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# ON THE STABILITY OF THE SOLITARY WAVES TO THE (GENERALIZED) KAWAHARA EQUATION

ANDRÉ KABAKOULA AND LUC MOLINET

ABSTRACT. In this paper we investigate the orbital stability of solitary waves to the (generalized) Kawahara equation (gKW) which is a fifth order dispersive equation. For some values of the power of the nonlinearity, we prove the orbital stability in the energy space  $H^2(\mathbb{R})$  of two branches of even solitary waves of gKW by combining the well-known spectral method introduced by Benjamin [3] with continuity arguments. We construct the first family of even solitons by applying the implicit function theorem in the neighborhood of the explicit solitons of gKW found by Dey et al. [8]. The second family consists of even travelling waves with low speeds. They are solutions of a constraint minimization problem on the line and rescaling of perturbations of the soliton of gKdV with speed 1.

## 1. INTRODUCTION

We consider the generalized Kawahara equation (gKW):

$$(1.1) \quad \partial_t u + u^p \partial_x u + \partial_x^3 u - \mu \partial_x^5 u = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R},$$

with initial data  $u(0) = u_0 \in H^2(\mathbb{R})$ , where  $p \in \mathbb{N}^*$  denotes the power of nonlinearity, and  $\mu > 0$  the parameter which control the fifth-order dispersion term. In the case  $p = 1$ , this equation has been derived by Kawahara [11] as a model for water waves with weak amplitude in the long-wave regime approximation for moderate values of surface tension and a Weber number close to 1/3. For such Weber numbers the usual description of long water waves via the Korteweg-de Vries (KdV) equation fails since the cubic term in the linear dispersion relation vanishes and fifth order dispersion becomes relevant at leading order. Note that positive values of the parameter  $\mu$  in (1.1) correspond to Weber numbers larger than 1/3.

The Cauchy problem associated to (1.1) is locally well-posed in  $H^2(\mathbb{R})$  (see for instance [1]). The  $H^2$ -solutions of (1.1) satisfy the following two conservation laws in time:

$$(1.2) \quad E_{p,\mu}(u(t)) = \int_{\mathbb{R}} \left[ \frac{\mu}{2} (\partial_x^2 u)^2(t) + \frac{1}{2} (\partial_x u)^2(t) - \frac{1}{(p+1)(p+2)} u^{p+2}(t) \right] = E_{\mu}(u_0) \quad (\text{energy})$$

and

$$(1.3) \quad V(u(t)) = \frac{1}{2} \int_{\mathbb{R}} u^2(t) = V(u_0) \quad (\text{mass}).$$

These conserved quantities enable to extend the solutions for all positive times so that (1.1) is actually globally well-posed in  $H^2(\mathbb{R})$ .

We are interested in solitary waves of gKW, i.e., the solutions to equation (1.1) of the form  $u(t, x) = \varphi_{c,p,\mu}(x - ct)$ , traveling to the right with the speed  $c > 0$ . Substituting  $u$  by  $\varphi_{c,p,\mu}$  in (1.1), integrating on  $\mathbb{R}$  with the assumption  $\partial_x^k \varphi_{c,p,\mu}(\pm\infty) = 0$  for  $k = 0, \dots, 4$ , we obtain

$$(1.4) \quad \mu \partial_x^4 \varphi_{c,p,\mu}(x) - \partial_x^2 \varphi_{c,p,\mu}(x) + c \varphi_{c,p,\mu}(x) = \frac{1}{p+1} \varphi_{c,p,\mu}^{p+1}(x), \quad \forall x \in \mathbb{R}.$$

In [8], Dey et al. compute explicit solutions to (1.4) that write :

$$(1.5) \quad \varphi_{c,p,\mu}(x) = \left[ \frac{(p+1)(p+4)(3p+4)c}{8(p+2)} \right]^{1/p} \operatorname{sech}^{4/p} \left[ \frac{p\sqrt{(p^2+4p+8)c}}{4(p+2)} x \right], \quad \forall x \in \mathbb{R},$$

with

$$(1.6) \quad c = \frac{2^2(p+2)^2}{(p^2+4p+8)^2\mu}.$$

and study their orbital stability. More generally, the existence of solutions to (1.1) with  $\mu > 0$  and has been proven in [13] by solving an associated constrained minimization problem. The orbital stability of the set of ground states to this minimization problem is also studied. Note that (1.6) is a fourth order differential equation and, up to our knowledge, no uniqueness result (up to symmetries) is known even for ground state solutions.

On the other hand, for  $\mu = 0$ , (1.7) becomes the well-known generalized Korteweg-de Vries equation (gKdV):

$$(1.7) \quad \partial_t u + u^p \partial_x u + \partial_x^3 u = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}$$

and thus, for  $\mu = 0$  in (1.4), we recover the equation of solitons of gKdV:

$$(1.8) \quad -\partial_x^2 \varphi_{c,p,0}(x) + c \varphi_{c,p,0}(x) = \frac{1}{p+1} \varphi_{c,p,0}^{p+1}(x), \quad \forall x \in \mathbb{R}.$$

Recall that, the solitons  $\varphi_{c,p,0}$  of gKdV are unique, up to translations, and to the transformation:  $\varphi_{c,p,0} \mapsto -\varphi_{c,p,0}$  if  $p$  is even. Moreover, they are explicitly defined by:

$$(1.9) \quad \varphi_{c,p,0}(x) = \left[ \frac{(p+1)(p+2)c}{2} \right]^{1/p} \operatorname{sech}^{2/p} \left[ \frac{p\sqrt{c}}{2} x \right], \quad \forall x \in \mathbb{R}.$$

In this paper, we start by constructing two branches of solutions to equation (1.4) with  $\mu = 1$  and establishing some uniqueness results on these solutions. Firstly, we fix  $p \in \mathbb{N}^*$ ,  $\mu = 1$  and  $c_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$ . By applying the Implicit Function Theorem in the neighborhood of the explicit solution  $\varphi_{c_p,p}$ , we construct a continuous in  $H^4(\mathbb{R})$  branch  $\{\varphi_{c,p}, c \in ]c_p - \delta_p, c_p + \delta_p[ \}$ , with  $0 < \delta_p \ll 1$ , of even solutions to (1.4). For each  $c \in ]c_p - \delta_p, c_p + \delta_p[$ ,  $\varphi_{c,p}$  is the unique even  $H^4(\mathbb{R})$  solution of (1.4) in some  $H^4$ -neighborhood of  $\varphi_{c_p,p}$ . Secondly, for all  $p \in \mathbb{N}^*$ ,  $c > 0$  and  $\mu > 0$ , following [13], we minimize in the even functions of  $H^2(\mathbb{R})$  the functional:

$$(1.10) \quad I_{c,\mu}(\psi) = \int_{\mathbb{R}} \left[ \frac{\mu}{2} (\partial_x^2 \psi)^2 + \frac{1}{2} (\partial_x \psi)^2 + \frac{c}{2} \psi^2 \right],$$

under the constraint:

$$(1.11) \quad K_p(\psi) = \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}} \psi^{p+2} = K_p(\varphi_{c,p,0}).$$

For  $\mu > 0$  small enough we prove the uniqueness of the associated even ground states by using ideas of [12]. In this way, we construct a family  $\{\varphi_{c,p,\mu}, 0 < \mu \ll 1\}$  of even solutions to (1.4) such that

$$(1.12) \quad \lim_{\mu \rightarrow 0^+} \|\varphi_{c,p,\mu} - \varphi_{c,p,0}\|_{H^1(\mathbb{R})} = 0,$$

where  $\varphi_{c,p,0}$  is defined by (1.8)-(1.9). Noticing that  $u$  is a solution to (1.4) with  $c = 1$ ,  $p = p_0 \geq 1$  and  $\mu > 0$  if and only if  $v = \mu^{1/p} u(\sqrt{\mu} \cdot)$  is a solution to (1.4) with  $c = \mu$ ,  $p = p_0$  and  $\mu = 1$ , we obtain a  $H^1$ -continuous branch

$$\{\varphi_{c,p,1}, c \in ]0, \delta'_p[ \} \quad \text{with} \quad 0 < \delta'_p \ll 1,$$

of even solutions to (1.4) (with  $\mu = 1$ ) traveling with low speeds. For each  $c \in ]0, \delta'_p[$ ,  $\varphi_{c,p}$  is the unique even solution of the constraint minimizing problem (1.15).

The main result of this paper is the orbital stability of the solitary waves that form these two branches. Note that this improves earlier results (see [13]) where the stability of the set of ground states is proven. Before stating our main result, let us recall the definition of orbital stability and define what we will call ground state solutions to (1.4).

**Definition 1.1** (Orbital Stability). *Let  $\phi \in H^2(\mathbb{R})$  be a solution of (1.4). We say that  $\phi$  is orbitally stable in  $H^2(\mathbb{R})$ , if for all  $\varepsilon > 0$ , there exists  $\delta_\varepsilon > 0$ , such that for all initial data  $u_0 \in H^2(\mathbb{R})$ , satisfying*

$$(1.13) \quad \|u_0 - \phi\|_{H^2(\mathbb{R})} \leq \delta_\varepsilon,$$

*the solution  $u \in C(\mathbb{R}_+, H^2(\mathbb{R}))$  of gKW emanating from  $u_0$  satisfies*

$$(1.14) \quad \sup_{t \in \mathbb{R}_+} \inf_{z \in \mathbb{R}} \|u(t, \cdot + z) - \phi\|_{H^2(\mathbb{R})} \leq \varepsilon.$$

In this paper, for  $s \geq 0$ , we set  $H_e^s(\mathbb{R}) = \{u \in H^s(\mathbb{R}) : u(-\cdot) = u(\cdot)\}$  with  $L_e^2(\mathbb{R}) = H_e^0(\mathbb{R})$ . This space, endowed with the metric of  $H^s(\mathbb{R})$  is an Hilbert space.

**Definition 1.2** (Even ground state solution). *We say that a solution to (1.4) is an even ground state solution to (1.4), if it is also a solution to the constraint minimization problem*

$$(1.15) \quad S_{c,p,\mu}^\beta = \inf \{I_{c,\mu}(\psi) : \psi \in H_e^2(\mathbb{R}), K_p(\psi) = \beta\}.$$

*for some  $\beta > 0$ .*

Our main results can be summarized as follows :

**Theorem 1.1.** *Let us fix  $\mu = 1$  and set  $c_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$  for any  $p \geq 1$ .*

*(i) For  $p \in \{1, 2, 3, 4\}$ , there exist  $\delta_p > 0$  such that for any  $c \in ]c_p - \delta_p, c_p + \delta_p[$ , (1.4) has a unique even solution in the ball of  $H^4(\mathbb{R})$  centered in  $\varphi_{c_p,p,1}$  (explicitly defined in (1.5)-(1.6)) with radius  $\delta_p$ .*

*These solutions form a curve of class  $C^1$  in  $H^4(\mathbb{R})$ , passing by  $\varphi_{c_p,p,1}$ , of even solitary waves to (1.1) which are all stable.*

*(ii) For  $p \in \{1, 2, 3\}$ , there exist  $\delta'_p > 0$  such that, for any  $c \in ]0, \delta'_p[$ , there exists a unique even ground state  $\varphi_{c,p,1}$  to (1.4). These solutions form a  $H^1$ -continuous curve of even solitary waves of (1.1) which are all stable.*

**Remark 1.1.** *If in Definition 1.2, we replace the requirement  $\beta > 0$  by  $\beta \neq 0$  then the uniqueness result for the even ground states to (1.13) with  $\mu = 1$  and  $c > 0$  small enough still holds but up to the symmetry  $v \mapsto -v$  in the case  $p$  even.*

The stability result will follow from a continuity argument together with a positivity property of the quadratic form associated with the second Fréchet derivative of the action functional at respectively  $\varphi_{1,p,\mu_p}$  and  $\varphi_{1,p,0}$  where  $\mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$ . Note that to prove this positivity property for  $\varphi_{1,p,\mu_p}$  we check numerically a sign condition on some  $L^2$ -scalar product linked to  $\varphi_{1,p,\mu_p}$ .

We will apply the following classical proposition for equations of form the  $\partial_t V'(u) = \partial_x E'(u)$  where  $E$  and  $V$  are conservation laws,  $V$  being quadratic.

**Proposition 1.1** (see for instance de Bouard [7]). *Let  $\phi_{c,p,\mu} \in H^2(\mathbb{R})$  be a solution of (1.4) traveling with the speed  $c > 0$ , and  $\mathcal{L}_{c,p,\mu}$  be the linearized operator associated to the second derivative of the action functional  $E_{p,\mu} + cV$  at  $\phi_{c,p}$ , defined by  $\mathcal{L}_{c,p,\mu}v = \mu \partial_x^4 v - \partial_x^2 v + cv - \phi_{c,p,\mu}^p v$ , for all  $v \in H^4(\mathbb{R})$ . Assume that there exists  $\delta > 0$  such that*

$$(1.16) \quad \langle \mathcal{L}_{c,p,\mu} v, v \rangle_{L^2} \geq \delta \|v\|_{H^2(\mathbb{R})}^2,$$

*for all  $v \in H^2(\mathbb{R})$  satisfying the orthogonalities*

$$(1.17) \quad \langle v, \phi_{c,p,\mu} \rangle_{L^2} = \langle v, \phi'_{c,p,\mu} \rangle_{L^2} = 0.$$

*Then,  $\phi_{c,p,\mu}$  is stable in  $H^2(\mathbb{R})$ .*

The proof of Proposition 1.1 follows from the theory developed by Benjamin [3], Bona [4] and Weinstein [19] which relies on the spectral properties of  $\mathcal{L}_{c,p,\mu}$  (see Section 2 for details).

For other stability results on solitons the similar models, see for instance: Karpman [9]-[10], Dey et al. [8], Levandosky [13]-[14], Bridges et al. [5], Pava [18] .

This paper is organized as follows. In Section 2, we prove the assertion (i) of Theorem 1.1 and, in Section 3, we prove the assertion (ii) of Theorem 1.1.

## 2. EXISTENCE AND STABILITY FOR THE BRANCH CROSSING THE EXPLICIT SOLITARY WAVES OF gKW

In this section, we prove the point (i) of Theorem 1.1. First, we observe that  $u$  is a solution to (1.1) with  $\mu > 0$  if and only if  $u_\mu(t, x) = \mu^{1/p} u(\mu^{3/2} t, \sqrt{\mu} x)$  is a solution to (1.1) with  $\mu = 1$ . In particular,  $\varphi$  is a solution to (1.4) with  $c = 1$  and  $\mu > 0$  if and only if  $\varphi_\mu = \mu^{1/p} \varphi(\sqrt{\mu} \cdot)$  is a solution to (1.4) with  $c = \mu$  and  $\mu = 1$ . This ensures that it is equivalent to prove the assertion (i) of Theorem 1.1 with  $\mu = \mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$  and  $c$  close to 1. For the numerical checking of the sign of some  $L^2$ -scalar product (see Subsection 2.4) we prefer to work with this last normalization in this section.

So, let  $p \in \mathbb{N}^*$  be fixed and set  $\mu = \mu_p = \frac{2^2(p+2)^2}{(p^2+4p+8)^2}$ , then the explicit solitary wave given by (1.5) travels with the speed  $c = 1$ . The strategy is as follows. By applying the Implicit Function Theorem, we will establish for all real  $c > 0$  sufficiently close to 1, the existence of even solitary waves  $\varphi_{c,p,\mu_p}$  of gKW traveling with speed  $c$  (see Lemma 2.2). Next, following Albert [2], we use that the hypotheses of Proposition 1.1 are fulfilled for the explicit solitary wave  $\varphi_{1,p,\mu_p}$  whenever a sign condition on some  $L^2$ -scalar product involving  $\varphi_{1,p,\mu_p}$  is satisfied. This sign condition is checked numerically in Subsection 2.4. Finally, arguing by continuity, we prove that the hypotheses of Proposition 1.1 are still fulfilled for  $c$  close enough to 1 which leads to the orbital stability of all the the family  $\{\varphi_{c,p,\mu_p}, c \in ]1 - \delta'_p, 1 + \delta'_p[ \} \subset H^4(\mathbb{R})$ , with  $0 < \delta'_p \ll 1$ .

We will divide the proof of Theorem 1.1 part (i) into several lemmas. We first study the spectral properties of the linearized operator  $\mathcal{L}_{1,p,\mu_p}$  associated with the second derivative of the action functional at the explicit soliton  $\varphi_{1,p,\mu_p}$  (defined in (1.5)-(1.6)).

### 2.1. Spectral properties of $\mathcal{L}_{1,p,\mu_p}$ .

**Lemma 2.1** (Spectral Properties of  $\mathcal{L}_{1,p,\mu_p}$ ). *Let  $p \in \mathbb{N}^*$ . We consider the unbounded operator  $\mathcal{L}_{1,p,\mu_p} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ , defined by:  $u \mapsto \mu_p \partial_x^4 u - \partial_x^2 u + u - \varphi_{1,p,\mu_p}^p u$ . We claim that  $\mathcal{L}_{1,p,\mu_p}$  possesses, among others, the following three crucial properties:*

- (P1) *The essential spectrum of  $\mathcal{L}_{1,p,\mu_p}$  is  $[1, +\infty[$  ;*
- (P2)  *$\mathcal{L}_{1,p,\mu_p}$  has only one negative eigenvalue  $\lambda_{1,p,\mu_p}$  which is simple;*
- (P3) *The kernel of  $\mathcal{L}_{1,p,\mu_p}$  is spanned by  $\varphi'_{1,p,\mu_p}$ .*

**Proof.** One can clearly see that  $\mathcal{L}_{1,p,\mu_p}$  is linear with domain  $H^4(\mathbb{R})$ , self-adjoint and closed on  $L^2(\mathbb{R})$ , since  $\|u\|_{H^4(\mathbb{R})} \lesssim \|u\|_{L^2(\mathbb{R})} + \|\mathcal{L}_{1,p,\mu_p} u\|_{L^2}$ . Moreover, it is a compact perturbation of  $\mu_p \partial_x^4 - \partial_x^2 + 1$ , since  $\varphi_{1,p,\mu_p}^p$  is smooth and decays exponentially to 0, and thus, its essential spectrum is given by  $\sigma_{ess}(\mathcal{L}_{1,p,\mu_p}) = [1, +\infty[$ .

According to Albert [2] (Theorem 3.2 and Lemma 10), to get the properties (P2)-(P3), it suffices to prove that:

$$(2.1) \quad \mathcal{F}(\varphi_{1,p,\mu_p})(\omega) > 0, \quad \forall \omega \in \mathbb{R}, \quad \text{and} \quad \frac{d^2}{d\omega^2} \log \mathcal{F}(\varphi_{1,p,\mu_p}^p)(\omega) < 0, \quad \forall \omega \in \mathbb{R}^*,$$

where  $\mathcal{F}$  denotes the spatial Fourier transform. Thanks to the computations done by Magnus and Oberhettinger [16] (p. 34), we first recall that

$$(2.2) \quad \mathcal{F}(\operatorname{sech}^\nu(\cdot))(\omega) = 2^{\nu-1} (\Gamma(\nu))^{-1} \left| \Gamma\left(\frac{\nu}{2} + i\frac{\omega}{2}\right) \right|^2 \geq 0, \quad \forall \omega \in \mathbb{R}, \quad \nu \in \mathbb{R}_+,$$

where  $\Gamma$  denotes the gamma function. On the other hand, using the usual Fourier transforms table, we establish that

$$(2.3) \quad \mathcal{F}(\operatorname{sech}^4(\cdot/2))(\omega) = \frac{2^4 \pi}{3!} \omega(\omega^2 + 1) \operatorname{cosech}(\pi\omega), \quad \forall \omega \in \mathbb{R}$$

and thus

$$(2.4) \quad \frac{d^2}{d\omega^2} \log \mathcal{F}(\operatorname{sech}^4(\cdot/2))(\omega) = \frac{2^4 \pi}{3!} \left( -\frac{1}{\omega^2} + \frac{2(1 - \omega^2)}{(1 + \omega^2)^2} + \frac{\pi^2}{\sinh^2(\pi\omega)} \right) < 0, \quad \forall \omega \in \mathbb{R}^*.$$

Then, combining (2.2)-(2.4), we get the sufficient condition (2.1). Thus  $\mathcal{L}_{1,p,\mu_p}$  has the spectral properties (P1)-(P3).

Let  $\lambda_{1,p,\mu_p}$  be the unique negative (simple) eigenvalue of  $\mathcal{L}_{1,p,\mu_p}$ , and let  $\chi_{1,p,\mu_p}$  be the, normalized in  $H^2(\mathbb{R})$ , eigenfunction associated with  $\lambda_{1,p,\mu_p}$ . One can notice that  $\{\varphi_{1,p,\mu_p}, \chi_{1,p,\mu_p}\} \subset (\text{Ker } \mathcal{L}_{1,p,\mu_p})^\perp$  for the usual inner product on  $L^2(\mathbb{R})$ , since  $\langle \chi_{1,p,\mu_p}, \varphi'_{1,p,\mu_p} \rangle_{L^2} = \lambda_{1,p,\mu_p}^{-1} \langle \chi_{1,p,\mu_p}, \mathcal{L}_{1,p,\mu_p} \varphi'_{1,p,\mu_p} \rangle_{L^2} = 0$ . Moreover, since  $\varphi_{1,p,\mu_p}$  is an even function, then  $\mathcal{L}_{1,p,\mu_p} \chi_{1,p,\mu_p}(\cdot) = \lambda_{1,p,\mu_p} \chi_{1,p,\mu_p}(\cdot)$ , and by uniqueness  $\chi_{1,p,\mu_p}(\cdot) = \chi_{1,p,\mu_p}(\cdot)$ . This implies that  $\chi_{1,p,\mu_p}$  is an even function.  $\square$

**2.2. Construction of the  $C^1$  branch of solitary waves.** We construct now the new solitons in a neighborhood of the explicit solitary wave  $\varphi_{1,p,\mu_p}$ .

**Lemma 2.2** (Existence of Solitons  $\varphi_{1,p,\mu_p}$  for  $c$  close to 1). *There exist  $\delta_p > 0$  and  $\tilde{\delta}_p > 0$  such that for any  $c > 0$  with  $|c - 1| < \delta_p$ , there exists a unique  $H^4(\mathbb{R})$  even solution  $\varphi_{c,p,\mu_p}$  of (1.4) in the ball of  $H^4(\mathbb{R})$  centered at  $\varphi_{1,p,\mu_p}$  with radius  $\tilde{\delta}_p > 0$ . Moreover, the function  $c \mapsto \varphi_{c,p,\mu_p}$  is of class  $C^1$  from  $]1 - \delta_p, 1 + \delta_p[$  into  $H^4(\mathbb{R})$ .*

**Proof.** We apply of the Implicit Function Theorem, (see for instance [17] for a similar application). Recall that we set  $H_e^s(\mathbb{R}) = \{u \in H^s(\mathbb{R}) : u(\cdot) = u(\cdot)\}$ . For  $\gamma > 0$ , we define the map  $T : ]1 - \gamma, 1 + \gamma[ \times H_e^4(\mathbb{R}) \rightarrow L_e^2(\mathbb{R})$ , by:  $(c, \psi) \mapsto \mu_p \partial_x^4 \psi - \partial_x^2 \psi + c\psi - \frac{1}{p+1} \psi^{p+1}$ .  $T$  is obviously of class  $C^1$  and, since  $\varphi_{1,p,\mu_p}$  satisfies (1.4)-(1.6), it holds

$$(2.5) \quad T(1, \varphi_{1,p,\mu_p}) = 0 \text{ and } \partial_\psi T(1, \varphi_{1,p,\mu_p}) = \mathcal{L}_{1,p,\mu_p} \Big|_{H_e^4(\mathbb{R})},$$

where  $\mathcal{L}_{1,p,\mu_p} \Big|_{H_e^4(\mathbb{R})}$  denotes the restriction of  $\mathcal{L}_{1,p,\mu_p}$  to  $H_e^4(\mathbb{R})$ .

It is easy to check that the spectrum of  $\mathcal{L}_{1,p,\mu_p} \Big|_{H_e^4(\mathbb{R})} : H_e^4(\mathbb{R}) \rightarrow L_e^2(\mathbb{R})$  is contained in the spectrum of  $\mathcal{L}_{1,p,\mu_p}$  and Lemma 2.1 (properties (P1) and (P3)) ensures that  $\sigma_{ess}(\mathcal{L}_{1,p,\mu_p}) = [1, +\infty[$  and  $\text{Ker } \mathcal{L}_{1,p,\mu_p} = \text{span} \{\varphi'_{1,p,\mu_p}\}$ . Since  $\varphi'_{1,p,\mu_p}$  is an odd function, we thus infer that  $0 \notin \sigma(\mathcal{L}_{1,p,\mu_p} \Big|_{H_e^4(\mathbb{R})})$  and thus  $\mathcal{L}_{1,p,\mu_p} \Big|_{H_e^4(\mathbb{R})}$  is an isomorphism.

Therefore, according to the Implicit Function Theorem, there exist  $\delta_p > 0$  and a  $C^1$  map  $R : ]1 - \delta_p, 1 + \delta_p[ \rightarrow H_e^4(\mathbb{R})$  which is uniquely determined such that:  $T(c, R(c)) = 0$  for all  $(c, R(c)) \in ]1 - \delta_p, 1 + \delta_p[ \times B_{H_e^4}(\varphi_{1,p,\mu_p}, \delta_p)$ , where  $B_{H_e^4}(\varphi_{1,p,\mu_p}, \delta_p)$  denotes the unit ball of  $H_e^4(\mathbb{R})$  centered at  $\varphi_{1,p,\mu_p}$  with radius  $\delta_p$ . This proves the lemma by setting  $\varphi_{c,p,\mu_p} = R(c)$  for  $c \in ]1 - \delta_p, 1 + \delta_p[$ .  $\square$

**Remark 2.1** (Some Important Properties of  $\varphi_{c,p}$  for  $c \sim 1$ ). Since  $\varphi_{c,p,\mu_p} \in H^4(\mathbb{R})$ , using the equation (1.4) of  $\varphi_{c,p,\mu_p}$ , and a classical bootstrap argument, we get that  $\varphi_{c,p,\mu_p} \in H^k(\mathbb{R})$  for all  $k \in \mathbb{N}$ . The well-known Sobolev embedding of  $H^{k+1}(\mathbb{R})$  into  $C^k(\mathbb{R})$  leads to

$$(2.6) \quad \|\varphi_{c,p,\mu_p}\|_{C^k([n,n+1])} \leq C_S \|\varphi_{c,p,\mu_p}\|_{H^{k+1}([n,n+1])} \rightarrow 0 \text{ as } n \rightarrow +\infty,$$

and thus  $\partial_x^k \varphi_{c,p,\mu_p}(\pm\infty) = 0$  for all  $k \in \mathbb{N}$ .

Now, for  $c$  close to 1, in view of the solutions to the linear asymptotic equation:  $\mu_p \partial_x^4 \psi - \partial_x^2 \psi + c\psi = 0$ , we infer that  $\varphi_{c,p,\mu_p}$  and its derivatives satisfy

$$(2.7) \quad |\partial_x^k \varphi_{1,p,\mu_p}(x)| \lesssim e^{-\sqrt{c}|x|} \text{ for } k \in \{0, 1, 2, 3, 4\}.$$

**2.3. Orbital stability result assuming  $\langle \mathcal{L}_{1,p,\mu_p}^{-1} \varphi_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} < 0$ .** Recall that, according to Lemma 2.1, the operator  $\mathcal{L}_{1,p,\mu_p}$  possesses the spectral properties (P1)-(P3). Then it is well known that if  $w \in H^2(\mathbb{R})$  satisfies  $\langle \mathcal{L}_{1,p,\mu_p} w, w \rangle_{L^2} < 0$  then (1.16) holds for any  $v \in H^2(\mathbb{R})$  such that  $\langle v, \varphi'_{c,p} \rangle_{L^2} = \langle v, \mathcal{L}_{1,p,\mu_p} w \rangle_{L^2} = 0$ . Now, since  $\varphi_{1,p,\mu_p}$  is even, it follows from the last subsection that

$\varphi_{1,p,\mu_p} \in \text{Im } \mathcal{L}_{1,p,\mu_p}$  and applying the above criterium with  $w \in \mathcal{L}_{1,p,\mu_p}^{-1} \varphi_{1,p,\mu_p}$ , we obtain that the hypotheses of Proposition 1.1 are satisfied as soon as

$$(2.8) \quad \mathcal{J}_p = \langle \mathcal{L}_{1,p,\mu_p}^{-1} \varphi_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2} < 0.$$

Note that the above quantity does not depend on the choice of the element of the preimage of  $\varphi_{1,p,\mu_p}$  since  $\varphi_{1,p,\mu_p} \in (\ker \mathcal{L}_{1,p,\mu_p})^\perp$ . In the next subsection we check numerically that this sign condition is fulfilled for  $p \in \{1, 2, 3, 4\}$ . In this subsection, we prove the orbital stability result assuming (2.8).

According to Proposition 1.1, it suffices to check that for any  $c$  close enough to 1 and any  $v \in H^2(\mathbb{R})$  such that  $\langle v, \varphi_{c,p} \rangle_{L^2} = \langle v, \varphi'_{c,p} \rangle_{L^2} = 0$ , there exists  $\delta_p > 0$  such that

$$(2.9) \quad \langle \mathcal{L}_{c,p} v, v \rangle_{L^2} \geq \delta_p \|v\|_{H^2}^2.$$

First, we notice that setting  $\gamma_{c,p} = \|\varphi_{c,p,\mu_p} - \varphi_{1,p,\mu_p}\|_{H^1}$  for  $|c-1| < \delta_p$ , it holds

$$(2.10) \quad \left| \langle v, \varphi_{1,p,\mu_p} \rangle_{L^2} \right| \leq \gamma_{c,p} \|v\|_{L^2(\mathbb{R})} \quad \text{and} \quad \left| \langle v, \varphi'_{1,p,\mu_p} \rangle_{L^2} \right| \leq \gamma_{c,p} \|v\|_{L^2(\mathbb{R})}.$$

Next, for all  $v \in H^2(\mathbb{R})$  satisfying the almost orthogonality conditions (2.10) with  $\gamma_{c,p}$  small enough, let us prove that  $\mathcal{L}_{1,p,\mu_p}$  is coercive in  $H^2(\mathbb{R})$ . We argue as, for instance, Côte et al ([6], Lemma 2.6). (2.10) clearly leads to

$$(2.11) \quad \left| \left\langle v, \frac{\varphi_{1,p,\mu_p}}{\|\varphi_{1,p,\mu_p}\|_{L^2(\mathbb{R})}} \right\rangle_{L^2} \right| + \left| \left\langle v, \frac{\varphi'_{1,p,\mu_p}}{\|\varphi'_{1,p,\mu_p}\|_{L^2(\mathbb{R})}} \right\rangle_{L^2} \right| \lesssim \gamma_{c,p} \|v\|_{H^2(\mathbb{R})}.$$

Now, we decompose  $v$  as follows

$$(2.12) \quad v = v_1 + a_1 \frac{\varphi_{1,p,\mu_p}}{\|\varphi_{1,p,\mu_p}\|_{L^2}} + a_2 \frac{\varphi'_{1,p,\mu_p}}{\|\varphi'_{1,p,\mu_p}\|_{L^2}} = v_1 + v_2,$$

with  $\langle v_1, \varphi_{1,p,\mu_p} \rangle_{L^2} = \langle v_1, \varphi'_{1,p,\mu_p} \rangle_{L^2} = 0$ . Then, combining (2.11) and (2.12), we infer that

$$(2.13) \quad |a_1| + |a_2| \lesssim \gamma_{c,p} \|v\|_{H^2(\mathbb{R})}.$$

Moreover, for  $k = 0, 1, 2$ , using that  $\langle \partial_x^k \varphi_{1,p,\mu_p}, \partial_x^k \varphi'_{1,p,\mu_p} \rangle_{L^2} = 0$ , it clearly holds

$$(2.14) \quad \|\partial_x^k v_2\|_{L^2(\mathbb{R})}^2 = a_1^2 \frac{\|\partial_x^k \varphi_{1,p,\mu_p}\|_{L^2(\mathbb{R})}^2}{\|\varphi_{1,p,\mu_p}\|_{L^2(\mathbb{R})}^2} + a_2^2 \frac{\|\partial_x^k \varphi'_{1,p,\mu_p}\|_{L^2(\mathbb{R})}^2}{\|\varphi'_{1,p,\mu_p}\|_{L^2(\mathbb{R})}^2}.$$

Then, combining (2.12)-(2.14), we deduce that there exists  $0 < \varepsilon_p < 1$  such that

$$(2.15) \quad \|v_1\|_{H^2(\mathbb{R})} \sim \|v\|_{H^2(\mathbb{R})} \quad \text{as soon as } \gamma_{c,p} < \varepsilon_p.$$

Next, we compute

$$(2.16) \quad \langle \mathcal{L}_{1,p,\mu_p} v, v \rangle_{L^2} = \langle \mathcal{L}_{1,p,\mu_p} v_1, v_1 \rangle_{L^2} + \langle \mathcal{L}_{1,p,\mu_p} v_2, v_2 \rangle_{L^2} + 2 \langle \mathcal{L}_{1,p,\mu_p} v_1, v_2 \rangle_{L^2}.$$

Note that, by construction,  $v_1$  is orthogonal in  $L^2(\mathbb{R})$  with  $\varphi_{1,p,\mu_p}$  and  $\varphi'_{1,p,\mu_p}$ . Therefore, according to the discussion in the beginning of this Subsection, there exists  $\delta_0 > 0$  such that

$$(2.17) \quad \langle \mathcal{L}_{1,p,\mu_p} v_1, v_1 \rangle_{L^2} \geq \delta_0 \|v_1\|_{H^2(\mathbb{R})}^2 \gtrsim \delta_0 \|v\|_{H^2(\mathbb{R})}^2.$$

On the other hand, using (2.13)-(2.15), it holds

$$(2.18) \quad \left| \langle \mathcal{L}_{1,p,\mu_p} v_2, v_2 \rangle_{L^2} \right| \lesssim |a_1|^2 + |a_2|^2 \lesssim (\gamma_{c,p})^2 \|v\|_{H^2(\mathbb{R})}^2$$

and

$$(2.19) \quad \left| \langle \mathcal{L}_{1,p,\mu_p} v_1, v_2 \rangle_{L^2} \right| \lesssim \|v_1\|_{H^2(\mathbb{R})} \|v_2\|_{H^2(\mathbb{R})} \lesssim \|v\|_{H^2(\mathbb{R})} (|a_1| + |a_2|) \lesssim \gamma_{c,p} \|v\|_{H^2(\mathbb{R})}^2.$$

Therefore, combining (2.16)-(2.19), we deduce that there exists  $\delta'_p > 0$  and  $\varepsilon_p > 0$  such that if  $\|\varphi_{c,p} - \varphi_{c_p,p}\|_{L^2} < \varepsilon_p$  then

$$(2.20) \quad \langle \mathcal{L}_{1,p,\mu_p} v, v \rangle_{L^2} \geq \delta'_p \|v\|_{H^2(\mathbb{R})}^2.$$

(2.9) follows immediately by noticing that

$$(2.21) \quad \begin{aligned} \langle \mathcal{L}_{c,p,\mu_p} v, v \rangle_{L^2} &= \langle \mathcal{L}_{1,p,\mu_p} v, v \rangle_{L^2} + (c-1) \|v\|_{L^2(\mathbb{R})}^2 - \left\langle v^2, \varphi_{c,p,\mu_p}^p - \varphi_{1,p,\mu_p}^p \right\rangle_{L^2} \\ &\geq \delta'_p \|v\|_{H^2(\mathbb{R})}^2 - (c-1 + K\varepsilon_p) \|v\|_{L^2(\mathbb{R})}^2 \\ &\geq \frac{\delta'_p}{2} \|v\|_{H^2(\mathbb{R})}^2, \end{aligned}$$

as soon as  $|c-1 + K\varepsilon_p| < \delta'_p/2$ . But the continuity of the branch of the  $\varphi_{c,p,\mu_p}$  in  $H^4(\mathbb{R})$  ensures that this is true as soon as  $c$  is close enough to 1 and we are done.

## 2.4. Numerical checking of the sign condition on $\langle \mathcal{L}_{1,p,\mu_p}^{-1} \varphi_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2}$ .

2.4.1. *Numerical computing of  $\mathcal{J}_1 = \langle \mathcal{L}_{1,1,\mu_1}^{-1} \varphi_{1,1,\mu_1}, \varphi_{1,1,\mu_1} \rangle_{L^2}$ .* Let  $\rho_1 \in \mathcal{L}_{1,1,\mu_1}^{-1} \varphi_{1,1}$ . Note that, since  $\mathcal{L}_{1,1,\mu_1}$  is a self-adjoint operator on  $L^2(\mathbb{R})$ , the value of  $\mathcal{J}_1$  does not depend on the choice of the element  $\rho_1$ . Moreover, using that  $\mathcal{L}_{1,1,\mu_1} \varphi_{1,1,\mu_1} = -\frac{1}{2} \varphi_{1,1,\mu_1}^2$ , we observe that

$$(2.22) \quad \langle \rho_1, \varphi_{1,1,\mu_1}^2 \rangle_{L^2} = -2 \|\varphi_{1,1,\mu_1}\|_{L^2(\mathbb{R})}^2 < 0.$$

From (2.22) and the exponential decay properties of  $\varphi_{1,1,\mu_1}$  and  $\rho_1$  (see (2.7)), we infer that  $\rho_1$  takes necessarily some negative values near the origin. Then, we can expect that  $\mathcal{J}_1$  is negative.

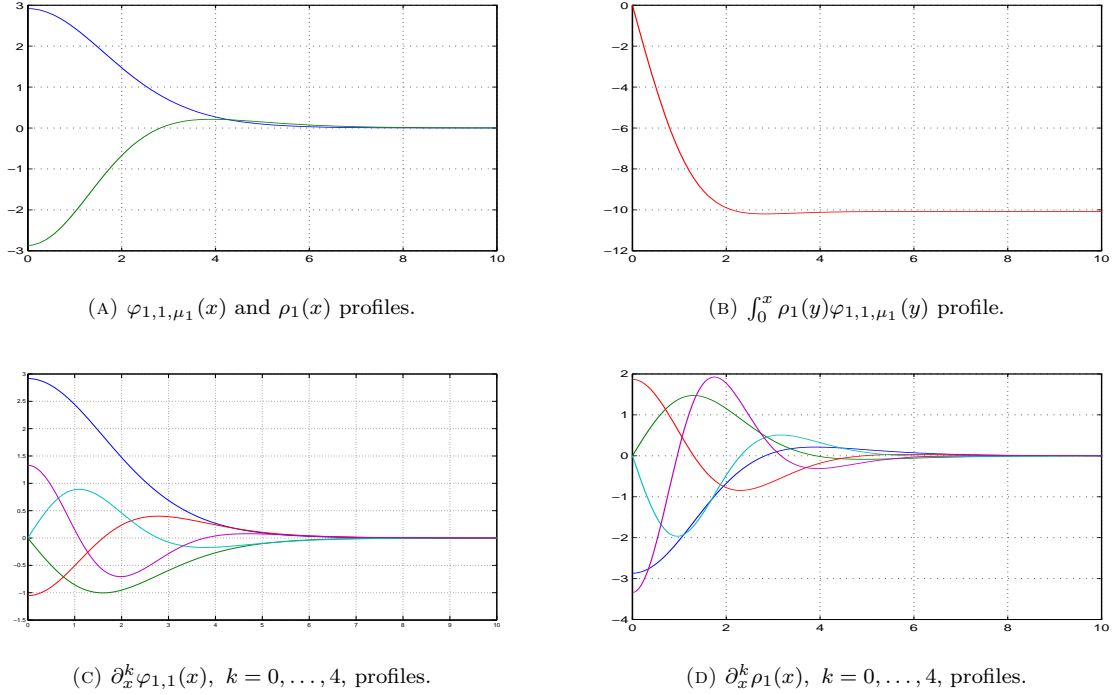
The goal is to solve numerically the following equation:

$$(2.23) \quad \partial_x^4 \rho_1 = \frac{13^2}{6^2} (\partial_x^2 \rho_1 - \rho_1 + \varphi_{1,1,\mu_1} \rho_1 + \varphi_{1,1,\mu_1}^3).$$

First, we remark that  $\rho_1(\cdot)$  and  $\rho_1(-\cdot)$  are both solutions of (2.23), since  $\varphi_{1,1,\mu_1}$  is an even function. Then, from (2.22) we deduce that  $\rho_1$  is also an even function. Moreover, we get that  $\partial_x^{2k} \rho_1$  is even and  $\partial_x^{2k+1} \rho_1$  is odd for all  $k \in \mathbb{N}$ . We can restrict ourself to study equation (2.23) on  $\mathbb{R}_+$ . We fix the domain to  $[0, r_{\max}]$ , with  $0 < r_{\max} < +\infty$ , and we rewrite equation (2.23) as a system of four first-order ODE. We set:  $\rho_{1,1} = y_1$ ,  $y_1' = y_2$ ,  $y_2' = y_3$ ,  $y_3' = y_4$  and  $y_4' = \frac{13^2}{6^2} (y_3 - y_1 + \varphi_{1,1,\mu_1} y_1 + \varphi_{1,1,\mu_1}^3)$ . Next, we choose the Robin type boundary conditions at point  $r_{\max}$ :  $y_1(r_{\max}) + y_2(r_{\max}) = 0$  and  $y_3(r_{\max}) + y_4(r_{\max}) = 0$ , and at point 0 we use the symmetry of  $\rho_{1,1}$ :  $y_2(0) = 0$  and  $y_4(0) = 0$ . Finally, in our numerical scheme, we take into account the exponential decay properties:  $\partial_x^k \varphi_{1,1,\mu_1}(x) \sim \partial_x^k \rho_1(x) \sim e^{-x}$ , for  $x \sim r_{\max}$ , and for all  $k \in \mathbb{N}$ . Therefore, computing with the MATLAB solver, the result is:  $\langle \rho_{1,1}, \varphi_{1,1} \rangle_{L^2([0, r_{\max}])} \approx -10.0787 < 0$  (see Fig. 1a-1b).

2.4.2. *Numerical values of  $\mathcal{J}_p = \langle \mathcal{L}_{1,p,\mu_p}^{-1} \varphi_{1,p,\mu_p}, \varphi_{1,p,\mu_p} \rangle_{L^2}$  for  $p \in \{2, 3, 4, 5\}$ .* Proceeding as in Subsection 2.3, Step 1, for  $p \in \{2, 3, 4, 5\}$ , we compute respectively that:  $\mathcal{J}_2 \approx -1.9325 < 0$ ,  $\mathcal{J}_3 \approx -0.5649 < 0$ ,  $\mathcal{J}_4 \approx -0.1443 < 0$  (see Fig. 2a-2b) and  $\mathcal{J}_5 \approx 0.0252 > 0$  (see Fig. 2c-2d). More precisely, we lose the sufficient condition of stability exactly for  $p_{crit} = 4.84$ . We observe that the same critical value appears in the stability analysis done by Bridges et al. [5] (see Section 8, p. 209-210) on the similar model, using the Evans functions approach. Finally, we point out that thanks to the term  $\partial_x^5 u$ , the family  $\{\varphi_{c,p,1}, c \in ]c_p - \delta_p, c_p + \delta_p[ \} \subset H^4(\mathbb{R})$ , with  $0 < \delta_p \ll 1$ , of even solitons of gKW remains stable in the critical gKdV case  $p = 4$ .



FIGURE 1. Numerical computation of  $\mathcal{J}_1$ 

### 3. EXISTENCE AND STABILITY OF SLOW SOLITONS OF gKW

In this section, we prove the point (ii) of Theorem 1.1. The idea is as follows. Around the functional equation  $E'_\mu(\varphi_{1,p,\mu}) + V'(\varphi_{c,p,\mu}) = 0$  (equivalent to equation (1.4)), we will construct a minimization problem on the line. By applying the Concentration-Compactness Principle [15], we will prove the existence of even solitary waves  $\varphi_{1,p,\mu}$  of gKW, traveling with the speed 1. Next, we will establish the strong convergence in  $H^1(\mathbb{R})$  of the family  $\{\varphi_{1,p,\mu}, 0 < \mu \ll 1\}$  to the solitons  $\varphi_{1,p} = \varphi_{1,p,0}$  of gKdV as  $\mu$  tends to  $0^+$  (see Subsection 3.1). Once again, we recall that the solitons  $\varphi_{c,p,0}$  of gKdV satisfy equation (1.4) with  $\mu = 0$ , i.e.:

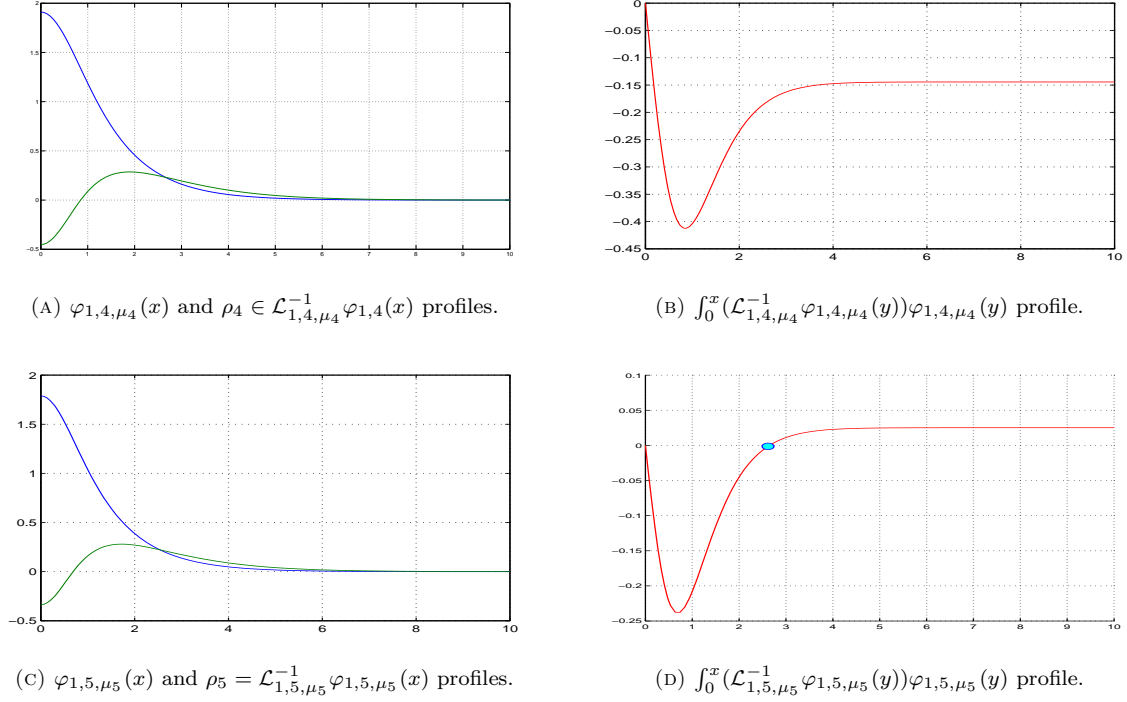
$$(3.1) \quad -\partial_x^2 \varphi_{c,p,0}(x) + c\varphi_{c,p,0}(x) = \frac{1}{p+1} \varphi_{c,p,0}^{p+1}(x), \quad \forall x \in \mathbb{R}.$$

They are unique, up to the translations, and the transformations:  $\varphi_{c,p} \mapsto -\varphi_{c,p}$  if  $p$  is even. Moreover, they are explicitly given by:

$$(3.2) \quad \varphi_{c,p}(x) = \left[ \frac{(p+1)(p+2)c}{2} \right]^{1/p} \operatorname{sech}^{2/p} \left[ \frac{p\sqrt{c}}{2} x \right], \quad \forall x \in \mathbb{R}.$$

Finally, using the well-known spectral properties of the linearized operator  $\mathcal{L}_{c,p,0} = -\partial_x^2(\cdot) + c(\cdot) - \varphi_{c,p,0}^p$  of gKdV around the solitons  $\varphi_{c,p,0}$ , and arguing as in Subsection 2.3, we will prove the stability in  $H^2(\mathbb{R})$  of the family  $\{\varphi_{1,p,\mu}, 0 < \mu \ll 1\}$  for the sub-critical gKdV nonlinearity  $p \in \{1, 2, 3\}$ . Note that we also obtain a uniqueness result on the solution to our minimization problem. After an appropriate rescaling, this enables us to construct a continuous branch of even solitary waves of gKW traveling with low speeds that are all orbitally stable .

**3.1. Existence and limit of even ground states of gKW as  $\mu$  tends to  $0^+$ .** In this subsection, we first prove the existence of even solitary waves  $\varphi_{1,p,\mu}$  of gKW for  $\mu > 0$ , by solving a minimization

FIGURE 2. Numerical computation of  $\mathcal{J}_p$  for  $p = 4, 5$ .

problem on the line. In this step we argue as in Levandosky [13] (Theorem 2.3). Second, we study the strong convergence in  $H^1(\mathbb{R})$  of the family  $\{\varphi_{1,p,\mu}, 0 < \mu \ll 1\}$  to the explicit solitons  $\varphi_{1,p}$  of gKdV (given in (3.2)) as  $\mu$  goes to  $0^+$  (see Lemma 3.1). Moreover, we establish the uniqueness of  $\varphi_{1,p,\mu}$  for  $0 < \mu \ll 1$  and  $p \in \{1, 2, 3\}$  (see Lemma 3.2).

**Lemma 3.1** (Existence and limit of  $\varphi_{1,p,\mu}$ ). *Let  $p \in \mathbb{N}^*$  be fixed. There exists a family  $\{\varphi_{1,p,\mu}, 0 < \mu \ll 1\} \subset H^4(\mathbb{R})$  of even solutions to (1.4) with  $c = 1$ , such that*

$$(3.3) \quad \lim_{\mu \rightarrow 0^+} \|\varphi_{1,p,\mu} - \varphi_{1,p,0}\|_{H^1(\mathbb{R})} = 0.$$

**Proof.** We split the proof in five steps.

*Step 1.* Minimization problem on  $\mathbb{R}$ .

We argue similarly as Levandosky [13] (Theorem 2.3). First, we set  $H_e^2(\mathbb{R}) = \{\psi \in H^2(\mathbb{R}) : \psi(-\cdot) = \psi(\cdot)\}$ . Clearly,  $H_e^2(\mathbb{R})$ , endowed with the scalar product of  $H^2(\mathbb{R})$ , is a Hilbert space since we have a continuous embedding of  $H^2(\mathbb{R})$  into  $C^1(\mathbb{R})$ . For all  $\psi \in H_e^2(\mathbb{R})$ , we define the functional associated with the linear part of (1.4) by:

$$(3.4) \quad I_\mu(\psi) = I_{1,\mu}(\psi) = \int_{\mathbb{R}} \left[ \frac{\mu}{2} (\partial_x^2 \psi)^2 + \frac{1}{2} (\partial_x \psi)^2 + \frac{1}{2} \psi^2 \right],$$

and the functional associated with the nonlinear part of (1.4) by:

$$(3.5) \quad K_p(\psi) = \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}} \psi^{p+2}.$$

One can see that  $I_\mu(\cdot)$  is coercive in  $H^2(\mathbb{R})$ , since for all  $\psi \in H^2(\mathbb{R})$ , we have

$$(3.6) \quad I_\mu(\psi) \geq \frac{\min\{\mu, 1\}}{2} \|\psi\|_{H^2(\mathbb{R})}^2.$$

Also, one can check that  $K_p(\cdot)$  is locally Lipschitz in  $L^{p+2}(\mathbb{R})$ . Indeed, for all  $(\psi, \phi) \in L^{p+2}(\mathbb{R}) \times L^{p+2}(\mathbb{R})$ , applying the Hölder inequality, we have

$$(3.7) \quad \begin{aligned} |K_p(\psi) - K_p(\phi)| &= \frac{1}{(p+1)(p+2)} \left| \int_{\mathbb{R}} (\psi^{p+2} - \phi^{p+2}) \right| \\ &\leq \frac{1}{(p+1)(p+2)} \|\psi - \phi\|_{L^{p+2}(\mathbb{R})} \sum_{k=0}^{p+1} \|\psi|^{p-k+1} |\phi|^k\|_{L^{\frac{p+2}{p+1}}(\mathbb{R})}, \end{aligned}$$

and applying the Young inequality, we have

$$(3.8) \quad \|\psi|^{p-k+1} |\phi|^k\|_{L^{\frac{p+2}{p+1}}(\mathbb{R})} \leq \frac{p-k+1}{p+1} \|\psi\|_{L^{p+2}(\mathbb{R})}^{p+2} + \frac{k}{p+1} \|\phi\|_{L^{p+2}(\mathbb{R})}^{p+2}.$$

Then combining (3.7) and (3.8), we obtain

$$(3.9) \quad |K_p(\psi) - K_p(\phi)| \leq C(\|\psi\|_{L^{p+2}(\mathbb{R})}, \|\phi\|_{L^{p+2}(\mathbb{R})}) \|\psi - \phi\|_{L^{p+2}(\mathbb{R})}.$$

Now, we will study the minimization of  $I_\mu(\cdot)$  in  $H_e^2(\mathbb{R})$  subject to the constraint  $K_p(\cdot) = K_p(\varphi_{1,p,0}) = \beta_p$  where  $\varphi_{1,p,0}$  is defined in (1.9). We chose this constraint to be sure that the family of minimizer (depending on the parameter  $\mu$ ) will converge to the solitons  $\varphi_{1,p,0}$  of gKdV (in a sense that we specify in Step 5). For  $\beta > 0$ , we define:

$$(3.10) \quad S_{p,\mu}^\beta = S_{1,p,\mu}^\beta = \inf \{I_\mu(\psi) : \psi \in H_e^2(\mathbb{R}), K_p(\psi) = \beta\}.$$

Since  $\varphi_{1,p} \in H_e^2(\mathbb{R})$ , and  $I_\mu(\cdot)$  is coercive in  $H_e^2(\mathbb{R})$ , one can see that  $S_{p,\mu}^\beta > 0$ . On the other hand, using the definition of  $I_\mu(\cdot)$  and  $K_p(\cdot)$ , one can check that

$$(3.11) \quad S_{p,\mu}^\beta = \left(\frac{\beta}{\beta_p}\right)^{\frac{2}{p+2}} S_{p,\mu}^{\beta_p}, \quad \forall \beta \in \mathbb{R}_+^*,$$

and then we infer the crucial sub-additivity property:

$$(3.12) \quad S_{p,\mu}^\beta + S_{p,\mu}^{\beta_p - \beta} > S_{p,\mu}^{\beta_p}, \quad \forall \beta \in (0, \beta_p).$$

Let us solve the constraint minimization problem (3.10) with  $\beta = \beta_p$ . We take  $(\psi_k)_{k \geq 1} \subset H$  a minimizing sequence of the problem (3.10), i.e., for all  $k \in \mathbb{N}^*$ , we have

$$(3.13) \quad K_p(\psi_k) = \beta_p \quad \text{and} \quad \lim_{k \rightarrow +\infty} |I_\mu(\psi_k) - S_{p,\mu}^\beta| = 0.$$

We claim that there exist a sub-sequence  $(\psi_j)_{j \geq 1}$  and a function  $\psi_\mu \in H_e^2(\mathbb{R})$ , such that  $\psi_j \rightarrow \psi_\mu$  strongly in  $H^2(\mathbb{R})$  as  $j \rightarrow +\infty$ . Moreover,  $\psi_\mu$  is a minimizer of the problem (3.10).

From the convergence of the energy (3.13), we deduce that there exists  $\varepsilon_0 > 0$ , such that

$$(3.14) \quad 0 < I_{c,\mu}(\psi_k) < S_{c,p,\mu} + \varepsilon,$$

with  $0 < \varepsilon < \varepsilon_0 \ll 1$ , and this implies that the sequence  $(\psi_k)_{k \geq 1}$  is bounded in  $H^2(\mathbb{R})$  (since  $I_{c,\mu}(\cdot)$  is equivalent to  $H^2(\mathbb{R})$ ). Since  $H$  is reflexive, there exist a sub-sequence  $(\psi_j)_{j \geq 1} \subset H$  and a function  $\psi_\mu \in H$  such that the following hold:

$$(3.15) \quad \psi_j \rightharpoonup \psi_\mu \text{ weakly in } H^2(\mathbb{R}),$$

$$(3.16) \quad \psi_j \rightarrow \psi_\mu \text{ strongly in } H_{loc}^1(\mathbb{R}),$$

$$(3.17) \quad \psi_j \rightarrow \psi_\mu \text{ a.e. on } \mathbb{R},$$

and

$$(3.18) \quad \|\psi_\mu\|_{H^2(\mathbb{R})}^2 \leq \liminf_{j \rightarrow +\infty} \|\psi_j\|_{H^2(\mathbb{R})}^2.$$

Now, we define a sequence of positive and even functions:

$$(3.19) \quad \Phi_k = |\partial_x^2 \psi_k|^2 + |\psi_k|^2, \quad \forall k \in \mathbb{N}^*.$$

From (3.14), one can see that  $(\Phi_k)_{k \geq 1}$  is bounded in  $L^1(\mathbb{R})$ . After extracting a sub-sequence, we may assume that  $\lim_{k \rightarrow +\infty} \int_{\mathbb{R}} \Phi_k = L < +\infty$ . By normalizing, we may assume further that  $\int_{\mathbb{R}} \Phi_k = L$  for all  $k \in \mathbb{N}^*$ . Then, by the Concentration-Compactness Lemma [13], there are three possibilities:

(a) Compactness: there exists  $(y_k)_{k \geq 1} \subset \mathbb{R}$ , such that for all  $\varepsilon > 0$ , there exists  $R_\varepsilon > 0$ , such that for all  $k \in \mathbb{N}^*$ ,

$$(3.20) \quad \int_{|x-y_k| \leq R_\varepsilon} \Phi_k \geq L - \varepsilon \Leftrightarrow \int_{|x-y_k| \geq R_\varepsilon} \Phi_k \leq \varepsilon.$$

(b) Vanishing: for every  $R > 0$ ,

$$(3.21) \quad \lim_{k \rightarrow +\infty} \sup_{y \in \mathbb{R}} \int_{|x-y| \leq R} \Phi_k = 0.$$

(c) Dichotomy: there exists  $l \in (0, L)$ , such that for all  $\varepsilon > 0$ , there exist  $R > 0$ ,  $R_k \rightarrow +\infty$ ,  $(y_k)_{k \geq 1}$  and  $k_0$ , such that for all  $k \geq k_0$ ,

$$(3.22) \quad \left| \int_{|x-y_k| \leq R} \Phi_k - l \right| \leq \varepsilon \text{ and } \left| \int_{R < |x-y_k| < R_k} \Phi_k \right| \leq \varepsilon.$$

First, let us assume that (a) holds, and we will prove that  $\psi_\mu$  is a minimizer of the problem (3.10). One can remark that  $|y_k| \leq R_{L/2}$  for all  $k \in \mathbb{N}^*$ , otherwise, there exists  $k_0$  such that  $\{|x - y_{k_0}| \leq R_{L/2}\} \subset \mathbb{R}^+$ . Then, using that  $\Phi_{k_0}$  is even, we get a contradiction:

$$(3.23) \quad \int_{\mathbb{R}} \Phi_{k_0} > 2 \int_{|x-y_{k_0}| < R_{L/2}} \Phi_{k_0} > L.$$

Thus, we can assume that  $y_k = 0$  for all  $k \in \mathbb{N}^*$  in hypothesis (3.20) by taking as radius  $R_{L/2} + R_\varepsilon$ . Now, by the compact embedding of  $H_{loc}^1(\mathbb{R})$  into  $L_{loc}^{p+2}(\mathbb{R})$ , and using (3.16), we obtain

$$(3.24) \quad \psi_j \rightarrow \psi_\mu \text{ strongly in } L_{loc}^{p+2}(\mathbb{R}).$$

On the other hand, using that  $\|\psi_j\|_{L^{p+2}(\mathbb{R})} \leq \frac{1}{2^{p/2(p+2)}} \|\psi_j\|_{H^1(\mathbb{R})}$ , and the compactness of  $(\Phi_j)_{j \geq 1}$ , one can easily check that the sequence  $(|\psi_j|^{p+2})_{j \geq 1}$  is also compact, i.e.

$$\lim_{R \rightarrow +\infty} \sup_{j \in \mathbb{N}} \int_{|x-y_j| > R} |\psi_j|^{p+2} = 0.$$

This clearly leads to

$$(3.25) \quad \psi_j \rightarrow \psi_\mu \text{ strongly in } L^{p+2}(\mathbb{R}).$$

Finally, by lower semi-continuity (3.18), we get  $I_\mu(\psi_\mu) \leq S_{p,\mu}^{\beta_p}$ , and from the fact that  $K_p(\cdot)$  is locally Lipschitz in  $L^{p+2}(\mathbb{R})$  (see (3.9)), and using (3.25), we get  $K_p(\psi_\mu) = \beta_p$ . Therefore,  $\psi_\mu$  is a minimizer of the problem (3.10).

Next, we suppose that (b) holds and prove that this leads to a contradiction. One can easily estimate that for all  $y \in \mathbb{R}$ ,

$$(3.26) \quad \left| \int_{|x-y| \leq 1} \psi_k^{p+2} \right| \lesssim \left( \int_{|x-y| \leq 1} \Phi_k \right)^{\frac{p+2}{2}}.$$

Then, using (b), for any fixed  $\varepsilon > 0$ , there exists  $k_\varepsilon \geq 0$  such that for  $k \geq k_\varepsilon$ ,

$$(3.27) \quad \left| \int_{|x-y| \leq 1} \psi_k^{p+2} \right| \lesssim \varepsilon^{\frac{p+1}{2}}.$$

Now, multiplying (3.27) by  $\frac{1}{(p+1)(p+2)}$  and summing over intervals centered at even integers, we obtain

$$(3.28) \quad K_p(\psi_k) \lesssim \varepsilon^{\frac{p-1}{2}} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

This contradicts the fact that  $K_p(\psi_k) = \beta_p$  for all  $k \in \mathbb{N}^*$ .

At last, we assume that (c) holds, and we will rule out this possibility. We define two smooth non negative test-functions ( $C^\infty(\mathbb{R})$ ) as follows:

$$(3.29) \quad \xi_1(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 2, \end{cases} \quad \text{and} \quad \xi_2(x) = \begin{cases} 1, & |x| \geq 1, \\ 0, & |x| \leq \frac{1}{2}. \end{cases}$$

Next, we construct two sequences  $\psi_{k,1}(x) = \xi_1(|x - y_k|/R)\psi_k(x)$  and  $\psi_{k,2}(x) = \xi_2(|x - y_k|/R_k)\psi_k(x)$ , for all  $x \in \mathbb{R}$  and  $k \in \mathbb{N}^*$ . Using the assumption (c) and the continuous embedding from  $H^1(\mathbb{R})$  into  $L^{p+2}(\mathbb{R})$ , one can check that for  $k \geq k_0$ ,

$$(3.30) \quad I_\mu(\psi_k) = I_\mu(\psi_{k,1}) + I_\mu(\psi_{k,2}) + O(\varepsilon)$$

and

$$(3.31) \quad K_p(\psi_k) = K_p(\psi_{k,1}) + K_p(\psi_{k,2}) + O(\varepsilon).$$

Since  $(\psi_k)_{k \geq 1}$  is uniformly bounded in  $H^2(\mathbb{R})$  and  $\|\psi_{k,i}\|_{H^2(\mathbb{R})} \leq \|\psi_k\|_{H^2(\mathbb{R})}$ , for  $i = 1, 2$ , it follows that  $(\psi_{k,i})_{k \geq 1}$ , for  $i = 1, 2$ , are also bounded uniformly in  $H^2(\mathbb{R})$ . Hence  $K_p(\psi_{k,i})$ , for  $i = 1, 2$ , are bounded and we may pass to a sub-sequence to define  $\beta_i(\varepsilon) = \lim_{k \rightarrow +\infty} K_p(\psi_{k,i})$ , for  $i = 1, 2$ . Since  $\beta_i(\varepsilon)$ , for  $i = 1, 2$ , are bounded uniformly, we can choose a sequence  $\varepsilon_j \rightarrow 0$  such that  $\beta_{p,i} = \lim_{j \rightarrow +\infty} \beta_i(\varepsilon_j) < \infty$ , for  $i = 1, 2$ . We clearly get  $\beta_{p,1} + \beta_{p,2} = \beta_p$ , and there are three cases to consider.

Case 1: if  $\beta_{p,1} \in (0, \beta_p)$ . Using (3.11) and (3.30), we compute that

$$(3.32) \quad \begin{aligned} I_\mu(\psi_k) &\geq S_{p,\mu}^{K_p(\psi_{k,1})} + S_{p,\mu}^{K_p(\psi_{k,2})} + O(\varepsilon_j) \\ &= \left[ \left( \frac{K_p(\psi_{k,1})}{\beta_p} \right)^{\frac{2}{p+2}} + \left( \frac{K_p(\psi_{k,2})}{\beta_p} \right)^{\frac{2}{p+2}} \right] S_{p,\mu}^{\beta_p} + O(\varepsilon_j). \end{aligned}$$

Letting  $k \rightarrow +\infty$ , and using that  $(\psi_k)_{k \geq 1}$  is a minimizing sequence, we obtain

$$(3.33) \quad S_{p,\mu}^{\beta_p} \geq \left[ \left( \frac{\beta_{p,1}}{\beta_p} \right)^{\frac{2}{p+2}} + \left( \frac{\beta_{p,2}}{\beta_p} \right)^{\frac{2}{p+2}} \right] S_{p,\mu}^{\beta_p} + O(\varepsilon_j).$$

Letting  $j \rightarrow +\infty$ , we get a contradiction:

$$(3.34) \quad S_{p,\mu}^{\beta_p} \geq \left[ \left( \frac{\beta_{p,1}}{\beta_p} \right)^{\frac{2}{p+2}} + \left( \frac{\beta_{p,2}}{\beta_p} \right)^{\frac{2}{p+2}} \right] S_{p,\mu}^{\beta_p} > S_{p,\mu}^{\beta_p}.$$

Case 2: if  $\beta_1 = 0$  (or equivalently  $\beta_1 = \beta_{c,p}$ ). Using the coercivity of  $I_\mu(\cdot)$  in  $H^2(\mathbb{R})$ , and the assumption (c), we have

$$(3.35) \quad \begin{aligned} I_\mu(\psi_{k,1}) &= \int_{\mathbb{R}} \left( |\partial_x^2 \psi_{k,1}|^2 + |\psi_{k,1}|^2 \right) \\ &= \int_{|x-y_k| \leq 2R} \left( |\partial_x^2 \psi_k|^2 + |\psi_k|^2 \right) + O(\varepsilon_j) \\ &= l + O(\varepsilon_j). \end{aligned}$$

Thus, combining (3.30) and (3.35), we get

$$(3.36) \quad I_\mu(\psi_k) \geq l + \left( \frac{K_p(\psi_{k,2})}{\beta_p} \right)^{\frac{2}{p+2}} S_{p,\mu}^{\beta_p} + O(\varepsilon_j).$$

Letting  $k, j \rightarrow +\infty$ , we get a contradiction:

$$(3.37) \quad S_{p,\mu}^{\beta_p} \geq l + S_{p,\mu}^{\beta_p} > S_{p,\mu}^{\beta_p}.$$

Case 3: if  $\beta_1 > \beta_{c,p}$  (or equivalently  $\beta_1 < 0$ ). Using the non negativity of  $I_\mu(\cdot)$ , we have

$$(3.38) \quad I_\mu(\psi_k) \geq I_\mu(\psi_{k,1}) + O(\varepsilon_j) \geq \left( \frac{K_p(\psi_{k,1})}{\beta_p} \right)^{\frac{2}{p+2}} S_{p,\mu}^{\beta_p} + O(\varepsilon_j).$$

As previously, letting  $k, j \rightarrow +\infty$ , once again, we get a contradiction:

$$(3.39) \quad S_{p,\mu}^{\beta_p} \geq \left(\frac{\beta_1}{\beta_p}\right)^{\frac{2}{p+2}} S_{p,\mu}^{\beta_p} > S_{p,\mu}^{\beta_p}.$$

Therefore, we deduce that the sequence  $(\psi_k)_{k \geq 1}$  is compact.

*Step 2. Euler-Lagrange equation.*

At this stage, we can write the Euler-Lagrange equation related to the minimization problem (3.10). Since  $\psi \mapsto I_p(\psi)$  and  $\psi \mapsto K_p(\psi)$  are obviously of class  $C^1$  in  $H^2(\mathbb{R})$ , the minimizer  $\psi_{p,\mu}$  satisfies the following functional equation:

$$(3.40) \quad \mu \langle \partial_x^2 \psi_{p,\mu}, \partial_x^2 \phi \rangle_{L^2} + \langle \partial_x \psi_{p,\mu}, \partial_x \phi \rangle_{L^2} + \langle \psi_{p,\mu}, \phi \rangle_{L^2} = \frac{\alpha_{p,\mu}}{p+1} \langle \psi_{p,\mu}^{p+1}, \phi \rangle_{L^2}, \quad \forall \phi \in H_c^2(\mathbb{R}),$$

where  $\alpha_{p,\mu} \in \mathbb{R}^*$  is a Lagrange multiplier. It is worth noticing that, equation (3.40) also holds with any odd test-function  $\phi \in H^2(\mathbb{R})$ , since then both members do cancel. Therefore  $\psi_{p,\mu}$  satisfies equation (3.40) for all  $\phi \in H^2(\mathbb{R})$ . Now, by a standard bootstrap argument, we get that  $\psi_{p,\mu} \in H^s(\mathbb{R})$  and  $\partial_x^s \psi_{p,\mu}(\pm\infty) = 0$  for all  $s \in \mathbb{N}$ . Therefore  $\psi_{p,\mu}$  is a strong solution of (3.40), i.e.:

$$(3.41) \quad \mu \partial_x^4 \psi_{p,\mu}(x) - \partial_x^2 \psi_{p,\mu}(x) + \psi_{p,\mu}(x) = \frac{\alpha_{p,\mu}}{p+1} \psi_{p,\mu}^{p+1}(x), \quad \forall x \in \mathbb{R}.$$

For  $0 < \mu < \frac{1}{4}$ , let us set:

$$(3.42) \quad s_1 = \pm \frac{\sqrt{1 - \sqrt{1 - 4\mu}}}{\sqrt{2\mu}} \quad \text{and} \quad s_2 = \pm \frac{\sqrt{1 + \sqrt{1 - 4\mu}}}{\sqrt{2\mu}}.$$

One can check that  $e^{s_1 x}$  and  $e^{s_2 x}$  are solutions of the linear asymptotic equation:  $\mu \partial_x^4 \psi - \partial_x^2 \psi + \psi = 0$ . Then, we infer the following exponential decay properties:

$$(3.43) \quad |\partial_x^s \psi_{p,\mu}(x)| \lesssim e^{-|x|}, \quad \text{for } |x| \gg 1, \quad s \in \mathbb{N}, \quad \text{and } 0 < \mu \ll 1.$$

*Step 3. Refined estimate on Lagrange multipliers.*

We will prove that the family  $\{\alpha_{p,\mu}, 0 < \mu < 1\}$  of Lagrange multipliers is bounded. Multiplying (3.41) by  $\psi_{p,\mu}$ , and integrating on space, we obtain

$$(3.44) \quad \alpha_{p,\mu} = \frac{2I_\mu(\psi_{p,\mu})}{(p+2)\beta_p} = \frac{2S_{p,\mu}^{\beta_p}}{(p+2)\beta_p} \leq \frac{2I_\mu(\varphi_{1,p,0})}{(p+2)\beta_p}$$

where  $\varphi_{1,p,0}$  is the soliton of the gKdV equation given by (1.9). From the uniqueness of  $\varphi_{1,p,0}$  (up to the symmetries of the equation) and the definition of  $\beta_p$ , we know that  $\varphi_{1,p}$  is the unique solution to the constraint minimization problem  $S_{p,0}^{\beta_p}$  which forces

$$(3.45) \quad \frac{2I_0(\varphi_{1,p,0})}{(p+2)\beta_p} = 1$$

and thus

$$(3.46) \quad \alpha_{p,\mu} \leq 1 + O(\mu).$$

This last estimate together with (3.44) clearly ensure that the family  $\{\psi_{p,\mu}, 0 < \mu \ll 1\}$  is bounded in  $H^1(\mathbb{R})$ .

*Step 4.  $H^2$ -boundedness of the family  $\{\psi_{p,\mu}, 0 < \mu < 1\}$ .*

We will prove that the family  $\{\psi_{p,\mu}, 0 < \mu < 1\}$  is bounded in  $H^2(\mathbb{R})$ . Applying the Fourier transformation on equation (3.41), we obtain for all  $\xi \in \mathbb{R}$ ,

$$(3.47) \quad (1 + |\xi|^2 + \mu |\xi|^4) \widehat{\psi_{p,\mu}}(\xi) = \frac{\alpha_{p,\mu}}{p+1} \widehat{\psi_{p,\mu}^{p+1}}(\xi).$$

Since  $\{\alpha_{p,\mu}, 0 < \mu < 1\}$  is bounded (from Step 3), it suffices to prove that the family  $\{\widehat{\psi_{p,\mu}^{p+1}}, 0 < \mu < 1\}$  is bounded in  $L^2(\mathbb{R})$ , and by the Plancherel-Parseval identity, this is equivalent to prove that the family  $\{\psi_{p,\mu}, 0 < \mu < 1\}$  is bounded in  $L^{2p+2}(\mathbb{R})$ . Thus, we compute

$$(3.48) \quad \begin{aligned} \|\psi_{p,\mu}\|_{L^{2p+2}(\mathbb{R})}^{2p+2} &\leq \|\psi_{p,\mu}\|_{L^\infty(\mathbb{R})}^{2p} \|\psi_{p,\mu}\|_{L^2(\mathbb{R})}^2 \\ &\leq 2^{-p} \|\psi_{p,\mu}\|_{H^1(\mathbb{R})}^{2p+2}, \end{aligned}$$

where we use the Sobolev embedding of  $H^1(\mathbb{R})$  into  $C^0(\mathbb{R})$  (with the best constant). Then, using that  $\{\psi_{p,\mu}, 0 < \mu < 1\}$  is bounded in  $H^1(\mathbb{R})$  (from Step 1), and (3.48), we obtain the desired result. Note that this boundedness, together with the fact that  $\varphi_{1,p,0}$  is a solution to the constraint minimization problem  $S_{p,0}^{\beta_p}$ , ensures that

$$(3.49) \quad \liminf_{\mu \rightarrow 0} I_\mu(\psi_{p,\mu}) = \liminf_{\mu \rightarrow 0} I_0(\psi_{p,\mu}) \geq I_0(\varphi_{1,p,0}).$$

Gathering this last estimate with (3.44)-(3.46), we get

$$(3.50) \quad \lim_{\mu \rightarrow 0} S_{p,\mu}^{\beta_p} = \lim_{\mu \rightarrow 0} I_\mu(\psi_{p,\mu}) = \lim_{\mu \rightarrow 0} I_0(\psi_{p,\mu}) = I_0(\varphi_{1,p,0}) \quad \text{and} \quad \lim_{\mu \rightarrow 0} \alpha_{p,\mu} = 1.$$

*Step 5.* Strong convergence in  $H^1(\mathbb{R})$  as  $\mu \rightarrow 0$  and construction of the family of solitary waves  $\{\varphi_{1,p,\mu}, 0 < \mu < 1\}$ .

Let  $(\mu_n)_{n \geq 1} \subset \mathbb{R}_+^*$  be a sequence which decreases to  $0^+$ , and for simplicity, let us rename  $\psi_{p,\mu_n} = \psi_n$  and  $\alpha_{p,\mu_n} = \alpha_{p,n}$ . Then, there exist a sub-sequence, still denoted by  $(\psi_n)_{n \geq 1}$ , that converges weakly in  $H_e^2(\mathbb{R})$  towards a function  $\psi_\infty \in H_e^2(\mathbb{R})$ . Passing to the limit in (3.41) by making use of (3.50), we get that  $\psi_\infty$  satisfies (1.13) with  $c = 1$ . Note that, to pass to the limit on the nonlinear part of equation (3.41) we deal as follows: using the pointwise convergence (3.17), the  $H^2$ -boundedness and the continuity of the function:  $y \mapsto y^{p+1}$ , one can check that

$$(3.51) \quad \psi_n^{p+1} \rightarrow \psi_\infty^{p+1} \text{ a.e. on } \mathbb{R}.$$

Then, applying Lebesgue's Dominated Convergence Theorem, it holds for all  $\phi \in H^2(\mathbb{R})$ ,

$$(3.52) \quad \lim_{n \rightarrow +\infty} \langle \psi_n^{p+1}, \phi \rangle_{L^2} = \langle \psi_\infty^{p+1}, \phi \rangle_{L^2}.$$

Now, the uniqueness result (up to symmetries) of the solutions to (1.13) ensures that  $\psi_\infty = \varphi_{1,p,0}$  whenever  $p$  is even and  $\psi_\infty = \mp \varphi_{1,p,0}$  whenever  $p$  is odd. Since, according to (3.50),

$$\lim_{n \rightarrow +\infty} I_0(\psi_n) = I_0(\mp \varphi_{1,p,0}) = I_0(\psi_\infty),$$

it follows that  $(\psi_n)_{n \geq 0}$  converges actually strongly to  $\psi_\infty$  in  $H^1(\mathbb{R})$  and thus  $K_p(\psi_\infty) = \beta_p > 0$ . This ensures that in all cases,  $\psi_\infty = \varphi_{1,p,0}$  and thus

$$(3.53) \quad \psi_\mu \rightarrow \varphi_{1,n,0} \text{ in } H^1(\mathbb{R}) \quad \text{as } \mu \rightarrow 0.$$

Finally, setting

$$\varphi_{1,p,\mu} = \alpha_{p,\mu}^{1/p} \psi_{p,\mu}, \quad \forall n \in \mathbb{N}^*.$$

It is easy to check that for any  $p \in \mathbb{N}^*$  and any  $\mu > 0$ ,  $\varphi_{1,p,\mu}$  is a solution of (1.4) with  $c = 1$ . Moreover, according to Step 4., (3.50) and (3.53), the family  $\{\varphi_{1,p,\mu}, 0 < \mu < 1\}$  is bounded in  $H^2(\mathbb{R})$  and

$$(3.54) \quad \varphi_{1,p,\mu} \rightarrow \varphi_{1,p,0} \text{ in } H^1(\mathbb{R}) \quad \text{as } \mu \rightarrow 0.$$

This completes the proof of the lemma.  $\square$

**3.2. Uniqueness of even ground states.** Note that the construction of the  $\varphi_{1,p,\mu}$  ensures that they are even ground state solutions (in the sense of Definition 1.2) to (1.13) with  $c = 1$ . The following lemma ensures that for  $\mu$  small enough, the even ground state solution to (1.13), with  $c = 1$ , is unique.

**Lemma 3.2** (Uniqueness of  $\varphi_{1,p,\mu}$  for  $0 < \mu \ll 1$ ). *Let  $p \in \{1, 2, 3\}$  be fixed and set  $c = 1$ . There exists  $\delta'_p > 0$  such that for all  $\mu \in ]0, \delta'_p[$ , there exists a unique even ground state solution  $\varphi_{1,p,\mu}$  to equation (1.4). Moreover, the map  $\mu \mapsto \varphi_{1,p,\mu}$  is-continuous from  $]0, \delta'_p[$  into  $H^1(\mathbb{R})$ .*

**Proof.** We fix  $p \in \{1, 2, 3\}$ . We first claim that for any  $\varepsilon > 0$  there exists  $\mu_\varepsilon > 0$  such that for any  $0 < \mu < \mu_\varepsilon$ , any even ground state solution  $\varphi$  to (1.4) with  $c = 1$  satisfies

$$(3.55) \quad \|\varphi - \varphi_{1,p,0}\|_{H^1} < \varepsilon.$$

To prove this claim we proceed by contradiction, assuming that there exist  $\varepsilon_0 > 0$ , a sequence  $(\mu_n)_{n \geq 0}$  of positive real numbers that converges to 0 and a sequence  $(\varphi_n)_{n \geq 0}$  of ground state solutions to (1.4) with  $c = 1$  and  $\mu = \mu_n$  such that

$$(3.56) \quad \|\varphi_n - \varphi_{1,p,0}\|_{H^1} \geq \varepsilon_0, \quad \forall n \in \mathbb{N}.$$

From (1.4) we infer that  $K_p(\varphi_n) = \frac{2}{p+2} I_{\mu_n}(\varphi_n) > 0$ . We set

$$\psi_n = \left( \frac{K_p(\varphi_{1,p,0})}{K_p(\varphi_n)} \right)^{\frac{1}{p+2}} \varphi_n$$

so that  $K_p(\psi_n) = K_p(\varphi_{1,p,0}) = \beta_p$ .  $(\psi_n)_{n \geq 0}$  is thus a sequence of solutions to  $S_{p,\mu_n}^{\beta_p}$  and as in the proof of Lemma 3.1 (Step 5.), it follows that  $(\psi)_{n \geq 0}$  is bounded in  $H^2(\mathbb{R})$  and  $\psi_n \rightarrow \varphi_{1,p,0}$  in  $H^1(\mathbb{R})$ . Therefore,

$$\varphi_n = \left( \frac{2I_\mu(\psi_n)}{(p+2)\beta_p} \right)^{1/p} \psi_n \rightarrow \varphi_{1,p,0} \quad \text{in } H^1(\mathbb{R})$$

which contradicts (3.56).

Let  $\varphi_{1,p,\mu}$  and  $\tilde{\varphi}_{1,p,\mu}$  be two ground ground states of (1.4) with  $c = 1$  and  $\mu > 0$ . Following the idea of Kenig et al. [12] (Proposition 3), we will prove, arguing by contradiction, that  $\tilde{w} = \varphi_{1,p,\mu} - \tilde{\varphi}_{1,p,\mu} = 0$  as soon as  $\mu$  is small enough. Note that, on account of (3.55), it holds  $\|\tilde{w}\|_{H^1} \leq \varepsilon(\mu)$  with  $\varepsilon(y) \rightarrow 0$  as  $y \rightarrow 0$ . We set  $F(x) = \frac{1}{p+1} x^{p+1}$  so that  $\tilde{w}$  satisfies

$$(3.57) \quad \mu \partial_x^4 \tilde{w} - \partial_x^2 \tilde{w} + \tilde{w} = F(\varphi_{1,p,\mu}) - F(\tilde{\varphi}_{1,p,\mu})$$

Denoting by  $\mathcal{L}_{1,p,\mu}$  the operator defined in Proposition 1.1 with  $\phi_{c,p,\mu} = \varphi_{1,p,\mu}$ , it holds

$$\mathcal{L}_{1,p,\mu} = \mu \partial_x^4 - \partial_x^2 + 1 - F'(\varphi_{1,p,\mu})$$

and thus

$$\mathcal{L}_{1,p,\mu} \tilde{w} = F(\varphi_{1,p,\mu}) - F(\tilde{\varphi}_{1,p,\mu}) + F'(\varphi_{1,p,\mu})(\varphi_{1,p,\mu} - \tilde{\varphi}_{1,p,\mu}) = \frac{F''(W_{p,\mu})}{2} \tilde{w}^2,$$

with, for all  $x \in \mathbb{R}$ ,  $W_{p,\mu}(x) \in [\varphi_{1,p,\mu}(x), \tilde{\varphi}_{1,p,\mu}(x)]$ . Therefore, assuming that  $\tilde{w} \neq 0$  and setting  $w = \frac{\tilde{w}}{\|\tilde{w}\|_{H^1}}$ , we get

$$(3.58) \quad \|\mathcal{L}_{1,p,\mu} w\|_{L^2(\mathbb{R})} \lesssim \frac{1}{\|\tilde{w}\|_{H^1}} \|\tilde{w}\|_{L^2} \|\tilde{w}\|_{H^1} \lesssim \|\tilde{w}\|_{L^2} \|w\|_{H^1}$$

so that

$$(3.59) \quad \begin{aligned} \left| \langle \mathcal{L}_{1,p,0} w, w \rangle_{L^2} \right| &= \left| \langle \mathcal{L}_{1,p,\mu} w, w \rangle_{L^2} + \langle w^2, \varphi_{1,p,\mu}^p - \varphi_{1,p,0}^p \rangle_{L^2} - \mu \|\partial_x^2 w\|_{L^2(\mathbb{R})}^2 \right| \\ &\lesssim \varepsilon(\mu) + \mu. \end{aligned}$$

On the other hand, it is well-known that for  $p \in \{1, 2, 3\}$ ,  $\frac{d}{dc}|_{c=1} \|\varphi_{c,p,0}\|_{L^2}^2 < 0$  which ensures that there exists  $\alpha_p > 0$  such that

$$\langle \mathcal{L}_{1,p,0} \phi, \phi \rangle \geq \alpha_p \|\phi\|_{H^1}^2$$



for all  $\phi \in H^1(\mathbb{R})$  that satisfies the orthogonality conditions :  $\langle \phi, \varphi'_{1,p,0} \rangle = \langle \phi, \varphi_{1,p,0} \rangle = 0$ . Now, since  $w$  is even and  $\varphi'_{1,p,\mu}$  is odd, we have the first orthogonality condition:

$$(3.60) \quad \langle w_n, \varphi'_{c,p} \rangle_{L^2} = 0.$$

Moreover, using that  $\mathcal{L}_{1,p,0} \partial_{c|c=1} \varphi_{c,p,0} = \varphi_{1,p,0}$  we get

$$(3.61) \quad \begin{aligned} |\langle w, \varphi_{1,p,0} \rangle_{L^2}| &= \left| \langle w, \mathcal{L}_{1,p,0} \partial_{c|c=1} \varphi_{c,p,0} \rangle_{L^2} \right| \\ &= \left| \langle \mathcal{L}_{1,p,0} w, \partial_{c|c=1} \varphi_{c,p,0} \rangle_{L^2} \right| \\ &\leq \left| \langle \mathcal{L}_{1,p,\mu} w, \partial_{c|c=1} \varphi_{c,p,0} \rangle_{L^2} \right| + \left| \langle w \partial_{c|c=1} \varphi_{c,p,0}, \varphi_{c,p,\mu}^p - \varphi_{1,p,0}^p \rangle_{L^2} \right| \\ &\quad + \mu \left| \langle \partial_x^2 w, \partial_x^2 \partial_{c|c=1} \varphi_{c,p,0} \rangle_{L^2} \right| \\ &\lesssim (\varepsilon(\mu) + \mu) \|w\|_{H^1} \end{aligned}$$

so that for  $\mu$  small enough,  $w$  is almost orthogonal to  $\varphi_{1,p,0}$ . Therefore, proceeding as in Subsection 2.3, we infer that for  $\mu > 0$  small enough,

$$\langle \mathcal{L}_{1,p,0} w, w \rangle \geq \frac{\alpha_p}{2} \|w\|_{H^1}^2 = \frac{\alpha_p}{2},$$

which contradicts (3.59) and proves the uniqueness result for  $0 < \mu < \delta'_p$ . Finally, once the uniqueness is proved, the convergence of  $\varphi_{1,p,\mu}$  towards  $\varphi_{1,p,\mu_0}$  as  $\mu \rightarrow \mu_0$  can be proved exactly as the convergence of  $\varphi_{1,p,\mu}$  towards  $\varphi_{1,p,0}$  as  $\mu \rightarrow 0$ .  $\square$

Now, we notice that  $v$  satisfies (1.4) with  $p \in \mathbb{N}^*$ ,  $c = 1$  and  $\mu > 0$  if and only if  $v_\mu = \mu^{1/p} v(\sqrt{\mu} \cdot)$  satisfies (1.4) with  $p \in \mathbb{N}^*$ ,  $c = \mu$  and  $\mu = 1$ . The following uniqueness result on the ground states to (1.4) with  $\mu = 1$  is thus a direct consequence of Lemma 3.2.

**3.3. End of the proof of Theorem 1.1.** Let  $p \in \{1, 2, 3\}$ . It remains to prove the orbital stability result. Since  $u$  is a solution to (1.1) if and only if  $u_\mu(t, x) = \mu^{1/p} u(\mu^{3/2} t, \sqrt{\mu} x)$  is a solution to (1.1) with  $\mu = 1$ , the orbital stability of  $\varphi_{\mu,p,1}$  is equivalent to the orbital stability of  $\varphi_{c,p,\mu}$ . To prove the orbital stability of this last ground state for  $\mu > 0$  small enough, we rely on a continuity argument as in the preceding section. First, by the  $H^1$ -convergence result (3.3), it is easy to check that for  $\mu > 0$  small enough, any  $v \in H^2(\mathbb{R})$  satisfying  $\langle v, \varphi_{1,p,\mu} \rangle = \langle v, \varphi'_{1,p,\mu} \rangle = 0$  is almost orthogonal in  $L^2(\mathbb{R})$  to  $\varphi_{1,p,0}$  and  $\varphi'_{1,p,0}$ . Second, recalling that, for  $p \in \{1, 2, 3\}$ ,  $\frac{d}{dc}|_{c=1} \|\varphi_{c,p,0}\|_{L^2}^2 < 0$  and arguing exactly as in Subsection 2.3, we obtain that  $\mathcal{L}_{1,p,0}$  is coercive in  $H^1(\mathbb{R})$  under these almost orthogonality conditions. It follows that

$$(3.62) \quad \begin{aligned} \langle \mathcal{L}_{1,p,\mu} v, v \rangle_{L^2} &= \mu \|\partial_x^2 v\|_{L^2(\mathbb{R})}^2 + \langle \mathcal{L}_{1,p} v, v \rangle_{L^2} - \langle v^2, \varphi_{1,p,\mu}^p - \varphi_{1,p,0}^p \rangle_{L^2} \\ &\geq \mu \|\partial_x^2 v\|_{L^2(\mathbb{R})}^2 + \frac{\alpha_p}{2} \|v\|_{H^1(\mathbb{R})}^2 - \varepsilon(\mu) \|v\|_{L^2(\mathbb{R})}^2 \\ &\geq \min\left\{\mu, \frac{\alpha_p}{2} - \varepsilon(\mu)\right\} \|v\|_{H^2(\mathbb{R})}^2, \end{aligned}$$

with  $\alpha_p > 0$  and  $\varepsilon(y) \rightarrow 0$  as  $y \rightarrow 0$ . By applying Proposition 1.1, we thus get that there exists  $\delta_p > 0$  such that for all  $\mu \in ]0, \delta_p[$ ,  $\varphi_{1,p,\mu}$  is orbitally stable. This completes the proof of the theorem.  $\square$

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