Simple Precoding Algorithms for Multi-User Relay Communications and Optimized Power Allocation

Jean-Pierre Cances, Vahid Meghdadi

Simple Precoding Algorithms for Multi-User Relay Communications and Optimized Power Allocation

J.P CANCES and V. MEGHDADI
*ENSIL, University of Limoges, Xlim/C2S2/ESTE UMR CNRS 7252
16, Rue Atlantis - Parc ESTER - BP 6804- 87068 Limoges cedex, France
e-mail meghdadi, cances@ensil.unilim.fr

Abstract: In this paper we propose new simple precoding solutions to be used at the relay station in the context of a multi-relay wireless network where the different mobile stations belong to the same network, i.e. we don't consider the case of competing users from different networks as in some recent works based on game theory. Our proposed algorithm is based on the Gram-Schmidt Orthonormalization since our goal is to mitigate the multi-user interference at each mobile station. The strength of this method is that it only requires the knowledge of all channel impulse responses from a given relay to all the mobile stations. In other words, to compute its precoding vectors, each relay does not need to know the channel impulse responses of the other relays. Using our algorithm, we find that, when compared to the centralized reference method where each mobile station benefit from the same diversity gain, some mobile stations will improve their diversity gain whilst the other ones will suffer from a loss. This constitutes a simple solution to supply different quality of service (QoS) in the case of a multi-services network. We propose then an optimized power allocation strategy at each relay and we give complete accurate performance derivations for the different studied contexts.

***Index Terms**:* Cooperative Diversity, Diversity Gain, Gram-Schmidt Algorithm, Multi-User Interference Mitigation, Expectation Maximization, Power Allocation Optimization.

I. Introduction

Emerging commercial multimedia applications require reliable wireless transmission links operating at high data rates. However, the limitations such as multipath fading and time variations, make this a challenging task. Cooperative relay combined with Multiple Input Multiple Output (MIMO) transmissions are attractive solutions to both extend the coverage of existing cellular networks and to improve the quality of wireless links. Recent relaying techniques, including cooperative relaying, show to be very promising in order to increase the reliability of telecommunication links and consequently to extend the wireless network coverage. That is why relaying schemes, especially cooperative networks, have become a very hot research area. In general, the relaying protocols are divided into two main categories [1][2]: Amplify-and-Forward (AF) and Decode-and-Forward (DF). In the AF case relay stations (RS's) amplify the received signal (which is the noised version of transmitted signal), and resend it towards its final destination [3][4]. In the DF case, the RS's decode the received signal then retransmit the information via a suitable processing to destination [5][6]. In the case of successful decoding in a relay, the noise effect from the first link (base station to relay

stations) is removed. To improve the performance, maximum ratio combining together with distributed beamforming are used to eliminate multiple access interference (MAI) and at the same time to maximize the signal to noise ratio (SNR) at receivers [7].

In this paper we will concentrate on the link between relays and mobile stations (MS). Since base stations (BS) and relays are fixed, we consider that it is easy to build reliable links between them using for example optical fiber. We will study the configuration where the RS's, using multiple transmit antennas, send the information corresponding to all the mobiles at the same time and at the same carrier frequency to all mobile stations using optimized precoding vectors. The objective is to maximize the signal to noise ratio at each MS and to mitigate the MAI. Furthermore, we will consider the case of slowly non-frequency-selective fading channels between RS's and MS's, enabling the MS's to feed back the RS's with their own Channel Impulse Response (CIR) thanks to a low-rate specific channel.

A lot of works have been already published in the open literature concerning the optimization of precoding vectors at relay places [8-12] for a single destination terminal. The optimization criterion is always the maximization of the SNR at the receive terminal. Among them, the paper of Zhihang & al [8] constitutes an outstanding reference since it copes with different realistic contexts including the case where only second order statistics of the channel are available. However, it does not take into account the multi-user case and one can remark that the majority of work is related to the AF context with only one destination terminal [9-10]. The system in [11] uses the precoding vectors to maximize the system performance of a MIMO single user system whilst [12] tries to use the same technique for the case of two distinct users. In fact, the literature concerning the usage of precoding vectors for the case of arbitrary number of MS's is not rich. One can cite the work of Shu & al [13] where the authors try to maximize the system capacity using MSE based precoders together with Particle Swarm Optimization (PSO) algorithm. However the proposed solution in [10] implies a complicated receiver structure at the MS which limits the practical interest of the work by imposing expensive structure to the end line users. At the same time, due to the complicated equivalent channel equations, very little analytic predictions are produced. In a former version of this study [14], we have proposed Zero-Forcing (ZF) equalization based precoders which offer always the same diversity gain at each

receive mobile station. This diversity gain is equal to the number of transmit antennas in the network minus the number of interfering users. This system will serve us as a reference to compare our results with the previous ones.

In the new presented work, we concentrate the complexity on the relay station side by computing precoding vectors which aim at cancelling MAI at MS's receiver side. This is done using the well known Gram-Schmidt orthonormalization process. The strength of this method is that it only requires the knowledge of CIR's from a given relay to all the mobile stations. In other words, to compute its precoding vectors, each relay does not need to know the CIR's of the other relays. Using this algorithm, we carefully study the potential diversity gain at the receivers and we show that, unlike in [14], the diversity gain is not constant at the receiver side and depends on the ranking of the mobile station used in the orthogonalization process. However, the diversity gain for the first MS's can be equal to the total number of transmit antennas in the network, i.e. it may be superior to the value we obtain in the reference system. Obviously this is obtained at the price of some diversity gain loss particularly for the last MS's included in the algorithm.

Furthermore, we study the power allocation policy to the relays when the total power is assumed limited. We obtain a rigorous mathematical analysis on the Symbol Error Rate (SER) at each MS in both cases i.e. when each MS benefits from optimized power and when the allocated power is chosen randomly. This mathematical derivation is obtained thanks to an accurate approximation of the p.d.f of the optimized SNR using an iterative Expectation-Maximization (EM) algorithm. The obtained results clearly demonstrate the advantage of optimizing the allocated power at the cost of a centralized strategy. This means that, in the case of optimized power allocation, each relay knows all the CIR's from all the relays in the network, adding an additional complexity in the signalling feedback channels. Hence, the contributions of this paper are threefold:

- A detailed study of the diversity gain is given with the corresponding theoretical Symbol Error Rate (SER) expressions.
- An accurate optimization of the allocated power to each relay is proposed with the corresponding BER expressions thanks to an iterative E.M based algorithm to find the SNR distribution.

- In the case of randomly distributed power we are able to derive theoretical SER expressions which enable us to compare with the optimized case.

The rest of the paper is organized as follows. The system model is described in part II together with the orthonormalization algorithm. The diversity gain is studied in Section III and the power allocation policy, together with the theoretical BER derivations, are given in part IV. Simulation results illustrating the performances of the proposed system are discussed in part V. Finally the main results are highlighted in the concluding part in Section VI.

II. System Model

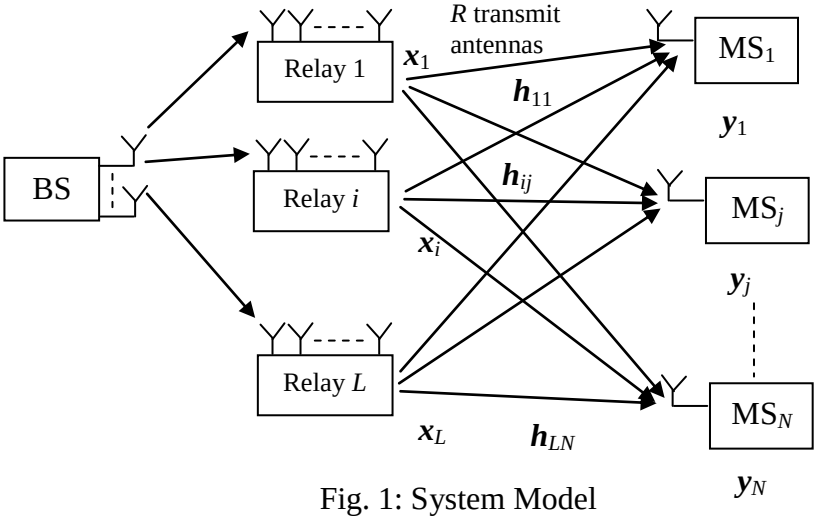


Fig. 1: System Model

The complete system model is depicted on Fig. 1. It includes one base station with M antennas which sends N symbols s_1 to s_N to N mobile stations MS_1 to MS_N via L fixed relays each equipped with R antennas. A two hop communication scheme is considered. In the first hop, the base station sends the signal to the relays. The relays will then decode the received signal and multiply it by some precoding vectors before transmitting them to mobile stations in the second hop. Since the base station and the relays are considered fixed, the communication between the BS and relays is considered to be perfect. In fact a wide variety of low-noise communication media such as optical fibers may be used. Moreover CRC may be employed to detect any possible errors in relays and to ask the base station to resend the missing information. As a result BS to RS link may be considered error-free. In the remainder of this paper we will focus on the second hop of the communication where L relays cooperate in sending each of the N data symbols to their intended mobile

stations. The link from the *i*th relay to the *j*th mobile station is a flat fading Rayleigh channel $\boldsymbol{h}_{i,j} \square C.N\ (0, \boldsymbol{I}_R)$ of size 1 × *R*. since each MS is equipped with one receive antenna. The channel state information is assumed to be known at the transmitter. We define the vector $\boldsymbol{s}$ of transmitted symbols as $\boldsymbol{s} = [s_1, s_2, ..., s_N]^T$ with s_1 to s_N being *N M*-PSK unitary symbols ($\left| s_j \right|^2 = 1$). Since the BS to RS links are considered to be error-free, we will assume that by the end of the first hop $\boldsymbol{s}$ is received correctly by all relays. Each relay will then multiply its received signals by a set of precoding vectors $\boldsymbol{w}_i^j$ each of size *R* × 1.

$$\boldsymbol{x}_i = \sum_{j=1}^{N} \boldsymbol{w}^i_{j|_{R\times 1}} . s_{j|_{1\times 1}} \quad i = 1, ..., L \qquad (1)$$

The relays then will send these signals to the MS's; the received signal at mobile station *j* can be written as:

$$\begin{aligned} y_k &= \sum_{i=1}^{L} \boldsymbol{h}^T_{i,k|_{1\times R}} . \boldsymbol{x}_{i|_{R\times 1}} + n_k \\ y_{k|_{1\times 1}} &= \sum_{i=1}^{L} \boldsymbol{h}^T_{i,k|_{1\times R}} . (\sum_{j=1}^{N} \boldsymbol{w}^i_{j|_{R\times 1}} . s_{j|_{1\times 1}}) + n_k \quad , k = 1, ..., N \end{aligned} \qquad (2)$$

By concatenating the different values $\boldsymbol{y}_k$ in a vector form: $\boldsymbol{y} = [y_1, y_2, ..., y_N]^T_{|_{N\times 1}}$, we obtain the following writing form:

$$\begin{aligned} y_1 &= \boldsymbol{h}^T_{1,1} . (\sum_{j=1}^{N} \boldsymbol{w}^1_j . s_j) + \boldsymbol{h}^T_{2,1} . (\sum_{j=1}^{N} \boldsymbol{w}^2_j . s_j) + ... + \boldsymbol{h}^T_{L,1} . (\sum_{j=1}^{N} \boldsymbol{w}^L_j . s_j) \\ y_k &= \boldsymbol{h}^T_{1,k} . (\sum_{j=1}^{N} \boldsymbol{w}^1_j . s_j) + \boldsymbol{h}^T_{2,k} . (\sum_{j=1}^{N} \boldsymbol{w}^2_j . s_j) + ... + \boldsymbol{h}^T_{L,k} . (\sum_{j=1}^{N} \boldsymbol{w}^L_j . s_j) \\ &\vdots \\ y_N &= \boldsymbol{h}^T_{1,N} . (\sum_{j=1}^{N} \boldsymbol{w}^1_j . s_j) + \boldsymbol{h}^T_{2,N} . (\sum_{j=1}^{N} \boldsymbol{w}^2_j . s_j) + ... + \boldsymbol{h}^T_{L,N} . (\sum_{j=1}^{N} \boldsymbol{w}^L_j . s_j) \end{aligned}$$

We can write it equivalently:

$$\boldsymbol{y} = (\sum_{i=1}^{L} \boldsymbol{H}_i . \boldsymbol{W}_i) . \boldsymbol{s} \qquad (3)$$

With:

$$\boldsymbol{H}_i = \begin{bmatrix} \boldsymbol{h}_{i,1} \\ \boldsymbol{h}_{i,2} \\ \vdots \\ \boldsymbol{h}_{i,N} \end{bmatrix}_{N\times R} , \boldsymbol{W}_i = [\boldsymbol{w}^i_1, \boldsymbol{w}^i_2, ..., \boldsymbol{w}^i_N]_{R\times N} \text{ and } i = 1, \ldots, L \qquad (4)$$

It is then possible to use a global matrix model since we can write:

$$\boldsymbol{y}=\begin{bmatrix} \boldsymbol{h}_{1,1}^T & \boldsymbol{h}_{2,1}^T & \cdots & \boldsymbol{h}_{L,1}^T \\ \boldsymbol{h}_{1,2}^T & \boldsymbol{h}_{2,2}^T & \cdots & \boldsymbol{h}_{L,2}^T \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{h}_{1,N}^T & \boldsymbol{h}_{2,N}^T & \cdots & \boldsymbol{h}_{L,N}^T \end{bmatrix}.\begin{bmatrix} \sum_{j=1}^{N} \boldsymbol{w}_j^1.s_j \\ \sum_{j=1}^{N} \boldsymbol{w}_j^2.s_j \\ \vdots \\ \sum_{j=1}^{N} \boldsymbol{w}_j^L.s_j \end{bmatrix}=\begin{bmatrix} \boldsymbol{h}_{1,1}^T & \boldsymbol{h}_{2,1}^T & \cdots & \boldsymbol{h}_{L,1}^T \\ \boldsymbol{h}_{1,2}^T & \boldsymbol{h}_{2,2}^T & \cdots & \boldsymbol{h}_{L,2}^T \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{h}_{1,N}^T & \boldsymbol{h}_{2,N}^T & \cdots & \boldsymbol{h}_{L,N}^T \end{bmatrix}_{|N\times R.L}.\begin{bmatrix} \boldsymbol{w}_1^1 & \boldsymbol{w}_2^1 & \cdots & \boldsymbol{w}_N^1 \\ \boldsymbol{w}_1^2 & \boldsymbol{w}_2^2 & \cdots & \boldsymbol{w}_N^2 \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{w}_1^L & \boldsymbol{w}_2^L & \cdots & \boldsymbol{w}_N^L \end{bmatrix}_{|R.L\times N}.\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix}_{|N\times 1}$$

$$\boldsymbol{y}=\boldsymbol{H}.\boldsymbol{W}.\boldsymbol{s}+\boldsymbol{n} \quad (5)$$

With:

$$\boldsymbol{H}=\begin{bmatrix} \boldsymbol{h}_{1,1}^T & \boldsymbol{h}_{2,1}^T & \cdots & \boldsymbol{h}_{L,1}^T \\ \boldsymbol{h}_{1,2}^T & \boldsymbol{h}_{2,2}^T & \cdots & \boldsymbol{h}_{L,2}^T \\ \vdots & \vdots & \cdots & \vdots \\ \boldsymbol{h}_{1,N}^T & \boldsymbol{h}_{2,N}^T & \cdots & \boldsymbol{h}_{L,N}^T \end{bmatrix}_{|N\times R.L} \quad \boldsymbol{W}=\begin{bmatrix} \boldsymbol{w}_1^1 & \boldsymbol{w}_2^1 & \cdots & \boldsymbol{w}_N^1 \\ \boldsymbol{w}_1^2 & \boldsymbol{w}_2^2 & \cdots & \boldsymbol{w}_N^2 \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{w}_1^L & \boldsymbol{w}_2^L & \cdots & \boldsymbol{w}_N^L \end{bmatrix}_{|R.L\times N} \text{ and } \boldsymbol{s}=\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix}_{|N\times 1}$$

2.1 **The reference system:**

In the reference system, using equations (3) or (5), the goal is to compute matrix $\boldsymbol{W}$ in order to obtain a diagonal matrix for: $\boldsymbol{H}.\boldsymbol{W}$. The *j*th element on the diagonal of $\boldsymbol{HW}$ determines the signal to noise ratio of the *j*th MS. In this paper we focus at first on the case where all MS's have the same SNR (i.e $\boldsymbol{HW} = g.\boldsymbol{I}_N$ with g being a positive real). We have in fact two possibilities depending on the fact that each relay knows completely the CIR's from all relays to the MS's or not.

Case 1: each relay knows all the CIR's from all relays to the MS's.

In this case, the system is mathematically equivalent to a relay with *R.L* antennas and we find the same behavior as a MIMO system with *R.L* transmit antennas. As stated before, in order to cancel the MAI while imposing equal received power for all MSs, we must find a precoding vector $\boldsymbol{W}$ that satisfies:

$$\boldsymbol{H}_{N\times L.R}.\boldsymbol{W}_{L.R\times N}=g.\boldsymbol{I}_N \quad (6)$$

For simplicity reason and without loosing any generality we assume $g = 1$ in the following. The solution to (6) exists if and only if $\boldsymbol{H}$ is full rank (i.e. $R.L \geq N$) and it is the Moore-Penrose pseudo inverse of $\boldsymbol{H}$. $\boldsymbol{W}_{L.R\times N}=\boldsymbol{H}^{\dagger}(\boldsymbol{H}.\boldsymbol{H}^{\dagger})^{-1}$. Then, if we scale the obtained solution to the available power *P*, we obtain:

$$\boldsymbol{W}_{L.R\times N} = \sqrt{P}.\frac{\boldsymbol{H}^{\dagger}(\boldsymbol{H}.\boldsymbol{H}^{\dagger})^{-1}}{\left\|\boldsymbol{H}^{\dagger}(\boldsymbol{H}.\boldsymbol{H}^{\dagger})^{-1}\right\|} \quad (7)$$

In this case, the received vector $\boldsymbol{y}$ is written as:

$$\begin{aligned} \boldsymbol{y} &= \boldsymbol{H}.\boldsymbol{W}.\boldsymbol{s} + \boldsymbol{n} = \sqrt{P}.\frac{\boldsymbol{H}.\boldsymbol{H}^{\dagger}(\boldsymbol{H}.\boldsymbol{H}^{\dagger})^{-1}}{\left\|\boldsymbol{H}^{\dagger}(\boldsymbol{H}.\boldsymbol{H}^{\dagger})^{-1}\right\|}.\boldsymbol{s} + \boldsymbol{n} \\ \boldsymbol{y} &= \frac{\sqrt{P}.\boldsymbol{s}}{\left\|\boldsymbol{H}^{\dagger}(\boldsymbol{H}.\boldsymbol{H}^{\dagger})^{-1}\right\|} + \boldsymbol{n} \end{aligned} \quad (8)$$

It is obvious from (8) that the *j*th element of $\boldsymbol{y}$ depends only on s_j and not on other data symbols. As a result the system can be considered as N parallel channels each one transmitting a symbol s_j.

Case 2: each relay knows only its own CIR

When each relay knows only the channel coefficients of the links between itself and the MS's, the precoding vector of relay i $\boldsymbol{W}_i$ must be calculated only as a function of the CIR between relay station RS_i and mobile station MS_j. That means that for all $i = 1,\ldots,L$ we have:

$$\boldsymbol{H}_i.\boldsymbol{W}_i = g_i.\boldsymbol{I}_N \quad i = 1,...,L \quad (9)$$

The equation (9) has a solution if $\boldsymbol{H}_i$ is full rank (i.e. if the number of antennas of each relay is greater than the number of receivers: $R \geq N$). Of course this criterion is much more difficult to obtain than its counterpart in the first case, but, on the other hand, relays do not need to be inter-connected to each other. Precoding vectors of the *i*th relay are given by:

$$\boldsymbol{W}_i = \sqrt{P_i}.\frac{\boldsymbol{H}_i^{\dagger}(\boldsymbol{H}_i.\boldsymbol{H}_i^{\dagger})^{-1}}{\left\|\boldsymbol{H}_i^{\dagger}(\boldsymbol{H}_i.\boldsymbol{H}_i^{\dagger})^{-1}\right\|} \quad (10)$$

With P_i satisfying the overall constraint power: $\sum_{i=1}^{L} P_i = P$. This time, we obtain for the received signal:

$$\boldsymbol{y} = \sum_{i=1}^{L} \mathbf{y}_i + \boldsymbol{n} \quad (11)$$

With $\mathbf{y}_{i|_{N\times 1}}$ is the vector contribution of *i*th relay in the received signal and is given by: $\mathbf{y}_i = \frac{\sqrt{P_i}.\boldsymbol{s}}{\left\|\boldsymbol{H}_i^{\dagger}(\boldsymbol{H}_i.\boldsymbol{H}_i^{\dagger})^{-1}\right\|}$.

2.2 **The new proposed system:**

To describe better the new propose system we can begin by the simpler case of two RS's communicating with two MS's a depicted on Fig. 2.

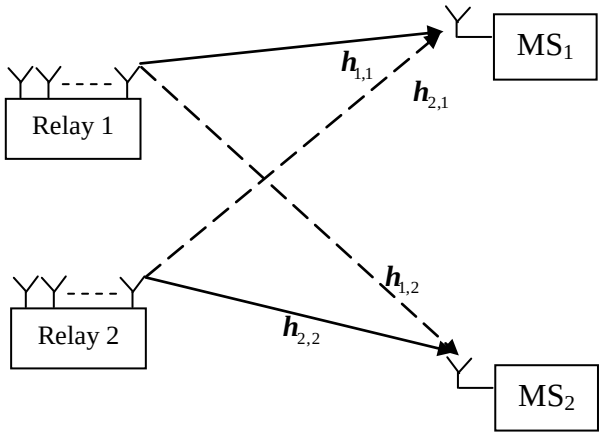


Fig. 2 : simplified relay network model

In this particular case, the received signal at the MS's can be written as:

$$\begin{aligned} y_1 &= (\boldsymbol{h}_{1,1}^T.\boldsymbol{w}_1^1 + \boldsymbol{h}_{2,1}^T.\boldsymbol{w}_1^2).s_1 + (\boldsymbol{h}_{1,1}^T.\boldsymbol{w}_2^1 + \boldsymbol{h}_{2,1}^T.\boldsymbol{w}_2^2).s_2 + n_1 \\ y_2 &= (\boldsymbol{h}_{1,2}^T.\boldsymbol{w}_1^1 + \boldsymbol{h}_{2,2}^T.\boldsymbol{w}_1^2).s_1 + (\boldsymbol{h}_{1,2}^T.\boldsymbol{w}_2^1 + \boldsymbol{h}_{2,2}^T.\boldsymbol{w}_2^2).s_2 + n_2 \end{aligned} \quad (12)$$

The goal, in order to compute precoding vectors $\boldsymbol{w}_1^1$, $\boldsymbol{w}_1^2$, $\boldsymbol{w}_2^1$ and $\boldsymbol{w}_2^2$ is to cancel interference i.e. that means that : $\boldsymbol{w}_2^1 \perp \boldsymbol{h}_{1,1}^T$, $\boldsymbol{w}_2^2 \perp \boldsymbol{h}_{2,1}^T$, $\boldsymbol{w}_1^1 \perp \boldsymbol{h}_{1,2}^T$ and : $\boldsymbol{w}_1^2 \perp \boldsymbol{h}_{2,2}^T$ while maximizing the SNR for the desired users i.e. maximizing $\boldsymbol{h}_{1,1}^T.\boldsymbol{w}_1^1 + \boldsymbol{h}_{2,1}^T.\boldsymbol{w}_1^2$, $\boldsymbol{h}_{1,2}^T.\boldsymbol{w}_2^1 + \boldsymbol{h}_{2,2}^T.\boldsymbol{w}_2^2$. This optimization problem is solved using the orthogonal projection theorem and yields to the following solution:

$$\boldsymbol{w}_2^1 = \frac{\Pi^{\perp}_{\boldsymbol{h}_{1,1}^*}.\boldsymbol{h}_{1,2}^*}{\left\| \Pi^{\perp}_{\boldsymbol{h}_{1,1}^*}.\boldsymbol{h}_{1,2}^* \right\|}, \boldsymbol{w}_1^2 = \frac{\Pi^{\perp}_{\boldsymbol{h}_{2,2}^*}.\boldsymbol{h}_{2,1}^*}{\left\| \Pi^{\perp}_{\boldsymbol{h}_{2,2}^*}.\boldsymbol{h}_{2,1}^* \right\|}, \boldsymbol{w}_1^1 = \frac{\Pi^{\perp}_{\boldsymbol{h}_{1,2}^*}.\boldsymbol{h}_{1,1}^*}{\left\| \Pi^{\perp}_{\boldsymbol{h}_{1,2}^*}.\boldsymbol{h}_{1,1}^* \right\|}, \boldsymbol{w}_2^2 = \frac{\Pi^{\perp}_{\boldsymbol{h}_{2,1}^*}.\boldsymbol{h}_{2,2}^*}{\left\| \Pi^{\perp}_{\boldsymbol{h}_{2,1}^*}.\boldsymbol{h}_{2,2}^* \right\|} \quad (13)$$

With: $\Pi_{\boldsymbol{X}}^{\perp} = \boldsymbol{I} - \boldsymbol{X}.(\boldsymbol{X}^H.\boldsymbol{X})^{-1}.\boldsymbol{X}^H$. Using definitions (13), it is straightforward to obtain:

$$\begin{aligned} \boldsymbol{h}_{1,1}^T.\boldsymbol{w}_2^1.s_2 &= \boldsymbol{h}_{1,1}^T.\frac{\Pi^{\perp}_{\boldsymbol{h}_{1,1}^*}.\boldsymbol{h}_{1,2}^*}{\left\| \Pi^{\perp}_{\boldsymbol{h}_{1,1}^*}.\boldsymbol{h}_{1,2}^* \right\|}.s_2 = \boldsymbol{h}_{1,1}^T.\frac{(\boldsymbol{I} - \boldsymbol{h}_{1,1}^*.(\boldsymbol{h}_{1,1}^{*H}.\boldsymbol{h}_{1,1}^*)^{-1}.\boldsymbol{h}_{1,1}^{*H})}{\left\| \Pi^{\perp}_{\boldsymbol{h}_{1,1}^*}.\boldsymbol{h}_{1,2}^* \right\|}.\boldsymbol{h}_{1,2}^*.s_2 \\ &= \frac{\boldsymbol{h}_{1,1}^T - \boldsymbol{h}_{1,1}^T.\boldsymbol{h}_{1,1}^*.(\boldsymbol{h}_{1,1}^T.\boldsymbol{h}_{1,1}^*)^{-1}.\boldsymbol{h}_{1,1}^T}{\left\| \Pi^{\perp}_{\boldsymbol{h}_{1,1}^*}.\boldsymbol{h}_{1,2}^* \right\|}.\boldsymbol{h}_{1,2}^*.s_2 = 0 \end{aligned} \quad (14)$$

And similarly, we obtain: $\boldsymbol{h}_{2,1}^T.\boldsymbol{w}_2^2.s_2 = 0$, $\boldsymbol{h}_{1,2}^T.\boldsymbol{w}_1^1.s_1 = 0$ and $\boldsymbol{h}_{2,2}^T.\boldsymbol{w}_1^2.s_1 = 0$. Concerning the maximization of the

desired signals we have the following theorem.

Theorem: Among all vectors $\boldsymbol{z}$ satisfying $\|\boldsymbol{z}\|=1$ and $\boldsymbol{X}^H.\boldsymbol{z}=0$, the vector $\boldsymbol{z}=\Pi_{\boldsymbol{X}}^{\perp}.\boldsymbol{y}$ maximizes the quantity: $\left|\boldsymbol{z}^H.\boldsymbol{y}\right|$. The proof is given just below.

Proof: If we take: $\Pi_{\boldsymbol{X}}^{\perp}=\boldsymbol{U}.\boldsymbol{U}^H$ with $\boldsymbol{U}^H.\boldsymbol{U}=\boldsymbol{I}$ and if we consider the vector $\boldsymbol{z}=\boldsymbol{U}\boldsymbol{p}$ for a given vector $\boldsymbol{p}$ then we have: $\left|\boldsymbol{z}^H.\boldsymbol{y}\right|=\left|\boldsymbol{y}^H.\boldsymbol{U}.\boldsymbol{p}\right|$ and: $\|\boldsymbol{z}\|=\|\boldsymbol{p}\|$. Thus, the quantity $\left|\boldsymbol{y}^H.\boldsymbol{U}.\boldsymbol{p}\right|$ is maximized under the constraint: $\|\boldsymbol{p}\|=1$ for $\boldsymbol{p}=\boldsymbol{U}^H.\boldsymbol{y}/\left\|\boldsymbol{U}^H.\boldsymbol{y}\right\|$ or, in other words, we obtain: $\boldsymbol{z}=\boldsymbol{U}\boldsymbol{p}=\Pi_{\boldsymbol{X}}^{\perp}.\boldsymbol{y}/\left\|\Pi_{\boldsymbol{X}}^{\perp}.\boldsymbol{y}\right\|$.

Applying this theorem to equations (12) enables us to affirm that $\boldsymbol{h}_{1,1}^T.\boldsymbol{w}_1^1+\boldsymbol{h}_{2,1}^T.\boldsymbol{w}_1^2$ and $\boldsymbol{h}_{1,2}^T.\boldsymbol{w}_2^1+\boldsymbol{h}_{2,2}^T.\boldsymbol{w}_2^2$ are maximized.

It is possible to generalize the former situation to our proposed cooperative relaying system of Fig. 1. We have the following equation:

$$
\begin{aligned}
&y_i=\sum_{k=1}^{L}\boldsymbol{h}_{k,i|_{1\times R}}^T.\boldsymbol{x}_{k|_{R\times 1}}+n_i\\
&y_i=\boldsymbol{h}_{1,i}^T.\boldsymbol{x}_1+\boldsymbol{h}_{2,i}^T.\boldsymbol{x}_2+\ldots+\boldsymbol{h}_{k,i}^T.\boldsymbol{x}_k+\ldots+\boldsymbol{h}_{L,i}^T.\boldsymbol{x}_L+n_i\\
&y_i=\boldsymbol{h}_{1,i}^T.(\boldsymbol{w}_1^1.s_1+\boldsymbol{w}_2^1.s_2+\ldots+\boldsymbol{w}_i^1.s_i+\ldots+\boldsymbol{w}_N^1.s_N)+\boldsymbol{h}_{2,i}^T.(\boldsymbol{w}_1^2.s_1+\boldsymbol{w}_2^2.s_2+\ldots+\boldsymbol{w}_i^2.s_i+\ldots+\boldsymbol{w}_N^2.s_N)+\ldots\\
&+\boldsymbol{h}_{k,i}^T.(\boldsymbol{w}_1^k.s_1+\boldsymbol{w}_2^k.s_2+\ldots+\boldsymbol{w}_i^k.s_i+\ldots+\boldsymbol{w}_N^k.s_N)+\ldots+\boldsymbol{h}_{L,i}^T.(\boldsymbol{w}_1^L.s_1+\boldsymbol{w}_2^L.s_2+\ldots+\boldsymbol{w}_i^L.s_i+\ldots+\boldsymbol{w}_N^L.s_N)+n_i\\
&y_i=(\boldsymbol{h}_{1,i}^T.\boldsymbol{w}_1^1+\boldsymbol{h}_{2,i}^T.\boldsymbol{w}_1^2+\ldots+\boldsymbol{h}_{j,i}^T.\boldsymbol{w}_1^j+\ldots+\boldsymbol{h}_{L,i}^T.\boldsymbol{w}_1^L).s_1+(\boldsymbol{h}_{1,i}^T.\boldsymbol{w}_2^1+\boldsymbol{h}_{2,i}^T.\boldsymbol{w}_2^2+\ldots+\boldsymbol{h}_{j,i}^T.\boldsymbol{w}_2^j+\ldots+\boldsymbol{h}_{L,i}^T.\boldsymbol{w}_2^L).s_2+\\
&+(\boldsymbol{h}_{1,i}^T.\boldsymbol{w}_i^1+\boldsymbol{h}_{2,i}^T.\boldsymbol{w}_i^2+\ldots+\boldsymbol{h}_{j,i}^T.\boldsymbol{w}_i^j+\ldots+\boldsymbol{h}_{L,i}^T.\boldsymbol{w}_i^L).s_i\ldots+(\boldsymbol{h}_{1,i}^T.\boldsymbol{w}_N^1+\boldsymbol{h}_{2,i}^T.\boldsymbol{w}_N^2+\ldots+\boldsymbol{h}_{j,i}^T.\boldsymbol{w}_N^j+\ldots+\boldsymbol{h}_{L,i}^T.\boldsymbol{w}_N^L).s_N+n_i
\end{aligned}
\qquad (15)
$$

Mitigating the MAI implies: $\boldsymbol{w}_k^l\perp\text{vect}(\boldsymbol{h}_{l,1}^T,\boldsymbol{h}_{l,2}^T,\ldots,\boldsymbol{h}_{l,k-1}^T,\boldsymbol{h}_{l,k+1}^T,\ldots,\boldsymbol{h}_{l,N}^T)$ for $k\in[1,N]$ and $l\in[1,L]$. In the same time, maximizing the received signal in the direction of vector: $\boldsymbol{h}_{l,k}$, entails the choice:

$$\boldsymbol{w}_k^l=\boldsymbol{h}_{l,k}^*-\text{proj}_{\boldsymbol{h}_{l,k}^*}(\text{vect}(\boldsymbol{h}_{l,1},\boldsymbol{h}_{l,2},\ldots,\boldsymbol{h}_{l,k-1},\boldsymbol{h}_{l,k+1},\ldots,\boldsymbol{h}_{l,N}))/\left\|\boldsymbol{h}_{l,k}^*-\text{proj}_{\boldsymbol{h}_{l,k}^*}(\text{vect}(\boldsymbol{h}_{l,1},\boldsymbol{h}_{l,2},\ldots,\boldsymbol{h}_{l,k-1},\boldsymbol{h}_{l,k+1},\ldots,\boldsymbol{h}_{l,N}))\right\|$$

Where the notation $\text{proj}_{\boldsymbol{h}_{l,k}^*}(\text{vect}(\boldsymbol{h}_{l,1},\boldsymbol{h}_{l,2},\ldots\boldsymbol{h}_{l,k-1},\boldsymbol{h}_{l,k+1},\ldots,\boldsymbol{h}_{l,N}))$ denotes the projection of vector $\boldsymbol{h}_{l,k}^*$ on the subspace spanned by vectors: $(\boldsymbol{h}_{l,1},\boldsymbol{h}_{l,2},\ldots\boldsymbol{h}_{l,k-1},\boldsymbol{h}_{l,k+1},\ldots,\boldsymbol{h}_{l,N})$.

The computation of this solution is easily done when we use the orthonormalization process of Gram-Schmidt. The obtain the computation of $\text{proj}_{\boldsymbol{h}_{l,k}^*}(\text{vect}(\boldsymbol{h}_{l,1},\boldsymbol{h}_{l,2},\ldots\boldsymbol{h}_{l,k-1},\boldsymbol{h}_{l,k+1},\ldots,\boldsymbol{h}_{l,N}))$, we suppose at first that the vector

family $(\boldsymbol{h}_{l,1},\boldsymbol{h}_{l,2},...\boldsymbol{h}_{l,k-1},\boldsymbol{h}_{l,k+1},...,\boldsymbol{h}_{l,N})$ is free; i.e. the subspace dimension of $\boldsymbol{V}_{\boldsymbol{h}}^{(l,k)}=\text{vect}(\boldsymbol{h}_{l,1},\boldsymbol{h}_{l,2},...\boldsymbol{h}_{l,k-1},\boldsymbol{h}_{l,k+1},...,\boldsymbol{h}_{l,N})$ is $R\times(N\text{-}1)$ or $\text{Det}(\boldsymbol{h}_{1,1},\boldsymbol{h}_{1,2},...,\boldsymbol{h}_{1,k-1},\boldsymbol{h}_{1,k+1},...,\boldsymbol{h}_{1,N})\neq 0$. This is equivalent to the condition: $N-1\leq R$ or : $N\leq R+1$. In this case, we can always extract an orthonormal free vector family $(\boldsymbol{e}_{l,1},\boldsymbol{e}_{l,2},...,\boldsymbol{e}_{l,k-1},\boldsymbol{e}_{l,k+1},...,\boldsymbol{e}_{l,N})$ from $\boldsymbol{V}_{\boldsymbol{h}}$. The proof is done by induction by constructing the new basis using the well known Gram-Schmidt theorem.

Proof: The first vector $\boldsymbol{e}_{l,1}$ is given by: $\boldsymbol{e}_{l,1}=\boldsymbol{h}_{l,1}/\left\|\boldsymbol{h}_{l,1}\right\|$. Then, we take:

$$\boldsymbol{e}_{l,2}=\frac{\boldsymbol{h}_{l,2}-\text{proj}_{\boldsymbol{e}_{l,1}}(\boldsymbol{h}_{l,2})}{\left\|\boldsymbol{h}_{l,2}-\text{proj}_{\boldsymbol{e}_{l,1}}(\boldsymbol{h}_{l,2})\right\|}=(\boldsymbol{h}_{l,2}-\frac{\left\langle\boldsymbol{e}_{l,1},\boldsymbol{h}_{l,2}\right\rangle}{\left\langle\boldsymbol{e}_{l,1},\boldsymbol{e}_{l,1}\right\rangle}.\boldsymbol{e}_{l,1})/\left\|\boldsymbol{h}_{l,2}-\text{proj}_{\boldsymbol{e}_{l,1}}(\boldsymbol{h}_{l,2})\right\|$$

$$\boldsymbol{e}_{l,2}=(\boldsymbol{h}_{l,2}-\left\langle\boldsymbol{e}_{l,1},\boldsymbol{h}_{l,2}\right\rangle.\boldsymbol{e}_{l,1})/\left\|\boldsymbol{h}_{l,2}-\left\langle\boldsymbol{e}_{l,1},\boldsymbol{h}_{l,2}\right\rangle.\boldsymbol{e}_{l,1}\right\|$$

And naturally, we obtain:

$$\boldsymbol{e}_{l,m}=\frac{\boldsymbol{h}_{l,m}-\sum_{j=1}^{m-1}\text{proj}_{\boldsymbol{e}_{l,j}}(\boldsymbol{h}_{l,j})}{\left\|\boldsymbol{h}_{l,m}-\sum_{j=1}^{m-1}\text{proj}_{\boldsymbol{e}_{l,j}}(\boldsymbol{h}_{l,j})\right\|}=(\boldsymbol{h}_{l,m}-\sum_{j=1}^{m-1}\left\langle\boldsymbol{e}_{l,j},\boldsymbol{h}_{l,j}\right\rangle.\boldsymbol{e}_{l,j})/\left\|\boldsymbol{h}_{l,m}-\sum_{j=1}^{m-1}\left\langle\boldsymbol{e}_{l,j},\boldsymbol{h}_{l,j}\right\rangle.\boldsymbol{e}_{l,j}\right\|$$

$$\vdots$$

$$\boldsymbol{e}_{l,N}=(\boldsymbol{h}_{l,N}-\sum_{j=1}^{N-1}\left\langle\boldsymbol{e}_{l,j},\boldsymbol{h}_{l,j}\right\rangle.\boldsymbol{e}_{l,j})/\left\|\boldsymbol{h}_{l,N}-\sum_{j=1}^{k-1}\left\langle\boldsymbol{e}_{l,j},\boldsymbol{h}_{l,j}\right\rangle.\boldsymbol{e}_{l,j}\right\|$$

Using this new vector family it is straightforward to express the precoding normalized vectors. We find in this case:

$$\boldsymbol{w}_k^l=\boldsymbol{h}_{l,k}^*-\text{proj}_{\boldsymbol{h}_{l,k}^*}(\text{vect}(\boldsymbol{e}_{l,1},\boldsymbol{e}_{l,2},...,\boldsymbol{e}_{l,k-1},\boldsymbol{e}_{l,k+1},...,\boldsymbol{e}_{l,N}))/\left\|\boldsymbol{h}_{l,k}^*-\text{proj}_{\boldsymbol{h}_{l,k}^*}(\text{vect}(\boldsymbol{e}_{l,1},\boldsymbol{e}_{l,2},...,\boldsymbol{e}_{l,k-1},\boldsymbol{e}_{l,k+1},...,\boldsymbol{e}_{l,N}))\right\|$$

$$\boldsymbol{w}_k^l=\frac{\boldsymbol{h}_{l,k}^*-\sum_{j=1,j\neq k}^{N}\left\langle\boldsymbol{h}_{l,k}^*,\boldsymbol{e}_{l,j}\right\rangle.\boldsymbol{e}_{l,j}}{\left\|\boldsymbol{h}_{l,k}^*-\sum_{j=1,j\neq k}^{N}\left\langle\boldsymbol{h}_{l,k}^*,\boldsymbol{e}_{l,j}\right\rangle.\boldsymbol{e}_{l,j}\right\|}\quad\text{for }\forall l\in[1,L]\text{ and }\forall k\in[1,N]\,. \tag{16}$$

With this choice we obtain the received signal at MS_i:

$$y_i = (\boldsymbol{h}_{1,i}^T.\boldsymbol{w}_i^1 + \boldsymbol{h}_{2,i}^T.\boldsymbol{w}_i^2 + ... + \boldsymbol{h}_{k,i}^T.\boldsymbol{w}_i^k + ... + \boldsymbol{h}_{L,i}^T.\boldsymbol{w}_i^L).s_i + n_i$$

$$y_i = (\boldsymbol{h}_{1,i}^T.\frac{\boldsymbol{h}_{1,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{1,i}^*, \boldsymbol{e}_{1,j} \right\rangle.\boldsymbol{e}_{1,j}}{\left\| \boldsymbol{h}_{1,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{1,i}^*, \boldsymbol{e}_{1,j} \right\rangle.\boldsymbol{e}_{1,j} \right\|} + \boldsymbol{h}_{2,i}^T.\frac{\boldsymbol{h}_{2,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{2,i}^*, \boldsymbol{e}_{2,j} \right\rangle.\boldsymbol{e}_{2,j}}{\left\| \boldsymbol{h}_{2,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{2,i}^*, \boldsymbol{e}_{2,j} \right\rangle.\boldsymbol{e}_{2,j} \right\|} ... + \boldsymbol{h}_{L,i}^T.\frac{\boldsymbol{h}_{L,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{L,i}^*, \boldsymbol{e}_{L,j} \right\rangle.\boldsymbol{e}_{L,j}}{\left\| \boldsymbol{h}_{L,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{L,i}^*, \boldsymbol{e}_{L,j} \right\rangle.\boldsymbol{e}_{L,j} \right\|}).s_i + n_i$$

Using additional vectors to complement the subspace: $\text{vect}(\boldsymbol{e}_{l,1}, \boldsymbol{e}_{l,2}, ..., \boldsymbol{e}_{l,k-1}, \boldsymbol{e}_{l,k+1}, ..., \boldsymbol{e}_{l,N})$, we obtain the complete basis: $\text{vect}(\boldsymbol{e}_{l,1}, \boldsymbol{e}_{l,2}, ..., \boldsymbol{e}_{l,k-1}, \boldsymbol{e}_{l,k}, \boldsymbol{e}_{l,k+1}, ..., \boldsymbol{e}_{l,N}, \boldsymbol{e}_{l,N+1}, ..., \boldsymbol{e}_{l,R})$ and we can write: $\boldsymbol{h}_{l,i}^* = \sum_{j=1}^{R} \left\langle \boldsymbol{h}_{l,i}^*, \boldsymbol{e}_{l,j} \right\rangle.\boldsymbol{e}_{l,j}$. We obtain then:

$$y_i = (\boldsymbol{h}_{1,i}^T.\frac{\sum_{j=1}^{R} \left\langle \boldsymbol{h}_{1,i}^*, \boldsymbol{e}_{1,j} \right\rangle.\boldsymbol{e}_{1,j} - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{1,i}^*, \boldsymbol{e}_{1,j} \right\rangle.\boldsymbol{e}_{1,j}}{\left\| \boldsymbol{h}_{1,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{1,i}^*, \boldsymbol{e}_{1,j} \right\rangle.\boldsymbol{e}_{1,j} \right\|} + ... + \boldsymbol{h}_{L,i}^T.\frac{\sum_{j=1}^{R} \left\langle \boldsymbol{h}_{L,i}^*, \boldsymbol{e}_{L,j} \right\rangle.\boldsymbol{e}_{L,j} - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{L,i}^*, \boldsymbol{e}_{L,j} \right\rangle.\boldsymbol{e}_{L,j}}{\left\| \boldsymbol{h}_{L,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{L,i}^*, \boldsymbol{e}_{L,j} \right\rangle.\boldsymbol{e}_{L,j} \right\|}).s_i + n_i$$

And (assuming that each precoding vector has a normalized power equal to one for simplicity):

$$y_i = (\boldsymbol{h}_{1,i}^T.\frac{\left\langle \boldsymbol{h}_{1,i}^*, \boldsymbol{e}_{1,i} \right\rangle.\boldsymbol{e}_{1,i} + \sum_{j=N+1}^{R} \left\langle \boldsymbol{h}_{1,i}^*, \boldsymbol{e}_{1,j} \right\rangle.\boldsymbol{e}_{1,j}}{\left\| \boldsymbol{h}_{1,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{1,i}^*, \boldsymbol{e}_{1,j} \right\rangle.\boldsymbol{e}_{1,j} \right\|} + ... + \boldsymbol{h}_{L,i}^T.\frac{\left\langle \boldsymbol{h}_{L,i}^*, \boldsymbol{e}_{L,i} \right\rangle.\boldsymbol{e}_{L,i} + \sum_{j=N+1}^{R} \left\langle \boldsymbol{h}_{L,i}^*, \boldsymbol{e}_{L,j} \right\rangle.\boldsymbol{e}_{L,j}}{\left\| \boldsymbol{h}_{L,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{L,i}^*, \boldsymbol{e}_{L,j} \right\rangle.\boldsymbol{e}_{L,j} \right\|}).s_i + n_i$$

$$y_i = [\frac{\left|\left\langle \boldsymbol{h}_{1,i}, \boldsymbol{e}_{1,i} \right\rangle\right|^2 + \sum_{j=N+1}^{R} \left|\left\langle \boldsymbol{h}_{1,i}, \boldsymbol{e}_{1,j} \right\rangle\right|^2}{\left\| \boldsymbol{h}_{1,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{1,i}^*, \boldsymbol{e}_{1,j} \right\rangle.\boldsymbol{e}_{1,j} \right\|} + ... + \frac{\left|\left\langle \boldsymbol{h}_{L,i}, \boldsymbol{e}_{L,i} \right\rangle\right|^2 + \sum_{j=N+1}^{R} \left|\left\langle \boldsymbol{h}_{L,i}, \boldsymbol{e}_{L,j} \right\rangle\right|^2}{\left\| \boldsymbol{h}_{L,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{L,i}^*, \boldsymbol{e}_{L,j} \right\rangle.\boldsymbol{e}_{L,j} \right\|}].s_i + n_i \qquad (17)$$

$$y_i = [[\left\| \boldsymbol{h}_{1,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{1,i}^*, \boldsymbol{e}_{1,j} \right\rangle.\boldsymbol{e}_{1,j} \right\|] + ... + [\left\| \boldsymbol{h}_{L,i}^* - \sum_{j=1,\,j\neq i}^{N} \left\langle \boldsymbol{h}_{L,i}^*, \boldsymbol{e}_{L,j} \right\rangle.\boldsymbol{e}_{L,j} \right\|]].s_i + n_i$$

<u>Remark:</u> To derive the precoding vectors we have supposed that the vector family $(\boldsymbol{h}_{l,1}, \boldsymbol{h}_{l,2}, ...\boldsymbol{h}_{l,k-1}, \boldsymbol{h}_{l,k+1}, ..., \boldsymbol{h}_{l,N})$ is free. If this condition is not met, we can always extract a free family vector from the subspace: $\boldsymbol{V}_{\boldsymbol{h}}^{(l,k)}$. Denoting $\boldsymbol{V}_{\boldsymbol{h},N(l)}^{(l,k)} = \text{vect}(\boldsymbol{h}_{l,1}, \boldsymbol{h}_{l,2}, ...\boldsymbol{h}_{l,k-1}, \boldsymbol{h}_{l,k+1}, ..., \boldsymbol{h}_{l,N(l)})$ with $N(l) < N$ this new free family vector, we can build a new orthonormalized vector family using the Gram-Schmidt algorithm.

III. Diversity gain study

3.1: **Diversity gain study**:

To obtain an accurate estimate of the potential diversity gain, we have to characterize the random variables:

$\lambda_{k,i}^2 = \left\| \boldsymbol{h}_{k,i}^* - \sum_{j=1, j\neq i}^{N} \left\langle \boldsymbol{h}_{k,i}^*, \boldsymbol{e}_{k,j} \right\rangle . \boldsymbol{e}_{k,j} \right\|^2 = \left| \left\langle \boldsymbol{h}_{k,i}, \boldsymbol{e}_{k,i} \right\rangle \right|^2 + \sum_{j=N+1}^{R} \left| \left\langle \boldsymbol{h}_{k,i}, \boldsymbol{e}_{k,j} \right\rangle \right|^2$. To do this, we come back to the Gram-Schmidt algorithm, we have:

$$\boldsymbol{e}_{k,i} = \frac{\boldsymbol{h}_{k,i} - \sum_{j=1}^{i-1} \text{proj}_{\boldsymbol{e}_{k,j}} (\boldsymbol{h}_{k,i})}{\left\| \boldsymbol{h}_{k,i} - \sum_{j=1}^{i-1} \text{proj}_{\boldsymbol{e}_{k,j}} (\boldsymbol{h}_{k,i}) \right\|} = \mu_{k,i} . (\boldsymbol{h}_{k,i} - \sum_{j=1}^{i-1} \left\langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \right\rangle . \boldsymbol{e}_{k,j}) \quad (18)$$

With: $\mu_{k,i} = \frac{1}{\left\| \boldsymbol{h}_{k,i} - \sum_{j=1}^{i-1} \text{proj}_{\boldsymbol{e}_{k,j}} (\boldsymbol{h}_{k,i}) \right\|}$. Then, using the orthogonal basis, we have: $\boldsymbol{h}_{k,i} = \sum_{j=1}^{R} \left\langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \right\rangle . \boldsymbol{e}_{k,j}$. Putting this into (18) this yields to:

$$\boldsymbol{e}_{k,i} = \mu_{k,i} . (\sum_{j=1}^{R} \left\langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \right\rangle . \boldsymbol{e}_{k,j} - \sum_{j=1}^{i-1} \left\langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \right\rangle . \boldsymbol{e}_{k,j}) = \mu_{k,i} . \sum_{j=i}^{R} \left\langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \right\rangle . \boldsymbol{e}_{k,j} \quad (19)$$

Equation (19) means that: $\left\langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \right\rangle = 0$ for $j > i$. Using this property, we can now simplify $\lambda_{k,i}^2$.

$$\lambda_{k,i}^2 = \mu_{k,i}^2 . \left\langle \boldsymbol{h}_{k,i}, \boldsymbol{h}_{k,i} - \sum_{j=1}^{i-1} \left\langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \right\rangle . \boldsymbol{e}_{k,j} \right\rangle = \mu_{k,i}^2 . [\left\langle \boldsymbol{h}_{k,i}, \boldsymbol{h}_{k,i} \right\rangle - \sum_{j=1}^{i-1} \left| \left\langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \right\rangle \right|^2] \quad (20)$$

With equation (20) we can now obtain the characterization of $\lambda_{k,i}^2$ by induction.

For *I* = 1, we have: $\boldsymbol{e}_{k,1} = \boldsymbol{h}_{k,1} / \left\| \boldsymbol{h}_{k,1} \right\|$ so: $\lambda_{k,1}^2 = \left| \left\langle \boldsymbol{h}_{k,1}, \boldsymbol{e}_{k,1} \right\rangle \right|^2 = \left| \left\langle \boldsymbol{h}_{k,1}, \frac{\boldsymbol{h}_{k,1}}{\left\| \boldsymbol{h}_{k,1} \right\|} \right\rangle \right|^2 = \left\| \boldsymbol{h}_{k,1} \right\|^2$. Since $\boldsymbol{h}_{k,1}$ is a vector of *R* complex Gaussian random components each having a null mean value and a variance equal to 0.5, the random variable $\lambda_{k,1}^2$ may be written as: $\lambda_{k,1}^2 = \sum_{i=1}^{R} \left| h_{k,1}^{(i)} \right|^2 = \sum_{i=1}^{R} [\text{Re}(h_{k,1}^{(i)})^2 + \text{Im}(h_{k,1}^{(i)})^2]$. It is straightforward to conclude in this case that: $\lambda_{k,1}^2$ is a chi-square variable with 2.*R* degrees of freedom.

For *I* = 2, we have : $\boldsymbol{e}_{k,2} = (\boldsymbol{h}_{k,2} - \left\langle \boldsymbol{e}_{k,1}, \boldsymbol{h}_{k,2} \right\rangle . \boldsymbol{e}_{k,1}) / \left\| \boldsymbol{h}_{k,2} - \left\langle \boldsymbol{e}_{k,1}, \boldsymbol{h}_{k,2} \right\rangle . \boldsymbol{e}_{k,1} \right\|$. Hence, we obtain:

$$\lambda_{k,2}^2 = \left|\left\langle \boldsymbol{h}_{k,2}, \boldsymbol{e}_{k,2}\right\rangle\right|^2 = \left|\left\langle \boldsymbol{h}_{k,2}, (\boldsymbol{h}_{k,2} - \left\langle \boldsymbol{e}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle.\boldsymbol{e}_{k,1}) / \left\|\boldsymbol{h}_{k,2} - \left\langle \boldsymbol{e}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle.\boldsymbol{e}_{k,1}\right\|\right\rangle\right|^2$$
$$= \mu_{k,2}^2.\left\langle \boldsymbol{h}_{k,2}, (\boldsymbol{h}_{k,2} - \left\langle \boldsymbol{e}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle.\boldsymbol{e}_{k,1})\right\rangle = \mu_{k,2}^2.[\left\langle \boldsymbol{h}_{k,2}, \boldsymbol{h}_{k,2}\right\rangle - \left|\left\langle \boldsymbol{e}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle\right|^2] \quad (21)$$
$$= \mu_{k,2}^2.[\left\langle \boldsymbol{h}_{k,2}, \boldsymbol{h}_{k,2}\right\rangle - \left|\left\langle \boldsymbol{h}_{k,1} / \left\|\boldsymbol{h}_{k,1}\right\|, \boldsymbol{h}_{k,2}\right\rangle\right|^2] = \mu_{k,2}^2.[\left\langle \boldsymbol{h}_{k,2}, \boldsymbol{h}_{k,2}\right\rangle - \frac{\left|\left\langle \boldsymbol{h}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle\right|^2}{\left\|\boldsymbol{h}_{k,1}\right\|^2}]$$

To calculate the term: $\left|\left\langle \boldsymbol{h}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle\right|^2$, we can write:

$$\left\langle \boldsymbol{h}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle = \sum_{i=1}^{R}[h_{k,1}^R(i,:) + j.h_{k,1}^I(i,:)].[h_{k,2}^R(i,:) + j.h_{k,2}^I(i,:)]$$
$$= \sum_{i=1}^{R}[h_{k,1}^R(i,:).h_{k,2}^R(i,:) - h_{k,1}^I(i,:).h_{k,2}^I(i,:) + j.(h_{k,1}^R(i,:).h_{k,2}^I(i,:) + h_{k,1}^I(i,:).h_{k,2}^R(i,:))] \quad (22)$$

And this yields to:

$$\frac{\left|\left\langle \boldsymbol{h}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle\right|^2}{\left\|\boldsymbol{h}_{k,1}\right\|^2} = \frac{\sum_{i=1}^{R}[h_{k,1}^R(i,:).h_{k,2}^R(i,:) - h_{k,1}^I(i,:).h_{k,2}^I(i,:)]^2 + [h_{k,1}^R(i,:).h_{k,2}^I(i,:) + h_{k,1}^I(i,:).h_{k,2}^R(i,:)]^2}{\sum_{i=1}^{R}[\left|h_{k,1}^R(i,:)\right|^2 + \left|h_{k,1}^I(i,:)\right|^2]}$$

Taking the mean of this expression and using the approximation [8]: $E[\frac{\left|\left\langle \boldsymbol{h}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle\right|^2}{\left\|\boldsymbol{h}_{k,1}\right\|^2}] \approx \frac{E[\left|\left\langle \boldsymbol{h}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle\right|^2]}{E[\left\|\boldsymbol{h}_{k,1}\right\|^2]}$, we arrive at:

$$E[\frac{\left|\left\langle \boldsymbol{h}_{k,1}, \boldsymbol{h}_{k,2}\right\rangle\right|^2}{\left\|\boldsymbol{h}_{k,1}\right\|^2}] \approx \frac{E[\sum_{i=1}^{R}[h_{k,1}^R(i,:).h_{k,2}^R(i,:) - h_{k,1}^I(i,:).h_{k,2}^I(i,:)]^2 + [h_{k,1}^R(i,:).h_{k,2}^I(i,:) + h_{k,1}^I(i,:).h_{k,2}^R(i,:)]^2]}{E[\sum_{i=1}^{R}[\left|h_{k,1}^R(i,:)\right|^2 + \left|h_{k,1}^I(i,:)\right|^2]]}$$
$$= \frac{\sum_{i=1}^{R} E[\left|h_{k,1}^R(i,:).h_{k,2}^R(i,:)\right|^2] + E[\left|h_{k,1}^I(i,:).h_{k,2}^I(i,:)\right|^2] + E[\left|h_{k,1}^R(i,:).h_{k,2}^I(i,:)\right|^2] + E[\left|h_{k,1}^I(i,:).h_{k,2}^R(i,:)\right|^2]}{\sum_{i=1}^{R} E[\left|h_{k,1}^R(i,:)\right|^2] + E[\left|h_{k,1}^I(i,:)\right|^2]} \quad (23)$$
$$= 1$$

This entails that the random variable $\lambda_{k,2}^2$ is a chi-square variable with 2.(*R*-1) degrees of freedom. Let then suppose that the random variable $\lambda_{k,i-1}^2$ is a chi-square random variable with 2.(*R*-(*i*-2)) degrees of freedom. In this case, we have:

$$\lambda_{k,i}^2 = \mu_{k,i}^2.[\langle \boldsymbol{h}_{k,i}, \boldsymbol{h}_{k,i} \rangle - \sum_{j=1}^{i-1} \left| \langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \rangle \right|^2] = \mu_{k,i}^2.[\langle \boldsymbol{h}_{k,i}, \boldsymbol{h}_{k,i} \rangle - \sum_{j=1}^{i-2} \left| \langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \rangle \right|^2 - \left| \langle \boldsymbol{e}_{k,i-1}, \boldsymbol{h}_{k,i} \rangle \right|^2] \quad (24)$$

Taking into account the former hypothesis on $\lambda_{k,i-1}^2$ we know that $\langle \boldsymbol{h}_{k,i}, \boldsymbol{h}_{k,i} \rangle - \sum_{j=1}^{i-2} \left| \langle \boldsymbol{e}_{k,j}, \boldsymbol{h}_{k,i} \rangle \right|^2$ is a chi-square variable with 2.(*R*-(*i*-2)) degrees of freedom, and subtracting the quantity $\left| \langle \boldsymbol{e}_{k,i-1}, \boldsymbol{h}_{k,i} \rangle \right|^2$ we will find exactly in the same way as in equation (23) that $\lambda_{k,i}^2$ is a chi-square random variable with 2.(*R*-(*i*-1)) degrees of freedom.

Conclusion: We have proved by induction that the random variable $\lambda_{k,i}^2$ is a chi-square random variable with 2.(*R*-(*i*-1)) degrees of freedom.

It is possible to check this property using the EM algorithm [15]. We want to identify the pdf of random variables $\lambda_{k,i}^2$ and we guess that Gamma law will be the best approximation. In this case, the pdf of $\lambda_{k,i}^2$ will be approximated by:

$$f_C(x) = \sum_{j=1}^{J} \pi_j . \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} . x^{\alpha_j - 1} . e^{-\beta_j . x} = \sum_{j=1}^{J} \pi_j . g(x, \alpha_j, \beta_j) \quad (25)$$

With: $g(x, \alpha_j, \beta_j) = \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} . x^{\alpha_j - 1} . e^{-\beta_j . x}$ denoting a Gamma law with parameters: (α_j, β_j). π_j represents the weight of $g(x, \alpha_j, \beta_j)$ in the complete mixture. We have, of course: $\pi_j > 0$ and $\sum_{j=1}^{J} \pi_j = 1$. The maximum likelihood (ML) estimate of the parameters $\boldsymbol{\theta} = \{\pi_j, \alpha_j, \beta_j, j = 1, ..., J\}$ based on the observations $\boldsymbol{\Xi} = \{\xi_i, i = 1, ..., n\}$ can be written s follows:

$$\begin{aligned} \hat{\boldsymbol{\theta}} &= \arg \max_{\boldsymbol{\theta}: \sum_{j=1}^{J} \pi_j = 1} \log p_{\boldsymbol{\theta}}(\boldsymbol{\Xi}) \\ &= \arg \max_{\boldsymbol{\theta}: \sum_{j=1}^{J} \pi_j = 1} \sum_{i=1}^{n} \log \sum_{j=1}^{J} \pi_j . g(\xi_i, \alpha_j, \beta_j) \end{aligned} \quad (26)$$

Direct solution to (26) is clearly of prohibitive complexity. The EM algorithm [16-17] is an iterative procedure for solving this ML estimation problem. In the EM algorithm the set of observation data ξ_i is termed as incomplete data. Starting from some initial estimate $\boldsymbol{\theta}^{(0)}$, the EM algorithm solves the ML estimation problem

(26) by the following iterative procedure:

- Expectation Step (E-step): Compute

$$Q(\boldsymbol{\theta}\left|\boldsymbol{\theta}^{(i)}\right.) = E_{\boldsymbol{\theta}^{(i)}}\left\{\log p_{\boldsymbol{\theta}}(\boldsymbol{X})\left|\boldsymbol{\Xi}\right.\right\} \quad (27)$$

- Maximization Step (M-step): Solve

$$\boldsymbol{\theta}^{(i+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}\left|\,\boldsymbol{\theta}^{(i)}\right.) \quad (28)$$

We define then the following hidden data $\boldsymbol{Z} = \{\mathbf{z}_i, i = 1,...,n\}$ where $\mathbf{z}_i$ is a *J*-dimensional indicator vector such that:

$$z_{i,j} = \begin{cases} 1, \text{ si } \xi_i \cong g(\alpha_j, \beta_j) \\ 0 \text{ otherwise} \end{cases} \quad (29)$$

The complete data is hen defined as: $\boldsymbol{X} \cong (\boldsymbol{\Xi}, \boldsymbol{Z})$. We have:

$$p_{\boldsymbol{\theta}}(\boldsymbol{\Xi}, \boldsymbol{Z}) = \prod_{i=1}^{n}\prod_{j=1}^{J}\left[\pi_j . g(\xi_i; \alpha_j, \beta_j)\right]^{z_{i,j}} \quad (30)$$

The log-likelihood of the complete data is then given by:

$$\begin{aligned} \log p_{\boldsymbol{\theta}}(\boldsymbol{\Xi}, \boldsymbol{Z}) = &\sum_{i=1}^{n}\sum_{j=1}^{J} z_{i,j}.\log \pi_j + \sum_{i=1}^{n}\sum_{j=1}^{J} z_{i,j}.[\alpha_j.\log(\beta_j) \;-\; \log(\Gamma(\alpha_j)) + ... \\ &+(\alpha_j - 1).\log(\xi_i) - \beta_j.\xi_i + C \end{aligned} \quad (31)$$

Where C is some constant. The E-step can then be calculated as follows:

$$Q(\boldsymbol{\theta}\left|\boldsymbol{\theta}'\right.) = E'_{\boldsymbol{\theta}}\left\{\log p_{\boldsymbol{\theta}}(\boldsymbol{\Xi}, \boldsymbol{Z})\left|\boldsymbol{\Xi}\right.\right\}$$

$$\begin{aligned} Q(\boldsymbol{\theta}\left|\boldsymbol{\theta}'\right.) = &\sum_{i=1}^{n}\sum_{j=1}^{J} \hat{z}_{i,j}.\log \pi_j + \sum_{i=1}^{n}\sum_{j=1}^{J} \hat{z}_{i,j}.[\alpha_j.\log(\beta_j) \;-\; \log(\Gamma(\alpha_j)) + ... \\ &+(\alpha_j - 1).\log(\xi_i) - \beta_j.\xi_i + C \end{aligned} \quad (32)$$

With :

$$\begin{aligned} \hat{z}_{i,j} &= E'_{\boldsymbol{\theta}}\left\{z_{i,j}\left|\boldsymbol{\Xi}, \boldsymbol{\theta}'\right.\right\} = P'_{\boldsymbol{\theta}}\left\{z_{i,j} = 1\left|\xi_i\right.\right\} \\ &= \frac{p'_{\boldsymbol{\theta}}(\xi_i\left|z_{i,j} = 1\right.).P'_{\boldsymbol{\theta}}(z_{i,j} = 1)}{\sum_{l=1}^{J} p'_{\boldsymbol{\theta}}(\xi_i\left|z_{i,l} = 1\right.).P'_{\boldsymbol{\theta}}(z_{i,l} = 1)} \\ &= \frac{\gamma(\xi_i; \alpha'_j, \beta'_j).\pi'_j}{\sum_{l=1}^{J}\gamma(\xi_i; \alpha'_j, \beta'_j).\pi'_l} \end{aligned} \quad (33)$$

For the M-step, we have to compute the following derivative functions:

$$\frac{\partial \log p_{\boldsymbol{\theta}}(\Xi, \boldsymbol{Z})}{\partial \alpha_j} = \sum_{i=1}^{n} \hat{z}_{i,j}.[\log(\beta_j) - \frac{\partial \Gamma(\alpha_j)/\partial \alpha_j}{\Gamma(\alpha_j)} + \log(\xi_i)] \quad (34)$$

With:

$$\frac{\partial \Gamma(\alpha_j)/\partial \alpha_j}{\Gamma(\alpha_j)} = \frac{\frac{\partial}{\partial \alpha_j}[\int_0^{+\infty} u^{\alpha_j - 1}.e^{-u}.du]}{\int_0^{+\infty} u^{\alpha_j - 1}.e^{-u}.du} = \frac{\int_0^{+\infty} \frac{\partial e^{(\alpha_j - 1).\ln(u)}}{\partial \alpha_j}.e^{-u}.du}{\int_0^{+\infty} u^{\alpha_j - 1}.e^{-u}.du} = \frac{\int_0^{+\infty} \ln(u).u^{\alpha_j - 1}.e^{-u}.du}{\int_0^{+\infty} u^{\alpha_j - 1}.e^{-u}.du}$$

$$= psi(0, \alpha_j) = \zeta(\alpha_j)$$

And:

$$\frac{\partial \log p_{\boldsymbol{\theta}}(\Xi, \boldsymbol{Z})}{\partial \beta_j} = \sum_{i=1}^{n} \hat{z}_{i,j}.[\alpha_j.\frac{1}{\beta_j} - \xi_i] = \frac{\alpha_j.\sum_{i=1}^{n} \hat{z}_{i,j} - \beta_j.\sum_{i=1}^{n} \xi_i.\hat{z}_{i,j}}{\beta_j}$$

Setting the above derivatives to zero, we obtain the set of non-linear equations:

$$\begin{aligned} &\frac{\partial \log p_{\boldsymbol{\theta}}(\Xi, \boldsymbol{Z})}{\partial \alpha_j} = 0 \;\; => \log(\beta_j).\sum_{i=1}^{n} \hat{z}_{i,j} = \sum_{i=1}^{n} (\zeta(\alpha_j) - \log(\xi_i)).\hat{z}_{i,j} \\ &\frac{\partial \log p_{\boldsymbol{\theta}}(\Xi, \boldsymbol{Z})}{\partial \beta_j} = 0 \;\; => \alpha_j = \frac{\beta_j.\sum_{i=1}^{n} \xi_i.\hat{z}_{i,j}}{\sum_{i=1}^{n} \hat{z}_{i,j}} \end{aligned} \quad (35)$$

Using the first equation in (35) we can express β_j as : $\log(\beta_j) = \frac{\sum_{i=1}^{n} (\zeta(\alpha_j) - \log(\xi_i)).\hat{z}_{i,j}}{\sum_{i=1}^{n} \hat{z}_{i,j}}$, so we obtain:

$$\begin{aligned} &\log(\beta_j) = \frac{\sum_{i=1}^{n} (\zeta(\alpha_j) - \log(\xi_i)).\hat{z}_{i,j}}{\sum_{i=1}^{n} \hat{z}_{i,j}} \\ &\alpha_j = \frac{\beta_j.\sum_{i=1}^{n} \xi_i.\hat{z}_{i,j}}{\sum_{i=1}^{n} \hat{z}_{i,j}} \end{aligned} \quad (36)$$

Substituting the value β_j from the first equation into the second one, we obtain the following equality:

$$\alpha_j = \frac{\beta_j . \sum_{i=1}^{n} \xi_i . \hat{z}_{i,j}}{\sum_{i=1}^{n} \hat{z}_{i,j}} = \frac{\exp([\zeta(\alpha_j) . \sum_{i=1}^{n} \hat{z}_{i,j} - \sum_{i=1}^{n} \log(\xi_i) . \hat{z}_{i,j}] / \sum_{i=1}^{n} \hat{z}_{i,j}) . \sum_{i=1}^{n} \xi_i . \hat{z}_{i,j}}{\sum_{i=1}^{n} \hat{z}_{i,j}} \quad (37)$$

We obtain an equation of the equivalent form: $x = f(x)$ which can be easily solved using the well known Newton-Raphson method.

For example, we test the E.M. based algorithm with J = 1 (Gamma pure law) to test the behavior of variables $\lambda_{k,i}^2$. We take R = 10, N = 5 and we found the following result for $\lambda_{k,1}^2$ on Fig. 3 and for $\lambda_{k,2}^2$ on Fig. 4 (green points denote experimental trials while blue ones represent the EM based approximated pdf). The curve on the right is obtained in log-log scale to compare the slopes of the two curves.

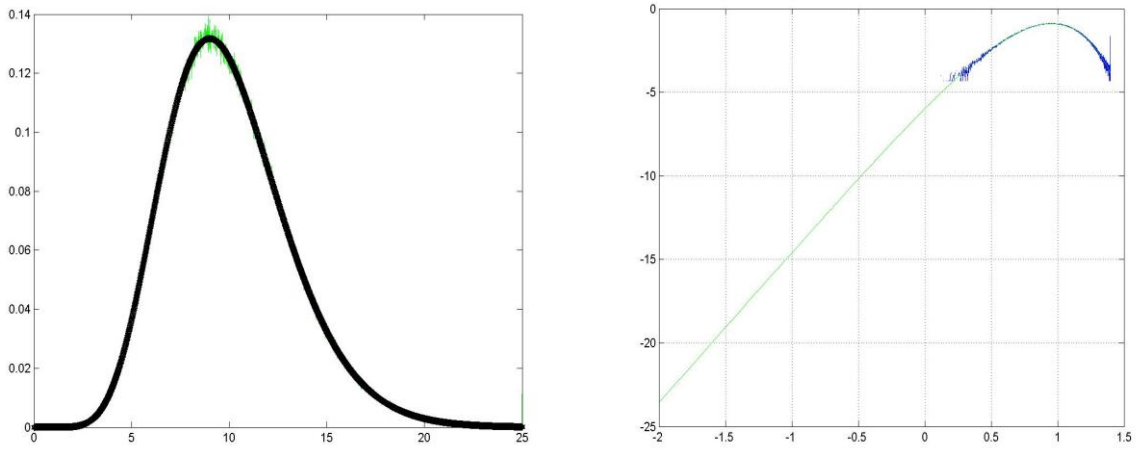


Fig. 3: p.d.f of $\lambda_{k,1}^2$ for R = 10, N = 5

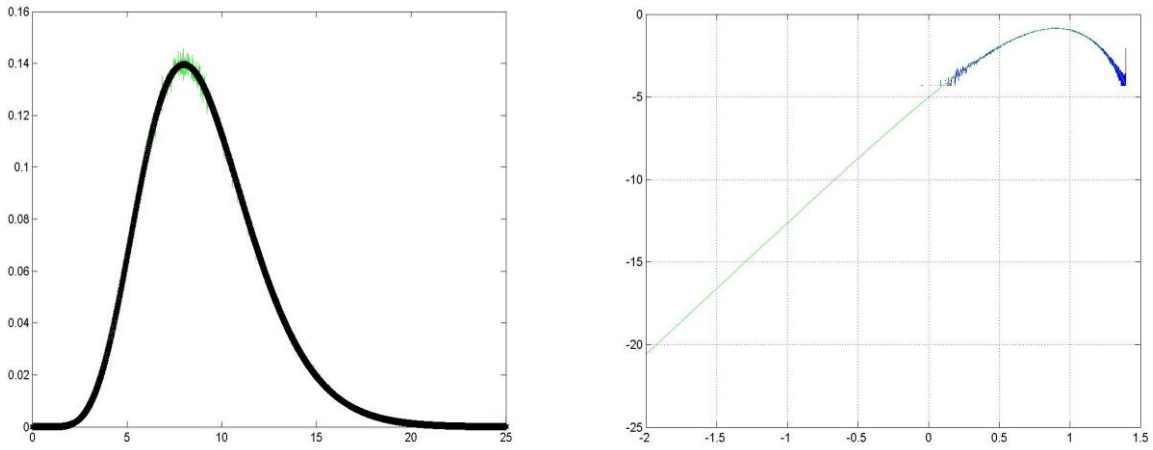


Fig. 4: p.d.f of $\lambda_{k,2}^2$ for R = 10, N = 5

We obtain in this case: $\alpha = 10$ and $\beta = 1$ for Fig. 3 and $\alpha = 9$ and $\beta = 1$ for Fig. 4. This means that we have

validated the fact that $\lambda_{k,i}^2$ follows a chi-square law with 2.(*R*-(*i*-1)) degrees of freedom (since each degrees of freedom contributes to 0.5 in the diversity order *D*, the diversity order is equal to $D = R-(i-1)-1$). Using this result, we are now ready to compute the potential diversity gain at each MS. We have the relationship:

$$y_i = [\sum_{k=1}^{L} \lambda_{k,i}].s_i + n_i \quad (38)$$

The SNR is then defined as: $\rho_i = \frac{[\sum_{k=1}^{L} \lambda_{k,i}]^2}{\sigma^2} = \frac{\sum_{k=1}^{L} \lambda_{k,i}^2 + \sum_{k}\sum_{j\neq k} \lambda_{k,i}.\lambda_{j,i}}{\sigma^2}$. It is clear that the random variable $Z_1 = \sum_{k=1}^{L} \lambda_{k,i}^2$ is a chi-square variable with 2.*L*.(*R*-(*i*-1)) degrees of freedom. The variable $Z = \sum_{k=1}^{L} \lambda_{k,i}$ is a sum of independent identically distributed (i.i.d) Rayleigh variables and it is difficult to obtain an accurate estimate of its p.d.f [18], so the study of the p.d.f of ρ_i is a complicated task too. In appendix I, we give a proof by induction that ρ_i has a diversity order equal to: *L*.(*R*-(*i*-1))-1. To confirm this, we use the E.M based algorithm described just before with a pure gamma law with p.d.f : $g(x,\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}.x^{\alpha-1}.e^{-\beta.x}$. For example, we obtain for *R* = 10, *N* = 5, *L* = 5 and: $\rho_2 = \frac{[\sum_{k=1}^{L} \lambda_{k,2}]^2}{\sigma^2}$, the values: $\alpha = 44.5, \beta = 0.2$ on Fig. 5 just below which yields to a diversity order of: 43.5. This confirms that the diversity order is nearly equal to: *L*.(*R*-(*i*-1))-1 = 5×(10-1)-1 = 44 in this case. In fact, for each scenario we tested with the E.M algorithm, we found a diversity order very close to: *L*.(*R*-(*i*-1)).

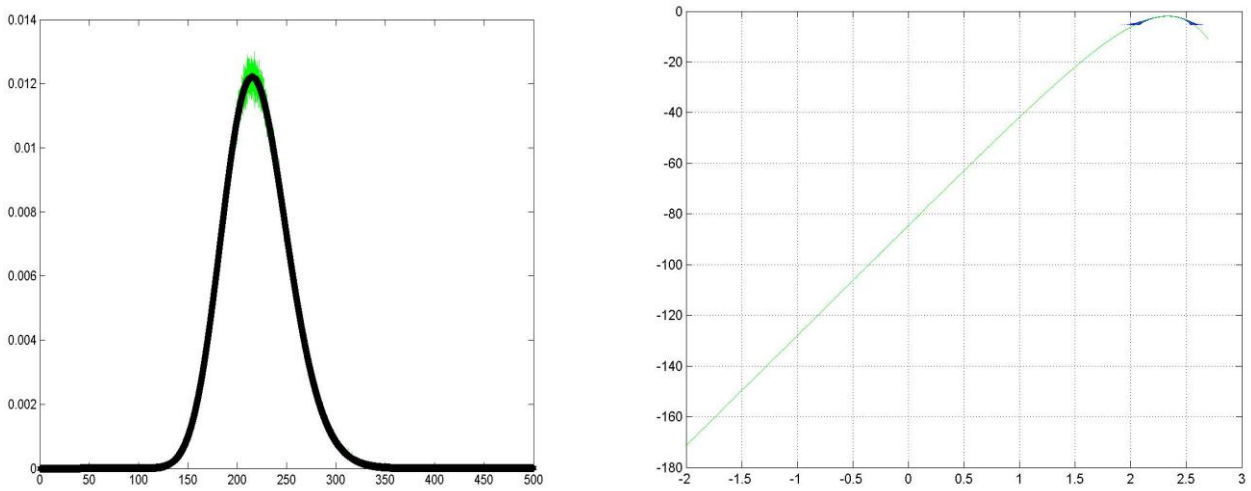


Fig. 5: p.d.f of ρ_i for $R = 10$, $N = 5$, $L = 5$
and its approximation with E.M based algorithm

IV. Optimized Power Allocation and theoretical performance derivations

4.1 **Power Allocation optimization**:

When we assume the solution exhibited in (17), we can define the average transmitted power of each relay as:

$$\begin{aligned} &< P_i >= E[\boldsymbol{x}_i^H.\boldsymbol{x}_i] = E[(\sum_{j=1}^{N}\boldsymbol{w}_j^i.s_j)^H(\sum_{j=1}^{N}\boldsymbol{w}_j^i.s_j)] = E[\sum_j\sum_k s_j^*\boldsymbol{w}_j^{iH}.\boldsymbol{w}_k^i.s_k] \\ &< P_i >= \sum_{k=1}^{N}E[s_k^*\boldsymbol{w}_k^{iH}.\boldsymbol{w}_k^i.s_k] = \sum_{k=1}^{N}E[\boldsymbol{w}_k^{iH}.\boldsymbol{w}_k^i].E[s_k^*.s_k] = \sum_{k=1}^{N}E[\boldsymbol{w}_k^{iH}.\boldsymbol{w}_k^i] = N \end{aligned} \quad (39)$$

Hence, when we normalize the transmitted power of each relay to one: $< P_i >=1$, the precoding vectors are written as:

$$\boldsymbol{w}_k^{'l} = \frac{1}{\sqrt{N}}.\boldsymbol{w}_k^l = \frac{1}{\sqrt{N}}.\frac{[\boldsymbol{h}_{l,k}^* - \sum_{j=1,j\neq k}^{N}\langle \boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,j}\rangle.\boldsymbol{e}_{l,j}]}{\left\|\boldsymbol{h}_{l,k}^* - \sum_{j=1,j\neq k}^{N}\langle \boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,j}\rangle.\boldsymbol{e}_{l,j}\right\|} \quad (40)$$

Hence, we obtain: $\lambda_{k,i}^{'} = \frac{1}{\sqrt{N}}.\lambda_{k,i}$ (see equation (20)).

In the case where each relay k has a transmit power P_k the received signal is written as:

$$y_i = [\sum_{k=1}^{L} \sqrt{P_k}.\lambda_{k,i}^{'}].s_i + n_i \quad (41)$$

The signal to noise ratio is equal to: $\rho_i = \frac{(\sum_{k=1}^{L} \sqrt{P_k}.\lambda_{k,i}^{'})^2}{\sigma^2}$. Using this and the approximation of the symbol error probability (SER) given in [19] by:

$$\begin{aligned} &P_{e_i}(\rho_i) = 2.\gamma.erfc(\sqrt{\xi.\rho_i}) - [\gamma.erfc(\sqrt{\xi.\rho_i})]^2 \\ &P_{e_i}(\rho_i) \approx 2.\gamma.erfc(\sqrt{\xi.\rho_i}) \quad \text{at the considered SNR's} \end{aligned} \quad (42)$$

We have $\gamma = 1 - 1/\sqrt{M}, \xi = 3/2.(M-1)$ for M-QAM constellations with $M = 2^k$ and $erfc(x) = \frac{2}{\sqrt{\pi}}.\int_{x}^{+\infty} e^{-t^2/2}.dt$ is the complementary error function. We can optimize the average SER for the complete set of mobile stations. In this case, the Power Allocation (P.A) problem can be described as:

$$\begin{cases} \min \ \ \bar{P}_e = (1/N).\sum_{i=1}^{N} P_{e_i}(\rho_i) = (1/N).2.\gamma.\sum_{i=1}^{N} erfc(\sqrt{\xi.\rho_i}) \\ \text{subject to}: \sum_{k=1}^{L} P_k = P_T = C^{ste} \end{cases} \quad (43)$$

Consequently, the cost function can be obtained as:

$$\begin{aligned} &J(P_1, P_2, ..., P_L) = \bar{P}_e + \eta.\left(\sum_{k=1}^{L} P_k - P_T\right) \\ &J(P_1, P_2, ..., P_L) = (2.\gamma/N).\sum_{i=1}^{N} erfc(\sqrt{\xi.\rho_i}) + \eta.\left(\sum_{k=1}^{L} P_k - P_T\right) \end{aligned} \quad (44)$$

We calculate the derivative : $\partial J / \partial P_k$, we obtain :

$$\begin{aligned} \partial J / \partial P_k &= 2.(\gamma/N).\sum_{i=1}^{N} \partial erfc(\sqrt{\xi}.(\sum_{k=1}^{L} \sqrt{P_k}.\lambda_{k,i}^{'})/\sigma) / \partial P_k + \eta \\ &= (-2.\gamma\sqrt{\xi}/N.\sigma.\sqrt{\pi.P_k}).\sum_{i=1}^{N} \lambda_{k,i}^{'}.\exp[-\xi.(\sum_{k=1}^{L} \sqrt{P_k}.\lambda_{k,i}^{'})^2/\sigma^2] + \eta \end{aligned} \quad (45)$$

Setting this derivative equal to zero, we obtain the equation:

$$\eta = (2.\gamma\sqrt{\xi}/N.\sigma.\sqrt{\pi.P_k}).\sum_{i=1}^{N} \lambda_{k,i}^{'}.\exp[-\xi.(\sum_{m=1}^{L} \sqrt{P_m}.\lambda_{m,i}^{'})^2/\sigma^2] \quad \forall k \in [1, L] \quad (46)$$

This yields to:

$$\sqrt{P_k} \propto \sum_{i=1}^{N} \lambda'_{k,i}.\exp[-\xi.(\sum_{m=1}^{L} \sqrt{P_m}.\lambda'_{m,i})^2 / \sigma^2] \quad (47)$$

So, we can write:

$$P_k = A^2.[\sum_{i=1}^{N} \lambda'_{k,i}.\exp[-\xi.(\sum_{m=1}^{L} \sqrt{P_m}.\lambda'_{m,i})^2 / \sigma^2]]^2 \quad (48)$$

Rewriting equation (47) for each relay station and summing all the different equalities, we arrive at:

$$\sum_{m=1}^{L} \sqrt{P_m}.\lambda'_{m,i} = A.\sum_{m=1}^{L}\sum_{i=1}^{N} \lambda'^2_{m,i}.\exp[-\xi.(\sum_{m=1}^{L} \sqrt{P_m}.\lambda'_{m,i})^2 / \sigma^2] \quad (49)$$

Using equation (47) we aim at first at calculating: $X_i = \sum_{m=1}^{L} \sqrt{P_m}.\lambda'_{m,i}$. X_i is the solution of the equation:

$$X_i = A.\sum_{m=1}^{L}\sum_{i=1}^{N} \lambda'^2_{m,i}.\exp[-\xi.X_i^2 / \sigma^2] \quad (50)$$

The constant A may be expressed as: $\sum_{k=1}^{L} P_k = P_T$ and so this yields to:

$$\begin{aligned}
&\sum_{k=1}^{L} A^2.[\sum_{i=1}^{N} \lambda'_{k,i}.\exp[-\xi.(\sum_{m=1}^{L} \sqrt{P_m}.\lambda'_{m,i})^2 / \sigma^2]]^2 = P_T \\
&A^2.\sum_{k=1}^{L}[\sum_{i=1}^{N} \lambda'_{k,i}.\exp[-\xi.(\sum_{m=1}^{L} \sqrt{P_m}.\lambda'_{m,i})^2 / \sigma^2]]^2 = P_T \\
&A^2.\sum_{k=1}^{L}[\sum_{i=1}^{N} \lambda'_{k,i}.\exp[-\xi.X_i^2 / \sigma^2]]^2 = P_T \\
&A = \sqrt{P_T} / \sqrt{\sum_{k=1}^{L}[\sum_{i=1}^{N} \lambda'_{k,i}.\exp[-\xi.X_i^2 / \sigma^2]]^2}
\end{aligned} \quad (51)$$

Finally, we obtain the equation:

$$X_i = \sqrt{P_T}.\sum_{m=1}^{L}\sum_{i=1}^{N} \lambda'^2_{m,i}.\exp[-\xi.X_i^2 / \sigma^2] / \sqrt{\sum_{k=1}^{L}[\sum_{i=1}^{N} \lambda'_{k,i}.\exp[-\xi.X_i^2 / \sigma^2]]^2} \quad i = 1,...,N$$

It is possible to solve this set of non-linear equations using the primitive function *fsolve*. As soon as the quantities X_i are found it is possible to compute the optimum power P_k:

$$P_k = P_T.\frac{[\sum_{i=1}^{N} \lambda'_{k,i}.\exp[-\xi.X_i^2 / \sigma^2]]^2}{\sum_{k=1}^{L}[\sum_{i=1}^{N} \lambda'_{k,i}.\exp[-\xi.X_i^2 / \sigma^2]]^2} \quad (52)$$

The calculation presented here is complicated to implement. To simplify it we can use an approximation at low SNR's. The solution found for the low SNR regime can be generalized at high SNR's as the simulation results highlight it on Section V. If we consider the low SNR regime we have $\exp[-\xi.(\sum_{k=1}^{L}\sqrt{P_k}.\lambda_{k,i}^{'})^2/\sigma^2]\approx 1$ and equation (48) reduces to:

$$P_k = A^2.[\sum_{i=1}^{N}\lambda_{k,i}^{'}]^2 \quad (53)$$

And the constraint on the overall transmit power yields to:

$$\sum_{k=1}^{L}P_k = A^2.\sum_{k=1}^{L}[\sum_{i=1}^{N}\lambda_{k,i}^{'}]^2 = P_T \Rightarrow A^2 = \frac{P_T}{\sum_{k=1}^{L}[\sum_{i=1}^{N}\lambda_{k,i}^{'}]^2} \quad (54)$$

And we found:

$$P_k = A^2.[\sum_{i=1}^{N}\lambda_{k,i}^{'}]^2 = \frac{P_T}{\sum_{k=1}^{L}[\sum_{i=1}^{N}\lambda_{k,i}^{'}]^2}.[\sum_{i=1}^{N}\lambda_{k,i}^{'}]^2 \quad (55)$$

In this case, the received signal at MS *I* takes the form:

$$y_i = (\sum_{k=1}^{L}\sqrt{P_k}.\lambda_{k,i}^{'}).s_i + n_i$$

$$y_i = (\sum_{k=1}^{L}\sqrt{\frac{P_T}{\sum_{k=1}^{L}[\sum_{j=1}^{N}\lambda_{k,j}^{'}]^2}.[\sum_{j=1}^{N}\lambda_{k,j}^{'}]^2.\lambda_{k,i}^{'}}).s_i + n_i$$

$$y_i = \sqrt{P_T}\frac{\left[\sum_{k=1}^{L}[\sum_{j=1}^{N}\lambda_{k,j}^{'}].\lambda_{k,i}^{'}\right]}{\sqrt{\sum_{k=1}^{L}[\sum_{j=1}^{N}\lambda_{k,j}^{'}]^2}}.s_i + n_i \quad (56)$$

4.2 **Theoretical SER performance derivations**:

We consider here two cases: the first one corresponds to the case of equal uniform power allocation whose diversity gain has already been studied at the end of Section III and the second one concerns the case of power optimization just given in the last section.

In the first case, we have demonstrated that a gamma law with p.d.f: $\gamma(x,\alpha,\beta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}.x^{\alpha-1}.e^{-\beta.x}$ was a good

approximation of the random variable: $[\sum_{k=1}^{L}\lambda_{k,i}^{'}]^2$ with: $\alpha-1=L.q_i-1=L.(R-(i-1))-1$ and $\beta=1/L$. The SNR is given by: $\rho_i=\frac{[\sum_{k=1}^{L}\lambda_{k,i}^{'}]^2}{\sigma^2}$ with $\sigma^2=P_u.10^{-SNR_{dB}/10}$ and: $P_u=E[(\sum_{k=1}^{L}\lambda_{k,i}^{'})^2]$. We have:

$$P_u \approx \frac{\beta^\alpha}{\Gamma(\alpha)}.\int_0^{+\infty} x.x^{\alpha-1}.e^{-\beta.x}.dx=\frac{\beta^\alpha}{\Gamma(\alpha)}.\int_0^{+\infty} x^\alpha.e^{-\beta.x}.dx=\frac{\beta^\alpha}{\Gamma(\alpha).\beta^{\alpha+1}}.\int_0^{+\infty} u^\alpha.e^{-u}.du$$
$$=\frac{1}{\Gamma(\alpha).\beta}.\Gamma(\alpha+1)=L.\alpha$$

Its p.d.f can be well approximated by:

$$h(x)=\sigma^2.\gamma(\sigma^2.x,\alpha,\beta)=\frac{\sigma^2.\beta^\alpha}{\Gamma(\alpha)}.[\sigma^2.x]^{\alpha-1}.e^{-\beta.\sigma^2.x}=\frac{\sigma^{2\alpha}.\beta^\alpha}{\Gamma(\alpha)}.x^{\alpha-1}.e^{-\beta.\sigma^2.x} \quad (57)$$

The average Symbol Error Rate (SER) at a given SNR or σ^2 value is then given by:

$$\bar{P}_e=\int_0^{+\infty} P_e(u).h(u).du=\frac{2.\gamma.\sigma^{2\alpha}.\beta^\alpha}{\Gamma(\alpha)}.\int_0^{+\infty} u^{\alpha-1}.e^{-\beta.\sigma^2.u}.erfc(\sqrt{\xi.u}).du \quad (58)$$

We have $\gamma=1-1/\sqrt{M}, \xi=3/2.(M-1)$ for M-QAM constellations with $M=2^k$.To calculate (58) we can remark that α-1 is an integer value i.e. $\alpha-1=n_i$. We calculate at first:

$$I_{\alpha,\beta}=\int_0^{+\infty} u^{\alpha-1}.e^{-\beta.\sigma^2.u}.erfc(\sqrt{\xi.u}).du$$

We set at first: $t=\beta.\sigma^2.u$, i.e. $dt=\beta.\sigma^2.du$; we obtain:

$$I_{\alpha,\beta}=\int_0^{+\infty}[\frac{t}{\beta.\sigma^2}]^{\alpha-1}.e^{-t}.erfc(\sqrt{\xi.t/\beta.\sigma^2}).\frac{dt}{\beta.\sigma^2}$$
$$=\frac{1}{[\beta.\sigma^2]^\alpha}.\int_0^{+\infty} t^{\alpha-1}.e^{-t}.erfc(\sqrt{\xi.t/\beta.\sigma^2}).dt \quad (59)$$

Then, we integrate by parts, setting:

$$\mathrm{v}=erfc(\sqrt{\xi.t/\beta.\sigma^2}); d\mathrm{v}=\frac{-2}{\sqrt{\pi}}.\sqrt{\xi/\beta.\sigma^2}.\frac{1}{2.\sqrt{t}}.e^{-\xi.t/\beta.\sigma^2}.dt=-\frac{1}{\sqrt{\pi}}.\sqrt{\xi/\beta.\sigma^2}.\frac{e^{-\xi.t/\beta.\sigma^2}}{\sqrt{t}}.dt$$

and $d\mathrm{w}=t^{\alpha-1}.e^{-t}.dt$. The primitive function of w is defined as: $\mathrm{w}(t)=-\sum_{k=0}^{\alpha-1}\frac{(\alpha-1)!}{k!}.t^k.e^{-t}$. We obtain then:

$$I_{\alpha,\beta} = \frac{1}{[\beta.\sigma^2]^{\alpha}}.\int_0^{+\infty} t^{\alpha-1}.e^{-t}.erfc(\sqrt{\xi.t/\beta.\sigma^2}).dt$$

$$= \frac{1}{[\beta.\sigma^2]^{\alpha}}.[(\alpha-1)! - \sum_{k=0}^{\alpha-1}\int_0^{+\infty}\frac{(\alpha-1)!}{k!}.t^k.e^{-t}.\frac{\sqrt{\xi/\beta.\sigma^2}}{\sqrt{\pi}}.\frac{e^{-\xi.t/\beta.\sigma^2}}{\sqrt{t}}.dt] \quad (60)$$

$$= \frac{(\alpha-1)!}{[\beta.\sigma^2]^{\alpha}}.[1 - \sum_{k=0}^{\alpha-1}\sqrt{\frac{\xi}{\beta.\sigma^2}}.\frac{1}{\sqrt{\pi}.k!}.\int_0^{+\infty} t^{k-1/2}.e^{-t(1+\xi/\beta.\sigma^2)}.dt]$$

To calculate integral form: $\int_0^{+\infty} t^{k-1/2}.e^{-t(1+\xi/\beta.\sigma^2)}.dt$, we set: $x = (1+\xi/\beta.\sigma^2).t$ i.e. $dx = (1+\xi/\beta.\sigma^2).dt$:

$$\int_0^{+\infty} t^{k-1/2}.e^{-t(1+1/\sigma^2)}.dt = \int_0^{+\infty}[x/(1+\xi/\beta.\sigma^2)]^{k-1/2}.e^{-x}.dx/(1+\xi/\beta.\sigma^2)$$

$$= 1/(1+\xi/\beta.\sigma^2)^{k+1/2}.\int_0^{+\infty} x^{k-1/2}.e^{-x}.dx$$

$$= 1/(1+\xi/\beta.\sigma^2)^{k+1/2}.\Gamma(k+1/2)$$

With: $\left(\Gamma(z) = \int_0^{+\infty} t^{z-1}.e^{-t}.dt\right)$, we have the following particular values:

$$\Gamma(n+\frac{1}{2}) = (n-\frac{1}{2}).(n-\frac{3}{2})...(n-\frac{2(n-1)+1}{2}).\Gamma(n-\frac{2(n-1)+1}{2}) = \frac{(2.n)!}{2^{2n}.n!}.\sqrt{\pi}$$

This yields to:

$$I_{\alpha,\beta} = \frac{(\alpha-1)!}{[\beta.\sigma^2]^{\alpha}}.[1 - \sum_{k=0}^{\alpha-1}\sqrt{\frac{\xi}{\beta.\sigma^2}}.\frac{1}{\sqrt{\pi}.k!}.\int_0^{+\infty} t^{k-1/2}.e^{-t(1+1/\sigma^2)}.dt]$$

$$= \frac{(\alpha-1)!}{[\beta.\sigma^2]^{\alpha}}.[1 - \sum_{k=0}^{\alpha-1}\sqrt{\frac{\xi}{\beta.\sigma^2}}.\frac{1}{\sqrt{\pi}.k!.(1+\xi/\beta.\sigma^2)^{k+1/2}}.\frac{(2.k)!}{2^{2k}.k!}.\sqrt{\pi}] \quad (60)$$

$$= \frac{(\alpha-1)!}{[\beta.\sigma^2]^{\alpha}}.[1 - \sum_{k=0}^{\alpha-1}\sqrt{\frac{\xi}{\beta.\sigma^2}}.\frac{(2.k)!}{2^{2k}.k!^2.(1+\xi/\beta.\sigma^2)^{k+1/2}}]$$

And, finally we have:

$$\bar{P}_e = \frac{2.\gamma.\sigma^{2\alpha}.\beta^{\alpha}}{\Gamma(\alpha)}.I_{\alpha,\beta}$$

$$= \frac{2.\gamma.\sigma^{2\alpha}.\beta^{\alpha}}{(\alpha-1)!}.\frac{(\alpha-1)!}{[\beta.\sigma^2]^{\alpha}}.[1 - \sum_{k=0}^{\alpha-1}\sqrt{\frac{\xi}{\beta.\sigma^2}}.\frac{(2.k)!}{2^{2k}.k!^2.(1+\xi/\beta.\sigma^2)^{k+1/2}}] \quad (61)$$

$$= 2.\gamma.[1 - \sqrt{\frac{\xi}{\beta.\sigma^2}}.\frac{1}{\sqrt{\xi/\beta.\sigma^2+1}}.\sum_{k=0}^{\alpha-1}\frac{(2.k)!}{2^{2k}.k!^2.(1+\xi/\beta.\sigma^2)^k}]$$

At high SNR's it's possible to simplify the expression of: $\bar{P}_e$, we have: $\sqrt{\frac{\xi/\beta.\sigma^2}{\xi/\beta.\sigma^2+1}} = \frac{1}{\sqrt{1+\sigma^2.\beta/\xi}} \approx 1-\frac{\sigma^2.\beta}{2.\xi}$

and $\frac{1}{(\xi/\beta.\sigma^2+1)^k} \approx \frac{\beta^k.\sigma^{2k}}{\xi^k}$, this entails the new expression:

$$\bar{P}_e = 2.\gamma.[1-(1-\frac{\sigma^2.\beta}{2.\xi}).\sum_{k=0}^{\alpha-1}\frac{(2.k)!.\beta^k.\sigma^{2.k}}{2^{2k}.\xi^k.k!^2}] \quad (62)$$

Conclusion: In the case where the relay radiates the same power, we have:

$$\bar{P}_{e,equal} = 2.\gamma.[1-\sqrt{\frac{\xi}{\beta.\sigma^2}}.\frac{1}{\sqrt{\xi/\beta.\sigma^2+1}}.\sum_{k=0}^{\alpha-1}\frac{(2.k)!}{2^{2k}.k!^2.(1+\xi/\beta.\sigma^2)^k}]$$

In the case of power optimization, we have: $\rho_i = \frac{\left[\sum_{k=1}^{L}[\sum_{j=1}^{N}\lambda'_{k,j}].\lambda'_{k,i}\right]^2}{\sigma^2.\sum_{k=1}^{L}[\sum_{j=1}^{N}\lambda'_{k,j}]^2}$. The p.d.f of ρ_i is difficult to track; we give in Appendix 2 the characteristics of the p.d.f of the product of two independent Rayleigh variables. Using this result, we give an approximate characterization of the p.d.f of $\sigma^2.\rho_i$ in Appendix 3. However, it is possible, once again, to use the E.M. based fitting algorithm to find, with a good precision, an approximated distribution for the term: $\delta_k = \frac{\left[\sum_{k=1}^{L}[\sum_{j=1}^{N}\lambda'_{k,j}].\lambda'_{k,i}\right]^2}{\sum_{k=1}^{L}[\sum_{j=1}^{N}\lambda'_{k,j}]^2}$. For example, using Gamma pure laws at each time, we have obtained the following result. In the case where L = 2, N = 3 and R = 8, we obtained the result just below for δ_1: α = 15.81, β = 1; i.e. we obtain a chi-square law with diversity order: $D_1 \approx R.L-1$.

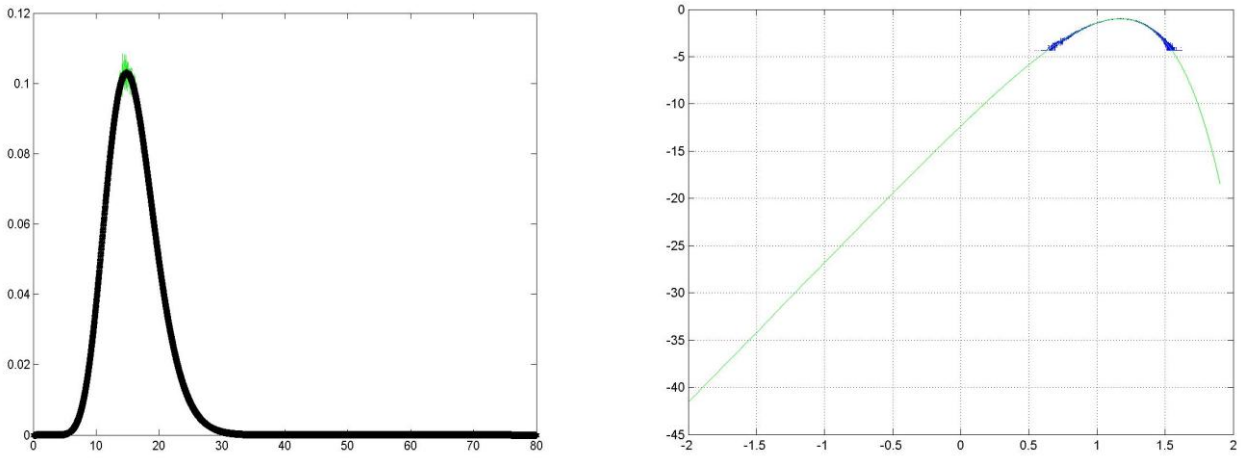


Fig. 6: characterization of the p.d.f of δ_1 with: L = 2, N = 3 and R = 8 (in log-log scale on the right side)

In the case where L = 2, N = 3 and R = 9, we obtained the result just below for δ_1: α = 17.81, β = 1; i.e. we obtain a chi-square law with diversity order: $D_1 \approx R.L-1$.

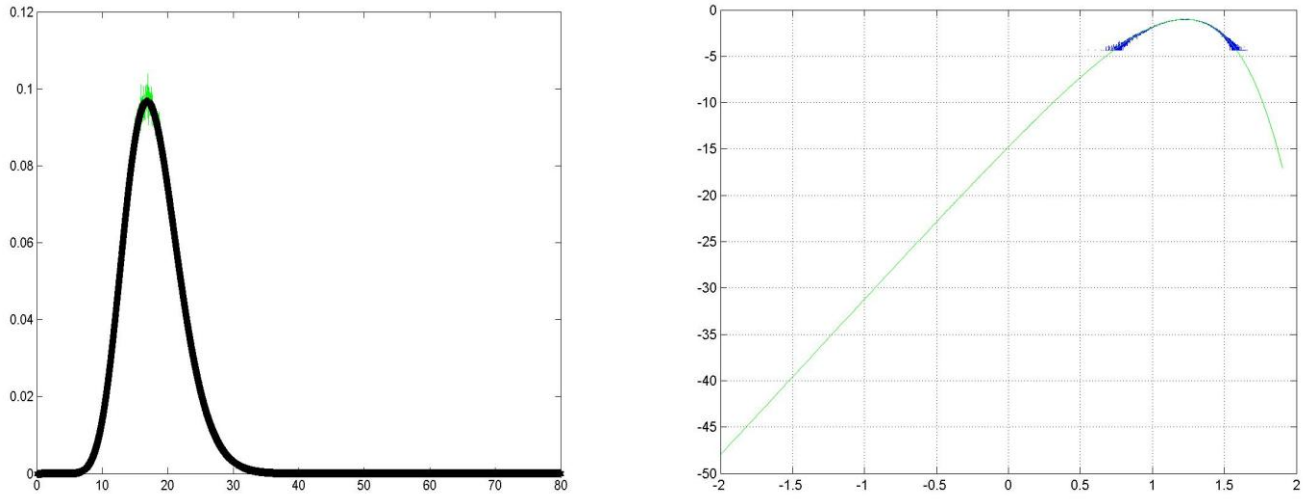


Fig. 7: characterization of the p.d.f of δ_1 with: L = 2, N = 3 and R = 9 (in log-log scale on the right side)

In the case where L = 3, N = 4 and R = 8, we obtained (Fig. 8) for δ_1: α = 23.71, β = 1; i.e. we obtain a chi-square law with diversity order: $D_1 \approx R.L-1$.

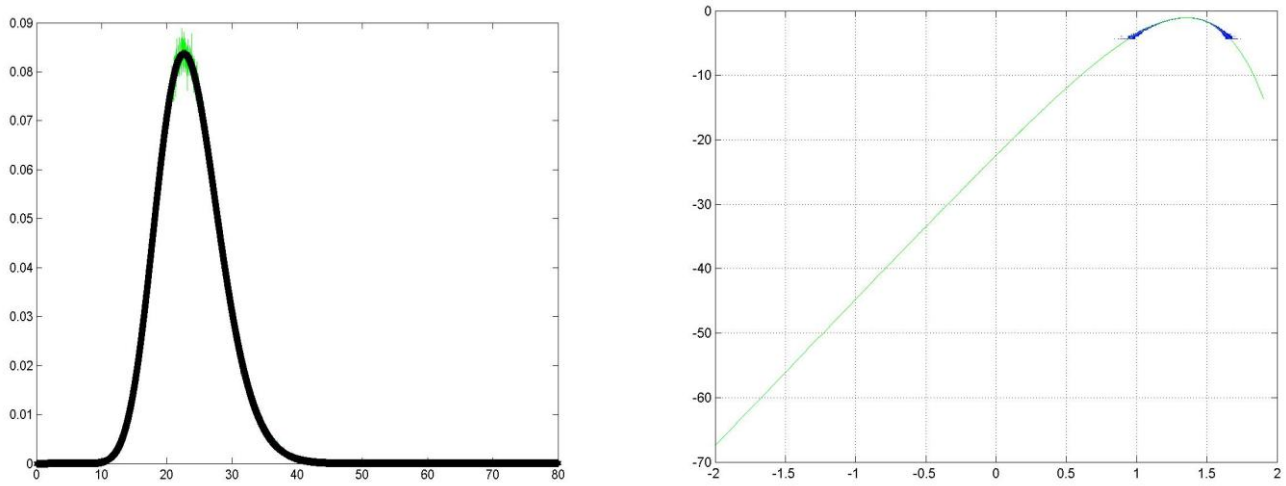


Fig. 8: characterization of the p.d.f of δ_1 with: L = 3, N = 4 and R = 8 (in log-log scale on the right side)

In the case where the ranking of the MS's is equal to two, we found the following result on Fig. 9. With L = 2, N = 3 and R = 8, we obtained for δ_2 : α = 13.83, β = 1; i.e. we obtain a chi-square law with diversity order: $D_2 \approx (R-1).L-1$.

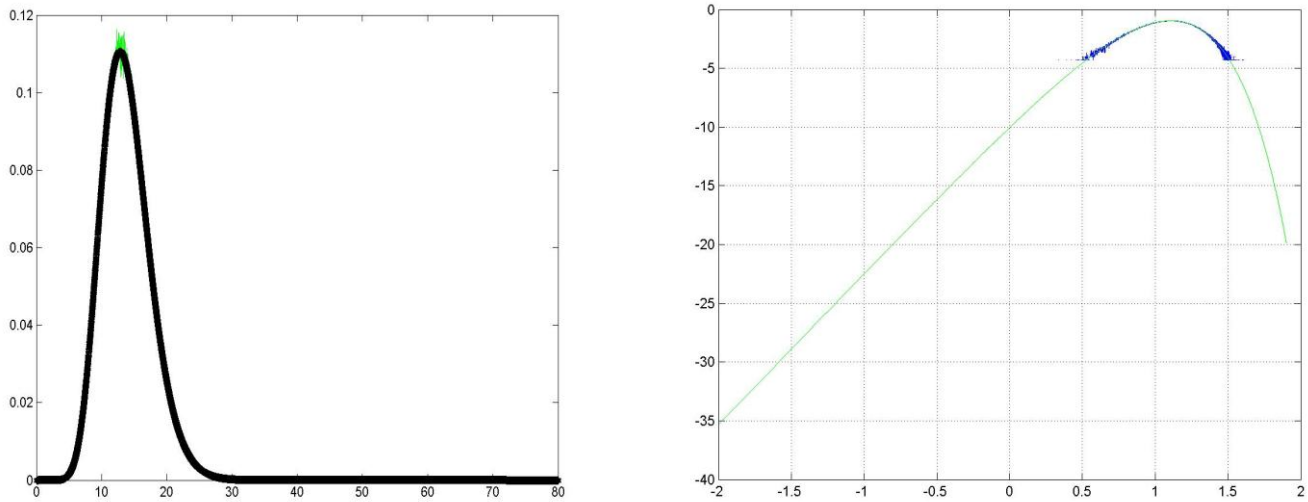


Fig. 9: characterization of the p.d.f of δ_2 with: L = 2, N = 3 and R = 8 (in log-log scale on the right side)

With L = 3, N = 4 and R = 8, we obtained (see Fig. 10) for δ_2 : α = 20.69, β = 1; i.e. we obtain a chi-square law with diversity order: $D_2 \approx (R-1).L-1$.

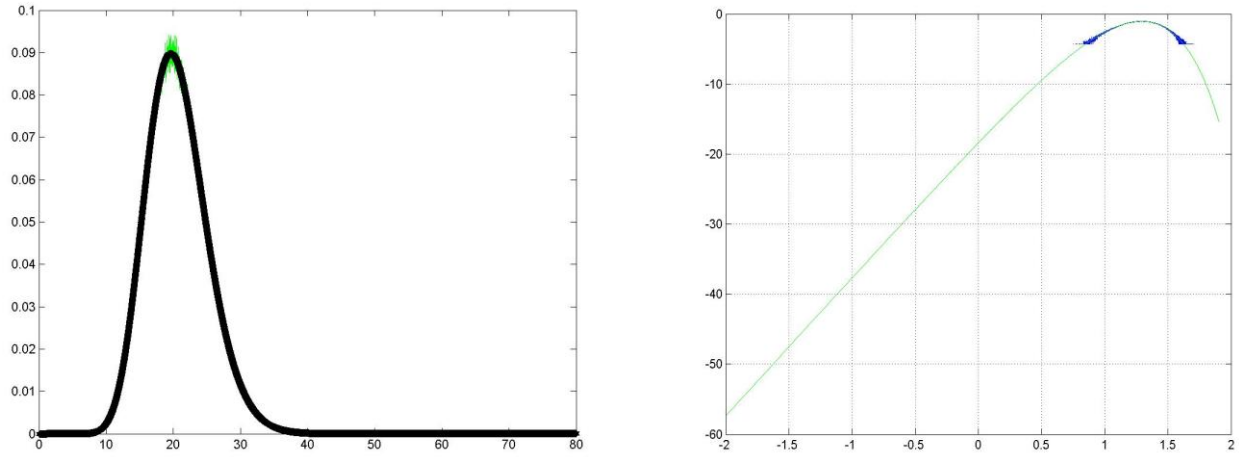


Fig. 10: characterization of the p.d.f of δ_2 with: $L = 3$, $N = 4$ and $R = 8$ (in log-log scale on the right side)

We can conclude experimentally that the variable δ_I is a chi-square random variable with diversity order: $D_i \approx [R-(i-1)].L-1 = q_i^{'}$. Its mean value is equal to:

$$P_u \approx \frac{1}{(q_i-1)!}.\int_0^{+\infty} x.x^{q_i^{'}-1}.\exp(-x).dx = \frac{1}{(q_i-1)!}.\int_0^{+\infty} x^{q_i^{'}}.\exp(-x).dx = \frac{q_i^{'}!}{(q_i^{'}-1)!} = q_i^{'}$$

In this case, it is straightforward to obtain the theoretical BER performance of this scheme. We have now:

$$\bar{P}_{e,opti} = 2.\gamma.[1-\sqrt{\frac{\xi}{\sigma^2}}.\frac{1}{\sqrt{\xi/\sigma^2+1}}.\sum_{k=0}^{\alpha-1}\frac{(2.k)!}{2^{2k}.k!^2.(1+\xi/\sigma^2)^k}] \quad (63)$$

It's in fact the same expression as in (61) but with: $\beta = 1$. Obviously, it's easy to check that: $\bar{P}_{e,opti} < \bar{P}_{e,equal}$.

Remark: There is still an interesting case to study. This situation is related to the case where we consider precoding vectors which are no more normalized. We have then:

$$\begin{aligned}\boldsymbol{w}_k^l &= \boldsymbol{h}_{l,k}^* - \sum_{j=1, j\neq k}^{N} \left\langle \boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,j} \right\rangle . \boldsymbol{e}_{l,j} \\ &= \sum_{j=1}^{R} \left\langle \boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,j} \right\rangle . \boldsymbol{e}_{l,j} - \sum_{j=1, j\neq k}^{N} \left\langle \boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,j} \right\rangle . \boldsymbol{e}_{l,j} \\ &= \left\langle \boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,k} \right\rangle . \boldsymbol{e}_{l,k} + \sum_{j=N+1}^{R} \left\langle \boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,j} \right\rangle . \boldsymbol{e}_{l,j}\end{aligned}$$

We have: $\left\langle \boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,j} \right\rangle = 0$ for $j > k$. Hence: $\boldsymbol{w}_k^l = \left\langle \boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,k} \right\rangle . \boldsymbol{e}_{l,k}$ and the average transmitted power of one relay station is written as: $< P_l > = \sum_{k=1}^{N} E[\boldsymbol{w}_k^{lH}.\boldsymbol{w}_k^l] = \sum_{k=1}^{N} E[<|\boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,k}|^2>]$. Since each random variable $|\boldsymbol{h}_{l,k}^*, \boldsymbol{e}_{l,k}|^2$ is a

chi-square random variable with $2.(R-(k-1))$ degrees of freedom, we have:

$$E[<\left|\boldsymbol{h}_{l,k}^{*}.e_{l,k}\right|^{2}>]=\frac{1}{(q_k-1)!}.\int_{0}^{+\infty}x.x^{q_k-1}.\exp(-x).dx=\frac{1}{(q_k-1)!}.\int_{0}^{+\infty}x^{q_k}.\exp(-x).dx=\frac{q_k!}{(q_k-1)!}=q_k$$

With: $q_k=R-(k-1)$. Finally, the transmitted power is equal to:

$$<P_l>=\sum_{k=1}^{N}q_k=\sum_{k=1}^{N}[R-(k-1)]=\psi(R,N) \quad (64)$$

In this case, to have an average transmitted power equal to one, we have to use precoding vectors of the form:

$$\boldsymbol{w}_k^{l}=\frac{\left\langle \boldsymbol{h}_{l,k}^{*},\boldsymbol{e}_{l,k}\right\rangle.\boldsymbol{e}_{l,k}}{\sqrt{\sum_{k=1}^{N}[R-(k-1)]}}=\frac{\left\langle \boldsymbol{h}_{l,k}^{*},\boldsymbol{e}_{l,k}\right\rangle.\boldsymbol{e}_{l,k}}{\sqrt{\psi(R,N)}} \quad (65)$$

Considering a non-uniform power allocation between the different relays, the received signal is equal to:

$$y_i=\frac{1}{\sqrt{\psi(R,N)}}[\sqrt{P_1}.\left|\left\langle \boldsymbol{h}_{1,i},\boldsymbol{e}_{1,i}\right\rangle\right|^{2}+...+\sqrt{P_2}.\left|\left\langle \boldsymbol{h}_{k,i},\boldsymbol{e}_{k,i}\right\rangle\right|^{2}+...+\sqrt{P_L}.\left|\left\langle \boldsymbol{h}_{L,i},\boldsymbol{e}_{L,i}\right\rangle\right|^{2}]].s_i+n_i \quad (66)$$

Since the derivations are heavy to obtain, we will simplify at first the studied context and we'll take: L = 2. We have then to characterize the p.d.f of the random variable: $Z_2=\sqrt{P_1}.\left|\left\langle \boldsymbol{h}_{1,i},\boldsymbol{e}_{1,i}\right\rangle\right|^{2}+\sqrt{P_2}.\left|\left\langle \boldsymbol{h}_{2,i},\boldsymbol{e}_{2,i}\right\rangle\right|^{2}$. This is equal to the convolution product of the pdf's of $\sqrt{P_1}.\left|\left\langle \boldsymbol{h}_{1,i},\boldsymbol{e}_{1,i}\right\rangle\right|^{2}$ and $\sqrt{P_2}.\left|\left\langle \boldsymbol{h}_{2,i},\boldsymbol{e}_{2,i}\right\rangle\right|^{2}$:

$$p_{Z_2}(x)=(1/\sqrt{P_1}.\Gamma(q_i)).(x/\sqrt{P_1})^{q_i-1}.\exp(-x/\sqrt{P_1})*(1/\sqrt{P_2}.\Gamma(q_i)).(x/\sqrt{P_2})^{q_i-1}.\exp(-x/\sqrt{P_2}) \quad (67)$$

With: $q_i=R-(i-1)$.

It is easier to use the Fourier Transform to calculate (67); we have: (setting: $\Theta_1=1/\sqrt{P_1},\Theta_2=1/\sqrt{P_2}$)

$$\begin{aligned}TF[p_{Z_2}(x)]=g_{Z_2}(v)&=\frac{(\Theta_1.\Theta_2)^{q_i}}{(q_i-1)!^{2}}.\frac{(q_i-1)!^{2}}{(\Theta_1+j.2.\pi.v)^{q_i}.(\Theta_2+j.2.\pi.v)^{q_i}}\\&=(\Theta_1.\Theta_2)^{q_i}.\frac{1}{(\Theta_1+j.2.\pi.v)^{q_i}.(\Theta_2+j.2.\pi.v)^{q_i}}\end{aligned} \quad (68)$$

The decomposition of this fractional function yields to:

$$g_{Z_2}(v) = (\Theta_1.\Theta_2)^{q_i}.\frac{1}{(\Theta_1 + j.2.\pi.v)^{q_i}.(\Theta_2 + j.2.\pi.v)^{q_i}}$$

$$= (\Theta_1.\Theta_2)^{q_i}.[\sum_{k=0}^{q_i-1}\frac{a_k}{(\Theta_1 + j.2.\pi.v)^{q_i-k}} + \frac{b_k}{(\Theta_2 + j.2.\pi.v)^{q_i-k}}]$$

We found:

$$a_k = \frac{(-1)^k}{(\Theta_2 - \Theta_1)^{q_i+k}}.\frac{q_i.(q_i+1)...(q_i+k-1)}{k!} = \frac{(-1)^k}{(\Theta_2 - \Theta_1)^{q_i+k}}.\frac{(q_i+k-1)!}{(q_i-1)!.k!} = \frac{(-1)^k.C_{q_i+k-1}^{q_i-1}}{(\Theta_2 - \Theta_1)^{q_i+k}}$$

$$b_k = \frac{(-1)^k.C_{q_i+k-1}^{q_i-1}}{(\Theta_1 - \Theta_2)^{q_i+k}}$$

<u>Remark</u>: if: $\Theta_2 = \Theta_1 = \Theta$, we obtain directly: $g_{Z_2}(v) = \Theta^{2.q_i}.\frac{1}{(\Theta + j.2.\pi.v)^{2.q_i}}$ and: $p_{Z_2}(x) = \Theta^{2.q_i}.\frac{x^{2.q_i-1}.e^{-\Theta.x}}{(2.q_i-1)!}$

In the case where: $\Theta_2 \neq \Theta_1$, we suppose that $\Theta_2 < \Theta_1$ without any loss of generality. This gives:

$$g_{Z_2}(v) = (\Theta_1.\Theta_2)^{q_i}.[\sum_{k=0}^{q_i-1}\frac{(-1)^k.C_{q_i+k-1}^{q_i-1}}{(\Theta_2 - \Theta_1)^{q_i+k}.(\Theta_1 + j.2.\pi.v)^{q_i-k}} + \frac{(-1)^k.C_{q_i+k-1}^{q_i-1}}{(\Theta_1 - \Theta_2)^{q_i+k}.(\Theta_2 + j.2.\pi.v)^{q_i-k}}]$$

$$= (\Theta_1.\Theta_2)^{q_i}.[\sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.(\frac{1}{(\Theta_2 - \Theta_1)^{q_i+k}.(\Theta_1 + j.2.\pi.v)^{q_i-k}} + \frac{1}{(\Theta_1 - \Theta_2)^{q_i+k}.(\Theta_2 + j.2.\pi.v)^{q_i-k}})]$$

and:

$$p_{Z_2}(x) = (\Theta_1.\Theta_2)^{q_i}.[\sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.(\frac{1}{(\Theta_2 - \Theta_1)^{q_i+k}}.\frac{x^{q_i-k-1}.e^{-\Theta_1.x}}{(q_i-k-1)!} + \frac{1}{(\Theta_1 - \Theta_2)^{q+k}}.\frac{x^{q_i-k-1}.e^{-\Theta_2.x}}{(q_i-k-1)!})]$$

$$p_{Z_2}(x) = (\Theta_1.\Theta_2)^{q_i}.[\sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.\frac{x^{q_i-k-1}}{(q_i-k-1)!}(\frac{e^{-\Theta_1.x}}{(\Theta_2 - \Theta_1)^{q_i+k}} + \frac{e^{-\Theta_2.x}}{(\Theta_1 - \Theta_2)^{q_i+k}})]$$

We have to calculate the p.d.f of: $\rho = Z_2^2/\sigma^2$, this yields to:

$$f_\rho(u) = p_{Z_2}(\sigma.\sqrt{u}).\sigma^2/(2.\sigma.\sqrt{u}) = p_{Z_2}(\sigma.\sqrt{u}).\sigma/(2.\sqrt{u})$$

$$f_\rho(u) = p_{Z_2}(\sigma.\sqrt{u}).\sigma/(2.\sqrt{u}) \quad (68)$$

$$f_\rho(u) = (\Theta_1.\Theta_2)^{q_i}.[\sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.\frac{\sigma^{q_i-k}.u^{(q_i-k)/2-1}}{2.(q_i-k-1)!}(\frac{e^{-\Theta_1.\sigma.\sqrt{u}}}{(\Theta_2 - \Theta_1)^{q_i+k}} + \frac{e^{-\Theta_2.\sigma.\sqrt{u}}}{(\Theta_1 - \Theta_2)^{q_i+k}})] \quad (69)$$

The computation of the average symbol error (SER) probability yields to:

$$P_e = \int_0^{+\infty} 2.\alpha.erfc(\sqrt{\xi.u}).f_\rho(u).du$$

$$P_e = \int_0^{+\infty} 2.\alpha.erfc(\sqrt{\xi.u}).(\Theta_1.\Theta_2)^{q_i}.[\sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.\frac{\sigma^{q_i-k}.u^{(q_i-k)/2-1}}{2.(q_i-k-1)!}(\frac{e^{-\Theta_1.\sigma.\sqrt{u}}}{(\Theta_2-\Theta_1)^{q_i+k}}+\frac{e^{-\Theta_2.\sigma.\sqrt{u}}}{(\Theta_1-\Theta_2)^{q_i+k}})].du$$

$$\begin{aligned} P_e = \sigma.\alpha.(\Theta_1.\Theta_2)^{q_i}.\sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.\frac{\sigma^{q_i-k}}{(q_i-k-1)!}[\int_0^{+\infty}\frac{u^{(q_i-k)/2-1}.e^{-\Theta_1\sigma.\sqrt{u}}}{(\Theta_2-\Theta_1)^{q_i+k}}.erfc(\sqrt{\xi.u}).du+... \\ ...+\int_0^{+\infty}\frac{u^{(q_i-k)/2-1}.e^{-\Theta_2\sigma.\sqrt{u}}}{(\Theta_1-\Theta_2)^{q_i+k}}.erfc(\sqrt{\xi.u}).du] \end{aligned} \quad (70)$$

$$P_e = \sigma.\alpha.(\Theta_1.\Theta_2)^{q_i}.\sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.\frac{\sigma^{q_i-k}}{(q_i-k-1)!}[\frac{I_{\Theta_1,\xi}^{q_i-k}}{(\Theta_2-\Theta_1)^{q_i+k}}+\frac{I_{\Theta_2,\xi}^{q_i-k}}{(\Theta_1-\Theta_2)^{q_i+k}}] \quad (71)$$

To obtain the final result, we have to calculate integrals of the form:

$$I_{\gamma,\xi}^{q-k} = \int_0^{+\infty} u^{(q-k)/2-1}.e^{-\gamma.\sigma.\sqrt{u}}.erfc(\sqrt{\xi.u}).du$$

We set: $x=\sqrt{u} \Rightarrow dx=\frac{1}{2.\sqrt{u}}.du \Rightarrow du=2.x.dx$; so we obtain:

$$I_{\gamma_1,\beta}^{q-k} = \int_0^{+\infty} x^{(q-k)-2}.e^{-\gamma.\sigma.x}.erfc(\sqrt{\xi}.x).2.x.dx$$

$$I_{\gamma_1,\beta}^{q-k} = 2.\int_0^{+\infty} x^{q-k-1}.e^{-\gamma.\sigma.x}.erfc(\sqrt{\xi}.x).dx = 2.J_{\gamma.\sigma,\xi}^{q-k}$$

We have now to calculate integrals of the form: $J_{\lambda,\xi}^{m} = \int_0^{+\infty} x^{m-1}.e^{-\lambda x}.erfc(\sqrt{\xi}.x).dx$; we integrate it by parts with:

$dv = x^{m-1}.e^{-\lambda x}.dx$; $v = -\sum_{k=0}^{m-1}\frac{(m-1)!}{k!.\lambda^{m-k}}.x^k.e^{-\lambda.x}$ and : $u = erfc(\sqrt{\xi}.x)$, hence : $du = \frac{-2.\sqrt{\xi}}{\sqrt{\pi}}.e^{-\xi.x^2}.dx$

$$J_{\lambda,\xi}^{m} = \int_0^{+\infty} x^{m-1}.e^{-\lambda x}.erfc(\sqrt{\xi}.x).dx$$

$$= [-erfc(\sqrt{\xi}.x).\sum_{k=0}^{m-1}\frac{(m-1)!}{k!.\lambda^{m-k}}.x^k.e^{-\lambda.x}]_0^{+\infty} - 2.\sqrt{\frac{\xi}{\pi}}\int_0^{+\infty}\sum_{k=0}^{m-1}\frac{(m-1)!}{k!.\lambda^{m-k}}.x^k.e^{-\lambda.x}.e^{-\xi.x^2}.dx$$

$$= \frac{(m-1)!}{\lambda^m} - 2.\sqrt{\frac{\xi}{\pi}}.(m-1)!\sum_{k=0}^{m-1}\frac{e^{\lambda^2/4.\xi}}{k!.\lambda^{m-k}}.\int_0^{+\infty} x^k.e^{-\xi.(x+\lambda/2.\xi)^2}.dx$$

With: $z=\sqrt{\xi}.(x+\lambda/2.\xi)$, it comes

$$\int_0^{+\infty} x^k.e^{-\xi.(x+\lambda/2.\xi)^2}.dx = \int_{\lambda/2.\sqrt{\xi}}^{+\infty} (\frac{z}{\sqrt{\xi}} - \lambda/2.\xi)^k e^{-z^2}.\frac{dz}{\sqrt{\xi}}$$

$$= \frac{1}{\sqrt{\xi}} \int_{\lambda/2.\sqrt{\xi}}^{+\infty} (\frac{z}{\sqrt{\xi}} - \lambda/2.\xi)^k e^{-z^2}.dz = \frac{1}{\sqrt{\xi}} \int_{\lambda/2.\sqrt{\xi}}^{+\infty} \sum_{n=0}^{k} C_k^n.(-1)^n.\frac{\lambda^n}{2^n.\xi^n}.\frac{z^{k-n}}{\xi^{(k-n)/2}} e^{-z^2}.dz$$

$$= \sum_{n=0}^{k} C_k^n.(-1)^n.\frac{\lambda^n}{2^n.\xi^{(k+n+1)/2}}.\int_{\lambda/2.\sqrt{\xi}}^{+\infty} z^{k-n}.e^{-z^2}.dz$$

And finally, the integral $\int_{\lambda/2.\sqrt{\xi}}^{+\infty} z^{k-n}.e^{-z^2}.dz$ can be expressed in terms of incomplete gamma function.

$$\int_{\lambda/2.\sqrt{\xi}}^{+\infty} z^{k-n}.e^{-z^2}.dz = \int_{\lambda^2/4.\xi}^{+\infty} x^{(k-n)/2}.e^{-x}.\frac{dx}{2.\sqrt{x}} = \frac{1}{2}.\int_{\lambda^2/4.\xi}^{+\infty} x^{(k-n-1)/2}.e^{-x}.dx = \frac{1}{2}.[\int_0^{+\infty} x^{(k-n-1)/2}.e^{-x}.dx - \int_0^{\lambda^2/4.\xi} x^{(k-n-1)/2}.e^{-x}.dx]$$

$$= \frac{1}{2}.\{\Gamma[(k-n+1)/2] - \Gamma_{inc}[\lambda^2/4.\xi, (k-n+1)/2]\}$$

With: $\Gamma_{inc}[x,n] = \int_0^x u^{n-1}.e^{-x}.dx$; we obtain:

$$J_{\lambda,\xi}^m = \frac{(m-1)!}{\lambda^m} - \sqrt{\frac{\xi}{\pi}}.(m-1)!\sum_{k=0}^{m-1} \frac{e^{\lambda^2/4.\xi}}{k!.\lambda^{m-k}}.\sum_{n=0}^{k} C_k^n.(-1)^n.\frac{\lambda^n}{2^n.\xi^{(k+n+1)/2}}.\{\Gamma[(k-n+1)/2] - \Gamma_{inc}[\lambda^2/4.\xi, (k-n+1)/2]\}$$

$$J_{\lambda,\xi}^m = \frac{(m-1)!}{\lambda^m}.[1 - \frac{1}{\sqrt{\pi}}.\sum_{k=0}^{m-1} \frac{e^{\lambda^2/4.\xi}.\lambda^k}{k!}.\sum_{n=0}^{k} C_k^n.(-1)^n.\frac{\lambda^n}{2^n.\xi^{(k+n)/2}}.\{\Gamma[(k-n+1)/2] - \Gamma_{inc}[\lambda^2/4.\xi, (k-n+1)/2]\}$$

And finally:

$$P_e = \alpha.(\Theta_1.\Theta_2)^{q_i}.\sum_{k=0}^{q_i-1} (-1)^k.C_{q_i+k-1}^{q_i-1}.\frac{2.\sigma^{q_i-k}}{(q_i-1-k)!}[\frac{J_{\Theta_1.\sigma,\xi}^{q_i-k}}{(\Theta_2-\Theta_1)^{q_i+k}} + \frac{J_{\Theta_2.\sigma,\xi}^{q_i-k}}{(\Theta_1-\Theta_2)^{q_i+k}}] \quad (72)$$

With:

$$J_{\lambda,\xi}^m = \frac{(m-1)!}{\lambda^m}.[1 - \frac{1}{\sqrt{\pi}}.\sum_{k=0}^{m-1} \frac{e^{\lambda^2/4.\xi}.\lambda^k}{k!}.\sum_{n=0}^{k} C_k^n.(-1)^n.\frac{\lambda^n}{2^n.\xi^{(k+n)/2}}.\{\Gamma[(k-n+1)/2] - \Gamma_{inc}[\lambda^2/4.\xi, (k-n+1)/2]\}$$

We need to calculate the average power at the receive side which is equal to: $P_u = E[Z_2^2]$ with

$Z_2 = \sqrt{P_1}.\left|\langle \boldsymbol{h}_{1,i}, \boldsymbol{e}_{1,i} \rangle\right|^2 + \sqrt{P_2}.\left|\langle \boldsymbol{h}_{2,i}, \boldsymbol{e}_{2,i} \rangle\right|^2$.

$$E[Z_2^2] = \int_0^{+\infty} (\Theta_1.\Theta_2)^{q_i}.x^2.[\sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.\frac{x^{q_i-k-1}}{(q_i-k-1)!}(\frac{e^{-\Theta_1.x}}{(\Theta_2-\Theta_1)^{q_i+k}}+\frac{e^{-\Theta_2.x}}{(\Theta_1-\Theta_2)^{q_i+k}})].dx$$

$$= (\Theta_1.\Theta_2)^{q_i}.\sum_{k=0}^{q_i-1}(-1)^k.\frac{C_{q_i+k-1}^{q_i-1}}{(q_i-k-1)!}.[\frac{1}{(\Theta_2-\Theta_1)^{q_i+k}}.\int_0^{+\infty} x^{q_i-k+1}e^{-\Theta_1.x}.dx+...$$

$$...+\frac{1}{(\Theta_1-\Theta_2)^{q_i+k}}.\int_0^{+\infty} x^{q_i-k+1}e^{-\Theta_2.x}.dx$$

$$= (\Theta_1.\Theta_2)^{q_i}.\sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.(q_i-k).(q_i-k+1).[\frac{1}{(\Theta_2-\Theta_1)^{q_i+k}.\Theta_1^{q_i-k+2}}+\frac{1}{(\Theta_1-\Theta_2)^{q_i+k}.\Theta_2^{q_i-k+2}}]$$

$$= \sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.(q_i-k).(q_i-k+1).[\frac{\Theta_1^{k-2}.\Theta_2^{q_i}+(-1)^{q_i+k}.\Theta_2^{k-2}.\Theta_1^{q_i}}{(\Theta_2-\Theta_1)^{q_i+k}}]$$

$$E[Z_2^2] = \sum_{k=0}^{q_i-1}(-1)^k.C_{q_i+k-1}^{q_i-1}.(q_i-k).(q_i-k+1).[\frac{\Theta_1^{k-2}.\Theta_2^{q_i}+(-1)^{q_i+k}.\Theta_2^{k-2}.\Theta_1^{q_i}}{(\Theta_2-\Theta_1)^{q_i+k}}] \quad (73)$$

With: $\Theta_1 = 1/\sqrt{P_1}, \Theta_2 = 1/\sqrt{P_2}$.

It is possible to generalize this result for a number L of relays. We have in this case the random variable:

$$Z_L = \sqrt{P_1}.\left|\left\langle \boldsymbol{h}_{1,i}, \boldsymbol{e}_{1,i} \right\rangle\right|^2 + \sqrt{P_2}.\left|\left\langle \boldsymbol{h}_{2,i}, \boldsymbol{e}_{2,i} \right\rangle\right|^2 + ... + \sqrt{P_L}.\left|\left\langle \boldsymbol{h}_{L,i}, \boldsymbol{e}_{L,i} \right\rangle\right|^2 \quad (74)$$

The Fourier transform of the p.d.f $p_{Z_L}(x)$ is equal to:

$$TF[p_{Z_L}(x)] = g_{Z_L}(v) = (\Theta_1.\Theta_2...\Theta_L)^{q_i}.\frac{1}{(\Theta_1 + j.2.\pi.v)^{q_i}.(\Theta_2 + j.2.\pi.v)^{q_i}...(\Theta_L + j.2.\pi.v)^{q_i}} \quad (75)$$

With: $\Theta_1 = 1/\sqrt{P_1}, \Theta_2 = 1/\sqrt{P_2}, ..., \Theta_L = 1/\sqrt{P_L}$

We have now:

$$g_{Z_2}(v) = (\Theta_1.\Theta_2...\Theta_L)^{q_i}.[\sum_{k=0}^{q_i-1}\frac{a_k^1}{(\Theta_1 + j.2.\pi.v)^{q_i-k}} + \frac{a_k^2}{(\Theta_2 + j.2.\pi.v)^{q_i-k}} + ... + \frac{a_k^L}{(\Theta_L + j.2.\pi.v)^{q_i-k}}] \quad (76)$$

To obtain the decomposition of $g_{Z_2}(v)$ into simple elements, we use the following property. If we consider the following function (obviously $x = j.2.\pi.v$ in our case)

$$\frac{1}{(\Theta_1 + x)^{q_i}.(\Theta_2 + x)^{q_i}...(\Theta_L + x)^{q_i}} = F_L(x) \quad (77)$$

We have:

$$(\Theta_i + x)^{q_i}.F_L(x) = \frac{1}{(\Theta_1 + x)^{q_i}...(\Theta_{i-1} + x)^{q_i}.(\Theta_{i+1} + x)^{q_i}...(\Theta_L + x)^{q_i}} = G_L(x) \quad (78)$$

Using the Taylor expansion of $G_L(x)$ for x around $-\Theta_i$, we obtain:

$$G_L(x) = G_L(-\Theta_i) + (x + \Theta_i).G_L^{'}(-\Theta_i) + \frac{(x+\Theta_i)^2}{2!}.G_L^{''}(-\Theta_i) + ... + \frac{(x+\Theta_i)^n}{n!}.G_L^{(n)}(-\Theta_i) + ... \quad (79)$$

With: $G_L^{(n)}(-\Theta_i) = [\frac{d^n G_L(x)}{dx^n}]_{x=-\Theta_i}$; in this case, we obtain:

$$F_L(x) = \frac{G_L(-\Theta_i)}{(\Theta_i + x)^{q_i}} + \frac{G_L^{'}(-\Theta_i)}{(\Theta_i + x)^{q_i - 1}} + \frac{G_L^{''}(-\Theta_i)}{2!.(\Theta_i + x)^{q_i - 2}} + ... + \frac{G_L^{(n)}(-\Theta_i)}{n!.(\Theta_i + x)^{q_i - n}} + ... + \frac{G_L^{(q_i - 1)}(-\Theta_i)}{(q_i - 1)!.(\Theta_i + x)}$$

To calculate these derivative values, we can use the following writing form:

$$\begin{aligned} 1/G_L(x) &= (\Theta_1 + x)^{q_i}...(\Theta_{i-1} + x)^{q_i}.(\Theta_{i+1} + x)^{q_i}...(\Theta_L + x)^{q_i} = [P_L(x)]^{q_i} \\ &= [(\Theta_1 + x)...(\Theta_{i-1} + x).(\Theta_{i+1} + x)...(\Theta_L + x)]^{q_i} \\ &= [x^{L-1} + \sigma_1(\Theta_{k=1:L,k\neq i}).x^{L-2} + ... + \sigma_k(\Theta_{k=1:L,k\neq i}).x^{L-1-k} + ... + \sigma_{L-1}(\Theta_{k=1:L,k\neq i})]^{q_i} \end{aligned} \quad (80)$$

With: $\Theta_{k=1:L,k\neq i} = (\Theta_1, \Theta_1, ..., \Theta_{i-1}, \Theta_{i+1}, ..., \Theta_L)$ and $\sigma_k(\Theta_{k=1:L,k\neq i}) = \sum_{1\le i_1 < i_2 < ... < i_k \le L, i_k \neq i} \Theta_{i_1}.\Theta_{i_2}...\Theta_{i_k}$. Using (80) it is not difficult to obtain the derivative values $G_L^{(n)}(-\Theta_i)$. We have, for example:

$$G_L(-\Theta_i) = \frac{1}{[(\Theta_1 - \Theta_i)...(\Theta_{i-1} - \Theta_i).(\Theta_{i+1} - \Theta_i)...(\Theta_L - \Theta_i)]^{q_i}}$$

$$G_L^{'}(-\Theta_i) = \frac{-q_i.[P_L^{'}(x)]_{x=\Theta_i}}{[(\Theta_1 - \Theta_i)...(\Theta_{i-1} - \Theta_i).(\Theta_{i+1} - \Theta_i)...(\Theta_L - \Theta_i)]^{q_i+1}}$$

$$G_L^{'}(-\Theta_i) = \frac{(-1)^n.q_i. \sum_{1\le i_1 < i_2 < .. i_k < ... < i_{L-1} \le L, i_k \neq i} [(\Theta_{i_1} - \Theta_i)...(\Theta_{i_k} - \Theta_i)...(\Theta_{i_{L-1}} - \Theta_i)]}{[(\Theta_1 - \Theta_i)...(\Theta_{i-1} - \Theta_i).(\Theta_{i+1} - \Theta_i)...(\Theta_L - \Theta_i)]^{q_i+1}}$$

For the derivation of $G_L^{(n)}(-\Theta_i)$, we have the following relationships:

$$G_L(x)=1/[P_L(x)]^{q_i},G_L^{'}(x)=\frac{-q_i.P_L^{'}(x)}{[P_L(x)]^{q_i+1}},G_L^{"}(x)=-q_i.[\frac{P_L^{"}(x)}{[P_L(x)]^{q_i+1}}-(q_i+1)\frac{P_L^{'}(x)}{[P_L(x)]^{q_i+2}}]$$

$$G_L^{"}(x)=-q_i.\frac{P_L^{"}(x)}{[P_L(x)]^{q_i+1}}+q_i.(q_i+1).\frac{P_L^{'}(x)}{[P_L(x)]^{q_i+2}}$$

$$G_L^{(3)}(x)=-q_i.[\frac{P_L^{(3)}(x)}{[P_L(x)]^{q_i+1}}-(q_i+1).\frac{P_L^{"}(x)}{[P_L(x)]^{q_i+2}}]+q_i.(q_i+1).[\frac{P_L^{"}(x)}{[P_L(x)]^{q_i+2}}-(q_i+2).\frac{P_L^{'}(x)}{[P_L(x)]^{q_i+3}}]$$

$$G_L^{(3)}(x)=-q_i.(q_i+1).(q_i+2).\frac{P_L^{'}(x)}{[P_L(x)]^{q_i+3}}+2.q_i.(q_i+1).\frac{P_L^{"}(x)}{[P_L(x)]^{q_i+2}}-q_i.\frac{P_L^{(3)}(x)}{[P_L(x)]^{q_i+1}}$$

$$G_L^{(4)}(x)=-q_i.(q_i+1).(q_i+2).[\frac{P_L^{"}(x)}{[P_L(x)]^{q_i+3}}-(q_i+3).\frac{P_L^{'}(x)}{[P_L(x)]^{q_i+4}}]+2.q_i.(q_i+1).[\frac{P_L^{(3)}(x)}{[P_L(x)]^{q_i+2}}-(q_i+2).\frac{P_L^{"}(x)}{[P_L(x)]^{q_i+3}}]$$
$$-q_i.[\frac{P_L^{(4)}(x)}{[P_L(x)]^{q_i+1}}-(q_i+1).\frac{P_L^{(3)}(x)}{[P_L(x)]^{q_i+2}}]$$

$$G_L^{(4)}(x)=q_i.(q_i+1).(q_i+2).(q_i+3).\frac{P_L^{'}(x)}{[P_L(x)]^{q_i+4}}-3.q_i.(q_i+1).(q_i+2)\frac{P_L^{"}(x)}{[P_L(x)]^{q_i+3}}+3.q_i.(q_i+1).\frac{P_L^{(3)}(x)}{[P_L(x)]^{q_i+2}}-q_i.\frac{P_L^{(4)}(x)}{[P_L(x)]^{q_i+1}}$$

$$G_L^{(5)}(x)=-q_i.(q_i+1).(q_i+2).(q_i+3).(q_i+4).\frac{P_L^{'}(x)}{[P_L(x)]^{q_i+5}}+4.q_i.(q_i+1).(q_i+2).(q_i+3).\frac{P_L^{"}(x)}{[P_L(x)]^{q_i+4}}+...$$
$$-6.q_i.(q_i+1).(q_i+2).\frac{P_L^{(3)}(x)}{[P_L(x)]^{q_i+3}}+4.q_i.(q_i+1).\frac{P_L^{(4)}(x)}{[P_L(x)]^{q_i+2}}-q_i.\frac{P_L^{(5)}(x)}{[P_L(x)]^{q_i+1}}$$

It is not difficult to obtain, by induction, a general expression:

$$G_L^{(n)}(x)=\sum_{k=0}^{n-1}C_{n-1}^k.(-1)^{k+1}.q_i.(q_i+1)...(q_i+k).\frac{P_L^{(n-k)}(x)}{[P_L(x)]^{q_i+1+k}} \quad (81)$$

With: $P_L^{(m)}(x)=\sum_{1\le i_1<i_2<..i_k<...<i_{L-m}\le L,i_k\ne i}[(\Theta_{i_1}+x)...(\Theta_{i_k}+x)...(\Theta_{i_{L-m}}+x)]$, taking into account (76), we arrive at:

$$a_k^i=\frac{G_L^{(k)}(-\Theta_i)}{k!}=\frac{1}{k!}.\sum_{k=0}^{n-1}C_{n-1}^k.(-1)^{k+1}.q_i.(q_i+1)...(q_i+k).\frac{P_L^{(n-k)}(-\Theta_i)}{[P_L(-\Theta_i)]^{q_i+1+k}}$$

$$a_k^i=\frac{1}{k!}.\sum_{k=0}^{n-1}C_{n-1}^k.(-1)^{k+1}.q_i.(q_i+1)...(q_i+k).\frac{\sum_{1\le i_1<i_2<..i_k<...<i_{L-n+k}\le L,i_k\ne i}[(\Theta_{i_1}-\Theta_i)...(\Theta_{i_k}-\Theta_i)...(\Theta_{i_{L-n+k}}-\Theta_i)]}{[(\Theta_1-\Theta_i)...(\Theta_{i-1}-\Theta_i).(\Theta_{i+1}-\Theta_i)...(\Theta_L-\Theta_i)]^{q_i+1+k}} \quad (81)$$

$$a_k^i=q_i.\sum_{k=0}^{n-1}C_{n-1}^k.C_{q_i+k}^{q_i}.(-1)^{k+1}.\frac{\sum_{1\le i_1<i_2<..i_k<...<i_{L-n+k}\le L,i_k\ne i}[(\Theta_{i_1}-\Theta_i)...(\Theta_{i_k}-\Theta_i)...(\Theta_{i_{L-n+k}}-\Theta_i)]}{[(\Theta_1-\Theta_i)...(\Theta_{i-1}-\Theta_i).(\Theta_{i+1}-\Theta_i)...(\Theta_L-\Theta_i)]^{q_i+1+k}}$$

It is now possible to derive the p.d.f $p_{Z_L}(x)$:

$$p_{Z_L}(x)=(\Theta_1.\Theta_2..\Theta_L)^{q_i}.[\sum_{k=0}^{q_i-1}a_k^1.\frac{x^{q_i-k-1}.e^{-\Theta_1.x}}{(q_i-k-1)!}+a_k^2.\frac{x^{q_i-k-1}.e^{-\Theta_2.x}}{(q_i-k-1)!}+...+a_k^L.\frac{x^{q_i-k-1}.e^{-\Theta_L.x}}{(q_i-k-1)!}] \quad (82)$$

And we eventually obtain:

$$P_e = 2.\alpha.(\Theta_1.\Theta_2...\Theta_L)^{q_i}.\sum_{k=0}^{q_i-1}\frac{\sigma^{q_i-k}}{(q_i-1-k)!}[a_k^1.J_{\Theta_1.\sigma,\xi}^{q_i-k}+a_k^2.J_{\Theta_2.\sigma,\xi}^{q_i-k}+...+a_k^L.J_{\Theta_L.\sigma,\xi}^{q_i-k}] \quad (83)$$

With:

$$J_{\lambda,\xi}^m=\frac{(m-1)!}{\lambda^m}.[1-\frac{1}{\sqrt{\pi}}.\sum_{k=0}^{m-1}\frac{e^{\lambda^2/4.\xi}.\lambda^k}{k!}.\sum_{n=0}^{k}C_k^n.(-1)^n.\frac{\lambda^n}{2^n.\xi^{(k+n)/2}}.\{\Gamma[(k-n+1)/2]-\Gamma_{inc}[\lambda^2/4.\xi,(k-n+1)/2]\}$$

V. Simulation Results

5.1 **The reference System**:

The reference system has been described in part 2.1 and in a former paper with a detailed study of the obtained diversity gain. We have the following equation:

$$\boldsymbol{y}=\boldsymbol{H}.\boldsymbol{W}.\boldsymbol{s}+\boldsymbol{n}=\sqrt{P}.\frac{\boldsymbol{H}.\boldsymbol{H}^{\dagger}(\boldsymbol{H}.\boldsymbol{H}^{\dagger})^{-1}}{\left\|\boldsymbol{H}^{\dagger}(\boldsymbol{H}.\boldsymbol{H}^{\dagger})^{-1}\right\|}.\boldsymbol{s}+\boldsymbol{n}$$

$$\boldsymbol{y}=\frac{\sqrt{P}.\boldsymbol{s}}{\left\|\boldsymbol{H}^{\dagger}(\boldsymbol{H}.\boldsymbol{H}^{\dagger})^{-1}\right\|}+\boldsymbol{n}$$

When we calculate the signal to noise ratio, using: $\|\boldsymbol{A}\|^2=\text{trace}(\boldsymbol{A}.\boldsymbol{A}^H)$, we obtain :

$$\left\|\ \boldsymbol{H}_1^{\dagger}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1}\right\|^2=\text{trace}\left(\boldsymbol{H}_1^{\dagger}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1}.(\boldsymbol{H}_1^{\dagger}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1})^{\dagger}\right)$$

$$\left\|\ \boldsymbol{H}_1^{\dagger}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1}\right\|^2=\text{trace}(\boldsymbol{H}_1^{\dagger}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-\dagger}.\boldsymbol{H}_1)$$

$$\left\|\ \boldsymbol{H}_1^{\dagger}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1}\right\|^2=\text{trace}((\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger}).(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1}(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-\dagger})$$

$$\left\|\ \boldsymbol{H}_1^{\dagger}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1}\right\|^2=\text{trace}((\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{\dagger}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-\dagger}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1})$$

$$\left\|\ \boldsymbol{H}_1^{\dagger}.(\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1}\right\|^2=\text{trace}((\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1}) \quad (84)$$

For the signal to noise ratio, this yields to:

$$\rho=\frac{1}{\sigma^2.\text{trace}([\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger}]^{-1})} \quad (85)$$

Since $\boldsymbol{H}_1$ is a matrix with N rows and $R.L$ columns, we have naturally $\boldsymbol{H}_1=\boldsymbol{H}_{N\times R.L}$ with: $N<R.L$. The p.d.f of the signal to noise ratio in (85) is difficult to track, particularly when $N>2$. However, using the well known Wishart distribution, it is possible to accurately obtain it in the case: $N=2$.

-If $N = 2$, we have the p.d.f of the non-ordered eigenvalues of the Wishart matrix: $\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger}$.

$$p(\lambda_1,\lambda_2)=\frac{1}{2}.K_{2,n}^{-1}.(\lambda_1.\lambda_2)^{n-2}.e^{-(\lambda_1+\lambda_2)}.(\lambda_1-\lambda_2)^2 \quad (86) \quad (n=R.L)$$

The normalization condition: $\iint p(\lambda_1,\lambda_2).d\lambda_1.d\lambda_2=\int\int\frac{1}{2}.K_{2,n}^{-1}.(\lambda_1.\lambda_2)^{n-2}.e^{-(\lambda_1+\lambda_2)}.(\lambda_1-\lambda_2)^2.d\lambda_1.d\lambda_2=1$, yields to:

$K_{2,n}=(n-1)!.(n-2)!.$

We have: $Z=\frac{1}{\text{trace}((\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger})^{-1})}=\frac{\lambda_1.\lambda_2}{\lambda_1+\lambda_2}=[\frac{1}{\lambda_1}+\frac{1}{\lambda_2}]^{-1}$; so the first task is to obtain the p.d.f of : $U=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}$.To do this, we set: $x=\frac{1}{\lambda_1}, y=\frac{1}{\lambda_2}$; the Jacobian of this transform is:

$$J(\lambda_1=\frac{1}{x},\lambda_2=\frac{1}{y})=\left|\text{Det}(\begin{bmatrix}\frac{\partial\lambda_1}{\partial x} & \frac{\partial\lambda_1}{\partial y}\\ \frac{\partial\lambda_2}{\partial x} & \frac{\partial\lambda_2}{\partial y}\end{bmatrix})\right|=\left|\text{Det}(\begin{bmatrix}-\frac{1}{x^2} & 0\\ 0 & -\frac{1}{y^2}\end{bmatrix})\right|=\frac{1}{x^2.y^2} \quad (87)$$

The p.d.f of the couple $(\frac{1}{\lambda_1},\frac{1}{\lambda_2})=(x,y)$ is then equal to:

$$\begin{aligned}p_{1/\lambda_1,1/\lambda_2}(x,y)&=\frac{1}{2}.K_{2,n}^{-1}.(\frac{1}{x.y})^{n-2}.e^{-(1/x+1/y)}.(\frac{1}{x}-\frac{1}{y})^2.\frac{1}{x^2.y^2}\\&=\frac{1}{2}.K_{2,n}^{-1}.(\frac{1}{x.y})^{n}.e^{-(1/x+1/y)}.(\frac{1}{x}-\frac{1}{y})^2\end{aligned} \quad (88)$$

Then we calculate the probability that: $\frac{1}{\lambda_1}+\frac{1}{\lambda_2}=\lambda$ or $x+y=\lambda$; this entails that: $(\frac{1}{\lambda_1},\frac{1}{\lambda_2})=(\frac{1}{\lambda_1},\lambda-\frac{1}{\lambda_1})=(x,\lambda-x)$.

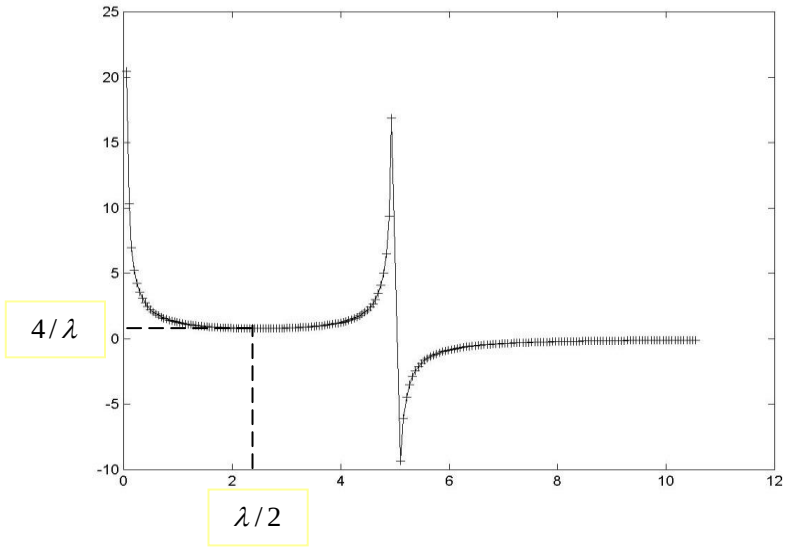
The corresponding p.d.f is:

$$p_{1/\lambda_1,1/\lambda_2}(x,\lambda-x)=\frac{1}{2}.K_{2,n}^{-1}.\frac{1}{x^n.(\lambda-x)^n}.e^{-\lambda/x.(\lambda-x)}.(\frac{1}{x}-\frac{1}{(\lambda-x)})^2 \quad (89)$$

To obtain the p.d.f of $U=\frac{1}{\lambda_1}+\frac{1}{\lambda_2}$, we have to average (89) over all the possible values of *x*. This yields to:

$$\begin{aligned}p_U(\lambda)&=\frac{1}{2}.K_{2,n}^{-1}.\int_0^{\lambda}\frac{1}{x^n.(\lambda-x)^n}.(\frac{1}{x}-\frac{1}{(\lambda-x)})^2.e^{-\lambda/x.(\lambda-x)}.dx\\p_U(\lambda)&=\frac{1}{2}.K_{2,n}^{-1}.\int_0^{\lambda}\frac{(\lambda-2.x)^2}{x^{n+2}.(\lambda-x)^{n+2}}.e^{-\lambda/x.(\lambda-x)}.dx=\frac{1}{2}.K_{2,n}^{-1}.M_n\end{aligned} \quad (90)$$

To compute this integral, we use the new variable: $u=\lambda/x.(\lambda-x)$; the graph of this function is given just below:



We obtain:

$$M_n = 2.\int_0^{\lambda/2} \frac{(\lambda-2.x)^2}{x^{n+2}.(\lambda-x)^{n+2}}.e^{-\lambda/x.(\lambda-x)}.dx = 2.\int_{4/\lambda}^{+\infty} \sqrt{\lambda^2-4.\lambda/u}.\frac{u^{n+2}}{\lambda^{n+2}}e^{-u}.\frac{2.\lambda.du}{u^2}$$
$$= 4.\int_{4/\lambda}^{+\infty} \sqrt{\lambda^2-4.\lambda/u}.\frac{u^n}{\lambda^{n+1}}.e^{-u}.du \qquad (91)$$

We set now: $u = \frac{4}{\lambda}.x$, to obtain:

$$M_n = \frac{4}{\lambda^{n+1}}.\int_1^{+\infty} (\frac{4}{\lambda})^n.x^n.[\sqrt{\lambda^2-\lambda^2/x}].e^{-4.x/\lambda}.\frac{4}{\lambda}.dx = \frac{2^{2n+4}}{\lambda^{2.n+1}}.\int_1^{+\infty} x^n.\sqrt{1-\frac{1}{x}}.e^{-4.x/\lambda}.dx \qquad (92)$$

To evaluate (92), we expand $\sqrt{1-\frac{1}{x}}$ for: $x>1$, this yields to:

$$(1-1/x)^{1/2} = 1-\sum_{n=1}^{+\infty} \frac{(2n-2)!}{2^{2.n-1}.(n-1)!.n!}.\frac{1}{x^n} \quad \text{for} \quad |1/x|<1 \qquad (93)$$

In this case, we obtain:

$$M_n = \frac{2^{2n+4}}{\lambda^{2.n+1}}.\int_1^{+\infty} x^n.\sqrt{1-\frac{1}{x}}.e^{-4.x/\lambda}.dx = \frac{2^{2n+4}}{\lambda^{2.n+1}}.\int_1^{+\infty} x^n.[1-\sum_{k=1}^{+\infty} \frac{(2k-2)!}{2^{2.k-1}.(k-1)!.k!}.\frac{1}{x^k}].e^{-4.x/\lambda}.dx$$
$$= \frac{2^{2n+4}}{\lambda^{2.n+1}}.[\int_1^{+\infty} x^n.e^{-4.x/\lambda}.dx - \sum_{k=1}^{+\infty} \frac{(2k-2)!}{2^{2.k-1}.(k-1)!.k!}.\int_1^{+\infty} x^{n-k}.e^{-4.x/\lambda}.dx]$$

$$M_n = \frac{2^{2n+4}}{\lambda^{2.n+1}}.[\int_1^{+\infty} x^n.e^{-4.x/\lambda}.dx - \sum_{k=1}^{n} \frac{(2k-2)!}{2^{2.k-1}.(k-1)!.k!}.\int_1^{+\infty} x^{n-k}.e^{-4.x/\lambda}.dx - \sum_{k=n+1}^{+\infty} \frac{(2k-2)!}{2^{2.k-1}.(k-1)!.k!}.\int_1^{+\infty} \frac{e^{-4.x/\lambda}}{x^{k-n}}.dx]$$
$$= \frac{2^{2n+4}}{\lambda^{2.n+1}}.[F_n(\lambda) - \sum_{k=1}^{n} \frac{(2k-2)!}{2^{2.k-1}.(k-1)!.k!}.F_{n-k}(\lambda) - \sum_{k=n+1}^{+\infty} \frac{(2k-2)!}{2^{2.k-1}.(k-1)!.k!}.G_{k-n}(\lambda)] \quad (94)$$

With:

$$F_m(\lambda) = \int_1^{+\infty} x^m.e^{-4.x/\lambda}.dx = \int_0^{+\infty} (1+u)^m.e^{-4.(u+1)/\lambda}.du = e^{-4/\lambda} \sum_{p=0}^{m} C_m^p.\int_0^{+\infty} u^p.e^{-4.u/\lambda}.du$$
$$= e^{-4/\lambda} \sum_{p=0}^{m} C_m^p.\int_0^{+\infty} \left[\frac{\lambda.x}{4}\right]^p e^{-x}.\frac{\lambda}{4}dx = e^{-4/\lambda} \sum_{p=0}^{m} C_m^p.\left[\frac{\lambda}{4}\right]^{p+1}.\int_0^{+\infty} x^p.e^{-x}.dx = e^{-4/\lambda} \sum_{p=0}^{m} C_m^p.\left[\frac{\lambda}{4}\right]^{p+1}.\Gamma(p+1)$$
$$= e^{-4/\lambda}.\sum_{p=0}^{m} \frac{m!}{p!.(m-p)!}.\frac{\lambda^{p+1}}{4^{p+1}}.p! = e^{-4/\lambda}.\sum_{p=0}^{m} \frac{m!}{(m-p)!}.\frac{\lambda^{p+1}}{4^{p+1}}$$

And:

$$G_m(\lambda) = e^{-4/\lambda}.[\frac{1}{m-1} - \frac{4}{\lambda.(m-1).(m-2)} + ... + (-1)^k.\frac{4^k}{\lambda^k.(m-1).(m-2)...(m-k-1)} + ...] + (-1)^{m-1}.\frac{4^{m-1}}{\lambda^{m-1}.(m-1)!}.EI_1(4/\lambda)$$

With:

$$EI_1(x) = \int_x^{+\infty} \frac{e^{-t}.dt}{t} = \int_1^{+\infty} \frac{e^{-xt}.dt}{t} = e^{-x}.\int_0^{+\infty} \frac{e^{-xt}}{1+t}.dt = \frac{e^{-x}}{x}.\int_0^{+\infty} \frac{e^{-xt}}{1+t/x}.dt = -\ln x - \gamma + \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}.x^n}{n.n!}$$

This gives the following result:

$$p_U(\lambda) = \frac{1}{2}.K_{2,n}^{-1}.M_n$$
$$= \frac{1}{2}.K_{2,n}^{-1}.\frac{2^{2n+4}}{\lambda^{2.n+1}}.[F_n(\lambda) - \sum_{k=1}^{n} \frac{(2k-2)!}{2^{2.k-1}.(k-1)!.k!}.F_{n-k}(\lambda) - \sum_{k=n+1}^{+\infty} \frac{(2k-2)!}{2^{2.k-1}.(k-1)!.k!}.G_{k-n}(\lambda)]$$

With: $K_{2,n} = (n-1)!.(n-2)!,\ F_m(\lambda) = e^{-4/\lambda}.\sum_{p=0}^{m} \frac{m!}{(m-p)!}.\frac{\lambda^{p+1}}{4^{p+1}}$ and :

$$G_m(\lambda) = e^{-4/\lambda}.[\frac{1}{m-1} - \frac{4}{\lambda.(m-1).(m-2)} + ... + (-1)^k.\frac{4^k}{\lambda^k.(m-1).(m-2)...(m-k-1)} + ...] + (-1)^{m-1}.\frac{4^{m-1}}{\lambda^{m-1}.(m-1)!}.EI_1(4/\lambda)$$

$$EI_1(x) = \int_x^{+\infty} \frac{e^{-t}.dt}{t} = \int_1^{+\infty} \frac{e^{-xt}.dt}{t} = e^{-x}.\int_0^{+\infty} \frac{e^{-xt}}{1+t}.dt = \frac{e^{-x}}{x}.\int_0^{+\infty} \frac{e^{-xt}}{1+t/x}.dt = -\ln x - \gamma + \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}.x^n}{n.n\ !}$$

However, the target p.d.f is those of $1/U$, not those of U; we have immediately: $p_{1/U}(\lambda) = \frac{1}{\lambda^2}.p_U(\frac{1}{\lambda})$. This gives the final result:

$$p_{1/U}(\lambda) = K_{2,n}^{-1}.2^{2n+3}.\lambda^{2.n-1}.[F_n(1/\lambda) - \sum_{k=1}^{n} \frac{(2k-2)!}{2^{2.k-1}.(k-1)!.k!}.F_{n-k}(1/\lambda) - \sum_{k=n+1}^{+\infty} \frac{(2k-2)!}{2^{2.k-1}.(k-1)!.k!}.G_{k-n}(1/\lambda)]$$

$$F_m(\frac{1}{\lambda}) = e^{-4.\lambda}.\sum_{p=0}^{m} \frac{m!}{(m-p)!}.\frac{1}{4^{p+1}.\lambda^{p+1}} \qquad (95)$$

$$G_m(\frac{1}{\lambda}) = e^{-4.\lambda}.[\frac{1}{m-1} - \frac{4.\lambda}{(m-1).(m-2)} + ... + (-1)^k.\frac{4^k.\lambda^k}{(m-1).(m-2)...(m-k-1)} + ...] + (-1)^{m-1}.\frac{4^{m-1}.\lambda^{m-1}}{(m-1)!}.EI_1(4.\lambda)$$

Using (95), it is straightforward to deduce the diversity order of the random variable $1/U$, this diversity order is due to the term $\lambda^{2.n-1}.F_n(1/\lambda)$ and is equal to: $2.n-1-n-1 = n-2 = R.L-N$. The corresponding term in the sum giving $p_{1/U}(\lambda)$ gives the approximate law which can be used to obtain the Symbol Error Rate (SER) performances. We have here:

$$p_{1/U}(\lambda) \approx_{\lambda \to 0} K_{2,n}^{-1}.2^{2n+3}.\lambda^{2.n-1}.e^{-4.\lambda}.\frac{1}{4^{n+1}.\lambda^{n+1}}$$

$$p_{1/U}(\lambda) \approx_{\lambda \to 0} \frac{1}{(n-2)!.(n-1)!}.2^{2n+3}.\lambda^{2.n-1}.e^{-4.\lambda}.\frac{1}{2^{2n+2}.\lambda^{n+1}} = \frac{2}{(n-2)!.(n-1)!}.\lambda^{n-2}.e^{-4.\lambda}$$

-If $N = 3$, we have the p.d.f of the non-ordered eigenvalues of the Wishart matrix: $\boldsymbol{H}_1.\boldsymbol{H}_1^{\dagger}$.

$$p(\lambda_1,\lambda_2,\lambda_3) = \frac{1}{3!.K_{3,n}}.\prod_{i=1}^{3}\lambda_i^{n-3}.e^{-\lambda_i}.\prod_{i<j}(\lambda_i-\lambda_j)^2$$
$$= \frac{1}{3!.K_{3,n}}.(\lambda_1.\lambda_2.\lambda_3)^{n-3}.e^{-(\lambda_1+\lambda_2+\lambda_3)}.[(\lambda_1-\lambda_2).(\lambda_1-\lambda_3).(\lambda_2-\lambda_3)]^2 \qquad (96)$$

The normalization condition yields: $K_{3,n} = (n-3)!.(n-2)!.(n-1)!$. We want to calculate the p.d.f of the random variable: $\frac{1}{U} = [\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}]^{-1}$. To do this, we first calculate the p.d.f of the random variables: $X = \frac{1}{\lambda_1}, Y = \frac{1}{\lambda_2}, Z = \frac{1}{\lambda_3}$. the Jacobian of this transform is:

$$J(\lambda_1 = \frac{1}{x}, \lambda_2 = \frac{1}{y}, \lambda_3 = \frac{1}{z}) = \left| \mathrm{Det}\left(\begin{bmatrix} \frac{\partial\lambda_1}{\partial x} & \frac{\partial\lambda_1}{\partial y} & \frac{\partial\lambda_1}{\partial z} \\ \frac{\partial\lambda_2}{\partial x} & \frac{\partial\lambda_2}{\partial y} & \frac{\partial\lambda_2}{\partial z} \\ \frac{\partial\lambda_3}{\partial x} & \frac{\partial\lambda_3}{\partial y} & \frac{\partial\lambda_3}{\partial z} \end{bmatrix} \right) \right| = \left| \mathrm{Det}\left(\begin{bmatrix} -\frac{1}{x^2} & 0 & 0 \\ 0 & -\frac{1}{y^2} & 0 \\ 0 & 0 & -\frac{1}{z^2} \end{bmatrix} \right) \right| = \frac{1}{x^2.y^2.z^2}$$

The joint p.d.f of (*X*, *Y*, *Z*) is thus:

$$p_{1/\lambda_1,1/\lambda_2,1/\lambda_3}(x,y,z)=\frac{1}{3!}.K_{3,n}^{-1}.(\frac{1}{x.y.z})^{n-3}.e^{-(1/x+1/y+1/z)}.(\frac{1}{x}-\frac{1}{y})^2.(\frac{1}{y}-\frac{1}{z})^2.(\frac{1}{x}-\frac{1}{z})^2.\frac{1}{x^2.y^2.z^2}$$
$$=\frac{1}{3!}.K_{3,n}^{-1}.(\frac{1}{x.y.z})^{n-1}.e^{-(1/x+1/y+1/z)}.(\frac{1}{x}-\frac{1}{y})^2.(\frac{1}{y}-\frac{1}{z})^2.(\frac{1}{x}-\frac{1}{z})^2$$

$$p_{1/\lambda_1,1/\lambda_2,1/\lambda_3}(x,y,z)=\frac{1}{3!}.K_{3,n}^{-1}.(\frac{y-x}{x.y})^2.(\frac{z-y}{y.z})^2.(\frac{z-x}{x.z})^2.(\frac{1}{x.y.z})^{n-1}.e^{-(1/x+1/y+1/z)}$$
$$p_{1/\lambda_1,1/\lambda_2,1/\lambda_3}(x,y,z)=\frac{1}{3!}.K_{3,n}^{-1}.(y-x)^2.(z-y)^2.(z-x)^2.\frac{e^{-(1/x+1/y+1/z)}}{(x.y.z)^{n+1}} \quad (98)$$

To obtain the p.d.f of U we have then to calculate: $\text{Proba}(X+Y+Z=\lambda)=\int p_{X,Y,Z}(x,y,\lambda-x-y).dx.dy$ i.e. we have to average $p_{X,Y,Z}(x,y,\lambda-x-y)$ over all possible values of (x,y). This may be written as:

$$p_U(\lambda)=\frac{1}{3!}.K_{3,n}^{-1}.\int_0^{\lambda}\int_0^{\lambda-x}\frac{e^{-[1/x+(\lambda-x)/y.(\lambda-x-y)]}}{x^{n-1}.y^{n-1}(\lambda-x-y)^{n-1}}.(\frac{1}{x}-\frac{1}{y})^2.(\frac{1}{y}-\frac{1}{\lambda-x-y})^2.(\frac{1}{x}-\frac{1}{\lambda-x-y})^2.dy.dx \quad (99)$$

We begin to integrate (99) with variable y; this yields:

$$p_U(\lambda,x)=\frac{1}{3!}.K_{3,n}^{-1}.\int_0^{\lambda-x}\frac{e^{-[1/x+(\lambda-x)/y.(\lambda-x-y)]}}{x^{n-1}.y^{n-1}(\lambda-x-y)^{n-1}}.(\frac{1}{x}-\frac{1}{y})^2.(\frac{1}{y}-\frac{1}{\lambda-x-y})^2.(\frac{1}{x}-\frac{1}{\lambda-x-y})^2.dy$$
$$p_U(\lambda,x)=\frac{1}{3!}.K_{3,n}^{-1}.e^{-1/x}.\int_0^{\lambda-x}\frac{e^{-(\lambda-x)/y.(\lambda-x-y)}}{y^{n-1}(\lambda-x-y)^{n-1}}.(\frac{1}{x}-\frac{1}{y})^2.(\frac{1}{y}-\frac{1}{\lambda-x-y})^2.(\frac{1}{x}-\frac{1}{\lambda-x-y})^2.dy$$
$$p_U(\lambda,x)=\frac{1}{3!}.K_{3,n}^{-1}.e^{-1/x}.\int_0^{\lambda_x}\frac{e^{-(\lambda-x)/y.(\lambda-x-y)}}{x^4.y^{n+3}(\lambda-x-y)^{n+3}}.(y-x)^2.(\lambda-x-2.y)^2.(\lambda-2.x-y)^2.dy \quad (100)$$
$$p_U(\lambda,x)=\frac{4.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4}.\int_0^{\lambda_x}\frac{e^{-\lambda_x/y.(\lambda_x-y)}}{y^{n+3}(\lambda_x-y)^{n+3}}.(y-x)^2.(y-\lambda_x/2)^2.(y-\mu_x)^2.dy$$

With: $\lambda_x=\lambda-x$ and $\mu_x=\lambda-2.x$. To obtain the result in (100), we use once again the new variable $u=\lambda_x/y.(\lambda_x-y)$; we obtain:

$$p_U(\lambda,x)=\frac{4.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4}.\int_0^{\lambda_x}\frac{e^{-\lambda_x/y.(\lambda_x-y)}}{y^{n+3}(\lambda_x-y)^{n+3}}.(y-x)^2.(y-\lambda_x/2)^2.(y-\mu_x)^2.dy$$
$$=\frac{4.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4}.[\int_0^{\lambda_x/2}\frac{e^{-\lambda_x/y.(\lambda_x-y)}}{y^{n+3}(\lambda_x-y)^{n+3}}.(y-x)^2.(y-\lambda_x/2)^2.(y-\mu_x)^2.dy+...$$
$$...+\int_{\lambda_x/2}^{\lambda_x}\frac{e^{-\lambda_x/y.(\lambda_x-y)}}{y^{n+3}(\lambda_x-y)^{n+3}}.(y-x)^2.(y-\lambda_x/2)^2.(y-\mu_x)^2.dy]$$

$$p_U(\lambda,x)=\frac{4.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4}.\frac{2}{\lambda_x^{n+2}}.\int_{4/\lambda_x}^{+\infty}u^{n+1}.e^{-u}.([\frac{\lambda_x-\sqrt{\lambda_x^2-4.\lambda_x/u}}{2}]-x)^2.\sqrt{(\lambda_x^2-4.\lambda_x/u)}.([\frac{\lambda_x-\sqrt{\lambda_x^2-4.\lambda_x/u}}{2}]-\mu_x)^2.du$$

$$p_U(\lambda,x)=\frac{8.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4.\lambda_x^{n+2}}.[\int_{4/\lambda_x}^{+\infty}u^{n+1}.e^{-u}.([\frac{\lambda_x-\sqrt{\lambda_x^2-4.\lambda_x/u}}{2}]-x)^2.\sqrt{(\lambda_x^2-4.\lambda_x/u)}.([\frac{\lambda_x-\sqrt{\lambda_x^2-4.\lambda_x/u}}{2}]-\mu_x)^2.du$$
$$+\int_{4/\lambda_x}^{+\infty}u^{n+1}.e^{-u}.([\frac{\lambda_x+\sqrt{\lambda_x^2-4.\lambda_x/u}}{2}]-x)^2.\sqrt{(\lambda_x^2-4.\lambda_x/u)}.([\frac{\lambda_x+\sqrt{\lambda_x^2-4.\lambda_x/u}}{2}]-\mu_x)^2.du]$$

$$p_U(\lambda,x)=\frac{8.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4.\lambda_x^{n+2}}.[J_{n,x}^{(1)}+J_{n,x}^{(2)}]$$

With: $J_{n,x}^{(1)}=\int_{4/\lambda_x}^{+\infty}u^{n+1}.e^{-u}.([\frac{\lambda_x-\sqrt{\lambda_x^2-4.\lambda_x/u}}{2}]-x)^2.\sqrt{(\lambda_x^2-4.\lambda_x/u)}.([\frac{\lambda_x-\sqrt{\lambda_x^2-4.\lambda_x/u}}{2}]-\mu_x)^2.du$ and a similar definition for $J_{n,x}^{(2)}$. Of course, the derivation of $J_{n,x}^{(1)}$ is a complicated task; however we are only interested in the diversity order estimation. To obtain it we proceed with integral $J_{n,x}^{(1)}$ setting: $u=\frac{4}{\lambda_x}.z$; it comes:

$$J_{n,x}^{(1)}=(\frac{4}{\lambda_x})^{n+2}.\int_1^{+\infty}z^{n+1}.e^{-4.z/\lambda_x}.([\frac{\lambda_x-\sqrt{\lambda_x^2-\lambda_x^2/z}}{2}]-x)^2.\sqrt{(\lambda_x^2-\lambda_x^2/z)}.([\frac{\lambda_x-\sqrt{\lambda_x^2-\lambda_x^2/z}}{2}]-\mu_x)^2.dz$$
$$J_{n,x}^{(1)}=\frac{4^{n+2}}{\lambda_x^{n+1}}.\int_1^{+\infty}z^{n+1}.e^{-4.z/\lambda_x}.(\lambda_x.[\frac{1-\sqrt{1-1/z}}{2}]-x)^2.\sqrt{1-1/z}.(\lambda_x.[\frac{1-\sqrt{1-1/z}}{2}]-\mu_x)^2.dz$$

To further proceed with $J_{n,x}^{(1)}$ we can expand the quantity: $\sqrt{1-1/z}$ for $|z|>1$; we have:

$$(1-1/z)^{1/2}=1-\sum_{n=1}^{+\infty}\frac{(2n-2)!}{2^{2.n-1}.(n-1)!.n!}.\frac{1}{z^n}\quad\text{for}\quad|z|>1$$

Hence, we obtain:

$$p_U(\lambda,x)=\frac{2^{2.n+5}.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4.\lambda_x^{2.n+3}}.[\int_1^{+\infty}z^{n+1}.e^{-4.z/\lambda_x}.(\lambda_x.[\frac{1-\sqrt{1-1/z}}{2}]-x)^2.\sqrt{1-1/z}.(\lambda_x.[\frac{1-\sqrt{1-1/z}}{2}]-\mu_x)^2.dz$$
$$+\int_1^{+\infty}z^{n+1}.e^{-4.z/\lambda_x}.(\lambda_x.[\frac{1+\sqrt{1-1/z}}{2}]-x)^2.\sqrt{1-1/z}.(\lambda_x.[\frac{1+\sqrt{1-1/z}}{2}]-\mu_x)^2.dz]$$

$$p_U(\lambda,x)=\frac{2^{2.n+5}.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4.\lambda_x^{2.n+3}}.[\int_1^{+\infty}z^{n+1}.e^{-4.z/\lambda_x}.[\frac{\lambda_x^2}{4}([1-\sqrt{1-1/z}])^2-\lambda_x^2.\frac{[1-\sqrt{1-1/z}]}{2}+x.\mu_x]^2.\sqrt{1-1/z}.dz$$
$$+\int_1^{+\infty}z^{n+1}.e^{-4.z/\lambda_x}.[\frac{\lambda_x^2}{4}([1+\sqrt{1-1/z}])^2-\lambda_x^2.\frac{[1+\sqrt{1-1/z}]}{2}+x.\mu_x]^2.\sqrt{1-1/z}.dz]$$

To obtain the diversity order it is sufficient to study one integral form:

$$p_U^{(1)}(\lambda,x)=\frac{2^{2.n+5}.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4.\lambda_x^{2.n+3}}.[\int_1^{+\infty} z^{n+1}.e^{-4.z/\lambda_x}.[\frac{\lambda_x^2}{4}([1-\sqrt{1-1/z}])^2-\lambda_x^2.\frac{[1-\sqrt{1-1/z}]}{2}+x.\mu_x]^2.\sqrt{1-1/z}.dz$$

$$\begin{aligned}p_U^{(1)}(\lambda,x)=\frac{2^{2.n+5}.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4.\lambda_x^{2.n+3}}.[\int_1^{+\infty} z^{n+1}.e^{-4.z/\lambda_x}.[\frac{\lambda_x^2}{4}.(\sum_{m=1}^{+\infty}\frac{(2m-2)!}{2^{2.m-1}.(m-1)!.m!}.\frac{1}{z^m})^2...\\ ...-\frac{\lambda_x^2}{2}.(\sum_{m=1}^{+\infty}\frac{(2m-2)!}{2^{2.m-1}.(m-1)!.m!}.\frac{1}{z^m})+x.\mu_x]^2.(1-\sum_{m=1}^{+\infty}\frac{(2m-2)!}{2^{2.m-1}.(m-1)!.m!}.\frac{1}{z^m})].dz\end{aligned} \quad (101)$$

We note: $a_m=\frac{(2m-2)!}{2^{2.m-1}.(m-1)!.m!}$ and $b_k=\sum_{p=0}^{k}a_p.a_{k-p}$, (101) is rewritten as:

$$\begin{aligned}p_U^{(1)}(\lambda,x)=\frac{2^{2.n+5}.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4.\lambda_x^{2.n+3}}.[\int_1^{+\infty} z^{n+1}.e^{-4.z/\lambda_x}.[\frac{\lambda_x^2}{4}.(\sum_{k=1}^{+\infty}b_k.\frac{1}{z^k})...\\ ...-\frac{\lambda_x^2}{2}.(\sum_{k=1}^{+\infty}a_k.\frac{1}{z^k})+x.\mu_x]^2.(1-\sum_{k=1}^{+\infty}a_k.\frac{1}{z^k})].dz\end{aligned} \quad (102)$$

Looking carefully at equation (102) it appears that we have two kinds of integral forms; the first one corresponds to the case of a positive exponent for z^k and the second one to the case of a negative exponent for z^k. To study the diversity order it is sufficient to take the term corresponding to the lowest exponent degree of λ_x. Hence, we have only to consider the following integral form:

$$I_U^{(1,n)}(x)=\frac{2^{2.n+5}.K_{3,n}^{-1}.e^{-1/x}}{3!.x^4.\lambda_x^{2.n+3}}.[x.\mu_x]^2.\int_1^{+\infty} z^{n+1}.e^{-4.z/\lambda_x}.dz$$

$$\text{-1}\quad I_U^{(1,n)}(x)=\frac{2^{2.n+1}.K_{3,n}^{-1}.e^{-1/x}}{3!.x^2.\lambda_x^{2.n+3}}.\mu_x^2.\int_1^{+\infty} z^{n+1}.e^{-4.z/\lambda_x}.dz$$

For the diversity order study, it is sufficient to keep on working with: $I_U^{(1,n)}(x)$. We have thus:

$$\int_1^{+\infty} z^{n+1}.e^{-4.z/\lambda_x}.dz=F_{n+1}(\lambda_x)=e^{-4/\lambda_x}.\sum_{p=0}^{n+1}\frac{(n+1)!}{(n+1-p)!}.\frac{\lambda_x^{p+1}}{4^{p+1}}$$

Plugging this expression into the definition of $I_U^{(1,n)}(x)$, we arrive at:

$$I_U^{(1,n)}(x)=\frac{2^{2.n+1}.K_{3,n}^{-1}.e^{-1/x}}{3!.x^2.\lambda_x^{2.n+3}}.\mu_x^2.\int_1^{+\infty} z^{n+1}.e^{-4.z/\lambda_x}.dz$$

$$I_U^{(1,n)}(x)=\frac{2^{2.n+1}.K_{3,n}^{-1}.e^{-1/x}}{3!.x^2.\lambda_x^{2.n+3}}.\mu_x^2.e^{-4/\lambda_x}.\sum_{p=0}^{n+1}\frac{(n+1)!}{(n+1-p)!}.\frac{\lambda_x^{p+1}}{4^{p+1}}$$

For the diversity order expression we only work now with the term:

$$J_U^{(1,n,p)}(x)=\frac{2^{2.n+1}.K_{3,n}^{-1}.e^{-1/x}}{3!.x^2.\lambda_x^{2.n+3}}.\mu_x^2.e^{-4/\lambda_x}.\frac{\lambda_x^{p+1}}{4^{p+1}}$$
$$J_U^{(1,n,p)}(x)=\frac{2^{2.n-2.p-1}.K_{3,n}^{-1}}{3!}.\frac{\mu_x^2.e^{-1/x}.e^{-4/\lambda_x}}{x^2.\lambda_x^{2.n+2-p}} \quad (103)$$

To conclude it is then necessary to integrate $J_U^{(1,n,p)}(x)$ over the set of possible values for *x*. We have:

$$J_U^{(1,k,p)}=\int_0^{\lambda}\frac{2^{2.n-2.p-1}.K_{3,n}^{-1}}{3!}.\frac{\mu_x^2.e^{-1/x}.e^{-4/\lambda_x}}{x^2.\lambda_x^{2.n+2-p}}.dx$$
$$J_U^{(1,k,p)}=\frac{2^{2.n-2.p-1}.K_{3,n}^{-1}}{3!}.\int_0^{\lambda}\frac{\mu_x^2.e^{-1/x}.e^{-4/(\lambda-x)}}{x^2.(\lambda-x)^{2.n+2-p}}.dx$$

We keep on working with the last integral form: $K(n,p)=\int_0^{\lambda}\frac{(\lambda-2.x)^2.e^{-1/x}.e^{-4/(\lambda-x)}}{x^2.(\lambda-x)^{2.n-p+2}}.dx$. To study it, we use the new variable: $u=\frac{4}{\lambda-x}+\frac{1}{x}=\frac{3.x+\lambda}{x.(\lambda-x)}$; this gives:

$$x=\frac{u.\lambda-3+/-\sqrt{9-10.u.\lambda+u^2.\lambda^2}}{2.u}=\frac{\lambda}{2}-\frac{3}{2.u}+/-\frac{1}{2}.\lambda.\sqrt{1-\frac{10}{\lambda.u}+\frac{9}{\lambda^2.u^2}}$$

And:

$$dx=\frac{3}{2.u^2}+/-\frac{1}{4}.\frac{\lambda}{\sqrt{1-10/\lambda.u+9/\lambda^2.u^2}}.(\frac{10}{\lambda^2.u^2}-\frac{18}{\lambda^2.u^3})$$

Using the decomposition:

$$K(n,p)=\int_0^{\lambda}\frac{(\lambda-2.x)^2.e^{-1/x}.e^{-4/(\lambda-x)}}{x^2.(\lambda-x)^{2.n-p+2}}.dx=\int_0^{\lambda/3}\frac{(\lambda-2.x)^2.e^{-1/x}.e^{-4/(\lambda-x)}}{x^2.(\lambda-x)^{2.n-p+2}}.dx+\int_{\lambda/3}^{\lambda}\frac{(\lambda-2.x)^2.e^{-1/x}.e^{-4/(\lambda-x)}}{x^2.(\lambda-x)^{2.n-p+2}}.dx$$
$$=K^{(1)}(n,p)+K^{(2)}(n,p)$$

$$K^{(1)}(n,p)=\int_0^{\lambda/3}\frac{(\lambda-2.x)^2.e^{-1/x}.e^{-4/(\lambda-x)}}{x^2.(\lambda-x)^{2.n-p+2}}.dx$$
$$=\int_{9/\lambda}^{+\infty}e^{-u}\frac{[3/u+\lambda.\sqrt{1-10/\lambda.u+9/\lambda^2.u^2}]^2.[u.\lambda+3+\sqrt{9-10.u.\lambda+u^2.\lambda^2}]^{p-2-2.n}}{(2.u)^{p-2.n-2}[u.\lambda-3-\sqrt{9-10.u.\lambda+u^2.\lambda^2}]^2}.[\frac{3}{2.u^2}-\frac{1}{2}.\frac{1}{u.\sqrt{\lambda^2.u^2-10\lambda.u+9}}.(5-\frac{9}{u})].du$$
$$=9.\int_{9/\lambda}^{+\infty}e^{-u}.\frac{[1+(u.\lambda/3).\sqrt{1-10/\lambda.u+9/\lambda^2.u^2}]^2.[u.\lambda+3+\sqrt{9-10.u.\lambda+u^2.\lambda^2}]^{p-2-2.n}}{(2.u)^{p-2.n}[u.\lambda-3-\sqrt{9-10.u.\lambda+u^2.\lambda^2}]^2}.[\frac{3}{2.u^2}-\frac{1}{2}.\frac{1}{u.\sqrt{\lambda^2.u^2-10\lambda.u+9}}.(5-\frac{9}{u})].du$$

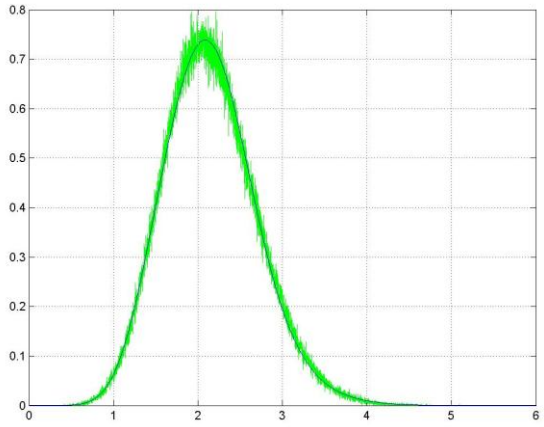
Then we use the new variable: $u=\frac{9}{\lambda}.x$; this gives:

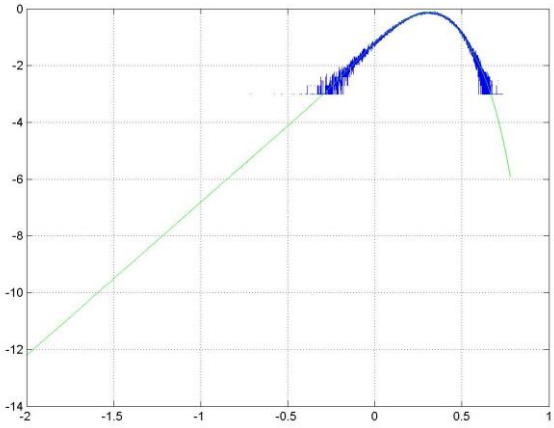
$$K^{(1)}(n,p)=9.\lambda^{p-2.n}\int_1^{+\infty}e^{-9.x/\lambda}.\frac{[1+3.x.\sqrt{1-10.x+9.x^2}]^2.[9.x+3+3.\sqrt{1-10.x+9.x^2}]^{p-2-2.n}}{(2.9.x)^{p-2.n}[9.x-3-3.\sqrt{1-10.x+9.x^2}]^2}.\frac{1}{2}[\frac{\lambda^2}{27.x^2}-\frac{\lambda}{27.x.\sqrt{1-10.x+9.x^2}}.(5-\frac{\lambda}{x})].\frac{9}{\lambda}.dx$$

Using this expression we can see that the lowest exponent for λ is : $\lambda^{p-2.n}$. Then, using the classical expansion: $e^{-9.x/\lambda} = \sum_{n=0}^{+\infty} \frac{(-1)^n.9^n.x^n}{n!.\lambda^n}$; it is then straightforward to conclude that $K^{(1)}(n,p)$ and $p_U(\lambda)$ can be written as: $K^{(1)}(n,p) \propto 1/\lambda^{2.n-p}$. However, the target p.d.f is those of $1/U$, not those of U; we have immediately: $p_{1/U}(\lambda) = \frac{1}{\lambda^2}.p_U(\frac{1}{\lambda})$. This gives the final result: $p_{1/U}(\lambda) \propto \frac{1}{\lambda^2}.\lambda^{2.n-p} \propto \frac{1}{\lambda^2}.\lambda^{2.n-(n+1)} \propto \lambda^{n-3}$ (when we take the highest value of p which is equal to $n + 1$). Hence we have proved that for $N = 3$ the diversity order is equal to: $n - 3$.

As we showed for $N = 3$, it is very difficult to find the exact law of the SNR when $N > 2$. However, we can approximate it by a mixture of Gamma laws following the expression given

of Nakagami laws. In appendix 5, we give the maximization step to determine the parameters of the Nakagami law mixture. For example, we found a mixture of six Nakagami laws for $R.L = 10$, $N = 3$ with the following results:





We found:

π = [0.0630 0.1084 0.1721 0.3285 0.1149 0.2132] ; ω = [4.1830 5.0961 5.9405 7.2067 6.5736 7.1543] ;

μ = [5.5000 5.9000 6.2800 6.0700 5.9800 3.1872].

Theoretical SER derivation:

We suppose $\mu = n + \varepsilon$ with $n \in N$, we have to calculate:

$$\bar{P}_e = \int_0^{+\infty} P_e(u).h(u).du = \frac{4.\gamma.\mu^{\mu}.\sigma^{4.\mu}}{\Gamma(\mu).\omega^{\mu}}.\int_0^{+\infty} u^{2.\mu-1}.e^{-\frac{\mu.\sigma^4}{\omega}.u^2} erfc(\sqrt{\xi.u}).du$$
$$= \frac{4.\gamma.\mu^{\mu}.\sigma^{4.\mu}}{\Gamma(\mu).\omega^{\mu}}.L \qquad (96)$$

With: $L = \int_0^{+\infty} u^{2.\mu-1}.e^{-\frac{\mu.\sigma^4}{\omega}.u^2} erfc(\sqrt{\xi.u}).du$; We have to find the primitive of:

$$f(x) = x^{2.\mu-1}.e^{-\frac{\mu.\sigma^4}{\omega}.x^2} = x^{2.(n+\varepsilon)-1}.e^{-\lambda.x^2} = x^{2.n-1+2.\varepsilon}.e^{-\lambda.x^2} .$$

This primitive can be written as:

$$F(x) = \int_0^x t^{2.n-1+2\varepsilon}.e^{-\lambda.t^2}.dt = \int_0^{\lambda.x^2} \left(\frac{w}{\lambda}\right)^{n-1/2+\varepsilon}.\frac{e^{-w}}{\sqrt{w}}.\frac{dw}{2.\sqrt{\lambda}} = \frac{1}{2.\lambda^{n+\varepsilon}} \int_0^{\lambda.x^2} w^{n-1+\varepsilon}.e^{-w}.dw .$$

The calculation of $I_{m-1} = \int_0^x u^{m-1+\varepsilon}.e^{-u}.du$ yields:

$$I_{m-1} = \int_0^x u^{m-1+\varepsilon}.e^{-u}.du = [-u^{m-1+\varepsilon}.e^{-u}]_0^x + (m-1+\varepsilon).\int_0^x u^{m-2+\varepsilon}.e^{-u}.du$$
$$I_{m-1} = -x^{m-1+\varepsilon}.e^{-x} + (m-1+\varepsilon).I_{m-2} = -x^{m-1+\varepsilon}.e^{-x} - (m-1+\varepsilon).x^{m-2+\varepsilon}.e^{-x} + (m-1+\varepsilon).(m-2+\varepsilon).I_{m-3}$$
$$I_{m-1} = -x^{m-1+\varepsilon}.e^{-x} - (m-1+\xi).x^{m-2+\varepsilon}.e^{-x} ... - (m-1+\varepsilon).(m-2+\varepsilon)...(m-k+1+\varepsilon).x^{m-k+\varepsilon}.e^{-x} ...$$
$$-(m-1+\varepsilon).(m-2+\varepsilon)...(m-(m-2)+\varepsilon).x^{1+\xi}.e^{-x} + (m-1+\varepsilon).(m-2+\varepsilon)...(m-(m-1)+\varepsilon).\int_0^x u^{\varepsilon}.e^{-u}.du$$

In this case, we can express $F(x)$ as:

$$F(x) = \frac{1}{2.\lambda^{n+\varepsilon}} \int_0^{\lambda.x^2} w^{n-1+\varepsilon}.e^{-w}.dw$$
$$= \frac{1}{2.\lambda^{n+\varepsilon}}.[-(\lambda.x^2)^{m-1+\varepsilon}.e^{-\lambda x^2} - (m-1+\varepsilon).(\lambda.x^2)^{m-2+\varepsilon}.e^{-\lambda x^2} ... - (m-1+\varepsilon).(m-2+\varepsilon)...(m-k+1+\varepsilon).(\lambda.x^2)^{m-k+\varepsilon}.e^{-\lambda x^2} ...$$
$$-(m-1+\varepsilon).(m-2+\varepsilon)...(m-(m-2)+\varepsilon).(\lambda.x^2)^{1+\varepsilon}.e^{-\lambda x^2} + (m-1+\varepsilon).(m-2+\varepsilon)...(m-(m-1)+\varepsilon).\int_0^{\lambda.x^2} u^{\varepsilon}.e^{-u}.du]$$

The last integral is equal to:

$$\int_0^{\lambda.x^2} u^{\varepsilon}.e^{-u}.du = \int_0^{\lambda.x^2} u^{\varepsilon-1+1}.e^{-u}.du = \Gamma(\lambda.x^2, \varepsilon+1) \text{ with: } \Gamma(x,m) = \int_0^x u^{m-1}.e^{-u}.du .$$

Using these preliminary results, we can integrate (96) by parts:

$$L=\int_0^{+\infty} u^{2.n-1+2.\varepsilon}.e^{-\lambda.u^2}.erfc(\sqrt{\xi.u}).du=[F(u).erfc(\sqrt{\xi.u})]_0^{+\infty}+\sqrt{\frac{\xi}{\pi}}.\int_0^{+\infty}F(u).\frac{e^{-\xi u}}{\sqrt{u}}.du$$

$$L=\sqrt{\frac{\xi}{\pi}}.\int_0^{+\infty}F(u).\frac{e^{-\xi u}}{\sqrt{u}}.du=\sqrt{\frac{\xi}{\pi}}.\int_0^{+\infty}\frac{1}{2.\lambda^{n+\varepsilon}}.[-(\lambda.u^2)^{m-1+\varepsilon}.e^{-\lambda u^2}-(m-1+\varepsilon).(\lambda.u^2)^{m-2+\varepsilon}.e^{-\lambda u^2}+...$$

$$-(m-1+\varepsilon).(m-2+\varepsilon)...(m-k+1+\varepsilon).(\lambda.u^2)^{m-k+\varepsilon}.e^{-\lambda u^2}+...$$

$$-(m-1+\varepsilon).(m-2+\varepsilon)...(m-(m-2)+\varepsilon).(\lambda.u^2)^{1+\varepsilon}.e^{-\lambda u^2}+...$$

$$+(m-1+\varepsilon).(m-2+\varepsilon)...(m-(m-1)+\varepsilon).\Gamma(\lambda u^2,\varepsilon+1)].\frac{e^{-\xi u}}{\sqrt{u}}.du]$$

With:

$$L_0=\int_0^{+\infty}\Gamma(\lambda u^2,\varepsilon+1).\frac{e^{-\beta.u}}{\sqrt{u}}.du=\sqrt{\frac{\pi}{\xi}}.\int_0^{+\infty}\Gamma(\lambda u^2,\varepsilon+1).\frac{d(erf(\sqrt{\xi.u}))}{du}.du$$

$$=\sqrt{\frac{\pi}{\xi}}.[[\Gamma(\lambda u^2,\varepsilon+1).erf(\sqrt{\xi.u})]_0^{+\infty}-\int_0^{+\infty}\frac{d\Gamma}{du}(\lambda u^2,\varepsilon+1)erf(\sqrt{\xi.u}).du$$

$$=\sqrt{\frac{\pi}{\xi}}.[\Gamma(\varepsilon+1)-\int_0^{+\infty}2.\lambda.u.(\lambda.u^2)^{\varepsilon}.e^{-\lambda.u^2}.erf(\sqrt{\xi.u}).du]$$

$$=\sqrt{\frac{\pi}{\xi}}.[\Gamma(\varepsilon+1)-2.\lambda^{\varepsilon+1}\int_0^{+\infty}u^{2.\varepsilon+1}.e^{-\lambda.u^2}.erf(\sqrt{\xi.u}).du]$$

To keep on calculating L_0, we expand the function $erf(\sqrt{\xi.u})$:

$$erf(\sqrt{\xi.u})=\frac{2}{\sqrt{\pi}}.\int_0^{\sqrt{\xi.u}}e^{-x^2}.dx=\frac{2}{\sqrt{\pi}}.\int_0^{\sqrt{\xi.u}}\sum_{n=0}^{+\infty}(-1)^n\frac{x^{2.n}}{n!}.dx=\frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}.u^{n+1/2}}{(2.n+1)}$$

Hence:

$$\int_0^{+\infty}u^{2.\varepsilon+1}.e^{-\lambda.u^2}.erf(\sqrt{\xi.u}).du=\frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\int_0^{+\infty}u^{2.\varepsilon+1}.e^{-\lambda.u^2}.u^{n+1/2}.du$$

$$=\frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\int_0^{+\infty}u^{n+2.\varepsilon+3/2}.e^{-\lambda.u^2}.du$$

$$=\frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\frac{1}{\sqrt{\lambda}}\int_0^{+\infty}(\frac{x}{\lambda})^{\varepsilon+n/2+3/4}.e^{-x}.\frac{dx}{2.\sqrt{x}}$$

$$=\frac{1}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1).\lambda^{\xi+n/2+5/4}}.\Gamma(n/2+\varepsilon+5/4)$$

Thus, we obtain:

$$L_0=\int_0^{+\infty}\Gamma(\lambda u^2,\varepsilon+1).\frac{e^{-\xi.u}}{\sqrt{u}}.du=\sqrt{\frac{\pi}{\xi}}.[\Gamma(\varepsilon+1)-2.\lambda^{\varepsilon+1}\frac{1}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1).\lambda^{\varepsilon+n/2+5/4}}.\Gamma(n/2+\varepsilon+5/4)]$$

To complete the result, we have to calculate:

$$L_{m-k} = \int_0^{+\infty} u^{2(m-k)+2\varepsilon}.\frac{e^{-\xi u}.e^{-\lambda u^2}}{\sqrt{u}}.du = \sqrt{\frac{\pi}{\xi}}.\int_0^{+\infty} u^{2(m-k)+2\varepsilon}.e^{-\lambda u^2}.\frac{d(erf(\sqrt{\xi.u}))}{du}.du$$

$$= \sqrt{\frac{\pi}{\xi}}.[[u^{2(m-k)+2\varepsilon}.e^{-\lambda u^2}.erf(\sqrt{\xi.u})]_0^{+\infty} + \ldots$$

$$- \int_0^{+\infty} erf(\sqrt{\xi.u}).e^{-\lambda u^2}.[(2.(m-k)+2.\varepsilon).u^{2(m-k)+2\varepsilon-1} - 2.\lambda.u.u^{2(m-k)+2\varepsilon}].du]$$

$$L_{m-k} = -2.\sqrt{\frac{\pi}{\xi}}.\int_0^{+\infty} erf(\sqrt{\xi.u}).e^{-\lambda u^2}.[(m-k+\varepsilon).u^{2(m-k)+2\varepsilon-1} - \lambda.u^{2(m-k)+2\varepsilon+1}].du]$$

This can be separated into two integral forms:

$$\int_0^{+\infty} u^{2(m-k)+2\varepsilon-1}.e^{-\lambda u^2}.erf(\sqrt{\xi.u}).du = \int_0^{+\infty} u^{2(m-k)+2\varepsilon-1}.e^{-\lambda u^2}.(\frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}.u^{n+1/2}}{(2.n+1)}).du$$

$$= \frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\int_0^{+\infty} u^{2(m-k)+2\varepsilon-1}.u^{n+1/2}.e^{-\lambda u^2}.du$$

$$= \frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\int_0^{+\infty} u^{2(m-k)+2\varepsilon+n-1/2}.e^{-\lambda u^2}.du$$

$$= \frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\frac{1}{\sqrt{\lambda}}.\int_0^{+\infty}(\frac{x}{\lambda})^{m-k+\varepsilon+n/2-1/4}.e^{-x}.\frac{dx}{2.\sqrt{x}}$$

$$= \frac{1}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\frac{\Gamma(m-k+\varepsilon+n/2+1/4)}{\lambda^{m-k+\varepsilon+n/2+1/4}}$$

And:

$$\int_0^{+\infty} u^{2(m-k)+2\varepsilon+1}.e^{-\lambda u^2}.erf(\sqrt{\xi.u}).du = \int_0^{+\infty} u^{2(m-k)+2\varepsilon+1}.e^{-\lambda u^2}.(\frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}.u^{n+1/2}}{(2.n+1)}).du$$

$$= \frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\int_0^{+\infty} u^{2(m-k)+2\varepsilon+1}.u^{n+1/2}.e^{-\lambda u^2}.du$$

$$= \frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\int_0^{+\infty} u^{2(m-k)+2\varepsilon+n+3/2}.e^{-\lambda u^2}.du$$

$$= \frac{2}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\frac{1}{\sqrt{\lambda}}.\int_0^{+\infty}(\frac{x}{\lambda})^{m-k+\varepsilon+n/2+3/4}.e^{-x}.\frac{dx}{2.\sqrt{x}}$$

$$= \frac{1}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\frac{\Gamma(m-k+\varepsilon+n/2+5/4)}{\lambda^{m-k+\varepsilon+n/2+5/4}}$$

We can now deduce L_{m-k}:

$$L_{m-k} = -2.\sqrt{\frac{\pi}{\xi}}.[(m-k+\varepsilon).\frac{1}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\frac{\Gamma(m-k+\varepsilon+n/2+1/4)}{\lambda^{m-k+\varepsilon+n/2+1/4}} + \ldots$$

$$- \lambda\frac{1}{\sqrt{\pi}}.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^{n+1/2}}{(2.n+1)}.\frac{\Gamma(m-k+\varepsilon+n/2+5/4)}{\lambda^{m-k+\varepsilon+n/2+5/4}}]$$

$$L_{m-k} = -2.[\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^n}{(2.n+1).\lambda^{m-k+\varepsilon+n/2+1/4}}.[(m-k+\varepsilon).\Gamma(m-k+\varepsilon+n/2+1/4) - \Gamma(m-k+\varepsilon+n/2+5/4)]$$

Finally, we obtain:

$$\begin{aligned}
&\bar{P}_e = \frac{4.\gamma.\mu^{\mu}.\sigma^{4.\mu}}{\Gamma(\mu).\omega^{\mu}}.L \\
&\bar{P}_e = \frac{4.\gamma.\mu^{\mu}.\sigma^{4.\mu}}{\Gamma(\mu).\omega^{\mu}}.\sqrt{\frac{\xi}{\pi}}.\frac{1}{2.\lambda^{n+\varepsilon}}[-\lambda^{n-1+\varepsilon}.L_{n-1} - (n-1+\varepsilon).\lambda^{n-2+\varepsilon}.L_{n-2} + ... \\
&-(n-1+\varepsilon).(n-2+\varepsilon)...(n-k+1+\varepsilon).\lambda^{n-k+\varepsilon}.L_{n-k} + ... \\
&-(n-1+\varepsilon).(n-2+\varepsilon)...(n-(n-2)+\varepsilon).\lambda^{1+\varepsilon}.L_1 + ... \\
&+(n-1+\varepsilon).(n-2+\varepsilon)...(n-(n-1)+\varepsilon).L_0]
\end{aligned} \tag{97}$$

With:

$$L_{n-k} = -2.[\sum_{i=0}^{+\infty}\frac{(-1)^i}{i!}.\frac{\xi^i}{(2.i+1).\lambda^{n-k+\varepsilon+i/2+1/4}}.[(n-k+\varepsilon).\Gamma(n-k+\varepsilon+i/2+1/4) - \Gamma(n-k+\varepsilon+i/2+5/4)] \tag{98}$$

Using the well known formula: $\Gamma(z+1) = z.\Gamma(z)$, we can simplify (98) into:

$$\begin{aligned}
L_{n-k} &= -2.[\sum_{i=0}^{+\infty}\frac{(-1)^i}{i!}.\frac{\xi^i}{(2.i+1).\lambda^{n-k+\varepsilon+i/2+1/4}}.[(n-k+\varepsilon).\Gamma(n-k+\varepsilon+i/2+1/4) - \\
&\qquad (n-k+\varepsilon+i/2+1/4).\Gamma(n-k+\varepsilon+i/2+1/4)]] \\
L_{n-k} &= -2.[\sum_{i=0}^{+\infty}\frac{(-1)^i}{i!}.\frac{\xi^i.\Gamma(n-k+\varepsilon+i/2+1/4)}{(2.i+1).\lambda^{n-k+\varepsilon+i/2+1/4}}.[(n-k+\varepsilon) - (n-k+\varepsilon+i/2+1/4)]] \\
L_{n-k} &= \frac{1}{2}.[\sum_{i=0}^{+\infty}\frac{(-1)^i}{i!}.\frac{\xi^i.\Gamma(n-k+\varepsilon+i/2+1/4)}{\lambda^{n-k+\varepsilon+i/2+1/4}}]
\end{aligned} \tag{99}$$

For $(n-k) \geq 1$ and :

$$\begin{aligned}
&L_0 = \int_0^{+\infty}\Gamma(\lambda x^2, \varepsilon+1).\frac{e^{-\xi.u}}{\sqrt{u}}.du = \sqrt{\frac{\pi}{\xi}}.[\Gamma(\varepsilon+1) - 2.\lambda^{\varepsilon+1}\frac{1}{\sqrt{\pi}}.\sum_{i=0}^{+\infty}\frac{(-1)^i}{i!}.\frac{\xi^{i+1/2}}{(2.i+1).\lambda^{\varepsilon+i/2+5/4}}.\Gamma(i/2+\varepsilon+5/4)] \\
&L_0 = \sqrt{\frac{\pi}{\xi}}.[\Gamma(\varepsilon+1) - \frac{2}{\sqrt{\pi}}.\sum_{i=0}^{+\infty}\frac{(-1)^i}{i!}.\frac{\xi^{i+1/2}}{(2.i+1).\lambda^{i/2+1/4}}.\Gamma(i/2+\varepsilon+5/4)] \\
&L_0 = \sqrt{\frac{\pi}{\xi}}.[\Gamma(\varepsilon+1) - \frac{2}{\sqrt{\pi}}.\sum_{i=0}^{+\infty}(\frac{-\xi}{\sqrt{\lambda}})^{i+1/2}\frac{1}{(2.i+1).i!}.\Gamma(i/2+\varepsilon+5/4)]
\end{aligned} \tag{100}$$

Remark: If $\mu = \varepsilon$, we obtain:

$$\bar{P}_e = \frac{2.\gamma.\lambda^{\mu-\varepsilon}}{\Gamma(\mu)}.[\sqrt{\frac{\pi}{\xi}}.\Gamma(\varepsilon) - 2.\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}.\frac{\xi^n}{(2.n+1).\lambda^{n/2+1/4}}.\Gamma(n/2+\varepsilon+1/4)] \tag{101}$$

Remark: At high SNR's, due to the high number of terms in the series expansion of $\bar{P}_e$, it may appear some divergence cases. To avoid this, we have developed an other bound for the SER working at high SNR's. This bound is obtained as follows:

$$\bar{P}_e = \int_0^{+\infty} P_e(u).h(u).du = \frac{4.\gamma.\mu^{\mu}.\sigma^{4.\mu}}{\Gamma(\mu).\omega^{\mu}}.\int_0^{+\infty} u^{2.\mu-1}.e^{-\frac{\mu.\sigma^4}{\omega}.u^2} erfc(\sqrt{\xi.u}).du$$

$$\bar{P}_e = \frac{4.\gamma.\mu^{\mu}.\sigma^{4.\mu}}{\Gamma(\mu).\omega^{\mu}}.[\ [\sum_{n=0}^{+\infty}\frac{(-1)^n.\mu^n.\sigma^{4.n}}{n!.\omega^n}.\frac{u^{2.(\mu+n)}}{2.(\mu+n)}.erfc(\sqrt{\xi.u})]_0^{+\infty} + ...$$
$$+\sum_{n=0}^{+\infty}\frac{(-1)^n.\mu^n.\sigma^{4.n}}{n!.\omega^n}.\sqrt{\frac{\xi}{\pi}}.\frac{1}{2.(\mu+n)}.\int_0^{+\infty} u^{2.(\mu+n)}.\frac{e^{-\xi.u}}{\sqrt{u}}.du$$

$$\bar{P}_e = \frac{4.\gamma.\mu^{\mu}.\sigma^{4.\mu}}{\Gamma(\mu).\omega^{\mu}}.[\ [\sum_{n=0}^{+\infty}\frac{(-1)^n.\mu^n.\sigma^{4.n}}{n!.\omega^n}.\frac{u^{2.(\mu+n)}}{2.(\mu+n)}.erfc(\sqrt{\xi.u})]_0^{+\infty} + ...$$
$$+\sum_{n=0}^{+\infty}\frac{(-1)^n.\mu^n.\sigma^{4.n}}{n!.\omega^n}.\sqrt{\frac{\xi}{\pi}}.\frac{1}{2.(\mu+n)}.\int_0^{+\infty} u^{2.(\mu+n)-1/2}.e^{-\xi.u}.du$$

$$\bar{P}_e = \frac{4.\gamma.\mu^{\mu}.\sigma^{4.\mu}}{\Gamma(\mu).\omega^{\mu}}.[\ [\sum_{n=0}^{+\infty}\frac{(-1)^n.\mu^n.\sigma^{4.n}}{n!.\omega^n}.\frac{u^{2.(\mu+n)}}{2.(\mu+n)}.erfc(\sqrt{\xi.u})]_0^{+\infty} + ...$$
$$+\sum_{n=0}^{+\infty}\frac{(-1)^n.\mu^n.\sigma^{4.n}}{n!.\omega^n}.\sqrt{\frac{\xi}{\pi}}.\frac{1}{2.\xi^{2.(\mu+n)+1/2}.(\mu+n)}.\int_0^{+\infty} t^{2.(\mu+n)-1/2}.e^{-t}.dt$$

We have: $\lim_{u\to\infty}\sum_{n=0}^{+\infty}\frac{(-1)^n.\mu^n.\sigma^{4.n}}{n!.\omega^n}.\frac{u^{2.(\mu+n)}}{2.(\mu+n)}.erfc(\sqrt{\xi.u}) = \lim_{u\to\infty}(\int_0^u t^{2.\mu-1}.e^{-\frac{\mu.\sigma^4}{\omega}.t^2}.dt).erfc(\sqrt{\xi.u})$

With:

$$\lim_{u\to\infty}(\int_0^u t^{2.\mu-1}.e^{-\frac{\mu.\sigma^4}{\omega}.t^2}.dt).erfc(\sqrt{\xi.u}) \le \lim_{u\to\infty}(\int_0^u t^{2.\mu-1}.dt).erfc(\sqrt{\xi.u})$$
$$= \lim_{u\to\infty}\frac{u^{2.\mu}}{2.\mu}.erfc(\sqrt{\xi.u}) = \lim_{u\to\infty}\frac{u^{2.\mu}}{2.\mu}.\frac{e^{-\xi.u}}{\sqrt{\pi}}.(\frac{1}{\xi.u} - \frac{1}{2.(\xi.u)^3} + \frac{3}{4.(\xi.u)^5} - \frac{5}{8.(\xi.u)^7}) = 0$$

Finally, we obtain the following bound at high SNR's:

$$\bar{P}_{e,high\ SNR's} = \frac{4.\gamma.\mu^{\mu}.\sigma^{4.\mu}}{\Gamma(\mu).\omega^{\mu}}.\sum_{n=0}^{+\infty}\frac{(-1)^n.\mu^n.\sigma^{4.n}}{n!.\omega^n}.\sqrt{\frac{\xi}{\pi}}.\frac{1}{2.\xi^{2.(\mu+n)+1/2}.(\mu+n)}.\Gamma(2.(\mu+n)+1/2)$$
$$\bar{P}_{e,,high\ SNR's} = \frac{2.\gamma.}{\sqrt{\pi}.\Gamma(\mu).}\sum_{n=0}^{+\infty}\frac{(-1)^n.\mu^{n+\mu}.\sigma^{4.(n+\mu)}}{n!.\omega^{(n+\mu)}}.\frac{\Gamma(2.(\mu+n)+1/2)}{\xi^{2.(\mu+n)}.(\mu+n)} \qquad (102)$$

5.2 **Simulation set-up and results:**

In this part, we give some simulation results to illustrate the accuracy of the proposed SER bounds and to test the efficacy of the optimization power allocation algorithm given in Section 4.2.

We consider at first the simplest case where we have $N = 2$ mobile stations in the network since it corresponds to a scenario where we have been able to completely characterize all the p.d.f's. We suppose that the total

available power of the relays P_T is normalized equal to 1 and that the relays have perfect CSI estimates. We consider the following cases:

1- Uniform power allocation between each transmit relay

2- Optimized power allocation (see Section 4.2) with two sub-cases

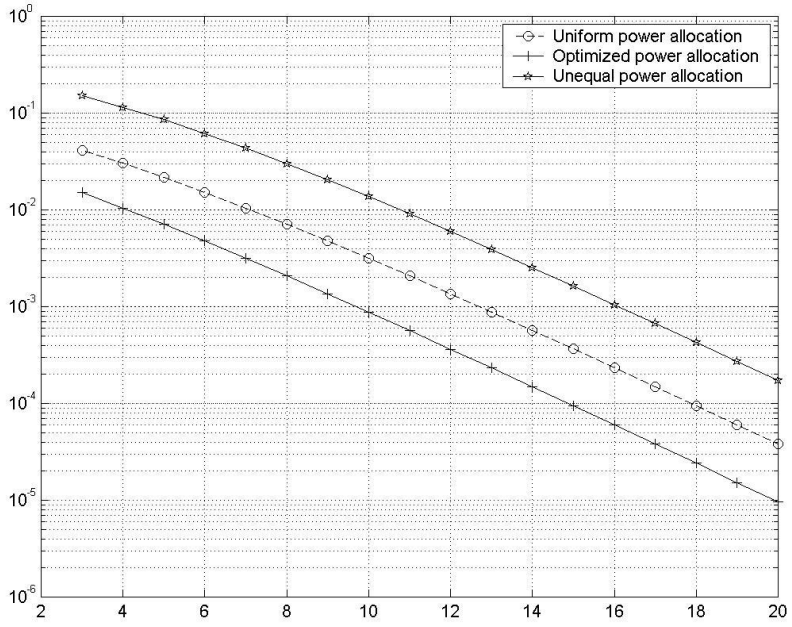
2.1- Optimum algorithm (see Section 4, equations (51-52))

2.2- Approximated solution (see Section 4, equations (55-56))

3- Zero forcing with complete CSI knowledge at each relay station (Section 5.1)

Each time the theoretical SER bound is available, we compare it with Monte-Carlo simulation results. We study the cases of QPSK and QAM-16.

3Zero forcing with complete CSI knowledge at each relay station (Section 5.1)



Appendix 1: Characterization of the square of the sum of i.i.d Rayleigh random variables

- We consider at first the case of the sum of two Rayleigh independent distributed random variables U_1 and U_2. Each of one has a p.d.f: ($q_i = R-(i-1)$)

$$g_{\lambda_{k,i}}(x) = \frac{2}{(q_i-1)!}.x^{2.q_i-1}.\exp(-x^2) \quad (1.1)$$

We calculate then:

$$\begin{aligned}
p^{(2)}_{U_1+U_2}(x) &= g_{\lambda_{k,i}}(x) * g_{\lambda_{p,i}}(x) = \int_0^x \frac{4}{(q_i-1)!^2}.u^{2.q_i-1}.\exp(-u^2).(x-u)^{2.q_i-1}.\exp(-(x-u)^2).du \\
&= \frac{4}{(q_i-1)!^2}.e^{-x^2}.\int_0^x u^{2.q_i-1}.(x-u)^{2.q_i-1}.e^{-2.[u^2-x.u]}.du \\
&= \frac{4}{(q_i-1)!^2}.e^{-x^2}.\int_0^x u^{2.q_i-1}.(x-u)^{2.q_i-1}.e^{-2.([u-x/2]^2-x^2/4)}.du \\
&= \frac{4}{(q_i-1)!^2}.e^{-x^2/2}.\int_0^x u^{2.q_i-1}.(x-u)^{2.q_i-1}.e^{-2.[u-x/2]^2}.du \\
&= \frac{4}{(q_i-1)!^2}.e^{-x^2/2}.\int_{-x/2}^{x/2} (t+x/2)^{2.q_i-1}.(x/2-t)^{2.q_i-1}.e^{-2.t^2}.dt \\
&= \frac{4}{(q_i-1)!^2}.e^{-x^2/2}.\int_{-x/2}^{x/2} (x^2/4-t^2)^{2.q_i-1}.e^{-2.t^2}.dt
\end{aligned} \quad (1.2)$$

We use the development: $(x^2/4-t^2)^m = \sum_{k=0}^{m} C_m^k(-1)^k.t^{m-k}.(x^2/4)^k$; with $m = 2.q_i - 1$, hence we obtain:

$$\begin{aligned}
p^{(2)}_{U_1+U_2}(x) &= g_{\lambda_{k,i}}(x) * g_{\lambda_{p,i}}(x) = \frac{4}{(q_i-1)!^2}.e^{-x^2/2}.\int_{-x/2}^{x/2} (x^2/4-t^2)^{2.q_i-1}.e^{-2.t^2}.dt \\
&= \frac{4}{(q_i-1)!^2}.\sum_{k=0}^{2.q_i-1} C^k_{2.q_i-1}(-1)^{k+1}.(x^2/4)^k.e^{-x^2/2}.\int_{-x/2}^{x/2} t^{2.(2.q_i-1-k)}.e^{-2.t^2}.dt \\
&= \frac{8}{(q_i-1)!^2}.\sum_{k=0}^{2.q_i-1} C^k_{2.q_i-1}(-1)^{k+1}.(x^2/4)^k.e^{-x^2/2}.\int_{0}^{x/2} t^{2.(2.q_i-1-k)}.e^{-2.t^2}.dt
\end{aligned} \quad (1.3)$$

We denote: $F_{2.q_i-1-k}(x) = \int_0^{x/2} t^{2.(2.q_i-1-k)}.e^{-2.t^2}.dt = \frac{1}{2^{2.q_i-k}}.\int_0^x u^{2.(2.q_i-1-k)}.e^{-u^2/2}.du$. We want now to compute the p.d.f of $(U_1+U_2)^2$. We have the relationship: $p^{(2)}_{(U_1+U_2)^2}(x^2).2.x.dx = p^{(2)}_{(U_1+U_2)}(x).dx$; this yields to:

$$p^{(2)}_{(U_1+U_2)^2}(x) = \frac{1}{2.\sqrt{x}}.p^{(2)}_{(U_1+U_2)}(\sqrt{x}) \quad (1.4)$$

Hence:

$$
\begin{aligned}
p^{(2)}_{(U_1+U_2)^2}(x) &= \frac{1}{2.\sqrt{x}}.p^{(2)}_{(U_1+U_2)}(\sqrt{x}) \\
&= \frac{1}{2.\sqrt{x}}.\frac{8}{(q_i-1)!^2}.\sum_{k=0}^{2.q_i-1} C^k_{2.q_i-1}(-1)^k.(x/4)^k.e^{-x/2}.\int_0^{\sqrt{x}/2} t^{2.(2.q_i-1-k)}.e^{-2.t^2}.dt \\
&= \frac{4}{(q_i-1)!^2}.\sum_{k=0}^{2.q_i-1} C^k_{2.q_i-1}\frac{(-1)^k}{4^k}.x^{k-1/2}.e^{-x/2}.\int_0^{\sqrt{x}/2} t^{2.(2.q_i-1-k)}.e^{-2.t^2}.dt \\
&= \frac{4}{(q_i-1)!^2}.\sum_{k=0}^{2.q_i-1} C^k_{2.q_i-1}\frac{(-1)^k}{4^k}.x^{k-1/2}.e^{-x/2}.F_{2.q_i-1-k}(\sqrt{x})
\end{aligned}
\qquad (1.5)
$$

With: $F_{2.q_i-1-k}(\sqrt{x}) = \frac{1}{2^{2.q_i-k}}.\int_0^{\sqrt{x}} u^{2.(2.q_i-1-k)}.e^{-u^2/2}.du = \frac{1}{2^{2.q_i+1-k}}.\int_0^{x} t^{2.q_i-1-k-1/2}.e^{-t/2}.dt = \frac{1}{2^{2.q_i+1-k}}.G_{2.q_i-1-k}(x)$ and

$G_k(x) = \int_0^x t^{k-1/2}.e^{-t/2}.dt$

$$
\begin{aligned}
p^{(2)}_{(U_1+U_2)^2}(x) &= \frac{4}{(q_i-1)!^2}.\sum_{k=0}^{2.q_i-1} C^k_{2.q_i-1}\frac{(-1)^k}{4^k}.x^{k-1/2}.e^{-x/2}.\frac{1}{2^{2.q_i+1-k}}.\int_0^x t^{2.q_i-1-k-1/2}.e^{-t/2}.dt \\
p^{(2)}_{(U_1+U_2)^2}(x) &= \frac{4}{(q_i-1)!^2}.\sum_{k=0}^{2.q_i-1} C^k_{2.q_i-1}\frac{(-1)^k}{2^{2.q_i+k+1}}.x^{k-1/2}.e^{-x/2}.G_{2.q_i-1-k}(x)
\end{aligned}
\qquad (1.6)
$$

To obtain the diversity gain, it is necessary to obtain the Taylor expansion of function: $G_k(x)$. We have obviously :

$$
G_{2.q_i-1-k}(x) = \int_0^x t^{2.q_i-1-k-1/2}.e^{-t/2}.dt = \sum_{n=0}^{+\infty}\frac{(-1)^n}{2^n.n!}\int_0^x t^{n+2.q_i-k-3/2}.dt = \sum_{n=0}^{+\infty}\frac{(-1)^n.x^{n+2.q_i-k-1/2}}{2^n.(n+2.q_i-k-1/2).n!}
$$

And :

$$
x^{k-1/2}.e^{-x/2}.G_{2.q_i-1-k}(x) = e^{-x/2}.\sum_{n=0}^{+\infty}\frac{(-1)^n.x^{n+2.q_i-1}}{2^n.(n+2.q_i-k-1/2).n!}
$$

Hence, the diversity order in this case (for the convolution of two Rayleigh random variables) is equal to: $D_2 = 2.q_i - 1.$

We will assume now that we have, at rank *n*, the following assumptions: the diversity order is equal to: $n.q_i - 1$ and the p.d.f around $x \approx 0$ may be well approximated by:

$$
p^{(n)}_{(Z_n)^2}(x) \approx K_n.e^{-x/n}.x^{n.q_i-1} \quad \text{with : } Z_n = \sum_{i=1}^{n} U_i
$$

We calculate then: $p_{Z_n}^{(n)}(x) = p_{(Z_n)^2}^{(n)}(x^2).2.x$, i.e. : $p_{Z_n}^{(n)}(x) = K_n.2.x.[x^2]^{n.q_i-1}.e^{-x^2/n} = 2.K_n.x^{2.n.q_i-1}.e^{-x^2/n}$. To obtain the result at rank n +1, we have now to compute the convolution product of $p_{Z_n}^{(n)}(x)$ with the p.d.f of U_{n+1} which is equal to : $\frac{2}{(q_i-1)!}.x^{2.q_i-1}.\exp(-x^2)$. We obtain then:

$$\begin{aligned} p_{Z_{n+1}}^{(n+1)}(x) &= 2.K_n.x^{2.n.q_i-1}.e^{-x^2/n} * \frac{2}{(q_i-1)!}.x^{2.q_i-1}.e^{-x^2} \\ &= \frac{4.K_n.}{(q_i-1)!}.x^{2.n.q_i-1}.e^{-x^2/n} * x^{2.q_i-1}.e^{-x^2} \\ &= K_{n+1}.x^{2.n.q_i-1}.e^{-x^2/n} * x^{2.q_i-1}.e^{-x^2} \end{aligned} \qquad (1.7)$$

$$\begin{aligned} & p_{Z_{n+1}}^{(n+1)}(x) = K_{n+1}.\int_0^x u^{2.n.q_i-1}.e^{-u^2/n}.(x-u)^{2.q_i-1}.e^{-(x-u)^2}.du \\ & = K_{n+1}.e^{-x^2}.\int_0^x u^{2.n.q_i-1}.(x-u)^{2.q_i-1}.e^{-(1+1/n).[u^2-2.n.x.u/(n+1)]}.du \\ & = K_{n+1}.e^{-x^2}.e^{x^2.n/n+1}.\int_0^x u^{2.n.q_i-1}.(x-u)^{2.q_i-1}.e^{-[(n+1)/n].[u-n.x/(n+1)]^2}.du \\ & = K_{n+1}.e^{-x^2/(n+1)}.\int_{-n.x/(n+1)}^{x/(n+1)} [t+n.x/(n+1)]^{2.n.q_i-1}.[x/(n+1)-t]^{2.q_i-1}.e^{-[(n+1)/n].t^2}.dt \end{aligned} \qquad (1.8)$$

We can calculate the polynomial function: $[t+n.x/(n+1)]^{2.n.q_i-1}.[x/(n+1)-t]^{2.q_i-1}$. We set: $\alpha_{n+1} = \frac{1}{n+1}$ and: $\beta_{n+1} = \frac{n}{n+1} = 1-\alpha_{n+1}$, we obtain in this case:

$$\begin{aligned} & P_{n+1}(t,x) = [t+n.x/(n+1)]^{2.n.q_i-1}.[x/(n+1)-t]^{2.q_i-1} \\ & = [t+\beta_{n+1}.x]^{2.n.q_i-1}.[\alpha_{n+1}.x-t]^{2.q_i-1} \\ & = [\sum_{k=0}^{2.n.q_i-1} C_{2.n.q_i-1}^k.t^k.(\beta_{n+1}.x)^{2.n.q_i-1-k}].[\sum_{k=0}^{2.q_i-1} C_{2.q_i-1}^k.(-1)^k.t^k.(\alpha_{n+1}.x)^{2.q_i-1-k}] \\ & = [\sum_{k=0}^{2.n.q_i-1} C_{2.n.q_i-1}^k.\beta_{n+1}^{2.n.q_i-1-k}.t^k.x^{2.n.q_i-1-k}].[\sum_{k=0}^{2.q_i-1} C_{2.q_i-1}^k.(-1)^k.\alpha_{n+1}^{2.q_i-1-k}.t^k.x^{2.q_i-1-k}] \\ & = [\sum_{k=0}^{2.n.q_i-1} a_k.t^k.x^{2.n.q_i-1-k}].[\sum_{m=0}^{2.q_i-1} b_m.t^m.x^{2.q_i-1-m}] \end{aligned}$$

With: $a_k = C_{2.n.q_i-1}^k.\beta_{n+1}^{2.n.q_i-1-k}$ and $b_m = (-1)^m.C_{2.q_i-1}^m.\alpha_{n+1}^{2.q_i-1-m}$; we obtain:

$$P_{n+1}(t,x) = \sum_{m=0}^{2.(n+1).q_i-1} [\sum_{k=0}^{m} a_k.b_{m-k}].t^m.x^{2.(n+1).q_i-2-m} \qquad (1.9)$$

Plugging this result into the calculation of $p_{Z_{n+1}}^{(n+1)}(x)$ (1.8), we have :

$$\begin{aligned} p_{Z_{n+1}}^{(n+1)}(x) &= K_{n+1}.e^{-x^2/(n+1)}.\int_{-n.x/(n+1)}^{x/(n+1)} P_{n+1}(t,x).e^{-[(n+1)/n].t^2}.dt \\ &= K_{n+1}.e^{-x^2/(n+1)}.\sum_{m=0}^{2.(n+1).q_i-1}[\sum_{k=0}^{m} a_k.b_{m-k}].x^{2.(n+1).q_i-2-m}.\int_{-n.x/(n+1)}^{x/(n+1)} t^m.e^{-[(n+1)/n].t^2}.dt \end{aligned} \quad (1.10)$$

We want now to compute the p.d.f of $(Z_{n+1})^2$. We have the relationship: $p_{(Z_{n+1})^2}^{(n+1)}(x^2).2.x.dx = p_{(Z_{n+1})}^{(n+1)}(x).dx$; this yields to:

$$p_{(Z_{n+1})^2}^{(n+1)}(x) = \frac{1}{2.\sqrt{x}}.p_{(Z_{n+1})}^{(n+1)}(\sqrt{x}) \quad (1.11)$$

$$\begin{aligned} p_{(Z_{n+1})^2}^{(n+1)}(x) &= \frac{1}{2.\sqrt{x}}.p_{(Z_{n+1})}^{(n+1)}(\sqrt{x}) \\ &= \frac{1}{2.\sqrt{x}}.K_{n+1}.e^{-x/(n+1)}.\sum_{m=0}^{2.(n+1).q_i-1}[\sum_{k=0}^{m} a_k.b_{m-k}].x^{(n+1).q_i-1-m/2}.\int_{-n.\sqrt{x}/(n+1)}^{\sqrt{x}/(n+1)} t^m.e^{-[(n+1)/n].t^2}.dt \\ &= \frac{K_{n+1}}{2}.e^{-x/(n+1)}.\sum_{m=0}^{2.(n+1).q_i-1}[\sum_{k=0}^{m} a_k.b_{m-k}].x^{(n+1).q_i-1-(m+1)/2}.[F(\sqrt{x}/(n+1)) - F(-n.\sqrt{x}/(n+1))] \end{aligned}$$

We have :

$$\begin{aligned} F(\sqrt{x}/(n+1)) &= \int_0^{\sqrt{x}/(n+1)} t^m.e^{-[(n+1)/n].t^2}.dt \\ &= \int_0^{x}[\frac{\sqrt{u}}{\sqrt{n+1}}]^m.e^{-u/n.(n+1)}.\frac{du}{2.\sqrt{u}.\sqrt{n+1}} \\ &= \frac{1}{2.(n+1)^{(m+1)/2}}\int_0^{x} u^{(m-1)/2}.e^{-u/n.(n+1)}.du \end{aligned} \quad (1.12)$$

And :

$$\begin{aligned} F(-n.\sqrt{x}/(n+1)) &= \int_0^{-n.\sqrt{x}/(n+1)} t^m.e^{-[(n+1)/n].t^2}.dt \\ &= \int_0^{x}[\sqrt{\frac{n}{n+1}}.u]^m.e^{-n.u/(n+1)}.\frac{n}{n+1}.\frac{du}{2.\sqrt{u}} \\ &= \frac{1}{2}.[\frac{n}{n+1}]^{m/2+1}.\int_0^{x} u^{(m-1)/2}.e^{-n.u/(n+1)}.du \end{aligned} \quad (1.13)$$

It is then straightforward to obtain the Taylor expansion of $F(\sqrt{x}/(n+1))$ and $F(-n.\sqrt{x}/(n+1))$; we have :

$$F(\sqrt{x}/(n+1)) = \frac{1}{2.(n+1)^{(m+1)/2}} \int_0^x u^{(m-1)/2}.e^{-u/n.(n+1)}.du$$
$$= \frac{1}{2.(n+1)^{(m+1)/2}} \sum_{k=0}^{+\infty} \frac{(-1)^k}{n^k.(n+1)^k.k!} \int_0^x u^{k+(m-1)/2}.du \qquad (1.14)$$
$$= \frac{1}{2.(n+1)^{(m+1)/2}} \sum_{k=0}^{+\infty} \frac{(-1)^k.x^{k+(m+1)/2}}{n^k.(n+1)^k.k!.[k+(m+1)/2]}$$

And:

$$F(-n.\sqrt{x}/(n+1)) = \frac{1}{2}.[\frac{n}{n+1}]^{m/2+1}.\int_0^x u^{(m-1)/2}.e^{-n.u/(n+1)}.du$$
$$= \frac{1}{2}.[\frac{n}{n+1}]^{m/2+1}.\sum_{k=0}^{+\infty} \frac{(-1)^k.n^k}{(n+1)^k.k!} \int_0^x u^{(m-1)/2+k}.du \qquad (1.15)$$
$$= \frac{1}{2}.[\frac{n}{n+1}]^{m/2+1}.\sum_{k=0}^{+\infty} \frac{(-1)^k.n^k.x^{(m+1)/2+k}}{(n+1)^k.k!.[k+(m+1)/2]}$$

So, finally, we obtain :

$$p_{(Z_{n+1})^2}^{(n+1)}(x) = \frac{K_{n+1}}{2}.e^{-x/(n+1)}.\{\sum_{m=0}^{2.(n+1).q_i-1} \frac{[\sum_{k=0}^{m} a_k.b_{m-k}]}{2.(n+1)^{(m+1)/2}}.x^{(n+1).q_i-1-(m+1)/2}.[\sum_{k=0}^{+\infty} \frac{(-1)^k.x^{k+(m+1)/2}}{n^k.(n+1)^k.k!.[k+(m+1)/2]}] \qquad (1.16)$$
$$- \sum_{m=0}^{2.(n+1).q_i-1} \frac{n^{m/2+1}[\sum_{k=0}^{m} a_k.b_{m-k}]}{2.(n+1)^{m/2+1}}.x^{(n+1).q_i-1-(m+1)/2}.[\sum_{k=0}^{+\infty} \frac{(-1)^k.n^k.x^{k+(m+1)/2}}{(n+1)^k.k!.[k+(m+1)/2]}]\}$$

The power of the current term x^i is equal to:

$$(n+1).q_i - 1 - (m+1)/2 + (m+1)/2 + k = (n+1).q_i - 1 + k \qquad (1.17)$$

In this case it is clear that the diversity order at rank $n+1$ is equal to: $(n+1).q_i - 1$.

Conclusion: The diversity order of a square sum of n i.i.d Rayleigh random variables, each having a diversity order equal to $2.q_i - 1$, is : $n.q_i - 1$.

Appendix 2: Characterization of the p.d.f of: $Z = \lambda_{k,i}.\lambda_{l,j} = U.V$

- We deal at first with the general case: $i \neq j$.

At first, we calculate the p.d.f of $\lambda_{k,i}$. Assuming that each real or imaginary part of each coefficient channel has a power 0.5, the p.d.f of $\lambda_{k,i}^2$ can be written as:

$$f_{\lambda_{k,i}^2}(x) = \frac{1}{\sigma_h^{2q_i}.2^{q_i}.\Gamma(q_i)}.x^{q_i-1}.\exp(-\frac{x}{2.\sigma_h^2}) \quad (2.1)$$

with : $\sigma_h^2 = 0.5$, this yields to : ($q_i = R-(i-1)$)

$$f_{\lambda_{k,i}^2}(x) = \frac{1}{(1/2)^{q_i}.2^{q_i}.(q_i-1)!}.x^{q_i-1}.\exp(-x) = \frac{1}{(q_i-1)!}.x^{q_i-1}.\exp(-x) \quad (2.2)$$

Denoting $g_{\lambda_{k,i}}(x)$ the pdf of $\lambda_{k,i}$, we have the following relationship:

$$f_{\lambda_{k,i}^2}(x).dx = g_{\lambda_{k,i}}(\sqrt{x}).d\sqrt{x} = \frac{1}{2}.g_{\lambda_{k,i}}(\sqrt{x}).\frac{1}{\sqrt{x}}.dx$$

which entails: $g_{\lambda_{k,i}}(\sqrt{x}) = 2.\sqrt{x}.f_{\lambda_{k,i}^2}(x)$ and $g_{\lambda_{k,i}}(x) = 2.x.f_{\lambda_{k,i}^2}(x^2) = \frac{2}{(q_i-1)!}.x^{2.q_i-1}.\exp(-x^2)$

So, we have demonstrated that:

$$g_{\lambda_{k,i}}(x) = \frac{2}{(q_i-1)!}.x^{2.q_i-1}.\exp(-x^2) \quad (2.3)$$

In the same way, we would obtain:

$$g_{\lambda_{l,j}}(x) = \frac{2}{(q_j-1)!}.x^{2.q_j-1}.\exp(-x^2) \quad (2.4) \quad \text{with: } q_j = R-(j-1)$$

We have now to find the p.d.f of the variable: $\lambda_{k,i}.\lambda_{l,j}$. To do this, we use the cumulative density function (c.d.f) and we set: $U = \lambda_{k,i}$ and $V = \lambda_{l,j}$. We have:

$$\text{Proba}(U.V \leq X) = \int_0^{+\infty} \text{Proba}(U \leq X/V|V).p_V(V).dV = \int_0^{+\infty}[\int_0^{X/V} p_U(y).dy].p_V(V).dV \quad (2.4)$$

Setting $y = z/V$ in the first integral, we obtain: $\int_0^{X/V} p_U(y).dy = \int_0^{X} p_U(z/V).\frac{dz}{V}$ and (2.4) changes into:

$$F_{U.V}(X) = \text{Proba}(U.V \le X) = \int_0^{+\infty} [\int_0^X p_U(z/V).\frac{dz}{V}].p_V(V).dV$$
$$= \int_0^{+\infty} [\int_0^X p_U(z/V).\frac{dz}{V}].p_V(V).dV = \int_0^{+\infty} [\int_0^X p_U(z/V).dz].\frac{p_V(V)}{V}.dV \quad (2.5)$$

This can be expressed as: $F_{U.V}(X) = = \int_0^{+\infty} [\int_0^X p_U(z/\lambda).dz].\frac{p_V(\lambda)}{\lambda}.d\lambda$.

We calculate at first:

$$\int_0^X p_U(z/\lambda).dz = \frac{2}{(q_i-1)!}.\int_0^X (z/\lambda)^{2.q_i-1}.\exp(-z^2/\lambda^2).dz$$
$$= \frac{2}{(q_i-1)!}.\int_0^{X^2} (\sqrt{u}/\lambda)^{2.q_i-1}.\exp(-u/\lambda^2).\frac{1}{2.\sqrt{u}}.du$$
$$= \frac{1}{(q_i-1)!}.\int_0^{X^2} \frac{u^{q_i-1}}{\lambda^{2.q_i-1}}.\exp(-u/\lambda^2).du \quad (2.6)$$
$$= \frac{\lambda}{(q_i-1)!}.\int_0^{X^2/\lambda^2} x^{q_i-1}.\exp(-x).dx = \frac{\lambda}{(q_i-1)!}.J_{q_i-1}(X^2/\lambda^2)$$

With: $J_n(x) = \int_0^x t^n.e^{-t}.dt$. Then, we use the classical result:

$$J_n(x) = \int_0^x t^n.e^{-t}.dt = [-t^n.e^{-t}]_0^x + n.\int_0^x t^{n-1}.e^{-t}.dt = -x^n.e^{-x} + n.I_{n-1}$$
$$= -e^{-x}.[x^n + n.x^{n-1} + n.(n-1).x^{n-2} + ... + n!] + n! \quad (2.7)$$
$$= n!.[1 - e^{-x}.(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + ... + \frac{x^{n-k}}{(n-k)!} + ... + x + 1)]$$

This yields to:

$$\int_0^X p_U(z/\lambda).dz = \frac{\lambda}{(q_i-1)!}.J_{q_i-1}(X^2/\lambda^2)$$
$$= \frac{\lambda}{(q_i-1)!}.(q_i-1)!.[1 - e^{-X^2/\lambda^2}.[\frac{(X^2/\lambda^2)^{q_i-1}}{(q_i-1)!} + \frac{(X^2/\lambda^2)^{q_i-2}}{(q_i-2)!} + ... + \frac{(X^2/\lambda^2)^{q_i-1-k}}{(q_i-1-k)!} + ... + X^2/\lambda^2 + 1]] \quad (2.8)$$
$$= \lambda.[1 - e^{-X^2/\lambda^2}.[\frac{(X^2/\lambda^2)^{q_i-1}}{(q_i-1)!} + \frac{(X^2/\lambda^2)^{q_i-2}}{(q_i-2)!} + ... + \frac{(X^2/\lambda^2)^{q_i-1-k}}{(q_i-1-k)!} + ... + X^2/\lambda^2 + 1]]$$

And finally, we obtain:

$$F_{U.V}(X) = \int_0^{+\infty} [\int_0^X p_U(z/\lambda).dz].\frac{p_V(\lambda)}{\lambda}.d\lambda$$
$$= \int_0^{+\infty} [1 - e^{-X^2/\lambda^2}.[\frac{(X^2/\lambda^2)^{q_i-1}}{(q_i-1)!} + \frac{(X^2/\lambda^2)^{q_i-2}}{(q_i-2)!} + ... + \frac{(X^2/\lambda^2)^{q_i-1-k}}{(q_i-1-k)!} + ... + X^2/\lambda^2 + 1]].\frac{2}{(q_j-1)!}.\lambda^{2.q_j-1}.\exp(-\lambda^2).d\lambda$$

$$= \int_0^{+\infty} [\frac{2.\lambda^{2.q_j-1}}{(q_j-1)!}.\exp(-\lambda^2).d\lambda - \frac{2.\lambda^{2.q_j-1}.e^{-[X^2/\lambda^2+\lambda^2]}}{(q_j-1)!}.[\frac{(X^2/\lambda^2)^{q_i-1}}{(q_i-1)!} + \frac{(X^2/\lambda^2)^{q_i-2}}{(q_i-2)!} + ... + \frac{(X^2/\lambda^2)^{q_i-1-k}}{(q_i-1-k)!} + ... + X^2/\lambda^2 + 1]].d\lambda$$

$$F_{U.V}(X) = 1 - \sum_{k=0}^{q_i-1} \int_0^{+\infty} \frac{2.\lambda^{2.q_j-1}.e^{-[X^2/\lambda^2+\lambda^2]}}{(q_j-1)!}.\frac{(X^2/\lambda^2)^{q_i-1-k}}{(q_i-1-k)!}.d\lambda$$

$$= 1 - \sum_{k=0}^{q_i-1} \frac{2.X^{2.(q_i-1-k)}}{(q_j-1)!.(q_i-1-k)!} \int_0^{+\infty} \lambda^{2.(q_j-q_i)+2.k+1}.e^{-[X^2/\lambda^2+\lambda^2]}.d\lambda \quad (2.9)$$

$$= 1 - \sum_{k=0}^{q_i-1} \frac{2.X^{2.(q_i-1-k)}}{(q_j-1)!.(q_i-1-k)!}.I_{2.(q_j-q_i)+2.k+1}(X)$$

With: $I_{2.(q_j-q_i)+2.k+1}(X) = \int_0^{+\infty} \lambda^{2.(q_j-q_i)+2.k+1}.e^{-[X^2/\lambda^2+\lambda^2]}.d\lambda$

We suppose at first that: $q_j > q_i$. The probability density function is the derivative of function: $F_{U.V}(X)$. In order to calculate this derivative, we have to compute at first:

$$\frac{\partial I_{2.(q_j-q_i)+2.k+1}(X)}{\partial X} = \int_0^{+\infty} \lambda^{2.(q_j-q_i)+2.k+1}.\frac{\partial}{\partial X}[e^{-(\lambda^2+X^2/\lambda^2)}].d\lambda = \int_0^{+\infty} \lambda^{2.(q_j-q_i)+2.k+1}.e^{-\lambda^2}.-\frac{2.X.e^{-X^2/\lambda^2}}{\lambda^2}.d\lambda$$

$$= -2.X.\int_0^{+\infty} \lambda^{2.(q_j-q_i)+2.k-1}.e^{-(\lambda^2+X^2/\lambda^2)}.d\lambda = -2.X.I_{2.(q_j-q_i)+2.k-1}(X)$$

We obtain then:

$$p_{U.V}(X) = \frac{\partial}{\partial X}[1 - \frac{2.X^{2.(q_i-1)}.I_{2.(q_j-q_i)+1}(X)}{(q_j-1)!.(q_i-1)!} - \frac{2.X^{2.(q_i-2)}.I_{2.(q_j-q_i)+3}(X)}{(q_j-1)!.(q_i-2)!} - ... - \frac{2.X^{2.(q_i-1-k)}.I_{2.(q_j-q_i)+2.k+1}(x)}{(q_j-1)!.(q_i-1-k)!} - ... - \frac{2.I_{2.q_j-1}(x)}{(q_j-1)!}]$$

$$= -\frac{4.(q_i-1).X^{2.(q_i-1)-1}.I_{2.(q_j-q_i)+1}(X)}{(q_j-1)!.(q_i-1)!} + \frac{2.X^{2.(q_i-1)}.2.X.I_{2.(q_j-q_i)-1}(X)}{(q_j-1)!.(q_i-1)!} - \frac{4.(q_i-2).X^{2.(q_i-2)-1}.I_{2.(q_j-q_i)+3}(X)}{(q_j-1)!(q_i-2)!} + \frac{2.X^{2.(q_i-2)}.2.X.I_{2.(q_j-q_i)+1}(X)}{(q_j-1)!(q_i-2)!} + ...$$

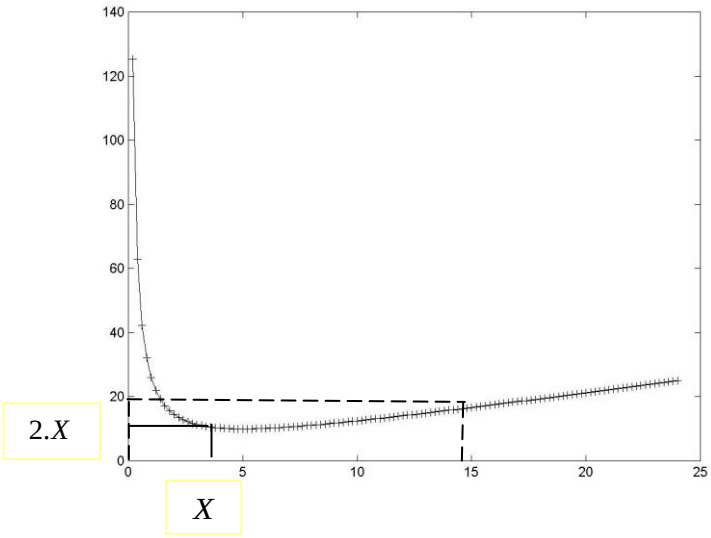
$$= -\frac{4.X^{2.(q_i-1)-1}.I_{2.(q_j-q_i)+1}(X)}{(q_j-1)!.(q_i-2)!} + \frac{4.X^{2.q_i-1}.I_{2.(q_j-q_i)-1}(X)}{(q_i-1)!(q_j-1)!} - \frac{4.X^{2.(q_i-2)-1}.I_3(X)}{(q_j-1)!(q_i-3)!} + \frac{4.X^{2.q_i-3}.I_1(X)}{(q_j-1)!(q_i-2)!} - ...$$

$$p_{U.V}(X) = \frac{4.X^{2.q_i-1}.I_{2.(q_j-q_i)-1}(X)}{(q_j-1)!.(q_i-1)!} \quad (2.10)$$

We have: $I_{2.(q_j-q_i)-1}(X) = \int_0^{+\infty} \lambda^{2.(q_j-q_i)-1}.e^{-[X^2/\lambda^2+\lambda^2]}.d\lambda$, setting $u = \lambda^2$, we obtain:

$$I_{2.(q_j-q_i)-1}(X) = \frac{1}{2}.\int_0^{+\infty} u^{q_j-q_i-1}.e^{-[X^2/u+u]}.du \quad (2.11)$$

To compute this integral we set: $z = u + \frac{X^2}{u}$, the graph of function $f(u) = u + \frac{X^2}{u}$ is drawn just below:



When we solve the equation $z = u + \frac{X^2}{u}$ => $u^2 - u.z + X^2 = 0$, we obtain the solutions:

- $u = \frac{1}{2}.[z - \sqrt{z^2 - 4.X^2}]$ for $u \in [0, X]$ and $du = \frac{1}{2}.[1 - \frac{z}{\sqrt{z^2 - 4.X^2}}].dz$
- $u = \frac{1}{2}.[z + \sqrt{z^2 - 4.X^2}]$ for $u \in [X, +\infty[$ et $du = \frac{1}{2}.[1 + \frac{z}{\sqrt{z^2 - 4.X^2}}].dz$

Then, we obtain:

$$I_{2.(q_j - q_i)-1}(X) = \frac{1}{2}.\int_{+\infty}^{2.X} \frac{1}{2^{q_j - q_i - 1}}.[z - \sqrt{z^2 - 4.X^2}]^{q_j - q_i - 1}.e^{-z}.\frac{1}{2}.[1 - \frac{z}{\sqrt{z^2 - 4.X^2}}].dz + ...$$

$$... + \frac{1}{2}.\int_{2.X}^{+\infty} \frac{1}{2^{q_j - q_i - 1}}.[z + \sqrt{z^2 - 4.X^2}]^{q_j - q_i - 1}.e^{-z}.\frac{1}{2}.[1 + \frac{z}{\sqrt{z^2 - 4.X^2}}].dz$$

$$I_{2.(q_j - q_i)-1}(X) = \frac{1}{2^{q_j - q_i + 1}}.[\int_{2.X}^{+\infty} [z - \sqrt{z^2 - 4.X^2}]^{q_j - q_i}.\frac{e^{-z}}{\sqrt{z^2 - 4.X^2}}.dz + \int_{2.X}^{+\infty} [z + \sqrt{z^2 - 4.X^2}]^{q_j - q_i}.\frac{e^{-z}}{\sqrt{z^2 - 4.X^2}}.dz]$$

This yields to:

$$I_{2.(q_j - q_i)-1}(X) = \frac{1}{2^{q_j - q_i + 1}}.[\int_{2.X}^{+\infty} [z - \sqrt{z^2 - 4.X^2}]^{q_j - q_i}.\frac{e^{-z}}{\sqrt{z^2 - 4.X^2}}.dz + \int_{2.X}^{+\infty} [z + \sqrt{z^2 - 4.X^2}]^{q_j - q_i}.\frac{e^{-z}}{\sqrt{z^2 - 4.X^2}}dz]$$

$$= \frac{1}{2^{q_j - q_i}}.\sum_{p=0}^{2.p \le q_j - q_i} C_{q_j - q_i}^{2.p}.\int_{2.X}^{+\infty} z^{q_j - q_i - 2.p}.[z^2 - 4.X^2]^p.\frac{e^{-z}}{\sqrt{z^2 - 4.X^2}}.dz$$

$$= \frac{1}{2^{q_j - q_i}}.\sum_{p=0}^{2.p \le q_j - q_i} C_{q_j - q_i}^{2.p}.\int_{2.X}^{+\infty} z^{q_j - q_i - 2.p}.[z^2 - 4.X^2]^{p-1/2}.e^{-z}.dz$$

The final variable change is when we set: $z = 2.X.u$, so we have:

$$I_{2.(q_j-q_i)-1}(X)=\frac{1}{2^{q_j-q_i}}.\sum_{p=0}^{2.p\le q_j-q_i}C_{q_j-q_i}^{2.p}.\int_{2.X}^{+\infty}z^{q_j-q_i-2.p}.[z^2-4.X^2]^{p-1/2}.e^{-z}.dz$$

$$=\frac{1}{2^{q_j-q_i}}.\sum_{p=0}^{2.p\le q_j-q_i}C_{q_j-q_i}^{2.p}.\int_{1}^{+\infty}(2.X.u)^{q_j-q_i-2.p}.2^{2.p-1}.X^{2.p-1}.(u^2-1)^{p-1/2}.e^{-2.X.u}.2.X.du$$

$$=\sum_{p=0}^{2.p\le q_j-q_i}C_{q_j-q_i}^{2.p}.X^{q_j-q_i}.\int_{1}^{+\infty}u^{q_j-q_i-2.p}.(u^2-1)^{p-1/2}.e^{-2.X.u}.du$$

$$=\sum_{p=0}^{2.p\le q_j-q_i}C_{q_j-q_i}^{2.p}.X^{q_j-q_i}.\int_{1}^{+\infty}u^{q_j-q_i-1}.(1-1/u^2)^{p-1/2}.e^{-2.X.u}.du$$

$$=\sum_{p=0}^{2.p\le q_j-q_i}C_{q_j-q_i}^{2.p}.X^{q_j-q_i}.K_{q_j-q_i}^{p}(X)$$

To conclude, we have to express integral values of the form:

$$K_p^{q_j-q_i}(X)=\int_1^{+\infty}u^{q_j-q_i-1}.[1-1/u^2]^{p-1/2}.e^{-2.X.u}.du \quad (2.12)$$

Since we have $1/u^2<1$ for $u\in[1,+\infty[$, it is possible to expand the function : $u\longrightarrow[1-1/u^2]^{p-1/2}$ over the integration interval:

$$[1-1/u^2]^{p-1/2}=1-\frac{(p-1/2)}{u^2}+\frac{(p-1/2).(p-3/2)}{2!.u^4}+...+(-1)^q\frac{(p-1/2).(p-3/2)...(p-1/2-q+1)}{q!.u^{2.q}}+...$$

With :

$$(k-1/2).(k-3/2)...(k-1/2-q+1)=\frac{(2.k-1).(2.k-3)...(2.k-2.q+1)}{2^q}$$

$$=\frac{(2.k).(2.k-1).(2.k-2).(2.k-3)...(2.k-2.q+1).(2.k-2.q)}{2^q.(2.k).(2.k-2).(2k-4)...(2.k-2.q)}$$

$$=\frac{(2.k).(2.k-1).(2.k-2).(2.k-3)...(2.k-2.q+1).(2.k-2.q)}{2^{2.q+1}.k.(k-1).(k-2)...(k-q)}$$

$$=\frac{(2.k).(2.k-1).(2.k-2).(2.k-3)...(2.k-2.q+1).(2.k-2.q).(2.k-2.q-1)!.(k-q-1)!}{2^{2.q+1}.k.(k-1).(k-2)...(k-q).(k-q-1)!.(2.k-2.q-1)!.}$$

$$=\frac{(2.k)!.(k-q-1)!}{2^{2.q+1}.k!.(2.k-2.q-1)!}$$

Thus, we obtain:

$$[1-1/u^2]^{p-1/2}=\sum_{n=0}^{+\infty}(-1)^n\frac{(2.p)!.(p-n-1)!}{2^{2.n+1}.n!.p!.(2.p-2.n-1)!}.\frac{1}{u^{2.n}} \quad (2.13)$$

Plugging (2.13) into (2.12), we obtain:

$$
\begin{aligned}
K_p^{q_j-q_i}(X) &= \int_1^{+\infty} u^{q_j-q_i-1}.[1-1/u^2]^{p-1/2}.e^{-2.X.u}.du \\
&= \sum_{r=0}^{+\infty}(-1)^r \frac{(2.p)!.(p-r-1)!}{2^{2.r+1}.r!.p!.(2.p-2.r-1)!}.\int_1^{+\infty}\frac{1}{u^{2.r}}.u^{q_j-q_i-1}.e^{-2.X.u}.du \\
&= \sum_{r=0}^{+\infty}(-1)^r \frac{(2.p)!.(p-r-1)!}{2^{2.r+1}.r!.p!.(2.p-2.r-1)!}.\int_1^{+\infty} u^{q_j-q_i-1-2.r}.e^{-2.X.u}.du \\
&= \sum_{r=0}^{+\infty}(-1)^r \frac{(2.p)!.(p-r-1)!}{2^{2.r+1}.r!.p!.(2.p-2.r-1)!}.L_{q_j-q_i-1-2.r}(X)
\end{aligned}
\qquad (2.14)
$$

With: $L_{q_j-q_i-1-2.r} = \int_1^{+\infty} u^{q_j-q_i-1-2.r}.e^{-2.X.u}.du$. Then, considering whether $q_j-q_i-1-2.r$ is positive or not, we have:

-1 if $q_j-q_i-1-2.r \geq 0$:

$$
\begin{aligned}
L_{q_j-q_i-1-2.r}(X) &= \int_1^{+\infty} u^{q_j-q_i-1-2.r}.e^{-2.X.u}.du = \int_0^{+\infty}(1+t)^{q_j-q_i-1-2.r}.e^{-2.X.(t+1)}.dt \\
&= e^{-2.X}.\sum_{k=0}^{q_j-q_i-1-2.r} C^k_{q_j-q_i-1-2.r}.\int_0^{+\infty} t^k.e^{-2.X.t}.dt = e^{-2.X}.\sum_{k=0}^{q_j-q_i-1-2.r} C^k_{q_j-q_i-1-2.r}.\frac{1}{[2.X]^{k+1}}\int_0^{+\infty} z^k.e^{-z}.dz \qquad (2.15) \\
&= e^{-2.X}.\sum_{k=0}^{q_j-q_i-1-2.r} C^k_{q_j-q_i-1-2.r}.\frac{\Gamma(k+1)}{[2.X]^{k+1}} = e^{-2.X}.\sum_{k=0}^{q_j-q_i-1-2.r} C^k_{q_j-q_i-1-2.r}.\frac{k!}{2^{k+1}.X^{k+1}}
\end{aligned}
$$

- 2 if $q_j-q_i-1-2.r<0$:

$$
L_{q_j-q_i-1-2.r}(X) = \int_1^{+\infty} u^{q_j-q_i-1-2.r}.e^{-2.X.u}.du = \int_1^{+\infty}\frac{e^{-2.X.u}}{u^{2.r+1+q_i-q_j}}.du = e^{-2.X}.\int_0^{+\infty}\frac{e^{-2.X.t}}{(1+t)^{2.r+1+q_i-q_j}}.dt \qquad (2.16)
$$

Using an induction process, it is possible to express this last integral with the exponential integral, we have:

$$
\begin{aligned}
L_n(X) &= \int_1^{+\infty}\frac{e^{-2.X.u}}{u^n}.du \\
&= e^{-2.X}.[\frac{1}{n-1}-\frac{2.X}{(n-1).(n-2)}+...+(-1)^k.\frac{2^k.X^k}{(n-1).(n-2)...(n-k-1)}+... \\
&\quad ...+(-1)^{n-1}.\frac{2^{n-1}.X^{n-1}}{(n-1).(n-2)...(n-(n-1))}.\int_0^{+\infty}\frac{e^{-2.X.u}}{1+u}.du] \\
&= e^{-2.X}.[\frac{1}{n-1}-\frac{2.X}{(n-1).(n-2)}+...+(-1)^k.\frac{2^k.X^k}{(n-1).(n-2)...(n-k-1)}+.....+(-1)^{n-1}.\frac{2^{n-1}.X^{n-1}}{(n-1)!}.EI_1(2.X)]
\end{aligned}
$$

With: $EI_1(X) = \int_X^{+\infty}\frac{e^{-t}.dt}{t} = \int_1^{+\infty}\frac{e^{-X.t}.dt}{t} = e^{-X}.\int_0^{+\infty}\frac{e^{-X.t}}{1+t}.dt = \frac{e^{-X}}{X}.\int_0^{+\infty}\frac{e^{-X.t}}{1+t/X}.dt = -\ln X-\gamma+\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}.X^n}{n.n\ !}$

In conclusion, we obtain:

$$p_{U.V}(X) = \frac{4.X^{2.q_i-1}.I_{2.(q_j-q_i)-1}(X)}{(q_j-1)!.(q_i-1)!} \quad (2.17)$$

$$I_{2.(q_j-q_i)-1}(X) = \sum_{p=0}^{2.p\le q_j-q_i} C_{q_j-q_i}^{2.p}.X^{q_j-q_i}.K_{q_j-q_i}^{p}(X)$$

With:

$$K_p^{q_j-q_i}(X) = \sum_{r=0}^{+\infty}(-1)^r \frac{(2.p)!.(p-r-1)!}{2^{2.r+1}.r!.p!.(2.p-2.r-1)!}.L_{q_j-q_i-1-2.r}(X)$$

$$= \sum_{r;q_j-q_i-1-2.r\ge 0}(-1)^r \frac{(2.p)!.(p-r-1)!}{2^{2.r+1}.r!.p!.(2.p-2.r-1)!}.L_{q_j-q_i-1-2.r}(X) + \sum_{r;q_j-q_i-1-2.r<0}(-1)^r \frac{(2.p)!.(p-r-1)!}{2^{2.r+1}.r!.p!.(2.p-2.r-1)!}.L_{q_j-q_i-1-2.r}(X)$$

We have:

- For $n \ge 0$:

$$L_n(X) = e^{-2.X}.\sum_{k=0}^{n} C_n^k.\frac{k!}{2^{k+1}.X^{k+1}}$$

- For $n < 0$:

$$L_n(X) = e^{-2.X}.[\frac{1}{n-1} - \frac{2.X}{(n-1).(n-2)} + ... + (-1)^k.\frac{2^k.X^k}{(n-1).(n-2)...(n-k-1)} + ... + (-1)^{n-1}.\frac{2^{n-1}.X^{n-1}}{(n-1)!}.EI_1(2.X)]$$

With: $EI_1(X) = \int_X^{+\infty}\frac{e^{-t}.dt}{t} = \int_1^{+\infty}\frac{e^{-X.t}.dt}{t} = e^{-X}.\int_0^{+\infty}\frac{e^{-X.t}}{1+t}.dt = \frac{e^{-X}}{X}.\int_0^{+\infty}\frac{e^{-X.t}}{1+t/X}.dt = -\ln X - \gamma + \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}.X^n}{n.n\ !}$

With γ denoting the Euler's constant: $\gamma \approx 0{,}5772$.

Using the development of $L_n(X)\ \forall n \ge 0$; it is easy to show that the diversity order of the product *U.V* is equal to: $q_i + q_j - 1$ (we have: $\forall n \ge 0,\ \lim_{x\to 0} x^n.\ln(x) = 0$); and that we have for $X \approx 0$: $p_{U.V}(X) \approx K.X^{q_i+q_j-1}.e^{-2.X}$.

The case where *j = i* is addressed now.

- Characterization of the p.d.f of: $Z = \lambda_{k,i}.\lambda_{l,i} = U.V$

At first, we calculate the p.d.f of $\lambda_{k,i}$. Assuming that each real or imaginary part of each coefficient channel has a power 0.5, the p.d.f of $\lambda_{k,i}^2$ can be written as:

$$f_{\lambda_{k,i}^2}(x) = \frac{1}{\sigma_h^{2q_i}.2^{q_i}.\Gamma(q_i)}.x^{q_i-1}.\exp(-\frac{x}{2.\sigma_h^2}) \quad (2.1)$$

with : $\sigma_h^2 = 0.5$, this yields to : ($q_i = R-(i-1)$)

$$f_{\lambda_{k,i}^2}(x) = \frac{1}{(1/2)^{q_i}.2^{q_i}.(q_i-1)!}.x^{q_i-1}.\exp(-x) = \frac{1}{(q_i-1)!}.x^{q_i-1}.\exp(-x) \quad (2.2)$$

Denoting $g_{\lambda_{k,i}}(x)$ the pdf of $\lambda_{k,i}$, we have the following relationship:

$$f_{\lambda_{k,i}^2}(x).dx = g_{\lambda_{k,i}}(\sqrt{x}).d\sqrt{x} = \frac{1}{2}.g_{\lambda_{k,i}}(\sqrt{x}).\frac{1}{\sqrt{x}}.dx$$

which entails: $g_{\lambda_{k,i}}(\sqrt{x}) = 2.\sqrt{x}.f_{\lambda_{k,i}^2}(x)$ and $g_{\lambda_{k,i}}(x) = 2.x.f_{\lambda_{k,i}^2}(x^2) = \frac{2}{(q_i-1)!}.x^{2.q_i-1}.\exp(-x^2)$

So, we have demonstrated that:

$$g_{\lambda_{k,i}}(x) = \frac{2}{(q_i-1)!}.x^{2.q_i-1}.\exp(-x^2) \quad (2.3)$$

We have now to find the p.d.f of the variable: $\lambda_{k,i}.\lambda_{l,i}$, assuming that $\lambda_{k,i}, \lambda_{l,i}$ are two independent identically distributed (i.i.d) random variables. To do this, we use the cumulative density function and we set: $U = \lambda_{k,i}$ and $V = \lambda_{l,i}$. We have:

$$\text{Proba}(U.V \le X) = \int_0^{+\infty} \text{Proba}(U \le X/V|V).p_V(V).dV = \int_0^{+\infty}[\int_0^{X/V} p_U(y).dy].p_V(V).dV \quad (2.4)$$

Setting $y = z/V$ in the first integral, we obtain: $\int_0^{X/V} p_U(y).dy = \int_0^{X} p_U(z/V).\frac{dz}{V}$ and (2.4) changes into:

$$\begin{aligned} F_{U.V}(X) &= \text{Proba}(U.V \le X) = \int_0^{+\infty}[\int_0^{X} p_U(z/V).\frac{dz}{V}].p_V(V).dV \\ &= \int_0^{+\infty}[\int_0^{X} p_U(z/V).\frac{dz}{V}].p_V(V).dV = \int_0^{+\infty}[\int_0^{X} p_U(z/V).dz].\frac{p_V(V)}{V}.dV \end{aligned} \quad (2.5)$$

This can be expressed as: $F_{U.V}(X) = = \int_0^{+\infty}[\int_0^{X} p_U(z/\lambda).dz].\frac{p_V(\lambda)}{\lambda}.d\lambda$.

We calculate at first:

$$\begin{aligned}
&\int_0^X p_U(z/\lambda).dz = \frac{2}{(q_i-1)!}.\int_0^X (z/\lambda)^{2.q_i-1}.\exp(-z^2/\lambda^2).dz \\
&= \frac{2}{(q_i-1)!}.\int_0^{X^2} (\sqrt{u}/\lambda)^{2.q_i-1}.\exp(-u/\lambda^2).\frac{1}{2.\sqrt{u}}.du \\
&= \frac{1}{(q_i-1)!}.\int_0^{X^2} \frac{u^{q_i-1}}{\lambda^{2.q_i-1}}.\exp(-u/\lambda^2).du \\
&= \frac{\lambda}{(q_i-1)!}.\int_0^{X^2/\lambda^2} x^{q_i-1}.\exp(-x).dx = \frac{\lambda}{(q_i-1)!}.I_{q_i-1}(X^2/\lambda^2)
\end{aligned} \tag{2.6}$$

With: $I_n(x) = \int_0^x t^n.e^{-t}.dt$. Then, we use the classical result:

$$\begin{aligned}
&I_n(x) = \int_0^x t^n.e^{-t}.dt = [-t^n.e^{-t}]_0^x + n.\int_0^x t^{n-1}.e^{-t}.dt = -x^n.e^{-x} + n.I_{n-1} \\
&= -e^{-x}.[x^n + n.x^{n-1} + n.(n-1).x^{n-2} + ... + n!] + n! \\
&= n!.[1 - e^{-x}.(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + ... + \frac{x^{n-k}}{(n-k)!} + ... + x + 1)]
\end{aligned} \tag{2.7}$$

This yields to:

$$\begin{aligned}
&\int_0^X p_U(z/\lambda).dz = \frac{\lambda}{(q_i-1)!}.I_{q_i-1}(X^2/\lambda^2) \\
&= \frac{\lambda}{(q_i-1)!}.(q_i-1)!.[1 - e^{-X^2/\lambda^2}.[\frac{(X^2/\lambda^2)^{q_i-1}}{(q_i-1)!} + \frac{(X^2/\lambda^2)^{q_i-2}}{(q_i-2)!} + ... + \frac{(X^2/\lambda^2)^{q_i-1-k}}{(q_i-1-k)!} + ... + X^2/\lambda^2 + 1]] \\
&= \lambda.[1 - e^{-X^2/\lambda^2}.[\frac{(X^2/\lambda^2)^{q_i-1}}{(q_i-1)!} + \frac{(X^2/\lambda^2)^{q_i-2}}{(q_i-2)!} + ... + \frac{(X^2/\lambda^2)^{q_i-1-k}}{(q_i-1-k)!} + ... + X^2/\lambda^2 + 1]]
\end{aligned} \tag{2.8}$$

And finally, we obtain:

$$\begin{aligned}
&F_{U.V}(X) = \int_0^{+\infty} [\int_0^X p_U(z/\lambda).dz].\frac{p_V(\lambda)}{\lambda}.d\lambda \\
&= \int_0^{+\infty} [1 - e^{-X^2/\lambda^2}.[\frac{(X^2/\lambda^2)^{q_i-1}}{(q_i-1)!} + \frac{(X^2/\lambda^2)^{q_i-2}}{(q_i-2)!} + ... + \frac{(X^2/\lambda^2)^{q_i-1-k}}{(q_i-1-k)!} + ... + X^2/\lambda^2 + 1]].\frac{2}{(q_i-1)!}.\lambda^{2.q_i-1}.\exp(-\lambda^2).d\lambda
\end{aligned}$$

$$= \int_0^{+\infty} [\frac{2.\lambda^{2.q_i-1}}{(q_i-1)!}.\exp(-\lambda^2).d\lambda - \frac{2.\lambda^{2.q_i-1}.e^{-[X^2/\lambda^2+\lambda^2]}}{(q_i-1)!}.[\frac{(X^2/\lambda^2)^{q_i-1}}{(q_i-1)!} + \frac{(X^2/\lambda^2)^{q_i-2}}{(q_i-2)!} + ... + \frac{(X^2/\lambda^2)^{q_i-1-k}}{(q_i-1-k)!} + ... + X^2/\lambda^2 + 1]].d\lambda$$

$$\begin{aligned}
&F_{U.V}(X) = 1 - \sum_{k=0}^{q_i-1} \int_0^{+\infty} \frac{2.\lambda^{2.q_i-1}.e^{-[X^2/\lambda^2+\lambda^2]}}{(q_i-1)!}.\frac{(X^2/\lambda^2)^{q_i-1-k}}{(q_i-1-k)!}.d\lambda \\
&= 1 - \sum_{k=0}^{q_i-1} \frac{2.X^{2.(q_i-1-k)}}{(q_i-1)!.(q_i-1-k)!} \int_0^{+\infty} \lambda^{2.k+1}.e^{-[X^2/\lambda^2+\lambda^2]}.d\lambda \\
&= 1 - \sum_{k=0}^{q_i-1} \frac{2.X^{2.(q_i-1-k)}}{(q_i-1)!.(q_i-1-k)!}.I_{2.k+1}(X)
\end{aligned} \tag{2.9}$$

The probability density function is the derivative of function: $F_{U,V}(X)$. In order to calculate this derivative, we have to compute at first:

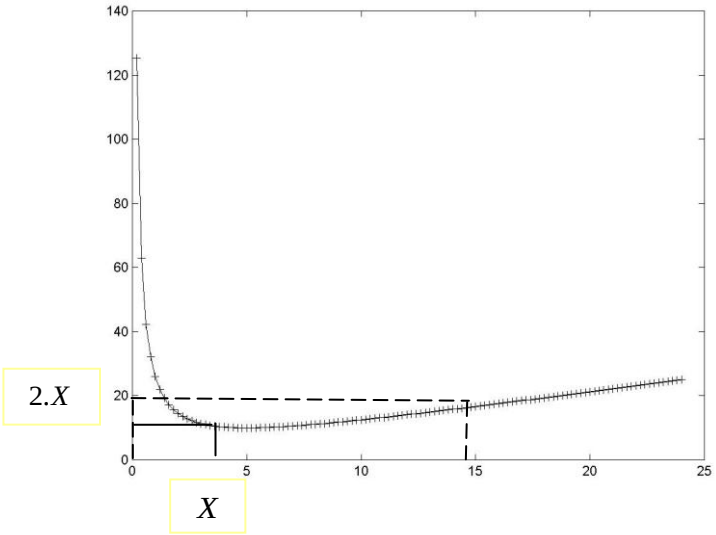
$$\frac{\partial I_{2.k+1}(X)}{\partial X} = \int_0^{+\infty} \lambda^{2.k+1}.\frac{\partial}{\partial X}[e^{-(\lambda^2+X^2/\lambda^2)}].d\lambda = \int_0^{+\infty} \lambda^{2.k+1}.e^{-\lambda^2}.-\frac{2.X.e^{-X^2/\lambda^2}}{\lambda^2}.d\lambda$$
$$= -2.X.\int_0^{+\infty} \lambda^{2.k-1}.e^{-(\lambda^2+X^2/\lambda^2)}.d\lambda = -2.X.I_{2.k-1}(X)$$

We obtain then:

$$p_{U,V}(X) = \frac{\partial}{\partial X}[1-\frac{2.X^{2.(q_i-1)}.I_1(X)}{(q_i-1)!^2}-\frac{2.X^{2.(q_i-2)}.I_3(X)}{(q_i-1)!.(q_i-2)!}-...-\frac{2.X^{2.(q_i-1-k)}.I_{2.k+1}(x)}{(q_i-1)!.(q_i-1-k)!}-...-\frac{2.I_{2.q_i-1}(x)}{(q_i-1)!}]$$
$$= -\frac{4.(q_i-1).X^{2.(q_i-1)-1}.I_1(X)}{(q_i-1)!^2}+\frac{2.X^{2.(q_i-1)}.2.X.I_{-1}(X)}{(q_i-1)!^2}-\frac{4.(q_i-2).X^{2.(q_i-2)-1}.I_3(X)}{(q_i-1)!(q_i-2)!}+\frac{2.X^{2.(q_i-2)}.2.X.I_1(X)}{(q_i-1)!(q_i-2)!}+...$$
$$= -\cancel{\frac{4.X^{2.(q_i-1)-1}.I_1(X)}{(q_i-1)!.(q_i-2)!}}+\frac{4.X^{2.q_i-1}.I_{-1}(X)}{(q_i-1)!^2}-\cancel{\frac{4.X^{2.(q_i-2)-1}.I_3(X)}{(q_i-1)!(q_i-3)!}}+\cancel{\frac{4.X^{2.q_i-1}.I_1(X)}{(q_i-1)!(q_i-2)!}}-...$$

$$p_{U,V}(X) = \frac{4.X^{2.q_i-1}.I_{-1}(X)}{(q_i-1)!^2} \quad (2.10)$$

We have: $I_{-1}(X) = \int_0^{+\infty} \frac{1}{\lambda}.e^{-[X^2/\lambda^2+\lambda^2]}.d\lambda$, setting $u=\lambda^2$, we obtain: $I_{-1}(X) = \frac{1}{2}.\int_0^{+\infty} \frac{e^{-[X^2/u+u]}}{u}.du$. To compute this integral we set: $z = u+\frac{X^2}{u}$, the graph of function $f(u) = u+\frac{X^2}{u}$ is drawn just below:



When we solve the equation $z = u+\frac{X^2}{u}$ => $u^2-u.z+X^2=0$, we obtain the solutions:

- $u = \frac{1}{2}.[z-\sqrt{z^2-4.X^2}]$ for $u \in [0, X]$ and $du = \frac{1}{2}.[1-\frac{z}{\sqrt{z^2-4.X^2}}].dz$

- $u = \frac{1}{2}.[z + \sqrt{z^2 - 4.X^2}]$ for $u \in [X, +\infty[$ et $du = \frac{1}{2}.[1 + \frac{z}{\sqrt{z^2 - 4.X^2}}].dz$

Then, we obtain:

$$I_{-1}(X) = \int_{2.X}^{+\infty} \frac{e^{-z}}{\sqrt{z^2 - 4.X^2}}.dz$$

The final variable is we set: $z = 2.X.u$, so that yields to:

$$I_{-1}(X) = \int_{1}^{+\infty} \frac{e^{-2.X.u}}{\sqrt{u^2 - 1}}.du = \int_{1}^{+\infty} \frac{e^{-2.X.u}}{u.\sqrt{1 - 1/u^2}}.du$$

With:

$$(1-x)^{\alpha} = 1 - \alpha.x + \frac{\alpha.(\alpha-1)}{2!}.x^2 + ... + (-1)^n \frac{\alpha.(\alpha-1)...(\alpha-n+1)}{n!}.x^n + ... \quad \text{pour} \quad |x| < 1$$

With $\alpha = -1/2$, we obtain:

$$\frac{\alpha.(\alpha-1)...(\alpha-n+1)}{n!} = \frac{-1/2.(-1/2-1)(-1/2-2)...(-1/2-n+1)}{n!}$$

$$= (-1)^n.\frac{1.3.5...(2.n-1)}{2^n.n!} = (-1)^n.\frac{1.2.3.4.5...(2.n-2).(2.n-1)}{2.4.6....(2n-2).2^n.n!} = (-1)^n.\frac{(2.n-1)!}{2^{2n-1}.(n-1)!n!}$$

$$\frac{1}{\sqrt{1-1/u^2}} = 1 + \sum_{n=1}^{+\infty} \frac{(2.n-1)!}{2^{2n-1}.(n-1)!n!}.\frac{1}{u^{2.n}}$$

And:

$$I_{-1}(X) = \int_{1}^{+\infty} \frac{e^{-2.X.u}}{u.\sqrt{1-1/u^2}}.du = \int_{1}^{+\infty} \frac{e^{-2.X.u}}{u}.[1 + \sum_{n=1}^{+\infty} \frac{(2.n-1)!}{2^{2n-1}.(n-1)!n!}.\frac{1}{u^{2.n}}].du$$

$$= \int_{1}^{+\infty} \frac{e^{-2.X.u}}{u}.du + \sum_{n=1}^{+\infty} \frac{(2.n-1)!}{2^{2n-1}.(n-1)!n!}.\int_{1}^{+\infty} \frac{e^{-2.X.u}}{u^{2.n+1}}.du \qquad (2.11)$$

$$= K_1(X) + \sum_{n=1}^{+\infty} \frac{(2.n-1)!}{2^{2n-1}.(n-1)!n!}.K_{2.n+1}(X)$$

We have:

$$K_{2.n+1}(X) = \int_{1}^{+\infty} \frac{e^{-2.X.u}}{u^{2.n+1}}.du = e^{-2.X}.[\frac{1}{2.n} - \frac{2.X}{2.n.(2.n-1)} + ... + (-1)^k.\frac{2^k.X^k}{2.n.(2.n-1)...(2.n-k)} + ...] + (-1)^{2.n}.\frac{2^{2.n}.X^{2.n}}{(2.n)!}.EI_1(2.X)$$

And: $EI_1(x) = \int_{x}^{+\infty} \frac{e^{-t}.dt}{t} = \int_{1}^{+\infty} \frac{e^{-xt}.dt}{t} = e^{-x}.\int_{0}^{+\infty} \frac{e^{-xt}}{1+t}.dt = \frac{e^{-x}}{x}.\int_{0}^{+\infty} \frac{e^{-xt}}{1+t/x}.dt = -\ln x - \gamma + \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}.x^n}{n.n\ !}$

In conclusion, we obtain:

$$p_{U.V}(X) = \frac{4.X^{2.q_i-1}.I_{-1}(X)}{(q_i-1)!^2}$$

$$I_{-1}(X) = \int_1^{+\infty} \frac{e^{-2.X.u}}{u.\sqrt{1-1/u^2}}.du = \int_1^{+\infty} \frac{e^{-2.X.u}}{u}.[1+\sum_{n=1}^{+\infty} \frac{(2.n-1)!}{2^{2n-1}.(n-1)!n!}.\frac{1}{u^{2.n}}].du$$
$$= \int_1^{+\infty} \frac{e^{-2.X.u}}{u}.du + \sum_{n=1}^{+\infty} \frac{(2.n-1)!}{2^{2n-1}.(n-1)!n!}.\int_1^{+\infty} \frac{e^{-2.X.u}}{u^{2.n+1}}.du \qquad (2.12)$$
$$= K_1(X) + \sum_{n=1}^{+\infty} \frac{(2.n-1)!}{2^{2n-1}.(n-1)!n!}.K_{2.n+1}(X)$$

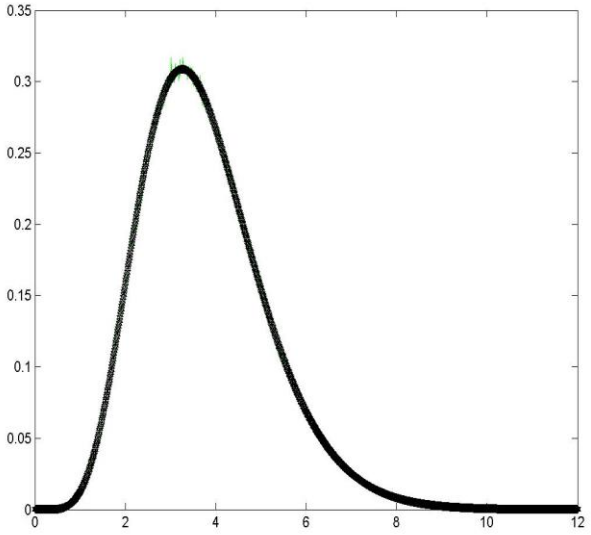
$$K_{2.n+1}(X) = \int_1^{+\infty} \frac{e^{-2.X.u}}{u^{2.n+1}}.du = e^{-2.X}.[\frac{1}{2.n} - \frac{2.X}{2.n.(2.n-1)} + ... + (-1)^k.\frac{2^k.X^k}{2.n.(2.n-1)...(2.n-k)} + ...] + (-1)^{2.n}.\frac{2^{2.n}.X^{2.n}}{(2.n)!}.EI_1(2.X)$$

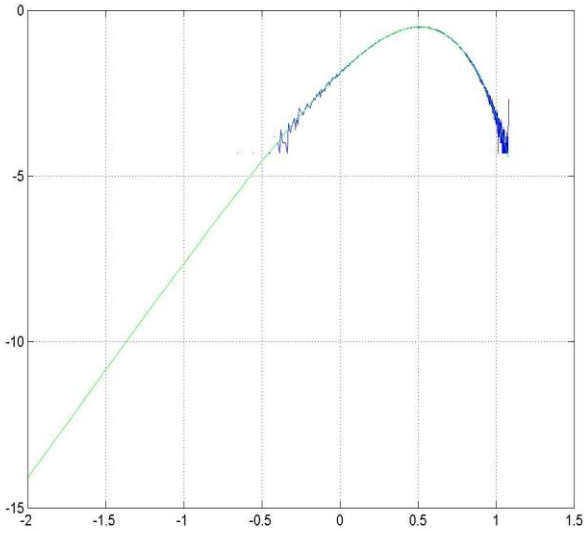
$$K_1(X) = EI_1(2.X) \text{ with: } EI_1(x) = \int_x^{+\infty} \frac{e^{-t}.dt}{t} = \int_1^{+\infty} \frac{e^{-xt}.dt}{t} = \frac{e^{-x}}{x}.\int_0^{+\infty} \frac{e^{-xt}}{1+t/x}.dt = -\ln x - \gamma + \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}.x^n}{n.n!}$$

With γ denoting the Euler's constant: $\gamma \approx 0{,}5772$.

Using the development of $K_{2.n+1}(X)$ $\forall n \geq 0$; it is easy to show that the diversity order of the product *U.V* is equal to: $2.q_i - 1$; and that we have for $X \approx 0$: $p_{U.V}(X) \approx K.x^{2.q_i-1}.e^{-2.x}$.

This result is confirmed by the E.M fitting algorithm; for example with *R* = 5 we have the following result for the variable: $\lambda_{1,2}.\lambda_{5,2}$.





We found: $\alpha \approx 7.6$ and $\beta \approx 2.0$; this confirms our theoretical derivation, i.e. $\alpha \approx 8.0$ and $\beta \approx 2.0$ in this case.

Appendix 3 : Characterization of the p.d.f of: $\delta_k = \frac{\left[\sum_{k=1}^{L}[\sum_{j=1}^{N}\lambda_{k,j}^{'}].\lambda_{k,i}^{'}\right]^2}{\sum_{k=1}^{L}[\sum_{j=1}^{N}\lambda_{k,j}^{'}]^2} = \frac{(V_i)^2}{U}$

-First, we characterize the diversity order of the numerator : $V_i = \sum_{k=1}^{L}[\sum_{j=1}^{N}\lambda_{k,j}^{'}].\lambda_{k,i}^{'} = \sum_{k=1}^{L}[\lambda_{k,i}^{'2} + \sum_{j=1, j\neq i}^{N}\lambda_{k,j}^{'}.\lambda_{k,i}^{'}]$. Using appendices 1 and 2, we know that each term $\lambda_{k,i}^{'2}$ has a diversity order equal to: $q_i - 1$ and that each term $\lambda_{k,j}^{'}.\lambda_{k,i}^{'}$ has a diversity order equal to: $q_i + q_j - 1$. However, due to the fact that the random variables in V_i are highly correlated, the diversity order study is difficult. It is easier to begin with a simpler case with L = 1 and N = 2; i.e. we have to study the distribution of : $V_{1,2} = \lambda_{11}^2 + \lambda_{11}.\lambda_{12}$. First of all, we calculate the c.d.f defined by:

$$
\begin{aligned}
&F_{V_{1,2}}(X) = \text{Proba}(V_{1,2} \le X) = \text{Proba}(\lambda_{11}^2 + \lambda_{11}.\lambda_{12} \le X) = \text{Proba}(\lambda_{11}.\lambda_{12} \le X - \lambda_{11}^2) \\
&= \int_0^{+\infty} \text{Proba}(\lambda_{12} \le X/\lambda_{11} - \lambda_{11} \,|\, \lambda_{11}).p_{\lambda_{11}}(\lambda_{11}).d\lambda_{11} = \int_0^{+\infty}[\int_0^{X/z-z} p_{\lambda_{12}}(y).dy].p_{\lambda_{11}}(z).dz \qquad (3.1) \\
&= \int_0^{+\infty}[\int_0^{X/z-z} p_{\lambda_{12}}(y).dy].p_{\lambda_{11}}(z).dz
\end{aligned}
$$

We calculate at first : $I(X,z) = \int_0^{X/z-z} p_{\lambda_{12}}(y).dy$, with : $p_{\lambda_{12}}(y) = \frac{2}{(q_2-1)!}.y^{2.q_2-1}.\exp(-y^2)$ and $q_2 = R-1$.

$$
\begin{aligned}
&I(X,z) = \int_0^{X/z-z} p_{\lambda_{12}}(y).dy = \frac{2}{(q_2-1)!}.\int_0^{X/z-z} y^{2.q_2-1}.\exp(-y^2).dy \\
&= \frac{2}{(q_2-1)!}.\int_0^{[X/z-z]^2} u^{q_2-1/2}.e^{-u}.\frac{du}{2.\sqrt{u}} = \frac{1}{(q_2-1)!}.\int_0^{[X/z-z]^2} u^{q_2-1}.e^{-u}.du
\end{aligned} \qquad (3.2)
$$

Using : $J_n(x) = \int_0^x t^n.e^{-t}.dt$ with : $J_n(x) = n!.[1-e^{-x}.(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + ... + \frac{x^{n-k}}{(n-k)!} + ... + x + 1)]$, we have :

$$
\begin{aligned}
&I(X,z) = \frac{1}{(q_2-1)!}.\int_0^{[X/z-z]^2} u^{q_2-1}.e^{-u}.du = \frac{1}{(q_2-1)!}.J_{q_2-1}([X/z-z]^2) \\
&= 1 - e^{-[X/z-z]^2}.(\frac{[X/z-z]^{2.(q_2-1)}}{(q_2-1)!} + \frac{[X/z-z]^{2.(q_2-2)}}{(q_2-2)!} + ... + \frac{[X/z-z]^{2.(q_2-1-k)}}{(q_2-1-k)!} + ... + [X/z-z]^2 + 1)
\end{aligned} \qquad (3.3)
$$

Plugging (3.3) into (3.1), we obtain:

$$
F_{V_{1,2}}(X) = \int_0^{+\infty}[1 - e^{-[X/z-z]^2}.(\frac{[X/z-z]^{2.(q_2-1)}}{(q_2-1)!} + \frac{[X/z-z]^{2.(q_2-2)}}{(q_2-2)!} + ... + \frac{[X/z-z]^{2.(q_2-1-k)}}{(q_2-1-k)!} + ... + [X/z-z]^2 + 1)].p_{\lambda_{11}}(z).dz \qquad (3.4)
$$

With : $p_{\lambda_{11}}(z)=\frac{2}{(q_1-1)!}.z^{2.q_1-1}.\exp(-z^2)$; we obtain :

$$F_{V_{1,2}}(X)=\int_0^{+\infty}[1-e^{-[X/z-z]^2}.(\frac{[X/z-z]^{2.(q_2-1)}}{(q_2-1)!}+\frac{[X/z-z]^{2.(q_2-2)}}{(q_2-2)!}+...+\frac{[X/z-z]^{2.(q_2-1-k)}}{(q_2-1-k)!}+...+[X/z-z]^2+1)].p_{\lambda_{11}}(z).dz$$

$$F_{V_{1,2}}(X)=1-\sum_{k=0}^{q_2-1}\frac{2}{(q_1-1)!.(q_2-1-k)!}.\int_0^{+\infty}[X/z-z]^{2.(q_2-1-k)}.e^{-[X/z-z]^2}.z^{2.q_1-1}.e^{-z^2}.dz$$

$$F_{V_{1,2}}(X)=1-\sum_{k=0}^{q_2-1}\frac{2}{(q_1-1)!.(q_2-1-k)!}.K_k(X,q_1,q_2)$$

With:

$$K_k(X,q_1,q_2)=\int_0^{+\infty}[X/z-z]^{2.(q_2-1-k)}.z^{2.q_1-1}.e^{-[X/z-z]^2}.e^{-z^2}.dz \quad (3.5)$$

The p.d.f of $V_{1,2}$ is the derivative of $F_{V_{1,2}}(X)$; we have :

$$p_{V_{1,2}}(X)=-\sum_{k=0}^{q_2-1}\frac{2}{(q_1-1)!.(q_2-1-k)!}.\frac{\partial K_k(X,q_1,q_2)}{\partial X} \quad (3.6)$$

With:

$$\frac{\partial K_k(X,q_1,q_2)}{\partial X}=2.(q_2-1-k)\int_0^{+\infty}[X/z-z]^{2.(q_2-1-k)-1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz+...$$
$$...-2.\int_0^{+\infty}[X/z-z]^{2.(q_2-1-k)+1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz \quad ...(3.7)$$

This yields to:

$$p_{V_{1,2}}(X)=-\sum_{k=0}^{q_2-1}\frac{2}{(q_1-1)!.(q_2-1-k)!}.\frac{\partial K_k(X,q_1,q_2)}{\partial X}$$
$$=-4.\sum_{k=0}^{q_2-2}\frac{1}{(q_1-1)!.(q_2-2-k)!}.\int_0^{+\infty}[X/z-z]^{2.(q_2-1-k)-1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz$$
$$+4.\sum_{k=0}^{q_2-1}\frac{1}{(q_1-1)!.(q_2-1-k)!}.\int_0^{+\infty}[X/z-z]^{2.(q_2-1-k)+1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz$$
$$=-4.\sum_{k=0}^{q_2-2}\frac{1}{(q_1-1)!.(q_2-1-(k+1))!}.\int_0^{+\infty}[X/z-z]^{2.(q_2-(k+1))-1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz$$
$$+4.\sum_{k=0}^{q_2-1}\frac{1}{(q_1-1)!.(q_2-1-k)!}.\int_0^{+\infty}[X/z-z]^{2.(q_2-k)-1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz$$

$$p_{V_{1,2}}(X) = -4.\sum_{k=1}^{q_2-1} \frac{1}{(q_1-1)!.(q_2-1-m)!}.\int_0^{+\infty} [X/z-z]^{2.(q_2-m)-1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz$$

$$+4.\sum_{k=0}^{q_2-1} \frac{1}{(q_1-1)!.(q_2-1-k)!}.\int_0^{+\infty} [X/z-z]^{2.(q_2-k)-1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz$$

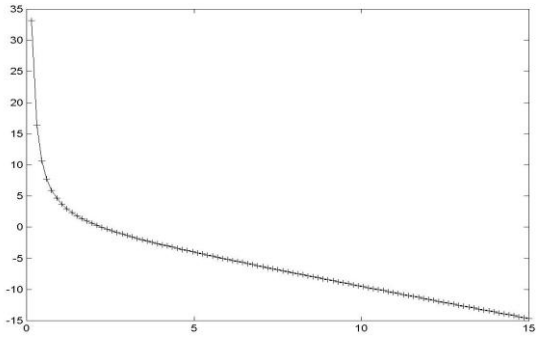
Finally, we have:

$$p_{V_{1,2}}(X) = \frac{4}{(q_1-1)!.(q_2-1)!}.\int_0^{+\infty} [X/z-z]^{2.q_2-1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz$$

$$p_{V_{1,2}}(X) = \frac{4}{(q_1-1)!.(q_2-1)!}.G_{q_1,q_2}(X) \quad (3.8)$$

To obtain the diversity order, it is necessary to study the behaviour of $G_{q_1,q_2}(X)$ around zero. To do this, we set:

$u = X/z - z$, the graph of function $f(z) = X/z - z$ is drawn just below:



We have: $du = -[X/z^2+1].dz$ and: $z = \frac{1}{2}.[-u+\sqrt{u^2+4.X}]$; so we obtain:

$$du = -\frac{8.X+2.u^2-2.u.\sqrt{u^2+4.X}}{4.X+2.u^2-2.u.\sqrt{u^2+4.X}}.dz \quad (3.9)$$

We can rewrite $G_{q_1,q_2}(X)$:

$$G_{q_1,q_2}(X) = \int_0^{+\infty} [X/z-z]^{2.q_2-1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz$$

$$= \int_{-\infty}^{+\infty} u^{2.q_2-1}.\frac{1}{2^{2.q_1-2}}.[-u+\sqrt{u^2+4.X}]^{2.q_1-2}.e^{-u^2}.e^{-[u^2/2+X-u/2.\sqrt{u^2+4.X}]}.\frac{[-u+\sqrt{u^2+4.X}]^2}{8.X+2.u^2-2.u.\sqrt{u^2+4.X}}.du \quad (3.10)$$

$$= \frac{1}{2^{2.q_1-2}}.\int_{-\infty}^{+\infty} u^{2.q_2-1}.\frac{[-u+\sqrt{u^2+4.X}]^{2.q_1}}{8.X+2.u^2-2.u.\sqrt{u^2+4.X}}.e^{-[3/2.u^2+X-u/2.\sqrt{u^2+4.X}]}.du$$

Of course, it is very difficult to calculate this integral but remember that we are only looking for the diversity order of the random variable $V_{1,2}$; i.e. the first non-null derivative of $p_{V_{1,2}}(X)$ around zero. To do this, we use the following approximation: $e^{-[3/2.u^2+X-u/2.\sqrt{u^2+4.X}]} \approx_{X\to 0} e^{-u^2}$ and we use the following development:

$$\frac{[-u+\sqrt{u^2+4.X}]^{2.q_1}}{8.X+2.u^2-2.u.\sqrt{u^2+4.X}} = \frac{u^{2.q_1}.[-1+\sqrt{1+4.X/u^2}]^{2.q_1}}{8.X+2.u^2-2.u^2.\sqrt{1+4.X/u^2}}$$
$$= \frac{u^{2.q_1}.[2.X/u^2+\vartheta(X)]^{2.q_1}}{8.X+2.u^2-2.u^2.[1+2.X/u^2+\vartheta(X)]} = \frac{4^{q_1-1}.X^{2.q_1-1}+\vartheta(X^{2.q_1-1})}{u^{2.q_1}} \quad (3.12)$$

It's then straightforward to deduce the diversity order of variable V_1: $D_1 = 2.q_1 - 1$. At this stage it is important to compare this result with the case where we consider independent variables. Using appendices 1 and 2, we know that each term $\lambda_{k,i}^{'2}$ has a diversity order equal to: $q_i - 1$ and that each term $\lambda_{k,j}^{'}.\lambda_{k,i}^{'}$ has a diversity order equal to: $q_i + q_j - 1$. So, if the terms λ_{11}^2 and $\lambda_{11}.\lambda_{12}$ were independent, they should exhibit a diversity order: $D = q_1 + q_2 - 1 + q_1 = 2.q_1 + q_2 - 1$. Compared to this ideal case on can see that we encounter a diversity loss of: $\Delta D = D - D_1 = q_2$.

It is possible to generalize this result, considering the random variable:

$$V_{1,n} = \lambda_{11}^2 + \lambda_{11}.\lambda_{12} + \lambda_{11}.\lambda_{13} + ... + \lambda_{11}.\lambda_{1n} \quad (3.13)$$

We calculate the c.d.f defined by:

$$F_{V_{1,n}}(X) = \text{Proba}(V_{1,n} \le X) = \text{Proba}(\lambda_{11}^2 + \lambda_{11}.\lambda_{12} + ... + \lambda_{11}.\lambda_{1n} \le X) = \text{Proba}(\lambda_{11}.\lambda_{12} + ... + \lambda_{11}.\lambda_{1n} \le X - \lambda_{11}^2)$$
$$= \int_0^{+\infty} \text{Proba}(\lambda_{12} + \lambda_{13} + ... + \lambda_{1n} \le X/\lambda_{11} - \lambda_{11} | \lambda_{11}).p_{\lambda_{11}}(\lambda_{11}).d\lambda_{11} = \int_0^{+\infty} [\int_0^{X/z-z} p_{\lambda_{12}+...+\lambda_{1n}}(y).dy].p_{\lambda_{11}}(z).dz \quad (3.14)$$
$$= \int_0^{+\infty} [\int_0^{X/z-z} p_{\lambda_{12}+...+\lambda_{1n}}(y).dy].p_{\lambda_{11}}(z).dz$$

We have demonstrated in Appendix 1 that around zero we have:

$$p_{\lambda_{12}+...+\lambda_{1n}}(y) \approx K_n.y^{2.[q_2+q_3+...+q_n]-1}.\exp(-y^2) \text{ and } q_i = R-(i-1).$$

We set: $r_n = q_2 + q_3 + ... + q_n$; we obtain thus (similarly as in equation (3.8)):

$$p_{V_{1,n}}(X) = K_n^{'}.\int_0^{+\infty}[X/z - z]^{2.r_n-1}.z^{2.q_1-2}.e^{-[X/z-z]^2}.e^{-z^2}.dz$$
$$p_{V_{1,n}}(X) = K_n^{'}.G_{q_1,r_n}(X) \qquad (3.15)$$

We can write $G_{q_1,r_n}(X)$:

$$G_{q_1,r_n}(X) = \frac{1}{2^{2.q_1-2}}.\int_{-\infty}^{+\infty}u^{2.r_n-1}.\frac{[-u+\sqrt{u^2+4.X}]^{2.q_1}}{8.X+2.u^2-2.u.\sqrt{u^2+4.X}}.e^{-[3/2.u^2+X-u/2.\sqrt{u^2+4.X}]}.du \qquad (3.16)$$

It is clear that the diversity order remains equal to: $D_1 = 2.q_1 - 1$.

In this case, it is straightforward to obtain the diversity order of the numerator $(V_i)^2$ since the quantities $\sum_{j=1}^{N}[\lambda_{m,j}^{'}].\lambda_{m,i}^{'}$ and $\sum_{j=1}^{N}[\lambda_{n,j}^{'}].\lambda_{n,i}^{'}$ are independent for $m \neq n$. With the results of Appendices 1 and 2 we conclude that $(V_i)^2$ exhibits a diversity order equal to: $R.q_i - 1$.

The diversity order of the quantity $\frac{(V_i)^2}{U}$ will be the same as those of $(V_i)^2$. Although we use here ratio of correlated variables, this may be justified in comparison with the case of the p.d.f of the ratio of two independent Chi-2 variables (see Appendix 4). We proved in Appendix 4 that the diversity order of the ratio of two independent random Chi-2 variables correspond to the diversity order of the numerator random variable and this completes the proof.

Appendix 4 : Characterization of the ratio of two independent Chi 2 random variables.

We want to calculate the p.d.f of U/V with:

$$p_U(\lambda) = \frac{1}{\Gamma(n)}.x^{n-1}.e^{-x} \text{ pour } x > 0 \quad \text{and} \quad p_V(\lambda) = \frac{1}{\Gamma(m)}.x^{m-1}.e^{-x} \text{ pour } x > 0$$

We calculate at first :

$$\text{Proba}(U/V \le X) = \int_0^{+\infty} \text{Proba}(U \le V.X \,|V).p_V(V).dV = \int_0^{+\infty} [\int_0^{V.X} p_U(y).dy].p_V(V).dV$$

We set, $y = V.z$; d'où : $\int_0^{V.X} p_U(y).dy = \int_0^{X} p_U(V.z).V.dz$; we obtain :

$$F_{U/V}(X) = \text{Proba}(U/V \le X) = \int_0^{+\infty} [\int_0^{V.X} p_U(y).dy].p_V(V).dV = \int_0^{+\infty} [\int_0^{X} p_U(V.z).V.dz].p_V(V).dV$$

$$= \int_0^{+\infty} [\int_0^{X} p_U(V.z).dz].V.p_V(V).dV$$

So, this yields to :

$$F_{U/V}(X) = = \int_0^{+\infty} [\int_0^{X} p_U(\lambda.z).dz].\lambda.p_V(\lambda).d\lambda \qquad (4.1)$$

We use the well known result :

$$I_n(x) = \int_0^{x} t^n.e^{-t}.dt = [-t^n.e^{-t}]_0^x + n.\int_0^{x} t^{n-1}.e^{-t}.dt = -x^n.e^{-x} + n.I_{n-1}$$

$$= -x^n.e^{-x} + n.[-x^{n-1}.e^{-x} + (n-1).I_{n-2}]$$

$$= -e^{-x}.[x^n + n.x^{n-1} + n.(n-1).x^{n-2} + ... + n!] + n!$$

$$= n!.[1 - e^{-x}.(\frac{x^n}{n!} + \frac{x^{n-1}}{(n-1)!} + ... + \frac{x^{n-k}}{(n-k)!} + ... + x + 1)]$$

We deduce : $J_n(x) = \int_0^{x} (\lambda.t)^n.e^{-\lambda.t}.dt = \frac{1}{\lambda}.\int_0^{\lambda.x} u^n.e^{-u}.du = \frac{1}{\lambda}.I_n(\lambda.x)$

We can then calculate : $\int_0^{X} p_U(\lambda.z).dz = \frac{\lambda^{n-1}}{\Gamma(n)}\int_0^{X} u^{n-1}.e^{-\lambda.u}.du = \frac{1}{\Gamma(n)}.J_{n-1}(X) = \frac{1}{\lambda.\Gamma(n)}.I_{n-1}(\lambda.X)$.

$$\int_0^{X} p_U(\lambda.z).dz = \frac{1}{\lambda.\Gamma(n)}.I_{n-1}(\lambda.X)$$

$$= \frac{1}{\lambda}.[1 - e^{-\lambda x}.(\frac{(\lambda X)^{n-1}}{(n-1)!} + \frac{(\lambda.X)^{n-2}}{(n-2)!} + ... + \frac{(\lambda.X)^{n-1-k}}{(n-1-k)!} + ... + \lambda.X + 1)] \qquad (4.2)$$

And we eventually obtain:

$$F_{U/V}(X)=\int_0^{+\infty}[\int_0^{X} p_U(\lambda.z).dz].\lambda.p_V(\lambda).d\lambda$$

$$=\int_0^{+\infty}\frac{1}{\lambda}.[1-e^{-\lambda X}.(\frac{(\lambda X)^{n-1}}{(n-1)!}+\frac{(\lambda.X)^{n-2}}{(n-2)!}+...+\frac{(\lambda.X)^{n-1-k}}{(n-1-k)!}+...+\lambda.X+1)].\lambda.p_V(\lambda).d\lambda$$

$$=\int_0^{+\infty}[1-e^{-\lambda X}.(\frac{(\lambda X)^{n-1}}{(n-1)!}+\frac{(\lambda.X)^{n-2}}{(n-2)!}+...+\frac{(\lambda.X)^{n-1-k}}{(n-1-k)!}+...+\lambda.X+1)].\frac{1}{\Gamma(m)}.\lambda^{m-1}.e^{-\lambda}d\lambda$$

$$=\frac{1}{\Gamma(m)}.\int_0^{+\infty}[\lambda^{m-1}.e^{-\lambda}-e^{-\lambda(X+1)}.(\frac{X^{n-1}.\lambda^{m+n-2}}{(n-1)!}+\frac{X^{n-2}.\lambda^{m+n-3}}{(n-2)!}+...+\frac{X^{n-1-k}.\lambda^{m+n-k-2}}{(n-1-k)!}+...+\lambda^{m}.X+\lambda^{m-1})].d\lambda$$

We have to calculate integrals of the type :

$$K_{n,m,k}=[1/(n-k-1)!].\int_0^{+\infty}\lambda^{m+n-k-2}.X^{n-k-1}.e^{-\lambda.(X+1)}.d\lambda$$

$$=[1/(n-k-1)!].X^{n-k-1}.\int_0^{+\infty}\lambda^{m+n-k-2}.e^{-\lambda.(X+1)}.d\lambda$$

We set : $z=\lambda.(X+1)$, soit $\lambda=z/(X+1)$ this gives:

$$K_{n,m,k}=[1/(n-k-1)!].X^{n-k-1}.\int_0^{+\infty}\lambda^{m+n-k-2}.e^{-\lambda.(X+1)}.d\lambda$$

$$=[1/(n-k-1)!].X^{n-k-1}.\int_0^{+\infty}[z/(X+1)]^{m+n-k-2}.e^{-z}.dz/(X+1)$$

$$K_{n,m,k}=\frac{X^{n-k-1}}{(n-1-k)!}.\frac{1}{(X+1)^{m+n-k-1}}\int_0^{+\infty}z^{m+n-k-2}.e^{-z}.dz=\frac{(m+n-k-2)!}{(n-1-k)!}.\frac{X^{n-k-1}}{(X+1)^{m+n-k-1}}\qquad(4.3)$$

We deduce :

$$F_{U/V}(X)=1-\frac{(m+n-2)!}{(m-1)!.(n-1)!}.\frac{X^{n-1}}{(X+1)^{m+n-1}}-...-\frac{(m+n-k-2)!}{(m-1)!.(n-1-k)!}.\frac{X^{n-k-1}}{(X+1)^{m+n-k-1}}-...-\frac{1}{(X+1)^{m}}\qquad(4.4)$$

The p.d.f of U/V is the derivative of $F_{U/V}(X)$; we have :

$$p_{U/V}(X)=\frac{\partial F_{U/V}(X)}{\partial X}$$

With :

$$\frac{(m+n-k-2)!}{(m-1)!.(n-1-k)!}.\frac{\partial}{\partial X}[\frac{X^{n-k-1}}{(X+1)^{m+n-k-1}}]$$

$$=\frac{(m+n-k-2)!}{(m-1)!.(n-1-k)!}.[(n-k-1).\frac{X^{n-k-2}}{(X+1)^{m+n-k-1}}-(m+n-k-1).\frac{X^{n-k-1}}{(X+1)^{m+n-k}}]$$

$$=\frac{(m+n-k-2)!}{(m-1)!.(n-k-2)!}.\frac{X^{n-k-2}}{(X+1)^{m+n-k-1}}-\frac{(m+n-k-1)!}{(m-1)!.(n-1-k)!}.\frac{X^{n-k-1}}{(X+1)^{m+n-k}}$$

We obtain :

$$p_{U/V}(X)=\frac{(m+n-1)!}{(m-1)!.(n-1)!}.\frac{X^{n-1}}{(X+1)^{m+n}}-\frac{(m+n-2)!}{(m-1)!.(n-2)!}.\frac{X^{n-2}}{(X+1)^{m+n-1}}+\frac{(m+n-2)!}{(m-1)!.(n-2)!}.\frac{X^{n-2}}{(X+1)^{m+n-1}}$$
$$-\frac{(m+n-3)!}{(m-1)!.(n-3)!}.\frac{X^{n-3}}{(X+1)^{m+n-2}}+\frac{(m+n-3)!}{(m-1)!.(n-3)!}.\frac{X^{n-3}}{(X+1)^{m+n-2}}-\frac{(m+n-4)!}{(m-1)!.(n-4)!}.\frac{X^{n-4}}{(X+1)^{m+n-3}}+...+$$
$$+...+\frac{m}{(X+1)^{m+1}}-\frac{m}{(X+1)^{m+1}}]$$

And finally :

$$p_{U/V}(X)=\frac{(m+n-1)!}{(m-1)!.(n-1)!}.\frac{X^{n-1}}{(X+1)^{m+n}}=\frac{\Gamma(n+m)}{\Gamma(m).\Gamma(n)}.\frac{X^{n-1}}{(1+X)^{n+m}} \qquad (4.5)$$

Formula (4.5) clearly shows that the diversity order of random variable U/V is equal to: n -1; i.e. the diversity order of U.

Appendix 5 : Characterization of SNR p.d.f for the reference system by means of a mixture of Nakagami laws.

Using the same notations as in Section 3.1 with the use of Gamma laws, we have now:

$$f(x) = \frac{2.\mu^{\mu}}{\Gamma(\mu).\omega^{\mu}} x^{2.\mu-1}.e^{-\frac{\mu}{\omega}.x^2} \quad , \mu > 0, \omega > 0 \qquad (5.1)$$

The mixture of Nakagami laws takes the following form:

$$f_C(x) = \sum_{j=1}^{J} \pi_j \frac{2.{\mu_j}^{\mu_j}}{\Gamma(\mu_j).{\omega_j}^{\mu_j}} x^{2.\mu_j-1}.e^{-\frac{\mu_j}{\omega_j}.x^2} \qquad (5.2)$$

We find now :

$$\begin{aligned} \log p_{\boldsymbol{\theta}}(\Xi, \boldsymbol{Z}) = \sum_{i=1}^{n}\sum_{j=1}^{J} \hat{z}_{i,j}.\log \pi_j + \sum_{i=1}^{n}\sum_{j=1}^{J} z_{i,j}.[\log(2.{\mu_j}^{\mu_j}) - \log(\Gamma(\mu_j)) + \ldots \\ -\mu_j.\log(\omega_j) + (2.\mu_j - 1).\log(\xi_i) - \frac{\mu_j}{\omega_j}.\xi_i^2] \end{aligned} \qquad (5.3)$$

The maximization step yields to :

$$\frac{\partial \log p_{\boldsymbol{\theta}}(\Xi, \boldsymbol{Z})}{\partial \omega_j} = \sum_{i=1}^{n} (-\frac{\mu_j}{\omega_j} + \frac{\mu_j}{\omega_j^2}.\xi_i^2).\hat{z}_{i,j}$$

$$\omega_j = \frac{\sum_{i=1}^{n} \xi_i^2.\hat{z}_{i,j}}{\sum_{i=1}^{n} \hat{z}_{i,j}} \qquad (5.4)$$

And :

$$\frac{\partial \log p_{\boldsymbol{\theta}}(\Xi, \boldsymbol{Z})}{\partial \mu_j} = \sum_{i=1}^{n} (\log(\mu_j) + 1 - \frac{\partial \Gamma(\mu_j)/\partial \mu_j}{\Gamma(\mu_j)} - \log(\omega_j) + 2.\log(\xi_i) - \frac{\xi_i^2}{\omega_j}).\hat{z}_{i,j}$$

$$= \sum_{i=1}^{n} (\log(\mu_j) + 1 - \zeta(\mu_j) - \log(\omega_j) + 2.\log(\xi_i) - \frac{\xi_i^2}{\omega_j}).\hat{z}_{i,j}$$

$$\frac{\partial \log p_{\boldsymbol{\theta}}(\Xi, \boldsymbol{Z})}{\partial \mu_j} = 0 \quad \Rightarrow \log(\mu_j) = \frac{\sum_{i=1}^{n} \hat{z}_{i,j}.(\log(\omega_j) - 1 + \zeta(\mu_j) - 2.\log(\xi_i) + \xi_i^2/\omega_j)}{\sum_{i=1}^{n} \hat{z}_{i,j}} \qquad (5.5)$$

With : $\zeta(\mu_j) = \frac{\int_0^{+\infty} \ln(u) u^{\mu_j}.e^{-u}.du}{\int_0^{+\infty} u^{\mu_j}.e^{-u}.du}$; and we solve (5.5) with the classical Newton Raphson method.