline{\varepsilon}^n_g(f)$ has the previous set as its subgraph.

$$G\left[\underline{\varepsilon}^n_g(f)\right] = \varepsilon_g[G(f)] \tag{21}$$

This definition, however, is not very operative, like all definitions based on the adjunction. It is therefore desirable to express this operator in a algebraic form. To do this, we will first write the relationship between each section X_i of $\left(\underline{\varepsilon}^n_g(f)\right)$ and the sections $X_j(f)$ et $X_j(g)$ of the functions f and g.

If a point x of G(g) at altitude i belongs to $X_i\left(\underline{\varepsilon}_g^n(f)\right)$, the cylinder $C_g(x, n)$ is included in G(f). Therefore, each section at altitude j $(j \leq i)$ of this cylinder corresponds to a geodesic ball of size n included in G(f), thus in $X_j(f)$. The intersection of the axis $]-\infty, x]$ of the cylinder $C_g(x, n)$ and of $X_j(f)$ thus belongs to the geodesic erosion of $X_j(f)$ in $X_j(g)$. We can therefore write:

$$X_i\left(\underline{\varepsilon}_g^n(f)\right) = \bigcap_{j \leq i}\left[\varepsilon^n_{X_j(g)}(X_j(f))\right] \tag{22}$$

To belong to the section X_i at altitude i of the geodesic erosion of f under g, a point x must also belong to all the set geodesic erosions of the sections $X_j(f)$ located at an altitude $j < i$ in the corresponding section $X_j(g)$.

We recognize, in this formulation, an already known technique to force an operator to be increasing when it is extended to the numerical domain. This approach is particularly used to extend the concepts of thinning and thickening [1]. Note that there are two ways for obtaining this result (Figure 5). We can either intersect all sections less than or equal to i or perform the union of all sections greater than or equal to i. In this present case, the first solution is obvious while the latter one prevails for the thinnings and thickenings (the first approach leads to the definition of other operators called over-thickening under-thinning, cf. [1, pages 125-127]).

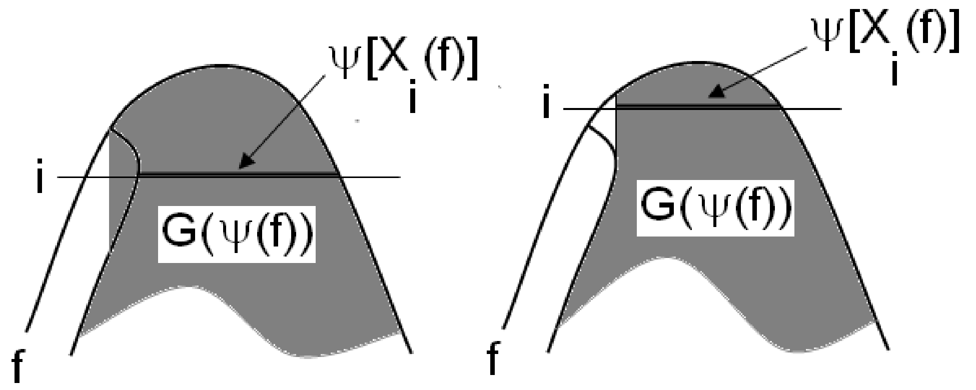

Figure 5 : Two approaches used to generate the subgraph of a function by stacking sets obtained with a non increasing set operator ψ. The procedure on the right (intersection of sections at an altitude lower than or equal to i) is used for the geodesic erosion. The procedure using the union of upper sections (on the left) is not currently used (it is used, however, for the thinnings).

Starting from the above formula, we can show that the elementary digital geodesic erosion of f under g can be written:

$$\underline{\varepsilon}_g(f) = [(f \vee m) \ominus B] \wedge g \tag{23}$$

where m is the indicator function defined by:

$m(x) = v_{max}$ if $f(x) = g(x)$

$m(x) = 0$ otherwise

(assuming f and g positive and bounded: $0 \leq f \leq g \leq v_{max}$).

The proof of this formula is quite long, it was postponed in Appendix A.

This operator is increasing and anti-extensive. It is increasing because, assuming $f \leq f'$, we have $m \leq m'$. Indeed, the locus of points where $f' = g$ is included in the set of points where $f = g$. It is also anti-extensive because:

$$(f \vee m) \ominus B \leq f \vee m \tag{24}$$
$$(f \vee m) \wedge g = (f \wedge g) \vee (g \wedge m) = f \vee (g \wedge m) = f \vee (f \wedge m) = f \tag{25}$$
Q.E.D.

Figure 6 illustrates the various stages of construction of this transformation. This operator produces a result identical to the set operator when applied to indicator functions.

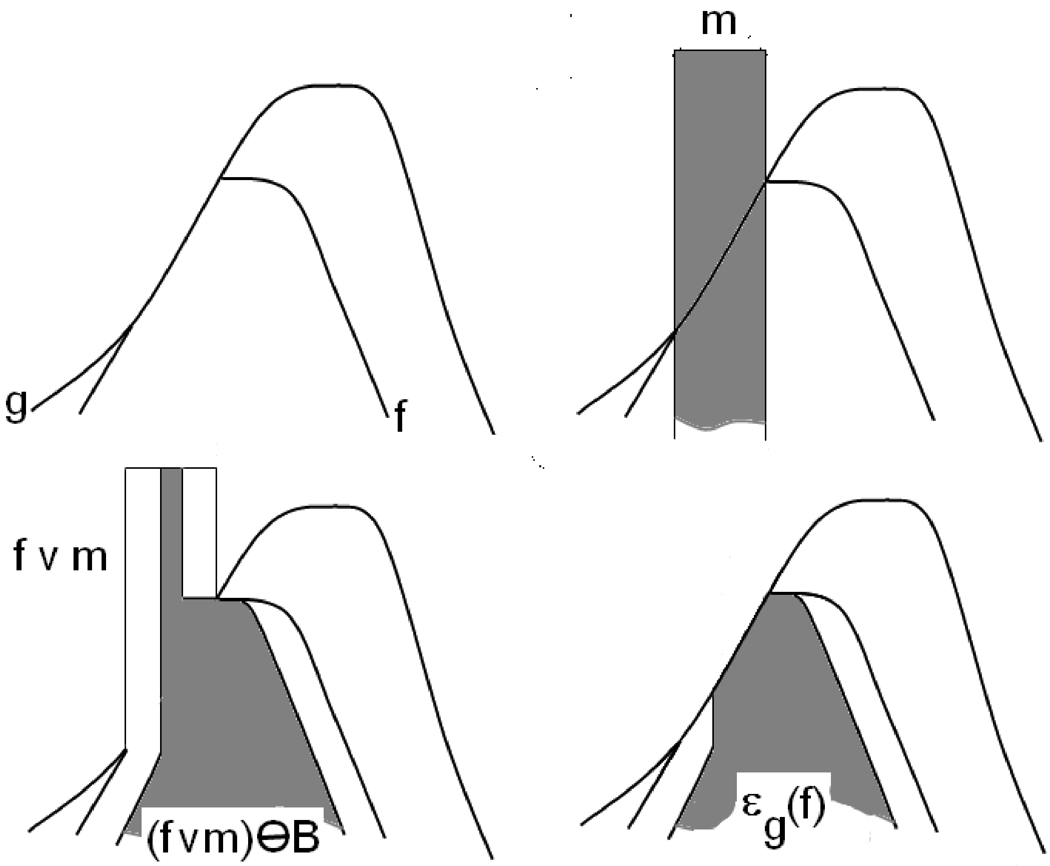

Figure 6 : Example of elementary geodesic erosion of a function f under a function g. Top, left: the initial functions. Top, right: the m function (indicator of the equality of f and g). Bottom, left: euclidean erosion of the function $f \vee m$. *Bottom, right: geodesic erosion of f under g.*

7. Geodesic opening and closing

By concatenating the operators $\delta_g(f)$ and $\underline{\varepsilon}_g(f)$, we can then define two new operators called geodesic opening and closing:

$$\gamma_g(f) = \delta_g.\underline{\varepsilon}_g(f) \tag{26}$$
$$\varphi_g(f) = \underline{\varepsilon}_g.\delta_g(f) \tag{27}$$

These operators are increasing. $\gamma_g(f)$ is anti-extensive. Indeed, the subgraph $G(\gamma_g)$ is made of the union of all the geodesic cylinders whose centers are included in $G\left(\underline{\varepsilon}_g\right)$. All thes cylinders are included in G(f). Hence we have $\gamma_g(f) \leq f$.
It can be shown, mutatis mutandis, that $\varphi_g(f)$ is extensive. These two operators are also idempotent. This can be easily proven in the same way as the proof for sets. This property comes from the increase and extensivity/anti-extensivity properties of the operators φ_g et γ_g.

This new geodesic erosion $\underline{\varepsilon}_g(f)$, defined in the same geodesic space as $\delta_g(f)$, makes it possible to define true openings and closings exhibiting properties and behavior which are similar to the set operators. Thus, in the previous example (Figure 1), with indicator functions instead of sets, the reader will verify that he/she obtains the same result.

8. Conclusion

This extension to the numerical case of the set geodesic erosion had, to my knowledge, never been proposed before. The problem raised by the surreptitious change of geodesic space had certainly been seen before but no solution had been provided. It is true that the geodesic opening and closing operators are not widely used. Thus, in the example given above of a "driven" merging of connected components, we can simply use a geodesic dilation to solve the problem in a satisfactory way. However, a new tool leads sooner or later to new ways to use it. It is thus possible to define more complex numerical operators (gradients, top-hat transforms, thickenings, thinnings, etc.) in geodesic spaces defined by functions. In particular, these operators could demonstrate their usefulness for applications where multiple imaging modalities are available.
Finally, this operator shows both the interest of adjunction to define a dual operator but also its limitations when, from this general definition, we need to design a more operative one.

9. References

[1] Beucher Serge, Segmentation d'Images et Morphologie Mathématique, PhD thesis, ENSMP, 1990.
[2] Beucher Serge, Geodesy and geodesic transformations, Mines ParisTech Summer Course, 2008 (http://cmm.ensmp.fr/~beucher/publi/Course2008_Geodesy_SB_eng.pdf).
[3] Heijmans Henk, Morphological Image Operators, Boston, M.A Academic,1994.
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10. Appendix A - algebraic formulation of the geodesic erosion of f under g

First recall some definitions and formulas linking the sections of a subgraph of a function and the algebraic addition and subtraction operators. These definitions are first given in the continuous case before transcribing them in the discrete case.

Let us call :

$X_\lambda(f) = \{x : f(x) \geq \lambda\}$

The (topological) closed section at level λ of the subgraph G(f).

In the same way:

$Y_\lambda(f) = \{x : f(x) > \lambda\}$

Is the (topological) open section at the same altitude.

We have:

$X_\lambda(f+g) = \{x : (f+g)(x) \geq \lambda\}$

$(f+g)(x) \geq \lambda \Rightarrow f(x) \geq \lambda \text{ et } g(x) \geq \lambda - \mu, \forall \mu$

$X_\lambda(f+g) = \bigcup_\mu \left[X_\lambda(f) \cap X_{\lambda-\mu}(g)\right]$

Or:

$X_\lambda(f+g) = \bigcup_\mu \left[X_{\lambda-\mu}(f) \cap X_\mu(g)\right]$

In the same way:

$X_\lambda(f-g) = \bigcup_\mu \left[X_{\lambda-\mu}(f) \cap X_\mu(-g)\right]$

But:

$X_\mu(-g) = \{x : g(x) \leq -\mu\} = Y^c_{-\mu}(g)$

$X_\lambda(f-g) = \bigcup_\mu \left[X_{\lambda-\mu}(f) \cap Y^c_{-\mu}(g)\right]$

$X_\lambda(f-g) = \bigcup_\mu \left[X_{\lambda+\mu}(f) \cap Y^c_{\mu}(g)\right]$

We can also write:

$Y_\lambda(f+g) = X^c_{-\lambda}(-f-g)$

$Y_\lambda(f+g) = \left[\bigcup_\mu \left[X_{-\lambda-\mu}(-f) \cap X_\mu(-g)\right]\right]^c$

$Y_\lambda(f+g) = \bigcap_\mu \left[X^c_{-\lambda-\mu}(-f) \cup X^c_\mu(-g)\right]$

$Y_\lambda(f+g) = \bigcap_\mu \left[Y_{\lambda+\mu}(f) \cup Y_{-\mu}(g)\right]$

That is:

$Y_\lambda(f+g) = \bigcap_\mu \left[Y_{\lambda-\mu}(f) \cup Y_\mu(g)\right]$

In the same way:

$Y_\lambda(f-g) = \bigcap_\mu \left[Y_{\lambda-\mu}(f) \cup Y_\mu(-g)\right]$

$Y_\lambda(f-g) = \bigcap_\mu \left[Y_{\lambda-\mu}(f) \cup X^c_{-\mu}(g)\right]$

$Y_\lambda(f-g) = \bigcap_\mu \left[Y_{\lambda+\mu}(f) \cup X^c_{\mu}(g)\right]$

In the discrete case (integer functions), we have:

$Y_i(f) = \{x : f(x) > i\} = \{x : f(x) \geq i+1\} = X_{i+1}(f)$

We can prove that:

$X_i(f+g) = \bigcup_j \left[X_{i-j}(f) \cap X_j(g)\right]$

$X_i(f-g) = \bigcup_j \left[X_{i+j}(f) \cap X^c_{j+1}(g)\right]$

Or:

$X_i(f-g) = \bigcap_j \left[X_{i+j}(f) \cup X^c_j(g)\right]$

Using formula (2), we can write:

$$\varepsilon_{X_j(g)}(X_j(f)) = \left[\left[X_j(f) \cup X_j^c(g)\right] \ominus B\right] \cap X_j(g)$$

The section at level i of the elementary numerical geodesic erosion $\underline{\varepsilon}_g(f)$ can be written:

$$X_i\left(\underline{\varepsilon}_g(f)\right) = \bigcap_{j \le i}\left[\left[\left[X_j(f) \cup X_j^c(g)\right] \ominus B\right] \cap X_j(g)\right]$$

$$X_i\left(\underline{\varepsilon}_g(f)\right) = \bigcap_{j \le i}\left[\left[X_j(f) \cup X_j^c(g)\right] \ominus B\right] \cap \left[\bigcap_{j \le i} X_j(g)\right]$$

$$X_i\left(\underline{\varepsilon}_g(f)\right) = \left[\bigcap_{j \le i}\left[\left[X_j(f) \cup X_j^c(g)\right] \ominus B\right]\right] \cap X_i(g)$$

We have $f \le g$. All the sections of the function m are identical and correspond to the same set containing all the points x where $f(x) = g(x)$. This set corresponds to the section $X_0(f - g)$:

$$X_0(f - g) = \bigcap_j \left[X_j(f) \cup X_j^c(g)\right]$$

Let us determine the section at level i of the function $f \vee m$. We have:

$$X_i(f \vee m) = X_i(f) \cup \left[\bigcap_j \left[X_j(f) \cup X_j^c(g)\right]\right]$$

That is:

$$X_i(f \vee m) = X_i(f) \cup \left[\left[\bigcap_{j \le i}\left[X_j(f) \cup X_j^c(g)\right]\right] \cap \left[\bigcap_{j > i}\left[X_j(f) \cup X_j^c(g)\right]\right]\right]$$

$$X_i(f \vee m) = \underbrace{\left[\left[\bigcap_{j \le i}\left[X_j(f) \cup X_j^c(g)\right]\right] \cup X_i(f)\right]}_{(a)} \cap \underbrace{\left[\bigcap_{j > i}\left[X_j(f) \cup X_j^c(g)\right] \cup X_i(f)\right]}_{(b)}$$

Term (a) can be written:

$$\bigcap_{j \le i}\left[X_j(f) \cup X_i(f) \cup X_j^c(g)\right] = \bigcap_{j \le i}\left[X_j(f) \cup X_j^c(g)\right] = \left[\bigcap_{j < i}\left[X_j(f) \cup X_j^c(g)\right]\right] \cap [X_i(f) \cup X_i^c(g)]$$

Term (b) is written:

$$\bigcap_{j > i}\left[X_j(f) \cup X_i(f) \cup X_j^c(g)\right] = \bigcap_{j > i}\left[X_j(f) \cup X_j^c(g)\right] = X_i(f) \cup \left[\bigcap_{j > i} X_j^c(g)\right] = X_i(f) \cup X_{i+1}^c(g)$$

Which gives:

$$X_i(f \vee m) = \left[\bigcap_{j < i}\left[X_j(f) \cup X_j^c(g)\right]\right] \cap [X_i(f) \cup X_i^c(g)]$$

$$X_i(f \vee m) = \bigcap_{j \le i}\left[X_j(f) \cup X_j^c(g)\right]$$

We can write:

$$X_i\left(\underline{\varepsilon}_g(f)\right) = \left[\bigcap_{j \le i}\left[\left[X_j(f) \cup X_j^c(g)\right] \ominus B\right]\right] \cap X_i(g)$$

$$X_i\left(\underline{\varepsilon}_g(f)\right) = \left[\left[\bigcap_{j \le i}\left[X_j(f) \cup X_j^c(g)\right]\right] \ominus B\right] \cap X_i(g)$$

$$X_i\left(\underline{\varepsilon}_g(f)\right) = [X_i(f \vee m) \ominus B] \cap X_i(g)$$

$$X_i\left(\underline{\varepsilon}_g(f)\right) = X_i[((f \vee m) \ominus B) \wedge g]$$

Which proves formula (23).

11. Appendix B - Summary of the various geodesic operators

This appendix summarizes the various elementary geodesic erosion and dilation operators available. We show that there is another dilation operator which has not been highlighted yet in this document. We also attempt to define operations allowing to go from an operator (eg erosion) to another (the dilation).
Only the numerical formulation of the operators will be considered. One can notice that despite a different formulation, the set geodesic erosion $\underline{\varepsilon}_X(Y)$ is identical to the numerical operator $\underline{\varepsilon}_g(f)$. Indeed, the mask M (equivalent for sets to the function m) corresponds to points in space which belong to both Y et X or to their complements. M is therefore the complement of the symmetrical difference Δ between X and Y:

$$M = (X \triangle Y)^c = (X \cap Y) \cup (X^c \cap Y^c) = Y \cup X^c \text{ because } Y \subset X$$

So, we have:

$$Y \cup M = Y \cup X^c$$

which is the initial definition of the elementary geodesic erosion for sets.

The following table summarizes the various elementary operations applicable to a function f in a geodesic space defined by g.

	Erosion	Dilation
$g \leq f$	$\varepsilon_g(f) = (f \ominus B) \vee g$	
$f \leq g$	$\underline{\varepsilon}_g(f) = [(f \vee m) \ominus B] \wedge g$ with $m = \{x : f = g\}$	$\delta_g(f) = (f \oplus B) \wedge g$

The analysis of this table brings two immediate observations:
- The erosion $\varepsilon_g(f)$ is deduced, as we already knew, from the dilation $\delta_g(f)$ by inversion. The two functions f and g are revolved, then the dilation is carried out. Finally the negation of the result provides the erosion. Writing $\overline{f}$ the inversion of a function, we have:

$$\left(\overline{f} \oplus B\right) \wedge \overline{g} = \left(\overline{f \ominus B}\right) \wedge \overline{g}$$

If we invert this result:

$$\overline{\left(\overline{f \ominus B}\right) \wedge \overline{g}} = (f \ominus B) \vee g = \varepsilon_g(f)$$

- The definition of an elementary geodesic dilation of f over g is missing. This definition would be the counterpart of the erosion $\varepsilon_g(f)$ when $f \geq g$. However, this dilation can be built by inversion starting from the erosion $\underline{\varepsilon}_g(f)$. It is denoted $\overline{\delta_g}(f)$ (beware of the confusion betweeen the highlighting of this operator and of the inversion). We have:

$$\overline{\delta_g}(f) = \overline{\left[\left(\overline{f} \vee m\right) \ominus B\right] \wedge \overline{g}} \;\; \text{with } m = \left\{x : \overline{f} = \overline{g}\right\}$$

That is:

$$\overline{\left[\left(\overline{f} \vee m\right) \ominus B\right] \wedge \overline{g}} = \left[\left(\overline{\overline{f} \vee m}\right) \oplus B\right] \vee g = [(f \wedge \overline{m}) \oplus B] \vee g$$

But:

$$\overline{m} = \left\{x : \overline{f} < \overline{g}\right\} = \{x : f > g\}$$

We arrive to the following definition:

$$\overline{\delta_g}(f) = [(f \wedge m) \oplus B] \vee g \text{ with } m = \{x : f > g\}$$

Figure 7 illustrates this operation.

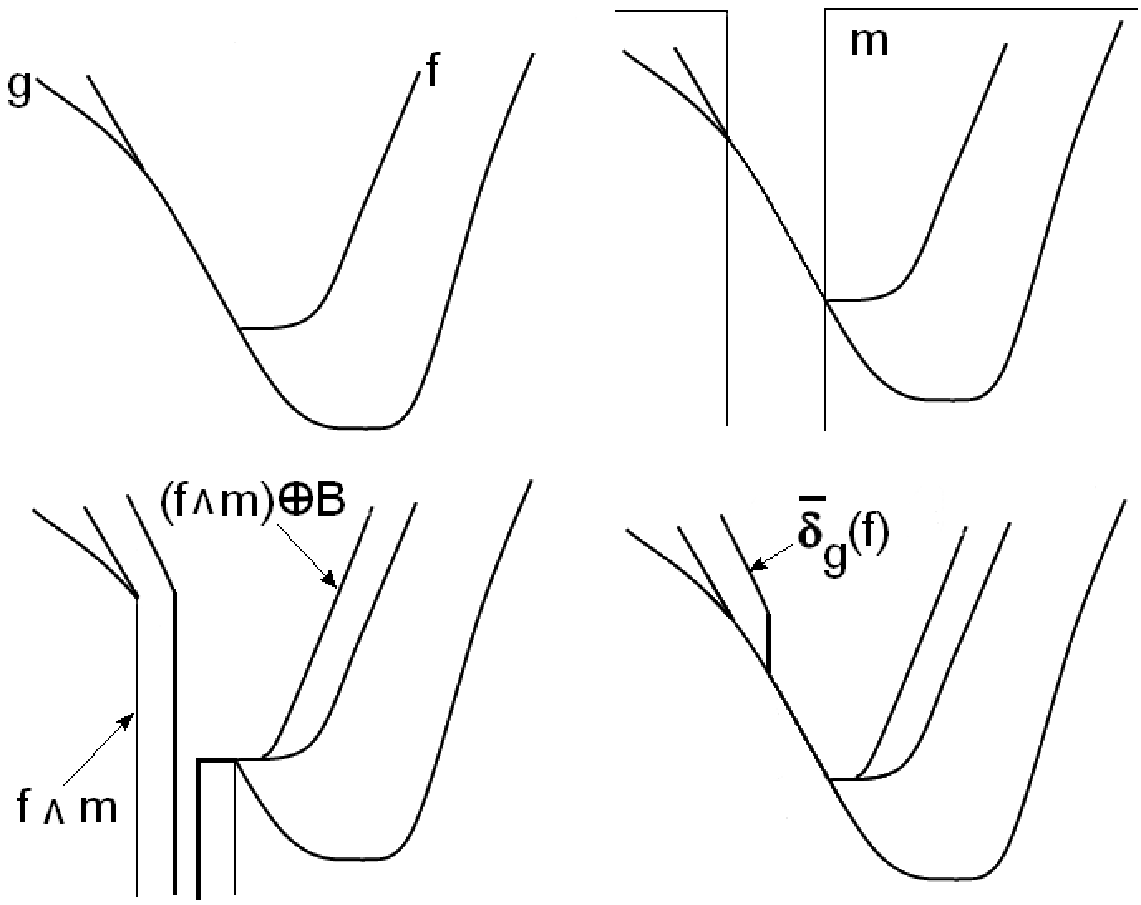

Figure 7 : elementary geodesic dilation of a function f over g.

The table of elementary geodesic operators can then be completed as follows:

	Erosion	Dilation
$g \leq f$	$\bar{\varepsilon}_g(f) = (f \ominus B) \vee g$	$\bar{\delta}_g(f) = [(f \wedge m) \oplus B] \vee g$ with $m = \{x : f > g\}$
$f \leq g$	$\underline{\varepsilon}_g(f) = [(f \vee m) \ominus B] \wedge g$ with $m = \{x : f = g\}$	$\underline{\delta}_g(f) = (f \oplus B) \wedge g$

We changed the notation of these operators to make it more consistent. The geodesic erosion and dilation of f under g are respectively are denoted $\underline{\varepsilon}_g(f)$ and $\underline{\delta}_g(f)$ while operators applied to f over g are denoted $\bar{\varepsilon}_g(f)$ and $\bar{\delta}_g(f)$.

Only the transformations $\underline{\varepsilon}_g(f)$ et $\bar{\delta}_g(f)$, on the one hand, together with $\bar{\varepsilon}_g(f)$ et $\underline{\delta}_g(f)$ on the other hand, are linked by an inversion operator. It is however not possible to concatenate these transformations to obtain openings and closings. Conversely, this is completely possible with $\underline{\varepsilon}_g(f)$ and $\underline{\delta}_g(f)$ or with $\bar{\varepsilon}_g(f)$ and $\bar{\delta}_g(f)$. We obtain two openings, one of f under g and the other of f over g:

$$\underline{\gamma}_g(f) = \underline{\delta}_g . \underline{\varepsilon}_g(f)$$
$$\bar{\gamma}_g(f) = \bar{\delta}_g . \bar{\varepsilon}_g(f)$$

Two equivalent definitions exist for the closing:

$$\overline{\varphi}_g(f) = \overline{\varepsilon}_g.\overline{\delta}_g(f)$$
$$\underline{\varphi}_g(f) = \underline{\varepsilon}_g.\underline{\delta}_g(f)$$

Nevertheless, it is not easy to exhibit a duality operator for building $\underline{\varepsilon}_g(f)$ from $\underline{\delta}_g(f)$ or $\overline{\varepsilon}_g(f)$ from $\overline{\delta}_g(f)$. This comes in particular from the fact that a mask m is involved in some definitions whereas it does not appear in others. We can however modify the previous definitions of $\overline{\varepsilon}_g(f)$ and $\underline{\delta}_g(f)$. These new definitions (see the table below) use slightly modified definitions of the masks in order to enhance the symmetries between the various operators.

	Erosion	Dilation
$g \leq f$	$\overline{\varepsilon}_g(f) = [(f \vee m) \ominus B] \vee g$ with $m = \{m : f < g\}$	$\overline{\delta}_g(f) = [(f \wedge m) \oplus B] \vee g$ with $m = \{x : f > g\}$
$f \leq g$	$\underline{\varepsilon}_g(f) = [(f \vee m) \ominus B] \wedge g$ with $m = \{x : f \geq g\}$	$\underline{\delta}_g(f) = [(f \wedge m) \oplus B] \wedge g$ with $m = \{x : f \leq g\}$

Of course, the masks introduced in $\overline{\varepsilon}_g(f)$ et $\underline{\delta}_g(f)$ are “fictitious” (in the erosion, it is empty and it covers the whole space in the dilation). However, these new definitions better show the internal structure of these operators and their similarities.
Nevertheless, achieving an erosion using a simple operator applied to the dilation in order to have a more operative bridge than the duality by adjunction between these two transforms is a difficult task, at least in the numerical case. Given that we have direct formulations to achieve these transformations, the interest of such a bridge is finally quite low..

12. Appendix C - Geodesic operators in the MAMBA Image library

This appendix is intended for the Mamba library users. The module "Geodesy" of the library has been modified. It now contains the newly defined elementary geodesic operators. Thus, there are two geodesic erosions and two geodesic dilations. These are:

- ***upperGeodesicErode*** corresponding to $\overline{\varepsilon}_g(f)$.
- ***lowerGeodesicErode*** corresponding to $\underline{\varepsilon}_g(f)$.
- ***upperGeodesicDilate*** corresponding to $\overline{\delta}_g(f)$.
- ***lowerGeodesicDilate*** corresponding to $\underline{\delta}_g(f)$.

These operators work with binary, 8-bit and 32-bit images. They are incorporated in the latest version of Mamba (version 1.1.1).

The former ***geodesicErode*** and ***geodesicDilate*** operators were kept for compatibility reasons with the previous version of Mamba. The ***geodesicDilate*** operator is nothing but an alias of ***lowerGeodesicDilate***, while ***geodesicErode*** corresponds either to ***lowerGeodesicErode*** or to ***upperGeodesicErode*** depending on the depth of the images, binary or greyscale.

Here is the Python source code of these new operators. You can find it in the geodesy.py module.

```
"""
This module provides a set of functions to perform geodesic computations using
Mamba based functions.

It includes build and dualbuild operations, geodesic erosion and dilation,
computation of maxima and minima...

it works with imageMb instances as defined in mamba.
"""

# Contributors: Serge BEUCHER, Nicolas BEUCHER

from mambaCore import ERR_BAD_DEPTH
import mamba
import mambaComposed as mC

def upperGeodesicDilate(imIn, imMask, imOut, n=1, se=mC.DEFAULT_SE):
    """
    Performs a upper geodesic dilation of image 'imIn' above 'imMask'.
    The result is put inside 'imOut', 'n' controls the size of the dilation.
    'se' specifies the type of structuring element used to perform the
    computation (DEFAULT_SE by default).

    Warning! 'imMask' and 'imOut' must be different.
    """

    mamba.logic(imIn, imMask, imOut, "sup")
    if imIn.getDepth() == 1:
```

```
        for i in range(n):
            mamba.diff(imOut, imMask, imOut)
            mC.dilate(imOut, imOut, se=se)
            mamba.logic(imMask, imOut, imOut, "sup")
    else:
        imWrk1 = mamba.imageMb(imIn)
        imWrk2 = mamba.imageMb(imIn, 1)
        for i in range(n):
            mamba.generateSupMask(imOut, imMask, imWrk2, True)
            mamba.convertByMask(imWrk2, imWrk1, 0, mamba.computeMaxRange(imWrk1)[1])
            mamba.logic(imOut, imWrk1, imOut, "inf")
            mC.dilate(imOut, imOut, se=se)
            mamba.logic(imOut, imMask, imOut, "sup")

def lowerGeodesicDilate(imIn, imMask, imOut, n=1, se=mC.DEFAULT_SE):
    """
    Performs a lower geodesic dilation of image 'imIn' below 'imMask'.
    The result is put inside 'imOut', 'n' controls the size of the dilation.
    'se' specifies the type of structuring element used to perform the
    computation (DEFAULT_SE by default).

    Warning! 'imMask' and 'imOut' must be different.
    """

    mamba.logic(imIn, imMask, imOut, "inf")
    for i in range(n):
        mC.dilate(imOut, imOut, se=se)
        mamba.logic(imMask, imOut, imOut, "inf")

def geodesicDilate(imIn, imMask, imOut, n=1, se=mC.DEFAULT_SE):
    """
    This operator is simply an alias of lowerGeodesicDilate. It is kept for
    compatibility reasons.
    """

    lowerGeodesicDilate(imIn, imMask, imOut, n, se=se)


def upperGeodesicErode(imIn, imMask, imOut, n=1, se=mC.DEFAULT_SE):
    """
    Performs a upper geodesic erosion of image 'imIn' above 'imMask'.
    The result is put inside 'imOut', 'n' controls the size of the erosion.
    'se' specifies the type of structuring element used to perform the
    computation (DEFAULT_SE by default).

    Warning! 'imMask' and 'imOut' must be different.
    """

    mamba.logic(imIn, imMask, imOut, "sup")
    for i in range(n):
        mC.erode(imOut, imOut, se=se)
        mamba.logic(imOut, imMask, imOut, "sup")

def lowerGeodesicErode(imIn, imMask, imOut, n=1, se=mC.DEFAULT_SE):
```

```
    """
    Performs a lower geodesic erosion of image 'imIn' under 'imMask'.
    The result is put inside 'imOut', 'n' controls the size of the erosion.
    'se' specifies the type of structuring element used to perform the
    computation (DEFAULT_SE by default).

    The binary lower geodesic erosion is realised using the fact that the dilation is the
    dual operation of the erosion.

    Warning! 'imMask' and 'imOut' must be different.
    """

    if imIn.getDepth() == 1:
        mamba.diff(imMask, imIn, imOut)
        lowerGeodesicDilate(imOut, imMask, imOut, n, se=se)
        mamba.diff(imMask, imOut, imOut)
    else:
        imWrk1 = mamba.imageMb(imIn)
        imWrk2 =mamba.imageMb(imIn, 1)
        mamba.logic(imIn, imMask, imOut, "inf")
        for i in range(n):
            mamba.generateSupMask(imOut, imMask, imWrk2, False)
            mamba.convertByMask(imWrk2, imWrk1, 0, mamba.computeMaxRange(imWrk1)[1])
            mamba.logic(imOut, imWrk1, imOut, "sup")
            mC.erode(imOut, imOut, se=se)
            mamba.logic(imOut, imMask, imOut, "inf")

def geodesicErode(imIn, imMask, imOut, n=1, se=mC.DEFAULT_SE):
    """
    This transformation is identical to the previous version and it has been kept for compatibilty
    purposes.

    Note that the binary and the greytone operators are different.
    """

    if imIn.getDepth() == 1:
        lowerGeodesicErode(imIn, imMask, imOut, n, se=se)
    else:
        upperGeodesicErode(imIn, imMask, imOut, n, se=se)
```