Optimal stopping with f -expectations: the irregular case

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OPTIMAL STOPPING WITH f-EXPECTATIONS: THE IRREGULAR CASE

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Abstract   We consider the optimal stopping problem with non-linear f-expectation (induced by a BSDE) without making any regularity assumptions on the reward process ξ. We show that the value family can be aggregated by an optional process Y. We characterize the process Y as the $\mathcal{E}^f$-Snell envelope of ξ. We also establish an infinitesimal characterization of the value process Y in terms of a Reflected BSDE with ξ as the obstacle. To do this, we first establish a comparison theorem for irregular RBSDEs. We give an application to the pricing of American options with irregular pay-off in an imperfect market model.

1. Introduction.   The *classical* optimal stopping probem with *linear expectations* has been largely studied. General results on the topic can be found in El Karoui (1981) ([11]) where no regularity assumptions on the reward process ξ are made.

In this paper, we are interested in a generalization of the classical optimal stopping problem where the linear expectation is replaced by a possibly non-linear functional, the so-called f-expectation (f-evaluation), induced by a BSDE with Lipschitz driver f. For a stopping time S such that $0 \leq S \leq T$ a.s. (where $T > 0$ is a fixed terminal horizon), we define

$$(1.1) \qquad V(S) := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau),$$

where $\mathcal{T}_{S,T}$ denotes the set of stopping times valued a.s. in $[S,T]$ and $\mathcal{E}^f_{S,\tau}(\cdot)$ denotes the conditional f-expectation/evaluation at time S when the terminal time is τ.

The above non-linear problem has been introduced in [13] in the case of a Brownian filtration and a continuous financial position/pay-off process ξ and applied to the (non-linear) pricing of American options. It has then attracted considerable interest, in particular, due to its links with dynamic risk measurement (cf., e.g., [3]). In the case of a financial position/payoff process ξ, only supposed to be right-continuous, this non-linear optimal stopping problem has been studied in [36] (the case of Brownian-Poisson filtration), and in [1] where the non-linear expectation is supposed to be convex. To the best of our knowledge, [16] is the first paper addressing the stopping problem (1.1) in the case of a non-right-continuous process ξ; in [16] the assumption of right-continuity of ξ is replaced by the weaker assumption of right- uppersemicontinuity (r.u.s.c.). In the present paper, we study problem (1.1) without making any regularity assumptions on ξ.

The complete lack of regularity of ξ allows for more flexibility in the modelling (compared to "the more regular cases").

The usual approach to address the classical optimal stopping problem (i.e., the case $f \equiv 0$ in (1.1)) is a *a direct approach*, based on a direct study of the value family $(V(S))_{S \in \mathcal{T}_{0,T}}$. An important step in this approach is the aggregation of the value family by an optional process. The approach used in the literature to address the non-linear case (where f is not necessarily equal to 0) is *an RBSDE-approach*, based on the study of a related Reflected BSDE and on linking directly the solution of the Reflected BSDE with the value family $(V(S), S \in \mathcal{T}_{0,T})$ (and thus avoiding, in particular, more technical aggregation questions). This approach requires at least the uppersemicontinuity of the reward process ξ (cf., e.g., [16], [36]) which we do not have here (cf. also Remark 6.9).

Neither of the two approaches is applicable in the general framework of the present paper and we adopt *a new approach which combines* some aspects of both the approaches. Our combined approach is the following: First, with the help of some results from the general theory of processes, we show that the value family $(V(S), S \in \mathcal{T}_{0,T})$ can be aggregated by a unique right-uppersemicontinuous optional process $(V_t)_{t \in [0,T]}$. We characterize the value process $(V_t)_{t \in [0,T]}$ as the $\mathcal{E}^f$-*Snell envelope* of ξ, that is, the smallest strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ. Then, we turn to establishing an infinitesimal characterization of the value process $(V_t)_{t \in [0,T]}$ in terms of a Reflected BSDE where the pay-off process ξ from (1.1) plays the role of a lower obstacle. We emphasize that this *RBSDE-part* of our approach is far from mimicking the one from the r.u.s.c. case; we have to rely to very different arguments here due to the complete irregularity of the process ξ.

Let us recall that Reflected BSDEs have been introduced by El Karoui et al. in the seminal paper [12] in the case of a Brownian filtration and a continuous obstacle, and then generalized to the case of a right-continuous obstacle and/or a larger stochastic basis than the Brownian one in [20], [5], [21], [14], [22], [36]. In [16], we have formulated a notion of Reflected BSDE in the case where the obstacle is only right-uppersemicontinuous (but possibly not right-continuous) and have shown existence and uniqueness of the solution. In the present paper, we show that the existence and uniqueness result from [16] still holds in the more general case, without any regularity assumptions on the obstacle. In the recent preprint [25], existence and uniqueness of the solution (in the Brownian framework) is shown by using a different approach, namely a penalization method.

We also establish a comparison result for RBSDEs with irregular obstacles. Due to the complete irregularity of the obstacles and the presence of jumps in the filtration, we are led to using an approach which differs from those existing in the literature on comparison of RBSDEs (cf. also Remark 5.8); in particular, we first prove a generalization of Gal'chouk-Lenglart's formula (cf. [15] and [29]) to the case of convex functions, which we then astutely apply in our framework. The comparison result together with the $\mathcal{E}^f$-Mertens decomposition for strong (r.u.s.c.) $\mathcal{E}^f$-supermartingales (cf. [16] or [4]), helps in the study of the non-linear operator $\mathcal{R}ef^f$ which maps a given (completely irregular) obstacle to the solution of the RBSDE with driver f. By using the properties of the operator $\mathcal{R}ef^f$, we show that $\mathcal{R}ef^f[\xi]$, that is, the (first component of the) solution to the Reflected BSDE with irregular obstacle ξ and driver f, is equal to the $\mathcal{E}^f$-*Snell*

envelope of ξ, from which we derive that it coincides with the value process $(V_t)_{t \in [0,T]}$ of problem (1.1).

Finally, we give a financial application to the problem of pricing of American options with irregular pay-off in an imperfect market model. In particular, we show that the superhedging price of the American option with irregular pay-off ξ is characterized as the solution of an associated RBSDE (where ξ is the lower obstacle). Some examples of digital American options are given as particular cases.

The rest of the paper is organized as follows: In Section 2 we give some preliminary definitions and some notation. In Section 3 we revisit the classical optimal stopping problem with irregular pay-off process ξ. We first give some general results such as aggregation, Mertens decomposition of the value process, Skorokhod conditions satisfied by the associated non decreasing processes; then, we characterize the value process of the classical problem in terms of the solution of a Reflected BSDE with irregular obstacle and driver f which does not depend on the solution. Section 4 is devoted to the first part of the study of the non-linear optimal stopping problem (1.1); in particular, we present the aggregation result and the Snell characterization. Section 5 is devoted to the study of the related Reflected BSDE with irregular obstacle; in particular, we prove existence and uniqueness of the solution for general Lipschitz driver f (Subsection 5.1), provide a comparison theorem (Subsection 5.3), and establish some useful properties of the non-linear operator $\mathcal{R}ef^f$ (Subsection 5.4). In Section 6 we present the infinitesimal characterization of the value of the non-linear optimal stopping problem (1.1) in terms of the solution of the RBSDE from Section 5. In Section 7 we give a financial application to the pricing of American options with irregular pay-off in an imperfect market model with jumps; we also give a useful corollary of the infinitesimal characterization, namely, *a priori estimates with universal constants* for RBSDEs with irregular obstacles.

2. Preliminaries. Let $T > 0$ be a fixed positive real number. Let $E = \mathbf{R}^n \setminus \{0\}, \mathscr{E} = \mathcal{B}(\mathbf{R}^n \setminus \{0\})$, which we equip with a σ-finite positive measure ν. Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a one-dimensional Brownian motion W and with an independent Poisson random measure $N(dt, de)$ with compensator $dt \otimes \nu(de)$. We denote by $\tilde{N}(dt, de)$ the compensated process, i.e. $\tilde{N}(dt, de) := N(dt, de) - dt \otimes \nu(de)$. Let $I\!F = \{\mathcal{F}_t \colon t \in [0,T]\}$ be the (complete) natural filtration associated with W and N. We denote by $\mathcal{P}$ (resp. $\mathcal{O}$) the predictable (resp. optional) σ-algebra on $\Omega \times [0,T]$. The notation $L^2(\mathcal{F}_T)$ stands for the space of random variables which are $\mathcal{F}_T$-measurable and square-integrable. For $t \in [0,T]$, we denote by $\mathcal{T}_{t,T}$ the set of stopping times τ such that $P(t \leq \tau \leq T) = 1$. More generally, for a given stopping time $\nu \in \mathcal{T}_{0,T}$, we denote by $\mathcal{T}_{\nu,T}$ the set of stopping times τ such that $P(\nu \leq \tau \leq T) = 1$.

We use also the following notation:

- L_ν^2 is the set of $(\mathscr{E}, \mathcal{B}(\mathbf{R}))$-measurable functions $\ell : E \to \mathbf{R}$ such that $\|\ell\|_\nu^2 := \int_E |\ell(e)|^2 \nu(de) < \infty$. For $\ell \in \mathcal{L}_\nu^2$, $\mathcal{k} \in \mathcal{L}_\nu^2$, we define $\langle \ell, \mathcal{k} \rangle_\nu := \int_E \ell(e) \mathcal{k}(e) \nu(de)$.
- $I\!H^2$ is the set of $\mathbf{R}$-valued predictable processes ϕ with $\|\phi\|_{I\!H^2}^2 := E \left[\int_0^T |\phi_t|^2 dt \right] < \infty$.
- $I\!H_\nu^2$ is the set of $\mathbf{R}$-valued processes $l : (\omega, t, e) \in (\Omega \times [0,T] \times E) \mapsto l_t(\omega, e)$ which are *predictable*, that is $(\mathcal{P} \otimes \mathscr{E}, \mathcal{B}(\mathbf{R}))$-measurable, and such that $\|l\|_{I\!H_\nu^2}^2 := E \left[\int_0^T \|l_t\|_\nu^2 dt \right] < \infty$.

As in [16], we denote by $\mathcal{S}^2$ the vector space of $\mathbf{R}$-valued optional (not necessarily cadlag) processes ϕ such that $\|\|\phi\|\|_{\mathcal{S}^2}^2 := E[\operatorname{ess\,sup}_{\tau \in \mathcal{T}_0} |\phi_\tau|^2] < \infty$. By Proposition 2.1 in [16], the mapping $\|\|\cdot\|\|_{\mathcal{S}^2}$ is a norm on the space $\mathcal{S}^2$, and $\mathcal{S}^2$ endowed with this norm is a Banach space.

Definition 2.1 (Driver, Lipschitz driver) *A function f is said to be a* driver *if*

- $f : \Omega \times [0,T] \times \mathbf{R}^2 \times L_\nu^2 \to \mathbf{R}$
  $(\omega, t, y, z, k) \mapsto f(\omega, t, y, z, k)$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L_\nu^2) -$ *measurable,*
- $E[\int_0^T f(t,0,0,0)^2 dt] < +\infty.$

A driver f is called a Lipschitz driver *if moreover there exists a constant $K \geq 0$ such that $dP \otimes dt$-a.e., for each $(y_1, z_1, k_1) \in \mathbf{R}^2 \times L_\nu^2$, $(y_2, z_2, k_2) \in \mathbf{R}^2 \times L_\nu^2$,*

$$|f(\omega, t, y_1, z_1, k_1) - f(\omega, t, y_2, z_2, k_2)| \leq K(|y_1 - y_2| + |z_1 - z_2| + \|k_1 - k_2\|_\nu).$$

Definition 2.2 (BSDE, conditional f-expectation) *We recall (cf. [2]) that, if f is a Lipschitz driver and if ξ is a square-integrable $\mathcal{F}_T$-measurable random variable, then there exists a unique solution $(X, \pi, l) \in \mathcal{S}^2 \times I\!H^2 \times I\!H_\nu^2$ to the following BSDE*

$$X_t = \xi + \int_t^T f(s, X_s, \pi_s, l_s) ds - \int_t^T \pi_s dW_s - \int_t^T \int_E l_s(e) \tilde{N}(ds, de) \text{ for all } t \in [0,T] \text{ a.s.}$$

For $t \in [0,T]$, the (non-linear) operator $\mathcal{E}_{t,T}^f(\cdot) : L^2(\mathcal{F}_T) \to L^2(\mathcal{F}_t)$ which maps a given terminal condition $\xi \in L^2(\mathcal{F}_T)$ to the position X_t (at time t) of the first component of the solution of the above BSDE is called conditional f-expectation at time t. *It is also well-known that this notion can be extended to the case where the (deterministic) terminal time T is replaced by a (more general) stopping time $\tau \in \mathcal{T}_{0,T}$, t is replaced by a stopping time S such that $S \leq \tau$ a.s. and the domain $L^2(\mathcal{F}_T)$ of the operator is replaced by $L^2(\mathcal{F}_\tau)$.*

We now pass to the notion of Reflected BSDE. Let $T > 0$ be a fixed terminal time. Let f be a driver. Let $\xi = (\xi_t)_{t \in [0,T]}$ be a left-limited process in $\mathcal{S}^2$.

Remark 2.1 *Let us note that in the following definitions and results* we can relax the assumption of existence of left limits for the obstacle ξ. *All the results still hold true provided we replace the process $(\xi_{t-})_{t \in]0,T]}$ by the process $(\underline{\xi}_t)_{t \in]0,T]}$ defined by $\underline{\xi}_t := \limsup_{s \uparrow t, s < t} \xi_s$, for all $t \in]0,T]$. We recall that $\underline{\xi}$ is a predictable process (cf. [7, Thm. 90, page 225]). We call the process $\underline{\xi}$ the left upper-semicontinuous envelope of ξ.*

Definition 2.3 (Reflected BSDE) *A process (Y, Z, k, A, C) is said to be a solution to the reflected BSDE with parameters (f, ξ), where f is a driver and ξ is a left-limited process in $\mathcal{S}^2$,*

if

$$(Y, Z, k, A, C) \in \mathcal{S}^2 \times I\!H^2 \times I\!H_\nu^2 \times \mathcal{S}^2 \times \mathcal{S}^2 \, and \, a.s. \, for \, all \, t \in [0, T]$$

(2.2)

$$Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, k_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E k_s(e) \tilde{N}(ds, de) + A_T - A_t + C_{T-} - C_{t-},$$

$Y_t \geq \xi_t$ *for all* $t \in [0, T]$ *a.s.,*

A is a nondecreasing right-continuous predictable process with $A_0 = 0$ *and such that*

(2.3)

$$\int_0^T \mathbf{1}_{\{Y_{t-} > \xi_{t-}\}} dA_t^c = 0 \ a.s. \ and \ (Y_{\tau-} - \xi_{\tau-})(A_\tau^d - A_{\tau-}^d) = 0 \ a.s. \ for \ all \ predictable \ \tau \in \mathcal{T}_{0,T},$$

C is a nondecreasing right-continuous adapted purely discontinuous process with $C_{0-} = 0$

(2.4)

and such that $(Y_\tau - \xi_\tau)(C_\tau - C_{\tau-}) = 0$ *a.s. for all* $\tau \in \mathcal{T}_{0,T}$.

Here A^c denotes the continuous part of the process A and A^d its discontinuous part.
Equations (2.3) and (2.4) are referred to as *minimality conditions* or *Skorokhod conditions*.

For real-valued random variables X and X_n, $n \in I\!N$, the notation "$X_n \uparrow X$" will stand for "the sequence (X_n) is nondecreasing and converges to X a.s.".

For a ladlag process ϕ, we denote by ϕ_{t+} and ϕ_{t-} the right-hand and left-hand limit of ϕ at t. We denote by $\Delta_+\phi_t := \phi_{t+} - \phi_t$ the size of the right jump of ϕ at t, and by $\Delta\phi_t := \phi_t - \phi_{t-}$ the size of the left jump of ϕ at t.

Remark 2.2 *If* (Y, Z, k, A, C) *is a solution to the RBSDE defined above, by (2.2), we have* $\Delta C_t = Y_t - Y_{t+}$, *which implies that* $Y_t \geq Y_{t+}$, *for all* $t \in [0, T)$. *Hence, Y is r.u.s.c. Moreover, from* $C_\tau - C_{\tau-} = -(Y_{\tau+} - Y_\tau)$, *combined with the Skorokhod condition (2.4), we derive* $(Y_\tau - \xi_\tau)(Y_{\tau+} - Y_\tau) = 0$, *a.s. for all* $\tau \in \mathcal{T}_{0,T}$. *This, together with* $Y_\tau \geq \xi_\tau$ *and* $Y_\tau \geq Y_{\tau+}$ *a.s., leads to* $Y_\tau = Y_{\tau+} \vee \xi_\tau$ *a.s. for all* $\tau \in \mathcal{T}_{0,T}$.

Definition 2.4 *Let* $\tau \in \mathcal{T}_0$. *An optional process* (ϕ_t) *is said to be* right upper-semicontinuous *(r.u.s.c.) along stopping times if for all stopping time* $\tau \in \mathcal{T}_0$ *and for all nonincreasing sequence of stopping times* (τ_n) *such that* $\tau^n \downarrow \tau$ *a.s. ,* $\phi_\tau \geq \limsup_{n \to \infty} \phi_{\tau_n}$ *a.s..*

3. The classical optimal stopping problem. Let $(\xi_t)_{t \in [0,T]}$ be a left-limited process belonging to $\mathcal{S}^2$, called the *reward process*. Let $f = (f_t)_{t \in [0,T]}$ be a predictable process with $E[\int_0^T f_t^2 dt] < +\infty$, called the *instantaneous reward process*. For each $S \in \mathcal{T}_{0,T}$, we define the value function $Y(S)$ at time S by

(3.5)
$$Y(S) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} E[\xi_\tau + \int_S^\tau f_u du \mid \mathcal{F}_S].$$

3.1. *General results.*

Lemma 3.1 (i) *There exists a ladlag optional process* $(Y_t)_{t \in [0,T]}$ *which aggregates the family* $(Y(S))_{S \in \mathcal{T}_{0,T}}$ *(i.e.* $Y_S = Y(S)$ *a.s. for all* $S \in \mathcal{T}_{0,T}$*).*

Moreover, the process $(Y_t + \int_0^t f_u du)_{t \in [0,T]}$ *is the smallest strong supermartingale greater than or equal to* $(\xi_t + \int_0^t f_u du)_{t \in [0,T]}$.

(ii) *We have* $Y_S = \xi_S \vee Y_{S+}$ *a.s. for all* S.

(iii) *For each* $S \in \mathcal{T}_{0,T}$ *and for each* $\lambda \in]0,1[$, *we set*

$$\tau_S^\lambda := \inf\{t \geq S\,,\, \lambda Y_t(\omega) \leq \xi_t\}.$$

The process $(Y_t + \int_0^t f_u du)_{t \in [0,T]}$ *is a martingale on* $[S, \tau_S^\lambda]$.

Proof.   These results follow from results of classical optimal stopping theory. For a sketch of the proof of the first two assertions, the reader is referred to the proof of Proposition A.5 in the Appendix of [16] (which still holds for a general process $\xi \in \mathcal{S}^2$). The last assertion corresponds to a result of optimal stopping theory (cf. [30], [11] or Lemma 2.7 in [26]). Its proof is based on a penalization method (used in convex analysis), introduced by Maingueneau (1978) (cf. the proof of Theorem 2 in [30]), which does not require any regularity assumption on the reward process ξ. $\square$

Remark 3.3 *It follows from* (ii) *in the above lemma that* $\Delta_+ Y_S = \mathbf{1}_{\{Y_S = \xi_S\}} \Delta_+ Y_S$ *a.s.*

Remark 3.4 *Let us note for further reference that Maingueneau's penalization approach for showing the martingale property on* $[S, \tau_S^\lambda]$ *(property (iii) in the above lemma) relies heavily on the convexity of the problem.*

Lemma 3.2 (i) *The value process* Y *of Lemma 3.1 belongs to* $\mathcal{S}^2$ *and admits the following (Mertens) decomposition:*

$$(3.6) \qquad Y_t = -\int_0^t f_u du + M_t - A_t - C_{t-} \text{ for all } t \in [0,T] \text{ a.s.,}$$

where M *is a square integrable martingale,* A *is a nondecreasing right-continuous predictable process such that* $A_0 = 0$, $E(A_T^2) < \infty$, *and* C *is a nondecreasing right-continuous adapted purely discontinuous process such that* $C_{0-} = 0$, $E(C_T^2) < \infty$.

(ii) *For each* $\tau \in \mathcal{T}_{0,T}$, *we have* $\Delta C_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}} \Delta C_\tau$ *a.s.*

(iii) *For each predictable* $\tau \in \mathcal{T}_{0,T}$, *we have* $\Delta A_\tau = \mathbf{1}_{\{Y_{\tau-} = \xi_{\tau-}\}} \Delta A_\tau$ *a.s.*

Proof.   By Lemma 3.1 (i), the process $(Y_t + \int_0^t f_u du)_{t \in [0,T]}$ is a strong supermartingale. Moreover, by using martingale inequalities, it can be shown that

$$(3.7) \qquad E[\operatorname*{ess\,sup}_{S \in \mathcal{T}_{0,T}} |Y_S|^2] \leq cE[X^2] \leq cT\|f\|_{\mathbb{H}^2}^2 + c\|\xi\|_{\mathcal{S}^2}^2.$$

Hence, the process $(Y_t + \int_0^t f_u du)_{t \in [0,T]}$ is in $\mathcal{S}^2$ (a fortiori, of class (D)). Applying Mertens decomposition for strong supermartingales of class (D) (cf., e.g., [8, Appendix 1, Thm.20, equalities (20.2)])gives the decomposition (3.6), where M is a cadlag uniformly integrable martingale, A is a nondecreasing right-continuous predictable process such that $A_0 = 0$, $E(A_T) < \infty$, and C is a nondecreasing right-continuous adapted purely discontinuous process such that $C_{0-} = 0$, $E(C_T) < \infty$. Based on some results of Dellacherie-Meyer [8] (cf., e.g., Theorem A.2 and Corollary A.1 in [16]), we derive that $A \in \mathcal{S}^2$ and $C \in \mathcal{S}^2$, which gives the assertion **(i)**.

Let $\tau \in \mathcal{T}_{0,T}$. By Remark 3.3 together with Mertens decomposition (3.6), we get $\Delta C_\tau = -\Delta_+ Y_\tau$ a.s. It follows that $\Delta C_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}} \Delta C_\tau$ a.s. , which corresponds to **(ii)**.

From Lemma 3.1 (iii) together with Mertens decomposition (3.6), it follows that, for each $S \in \mathcal{T}_{0,T}$ and for each $\lambda \in]0, 1[$, we have

$$(3.8) \qquad\qquad A_S = A_{\tau_S^\lambda} \quad \text{a.s.}$$

Assertion **(iii)** (concerning the jumps of A) is due to El Karoui ([11, Proposition 2.34]). Its proof is based on the equality (3.8). $\qquad\square$

The following minimality property is well-known from the literature in the "more regular" cases (cf., e.g., [27] for the right-uppersemicontinuous case). In the case of completely irregular ξ, this minimality property was not explicitly available. Only recently, it was proved by [25] (cf. Proposition 3.7) in the Brownian framework. Here, we generalize the result of [25] by using different analytic arguments.

Lemma 3.3 *The continuous part A^c of A satisfies the equality $\int_0^T \mathbf{1}_{\{Y_{t-} > \xi_{t-}\}} dA_t^c = 0$ a.s.*

Proof. As for the discontinuous part of A, the proof is based on Lemma 3.1 (iii) , and also on some analytic arguments similar to those used in the proof of Theorem D13 in Karatzas and Shreve (1998) ([24]).

We have to show that $\int_0^T (Y_{t-} - \xi_{t-}) dA_t^c = 0$ a.s.

Lemma 3.1 (iii) yields that for each $S \in \mathcal{T}_{0,T}$ and for each $\lambda \in]0, 1[$, we have $A_S = A_{\tau_S^\lambda}$ a.s. Without loss of generality, we can assume that for each ω, the map $t \mapsto A_t^c(\omega)$ is continuous, that the maps $t \mapsto Y_t(\omega)$ and $t \mapsto \xi_t(\omega)$ are left-limited, and that, for all $\lambda \in]0, 1[\cap \mathbb{Q}$ and $t \in [0, T[\cap \mathbb{Q}$, we have $A_t(\omega) = A_{\tau_t^\lambda}(\omega)$.

Let us denote by $\mathcal{J}(\omega)$ the set on which the nondecreasing function $t \mapsto A_t^c(\omega)$ is "flat":

$$\mathcal{J}(\omega) := \{t \in]0, T[, \ \exists \delta > 0 \ \text{with} \ A_{t-\delta}^c(\omega) = A_{t+\delta}^c(\omega)\}$$

The set $\mathcal{J}(\omega)$ is clearly open and hence can be written as a countable union of disjoint intervals: $\mathcal{J}(\omega) = \cup_i]\alpha_i(\omega), \beta_i(\omega)[$. We consider

$$(3.9) \qquad \hat{\mathcal{J}}(\omega) := \cup_i]\alpha_i(\omega), \beta_i(\omega)] = \{t \in]0, T] , \ \exists \delta > 0 \ \text{with} \ A_{t-\delta}^c(\omega) = A_t^c(\omega)\}.$$

We have $\int_0^T \mathbf{1}_{\hat{\mathcal{J}}(\omega)} dA_t^c(\omega) = \sum_i (A_{\beta_i(\omega)}^c(\omega) - A_{\alpha_i(\omega)}^c(\omega)) = 0$. Hence, the nondecreasing function $t \mapsto A_t^c(\omega)$ is "flat" on $\hat{\mathcal{J}}(\omega)$. We now introduce

$$\mathcal{K}(\omega) := \{t \in]0, T] \ \text{s.t.} \ Y_{t-}(\omega) > \xi_{t-}(\omega)\}$$

We next show that for almost every ω, $\mathcal{K}(\omega) \subset \hat{\mathcal{J}}(\omega)$, which clearly provides the desired result. Let $t \in \mathcal{K}(\omega)$. Let us prove that $t \in \hat{\mathcal{J}}(\omega)$. By (3.9), we thus have to show that there exists $\delta > 0$ such that $A_{t-\delta}^c(\omega) = A_t^c(\omega)$. Since $t \in \mathcal{K}(\omega)$, we have $Y_{t-}(\omega) > \xi_{t-}(\omega)$. Hence, there exists $\delta > 0$ and $\lambda \in]0, 1[\cap \mathbb{Q}$ such that $t - \delta \in [0, T[\cap \mathbb{Q}$ and for each $r \in [t - \delta, t[$, $\lambda Y_r(\omega) > \xi_r(\omega)$. By definition of $\tau_{t-\delta}^\lambda(\omega)$, it follows that $\tau_{t-\delta}^\lambda(\omega) \geq t$. Now, we have $A_{\tau_{t-\delta}^\lambda}^c(\omega) = A_{t-\delta}^c(\omega)$. Since the map $s \mapsto A_s^c(\omega)$ is nondecreasing, we derive that $A_t^c(\omega) = A_{t-\delta}^c(\omega)$, which implies that $t \in \hat{\mathcal{J}}(\omega)$. We thus have $\mathcal{K}(\omega) \subset \hat{\mathcal{J}}(\omega)$, which completes the proof. $\qquad\square$

Remark 3.5 *We see from the above proofs that Lemmas 3.2 and 3.3 also hold true in the case of a general filtration assumed to satisfy the usual hypotheses. We note also that the martingale property from assertion (iii) of Lemma 3.1 is crucial for the proof of the minimality conditions for the process A (namely, for the proofs of Lemma 3.2 assertion (iii), and for Lemma 3.3).*

3.2. *Characterization of the value function as the solution of an RBSDE.* Using Lemmas 3.2 and 3.3, we show that the value process Y of the optimal stopping problem (3.5) solves the RBSDE from Definition 2.3 with parameters the driver process (f_t) and the obstacle (ξ_t), and that, moreover, Y is the unique solution of the RBSDE. We thus have an "infinitesimal characterization" of the value process Y.

Theorem 3.1 *Let Y be the value process of the optimal stopping problem (3.5). Let A and C be the non decreasing processes associated with the Mertens decomposition (3.6) of Y. There exists a unique pair $(Z, k) \in I\!H^2 \times I\!H^2_\nu$ such that the process (Y, Z, k, A, C) is a solution of the RBSDE from Definition 2.3 associated with the driver process $f(\omega, t, y, z, \kappa) = f_t(\omega)$ and the obstacle (ξ_t). Moreover, the solution of this RBSDE is unique.*

Proof.   The proof relies on the above lemmas and also on the *a priori* estimates from Lemma 8.1 of the Appendix.

By Lemma 3.1 (ii), the value process Y corresponding to the optimal stopping problem (3.5) satisfies $Y_T = Y(T) = \xi_T$ a.s. and $Y_t \geq \xi_t$, $0 \leq t \leq T$, a.s. By Lemma 3.2 **(ii)**, the process C of the Mertens decomposition of Y (3.6) satisfies the minimality condition (2.4). Moreover, by Lemma 3.2 **(iii)** and Lemma 3.3, the process A satisfies the minimality condition (2.3). By the martingale representation theorem (cf., e.g., Lemma 2.3 in [39]) there exists a unique predictable process $Z \in I\!H^2$ and a unique *predictable* $k \in I\!H^2_\nu$ such that $dM_t = Z_t dW_t + \int_E k_t(e)\tilde{N}(dt, de)$. The process (Y, Z, k, A, C) is thus a solution of the RBSDE (2.3) associated with the driver process (f_t) and with the obstacle ξ.

It remains to show the uniqueness of the solution. Using the *a priori estimates* from Lemma 8.1 of the Appendix, together with classical arguments of the theory of BSDEs, we obtain the desired result (for details, see step 5 of the proof of Lemma 3.3 in [16]).   □

4. Optimal stopping with non-linear f-expectation and irregular pay-off . Let $(\xi_t)_{t \in [0,T]}$ be a left-limited process in $\mathcal{S}^2$. Let f be a Lipschitz driver satisfying Assumption 4.1. For each $S \in \mathcal{T}_{0,T}$, we consider the random variable

(4.10) $$V(S) := \text{ess} \sup_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau).$$

As mentioned in the introduction, the above optimal stopping problem has been largely studied: in [13], and in [3], in the case of a continuous pay-off process ξ; in [36] and [1] in the case of a right-continuous pay-off; and recently in [16] in the case of a right-uppersemicontinuous pay-off process ξ. In this section, we do not make any regularity assumptions on ξ (cf. also Remark 2.1).

We make the following assumption on the driver (cf., e.g., Theorem 4.2 in [35]).

Assumption 4.1 *Assume that $dP \otimes dt$-a.e. for each $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L_\nu^2)^2$,*

$$f(t, y, z, k_1) - f(t, y, z, k_2) \geq \langle \theta_t^{y,z,k_1,k_2}, k_1 - k_2 \rangle_\nu,$$

with

$$\theta : [0, T] \times \Omega \times \mathbb{R}^2 \times (L_\nu^2)^2 \to L_\nu^2 \, ; \, (\omega, t, y, z, k_1, k_2) \mapsto \theta_t^{y,z,k_1,k_2}(\omega, \cdot)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}((L_\nu^2)^2)$-measurable, satisfying $\|\theta_t^{y,z,k_1,k_2}(\cdot)\|_\nu \leq K$ for all $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L_\nu^2)^2$, $dP \otimes dt$-a.e. , where K is a positive constant, and such that

(4.11) $$\theta_t^{y,z,k_1,k_2}(e) \geq -1,$$

for all $(y, z, k_1, k_2) \in \mathbb{R}^2 \times (L_\nu^2)^2$, $dP \otimes dt \otimes d\nu(e) -$ a.e.

The above assumption is satisfied if, for example, f is of class $\mathcal{C}^1$ with respect to k such that $\nabla_k f$ is bounded (in L_ν^2) and $\nabla_k f \geq -1$ (cf. Proposition A.2. in [9]).

We recall that under Assumption 4.1 on the driver f, the functional $\mathcal{E}_{S,\tau}^f(\cdot)$ is nondecreasing (cf. [35, Thm. 4.2]).

If we interpret ξ as a financial position process and $-\mathcal{E}^f(\cdot)$ as a dynamic risk measure (cf.,e.g., [33], [37]), then (up to a minus sign) $V(S)$ can be seen as the minimal risk at time S. As also mentioned in the introduction, the absence of regularity allows for more flexibility in the modelling. If, for instance, we consider a situation where the jump times of the Poisson random measure model times of default (which, being totally inaccessible, cannot be foreseen), then, the complete lack of regularity allows to take into account an immediate non-smooth, positive or negative, impact on ξ after the default occurs.

If we interpret ξ as a payoff process, and $\mathcal{E}^f(\cdot)$ as a non linear pricing rule, then the optimal stopping problem (4.10) is related to the (non linear) pricing problem of the American option with payoff ξ. The absence of regularity allows us to deal with the case of American options with irregular payoffs, such as American digital options (cf. Section 7.1 for details).

4.1. *Preliminary results on the value family.* Let us first introduce the definition of an admissible family of random variables indexed by stopping times in $\mathcal{T}_{0,T}$ (or $\mathcal{T}_{0,T}$-system in the vocabulary of Dellacherie and Lenglart [6]).

Definition 4.5 *We say that a family $U = (U(\tau), \tau \in \mathcal{T}_{0,T})$ is admissible if it satisfies the following conditions*

1.   *for all $\tau \in \mathcal{T}_{0,T}$, $U(\tau)$ is a real-valued $\mathcal{F}_\tau$-measurable random variable.*
2.   *for all $\tau, \tau' \in \mathcal{T}_{0,T}$, $U(\tau) = U(\tau')$ a.s. on $\{\tau = \tau'\}$.*

Moreover, we say that an admissible family U is square-integrable if for all $\tau \in \mathcal{T}_{0,T}$, $U(\tau)$ is square-integrable.

Lemma 4.4 (*Admissibility of the family V*) *The family $V = (V(S), S \in \mathcal{T}_{0,T})$ defined in (4.10) is a square-integrable admissible family.*

The proof uses arguments similar to those used in the "classical" case of linear expectations (cf., e.g., [28]), combined with some properties of f-expectations.

Proof: For each $S \in \mathcal{T}_{0,T}$, V(S) is an $\mathcal{F}_S$-measurable square-integrable random variable, due to the definitions of the conditional f-expectation and of the essential supremum (cf. [31]). Let us prove Property 2 of the definition of admissibility. Let S and S' be two stopping times in $\mathcal{T}_{0,T}$. We set $A := \{S = S'\}$ and we show that $V(S) = V(S')$, P-a.s. on A. For each $\tau \in \mathcal{T}_{S,T}$, we set $\tau_A := \tau \mathbf{1}_A + T \mathbf{1}_{A^c}$. We have $\tau_A \geq S'$ a.s. By using the fact that $S = S'$ a.s. on A, the fact that $\tau_A = \tau$ a.s. on A, and a standard property of conditional f-expectations (cf., e.g., Proposition A.3 in [18]), we obtain

$$\mathbf{1}_A \mathcal{E}^f_{S,\tau}[\xi_\tau] = \mathbf{1}_A \mathcal{E}^f_{S',\tau}[\xi_\tau] = \mathcal{E}^{f^\tau \mathbf{1}_A}_{S',T}[\xi_\tau \mathbf{1}_A] = \mathcal{E}^{f^{\tau_A}\mathbf{1}_A}_{S',T}[\xi_{\tau_A}\mathbf{1}_A] = \mathbf{1}_A \mathcal{E}^f_{S',\tau_A}[\xi_{\tau_A}] \leq \mathbf{1}_A V(S'),$$

where $f^\tau(t,y,z,\kappa) := f(t,y,z,\kappa)\mathbf{1}_{\{\mathbf{t} \leq \tau\}}$. By taking the ess sup over $\mathcal{T}_{S,T}$ on both sides, we get $\mathbf{1}_A V(S) \leq \mathbf{1}_A V(S')$. We obtain the converse inequality by interchanging the roles of S and S'. $\square$

Lemma 4.5 (Optimizing sequence) *For each $S \in \mathcal{T}_{0,T}$, there exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times in $\mathcal{T}_{S,T}$ such that the sequence $(\mathcal{E}^f_{S,\tau_n}(\xi_{\tau_n}))_{n \in \mathbb{N}}$ is nondecreasing and*

$$V(S) = \lim_{n \to \infty} \uparrow \mathcal{E}^f_{S,\tau_n}(\xi_{\tau_n}) \quad \text{a.s.}$$

Proof: Due to a classical result on essential suprema (cf. [31]), it is sufficient to show that, for each $S \in \mathcal{T}_{0,T}$, the family $(\mathcal{E}_{S,\tau}(\xi_\tau), \ \tau \in \mathcal{T}_{S,T})$ is stable under pairwise maximization. Let us fix $S \in \mathcal{T}_{0,T}$. Let $\tau \in \mathcal{T}_{S,T}$ and $\tau' \in \mathcal{T}_{S,T}$. We define $A := \{\mathcal{E}^f_{S,\tau'}(\xi_{\tau'}) \leq \mathcal{E}^f_{S,\tau}(\xi_\tau)\}$. The set A is in $\mathcal{F}_S$. We define $\nu := \tau \mathbf{1}_A + \tau' \mathbf{1}_{A^c}$. We have $\nu \in \mathcal{T}_{S,T}$. We compute $\mathbf{1}_A \mathcal{E}^f_{S,\nu}(\xi_\nu) = \mathcal{E}^{f^\nu \mathbf{1}_A}_{S,T}(\xi_\nu \mathbf{1}_A) = \mathcal{E}^{f^\tau \mathbf{1}_A}_{S,T}(\xi_\tau \mathbf{1}_A) = \mathbf{1}_A \mathcal{E}^f_{S,\tau}(\xi_\tau)$ a.s. Similarly, we show $\mathbf{1}_{A^c} \mathcal{E}^f_{S,\nu}(\xi_\nu) = \mathbf{1}_{A^c} \mathcal{E}^f_{S,\tau'}(\xi_{\tau'})$. It follows that $\mathcal{E}^f_{S,\nu}(\xi_\nu) = \mathcal{E}^f_{S,\tau}(\xi_\tau)\mathbf{1}_A + \mathcal{E}^f_{S,\tau'}(\xi_{\tau'})\mathbf{1}_{A^c} = \mathcal{E}^f_{S,\tau}(\xi_\tau) \vee \mathcal{E}^f_{S,\tau'}(\xi_{\tau'})$, which shows the stability under pairwise maximization and concludes the proof. $\square$

We need two more definitions.

Definition 4.6 ($\mathcal{E}^f$-supermartingale family) *An admissible square-integrable family $U := (U(S), \ S \in \mathcal{T}_{0,T})$ is said to be a strong $\mathcal{E}^f$-supermartingale family if for all $S, S' \in \mathcal{T}_{0,T}$ such that $S \leq S'$ a.s.,*

$$\mathcal{E}^f_{S,S'}(U(S')) \leq U(S) \quad a.s.$$

Definition 4.7 (Right-uppersemicontinuous family) *An admissible family $U := (U(S), \ S \in \mathcal{T}_{0,T})$ is said to be a right-uppersemicontinuous (along stopping times) family if, for all (τ_n) non-increasing sequence in $\mathcal{T}_{0,T}$, $U(\tau) \geq \limsup_{n \to \infty} U(\tau_n)$ a.s. on $\{\tau = \lim \downarrow \tau_n\}$.*

The following lemma gives a link between the previous two notions.

Lemma 4.6 *Let $U := (U(S), \ S \in \mathcal{T}_{0,T})$ be a strong $\mathcal{E}^f$-supermartingale family. Then, $(U(S), \ S \in \mathcal{T}_{0,T})$ is a right-uppersemicontinuous (along stopping times) family in the sense of Definition 4.7.*

Proof: Let $\tau \in \mathcal{T}_{0,T}$ and let $(\tau_n) \in \mathcal{T}_{0,T}^{I\!N}$ be a nonincreasing sequence of stopping times such that $\lim_{n \to +\infty} \tau_n = \tau$ a.s. and for all $n \in I\!N$, $\tau_n > \tau$ a.s. on $\{\tau < T\}$, and such that $\lim_{n \to +\infty} U(\tau_n)$ exists a.s. As U is an $\mathcal{E}^f$-supermartingale family and as the sequence (τ_n) is nonincreasing, we have $\mathcal{E}^f_{\tau,\tau_n}(U(\tau_n)) \leq \mathcal{E}^f_{\tau,\tau_{n+1}}(U(\tau_{n+1})) \leq U(\tau)$ a.s. Hence, the sequence $(\mathcal{E}^f_{\tau,\tau_n}(U(\tau_n)))_n$ is nondecreasing and $U(\tau) \geq \lim \uparrow \mathcal{E}^f_{\tau,\tau_n}(U(\tau_n))$. This inequality, combined with the property of continuity of BSDEs with respect to terminal time and terminal condition (cf. [35, Prop. A.6]) gives

$$U(\tau) \geq \lim_{n \to +\infty} \mathcal{E}^f_{\tau,\tau_n}(U(\tau_n)) = \mathcal{E}^f_{\tau,\tau}(\lim_{n \to +\infty} U(\tau_n)) = \lim_{n \to +\infty} U(\tau_n) \quad \text{a.s.}$$

By Lemma 5 of Dellacherie and Lenglart [6] [1], the family $(U(S))$ is thus right-uppersemicontinuous (along stopping times).

$\square$

Theorem 4.2 *The value family $V = (V(S), \ S \in \mathcal{T}_{0,T})$ defined in (4.10) is a strong $\mathcal{E}^f$-supermartingale family. In particular, $V = (V(S), \ S \in \mathcal{T}_{0,T})$ is a right-uppersemicontinuous (along stopping times) family in the sense of Definition 4.7.*

Proof: We know from Lemma 4.4 that $V = (V(S), S \in \mathcal{T}_{0,T})$ is a square-integrable admissible family. Let $S \in \mathcal{T}_{0,T}$ and $S' \in \mathcal{T}_{S,T}$. We will show that $\mathcal{E}^f_{S,S'}(V(S')) \leq V(S)$ a.s., which will prove that V is a strong $\mathcal{E}^f$-supermartingale family. By Lemma 4.5, there exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times such that $\tau_n \geq S'$ a.s. and $V(S') = \lim_{n \to \infty} \uparrow \mathcal{E}^f_{S',\tau_n}(\xi_{\tau_n})$ a.s. By using this equality, the property of continuity of BSDEs, and the consistency of conditional f-expectation, we get

$$\mathcal{E}^f_{S,S'}(V(S')) = \mathcal{E}^f_{S,S'}(\lim_{n \to \infty} \uparrow \mathcal{E}^f_{S',\tau_n}(\xi_{\tau_n})) = \lim_{n \to \infty} \mathcal{E}^f_{S,S'}(\mathcal{E}^f_{S',\tau_n}(\xi_{\tau_n})) = \lim_{n \to \infty} \mathcal{E}^f_{S,\tau_n}(\xi_{\tau_n}) \leq V(S).$$

We conclude that V is a strong $\mathcal{E}^f$-supermartingale family. This property, together with Lemma 4.6, gives the property of right-uppersemicontinuity (along stopping times) of the family V. The proof is thus completed. $\square$

4.2. *Aggregation and Snell characterization.* We now show the following result, which generalizes some results of classical optimal stopping theory (more precisely, the assertion (i) from Lemma 3.1) to the case of an optimal stopping problem with f-expectation.

Theorem 4.3 (Aggregation and Snell characterization) *There exists a unique right- uppersemicontinuous optional process, denoted by $(V_t)_{t \in [0,T]}$, which aggregates the value family $V = (V(S), \ S \in \mathcal{T}_{0,T})$. Moreover, $(V_t)_{t \in [0,T]}$ is the $\mathcal{E}^f$-Snell envelope of the pay-off process ξ, that is, the smallest strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ.*

The proof of this theorem relies on the preliminary resuts on the value family $V = (V(S), \ S \in \mathcal{T}_{0,T})$ presented in the previous subsection.

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[1] The chronology Θ (in the vocabulary and notation of [6]) which we work with here is the chronology of all stopping times, that is, $\Theta = \mathcal{T}_{0,T}$; hence $[\Theta] = \Theta = \mathcal{T}_{0,T}$.

Proof: By Theorem 4.2, the value family $V = (V(S), \ S \in \mathcal{T}_{0,T})$ is a right-uppersemicontinuous family (or a right-uppersemicontinuous $\mathcal{T}_{0,T}$-system in the vocabulary of Dellacherie-Lenglart [6]). Applying Theorem 4 of Dellacherie-Lenglart ([6]), gives the existence of a unique (up to indistinguishability) right-uppersemicontinuous optional process $(V_t)_{t \in [0,T]}$ which *aggregates* the value family $(V(S), \ S \in \mathcal{T}_{0,T})$. From this aggregation property, namely the property $V_S = V(S)$ a.s. for each $S \in \mathcal{T}_{0,T}$, and from Theorem 4.2, we deduce that the process $(V_t)_{t \in [0,T]}$ is a strong $\mathcal{E}^f$-supermartingale. Moreover, $V_t \geq \xi_t$, for all $t \in [0,T]$, a.s. Indeed, due to the definition of the family $(V(S), \ S \in \mathcal{T}_{0,T})$ and to the aggregation result, we have $V_S \geq \xi_S$ a.s. for each $S \in \mathcal{T}_{0,T}$. We deduce that $V_t \geq \xi_t$, for all $t \in [0,T]$, a.s., by applying a well-known result from the general theory of processes (cf. ([7, Theorem IV.84])

Let us now prove that the process $(V_t)_{t \in [0,T]}$ is *the smallest* strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ. Let $(V'_t)_{t \in [0,T]}$ be a strong $\mathcal{E}^f$-supermartingale such that $V'_t \geq \xi_t$, for all $t \in [0,T]$, a.s. Let $S \in \mathcal{T}_{0,T}$. We have $V'_\tau \geq \xi_\tau$ a.s. for all $\tau \in \mathcal{T}_{S,T}$. Hence, $\mathcal{E}^f_{S,\tau}(V'_\tau) \geq \mathcal{E}^f_{S,\tau}(\xi_\tau)$ a.s., where we have used the monotonicity of the conditional f-expectation. On the other hand, by using the $\mathcal{E}^f$-supermartingale property of the process $(V'_t)_{t \in [0,T]}$, we have $V'_S \geq \mathcal{E}^f_{S,\tau}(V'_\tau)$ a.s. for all $\tau \in \mathcal{T}_{S,T}$. Hence, $V'_S \geq \mathcal{E}^f_{S,\tau}(\xi_\tau)$ a.s. for all $\tau \in \mathcal{T}_{S,T}$. By taking the essential supremum over $\tau \in \mathcal{T}_{S,T}$ in the inequality, we get $V'_S \geq \text{ess sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau) = V_S$ a.s. Note that the last equality in the above computation is due to the definition of $V(S)$ and to the aggregation result. We have thus obtained $V'_S \geq V_S$ a.s., which (as S is arbitrary in $\mathcal{T}_{0,T}$) leads to $V'_t \geq V_t$, for all $t \in [0,T]$, a.s., due to the same well-known result from the general theory of processes as above. □

5. Non-linear Reflected BSDE with complete irregular obstacle. Comparison theorem. Our aim now is to establish an infinitesimal characterization for the non-linear problem (4.10) in terms of the solution of a non-linear RBSDE (thus generalizing Theorem 3.1 from the classical linear case to the non-linear case). In order to do so, we need to establish first some results on non-linear RBSDEs with completely irregular obstacles, in particular, a comparison result for such RBSDEs. This section is devoted to these results. This extends and completes our work from [16], where an assumption of right-uppersemicontinuity on the obstacle is made. Let us note that the proof of the comparison theorem from [16] cannot be adapted to the completely irregular framework considered here; instead, we rely on a Tanaka-type formula for strong (irregular) semimartingales which we establish.

Remark 5.6 *One might wonder whether the infinitesimal characterization for the non-linear optimal stopping problem* (4.10) *can be obtained by a direct study of the value process* (V_t) *of problem* (4.10), *similarly to what was done in the classical linear case in Section 3. In the classical case, we applied Mertens decomposition for* (V_t); *then, we showed directly the minimality properties for the processes* A^d *and* A^c *(cf. Lemmas 3.2 and 3.3) by using the martingale property on the interval* $[S, \tau_S^\lambda]$ *from Lemma 3.1(iii), which itself relies on Maingueneau's penalization approach (cf. also Remarks 3.5 and 3.4). In the non-linear case, Mertens decomposition is generalized by the* $\mathcal{E}^f$-*Mertens decomposition (cf. Proposition 8.2 in the Appendix). However, the analogue in the non-linear case of the martingale property of Lemma 3.1[(iii)] (namely, the* $\mathcal{E}^f$-*martingale property) cannot be obtained via Maingueneau's approach due to the non-convexity of the functional*

$\mathcal{E}^f$.

5.1. *Existence and uniqueness of the solution of the RBSDE.* In Theorem 3.1, we have shown that, in the case where the driver does not depend on y, z, and k, the RBSDE from Definition 2.3 admits a unique solution. Using this theorem and the same arguments as in [16], we derive the following existence and uniqueness result in the case of a general Lipschitz driver f. We note that Assumption 4.1 is not required for this result.

Theorem 5.4 (Existence and uniqueness) *Let ξ be a left-limited [2] process in $\mathcal{S}^2$ and let f be a Lipschitz driver. The RBSDE with parameters (f, ξ) from Definition 2.3 admits a unique solution $(Y, Z, k, A, C) \in \mathcal{S}^2 \times I\!H^2 \times I\!H^2_\nu \times \mathcal{S}^2 \times \mathcal{S}^2$.*

Proof.   The proof relies on the existence and uniqueness result for RBSDEs with a driver which does not depend on the solution (Theorem 3.1), the a priori estimates from Lemma 8.1 of the Appendix, and a fixed point theorem. For details, the reader is referred to the proof of Theorem 3.4 in [16]. □

Remark 5.7 *In [25] the above existence and uniqueness result is shown (in a Brownian framework) by using a penalization method. Our approach provides an alternative proof of this result.*

5.2. *Tanaka-type formula.* The following lemma will be used in the proof of the comparison theorem for RBSDEs with irregular obstacles. The lemma can be seen as an extension of Theorem 66 of [34, Chapter IV] from the case of right-continuous semimartingales to the more general case of strong optional semimartingales.

Lemma 5.7 (Tanaka-type formula) *Let X be a (real-valued) strong optional semimartingale with decomposition $X = X_0 + M + A + B$, where M is a local (cadlag) martingale, A is a right-continuous adapted process of finite variation such that $A_0 = 0$, B is a left-continuous adapted purely discontinuous process of finite variation such that $B_0 = 0$. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be a convex function. Then, $f(X)$ is a strong optional semimartingale. Moreover, denoting by f' the left-hand derivative of the convex function f, we have*

$$f(X_t) = f(X_0) + \int_{]0,t]} f'(X_{s-})d(A_s + M_s) + \int_{[0,t[} f'(X_s)dB_{s+} + K_t,$$

where K is a nondecreasing adapted process such that

$$\Delta K_t = f(X_t) - f(X_{t-}) - f'(X_{t-})\Delta X_t \text{ and } \Delta_+ K_t = f(X_{t+}) - f(X_t) - f'(X_t)\Delta_+ X_t.$$

Note that the process K in the above lemma is in general neither left-continuous nor right-continuous.

Proof: Our proof follows the proof of Theorem 66 of [34, Chapter IV] with suitable changes.

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[2]By Remark 2.1, this result still holds for a completely irregular payoff (not necessarily left-limited).

Step 1. We assume that X is bounded; more precisely, we assume that there exists $N \in \mathbb{N}$ such that $|X| \leq N$. We know (cf. [34]) that there exists a sequence (f_n) of twice continuously differentiable convex functions such that (f_n) converges to f, and (f'_n) converges to f' from below. By applying Gal'chouk-Lenglart's formula (cf., e.g., Theorem A.3 in [16]) to $f_n(X_t)$, we obtain for all $\tau \in \mathcal{T}_{0,T}$

$$(5.12) \quad f_n(X_\tau) = f_n(X_0) + \int_{]0,\tau]} f'_n(X_{s-})d(A_s + M_s) + \int_{[0,\tau[} f'_n(X_s)dB_{s+} + K^n_\tau, \text{ a.s., where}$$

$$(5.13)$$
$$K^n_\tau := \sum_{0<s\leq\tau} \left[f_n(X_s) - f_n(X_{s-}) - f'_n(X_{s-})\Delta X_s \right] + \sum_{0\leq s<\tau} \left[f_n(X_{s+}) - f_n(X_s) - f'_n(X_s)\Delta_+ X_s \right]$$
$$+ \frac{1}{2} \int_{]0,\tau]} f''_n(X_{s-})d\langle M^c, M^c\rangle_s \text{ a.s.}$$

We show that (K^n_τ) is a convergent sequence by showing that the other terms in Equation (5.12) converge. The convergence $\int_{]0,\tau]} f'_n(X_{s-})d(A_s + M_s) \xrightarrow[n\to\infty]{} \int_{]0,\tau]} f'(X_{s-})d(A_s + M_s)$ is shown by using the same arguments as in the proof of [34, Thorem 66, Ch. IV]. The convergence of the term $\int_{[0,\tau[} f'_n(X_s)dB_{s+}$, which is specific to the non-right-continuous case, is shown by using dominated convergence. We conclude that (K^n_τ) converges and we set $K_\tau := \lim_{n\to\infty} K^n_\tau$. The process (K_t) is adapted as the limit of adapted processes. Moreover, we have from Eq. (5.13) and from the convexity of f_n that, for each n, K^n_t is nondecreasing in t. Hence, the limit K_t is nondecreasing.

Step 2. We treat the general case where X is not necessarily bounded by using a localization argument similar to that used in [34, Th. 66, Ch. IV].

$\square$

5.3. *Comparison theorem.*

Theorem 5.5 (Comparison) *Let $\xi \in \mathcal{S}^2$, $\xi' \in \mathcal{S}^2$ be two left-limited* [3] *processes. Let f and f' be Lipschitz drivers satisfying Assumption 4.1. Let (Y, Z, k, A, C) (resp. (Y', Z', k', A', C')) be the solution of the RBSDE associated with obstacle ξ (resp. ξ') and with driver f (resp. f'). If $\xi_t \leq \xi'_t$, $0 \leq t \leq T$ a.s. and $f(t, Y'_t, Z'_t, k'_t) \leq f'(t, Y'_t, Z'_t, k'_t)$, $0 \leq t \leq T$ $dP \otimes dt$-a.s., then, $Y_t \leq Y'_t$, $0 \leq t \leq T$ a.s.*

Proof: We set $\bar{Y}_t = Y_t - Y'_t$, $\bar{Z}_t = Z_t - Z'_t$, $\bar{k}_t = k_t - k'_t$, $\bar{A}_t = A_t - A'_t$, $\bar{C}_t = C_t - C'_t$ and $\bar{f}_t = f(t, Y_{t-}, Z_t, k_t) - f'(t, Y'_{t-}, Z'_t, k'_t)$. Then,

$$-d\bar{Y}_t = \bar{f}_t dt + d\bar{A}_t + d\bar{C}_{t-} - \bar{Z}_t dW_t - \int_E \bar{k}_t(e)\tilde{N}(dt, de), \quad \bar{Y}_T = 0.$$

---

[3]By Remark 2.1, this result still holds for a completely irregular payoff (not necessarily left-limited).

Applying Lemma 5.7 to the positive part of $\bar{Y}_t$, we obtain

$$
\begin{aligned}
(5.14) \quad \bar{Y}_t^+ ={}& -\int_{]t,T]} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{Z}_s dW_s - \int_{]t,T]} \int_E \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{k}_s(e) \tilde{N}(ds,de) + \int_{]t,T]} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{f}_s ds \\
& + \int_{]t,T]} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} d\bar{A}_s + \int_{[t,T[} \mathbf{1}_{\{\bar{Y}_s>0\}} d\bar{C}_s + (K_t - K_T).
\end{aligned}
$$

We set $\delta_t := \frac{f(t,Y_{t-},Z_t,k_t)-f(t,Y'_{t-},Z_t,k_t)}{Y_{t-}-Y'_{t-}} \mathbf{1}_{\{\bar{Y}_{t-}\neq 0\}}$ and $\beta_t := \frac{f(t,Y'_{t-},Z_t,k_t)-f(t,Y'_{t-},Z'_t,k_t)}{Z_t-Z'_t} \mathbf{1}_{\{\bar{Z}_t\neq 0\}}$. Due to the Lipschitz-continuity of f, the processes δ and β are bounded. We note that $\bar{f}_t = \delta_t \bar{Y}_t + \beta_t \bar{Z}_t + f(Y'_{t-},Z'_t,k_t) - f(Y'_{t-},Z'_t,k'_t) + \varphi_t$, where $\varphi_t := f(Y'_{t-},Z'_t,k'_t) - f'(Y'_{t-},Z'_t,k'_t)$. Using this, together with Assumption 4.1, we obtain

$$
(5.15) \quad \bar{f}_t \leq \delta_t \bar{Y}_t + \beta_t \bar{Z}_t + \langle \gamma_t, \bar{k}_t \rangle_\nu, + \varphi_t \quad 0 \leq t \leq T, \quad dP \otimes dt - \text{a.e.},
$$

where we have set $\gamma_t := \theta_t^{Y'_{t-},Z'_t,k'_t,k_t}$.
For $\tau \in \mathcal{T}_{0,T}$, let $\Gamma_{\tau,\cdot}$ be the unique solution of the following forward SDE

$$
(5.16) \quad d\Gamma_{\tau,s} = \Gamma_{\tau,s-} \left[\delta_s ds + \beta_s dW_s + \int_E \gamma_s(e) \tilde{N}(ds,de) \right]; \quad \Gamma_{\tau,\tau} = 1.
$$

To simplify the notation, we denote $\Gamma_{\tau,s}$ by Γ_s for $s \geq \tau$.
By applying Gal'chouk-Lenglart's formula to the product $(\Gamma_t \bar{Y}_t^+)$ we get

$$
\begin{aligned}
(5.17) \quad \Gamma_\tau \bar{Y}_\tau^+ ={}& -\int_\tau^\theta \Gamma_{s-}(\mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{Z}_s + \bar{Y}_{s-}^+ \beta_s) dW_s - \int_\tau^\theta \Gamma_s(\bar{Y}_{s-}^+ \delta_s + \bar{Z}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \beta_s - \bar{f}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}}) ds \\
& + \int_\tau^\theta \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} d\bar{A}_s^c + \sum_{\tau \leq s \leq \theta} \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \Delta \bar{A}_s - \int_\tau^\theta \Gamma_{s-} dK_s^c - \int_\tau^\theta \Gamma_{s-} dK_s^{d,-} \\
& + \int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} d\bar{C}_s - \int_\tau^\theta \Gamma_s dK_s^{d,+} - \int_\tau^\theta \int_E \Gamma_{s-}(\bar{k}_s(e) \mathbf{1}_{\{\bar{Y}_{s-}>0\}} + \bar{Y}_{s-}^+ \gamma_s(e)) \tilde{N}(ds,de) \\
& - \sum_{\tau \leq s \leq \theta} \Delta \Gamma_s \Delta \bar{Y}_s^+.
\end{aligned}
$$

Note that by (5.16), $\Gamma_\tau = 1$, which gives that $\Gamma_\tau \bar{Y}_\tau^+ = \bar{Y}_\tau^+$. Moreover, we have $\int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} d\bar{C}_s = \int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} dC_s - \int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} dC'_s$. For the first term, it holds $\int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} dC_s = 0$. Indeed, $\{\bar{Y}_s > 0\} = \{Y_s > Y'_s\} \subset \{Y_s > \xi_s\}$ (as $Y'_s \geq \xi'_s \geq \xi_s$). This, together with the Skorokhod condition for C gives the equality. For the second term, it holds $-\int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} dC'_s \leq 0$, as $\Gamma \geq 0$ and dC' is a nonnegative measure. Hence, $\int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s>0\}} d\bar{C}_s \leq 0$. Similarly, we obtain $\int_\tau^\theta \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} d\bar{A}_s^c \leq 0$. We also have $-\int_\tau^\theta \Gamma_{s-} dK_s^c \leq 0$ and $-\int_\tau^\theta \Gamma_s dK_s^{d,+} \leq 0$. Hence,

$$
\begin{aligned}
(5.18) \quad \bar{Y}_\tau^+ \leq{}& -\int_\tau^\theta \Gamma_{s-}(\mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{Z}_s + \bar{Y}_{s-}^+ \beta_s) dW_s - \int_\tau^\theta \Gamma_s(\bar{Y}_{s-}^+ \delta_s + \bar{Z}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \beta_s - \bar{f}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}}) ds \\
& + \sum_{\tau \leq s \leq \theta} \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \Delta \bar{A}_s - \int_\tau^\theta \Gamma_{s-} dK_s^{d,-} - \int_\tau^\theta \int_E \Gamma_{s-}(\bar{k}_s(e) \mathbf{1}_{\{\bar{Y}_{s-}>0\}} + \bar{Y}_{s-}^+ \gamma_s(e)) \tilde{N}(ds,de) \\
& - \sum_{\tau \leq s \leq \theta} \Delta \Gamma_s \Delta \bar{Y}_s^+.
\end{aligned}
$$

We compute the last term $\sum_{\tau \leq s \leq \theta} \Delta\Gamma_s \Delta\bar{Y}_s^+$.

Let (p_s) be the point process associated with the Poisson random measure N (cf. [8, VIII Section 2. 67], or [23, Section III §d]). We have $\Delta\Gamma_s = \Gamma_{s-}\gamma_s(p_s)$ and $\Delta\bar{Y}_s^+ = \mathbf{1}_{\{\bar{Y}_{s-}>0\}}\bar{k}_s(p_s) - \mathbf{1}_{\{\bar{Y}_{s-}>0\}}\Delta\bar{A}_s + \Delta K_s^{d,-}$. Hence,

$$
\begin{aligned}
\sum_{\tau \leq s \leq \theta} \Delta\Gamma_s \Delta\bar{Y}_s^+ &= \sum_{\tau \leq s \leq \theta} \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\gamma_s(p_s)\bar{k}_s(p_s) - \sum_{\tau \leq s \leq \theta} \Gamma_{s-}\gamma_s(p_s)(\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\Delta\bar{A}_s - \Delta K_s^{d,-}) \\
&= \int_\tau^\theta \int_E \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\gamma_s(e)\bar{k}_s(e)N(ds,de) - \sum_{\tau \leq s \leq \theta} \Gamma_{s-}\gamma_s(p_s)(\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\Delta\bar{A}_s - \Delta K_s^{d,-}) \\
&= \int_\tau^\theta \int_E \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\gamma_s(e)\bar{k}_s(e)\tilde{N}(ds,de) + \int_\tau^\theta \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\langle\gamma_s,\bar{k}_s\rangle_\nu ds \\
&\quad - \sum_{\tau \leq s \leq \theta} \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\gamma_s(p_s)\Delta\bar{A}_s + \sum_{\tau \leq s \leq \theta} \Gamma_{s-}\gamma_s(p_s)\Delta K_s^{d,-}.
\end{aligned}
\tag{5.19}
$$

By plugging this expression in equation (5.18) and by putting together the terms in "ds", the terms in "$dK_s^{d,-}$", and the terms in "$\Delta\bar{A}_s$", we get

$$
\begin{aligned}
\bar{Y}_\tau^+ &\leq -\int_\tau^\theta \Gamma_{s-}(\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\bar{Z}_s + \bar{Y}_{s-}^+\beta_s)dW_s \\
&\quad -\int_\tau^\theta \Gamma_{s-}(\bar{Y}_{s-}^+\delta_s + \bar{Z}_s\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\beta_s + \mathbf{1}_{\{\bar{Y}_{s-}>0\}}\langle\gamma_s,\bar{k}_s\rangle_\nu - \bar{f}_s\mathbf{1}_{\{\bar{Y}_{s-}>0\}})ds \\
&\quad + \sum_{\tau \leq s \leq \theta} \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}(1+\gamma_s(p_s))\Delta\bar{A}_s - \sum_{\tau \leq s \leq \theta} \Gamma_{s-}(1+\gamma_s(p_s))\Delta K_s^{d,-} \\
&\quad -\int_\tau^\theta \int_E \Gamma_{s-}(\bar{k}_s(e)\mathbf{1}_{\{\bar{Y}_{s-}>0\}} + \bar{Y}_{s-}^+\gamma_s(e) + \mathbf{1}_{\{\bar{Y}_{s-}>0\}}\gamma_s(e)\bar{k}_s(e))\tilde{N}(ds,de).
\end{aligned}
\tag{5.20}
$$

We have $-\int_\tau^\theta \Gamma_{s-}(\bar{Y}_{s-}^+\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\delta_s + \bar{Z}_s\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\beta_s + \mathbf{1}_{\{\bar{Y}_{s-}>0\}}\langle\gamma_s,\bar{k}_s\rangle_\nu - \bar{f}_s\mathbf{1}_{\{\bar{Y}_{s-}>0\}})ds \leq \int_\tau^\theta \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\varphi_s ds$, due to the inequality (5.15). The term $-\sum_{\tau \leq s \leq \theta} \Gamma_{s-}(1+\gamma_s(p_s))\Delta K_s^{d,-}$ is nonpositive, as $1+\gamma_s \geq 0$ by Assumption 4.1. The term $\sum_{\tau \leq s \leq \theta} \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}(1+\gamma_s(p_s))\Delta\bar{A}_s$ is nonpositive, due to $1+\gamma_s \geq 0$, to the Skorokhod condition for ΔA_s and to $\Delta A'_s \geq 0$ (the details are similar to those for $d\bar{C}$ in the reasoning above). By classical arguments (using Burkholder-Davis-Gundy inequalities), the stochastic integrals "with respect to dW_s" and "with respect to $\tilde{N}(ds,de)$" are equal to zero in expectation. Moreover, the term $\int_\tau^\theta \Gamma_{s-}\mathbf{1}_{\{\bar{Y}_{s-}>0\}}\varphi_s ds$ is nonpositive, as $\varphi_s = f(Y'_s, Z'_s, k'_s) - f'(Y'_s, Z'_s, k'_s) \leq 0$ $dP \otimes ds$-a.s. by the assumptions of the theorem. We conclude that $E[\bar{Y}_\tau^+] \leq 0$, which implies $\bar{Y}_\tau^+ = 0$ a.s. The proof is thus complete. $\qquad\square$

Remark 5.8 *Note that due to the irregularity of the obstacles, together with the presence of jumps, we cannot adopt the approaches used up to now in the literature (see e.g. [12], [5], [36] and [16]) to show the comparison theorem for our RBSDE.*

5.4. *Non-linear operator induced by an RBSDE with irregular obstacle.* We introduce the non-linear operator $\mathcal{R}ef^f$ (associated with a given non-linear driver f) and provide some useful properties. In particular, we show that this non-linear operator coincides with the $\mathcal{E}^f$-Snell envelope operator (cf. Theorem 5.6).

Definition 5.8 (Non-linear operator $\mathcal{R}ef^f$) *Let f be a Lipschitz driver. For a process $(\xi_t) \in \mathcal{S}^2$, we denote by $\mathcal{R}ef^f[\xi]$ the first component of the solution to the Reflected BSDE with (lower) barrier ξ and with Lipschitz driver f.*

The operator $\mathcal{R}ef^f[\cdot]$ is well-defined due to Theorem 5.4 and to Remark 2.1. Moreover, $\mathcal{R}ef^f[\cdot]$ is valued in $\mathcal{S}^{2,rusc}$, where $\mathcal{S}^{2,rusc} := \{\phi \in \mathcal{S}^2 : \phi \text{ is r.u.s.c.}\}$ (cf. Remark 2.2). In the following proposition we give some properties of the operator $\mathcal{R}ef^f$. Note that equalities (resp. inequalities) between processes are to be understood in the "up to indistinguishability"-sense.

We recall the notion of a strong $\mathcal{E}^f$-supermartingale.

Definition 5.9 *Let ϕ be a process in $\mathcal{S}^2$. Let f be a Lipschitz driver. The process ϕ is said to be a strong $\mathcal{E}^f$-supermartingale (resp. a strong $\mathcal{E}^f$-martingale) , if $\mathcal{E}^f_{\sigma,\tau}(\phi_\tau) \leq \phi_\sigma$ a.s. (resp. $\mathcal{E}^f_{\sigma,\tau}(\phi_\tau) = \phi_\sigma$ a.s.) on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_{0,T}$.*

Using the above comparison theorem and the $\mathcal{E}^f$-Mertens decomposition for strong (r.u.s.c.) $\mathcal{E}^f$-supermartingales (cf. Proposition 8.2 in the Appendix), we show that the operator $\mathcal{R}ef^f$ satisfies the following properties.

Proposition 5.1 (Properties of the operator $\mathcal{R}ef^f$) *Let f be a Lipschitz driver satisfying Assumption 4.1. The operator $\mathcal{R}ef^f : S^2 \to S^{2,rusc}$, defined in Definition 5.8, has the following properties:*

1. *The operator $\mathcal{R}ef^f$ is nondecreasing, that is, for $\xi, \xi' \in S^2$ such that $\xi \leq \xi'$ we have $\mathcal{R}ef^f[\xi] \leq \mathcal{R}ef^f[\xi']$.*
2. *If $\xi \in S^2$ is a (r.u.s.c.) strong $\mathcal{E}^f$-supermartingale, then $\mathcal{R}ef^f[\xi] = \xi$.*
3. *For each $\xi \in S^2$, $\mathcal{R}ef^f[\xi]$ is a strong $\mathcal{E}^f$-supermartingale and satisfies $\mathcal{R}ef^f[\xi] \geq \xi$.*

Proof: The first assertion follows from our comparison theorem for reflected BSDEs with irregular obstacles (Theorem 5.5).

Let us prove the second assertion. Let ξ be a (r.u.s.c.) strong $\mathcal{E}^f$-supermartingale in $\mathcal{S}^2$. By definition of $\mathcal{R}ef^f$, we have to show that ξ is the solution of the reflected BSDE associated with driver f and obstacle ξ. By the $\mathcal{E}^f$-Mertens decomposition for strong (r.u.s.c.) $\mathcal{E}^f$-supermartingales shown in [16] (cf. Proposition 8.2 in the Appendix of the present paper), together with the martingale representation theorem, there exists $(Z, k, A, C) \in \mathbb{H}^2 \times \mathbb{H}^2_\nu \times \mathcal{S}^2 \times \mathcal{S}^2$ such that a.s. for all $t \in [0, T]$,

$$\xi_t = \xi_T + \int_t^T f(s, \xi_s, Z_s, k_s)ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbf{E}} k_s(e)\tilde{N}(ds, de) + A_T - A_t + C_{T-} - C_{t-},$$

where A is predictable right-continuous nondecreasing with $A_0 = 0$, and C is adapted right-continuous nondecreasing and purely discontinuous, with $C_{0-} = 0$. Moreover, the Skorokhod conditions (for RBSDEs) are here trivially satisfied. Hence, $\xi = \mathcal{R}ef^f[\xi]$, which is the desired conclusion.

It remains to show the third assertion. By definition, the process $\mathcal{R}ef^f[\xi]$ is equal to Y, where (Y, Z, K, A, C) is the solution our reflected BSDE. Hence, $\mathcal{R}ef^f[\xi] = Y$ admits the decomposition (8.33), which, by Proposition 8.2, implies that $\mathcal{R}ef^f[\xi] = Y$ is a strong $\mathcal{E}^f$-supermartingale. Moreover, by definition, $\mathcal{R}ef^f[\xi] = Y$ is greater than or equal to the obstacle ξ. $\qquad \square$

In the following theorem, we characterize $\mathcal{R}ef^f[\xi]$, that is, the first component of the solution of the RBSDE with irregular obstacle ξ, in terms of the smallest strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ.

Theorem 5.6 (The operator $\mathcal{R}ef^f$ and the $\mathcal{E}^f$- Snell envelope operator) *Let $(\xi_t, 0 \leq t \leq T)$ be a left-limited [4] process in $\mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption 4.1. The first component $Y = \mathcal{R}ef^f[\xi]$ of the solution to the reflected BSDE with parameters (ξ, f) coincides with the $\mathcal{E}^f$-Snell envelope of ξ, that is, the smallest strong $\mathcal{E}^f$-supermartingale greater than or equal to ξ.*

Proof: The proof relies on the properties of the operator $\mathcal{R}ef^f$ from the above Proposition 5.1. By the third assertion of Proposition 5.1, the process $Y = \mathcal{R}ef^f[\xi]$ is a strong $\mathcal{E}^f$-supermartingale satisfying $Y \geq \xi$. It remains to show the minimality property. Let Y' be a strong $\mathcal{E}^f$-supermartingale such that $Y' \geq \xi$. We have $\mathcal{R}ef^f[Y'] \geq \mathcal{R}ef^f[\xi]$, due to the nondecreasingness of the operator $\mathcal{R}ef^f$ (cf. Proposition 5.1, 1st assertion). On the other hand, $\mathcal{R}ef^f[Y'] = Y'$ (due to Proposition 5.1, 2nd assertion) and $\mathcal{R}ef^f[\xi] = Y$. Hence, $Y' \geq Y$, which is the desired conclusion.   □

In the case of a right-continuous obstacle ξ the above characterization has been established in [36]; it has been generalized to the case of a right-upper-semicontinuous obstacle in [16, Prop. 4.4]. Let us note however that the arguments of the proofs given in [36] and in [16] cannot be adapted to our general framework.

6. Infinitesimal characterization in terms of an RBSDE.
The following theorem is a direct consequence of Theorem 5.6 and Theorem 4.3. It gives "an infinitesimal characterization" of the value process $(V_t)_{t \in [0,T]}$ of the non-linear problem (4.10).

Theorem 6.7 (Characterization in terms of an RBSDE) *Let $(\xi_t, 0 \leq t \leq T)$ be a left-limited [5] process in $\mathcal{S}^2$ and let f be a Lipschitz driver satisfying Assumption 4.1. The value process $(V_t)_{t \in [0,T]}$ aggregating the family $V = (V(S), S \in \mathcal{T}_{0,T})$ defined by (4.10) coincides (up to indistinguishability) with the first component $(Y_t)_{t \in [0,T]}$ of the solution of our RBSDE with driver f and obstacle ξ. In other words, we have, for all $S \in \mathcal{T}_{0,T}$,*

$$(6.21) \qquad Y_S = V_S = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}^f_{S,\tau}(\xi_\tau) \ a.s.$$

By using this theorem, we derive the following corollary, which generalizes some results of classical optimal stopping theory (more precisely, the assertions (ii) and (iii) from Lemma 3.1) to the case of an optimal stopping problem with (non-linear) f-expectation.

Corollary 6.1 *The value process of our optimal stopping problem (6.21), which is equal to the first component (Y_t) of the solution of our RBSDE, satisfies the following properties:*
(i) For each $S \in \mathcal{T}_{0,T}$, we have:

$$Y_S = \xi_S \vee Y_{S+} \quad a.s.$$

---

[4] By Remark 2.1, this result still holds for a completely irregular payoff (not necessarily left-limited).

[5] By Remark 2.1, this result still holds for a completely irregular payoff (not necessarily left-limited).

(ii) For each $S \in \mathcal{T}_{0,T}$ and for each $\lambda \in (0,1)$, we set

$$(6.22) \qquad \tau_S^\lambda := \inf\{t \geq S, \lambda Y_t \leq \xi_t\}.$$

The value process (Y_t) is an $\mathcal{E}^f$-martingale on $[S, \tau_S^\lambda]$.

Proof: By Theorem 6.7, the value process V is equal to Y, where (Y, Z, k, A, C) is the solution or our RBSDE.

The first assertion follows from Remark 2.2. Let us show the second assertion. We note that (Y, Z, k, A, C) is also the solution of the RBSDE from Definition 2.3 associated with the obstacle (ξ_t) and the driver process $g_t(\omega) := f(t, \omega, Y_t(\omega), Z_t(\omega), k_t(\omega))$. By Theorem 3.1, we derive that (Y_t) is equal to the value process of the classical optimal stopping problem (3.5) associated with the instantaneous reward process (g_t). By applying the assertion (iii) from Lemma 3.1, the process $(Y_t + \int_0^t g_u du)_{t \in [0,T]}$ is thus a martingale on $[S, \tau_S^\lambda]$. Since A and C are equal to the non decreasing processes of the Mertens decomposition of the strong supermartingale $(Y_t + \int_0^t g_u du)_{t \in [0,T]}$, we derive that $A_S = A_{\tau_S^\lambda}$ a.s. and $C_{S-} = C_{(\tau_S^\lambda)-}$ a.s. Hence, Y is the solution on $[S, \tau_S^\lambda]$ of the BSDE associated with driver f, terminal time τ_S^λ and terminal condition $Y_{\tau_S^\lambda}$. The process (Y_t) is thus an $\mathcal{E}^f$-martingale on $[S, \tau_S^\lambda]$, which completes the proof. $\qquad \square$

Corollary 6.2 *We assume that the process (ξ_t) is right-uppersemicontinuous (r.u.s.c.). The value process of the optimal stopping problem (6.21), which is equal to the solution (Y_t) of our RBSDE, satisfies the following property: for each $S \in \mathcal{T}_{0,T}$ and for each $\lambda \in]0,1[$,*

$$(6.23) \qquad \lambda Y_{\tau_S^\lambda} \leq \xi_{\tau_S^\lambda} \quad \text{a.s.},$$

where τ_S^λ is defined by (6.22). Moreover, the stopping time τ_S^λ satisfies

$$(6.24) \qquad Y_S \leq \mathcal{E}_{S,\tau_S^\lambda}^f(\xi_{\tau_S^\lambda}) + \varepsilon_S(\lambda) \quad a.s. \ ,$$

where $\lim_{\lambda \to 1} \varepsilon_S(\lambda) = 0$ a.s. In other words, τ_S^λ is an $\varepsilon_S(\lambda)$-optimal stopping time for problem (6.21).

Proof: By Theorem 6.7, the value process V is equal to Y, where (Y, Z, k, A, C) be the solution or our RBSDE. The proof of the inequality (6.23) is similar to that of [16, Lemma 4.1(i)]. We give again the arguments here in order to emphasize the important role of the right-uppersemicontinuity assumption in this result. By way of contradiction, we suppose $P(\lambda Y_{\tau_S^\lambda} > \xi_{\tau_S^\lambda}) > 0$. By the Skorokhod condition for C, we have $\Delta C_{\tau_S^\lambda} = C_{\tau_S^\lambda} - C_{(\tau_S^\lambda)-} = 0$ on the set $\{\lambda Y_{\tau_S^\lambda} > \xi_{\tau_S^\lambda}\}$. On the other hand, due to Remark 2.2, $\Delta C_{\tau_S^\lambda} = Y_{\tau_S^\lambda} - Y_{(\tau_S^\lambda)+}$. Thus, $Y_{\tau_S^\lambda} = Y_{(\tau_S^\lambda)+}$ on the set $\{\lambda Y_{\tau_S^\lambda} > \xi_{\tau_S^\lambda}\}$. Hence,

$$(6.25) \qquad \lambda Y_{(\tau_S^\lambda)+} > \xi_{\tau_S^\lambda} \text{ on the set } \{\lambda Y_{\tau_S^\lambda} > \xi_{\tau_S^\lambda}\}.$$

We will obtain a contradiction with this statement. Let us fix $\omega \in \Omega$. By definition of $\tau_S^\lambda(\omega)$, there exists a non-increasing sequence $(t_n) = (t_n(\omega)) \downarrow \tau_S^\lambda(\omega)$ such that $\lambda Y_{t_n}(\omega) \leq \xi_{t_n}(\omega)$,

for all $n \in I\!N$. Hence, $\lambda \limsup_{n\to\infty} Y_{t_n}(\omega) \leq \limsup_{n\to\infty} \xi_{t_n}(\omega)$. As the process ξ is *right-uppersemicontinuous* , we have $\limsup_{n\to\infty} \xi_{t_n}(\omega) \leq \xi_{\tau_S^\lambda}(\omega)$. On the other hand, as $(t_n(\omega)) \downarrow \tau_S^\lambda(\omega)$, we have $\limsup_{n\to\infty} Y_{t_n}(\omega) = Y_{(\tau_S^\lambda)+}(\omega)$. Thus, $\lambda Y_{(\tau_S^\lambda)+}(\omega) \leq \xi_{\tau_S^\lambda}(\omega)$, which is in contradiction with (6.25). We conclude that $\lambda Y_{\tau_S^\lambda} \leq \xi_{\tau_S^\lambda}$ a.s.

Let us now show the inequality (6.24). The arguments are classical. Since by Corollary 6.1 (ii), the value process (Y_t) is an $\mathcal{E}^f$-martingale on $[S, \tau_S^\lambda]$, we get $Y_S = \mathcal{E}^f_{S,\tau_S^\lambda}(Y_{\tau_S^\lambda})$ a.s. By the inequality (6.23), together with the monoticity property of the conditional f-expectation and the *a priori estimates* for BSDEs (cf. [35]), we derive that

$$Y_S = \mathcal{E}^f_{S,\tau_S^\lambda}(Y_{\tau_S^\lambda}) \leq \mathcal{E}^f_{S,\tau_S^\lambda}(\frac{\xi_{\tau_S^\lambda}}{\lambda}) \leq \mathcal{E}^f_{S,\tau_S^\lambda}(\xi_{\tau_S^\lambda}) + (\frac{1}{\lambda} - 1)\,\alpha_S \quad \text{a.s. ,}$$

with $\alpha_S := CE[\text{ess sup}_{\tau \in \mathcal{T}_{S,T}} \xi_\tau^2 \,|\, \mathcal{F}_S]^{\frac{1}{2}}$, where C is a positive constant which depends only on T and the Lipschitz constant K of the driver f. We thus obtain the desired result with $\varepsilon_S(\lambda) := (\frac{1}{\lambda} - 1)\,\alpha_S$, which ends the proof. $\qquad \square$

Remark 6.9 *In the general case where the process (ξ_t) is not r.u.s.c. , the inequality $\lambda Y_{\tau_S^\lambda} \leq \xi_{\tau_S^\lambda}$ (i.e. inequality (6.23)) does not necessarily hold (not even in the simplest case of linear expectations; cf., e.g., [11]). Let us emphasize that this fact leads to some important technical difficulties in the treatment of the completely irregular case with respect to the "more regular" cases. In particular, this prevents us from adopting here the approach used in [16] (in the r.u.s.c. case) to prove the infinitesimal characterization of the value process of the non-linear optimal stopping problem in terms of the solution of an RBSDE. Thus, in the general framework of the present paper, we proceed differently: First, we apply a direct approach to the non-linear optimal stopping problem (4.10) which consists in showing that the value family $(V(S))_{S\in\mathcal{T}_{0,T}}$ can be aggregated by an optional process $(V_t)_{t\in[0,T]}$ and, then, in characterizing (V_t) as the $\mathcal{E}^f$-Snell envelope of the (completely irregular) pay-off process (ξ_t). On the other hand, we apply an RBSDE-approach which consists in establishing some results on RBSDEs with irregular obstacles, in particular a comparison theorem and some properties of the operator $\mathcal{R}ef^f$ [6], and then in using these properties to show that the solution (Y_t) of the RBSDE is the $\mathcal{E}^f$-Snell envelope of the obstacle. We deduce from those two approaches that (Y_t) and (V_t) coincide, which gives an infinitesimal characterization for the value process (V_t).*

Note that, in the r.u.s.c. case (cf. [16]), this characterization is shown by using only an RBSDE approach. More precisely, it is shown that the solution Y of the RBSDE satisfies the property (ii) of Corollary 6.1 as well as the inequality (6.23) (which is true due to the assumption of r.u.s.c. on ξ), from which we directly derive the characterization (cf. Th. 4.2 in [16]). [7]

Finally, let us briefly summarize some of the results for the non-linear optimal stopping problem (4.10):

  i) For any left-limited (without loss of generality due to Remark 2.1) reward process $\xi \in \mathcal{S}^2$, we have the infinitesimal characterization $V_t = Y_t = \mathcal{R}ef^f_t[\xi]$, for all t, a.s. (Theorem 6.7).

---

[6]We underline that the proof of these properties (cf. Proposition 5.1) relies on the $\mathcal{E}^f$-Mertens decomposition for strong (r.u.s.c.) $\mathcal{E}^f$-supermartingales (cf. Proposition 8.2).

[7]Note that in the r.u.s.c. case, the comparison theorem is deduced as an almost immediate corollary of this characterization (cf. Th. 5.3 in [16]).

ii) If, moreover, ξ is right-uppersemicontinuous, then, for any $S \in \mathcal{T}_{0,T}$, for any $\lambda \in (0,1)$, there exists an $\varepsilon_S(\lambda)-$ optimal stopping time for the problem at time S (Corollary 6.2, Eq. (6.24)).

iii) If, moreover, ξ is also left-uppersemicontinuous along stopping times, then, for any $S \in \mathcal{T}_{0,T}$, there exists an optimal stopping time for the problem at time S (cf. [16, Proposition 4.3]).

7. Applications of Theorem 6.7.

7.1. *Application to American options with a completely irregular payoff.* In the following example, we set $E := \mathbf{R}$, $\nu(de) := \lambda \delta_1(de)$, where λ is a positive constant, and where δ_1 denotes the Dirac measure at 1. The process $N_t := N([0,t] \times \{1\})$ is then a Poisson process with parameter λ, and we have $\tilde{N}_t := \tilde{N}([0,t] \times \{1\}) = N_t - \lambda t$.

We consider a financial market which consists of one risk-free asset, whose price process S^0 satisfies $dS_t^0 = S_t^0 r_t dt$, and two risky assets with price processes S^1, S^2 satisfying the following dynamics:

$$dS_t^1 = S_{t-}^1 [\mu_t^1 dt + \sigma_t^1 dW_t + \beta_t^1 d\tilde{N}_t]; \quad dS_t^2 = S_{t-}^2 [\mu_t^2 dt + \sigma_t^2 dW_t + \beta_t^2 d\tilde{N}_t].$$

We suppose that the processes $\sigma^1, \sigma^2, \beta^1, \beta^2, r, \mu^1, \mu^2$ are predictable and bounded, with $\beta_t^i > -1$ for $i = 1, 2$. Let $\mu_t := (\mu^1, \mu^2)'$ and let $\Sigma_t := (\sigma_t, \beta_t)$ be the 2×2-matrix with first column $\sigma_t := (\sigma_t^1, \sigma_t^2)'$ and second column $\beta_t := (\beta_t^1, \beta_t^2)'$. We suppose that Σ_t is invertible and that the coefficients of Σ_t^{-1} are bounded.

We consider an agent who can invest his/her initial wealth $x \in \mathbf{R}$ in the three assets.

For $i = 1, 2$, we denote by φ_t^i the amount invested in the ith risky asset. A process $\varphi = (\varphi^1, \varphi^2)'$ belonging to $\mathbb{H}^2 \times \mathbb{H}_\nu^2$ will be called a *portfolio strategy*.

The value of the associated portfolio (or *wealth*) at time t is denoted by $X_t^{x,\varphi}$ (or simply by X_t). In the case of a perfect market, we have

$$dX_t = (r_t X_t + \varphi_t^1(\mu_t^1 - r_t) + \varphi_t^2(\mu_t^2 - r_t))dt + (\varphi_t^1 \sigma_t^1 + \varphi_t^2 \sigma_t^2)dW_t + (\varphi_t^1 \beta_t^1 + \varphi_t^2 \beta_t^2)d\tilde{N}_t$$
$$= (r_t X_t + \varphi_t'(\mu_t - r_t \mathbf{1}))dt + \varphi_t' \sigma_t dW_t + \varphi_t' \beta_t d\tilde{N}_t,$$

where $\mathbf{1} = (1, 1)'$. More generally, we will suppose that there may be some imperfections in the market, taken into account via the *nonlinearity* of the dynamics of the wealth and encoded in a Lipschitz driver f satisfying Assumption 4.1 (cf. [13] or [10] for some examples). More precisely, we suppose that the *wealth* process $X_t^{x,\varphi}$ (also X_t) satisfies the forward differential equation:

(7.26) $$-dX_t = f(t, X_t, \varphi_t'\sigma_t, \varphi_t'\beta_t)dt - \varphi_t'\sigma_t dW_t - \varphi_t'\beta_t d\tilde{N}_t, \; ; \; X_0 = x,$$

or, equivalently, setting $Z_t = \varphi_t'\sigma_t$ and $k_t = \varphi_t'\beta_t$,

(7.27) $$-dX_t = f(t, X_t, Z_t, k_t)dt - Z_t dW_t - k_t d\tilde{N}_t; \; X_0 = x.$$

Note that $(Z_t, k_t) = \varphi_t'\Sigma_t$, which is equivalent to $\varphi_t' = (Z_t, k_t)\Sigma_t^{-1}$.

This model includes the case of a perfect market, for which f is given by

$$f(t, y, z, k) = -r_t y - (z, k)\Sigma_t^{-1}(\mu_t - r_t \mathbf{1}),$$

supposed to satisfy $\partial_k f \geq -\lambda$ (which corresponds to Assumption 4.1 in this case).

Remark 7.10 *Note that the wealth process $X^{x,\varphi}$ is an $\mathcal{E}^f$-martingale, since $X^{x,\varphi}$ is the solution of the BSDE with driver f, terminal time T and terminal condition $X_T^{x,\varphi}$.*

Let us consider an American option associated with terminal time T and payoff given by a process $(\xi_t) \in \mathcal{S}^2$. As is usual in the literature, the option's *superhedging price* at time 0, denoted by u_0, is defined as the minimal initial wealth enabling the seller to invest in a portfolio whose value is greater than or equal to the payoff of the option at all times. More precisely, for each initial wealth x, we denote by $\mathcal{A}(x)$ the set of all portfolio strategies $\varphi \in \mathbb{H}^2 \times \mathbb{H}_\nu^2$ such that $X_t^{x,\varphi} \geq \xi_t$, for all $t \in [0, T]$ a.s. The *superhedging price* of the American option is thus defined by

$$(7.28) \qquad\qquad u_0 := \inf\{x \in \mathbf{R}, \ \exists \varphi \in \mathcal{A}(x)\}.^8$$

Using the infinitesimal characterization of the value function (4.10) (cf. Theorem 6.7), we show the following characterizations of the superhedging price u_0, as well as the existence of a superhedging strategy.

Proposition 7.2 *Let (ξ_t) be an irregular left-limited process belonging to $\mathcal{S}^2$.*
(i) The superhedging price u_0 *of the American option with payoff (ξ_t) is equal to the value function $V(0)$ of our optimal stopping problem (1.1) at time 0, that is*

$$(7.29) \qquad\qquad u_0 = \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}_{0,\tau}^f(\xi_\tau).$$

(ii) We have $u_0 = Y_0$, where (Y, Z, k, A, C) is the solution of the reflected BSDE (2.2).
(iii) The portfolio strategy $\hat{\varphi}$, defined by $\hat{\varphi}_t{}' = (Z_t, k_t)\Sigma_t^{-1}$, is a superhedging strategy, that is, belongs to $\mathcal{A}(u_0)$.
(iv) If (ξ_t) is right-uppersemicontinuous, then for each $\lambda \in (0, 1)$, the stopping time $\tau^\lambda := \inf\{t \geq 0, \lambda Y_t \leq \xi_t\}$ is an $\varepsilon(\lambda)$-optimal stopping time for (7.29).
(v) If, moreover, (ξ_t) is also left-upper-semicontinuous along stopping times, then the stopping time $\tau^* := \inf\{t \geq 0, Y_t = \xi_t\}$ is an optimal exercise time *for the American option, in the sense that it attains the supremum in (7.29).*

Remark 7.11 *By Remark 2.1, this result still holds for a completely irregular payoff (not necessarily left-limited).*

Remark 7.12 *In the case of a perfect market ($f \equiv 0$) and a regular pay-off, the above result (in particular assertion (ii)) reduces to a well-known result form the literature (cf., e.g., [19]). Even in the case of a perfect market, our result for completely irregular pay-off is new.*

Proof: The proof of the three first assertions rely on Theorem 6.7 and similar arguments to those in [10] (in the case of game options with RCLL payoffs and default).

Note that, by Theorem 6.7, we have $\sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}_{0,\tau}^f(\xi_\tau) = Y_0$. In order to prove the three first assertions of the above theorem, it is thus sufficient to show that $u_0 = Y_0$ and $\hat{\varphi} \in \mathcal{A}(Y_0)$.

We first show that $\hat{\varphi} \in \mathcal{A}(Y_0)$. By (7.27), the value $X^{Y_0,\hat{\varphi}}$ of the portfolio associated with initial wealth Y_0 and strategy $\hat{\varphi}$ satisfies:

$$X_t^{Y_0,\hat{\varphi}} = Y_0 - \int_0^t f(s, X_s^{Y_0,\hat{\varphi}}, Z_s, k_s)ds + h_t, \ 0 \leq t \leq T,$$

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⁸As shown in assertion (iii) of Proposition 7.2, the infimum in (7.28) is always attained.

where $h_t := \int_0^t Z_s dW_s + \int_0^t k_s d\tilde{N}_s$. Moreover, since Y is the solution of the reflected BSDE (2.2), we have

$$Y_t = Y_0 - \int_0^t f(s, Y_s, Z_s, k_s)ds + h_t - A_t - C_{t^-}, \ 0 \le t \le T \quad \text{a.s.}$$

Applying the comparison result for forward differential equations, we derive that $X_t^{Y_0, \hat{\varphi}} \ge Y_t$, for all $t \in [0, T]$ a.s. Since $Y_t \ge \xi_t$, we thus get $X_t^{Y_0, \hat{\varphi}} \ge \xi_t$ for all $t \in [0, T]$ a.s. It follows that $\hat{\varphi} \in \mathcal{A}(Y_0)$.

We now show that $Y_0 = u_0$. Since $\hat{\varphi} \in \mathcal{A}(Y_0)$, by definition of u_0 (cf. (7.28)), we derive that $Y_0 \ge u_0$. Let us now show that $u_0 \ge Y_0$. Let $x \in \mathbf{R}$ be such that there exists a strategy $\varphi \in \mathcal{A}(x)$. We show that $x \ge Y_0$. Since $\varphi \in \mathcal{A}(x)$, we have $X_t^{x, \varphi} \ge \xi_t$, for all $t \in [0, T]$ a.s. For each $\tau \in \mathcal{T}$ we thus get the inequality $X_\tau^{x, \varphi} \ge \xi_\tau$ a.s. By the non decreasing property of $\mathcal{E}^f$ together with the $\mathcal{E}^f$-martingale property of $X^{x, \varphi}$ (cf. Remark 7.10), we thus get $x = \mathcal{E}_{0,\tau}^f(X_\tau^{x, \varphi}) \ge \mathcal{E}_{0,\tau}^f(\xi_\tau)$. By taking the supremum over $\tau \in \mathcal{T}_{0,T}$, we derive that $x \ge \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{E}_{0,\tau}^f(\xi_\tau) = Y_0$, where the equality holds by Theorem 6.7. By definition of u_0 as an infimum (cf (7.28)), we get $u_0 \ge Y_0$, which, since $Y_0 \ge u_0$, yields the equality $u_0 = Y_0$. We have thus shown the three first assertions of the proposition. The fourth assertion follows from Corollary 6.2. The last assertion follows from [16, Proposition 4.3]. The proof is thus complete. $\square$

We now give some examples of American options with completely irregular pay-off.

Example 7.1 *We consider a pay-off process (ξ_t) of the form $\xi_t := h(S_t^1)$, for $t \in [0, T]$, where $h : \mathbf{R} \to \mathbf{R}$ is a (possibly irregular) Borel function such that $(h(S_t^1)) \in \mathcal{S}^2$. In general, the pay-off (ξ_t) is a completely irregular process. By the first two statements of Proposition 7.2, the superhedging price of the American option is equal to the value function of the optimal stopping problem (7.29), and is also characterized as the solution of the reflected BSDE (2.2) with obstacle $\xi_t = h(S_t^1)$.*

If h is an uppersemicontinuous function on $\mathbf{R}$, then the process (ξ_t) is right-u.s.c. and also left-u.s.c. along stopping times. The right-uppersemicontinuity of (ξ_t) follows from the fact that the process S^1 is right-continuous; the left-uppersemicontinuity along stopping times of (ξ_t) follows from the fact that S^1 jumps only at totally inaccessible stopping times. In virtue of Proposition 7.2, last statement, there exists in this case an optimal exercise time *for the American option with payoff $\xi_t = h(S_t^1)$.*

The particular case where $\xi_t := \mathbf{1}_B(S_t^1)$, for $t \in [0, T]$, with B a Borel set in $\mathbf{R}$ corresponds to the pay-off of an American digital option, which is a completely irregular process in general. For example, if $B = [K, +\infty[$ (American digital call option) then the function $\mathbf{1}_B$ is u.s.c. on $\mathbf{R}$. The corresponding payoff process $\xi_t := \mathbf{1}_{S_t^1 \ge K}$ is thus r.u.s.c and left-u.s.c. along stopping times in this case, which implies the existence of an optimal exercise time. If $B =]-\infty, K[$ (American digital put option), the corresponding payoff $\xi_t := \mathbf{1}_{S_t^1 < K}$ is not r.u.s.c. We note that the pay-off of the American digital call and put options is in general neither left-limited nor right-limited.

There are also more "sophisticated" types of digital American options, such as an American digital call option with lower barrier L, for which the payoff is of the form: $\xi_t := \mathbf{1}_{S_t^1 \ge K} \mathbf{1}_{\inf_{0 \le s \le t} S_s^1 > L}$. Note that in this case, the payoff process (ξ_t) is not right u.s.c.

7.2. *An application to RBSDEs.* The characterization (Theorem 6.7) is also useful in the theory of RBSDEs in itself: it allows us to obtain a priori estimates with universal constants for RBSDEs with completely irregular obstacles.

Proposition 7.3 (A priori estimates with universal constants) *Let ξ and ξ' be two left-limited [9] processes in $\mathcal{S}^2$. Let f and f' be two Lipschitz drivers satisfying Assumption 4.1 with common Lipschitz constant $K > 0$. Let (Y, Z, k) (resp. (Y', Z', k')) be the three first components of the solution of the reflected BSDE associated with driver f (resp. f') and obstacle ξ (resp. ξ'). Let $\overline{Y} := Y - Y'$, $\overline{\xi} := \xi - \xi'$.*

Let $\delta f_s := f'(s, Y'_s, Z'_s, k'_s) - f(s, Y'_s, Z'_s, k'_s)$. Let $\eta, \beta > 0$ with $\beta \geq \dfrac{3}{\eta} + 2K$ and $\eta \leq \dfrac{1}{K^2}$. For each $S \in \mathcal{T}_{0,T}$, we have

$$(7.30) \qquad \overline{Y}_S^2 \leq e^{\beta(T-S)} E[\underset{\tau \in \mathcal{T}_{S,T}}{\mathrm{ess\,sup}}\, \overline{\xi}_\tau^2 | \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)}(\delta f_s)^2 ds | \mathcal{F}_S] \quad \text{a.s.}$$

Proof: The proof is divided into two steps.

Step 1: For each $\tau \in \mathcal{T}_{0,T}$, let $(X^\tau, \pi^\tau, l^\tau)$ (resp. $(X'^\tau, \pi'^\tau, l'^\tau)$) be the solution of the BSDE associated with driver f (resp. f'), terminal time τ and terminal condition ξ_τ (resp. ξ'_τ). Set $\overline{X}^\tau := X^\tau - X'^\tau$. By an estimate on BSDEs (cf. Proposition $A.4$ in [35]), we have

$$(\overline{X}_S^\tau)^2 \leq e^{\beta(T-S)} E[\overline{\xi}^2 | \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)}[(f - f')(s, X'^\tau_s, \pi'^\tau_s, l'^\tau_s)]^2 ds | \mathcal{F}_S] \quad \text{a.s.}$$

from which we derive

$$(7.31) \qquad (\overline{X}_S^\tau)^2 \leq e^{\beta(T-S)} E[\underset{\tau \in \mathcal{T}_{S,T}}{\mathrm{ess\,sup}}\, \overline{\xi}_\tau^2 | \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)}(\overline{f}_s)^2 ds | \mathcal{F}_S] \quad \text{a.s.,}$$

where $\overline{f}_s := \sup_{y,z,k}|f(s, y, z, k) - f'(s, y, z, k)|$. Now, by Theorem 6.7, we have $Y_S = \mathrm{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} X_S^\tau$ a.s. and $Y'_S = \mathrm{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} X'^\tau_S$ a.s. We thus get $|\overline{Y}_S| \leq \mathrm{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} |\overline{X}_S^\tau|$ a.s. By the inequality (7.31), we derive

$$\overline{Y}_S^2 \leq e^{\beta(T-S)} E[\underset{\tau \in \mathcal{T}_{S,T}}{\mathrm{ess\,sup}}\, \overline{\xi}_\tau^2 | \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)}(\overline{f}_s)^2 ds | \mathcal{F}_S] \quad \text{a.s.}$$

Step 2: Note that (Y', Z', k') is the solution the RBSDE associated with obstacle ξ' and driver $f(t, y, z, k) + \delta f_t$. By applying the result of Step 1 to the driver $f(t, y, z, k)$ and the driver $f(t, y, z, k) + \delta f_t$ (instead of f'), we get the desired result. $\qquad\qquad \square$

Remark 7.13 *The previous proposition illustrates the relevance of the characterization of the solution of the non-linear RBSDE with irregular obstacle as the value of the non-linear optimal stopping problem (4.10), as established in Theorem 6.7.*

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[9]without loss of generality.

8. Appendix. We give *a priori* estimates for RBSDEs with completely irregular obstacles.

Lemma 8.1 (A priori estimates) *Let* $(Y^1, Z^1, k^1, A^1, C^1) \in \mathcal{S}^2 \times I\!\!H^2 \times I\!\!H_\nu^2 \times \mathcal{S}^2 \times \mathcal{S}^2$ *(resp.* $(Y^2, Z^2, k^2, A^2, C^2) \in \mathcal{S}^2 \times I\!\!H^2 \times I\!\!H_\nu^2 \times \mathcal{S}^2 \times \mathcal{S}^2$*) be a solution to the RBSDE associated with driver* $f^1(\omega, t)$ *(resp.* $f^2(\omega, t)$*) and with obstacle* $\xi \in \mathcal{S}^2$*. There exists* $c > 0$ *such that for all* $\varepsilon > 0$*, for all* $\beta \geq \frac{1}{\varepsilon^2}$ *we have*

$$\|k^1 - k^2\|_{\nu,\beta}^2 \leq \varepsilon^2 \|f^1 - f^2\|_\beta^2; \qquad \|Z^1 - Z^2\|_\beta^2 \leq \varepsilon^2 \|f^1 - f^2\|_\beta^2;$$

(8.32)
$$\||Y^1 - Y^2\||_\beta^2 \quad \leq \quad 4\varepsilon^2(1 + 6c^2)\|f^1 - f^2\|_\beta^2.$$

Proof: The result was proved in [16, Lemma 3.2] in the case of an r.u.s.c. obstacle ξ. The proof of [16] still holds in our framework and is therefore omitted. $\qquad \square$

We recall the $\mathcal{E}^f$-Mertens decomposition of (r.u.s.c.) strong $\mathcal{E}^f$-supermartingales provided in [16], which is a crucial result used in the present paper.

Proposition 8.2 ($\mathcal{E}^f$-Mertens decomposition) *Let* (Y_t) *be a process in* $\mathcal{S}^2$*. Let* f *be a predictable Lipschitz driver satisfying Assumption 4.1. The process* (Y_t) *is a strong* $\mathcal{E}^f$*-supermartingale if and only if there exists a nondecreasing right-continuous predictable process* A *in* $\mathcal{S}^2$ *with* $A_0 = 0$ *and a nondecreasing right-continuous adapted purely discontinuous process* C *in* $\mathcal{S}^2$ *with* $C_{0-} = 0$*, as well as two processes* $Z \in I\!\!H^2$ *and* $k \in \mathbb{H}_\nu^2$*, such that a.s. for all* $t \in [0, T]$*,*
(8.33)
$$Y_t = Y_T + \int_t^T f(s, Y_s, Z_s, k_s)ds + A_T - A_t + C_{T-} - C_{t-} - \int_t^T Z_s dW_s - \int_t^T \int_E k_s(e) \tilde{N}(ds, de).$$

This decomposition is unique.

Remark 8.14 *From this property, it follows that a strong* $\mathcal{E}^f$*-supermartingale in* $\mathcal{S}^2$ *is necessarily r.u.s.c.*

Recall that this result is shown in [16] (cf. Theorem 5.2 in [16]) by using the characterization (cf. Theorem 4.2 in [16]) of the solution of the RBSDE with an r.u.s.c. obstacle as the value function of the non-linear optimal stopping problem (4.10).

The above $\mathcal{E}^f$-Mertens decomposition was also shown in [4] (at the same time as in [16]) in a Brownian framework by using a different approach.

9. Complements: The strict value. In this section we give some complements on a closely related (non-linear) optimal stopping problem.

Let S be a stopping time in $\mathcal{T}_{0,T}$. We denote by $\mathcal{T}_{S^+}$ the set of stopping times $\tau \in \mathcal{T}_{0,T}$ with $\tau > S$ a.s. on $\{S < T\}$ and $\tau = T$ a.s. on $\{S = T\}$. *The strict value* $V^+(S)$ (at time S) of the non-linear optimal stopping problem is defined by

(9.34)
$$V^+(S) := \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_{S^+}} \mathcal{E}^f_{S,\tau}(\xi_\tau).$$

We note that $V^+(S) = \xi_T$ a.s. on $\{S = T\}$.
Using the same arguments as for the value family $(V(S))_{S \in \mathcal{T}_{0,T}}$, we show that

Proposition 9.1 *The strict value family* $(V^+(S))_{S \in \mathcal{T}_{0,T}}$ *is a strong* $\mathcal{E}^f$*-supermartingale family. There exists a unique right-uppersemicontinuous optional process, denoted by* $(V_t^+)_{t \in [0,T]}$*, which aggregates the family* $(V^+(S))_{S \in \mathcal{T}_{0,T}}$*. The process* $(V_t^+)_{t \in [0,T]}$ *is a strong* $\mathcal{E}^f$*-supermartingale.*

The following theorem connects the above strict value process $(V_t^+)_{t \in [0,T]}$ with the process of right-limits $(V_{t+})_{t \in [0,T]}$, where (V_t) denotes as before the value process of our non-linear problem (4.10).

Theorem 9.2 (i) *The strict value process* (V_t^+) *is right-continuous.*
(ii) *For all* $S \in \mathcal{T}_{0,T}$*,* $V_S^+ = V_{S+}$ *a.s.*
(iii) *For all* $S \in \mathcal{T}_{0,T}$*,* $V_S = V_S^+ \vee \xi_S$ *a.s.*

The proof of the theorem uses the following preliminary result which states that the strict value process (V_t^+) is right-continuous along stopping times in $\mathcal{E}^f$-conditional expectation.

Lemma 9.8 (Right-continuity along stopping times in $\mathcal{E}^f$**-conditional expectation)** *The strict value process* (V_t^+) *is right-continuous along stopping times in* $\mathcal{E}^f$*-expectation, in the sense that for each* $\theta \in \mathcal{T}_{0,T}$*, and for each sequence of stopping times* $(\theta_n)_{n \in \mathbb{N}}$ *belonging to* $\mathcal{T}_{0,T}$ *such that* $\theta_n \downarrow \theta$*, we have*

$$(9.35) \qquad \lim_{n \to \infty} \uparrow \mathcal{E}^f_{\theta,\theta_n}(V_{\theta_n}^+) = V_\theta^+ \quad \text{a.s.}$$

For the proof, we recall the following classical statement:

Remark 9.15 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. Let* $A \in \mathcal{F}$*. Let* (X_n) *be a sequence of real valued random variables. Suppose that* (X_n) *converges a.s. on* A *to a random variable* X*. Then, for each* $\varepsilon > 0$*,* $\lim_{n \to +\infty} P(\{|X - X_n| < \varepsilon\} \cap A) = P(A)$*.*
From this property, it follows that for each $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geq n_0$*,* $P(\{|X - X_n| < \varepsilon\} \cap A) \geq \frac{P(A)}{2}$*.*

Proof of Lemma 9.8: Let $n \in \mathbb{N}$. By the consistency property of $\mathcal{E}^f$, we have

$$(9.36) \qquad \mathcal{E}^f_{\theta,\theta_n}(V_{\theta_n}^+) = \mathcal{E}^f_{\theta,\theta_{n+1}}\left(\mathcal{E}^f_{\theta_{n+1},\theta_n}(V_{\theta_n}^+)\right) \quad \text{a.s.}$$

Now, since the process (V_t^+) is a strong $\mathcal{E}^f$- supermartingale, we have $\mathcal{E}^f_{\theta_{n+1},\theta_n}(V_{\theta_n}^+) \leq V_{\theta_{n+1}}^+$ a.s. Using this inequality, together with equality (9.36) and the monotonicity of $\mathcal{E}^f_{\theta,\theta_{n+1}}$, we obtain

$$\mathcal{E}^f_{\theta,\theta_n}(V_{\theta_n}^+) \leq \mathcal{E}^f_{\theta,\theta_{n+1}}(V_{\theta_{n+1}}^+) \quad \text{a.s.}$$

Since this inequality holds for each $n \in \mathbb{N}$, we derive that the sequence of random variables $\left(\mathcal{E}^f_{\theta,\theta_n}(V_{\theta_n}^+)\right)_{n \in \mathbb{N}}$ is nondecreasing. Moreover, since the process (V_t^+) is a strong $\mathcal{E}^f$- supermartingale, we have $\mathcal{E}^f_{\theta,\theta_n}(V_{\theta_n}^+) \leq V_\theta^+$ a.s. for each $n \in \mathbb{N}$. By taking the limit as n tend to $+\infty$, we thus get

$$\lim_{n \to \infty} \uparrow \mathcal{E}^f_{\theta,\theta_n}(V_{\theta_n}^+) \leq V_\theta^+ \quad \text{a.s.}$$

It remains to show the converse inequality:

$$(9.37) \qquad \lim_{n \to \infty} \uparrow \mathcal{E}^f_{\theta,\theta_n}(V_{\theta_n}^+) \geq V_\theta^+ \quad \text{a.s.}$$

Suppose, by way of contradiction, that this inequality does not hold. Then, there exists a constant $\alpha > 0$ such that the event A defined by

$$A := \{\lim_{n\to\infty} \uparrow \mathcal{E}_{\theta,\theta_n}^f(V_{\theta_n}^+) \leq V_\theta^+ - \alpha\}$$

satisfies $P(A) > 0$. By definition of A, we have

$$(9.38) \qquad \lim_{n\to\infty} \uparrow \mathcal{E}_{\theta,\theta_n}^f(V_{\theta_n}^+) + \alpha \leq V_\theta^+ \quad \text{a.s. on } A.$$

As for the value function, there exists an optimizing sequence $(\tau_p)_{p\in\mathbb{N}}$ for the strict value function V_θ^+, that is, such that, for each $p \in \mathbb{N}$, $\tau_p \in \mathcal{T}_{\theta+}$, and such that

$$V_\theta^+ = \lim_{p\to\infty} \uparrow \mathcal{E}_{\theta,\tau_p}^f(\xi_{\tau_p}) \quad \text{a.s.}$$

By Remark 9.15 (applied with $\varepsilon = \frac{\alpha}{2}$), we derive that there exists $p_0 \in \mathbb{N}$ such that the event B defined by

$$B := \{V_\theta^+ \leq \mathcal{E}_{\theta,\tau_{p_0}}^f(\xi_{\tau_{p_0}}) + \frac{\alpha}{2}\} \cap A$$

satisfies $P(B) \geq \frac{P(A)}{2}$. Denoting τ_{p_0} by θ', we have

$$V_\theta^+ \leq \mathcal{E}_{\theta,\theta'}^f(\xi_{\theta'}) + \frac{\alpha}{2} \quad \text{a.s. on } B.$$

By the inequality (9.38), we derive that

$$(9.39) \qquad \lim_{n\to\infty} \uparrow \mathcal{E}_{\theta,\theta_n}^f(V_{\theta_n}^+) + \frac{\alpha}{2} \leq \mathcal{E}_{\theta,\theta'}^f(\xi_{\theta'}) \quad \text{a.s. on } B.$$

Let us first consider the simpler case where $\theta < T$ a.s.
In this case, since $\theta' \in \mathcal{T}_{\theta+}$, we have $\theta' > \theta$ a.s. Hence, we have $\Omega = \cup_{n\in\mathbb{N}} \uparrow \{\theta' > \theta_n\}$ a.s.
Define the stopping time $\overline{\theta}_n := \theta' \mathbf{1}_{\{\theta' > \theta_n\}} + T \mathbf{1}_{\{\theta' \leq \theta_n\}}$. We note that $\overline{\theta}_n \in \mathcal{T}_{\theta_n+}$ for each $n \in \mathbb{N}$.
Moreover, $\lim_{n\to\infty} \overline{\theta}_n = \theta'$ a.s. and $\lim_{n\to\infty} \xi_{\overline{\theta}_n} = \xi_{\theta'}$ a.s. By the continuity property of $\mathcal{E}^f$ with respect to terminal condition and terminal time, we get

$$\lim_{n\to\infty} \mathcal{E}_{\theta,\overline{\theta}_n}^f(\xi_{\overline{\theta}_n}) = \mathcal{E}_{\theta,\theta'}^f(\xi_{\theta'}) \quad \text{a.s.}$$

By Remark 9.15, we derive that there exists $n_0 \in \mathbb{N}$ such that the event C defined by

$$C := \{|\mathcal{E}_{\theta,\theta'}^f(\xi_{\theta'}) - \mathcal{E}_{\theta,\overline{\theta}_{n_0}}^f(\xi_{\overline{\theta}_{n_0}})| \leq \frac{\alpha}{4}\} \cap B$$

satisfies $P(C) > 0$. By the inequality (9.39), we derive that

$$(9.40) \qquad \lim_{n\to\infty} \uparrow \mathcal{E}_{\theta,\theta_n}^f(V_{\theta_n}^+) + \frac{\alpha}{4} \leq \mathcal{E}_{\theta,\overline{\theta}_{n_0}}^f(\xi_{\overline{\theta}_{n_0}}) \quad \text{a.s. on } C.$$

Now, by the consistency of $\mathcal{E}^f$, we have

$$\mathcal{E}_{\theta,\overline{\theta}_{n_0}}^f(\xi_{\overline{\theta}_{n_0}}) = \mathcal{E}_{\theta,\theta_{n_0}}^f\left(\mathcal{E}_{\theta_{n_0},\overline{\theta}_{n_0}}^f(\xi_{\overline{\theta}_{n_0}})\right) \leq \mathcal{E}_{\theta,\theta_{n_0}}^f(V_{\theta_{n_0}}^+) \quad \text{a.s.,}$$

where the last inequality follows from the fact that $\overline{\theta}_{n_0} \in \mathcal{T}_{\theta_{n_0}+}$ and from the definition of $V_{\theta_{n_0}}^+$. By (9.40), we thus derive that

$$\lim_{n \to \infty} \uparrow \mathcal{E}_{\theta,\theta_n}^f(V_{\theta_n}^+) + \frac{\alpha}{4} \leq \mathcal{E}_{\theta,\theta_{n_0}}^f(V_{\theta_{n_0}}^+) \quad \text{a.s. on } C,$$

which gives a contradiction. Hence, the desired inequality (9.37) holds.

Let us now consider a general $\theta \in \mathcal{T}_{0,T}$.
On the set $\{\theta = T\}$, we have $\theta_n = \theta$ a.s. for all n. Hence, on $\{\theta = T\}$, we have $\lim_{n \to \infty} \mathcal{E}_{\theta,\theta_n}^f(V_{\theta_n}^+) = V_\theta^+$ a.s. On the set $\{\theta < T\}$, using the same arguments as above with $\overline{\theta}_n = \theta' \mathbf{1}_{\{\theta' > \theta_n\} \cap \{T > \theta\}} + T \mathbf{1}_{\{\theta' \leq \theta_n\} \cup \{T = \theta\}}$, we show the inequality (9.37). The proof is thus complete. $\qquad\square$

We are now ready to prove the theorem.

Proof of Theorem 9.2: The proof of (i) is based on the previous Lemma 9.8 and on a result from the general theory of processes. Let $S \in \mathcal{T}_{0,T}$ and let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_{S+}$ with $\lim \downarrow S_n = S$ a.s. By applying Lemma 9.8 and the continuity property of $\mathcal{E}^f$-expectations with respect to the terminal condition and to the terminal time, we get

$$V_S^+ = \lim_{n \to \infty} \mathcal{E}_{S,S_n}^f(V_{S_n}^+) = \mathcal{E}_{S,S}^f(\lim_{n \to \infty} V_{S_n}^+) = \lim_{n \to \infty} V_{S_n}^+,$$

where we have used that $\lim_{n \to \infty} V_{S_n}^+$ exists, as (V_t^+) is a strong $\mathcal{E}^f$-supermartingale, and hence has right limits. The above equality shows that the process (V_t^+) is right-continuous along stopping times. By Proposition 2 in [6], we conclude that (V_t^+) is right-continuous.

We now show (ii). Let $S \in \mathcal{T}_{0,T}$. Let (S_n) be a non-increasing sequence of stopping times in $\mathcal{T}_{S+}$ with $\lim \downarrow S_n = S$ a.s. We know that $V_\tau \geq V_\tau^+$ a.s., for all $\tau \in \mathcal{T}_{0,T}$. Hence, $V_{S_n} \geq V_{S_n}^+$ a.s., for all n. We derive that $\lim_{n \to \infty} V_{S_n} \geq \lim_{n \to \infty} V_{S_n}^+$ a.s. Using this and the right-continuity of V^+ established in (i), gives $V_{S+} \geq V_S^+$ a.s. In order to show the converse inequality, we first show

(9.41) $$\mathcal{E}_{S,S^n}^f(V_{S_n}) \leq V_S^+ \quad \text{a.s. for all } n.$$

We fix n and we take $(\tau^p) \in \mathcal{T}_{S_n}$ an optimizing sequence for the problem with value V_{S_n}, i.e. $V_{S_n} = \lim_{p \to \infty} \mathcal{E}_{S_n,\tau_p}^f(\xi_{\tau_p})$. We have

(9.42) $$\mathcal{E}_{S,S_n}^f(V_{S_n}) = \mathcal{E}_{S,S_n}^f(\lim_{p \to \infty} \mathcal{E}_{S_n,\tau_p}^f(\xi_{\tau_p})) = \lim_{p \to \infty} \mathcal{E}_{S,S_n}^f(\mathcal{E}_{S_n,\tau_p}^f(\xi_{\tau_p})) \text{ a.s.},$$

where we have used the continuity property of $\mathcal{E}_{S,S^n}^f(\cdot)$ with respect to the terminal condition (recall that here n is fixed). Using the consistency property of $\mathcal{E}^f$-expectations, we get $\mathcal{E}_{S,S_n}^f(\mathcal{E}_{S_n,\tau_p}^f(\xi_{\tau_p})) = \mathcal{E}_{S,\tau_p}^f(\xi_{\tau_p}) \leq V_S^+$ a.s. (where for the inequality we have used that $\tau_p \in \mathcal{T}_{S+}$). From this, together with equation (9.42), we derive the desired inequality (9.41). From inequality (9.41), together with the continuity of $\mathcal{E}^f$-expectations with respect to the terminal time and the terminal condition, we derive $V_S^+ \geq \lim_{n \to \infty} \mathcal{E}_{S,S^n}^f(V_{S_n}) = \mathcal{E}_{S,S}^f(V_{S+}) = V_{S+}$ a.s. Hence, $V_S^+ \geq V_{S+}$ a.s., which, together with the previously shown converse inequality, proves the equality $V_{S+} = V_S^+$ a.s.
Statement (iii) is a direct consequence of part (ii) (which we have just shown), together with Corollary 6.1(i).

$\qquad\square$

Remark 9.16 *By the same arguments as those of the proof of statement (i) in the above Theorem 9.2, the following general statement can be shown: A strong $\mathcal{E}^f$-supermartingale is right-continuous if it is right-continuous along stopping times in $\mathcal{E}^f$-conditional expectation.*

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