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# Optimal stopping with $f$ -expectations: the irregular case

Miryana Grigorova <sup>\*</sup>     Peter Imkeller <sup>†</sup>     Youssef Ouknine <sup>‡</sup>  
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## Abstract

We consider the optimal stopping problem with non-linear  $f$ -expectation (induced by a BSDE) without making any regularity assumptions on the pay-off process  $\xi$ . We show that the value family can be aggregated by an optional process  $Y$ . We characterize the process  $Y$  as the  $\mathcal{E}^f$ -Snell envelope of  $\xi$ . We also establish an infinitesimal characterization of the value process  $Y$  in terms of a Reflected BSDE with  $\xi$  as the obstacle. This characterization is established by first showing existence and uniqueness for the Reflected BSDE with irregular obstacle and also a comparison theorem.

## 1 Introduction

The *classical* optimal stopping problem with *linear expectations* has been largely studied. General results on the topic can be found in El Karoui (1981) ([8]) where no regularity assumptions on the pay-off process  $\xi$  are made.

In this paper, we are interested in a generalization of the classical optimal stopping problem where the linear expectation is replaced by a possibly non-linear functional, the so-called  $f$ -expectation ( $f$ -evaluation), induced by a BSDE with Lipschitz driver  $f$ . For a stopping time  $S$  such that  $0 \leq S \leq T$  a.s. (where  $T > 0$  is a fixed terminal horizon), we define

$$V(S) := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}_{S,\tau}^f(\xi_\tau), \quad (1.1)$$

where  $\mathcal{T}_{S,T}$  denotes the set of stopping times valued a.s. in  $[S, T]$  and  $\mathcal{E}_{S,\tau}^f(\cdot)$  denotes the  $f$ -conditional expectation/evaluation at time  $S$  when the terminal time is  $\tau$ .

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The above non-linear problem has been introduced in [10] in the case of a Brownian filtration and a continuous pay-off process  $\xi$ . The problem has been generalized to the case of a Wiener-Poisson filtration and a right-continuous pay-off process  $\xi$  in [32]. To the best of our knowledge, [13] is the first paper dealing with the stopping problem (1.1) in the case of a non-right-continuous pay-off process  $\xi$ . In [13] the usual assumption of right-continuity of  $\xi$  is replaced by the weaker assumption of right-uppersemicontinuity. In the present paper, we study problem (1.1) without making any regularity assumptions on  $\xi$ . Due to the lack of regularity the usual approach (cf., e.g., [32], [13]) of linking directly the value family  $(V(S))$  with the solution of a suitably defined Reflected BSDE (and thus avoiding more technical aggregation questions) is no longer applicable. We are thus led to following a different approach: With the help of some results from the general theory of processes, we show that the value family  $(V(S))$  can be aggregated by a unique right-uppersemicontinuous optional process  $(V_t)_{t \in [0, T]}$ . We characterize  $(V_t)_{t \in [0, T]}$  as the  $\mathcal{E}^f$ -Snell envelope of  $\xi$ , that is, the smallest strong  $\mathcal{E}^f$ -supermartingale greater than or equal to  $\xi$ . We then turn to establishing an infinitesimal characterization of the value process  $(V_t)$  in terms of a Reflected BSDE where the pay-off process  $\xi$  from (1.1) plays the role of a lower obstacle.

Let us recall that reflected BSDEs have been introduced by El Karoui et al. in the seminal paper [9] in the case of a Brownian filtration and a continuous obstacle, and then generalized to the case of a right-continuous obstacle and/or a larger stochastic basis than the Brownian one in [16], [3], [17], [11], [18], [32]. In [13], we have formulated a notion of Reflected BSDE in the case where the obstacle is only right-uppersemicontinuous (but possibly not right-continuous) and have shown existence and uniqueness of the solution. In the present paper, we show that the existence and uniqueness result from [13] still holds in the more general case, without any regularity assumptions on the obstacle. In the recent preprint [21], existence and uniqueness of the solution (in the Brownian framework) is shown by using a penalization method. The proof which we give here is different from the proof of [21]; we rely on an explicit characterization of the solution in the case where the driver  $f$  does not depend on  $y, z$ , and  $\xi$ , and on Banach fixed point theorem in the case of a general Lipschitz driver  $f$ . We also establish a comparison result for RBSDEs with irregular obstacles. Due to the irregularity of the obstacles and the presence of jumps in the filtration, we are led to using an approach which differs from those existing in the literature on comparison of RBSDEs (cf. also Remarks 4.3 and 5.5); in particular, we first prove a generalization of Gal'chouk-Lenglart's formula (cf. [12] and [25]) to the case of convex functions, which we then astutely apply in our framework. With the help of our comparison result, we prove that the (first component of the) solution to the Reflected BSDE with irregular obstacle  $\xi$  (and driver  $f$ ) coincides with the value process  $V$  of problem (1.1), which gives us the desired infinitesimal characterization for  $V$ .

The rest of the paper is organized as follows: In Section 2 we give some preliminary definitions and some notation. In Section 3 we revisit the classical optimal stopping problem with irregular pay-off process  $\xi$ . We first give some general results such as aggregation, Mertens decomposition of the value process, minimality of the non-decreasing processes of the Mertens

decomposition; then, we characterize the value process of the classical problem in terms of the solution of a Reflected BSDE with irregular obstacle and driver  $f$  which does not depend on the solution. In Section 4 we prove existence and uniqueness of the solution to the Reflected BSDE with irregular obstacle in the case of a general Lipschitz driver  $f$  (Subsection 4.1), and we establish a comparison theorem for such RBSDEs (Subsection 4.2). Section 5 is devoted to the study of the non-linear optimal stopping problem (1.1); in particular, we present the aggregation result, the Snell characterization, and the infinitesimal characterization in terms of the solution of the RBSDE from Section 4; we also give a financial motivation of the problem. The Appendix contains a useful corollary of the infinitesimal characterization, namely *a priori estimates with universal constants* for RBSDEs with irregular obstacles.

## 2 Preliminaries

Let  $T > 0$  be a fixed positive real number. Let  $(E, \mathcal{E})$  be a measurable space equipped with a  $\sigma$ -finite positive measure  $\nu$ . Let  $(\Omega, \mathcal{F}, P)$  be a probability space equipped with a one-dimensional Brownian motion  $W$  and with an independent Poisson random measure  $N(dt, de)$  with compensator  $dt \otimes \nu(de)$ . We denote by  $\tilde{N}(dt, de)$  the compensated process, i.e.  $\tilde{N}(dt, de) := N(dt, de) - dt \otimes \nu(de)$ . Let  $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$  be the (complete) natural filtration associated with  $W$  and  $N$ . For  $t \in [0, T]$ , we denote by  $\mathcal{T}_{t,T}$  the set of stopping times  $\tau$  such that  $P(t \leq \tau \leq T) = 1$ . More generally, for a given stopping time  $\nu \in \mathcal{T}_0$ , we denote by  $\mathcal{T}_{\nu,T}$  the set of stopping times  $\tau$  such that  $P(\nu \leq \tau \leq T) = 1$ .

We use also the following notation:

- $\mathcal{P}$  (resp.  $\mathcal{O}$ ) is the predictable (resp. optional)  $\sigma$ -algebra on  $\Omega \times [0, T]$ .
- Prog is the progressive  $\sigma$ -algebra on  $\Omega \times [0, T]$ .
- $\mathcal{B}(\mathbf{R})$  (resp.  $\mathcal{B}(\mathbf{R}^2)$ ) is the Borel  $\sigma$ -algebra on  $\mathbf{R}$  (resp.  $\mathbf{R}^2$ ).
- $L^2(\mathcal{F}_T)$  is the set of random variables which are  $\mathcal{F}_T$ -measurable and square-integrable.
- $L_\nu^2$  is the set of  $(\mathcal{E}, \mathcal{B}(\mathbf{R}))$ -measurable functions  $\ell : E \rightarrow \mathbf{R}$  such that  $\|\ell\|_\nu^2 := \int_E |\ell(e)|^2 \nu(de) < \infty$ . For  $\ell \in \mathcal{L}_\nu^2$ ,  $\kappa \in \mathcal{L}_\nu^2$ , we define  $\langle \ell, \kappa \rangle_\nu := \int_E \ell(e) \kappa(e) \nu(de)$ .
- $\mathcal{B}(L_\nu^2)$  is the Borel  $\sigma$ -algebra on  $L_\nu^2$ .
- $\mathbb{H}^2$  is the set of  $\mathbf{R}$ -valued predictable processes  $\phi$  with  $\|\phi\|_{\mathbb{H}^2}^2 := E \left[ \int_0^T |\phi_t|^2 dt \right] < \infty$ .
- $\mathbb{H}_\nu^2$  is the set of  $\mathbf{R}$ -valued processes  $l : (\omega, t, e) \in (\Omega \times [0, T] \times E) \mapsto l_t(\omega, e)$  which are *predictable*, that is  $(\mathcal{P} \otimes \mathcal{E}, \mathcal{B}(\mathbf{R}))$ -measurable, and such that  $\|l\|_{\mathbb{H}_\nu^2}^2 := E \left[ \int_0^T \|l_t\|_\nu^2 dt \right] < \infty$ .

As in [13], we denote by  $\mathcal{S}^2$  the vector space of  $\mathbf{R}$ -valued optional (not necessarily cadlag) processes  $\phi$  such that  $\|\phi\|_{\mathcal{S}^2}^2 := E[\text{ess sup}_{\tau \in \mathcal{T}_0} |\phi_\tau|^2] < \infty$ . By Proposition 2.1 in [13], the mapping  $\|\cdot\|_{\mathcal{S}^2}$  is a norm on the space  $\mathcal{S}^2$ , and  $\mathcal{S}^2$  endowed with this norm is a Banach space.

**Definition 2.1 (Driver, Lipschitz driver)** *A function  $f$  is said to be a driver if*

- $f : \Omega \times [0, T] \times \mathbf{R}^2 \times L_\nu^2 \rightarrow \mathbf{R}$   
 $(\omega, t, y, z, \kappa) \mapsto f(\omega, t, y, z, \kappa)$  is  $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^2) \otimes \mathcal{B}(L_\nu^2)$ -measurable,
- $E[\int_0^T f(t, 0, 0, 0)^2 dt] < +\infty$ .

A driver  $f$  is called a Lipschitz driver if moreover there exists a constant  $K \geq 0$  such that  $dP \otimes dt$ -a.e., for each  $(y_1, z_1, \kappa_1) \in \mathbf{R}^2 \times L_\nu^2$ ,  $(y_2, z_2, \kappa_2) \in \mathbf{R}^2 \times L_\nu^2$ ,

$$|f(\omega, t, y_1, z_1, \kappa_1) - f(\omega, t, y_2, z_2, \kappa_2)| \leq K(|y_1 - y_2| + |z_1 - z_2| + \|\kappa_1 - \kappa_2\|_\nu).$$

Let  $T > 0$  be a fixed terminal time. Let  $f$  be a driver. Let  $\xi = (\xi_t)_{t \in [0, T]}$  be a left-limited process in  $\mathcal{S}^2$ .

**Remark 2.1** *Let us note that in the following definitions and results we can relax the assumption of existence of left limits for the obstacle  $\xi$ . All the results still hold true provided we replace the process  $(\xi_{t-})_{t \in [0, T]}$  by the process  $(\underline{\xi}_t)_{t \in [0, T]}$  defined by  $\underline{\xi}_t := \limsup_{s \uparrow t, s < t} \xi_s$ , for all  $t \in ]0, T[$ . We recall that  $\underline{\xi}$  is a predictable process (cf. [5, Thm. 90, page 225]). We call the process  $\underline{\xi}$  the left upper-semicontinuous envelope of  $\xi$ .*

**Definition 2.2** *A process  $(Y, Z, k, A, C)$  is said to be a solution to the reflected BSDE with parameters  $(f, \xi)$ , where  $f$  is a driver and  $\xi$  is a left-limited process in  $\mathcal{S}^2$ , if*

$(Y, Z, k, A, C) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_\nu^2 \times \mathcal{S}^2 \times \mathcal{S}^2$  and a.s. for all  $t \in [0, T]$

$$Y_t = \xi_T + \int_t^T f(s, Y_s, Z_s, k_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_E k_s(e) \tilde{N}(ds, de) + A_T - A_t + C_{T-} - C_{t-}, \quad (2.2)$$

$$Y_t \geq \xi_t \text{ for all } t \in [0, T] \text{ a.s.}, \quad (2.3)$$

$A$  is a nondecreasing right-continuous predictable process with  $A_0 = 0$  and such that

$$\int_0^T \mathbf{1}_{\{Y_t > \xi_t\}} dA_t^c = 0 \text{ a.s. and } (Y_{\tau-} - \xi_{\tau-})(A_\tau^d - A_{\tau-}^d) = 0 \text{ a.s. for all predictable } \tau \in \mathcal{T}_{0, T}, \quad (2.4)$$

$C$  is a nondecreasing right-continuous adapted purely discontinuous process with  $C_{0-} = 0$

$$\text{and such that } (Y_\tau - \xi_\tau)(C_\tau - C_{\tau-}) = 0 \text{ a.s. for all } \tau \in \mathcal{T}_{0, T}. \quad (2.5)$$

Here  $A^c$  denotes the continuous part of the process  $A$  and  $A^d$  its discontinuous part. Equations (2.4) and (2.5) are referred to as *minimality conditions* or *Skorokhod conditions*.

For real-valued random variables  $X$  and  $X_n$ ,  $n \in \mathbb{N}$ , the notation " $X_n \uparrow X$ " will stand for "the sequence  $(X_n)$  is nondecreasing and converges to  $X$  a.s."

For a *ladlag* process  $\phi$ , we denote by  $\phi_{t+}$  and  $\phi_{t-}$  the right-hand and left-hand limit of  $\phi$  at  $t$ . We denote by  $\Delta_+ \phi_t := \phi_{t+} - \phi_t$  the size of the right jump of  $\phi$  at  $t$ , and by  $\Delta \phi_t := \phi_t - \phi_{t-}$  the size of the left jump of  $\phi$  at  $t$ .

**Definition 2.3** *Let  $\tau \in \mathcal{T}_0$ . An optional process  $(\phi_t)$  is said to be right upper-semicontinuous (r.u.s.c.) along stopping times if for all stopping time  $\tau \in \mathcal{T}_0$  and for all nonincreasing sequence of stopping times  $(\tau_n)$  such that  $\tau_n \downarrow \tau$  a.s.,  $\phi_\tau \geq \limsup_{n \rightarrow \infty} \phi_{\tau_n}$  a.s..*

### 3 The classical optimal stopping problem

Let  $(\xi_t)_{t \in [0, T]}$  be a left-limited process belonging to  $\mathcal{S}^2$ , called the *reward process*. Let  $f = (f_t)_{t \in [0, T]}$  be a predictable process with  $E[\int_0^T f_t^2 dt] < +\infty$ , called the *instantaneous reward process*. For each  $S \in \mathcal{T}_{0, T}$ , we define the value function  $Y(S)$  at time  $S$  by

$$Y(S) := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S, T}} E[\xi_\tau + \int_S^\tau f_u du \mid \mathcal{F}_S]. \quad (3.6)$$

#### 3.1 General results

**Lemma 3.1 (i)** *There exists a *ladlag* optional process  $(Y_t)_{t \in [0, T]}$  which aggregates the family  $(Y(S))_{S \in \mathcal{T}_{0, T}}$  (i.e.  $Y_S = Y(S)$  a.s. for all  $S \in \mathcal{T}_{0, T}$ ).*

*Moreover, the process  $(Y_t + \int_0^t f_u du)_{t \in [0, T]}$  is the smallest strong supermartingale greater than or equal to  $(\xi_t + \int_0^t f_u du)_{t \in [0, T]}$ .*

**(ii)** *We have  $Y_S = \xi_S \vee Y_{S+}$  a.s. for all  $S$ .*

**(iii)** *For each  $S \in \mathcal{T}_{0, T}$  and for each  $\lambda \in ]0, 1[$ , we set*

$$\tau_S^\lambda := \inf\{t \geq S, \lambda Y_t(\omega) \leq \xi_t\}.$$

*The process  $(Y_t + \int_0^t f_u du)_{t \in [0, T]}$  is a martingale on  $[S, \tau_S^\lambda]$ .*

*Proof.* These results follow from results of classical optimal stopping theory. For a sketch of the proof of the first two assertions, the reader is referred to the proof of Proposition A.5 in the Appendix of [13] (which still holds for a general process  $\xi \in \mathcal{S}^2$ ). The last assertion corresponds to a result of optimal stopping theory (cf. [26], [8] or Lemma 2.7 in [22]). Its proof is based on a penalization method, introduced by Maingueneau (1978) ([26]), which does not require any regularity assumption on the reward process  $\xi$ .  $\square$

**Remark 3.2** It follows from (ii) in the above lemma that  $\Delta_+ Y_S = \mathbf{1}_{\{Y_S = \xi_S\}} \Delta_+ Y_S$  a.s.

**Theorem 3.1 (i)** The value process  $Y$  of Lemma 3.1 belongs to  $\mathcal{S}^2$  and admits the following (Mertens) decomposition:

$$Y_t = - \int_0^t f_u du + M_t - A_t - C_{t-} \text{ for all } t \in [0, T] \text{ a.s.}, \quad (3.7)$$

where  $M$  is a square integrable martingale,  $A$  is a nondecreasing right-continuous predictable process such that  $A_0 = 0$ ,  $E(A_T^2) < \infty$ , and  $C$  is a nondecreasing right-continuous adapted purely discontinuous process such that  $C_{0-} = 0$ ,  $E(C_T^2) < \infty$ .

(ii) For each  $\tau \in \mathcal{T}_{0,T}$ , we have  $\Delta C_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}} \Delta C_\tau$  a.s.

(iii) For each predictable  $\tau \in \mathcal{T}_{0,T}$ , we have  $\Delta A_\tau = \mathbf{1}_{\{Y_{\tau-} = \xi_{\tau-}\}} \Delta A_\tau$  a.s.

(iv) The continuous part  $A^c$  of  $A$  satisfies the equality  $\int_0^T \mathbf{1}_{\{Y_t > \xi_t\}} dA_t^c = 0$  a.s.

Proof. By Lemma 3.1 (i), the process  $(Y_t + \int_0^t f_u du)_{t \in [0, T]}$  is a strong supermartingale. Moreover, it can be shown (cf. [13]) that

$$E[\text{ess sup}_{S \in \mathcal{T}_{0,T}} |Y_S|^2] \leq cE[X^2] \leq cT \|f\|_{H^2}^2 + c\|\xi\|_{S^2}^2, \quad (3.8)$$

Hence, the process  $(Y_t + \int_0^t f_u du)_{t \in [0, T]}$  is in  $\mathcal{S}^2$  (a fortiori, of class (D)). Applying Mertens decomposition for strong supermartingales of class (D) (cf., e.g., [6, Theorem 20, page 429, combined with Remark 3(b), page 205] and [6, Appendix 1, Thm.20, equalities (20.2)]) gives the decomposition (3.7), where  $M$  is a cadlag uniformly integrable martingale,  $A$  is a nondecreasing right-continuous predictable process such that  $A_0 = 0$ ,  $E(A_T) < \infty$ , and  $C$  is a nondecreasing right-continuous adapted purely discontinuous process such that  $C_{0-} = 0$ ,  $E(C_T) < \infty$ . By applying the same arguments as in the proof of Lemma 3.3 (step 3) in [13], we show that  $A \in \mathcal{S}^2$  and  $C \in \mathcal{S}^2$ , which gives the assertion (i).

Let  $\tau \in \mathcal{T}_{0,T}$ . By Remark 3.2 together with Mertens decomposition (3.7), we get  $\Delta C_\tau = -\Delta_+ Y_\tau$  a.s. It follows that  $\Delta C_\tau = \mathbf{1}_{\{Y_\tau = \xi_\tau\}} \Delta C_\tau$  a.s., which corresponds to (ii).

From Lemma 3.1 (iii) together with Mertens decomposition (3.7), it follows that, for each  $S \in \mathcal{T}_{0,T}$  and for each  $\lambda \in ]0, 1[$ , we have

$$A_S = A_{\tau_S^\lambda} \text{ a.s.} \quad (3.9)$$

Assertion (iii) (concerning the jumps of  $A$ ) is due to El Karoui ([8]). Its proof is based on the equality (3.9). For details, the reader is also referred to the proof of Proposition 2.34 in [8].

Let us now show the assertion (iv). As for the discontinuous part of  $A$ , the proof is based on the equality (3.9), and also on some analytic arguments similar to those used in the proof of Theorem D13 in Karatzas and Shreve (1998) ([20]).

We have to show that  $\int_0^T (Y_t - \xi_t) dA_t^c = 0$  a.s., or, equivalently, that  $\int_0^T (Y_{t-} - \xi_{t-}) dA_t^c = 0$  a.s. (since  $A^c$  is continuous). Without loss of generality, we can assume that for each  $\omega$ , the map  $t \mapsto A_t^c(\omega)$  is continuous, that the maps  $t \mapsto Y_t(\omega)$  and  $t \mapsto \xi_t(\omega)$  are left-limited, and that, for all  $\lambda \in ]0, 1[ \cap \mathbb{Q}$  and  $t \in [0, T[ \cap \mathbb{Q}$ , we have  $A_t(\omega) = A_{\tau_t^\lambda}(\omega)$ . Let us denote by  $\mathcal{J}(\omega)$  the set on which the nondecreasing function  $t \mapsto A_t^c(\omega)$  is "flat":

$$\mathcal{J}(\omega) := \{t \in ]0, T[, \exists \delta > 0 \text{ with } A_{t-\delta}^c(\omega) = A_{t+\delta}^c(\omega)\}$$

The set  $\mathcal{J}(\omega)$  is clearly open and hence can be written as a countable union of disjoint intervals:  $\mathcal{J}(\omega) = \cup_i ]\alpha_i(\omega), \beta_i(\omega)[$ . We consider

$$\hat{\mathcal{J}}(\omega) := \cup_i ]\alpha_i(\omega), \beta_i(\omega)] = \{t \in ]0, T], \exists \delta > 0 \text{ with } A_{t-\delta}^c(\omega) = A_t^c(\omega)\}. \quad (3.10)$$

We have  $\int_0^T \mathbf{1}_{\hat{\mathcal{J}}(\omega)} dA_t^c(\omega) = \sum_i (A_{\beta_i(\omega)}^c(\omega) - A_{\alpha_i(\omega)}^c(\omega)) = 0$ . Hence, the nondecreasing function  $t \mapsto A_t^c(\omega)$  is "flat" on  $\hat{\mathcal{J}}(\omega)$ . We now introduce

$$\mathcal{K}(\omega) := \{t \in ]0, T] \text{ s.t. } Y_{t-}(\omega) > \xi_{t-}(\omega)\}$$

We next show that for almost every  $\omega$ ,  $\mathcal{K}(\omega) \subset \hat{\mathcal{J}}(\omega)$ , which clearly provides the desired result. Let  $t \in \mathcal{K}(\omega)$ . Let us prove that  $t \in \hat{\mathcal{J}}(\omega)$ . By (3.10), we thus have to show that there exists  $\delta > 0$  such that  $A_{t-\delta}^c(\omega) = A_t^c(\omega)$ . Since  $t \in \mathcal{K}(\omega)$ , we have  $Y_{t-}(\omega) > \xi_{t-}(\omega)$ . Hence, there exists  $\delta > 0$  and  $\lambda \in ]0, 1[ \cap \mathbb{Q}$  such that  $t - \delta \in [0, T[ \cap \mathbb{Q}$  and for each  $r \in [t - \delta, t[$ ,  $\lambda Y_r(\omega) > \xi_r(\omega)$ . By definition of  $\tau_{t-\delta}^\lambda(\omega)$ , it follows that  $\tau_{t-\delta}^\lambda(\omega) \geq t$ . Now, we have  $A_{\tau_{t-\delta}^\lambda}^c(\omega) = A_{t-\delta}^c(\omega)$ . Since the map  $s \mapsto A_s^c(\omega)$  is nondecreasing, we derive that  $A_t^c(\omega) = A_{t-\delta}^c(\omega)$ , which implies that  $t \in \hat{\mathcal{J}}(\omega)$ . We thus have  $\mathcal{K}(\omega) \subset \hat{\mathcal{J}}(\omega)$ , which completes the proof.  $\square$

*Remark 3.1* We see from the above proof that Theorem 3.1 also holds true in the case of a general filtration assumed to satisfy the usual hypotheses.

*Remark 3.2* The assertion (iv) of Theorem 3.1 generalizes a result shown in [21] (cf. Proposition 7.3) in a Brownian framework. The proof given in [21] uses analytic arguments which are different from the ones used in our proof.

## 3.2 Characterization of the value function as the solution of an RBSDE

Using Theorem 3.1, we show that the value process  $Y$  of the optimal stopping problem (3.6) solves the RBSDE from Definition 2.2 with parameters the driver process  $(f_t)$  and the obstacle  $(\xi_t)$ , and that, moreover,  $Y$  is the unique solution of the RBSDE. We thus have an "infinitesimal characterization" of the value process  $Y$ .



**Theorem 3.2** *Let  $Y$  be the value process of the optimal stopping problem (3.6). Let  $A$  and  $C$  be the non decreasing processes associated with the Mertens decomposition (3.7) of  $Y$ . There exists a unique pair  $(Z, k) \in \mathbb{H}^2 \times \mathbb{H}_V^2$  such that the process  $(Y, Z, k, A, C)$  is a solution of the RBSDE from Definition 2.2 associated with the driver process  $f(\omega, t, y, z, \kappa) = f_t(\omega)$  and the obstacle  $(\xi_t)$ . Moreover, the solution of this RBSDE is unique.*

*Proof.* The proof relies on Theorem 3.1 and the same arguments as in the proof given in the case of a right-uppersemicontinuous (r.u.s.c.) obstacle  $\xi$  (cf. [13]).

By Lemma 3.1 (ii), the value process  $Y$  corresponding to the optimal stopping problem (3.6) satisfies  $Y_T = Y(T) = \xi_T$  a.s. and  $Y_t \geq \xi_t$ ,  $0 \leq t \leq T$ , a.s. By Theorem 3.1 (ii), the process  $C$  of the Mertens decomposition of  $Y$  (3.7) satisfies the minimality condition (2.5). Moreover, by Theorem 3.1 (iii) and (iv), the process  $A$  satisfies the minimality condition (2.4). By the martingale representation theorem (cf., e.g., Lemma 2.3 in [35]) there exists a unique predictable process  $Z \in \mathbb{H}^2$  and a unique predictable  $k \in \mathbb{H}_V^2$  such that  $dM_t = Z_t dW_t + \int_E k_t(e) \tilde{N}(dt, de)$ . The process  $(Y, Z, k, A, C)$  is thus a solution of the RBSDE (2.2) associated with the driver process  $(f_t)$  and with the obstacle  $\xi$ .

Let us show the uniqueness of the solution. We note that the *a priori estimates* provided in [13] (cf. [13, Lemma 3.2]) still hold in our framework. Using these estimates and the same arguments as in step 5 of the proof of Lemma 3.3 in [13], we obtain the desired result.  $\square$

## 4 Reflected BSDE with a non-linear driver and irregular obstacle

### 4.1 Existence and uniqueness of the solution

In Theorem 3.2 of the previous section, we have shown that, in the case where the driver does not depend on  $y, z$ , and  $\kappa$ , the RBSDE from Definition 2.2 admits a unique solution. Using this theorem and the same arguments as in [13], we derive the following existence and uniqueness result in the case of a general Lipschitz driver  $f$ .

**Theorem 4.3 (Existence and uniqueness of the solution of the RBSDE)** *Let  $\xi$  be a left-limited process in  $\mathcal{S}^2$  and let  $f$  be a Lipschitz driver. The RBSDE with parameters  $(f, \xi)$  from Definition 2.2 admits a unique solution  $(Y, Z, k, A, C) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}_V^2 \times \mathcal{S}^2 \times \mathcal{S}^2$ .*

*Proof.* The proof relies on the existence and uniqueness result for RBSDEs with a driver which does not depend on the solution (Theorem 3.2), the a priori estimates provided in [13] (cf. [13, Lemma 3.2]), which still hold in our framework, and a fixed point theorem. For details, the reader is referred to the proof of Theorem 3.4 in [13].  $\square$

*Remark 4.1* *In [21] the above existence and uniqueness result is shown by using a penalization method. Our approach provides an alternative proof of this result.*

## 4.2 Comparison theorem

The following lemma will be used in the proof of the comparison theorem for RBSDEs with irregular obstacles. The lemma can be seen as an extension of Theorem 66 of [30, Chapter IV] from the case of right-continuous semimartingales to the more general case of strong optional semimartingales.

**Lemma 4.2** *Let  $X$  be a (real-valued) strong optional semimartingale with decomposition  $X = X_0 + M + A + B$ , where  $M$  is a local (cadlag) martingale,  $A$  is a right-continuous adapted process of finite variation such that  $A_0 = 0$ ,  $B$  is a left-continuous adapted purely discontinuous process of finite variation such that  $B_0 = 0$ . Let  $f : \mathbf{R} \rightarrow \mathbf{R}$  be a convex function. Then,  $f(X)$  is a strong optional semimartingale. Moreover, denoting by  $f'$  the left-hand derivative of the convex function  $f$ , we have*

$$f(X_t) = f(X_0) + \int_{]0,t[} f'(X_{s-})d(A_s + M_s) + \int_{[0,t[} f'(X_s)dB_{s+} + K_t,$$

where  $K$  is a nondecreasing adapted process such that

$$\Delta K_t = f(X_t) - f(X_{t-}) - f'(X_{t-})\Delta X_t \text{ and } \Delta_+ K_t = f(X_{t+}) - f(X_t) - f'(X_t)\Delta_+ X_t.$$

Note that the process  $K$  in the above lemma is in general neither left-continuous nor right-continuous.

**Proof:** Our proof follows the proof of Theorem 66 of [30, Chapter IV] with suitable changes.

**Step 1** We assume that  $X$  is bounded; more precisely, we assume that there exists  $N \in \mathcal{N}$  such that  $|X| \leq N$ . We know (cf. [30]) that there exists a sequence  $(f_n)$  of twice continuously differentiable convex functions such that  $(f_n)$  converges to  $f$ , and  $(f'_n)$  converges to  $f'$  from below. By applying Gal'chouk-Lenglart's formula (cf. [12] and [25]) to  $f_n(X_t)$ , we obtain for all  $\tau \in \mathcal{T}_{0,T}$

$$f_n(X_\tau) = f_n(X_0) + \int_{]0,\tau[} f'_n(X_{s-})d(A_s + M_s) + \int_{[0,\tau[} f'_n(X_s)dB_{s+} + K_\tau^n, \text{ a.s., where } \quad (4.11)$$

$$\begin{aligned} K_\tau^n := & \sum_{0 < s \leq \tau} [f_n(X_s) - f_n(X_{s-}) - f'_n(X_{s-})\Delta X_s] + \sum_{0 \leq s < \tau} [f_n(X_{s+}) - f_n(X_s) - f'_n(X_s)\Delta_+ X_s] \\ & + \frac{1}{2} \int_{]0,\tau[} f''_n(X_{s-})d\langle M^c, M^c \rangle_s \text{ a.s.} \end{aligned} \quad (4.12)$$

We show that  $(K_\tau^n)$  is a convergent sequence by showing that the other terms in Equation (4.11) converge. The convergence  $\int_{]0,\tau[} f'_n(X_{s-})d(A_s + M_s) \xrightarrow{n \rightarrow \infty} \int_{]0,\tau[} f'(X_{s-})d(A_s + M_s)$  is shown by using the same arguments as in the proof of [30, Thorem 66, Ch. IV]. The convergence of the term  $\int_{[0,\tau[} f'_n(X_s)dB_{s+}$ , which is specific to the non-right-continuous case,

is shown by using dominated convergence. We conclude that  $(K_\tau^n)$  converges and we set  $K_\tau := \lim_{n \rightarrow \infty} K_\tau^n$ . The process  $(K_t)$  is adapted as the limit of adapted processes. Moreover, we have from Eq. (4.12) and from the convexity of  $f_n$  that, for each  $n$ ,  $K_t^n$  is nondecreasing in  $t$ . Hence, the limit  $K_t$  is nondecreasing.

**Step 2** We treat the general case where  $X$  is not necessarily bounded by using a localization argument similar to that used in [30, Th. 66, Ch. IV]. □

We make the following assumption on the driver (cf., e.g., Theorem 4.2 in [31]). The assumption will be used, in particular, in the proof of the comparison theorem for RBSDE.

**Assumption 4.1** *Assume that  $dP \otimes dt$ -a.e. for each  $(y, z, \kappa_1, \kappa_2) \in \mathbb{R}^2 \times (L_\nu^2)^2$ ,*

$$f(t, y, z, \kappa_1) - f(t, y, z, \kappa_2) \geq \langle \theta_t^{y, z, \kappa_1, \kappa_2}, \kappa_1 - \kappa_2 \rangle_\nu,$$

with

$$\theta : [0, T] \times \Omega \times \mathbb{R}^2 \times (L_\nu^2)^2 \rightarrow L_\nu^2; (\omega, t, y, z, \kappa_1, \kappa_2) \mapsto \theta_t^{y, z, \kappa_1, \kappa_2}(\omega, \cdot)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}((L_\nu^2)^2)$ -measurable, satisfying  $\|\theta_t^{y, z, \kappa_1, \kappa_2}(\cdot)\|_\nu \leq K$  for all  $(y, z, \kappa_1, \kappa_2) \in \mathbb{R}^2 \times (L_\nu^2)^2$ ,  $dP \otimes dt$ -a.e., where  $K$  is a positive constant, and such that

$$\theta_t^{y, z, \kappa_1, \kappa_2}(e) \geq -1, \tag{4.13}$$

for all  $(y, z, \kappa_1, \kappa_2) \in \mathbb{R}^2 \times (L_\nu^2)^2$ ,  $dP \otimes dt \otimes d\nu(e) - \text{a.e.}$

The above assumption is satisfied if, for example,  $f$  is of class  $\mathcal{C}^1$  with respect to  $k$  such that  $\nabla_k f$  is bounded (in  $L_\nu^2$ ) and  $\nabla_k f \geq -1$  (cf. Proposition A.2. in [7]).

**Theorem 4.2 (Comparison)** *Let  $\xi \in \mathcal{S}^2$ ,  $\xi' \in \mathcal{S}^2$  be two left-limited processes. Let  $f$  be a Lipschitz driver satisfying Assumption 4.1. Let  $(Y, Z, k, A, C)$  (resp.  $(Y', Z', k', A', C')$ ) be the solution of the RBSDE associated with obstacle  $\xi$  (resp.  $\xi'$ ) and with driver  $f$ . If  $\xi_t \leq \xi'_t$ ,  $0 \leq t \leq T$  a.s., then  $Y_t \leq Y'_t$ ,  $0 \leq t \leq T$  a.s.*

**Proof:** We set  $\bar{Y}_t = Y_t - Y'_t$ ,  $\bar{Z}_t = Z_t - Z'_t$ ,  $\bar{k}_t = k_t - k'_t$ ,  $\bar{A}_t = A_t - A'_t$ ,  $\bar{C}_t = C_t - C'_t$ , and  $\bar{f}_t = f(t, Y_{t-}, Z_t, k_t) - f(t, Y'_{t-}, Z'_t, k'_t)$ . Then,

$$-d\bar{Y}_t = \bar{f}_t dt + d\bar{A}_t + d\bar{C}_{t-} - \bar{Z}_t dW_t - \int_E \bar{k}_t(e) \tilde{N}(dt, de), \quad \bar{Y}_T = 0.$$

Applying Lemma 4.2 to the positive part of  $\bar{Y}_t$ , we obtain

$$\begin{aligned} \bar{Y}_t^+ &= - \int_{]t, T]} \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} \bar{Z}_s dW_s - \int_{]t, T]} \int_E \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} \bar{k}_s(e) \tilde{N}(ds, de) + \int_{]t, T]} \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} \bar{f}_s ds \\ &+ \int_{]t, T]} \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} d\bar{A}_s + \int_{]t, T]} \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} d\bar{C}_s + (K_t - K_T). \end{aligned} \tag{4.14}$$

We set  $\delta_t := \frac{f(t, Y_{t-}, Z_t, k_t) - f(t, Y'_{t-}, Z_t, k_t)}{Y_{t-} - Y'_{t-}} \mathbf{1}_{\{\bar{Y}_{t-} \neq 0\}}$  and  $\beta_t := \frac{f(t, Y'_{t-}, Z_t, k_t) - f(t, Y'_{t-}, Z'_t, k_t)}{Z_t - Z'_t} \mathbf{1}_{\{\bar{Z}_t \neq 0\}}$ . Due to the Lipschitz-continuity of  $f$ , the processes  $\delta$  and  $\beta$  are bounded. We note that  $\bar{f}_t = \delta_t \bar{Y}_t + \beta_t \bar{Z}_t + f(Y'_{t-}, Z'_t, k_t) - f(Y'_{t-}, Z_t, k_t)$ . Using this, together with Assumption 4.1, we obtain

$$\bar{f}_t \leq \delta_t \bar{Y}_t + \beta_t \bar{Z}_t + \langle \gamma_t, \bar{k}_t \rangle_\nu, \quad 0 \leq t \leq T, \quad dP \otimes dt - \text{a.e.}, \quad (4.15)$$

where we have set  $\gamma_t := \theta_t^{Y'_{t-}, Z'_t, k'_t, k_t}$ .

For  $t \in [0, T]$ , let  $\Gamma_{t,\cdot}$  be the unique solution of the following forward SDE

$$d\Gamma_{\tau,s} = \Gamma_{\tau,s-} \left[ \delta_s ds + \beta_s dW_s + \int_E \gamma_s(e) \tilde{N}(ds, de) \right]; \quad \Gamma_{\tau,\tau} = 1. \quad (4.16)$$

To simplify the notation, we denote  $\Gamma_{t,s}$  by  $\Gamma_s$  for  $s \geq t$ .

By applying Gal'chouk-Lenglart's formula to the product  $(\Gamma_t \bar{Y}_t^+)$  we get

$$\begin{aligned} \Gamma_\tau \bar{Y}_\tau^+ &= - \int_\tau^\theta \Gamma_{s-} (\mathbf{1}_{\{\bar{Y}_{s-} > 0\}} \bar{Z}_s + \bar{Y}_{s-}^+ \beta_s) dW_s - \int_\tau^\theta \Gamma_s (\bar{Y}_{s-}^+ \delta_s + \bar{Z}_s \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} \beta_s - \bar{f}_s \mathbf{1}_{\{\bar{Y}_{s-} > 0\}}) ds \\ &\quad + \int_\tau^\theta \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} d\bar{A}_s - \int_\tau^\theta \Gamma_{s-} dK_s^c - \int_\tau^\theta \Gamma_{s-} dK_s^{d,-} + \int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s > 0\}} d\bar{C}_s \\ &\quad - \int_\tau^\theta \Gamma_s dK_s^{d,+} - \int_\tau^\theta \int_E \Gamma_{s-} (\bar{k}_s(e) \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} + \bar{Y}_{s-}^+ \gamma_s(e)) \tilde{N}(ds, de) - \sum_{\tau \leq s \leq \theta} \Delta \Gamma_s \Delta \bar{Y}_s^+. \end{aligned} \quad (4.17)$$

We have  $\int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s > 0\}} d\bar{C}_s = \int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s > 0\}} dC_s - \int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s > 0\}} dC'_s$ . For the first term, it holds  $\int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s > 0\}} dC_s = 0$ . Indeed,  $\{\bar{Y}_s > 0\} = \{Y_s > Y'_s\} \subset \{Y_s > \xi_s\}$  (as  $Y'_s \geq \xi'_s \geq \xi_s$ ). This, together with the Skorokhod condition for  $C$  gives the equality. For the second term, it holds  $-\int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s > 0\}} dC'_s \leq 0$ , as  $\Gamma \geq 0$  and  $dC'$  is a nonnegative measure. Hence,  $\int_\tau^\theta \Gamma_s \mathbf{1}_{\{\bar{Y}_s > 0\}} d\bar{C}_s \leq 0$ . Similarly, we obtain  $\int_\tau^\theta \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} d\bar{A}_s \leq 0$ . We also have  $-\int_\tau^\theta \Gamma_{s-} dK_s^c \leq 0$  and  $-\int_\tau^\theta \Gamma_{s-} dK_s^{d,-} \leq 0$ . Hence,

$$\begin{aligned} \Gamma_\tau \bar{Y}_\tau^+ &\leq - \int_\tau^\theta \Gamma_{s-} (\mathbf{1}_{\{\bar{Y}_{s-} > 0\}} \bar{Z}_s + \bar{Y}_{s-}^+ \beta_s) dW_s - \int_\tau^\theta \Gamma_s (\bar{Y}_{s-}^+ \delta_s + \bar{Z}_s \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} \beta_s - \bar{f}_s \mathbf{1}_{\{\bar{Y}_{s-} > 0\}}) ds \\ &\quad - \int_\tau^\theta \Gamma_{s-} dK_s^{d,-} - \int_\tau^\theta \int_E \Gamma_{s-} (\bar{k}_s(e) \mathbf{1}_{\{\bar{Y}_{s-} > 0\}} + \bar{Y}_{s-}^+ \gamma_s(e)) \tilde{N}(ds, de) - \sum_{\tau \leq s \leq \theta} \Delta \Gamma_s \Delta \bar{Y}_s^+. \end{aligned} \quad (4.18)$$

We compute the last term  $\sum_{\tau \leq s \leq \theta} \Delta \Gamma_s \Delta \bar{Y}_s^+$ .

Let  $(p_s)$  be the point process associated with the Poisson random measure  $N$  (cf. [6, VIII Section 2. 67], or [19, Section III §d]).

We have  $\Delta\Gamma_s = \Gamma_{s-}\gamma_s(p_s)$  and  $\Delta\bar{Y}_s^+ = \mathbf{1}_{\{\bar{Y}_{s-}>0\}}\bar{k}_s(p_s) - \mathbf{1}_{\{\bar{Y}_{s-}>0\}}\Delta\bar{A}_s + \Delta K_s^{d,-}$ . Hence,

$$\begin{aligned}
\sum_{\tau \leq s \leq \theta} \Delta\Gamma_s \Delta\bar{Y}_s^+ &= \sum_{\tau \leq s \leq \theta} \left( \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \gamma_s(p_s) \bar{k}_s(p_s) + \Gamma_{s-} \gamma_s(p_s) \Delta K_s^{d,-} \right) \\
&= \int_{\tau}^{\theta} \int_{\mathbf{R}^*} \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \gamma_s(e) \bar{k}_s(e) N(ds, de) + \sum_{\tau \leq s \leq \theta} \Gamma_{s-} \gamma_s(p_s) \Delta K_s^{d,-} \\
&= \int_{\tau}^{\theta} \int_{\mathbf{R}^*} \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \gamma_s(e) \bar{k}_s(e) \tilde{N}(ds, de) + \int_{\tau}^{\theta} \Gamma_{s-} \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \langle \gamma_s, \bar{k}_s \rangle_{\nu} ds \\
&\quad + \sum_{\tau \leq s \leq \theta} \Gamma_{s-} \gamma_s(p_s) \Delta K_s^{d,-}.
\end{aligned} \tag{4.19}$$

By plugging this expression in equation (4.18) and by putting together the terms in "ds" and in " $dK_s^{d,-}$ ", we get

$$\begin{aligned}
\Gamma_{\tau} \bar{Y}_{\tau}^+ &\leq - \int_{\tau}^{\theta} \Gamma_{s-} (\mathbf{1}_{\{\bar{Y}_{s-}>0\}} \bar{Z}_s + \bar{Y}_{s-}^+ \beta_s) dW_s \\
&\quad - \int_{\tau}^{\theta} \Gamma_{s-} (\bar{Y}_{s-}^+ \delta_s + \bar{Z}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \beta_s + \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \langle \gamma_s, \bar{k}_s \rangle_{\nu} - \bar{f}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}}) ds \\
&\quad - \sum_{\tau \leq s \leq \theta} \Gamma_{s-} (1 + \gamma_s(p_s)) \Delta K_s^{d,-} \\
&\quad - \int_{\tau}^{\theta} \int_{\mathbf{R}^*} \Gamma_{s-} (\bar{k}_s(e) \mathbf{1}_{\{\bar{Y}_{s-}>0\}} + \bar{Y}_{s-}^+ \gamma_s(e) + \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \gamma_s(e) \bar{k}_s(e)) \tilde{N}(ds, de).
\end{aligned} \tag{4.20}$$

We have  $-\int_{\tau}^{\theta} \Gamma_{s-} (\bar{Y}_{s-}^+ \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \delta_s + \bar{Z}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \beta_s + \mathbf{1}_{\{\bar{Y}_{s-}>0\}} \langle \gamma_s, \bar{k}_s \rangle_{\nu} - \bar{f}_s \mathbf{1}_{\{\bar{Y}_{s-}>0\}}) ds \leq 0$  due to the inequality 4.15. The term  $-\sum_{\tau \leq s \leq \theta} \Gamma_{s-} (1 + \gamma_s(p_s)) \Delta K_s^{d,-}$  is nonpositive, as  $1 + \gamma_s \geq 0$  by Assumption 4.1. The stochastic integrals "with respect to  $dW_s$ " and "with respect to  $\tilde{N}(ds, de)$ " are equal to zero in expectation. We conclude that  $E[\Gamma_{\tau} \bar{Y}_{\tau}^+] \leq 0$ , which (as  $\Gamma_{\tau} = 1$ ) implies  $\bar{Y}_{\tau}^+ = 0$  a.s. The proof is thus complete.  $\square$

*Remark 4.3* Note that due to the irregularity of the obstacles, together with the presence of jumps, we cannot adopt the approaches used up to now in the literature (see e.g. [9], [3], [32] and [13]) to show the comparison theorem for our RBSDE (cf. also Remark 5.5 for additional comments).

**Definition 4.4** Let  $f$  be a Lipschitz driver. For a left-limited process  $(\phi_t) \in \mathcal{S}^2$ , we denote by  $\mathcal{R}ef^f[\phi]$  the first component of the solution to the Reflected BSDE with (lower) barrier  $\phi$  and with Lipschitz driver  $f$ .

The operator  $\mathcal{R}ef^f[\cdot]$  is well-defined due to Theorem 4.3 and to Remark 2.1. Moreover,  $\mathcal{R}ef^f[\cdot]$  is valued in  $\mathcal{S}^{2,rusc}$ , where  $\mathcal{S}^{2,rusc} := \{\phi \in \mathcal{S}^2 : \phi \text{ is right-uppersemicontinuous (r.u.s.c.)}\}$ ; this is due to Equation (2.2). We give some properties of the operator  $\mathcal{R}ef^f$  in the following proposition. Note that equalities (resp. inequalities) between processes are to be understood in the "up to indistinguishability"-sense.

**Lemma 4.5** *Let  $f$  be a Lipschitz driver satisfying Assumption 4.1. The operator  $\mathcal{R}ef^f : S^2 \rightarrow S^{2,rusc}$ , defined in Definition 4.4, has the following properties:*

1. *The operator  $\mathcal{R}ef^f$  is nondecreasing, that is, for  $\xi, \xi' \in S^2$  such that  $\xi \leq \xi'$  we have  $\mathcal{R}ef^f[\xi] \leq \mathcal{R}ef^f[\xi']$ .*
2. *If  $\xi \in S^2$  is a (r.u.s.c.) strong  $\mathcal{E}^f$ -supermartingale, then  $\mathcal{R}ef^f[\xi] = \xi$ .*
3. *For each  $\xi \in S^2$ ,  $\mathcal{R}ef^f[\xi]$  is a strong  $\mathcal{E}^f$ -supermartingale and satisfies  $\mathcal{R}ef^f[\xi] \geq \xi$ .*

**Remark 4.3** *We recall that a strong supermartingale in  $\mathcal{S}^2$  is necessarily r.u.s.c. (cf., e.g., Remark A.16 in [13]).*

**Proof:** The first assertion follows from our comparison theorem for reflected BSDEs with irregular obstacles (Theorem 4.2).

Let us prove the second assertion. Let  $\xi$  be a (r.u.s.c.) strong  $\mathcal{E}^f$ -supermartingale in  $\mathcal{S}^2$ . By definition of  $\mathcal{R}ef^f$ , we have to show that  $\xi$  is the solution of the reflected BSDE associated with driver  $f$  and obstacle  $\xi$ . By the  $\mathcal{E}^f$ -Mertens decomposition for strong (r.u.s.c.)  $\mathcal{E}^f$ -supermartingales shown in Theorem 5.2 in [13] (cf. also [2]), together with the martingale representation theorem, there exists  $(Z, k, A, C) \in \mathbb{H}^2 \times \mathbb{H}_\nu^2 \times \mathcal{S}^2 \times \mathcal{S}^2$  such that a.s. for all  $t \in [0, T]$ ,

$$\xi_t = \xi_T + \int_t^T f(s, Y_s, Z_s, k_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathbf{E}} k_s(e) \tilde{N}(ds, de) + A_T - A_t + C_{T-} - C_{t-},$$

where  $A$  is predictable right-continuous nondecreasing with  $A_0 = 0$ , and  $C$  is adapted right-continuous nondecreasing and purely discontinuous, with  $C_{0-} = 0$ . Moreover, the Skorokhod conditions (for RBSDEs) are here trivially satisfied. Hence,  $\xi = \mathcal{R}ef^f[\xi]$ , which is the desired conclusion.

The third assertion follows directly from the definition of the solution of a reflected BSDE and from Proposition A.4 in [13].  $\square$

In the following theorem, we characterize the first component of the solution of the RBSDE with irregular obstacle  $\xi$  in terms of the smallest strong  $f$ -supermartingale greater than or equal to  $\xi$ . In the case of a right-continuous obstacle  $\xi$  this characterization has been established in [32]; it has been generalized to the case of a right-upper-semicontinuous obstacle in [13, Prop. 4.4].

**Theorem 4.4** *Let  $(\xi_t, 0 \leq t \leq T)$  be a left-limited process in  $\mathcal{S}^2$  and let  $f$  be a Lipschitz driver satisfying Assumption 4.1. Let  $(Y, Z, k, A, C)$  be the solution to the reflected BSDE with parameters  $(\xi, f)$ . The first component  $Y$  of the solution is the  $\mathcal{E}^f$ -Snell envelope of  $\xi$ , that is, the smallest strong  $\mathcal{E}^f$ -supermartingale greater than or equal to  $\xi$ .*

**Proof:** By the previous Lemma 4.5, third assertion, the process  $Y$  is a strong  $\mathcal{E}^f$ -supermartingale satisfying  $Y \geq \xi$ . It remains to show the minimality property. Let  $Y'$  be a strong  $\mathcal{E}^f$ -supermartingale such that  $Y' \geq \xi$ . We have  $\mathcal{R}ef^f[Y'] \geq \mathcal{R}ef^f[\xi]$ , due to the nondecreasingness of the operator  $\mathcal{R}ef^f$  (cf. Lemma 4.5, 1st assertion). On the other hand,  $\mathcal{R}ef^f[Y'] = Y'$  (due to Lemma 4.5, 2nd assertion) and  $\mathcal{R}ef^f[\xi] = Y$  (by definition of  $Y$  and of the operator  $\mathcal{R}ef^f$ ). Hence,  $Y' \geq Y$ , which is the desired conclusion.  $\square$

## 5 Optimal stopping with non-linear expectation and irregular pay-off

Let  $(\xi_t)_{t \in [0, T]}$  be a left-limited process in  $\mathcal{S}^2$ . Let  $f$  be a Lipschitz driver satisfying Assumption 4.1.

For each  $S \in \mathcal{T}_{0, T}$ , we consider the random variable

$$V(S) := \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{S, T}} \mathcal{E}_{S, \tau}^f(\xi_\tau). \quad (5.21)$$

As mentioned in the introduction, the above optimal stopping problem has been studied in [10] in the case of a continuous pay-off process  $\xi$  and a Brownian filtration, then in [32] in the case of a right-continuous pay-off  $\xi$ , and in [13] in the case of a pay-off process which is only right-upper-semicontinuous. Here, we do not make any regularity assumptions on  $\xi$  (cf. also Remark 2.1).

Under Assumption 4.1 on the driver  $f$ , the functional  $\mathcal{E}_{S, \tau}^f(\cdot)$  is nondecreasing (cf. [31, Thm. 4.2]). If we interpret  $\xi_\tau$  as the profit and loss of a financial position at time  $\tau$  and the functional  $\mathcal{E}_{S, \tau}^f(\cdot)$  as a (nondecreasing) risk measure (cf., e.g., [29], [33]), then (up to a minus sign)  $V(S)$  can be seen as the minimal risk at time  $S$ . The fact that we do not impose any regularity assumption on the process  $\xi$  allows for more flexibility in the modelling (compared to "the more regular cases"). In particular, our general framework allows for situations where economic and/or financial shocks affect the financial position  $\xi$  immediately after their occurrence resulting in a (positive or negative) right-hand jump of  $\xi$ . If, for instance, we place ourselves in a situation where the jump times of the Poisson random measure model times of default (which, being totally inaccessible, cannot be foreseen), it might be plausible to allow for an *immediate*, but *non-smooth*, impact on  $\xi$  *after* the default occurs.

### 5.1 Preliminary results on the value family

Let us first introduce the definition of an admissible family of random variables indexed by stopping times in  $\mathcal{T}_{0, T}$  (or  $\mathcal{T}_{0, T}$ -system in the vocabulary of Dellacherie and Lenglart [4]).

**Definition 5.1** We say that a family  $U = (U(\tau), \tau \in \mathcal{T}_{0,T})$  is admissible if it satisfies the following conditions

1. for all  $\tau \in \mathcal{T}_{0,T}$ ,  $U(\tau)$  is a real-valued  $\mathcal{F}_\tau$ -measurable random variable.
2. for all  $\tau, \tau' \in \mathcal{T}_{0,T}$ ,  $U(\tau) = U(\tau')$  a.s. on  $\{\tau = \tau'\}$ .

Moreover, we say that an admissible family  $U$  is square-integrable if for all  $\tau \in \mathcal{T}_{0,T}$ ,  $U(\tau)$  is square-integrable.

**Lemma 5.3 (Admissibility of the family  $V$ )** The family  $V = (V(S), S \in \mathcal{T}_{0,T})$  defined in (5.21) is a square-integrable admissible family.

The proof uses arguments similar to those used in the "classical" case of linear expectations (cf., e.g., [24]), combined with some properties of  $f$ -expectations.

**Proof:** For each  $S \in \mathcal{T}_{0,T}$ ,  $V(S)$  is an  $\mathcal{F}_S$ -measurable square-integrable random variable, due to the definitions of the  $f$ -conditional expectation and of the essential supremum (cf. [27]). Let us prove Property 2 of the definition of admissibility. Let  $S$  and  $S'$  be two stopping times in  $\mathcal{T}_{0,T}$ . We set  $A := \{S = S'\}$  and we show that  $V(S) = V(S')$ ,  $P$ -a.s. on  $A$ . For each  $\tau \in \mathcal{T}_{S,T}$ , we set  $\tau_A := \tau \mathbf{1}_A + T \mathbf{1}_{A^c}$ . We have  $\tau_A \geq S'$  a.s. By using the fact that  $S = S'$  a.s. on  $A$ , the fact that  $\tau_A = \tau$  a.s. on  $A$ , and a standard property of  $f$ -conditional expectations (cf., e.g., Proposition A.3 in [15]), we obtain

$$\mathbf{1}_A \mathcal{E}_{S,\tau}^g[\xi_\tau] = \mathbf{1}_A \mathcal{E}_{S',\tau}^g[\xi_\tau] = \mathcal{E}_{S',T}^{g^{\tau \mathbf{1}_A}}[\xi_\tau \mathbf{1}_A] = \mathcal{E}_{S',T}^{g^{\tau_A \mathbf{1}_A}}[\xi_{\tau_A} \mathbf{1}_A] = \mathbf{1}_A \mathcal{E}_{S',\tau_A}^g[\xi_{\tau_A}] \leq \mathbf{1}_A V(S').$$

By taking the esssup over  $\mathcal{T}_{S,T}$  on both sides, we get  $\mathbf{1}_A V(S) \leq \mathbf{1}_A V(S')$ . We obtain the converse inequality by interchanging the roles of  $S$  and  $S'$ . □

**Lemma 5.4 (Optimizing sequence)** For each  $S \in \mathcal{T}_{0,T}$ , there exists a sequence  $(\tau_n)_{n \in \mathbb{N}}$  of stopping times in  $\mathcal{T}_{S,T}$  such that the sequence  $(\mathcal{E}_{S,\tau_n}^f(\xi_{\tau_n}))_{n \in \mathbb{N}}$  is nondecreasing and

$$V(S) = \lim_{n \rightarrow \infty} \uparrow \mathcal{E}_{S,\tau_n}^f(\xi_{\tau_n}) \quad \text{a.s.}$$

**Proof:** Due to a classical result on essential suprema (cf. [27]), it is sufficient to show that, for each  $S \in \mathcal{T}_{0,T}$ , the family  $(\mathcal{E}_{S,\tau}(\xi_\tau), \tau \in \mathcal{T}_{S,T})$  is stable under pairwise maximization. Let us fix  $S \in \mathcal{T}_{0,T}$ . Let  $\tau \in \mathcal{T}_{S,T}$  and  $\tau' \in \mathcal{T}_{S,T}$ . We define  $A := \{\mathcal{E}_{S,\tau'}^f(\xi_{\tau'}) \leq \mathcal{E}_{S,\tau}^f(\xi_\tau)\}$ . The set  $A$  is in  $\mathcal{F}_S$ . We define  $\nu := \tau \mathbf{1}_A + \tau' \mathbf{1}_{A^c}$ . We have  $\nu \in \mathcal{T}_{S,T}$ . We compute  $\mathbf{1}_A \mathcal{E}_{S,\nu}^f(\xi_\nu) = \mathcal{E}_{S,T}^{f^\nu \mathbf{1}_A}(\xi_\nu \mathbf{1}_A) = \mathcal{E}_{S,T}^{f^\tau \mathbf{1}_A}(\xi_\tau \mathbf{1}_A) = \mathbf{1}_A \mathcal{E}_{S,\tau}^f(\xi_\tau)$  a.s. Similarly, we show  $\mathbf{1}_{A^c} \mathcal{E}_{S,\nu}^f(\xi_\nu) = \mathbf{1}_{A^c} \mathcal{E}_{S,\tau'}^f(\xi_{\tau'})$ . It follows that  $\mathcal{E}_{S,\nu}^f(\xi_\nu) = \mathcal{E}_{S,\tau}^f(\xi_\tau) \mathbf{1}_A + \mathcal{E}_{S,\tau'}^f(\xi_{\tau'}) \mathbf{1}_{A^c} = \mathcal{E}_{S,\tau}^f(\xi_\tau) \vee \mathcal{E}_{S,\tau'}^f(\xi_{\tau'})$ , which shows the stability under pairwise maximization and concludes the proof. □

We need two more definitions.



**Definition 5.2 ( $\mathcal{E}^f$ -supermartingale family)** An admissible square-integrable family  $U := (U(S), S \in \mathcal{T}_{0,T})$  is said to be a strong  $\mathcal{E}^f$ -supermartingale family if for all  $S, S' \in \mathcal{T}_{0,T}$  such that  $S \leq S'$  a.s.,

$$\mathcal{E}_{S,S'}^f(U(S')) \leq U(S) \quad \text{a.s.}$$

**Definition 5.3 (Right-uppersemicontinuous family)** An admissible family  $U := (U(S), S \in \mathcal{T}_{0,T})$  is said to be a right-uppersemicontinuous (along stopping times) family if, for all  $(\tau_n)$  nonincreasing sequence in  $\mathcal{T}_{0,T}$ ,  $U(\tau) \geq \limsup_{n \rightarrow \infty} U(\tau_n)$  a.s. on  $\{\tau = \lim \downarrow \tau_n\}$ .

The following lemma gives a link between the previous two notions.

**Lemma 5.5** Let  $U := (U(S), S \in \mathcal{T}_{0,T})$  be a strong  $\mathcal{E}^f$ -supermartingale family. Then,  $(U(S), S \in \mathcal{T}_{0,T})$  is a right-uppersemicontinuous (along stopping times) family in the sense of Definition 5.3.

**Proof:** Let  $(\tau_n)$  and  $\tau$  be as in the above Definition 5.3. As  $U$  is a strong  $\mathcal{E}^f$ -supermartingale family and as the sequence  $(\tau_n)$  is nonincreasing, we have  $\mathcal{E}_{\tau,\tau_n}^f(U(\tau_n)) \leq \mathcal{E}_{\tau,\tau_{n+1}}^f(U(\tau_{n+1})) \leq U(\tau)$  a.s. Hence, the sequence  $(\mathcal{E}_{\tau,\tau_n}^f(U(\tau_n)))_n$  is nondecreasing and  $U(\tau) \geq \lim \uparrow \mathcal{E}_{\tau,\tau_n}^f(U(\tau_n))$ . This inequality, combined with the property of continuity of BSDEs with respect to terminal time and terminal condition (cf. [31, Prop. A.6]) gives

$$U(\tau) \geq \lim_{n \rightarrow +\infty} \mathcal{E}_{\tau,\tau_n}^f(U(\tau_n)) = \mathcal{E}_{\tau,\tau}^f(\lim_{n \rightarrow +\infty} U(\tau_n)) = \lim_{n \rightarrow +\infty} U(\tau_n) \quad \text{a.s.}$$

We conclude that the family  $(U(S))$  is a right-uppersemicontinuous (along stopping times) family in the sense of Definition 5.3.  $\square$

**Theorem 5.5** The value family  $V = (V(S), S \in \mathcal{T}_{0,T})$  defined in (5.21) is a strong  $\mathcal{E}^f$ -supermartingale family. In particular,  $V = (V(S), S \in \mathcal{T}_{0,T})$  is a right-uppersemicontinuous (along stopping times) family in the sense of Definition 5.3.

**Proof:** We know from Lemma 5.3 that  $V = (V(S), S \in \mathcal{T}_{0,T})$  is a square-integrable admissible family. Let  $S \in \mathcal{T}_{0,T}$  and  $S' \in \mathcal{T}_{S,T}$ . We will show that  $\mathcal{E}_{S,S'}^f(V(S')) \leq V(S)$  a.s., which will prove that  $V$  is a strong  $\mathcal{E}^f$ -supermartingale family. By Lemma 5.4, there exists a sequence  $(\tau_n)_{n \in \mathbb{N}}$  of stopping times such that  $\tau_n \geq S'$  a.s. and  $V(S') = \lim_{n \rightarrow \infty} \uparrow \mathcal{E}_{S',\tau_n}^f(\xi_{\tau_n})$  a.s. By using this equality, the property of continuity of BSDEs, and the consistency of the  $f$ -conditional expectation, we get

$$\mathcal{E}_{S,S'}^f(V(S')) = \mathcal{E}_{S,S'}^f(\lim_{n \rightarrow \infty} \uparrow \mathcal{E}_{S',\tau_n}^f(\xi_{\tau_n})) = \lim_{n \rightarrow \infty} \mathcal{E}_{S,S'}^f(\mathcal{E}_{S',\tau_n}^f(\xi_{\tau_n})) = \lim_{n \rightarrow \infty} \mathcal{E}_{S,\tau_n}^f(\xi_{\tau_n}) \leq V(S).$$

We conclude that  $V$  is a strong  $\mathcal{E}^f$ -supermartingale family. This property, together with Lemma 5.5, gives the property of right-uppersemicontinuity (along stopping times) of the family  $V$ . The proof is thus completed.  $\square$

## 5.2 Aggregation and Snell characterization

**Definition 5.4** *Let  $\phi$  a process in  $\mathcal{S}^2$ . Let  $f$  be a Lipschitz driver. The process  $\phi$  is said to be a strong  $\mathcal{E}^f$ -supermartingale, if  $\mathcal{E}_{\sigma,\tau}^f(\phi_\tau) \leq \phi_\sigma$  a.s. on  $\sigma \leq \tau$ , for all  $\sigma, \tau \in \mathcal{T}_{0,T}$ .*

The notion of a strong  $\mathcal{E}^f$ -martingale is defined similarly.

We now show the following result, which generalizes some results of classical optimal stopping theory (more precisely, the assertion (i) from Lemma 3.1) to the case of an optimal stopping problem with  $f$ -expectation.

**Theorem 5.6 (Aggregation and Snell characterization)** *There exists a unique right-uppersemicontinuous optional process, denoted by  $(V_t)_{t \in [0,T]}$ , which aggregates the value family  $V = (V(S), S \in \mathcal{T}_{0,T})$ . Moreover,  $(V_t)_{t \in [0,T]}$  is the  $\mathcal{E}^f$ -Snell envelope of the pay-off process  $\xi$ , that is, the smallest strong  $\mathcal{E}^f$ -supermartingale greater than or equal to  $\xi$ .*

The proof of this theorem relies on the preliminary results on the value family  $V = (V(S), S \in \mathcal{T}_{0,T})$  presented in the previous subsection.

**Proof:** By Theorem 5.5, the value family  $V = (V(S), S \in \mathcal{T}_{0,T})$  is a right-uppersemicontinuous family (or a right-uppersemicontinuous  $\mathcal{T}_{0,T}$ -system in the vocabulary of Dellacherie-Lenglart [4]). Applying Theorem 4 of Dellacherie-Lenglart ([4]), gives the existence of a unique (up to indistinguishability) right-uppersemicontinuous optional process  $(V_t)_{t \in [0,T]}$  which *aggregates* the value family  $(V(S), S \in \mathcal{T}_{0,T})$ . From this aggregation property, namely the property  $V_S = V(S)$  a.s. for each  $S \in \mathcal{T}_{0,T}$ , and from Theorem 5.5, we deduce that the process  $(V_t)_{t \in [0,T]}$  is a strong  $\mathcal{E}^f$ -supermartingale. Moreover,  $V_t \geq \xi_t$ , for all  $t \in [0, T]$ , a.s. Indeed, due to the definition of the family  $(V(S), S \in \mathcal{T}_{0,T})$  and to the aggregation result, we have  $V_S \geq \xi_S$  a.s. for each  $S \in \mathcal{T}_{0,T}$ . We deduce that  $V_t \geq \xi_t$ , for all  $t \in [0, T]$ , a.s., by applying a well-known result from the general theory of processes (cf. ([5, Theorem IV.84])

Let us now prove that the process  $(V_t)_{t \in [0,T]}$  is *the smallest* strong  $\mathcal{E}^f$ -supermartingale greater than or equal to  $\xi$ . Let  $(V'_t)_{t \in [0,T]}$  be a strong  $\mathcal{E}^f$ -supermartingale such that  $V'_t \geq \xi_t$ , for all  $t \in [0, T]$ , a.s. Let  $S \in \mathcal{T}_{0,T}$ . We have  $V'_\tau \geq \xi_\tau$  a.s. for all  $\tau \in \mathcal{T}_{S,T}$ . Hence,  $\mathcal{E}_{S,\tau}^f(V'_\tau) \geq \mathcal{E}_{S,\tau}^f(\xi_\tau)$  a.s., where we have used the monotonicity of the  $f$ -conditional expectation. On the other hand, by using the  $\mathcal{E}^f$ -supermartingale property of the process  $(V'_t)_{t \in [0,T]}$ , we have  $V'_S \geq \mathcal{E}_{S,\tau}^f(V'_\tau)$  a.s. for all  $\tau \in \mathcal{T}_{S,T}$ . Hence,  $V'_S \geq \mathcal{E}_{S,\tau}^f(\xi_\tau)$  a.s. for all  $\tau \in \mathcal{T}_{S,T}$ . By taking the essential supremum over  $\tau \in \mathcal{T}_{S,T}$  in the inequality, we get  $V'_S \geq \text{ess sup}_{\tau \in \mathcal{T}_{S,T}} \mathcal{E}_{S,\tau}^f(\xi_\tau) = V_S$  a.s. Note that the last equality in the above computation is due to the definition of  $V(S)$  and to the aggregation result. We have thus obtained  $V'_S \geq V_S$  a.s., which (as  $S$  is arbitrary in  $\mathcal{T}_{0,T}$ ) leads to  $V'_t \geq V_t$ , for all  $t \in [0, T]$ , a.s., due to the same well-known result from the general theory of processes as above.  $\square$

### 5.3 Characterization in terms of an RBSDE

The following theorem is a direct consequence of Theorem 4.4 and Theorem 5.6. It gives "an infinitesimal characterization" of the value process  $(V_t)_{t \in [0, T]}$ .

**Theorem 5.7 (Characterization in terms of an RBSDE)** *The value process  $(V_t)_{t \in [0, T]}$  aggregating the family  $V = (V(S), S \in \mathcal{T}_{0, T})$  defined by (5.21) coincides (up to indistinguishability) with the first component  $(Y_t)_{t \in [0, T]}$  of the solution of our RBSDE with driver  $f$  and obstacle  $\xi$ . In other words, we have, for all  $S \in \mathcal{T}_{0, T}$ ,*

$$Y_S = V_S = \text{ess sup}_{\tau \in \mathcal{T}_{S, T}} \mathcal{E}_{S, \tau}^f(\xi_\tau) \text{ a.s.} \quad (5.22)$$

By using this theorem, we derive the following corollary, which generalizes some results of classical optimal stopping theory (more precisely, the assertions (ii) and (iii) from Lemma 3.1) to the case of an optimal stopping problem with (non-linear)  $f$ -expectation.

**Corollary 5.1** *The value process of our optimal stopping problem (5.22), which is equal to the first component  $(Y_t)$  of the solution of our RBSDE, satisfies the following properties:*

(i) *For each  $S \in \mathcal{T}_{0, T}$ , we have:*

$$Y_S = \xi_S \vee Y_{S+} \quad \text{a.s.}$$

(ii) *For each  $S \in \mathcal{T}_{0, T}$  and for each  $\lambda \in ]0, 1[$ , we set*

$$\tau_S^\lambda := \inf\{t \geq S, \lambda Y_t(\omega) \leq \xi_t\}. \quad (5.23)$$

*The value process  $(Y_t)$  is an  $\mathcal{E}^f$ -martingale on  $[S, \tau_S^\lambda]$ .*

**Proof:** Let  $(Y, Z, k, A, C)$  be the solution to the RBSDE from Definition 2.2 associated with the obstacle  $(\xi_t)$  and the driver  $f$ . We note that  $(Y, Z, k, A, C)$  is also the solution of the RBSDE from Definition 2.2 associated with the obstacle  $(\xi_t)$  and the driver process  $g_t(\omega) := f(t, \omega, Y_t(\omega), Z_t(\omega), k_t(\omega))$ . From this observation and Lemma 3.1 (ii), we deduce the first assertion. Let us show the second one. By Theorem 3.2, we derive that  $(Y_t)$  is equal to the value process of the classical optimal stopping problem (3.6) associated with the instantaneous reward process  $(g_t)$ . By applying the assertion (iii) from Lemma 3.1, the process  $(Y_t + \int_0^t g_u du)_{t \in [0, T]}$  is thus a martingale on  $[S, \tau_S^\lambda]$ . Since  $A$  and  $C$  are equal to the non decreasing processes of the Mertens decomposition of the strong supermartingale  $(Y_t + \int_0^t g_u du)_{t \in [0, T]}$ , we derive that  $A_S = A_{\tau_S^\lambda}$  a.s. and  $C_{S-} = C_{(\tau_S^\lambda)-}$  a.s. Hence,  $Y$  is the solution on  $[S, \tau_S^\lambda]$  of the BSDE associated with driver  $f$ , terminal time  $\tau_S^\lambda$  and terminal condition  $Y_{\tau_S^\lambda}$ . The process  $(Y_t)$  is thus an  $\mathcal{E}^f$ -martingale on  $[S, \tau_S^\lambda]$ , which completes the proof.  $\square$

**Remark 5.4** *Alternatively, we can show the assertion (ii) from the above corollary by using arguments similar to those used in the proof of the assertion (ii) from Lemma 4.1 in [13].*

**Corollary 5.2** *If we assume that the process  $(\xi_t)$  is right-uppersemicontinuous (r.u.s.c.), then the value process  $Y$  of the optimal stopping problem (5.22) satisfies the following property: for each  $S \in \mathcal{T}_{0,T}$  and for each  $\lambda \in ]0, 1[$ ,*

$$\lambda Y_{\tau_S^\lambda} \leq \xi_{\tau_S^\lambda} \quad \text{a.s.}, \quad (5.24)$$

where  $\tau_S^\lambda$  is defined by (5.23). Moreover, the stopping time  $\tau_S^\lambda$  satisfies

$$Y_S \leq \mathcal{E}_{S, \tau_S^\lambda}^f(\xi_{\tau_S^\lambda}) + \varepsilon_S(\lambda) \quad \text{a.s.}, \quad (5.25)$$

where  $\lim_{\lambda \rightarrow 1} \varepsilon_S(\lambda) = 0$  a.s. In other words,  $\tau_S^\lambda$  is an  $\varepsilon_S(\lambda)$ -optimal stopping time for problem (5.22).

**Proof:** By using similar arguments to those used in the proof of the assertion (i) from Lemma 4.1 in [13], we show that the inequality (5.24) holds. Let us now show the inequality (5.25). The arguments are classical. Since by Corollary 5.1 (ii), the value process  $(Y_t)$  is an  $\mathcal{E}^f$ -martingale on  $[S, \tau_S^\lambda]$ , we get  $Y_S = \mathcal{E}_{S, \tau_S^\lambda}^f(Y_{\tau_S^\lambda})$  a.s. By the inequality (5.24), together with the monotonicity property of the  $f$ -conditional expectation and the *a priori estimates* for BSDEs (cf. [31]), we derive that

$$Y_S = \mathcal{E}_{S, \tau_S^\lambda}^f(Y_{\tau_S^\lambda}) \leq \mathcal{E}_{S, \tau_S^\lambda}^f\left(\frac{\xi_{\tau_S^\lambda}}{\lambda}\right) \leq \mathcal{E}_{S, \tau_S^\lambda}^f(\xi_{\tau_S^\lambda}) + \left(\frac{1}{\lambda} - 1\right) \alpha_S \quad \text{a.s.},$$

with  $\alpha_S := CE[\text{ess sup}_{\tau \in \mathcal{T}_{S,T}} \xi_\tau^2 | \mathcal{F}_S]^{\frac{1}{2}}$ , where  $C$  is a positive constant which depends only on  $T$  and the Lipschitz constant  $K$  of the driver  $f$ . We thus obtain the desired result with  $\varepsilon_S(\lambda) := \left(\frac{1}{\lambda} - 1\right) \alpha_S$ , which ends the proof.  $\square$

*Remark 5.5* *In the general case where the process  $(\xi_t)$  is not r.u.s.c., the inequality (5.24) (that is, the inequality  $\lambda Y_{\tau_S^\lambda} \leq \xi_{\tau_S^\lambda}$ ) does not necessarily hold (cf., e.g., the classical optimal stopping problem in [8]).*

*Let us emphasize that this introduces some important technical difficulties in the treatment of the irregular case with respect to the "more regular" cases.*

*In particular, this lack of regularity prevents us from adopting here the approach used in [13] (in the r.u.s.c. case) to prove the infinitesimal characterization of the value process of the (non-linear) optimal stopping problem in terms of the solution of an RBSDE. In the general framework of the present paper, we are thus led to using a different approach: we first establish a comparison theorem for RBSDEs, which we then use to prove the infinitesimal characterization (whereas in [13], the characterization is shown directly by using inequality (5.24), and the comparison theorem is deduced as an almost immediate corollary).*

*Also, as the inequality (5.24) does not necessarily hold in our case, we cannot adapt the approach used in [3] (in the case of right-continuous obstacles) to prove our comparison theorem, and we use a different approach.*

## 6 Appendix

The infinitesimal characterization of the previous section (Theorem 5.7) also allows us to obtain a priori estimates with universal constants for RBSDEs with irregular obstacles.

**Proposition 6.1 (A priori estimates for RBSDEs)** *Let  $\xi^1, \xi^2$  be two left-limited processes in  $\mathcal{S}^2$ . Let  $f^1, f^2$  be Lipschitz drivers satisfying Assumption 4.1 with common Lipschitz constant  $K > 0$ . For  $i = 1, 2$ , let  $(Y^i, Z^i, k^i)$  be the three first components of the solution of the RBSDE associated with driver  $f^i$  and obstacle  $\xi^i$ .*

*Let  $\bar{Y} := Y^1 - Y^2$ ,  $\bar{\xi} := \xi^1 - \xi^2$ . Let  $\eta, \beta > 0$  with  $\beta \geq \frac{3}{\eta} + 2K$  and  $\eta \leq \frac{1}{K^2}$ . Let  $\delta f_s = f^2(s, Y_s^2, Z_s^2, k_s^2) - f^1(s, Y_s^2, Z_s^2, k_s^2)$ . For each  $S \in \mathcal{T}_{0,T}$ , we have*

$$\bar{Y}_S^2 \leq e^{\beta(T-S)} E[\text{ess sup}_{\tau \in \mathcal{T}_{S,T}} \bar{\xi}_\tau^2 | \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)} (\delta f_s)^2 ds | \mathcal{F}_S] \quad \text{a.s.} \quad (6.26)$$

**Proof:** The proof is divided into two steps.

**Step 1:** For  $i = 1, 2$  and for each  $\tau \in \mathcal{T}_0$ , let  $(X^{i,\tau}, \pi^{i,\tau}, l^{i,\tau})$  be the solution of the BSDE associated with driver  $f^i$ , terminal time  $\tau$  and terminal condition  $\xi_\tau$ . Set  $\bar{X}^\tau := X^{1,\tau} - X^{2,\tau}$ . By an estimate on BSDEs (see Proposition A.4 in [31]), we have

$$(\bar{X}_S^\tau)^2 \leq e^{\beta(T-S)} E[\bar{\xi}^2 | \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)} [(f^1 - f^2)(s, X_s^{2,\tau}, \pi_s^{2,\tau}, l_s^{2,\tau})]^2 ds | \mathcal{F}_S] \quad \text{a.s.}$$

from which we derive that

$$(\bar{X}_S^\tau)^2 \leq e^{\beta(T-S)} E[\text{ess sup}_{\tau \in \mathcal{T}_{S,T}} \bar{\xi}_\tau^2 | \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)} (\bar{f}_s)^2 ds | \mathcal{F}_S] \quad \text{a.s.} \quad (6.27)$$

where  $\bar{f}_s := \sup_{y,z,k} |f^1(s, y, z, k) - f^2(s, y, z, k)|$ . Now, by Theorem 5.7, we have  $Y_S^i = \text{ess sup}_{\tau \in \mathcal{T}_{S,T}} X_S^{i,\tau}$  a.s. for  $i = 1, 2$ . We thus get  $|\bar{Y}_S| \leq \text{ess sup}_{\tau \in \mathcal{T}_{S,T}} |\bar{X}_S^\tau|$  a.s. By the inequality (6.27), we derive that

$$\bar{Y}_S^2 \leq e^{\beta(T-S)} E[\text{ess sup}_{\tau \in \mathcal{T}_{S,T}} \bar{\xi}_\tau^2 | \mathcal{F}_S] + \eta E[\int_S^T e^{\beta(s-S)} (\bar{f}_s)^2 ds | \mathcal{F}_S] \quad \text{a.s.}$$

**Step 2:** Note that  $(Y^2, Z^2, k^2)$  is the solution the RBSDE associated with obstacle  $\xi^2$  and driver  $f^1(t, y, z, k) + \delta f_t$ . By applying the result of Step 1 to the driver  $f^1(t, y, z, k)$  and the driver  $f^1(t, y, z, k) + \delta f_t$  (instead of  $f^2$ ), we get the desired result.  $\square$

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