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PATRICK GÉRARD, ENNO LENZMANN, OANA POCOVNICU, AND PIERRE RAPHAËL

Abstract. We consider the focusing cubic half-wave equation on the real line

$$i\partial_t u + |D|u = |u|^2 u, \quad \widehat{|D|u}(\xi) = |\xi|\hat{u}(\xi), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

We construct an asymptotic global-in-time compact two-soliton solution with arbitrarily small L^2-norm which exhibits the following two regimes: (i) a transient turbulent regime characterized by a dramatic and explicit growth of its H^1-norm on a finite time interval, followed by (ii) a saturation regime in which the H^1-norm remains stationary large forever in time.

Contents

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1. Introduction

In this paper we consider the L^2-critical focusing half-wave equation on $\mathbb{R}$:

$$\text{(Half-wave)} \quad \begin{cases} i\partial_t u + |D|u = |u|^2 u \\ u_{|t=0} = u_0 \in H^{\frac{1}{2}}(\mathbb{R}) \end{cases}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \quad u(t,x) \in \mathbb{C}, \qquad (1.1)$$

where we use the pseudo–differential operators

$$D = -i\partial_x, \quad \widehat{|D|f}(\xi) = |\xi|\widehat{f}(\xi).$$

Evolution problems with nonlocal dispersion such as (1.1) naturally arise in various physical settings, including continuum limits of lattice systems [25], models for wave turbulence [6, 30], and gravitational collapse [10, 12]. The phenomenon that we study in this paper is the growth of high Sobolev norms in infinite dimensional Hamiltonian systems, which has attracted considerable attention over the past twenty years [2, 49, 30, 4, 52, 6, 7, 13, 43, 21, 19, 22, 23, 20, 17] . The aim of this paper is to develop a robust approach for constructing solutions whose high Sobolev norms grow over time, based on multisolitary wave interactions. In particular, we construct an asymptotic two-soliton solution of (1.1) that exhibits the following two regimes: (i) a transient turbulent regime characterized by a dramatic and explicit growth of its H^1-norm on a finite time interval, followed by (ii) a saturation regime in which the H^1-norm remains stationary large forever in time.

1.1. The focusing cubic half-wave equation.

Let us recall the main qualitative features of the half-wave model (1.1). The Cauchy problem is locally well-posed in $H^{\frac{1}{2}}$, see [15, 26], and for all $u_0 \in H^{\frac{1}{2}}$, there exists a unique solution $u \in \mathcal{C}([0,T), H^{\frac{1}{2}})$ with the blow up alternative

$$T < +\infty \text{ implies } \lim_{t \uparrow T} \|u(t)\|_{H^{\frac{1}{2}}} = +\infty. \qquad (1.2)$$

Moreover, additional H^s-regularity on the data, $s > \frac{1}{2}$, is propagated by the flow. The Hamiltonian model (1.1) admits three conservation laws:

$$\text{Mass}: \quad \int |u(t,x)|^2 dx = \int |u_0(x)|^2 dx$$

$$\text{Momentum}: \quad \text{Re}\left(\int Du\overline{u}(t,x)dx\right) = \text{Re}\left(\int Du_0\overline{u_0}(x)dx\right)$$

$$\text{Energy}: \quad E(u(t)) := \frac{1}{2}\int ||D|^{\frac{1}{2}}u|^2(t,x)dx - \frac{1}{4}\int |u|^4(t,x)dx = E(u_0).$$

The scaling symmetry

$$u_\lambda(t,x) = \lambda^{\frac{1}{2}}u(\lambda^2 t, \lambda x)$$

leaves the L^2-norm invariant

$$\|u_\lambda(t,\cdot)\|_{L^2} = \|u(\lambda^2 t,\cdot)\|_{L^2}$$

and hence the problem is L^2-critical.

By a standard variational argument, the best constant in the Gagliardo-Nirenberg inequality

$$\|u\|_{L^4}^4 \lesssim \||D|^{\frac{1}{2}}u\|_{L^2}^2\|u\|_{L^2}^2, \quad \forall u \in H^{\frac{1}{2}},$$

is attained on the *unique* positive even ground state solution to

$$|D|Q + Q - Q^3 = 0.$$

Note that the uniqueness of Q is a nontrivial claim, recently obtained in [11]. This implies the lower bound

$$E(u) \geq \frac{1}{2}\left[1 - \frac{\|u\|_{L^2}^2}{\|Q\|_{L^2}^2}\right]\int_{\mathbb{R}} ||D|^{\frac{1}{2}}u|^2 dx, \quad \forall u \in H^{\frac{1}{2}}. \tag{1.3}$$

Using the conservation of mass and energy, it then follows for $u_0 \in H^{\frac{1}{2}}$ with $\|u_0\|_{L^2} < \|Q\|_{L^2}$ that

$$\|u(t)\|_{H^{\frac{1}{2}}} \leqslant C(\|u_0\|_{L^2}, E(u_0)), \quad \forall t \in \mathbb{R}. \tag{1.4}$$

Combining this with (1.2), one obtains the global existence criterion:

$$u_0 \in H^{\frac{1}{2}} \text{ and } \|u_0\|_{L^2} < \|Q\|_{L^2} \text{ imply } T = +\infty. \tag{1.5}$$

This criterion is sharp as there exist minimal mass finite energy finite time blow up solutions, see [26]. In this paper we will only consider solutions with $u_0 \in H^1$ of arbitrarily small mass, which are hence global-in-time $u \in \mathcal{C}(\mathbb{R}, H^1)$.

1.2. **Growth of high Sobolev norms.** One of the main topics in the study of nonlinear Hamiltonian PDEs is the long time behaviour of global-in-time solutions. A possible type of behavior, that attracted significant attention over the last twenty years, is the so called *forward energy cascade* phenomenon. This phenomenon refers to the *conserved* energy of global-in-time solutions moving from low-frequency concentration zones to high-frequency ones over time. One way to illustrate it is the growth of high Sobolev norms:

$$\|u(t)\|_{H^s} = \left(\int \langle\xi\rangle^{2s}|\hat{u}(t,\xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

Indeed, for sufficiently large $s > 0$, above the level of regularity of the conserved Hamiltonian, the growth over time of $\|u(t)\|_{H^s}$ indicates that the Fourier transform $\hat{u}(t,\xi)$ is supported on higher and higher frequencies ξ as the time t increases. To the best of the authors' knowledge, all the rigorous mathematical analysis that

has been done on the forward energy cascade focuses on finding infinite dimensional Hamiltonian PDEs that admit examples of solutions exhibiting growth of high Sobolev norms. A lot of the results available are in the context of nonlinear Schrödinger equations (NLS). In particular, for the defocusing cubic nonlinear Schrödinger equation on $\mathbb{T}^2$, Bourgain [5] asked whether there exist solutions u with initial condition $u_0 \in H^s(\mathbb{T}^2)$, $s > 1$, such that

$$\limsup_{t\to\infty} \|u(t)\|_{H^s} = \infty.$$

Despite attracting considerable attention, this question remains unanswered.

The forward energy cascade phenomenon also appears in the physical theory of *wave* (*weak*) *turbulence*. This is a theory in plasma physics and water waves, based on pioneering work of Zakharov from the 1960s, with many similarities to Kolmogorov's theory of hydrodynamical turbulence. It can be loosely defined as the "out-of-equilibrium statistics of random nonlinear waves" (see [21]). Even though wave turbulence refers to a statistical description of solutions and not to single solutions, and even though this theory does not yet have a rigorous mathematical justification, it is believed that exhibiting examples of solutions whose high Sobolev norms grow over time is a first step and a minimal necessary condition for wave turbulence. As far as the authors are aware, all mathematically rigorous results that are available are in this spirit, and so is the main result of this paper.

In the following, we briefly mention some of the references in the literature regarding the growth of high Sobolev norms for nonlinear Hamiltonian PDEs. First, in the context of NLS, polynomial-in-time *upper* bounds on the growth of $\|u(t)\|_{H^s}$,

$$\|u(t)\|_{H^s} \lesssim \langle t\rangle^{c(s-1)}, \quad s > 1,$$

were obtained; see Bourgain [2, 5], Staffilani [49], Sohinger [47, 48], Colliander, Kwon, and Oh [8].

The first examples of Hamiltonian PDEs (nonlinear Schrödinger equations and nonlinear wave equations) that admit solutions with energy transfer were constructed by Bourgain [1, 2, 3]. However, these examples do not deal with standard NLS or NLW, but with modifications of these specifically designed to exhibit infinite growth of high Sobolev norms (these are PDEs involving, instead of the Laplace operator, a perturbation of it, or PDEs with a suitably chosen nonlocal nonlinearity). In [29], Kuksin considered small dispersion cubic NLS and proved that generic solutions grow larger than a negative power of the dispersion. A seminal result is that by Colliander, Keel, Staffilani, Takaoka, and Tao [7] who proved arbitrarily large growth of high Sobolev norms in finite time for the defocusing cubic NLS on $\mathbb{T}^2$. More precisely, given $s > 1$, $\varepsilon \ll 1$, and $K \gg 1$, they constructed a solution u such that

$$\|u(0)\|_{H^2} \leqslant \varepsilon \quad \text{and} \quad \|u(T)\|_{H^s} \geqslant K,$$

for some finite time $T > 0$. The influential result in [7], especially their intricate combinatorial construction, was refined and generalized to various other settings [21, 19, 18, 22, 20, 23]. In particular, in [22], an example of infinite growth of high Sobolev norms was obtained for the defocusing cubic NLS on $\mathbb{R} \times \mathbb{T}^d$, $d \geqslant 2$. For the cubic NLS on $\mathbb{T}^2$, however, the fate of the solution u after the growth time T remains unknown.

For the cubic half-wave equation, due to mass and energy conservation, the $H^{\frac{1}{2}}$-norm of solutions with initial data in $H^{\frac{1}{2}}$ is uniformly bounded in time, both for the defocusing equation, as well as for the focusing equation with initial data of sufficiently small mass $\|u(0)\|_{L^2} < \|Q\|_{L^2}$ (see (1.4) above). However, in the spirit

of [7], arbitrarily large growth in finite time of higher Sobolev norms — H^s-norms with $s > 1/2$ — was proved on $\mathbb{R}$ in [44][1] and on $\mathbb{T}$ in [15]. As in [7], the behaviour of the solutions that exhibit growth remains unknown after some finite time, which is what motivated our work in the present paper. The results in [15, 44] are based on information on the totally resonant model associated with the cubic half-wave equation, namely the Szegő equation. Infinite growth of high Sobolev norms for solutions of the Szegő equation was obtained on $\mathbb{R}$ in [43] and on $\mathbb{T}$ in [17]. Moreover, on $\mathbb{T}$, this was shown [17] to be a generic phenomenon, displaying infinitely many *forward* and *backward* energy cascades. Also notice that long time divergence of high Sobolev norms was also obtained for a perturbation of the cubic Szegő equation on $\mathbb{T}$ in [51]. We present below the key features of the Szegő equation and its relation to the cubic half-wave equation.

1.3. **The Szegő program.** Applying the Szegő projector Π_+ of L^2 onto nonnegative Fourier modes:

$$\widehat{\Pi^+ u}(\xi) = \mathbf{1}_{\xi>0}\hat{u}(\xi),$$

the half-wave equation (1.1) becomes

$$\begin{cases} i(\partial_t u_+ - \partial_x u_+) = \Pi^+(|u|^2 u) \\ i(\partial_t u_- + \partial_x u_-) = (I - \Pi^+)(|u|^2 u) \\ u_+ := \Pi^+ u \ , \ u_- := (I - \Pi^+)u \ . \end{cases}$$

For small data in the range of Π_+ and of norm $\varepsilon \ll 1$ in a sufficiently regular Sobolev space one can show [15, 44] that, for times of order $\varepsilon^{-2}|\log \varepsilon|$, an approximation of the half-wave flow is given by the cubic Szegő equation

$$\begin{cases} i\partial_t u = \Pi^+(|u|^2 u) \\ u_{|t=0} = u_0 \in H^{\frac{1}{2}}. \end{cases} \tag{1.6}$$

The Szegő equation can be understood as the *totally* resonant model associated to (1.1). It is still a nonlinear Hamiltonian model, well-posed in $H^{\frac{1}{2}}$, and the conservation of mass and momentum implies that all $H^{\frac{1}{2}}$-solutions are global-in-time and

$$\|u(t)\|_{H^{\frac{1}{2}}} \simeq \|u(0)\|_{H^{\frac{1}{2}}}, \quad \forall t \in \mathbb{R}.$$

A spectacular feature of the cubic Szegő equation discovered in [13] is its *complete integrability* in the sense of the existence of a Lax pair, which in particular allows for the derivation of explicit families of special solutions of either multisolitary waves or breather-type, both on the line and on the torus, see [42, 43, 13, 14, 16, 17]. The complete integrability implies the conservation of infinitely many conservation laws which, however, roughly speaking, all live at the $H^{\frac{1}{2}}$-level of regularity only.

In [43], Pocovnicu exhibits for the flow on the line, one of the very first explicit examples of growth of high Sobolev norms for a nonlinear infinite dimensional Hamiltonian model:

$$\|u(t)\|_{H^{\frac{1}{2}}} \lesssim 1, \quad \lim_{t\to+\infty} \|u(t)\|_{H^1} = +\infty \ \text{ as } \ t \to +\infty.$$

The analysis in [43] is based on the explicit computation of a two-soliton solution for the cubic Szegő flow, relying on complete integrability.[2] Indeed, as observed in

[1] In [44], only a *relative* growth of high Sobolev norms was obtained, $\frac{\|u(T_\varepsilon)\|_{H^s}}{\|u(0)\|_{H^s}} \to \infty$ as $\varepsilon \to 0$ for some $T_\varepsilon \gg 1$. However, this readily yields arbitrary large growth in finite time via an L^2-invariant scaling argument. Secondly, the result in [44] is stated for the defocusing half-wave equation, but essentially the same proof works for the focusing half-wave equation with initial data of small mass.

[2] The key property that triggers growth of high Sobolev norms $\|u(t)\|_{H^s} \sim t^{2s-1}$, $s > \frac{1}{2}$, is that the Hankel operator H_u in the Lax pair of the Szegő equation has a multiple (double) eigenvalue.

[42], (1.6) admits a traveling wave solution

$$u(t,x) = Q^+(x-t)e^{-it} \quad \text{with} \quad Q^+(x) := \frac{2}{2x+i}. \tag{1.7}$$

Using complete integrability formulas, an exact two-soliton can be computed:

$$u(t,x) = \alpha_1(t)Q^+\left(\frac{x-x_1(t)}{\kappa_1(t)}\right)e^{-i\gamma_1(t)} + \alpha_2(t)Q^+\left(\frac{x-x_2(t)}{\kappa_2(t)}\right)e^{-i\gamma_2(t)},$$

with the asymptotic behavior on the manifold of solitary waves,

$$\begin{cases} \alpha_1(t) \sim 1, & \kappa_1(t) \sim 1-\eta, \quad x_1(t) \sim (1-\eta)t, \quad 0 < \eta \ll 1 \\ \alpha_2(t) \sim 1, & \kappa_2(t) \sim \frac{1}{t^2}, \quad x_2(t) \sim t. \end{cases} \tag{1.8}$$

In particular, this two-soliton exhibits growth of high Sobolev norms over time $\|u(t)\|_{H^s} \sim t^{2s-1}$, $s > \frac{1}{2}$, and the mechanism of growth is the concentration of the second bubble $k_2(t) \sim \frac{1}{t^2}$.

The full dynamical system underlying two-solitons for the Szegő equation and the associated codimension one set of turbulent initial data is revisited in details in Appendix B.

Combining the growth of high Sobolev norms for a two-soliton of the Szegő equation on $\mathbb{R}$ [43] discussed above, with a long time approximation theorem relating the Szegő model and the half-wave equation, yields the following arbitrarily large growth in finite time result for the half-wave equation:

Theorem 1.1 ([44]). *Let $0 < \varepsilon \ll 1$. There exists a solution of the (focusing/defocusing) cubic half-wave equation on $\mathbb{R}$ and there exists $T \sim e^{\frac{c}{\varepsilon^3}}$ such that*

$$\|u(0)\|_{H^1} = \varepsilon \quad \text{and} \quad \|u(T)\|_{H^1} \geqslant \frac{1}{\varepsilon} \gg 1.$$

As in [7], the behaviour of the turbulent solution in the above theorem after the time T remains unknown. In this paper, we construct a turbulent solution of (1.1) that we can control *for all future times*. Furthermore, our aim in this paper is to develop a robust approach to compute turbulent regimes based on multisolitary wave interactions, avoiding on purpose complete integrability tools.

1.4. **Mass-subcritical traveling waves.** As observed in [26] following [11], the half-wave problem (1.1) admits *mass-subcritical* small speed traveling waves[3]

$$u_\beta(t,x) = Q_\beta\left(\frac{x-\beta t}{1-\beta}\right)e^{-it}, \quad \frac{|D|-\beta D}{1-\beta}Q_\beta + Q_\beta - |Q_\beta|^2 Q_\beta = 0, \tag{1.9}$$

with

$$\lim_{\beta\to 0} Q_\beta = Q, \quad \|Q_\beta\|_{L^2} < \|Q\|_{L^2}.$$

An elementary but spectacular observation is that these traveling waves in fact exist for all $|\beta| < 1$ and converge in the *singular relativistic limit* $\beta \to 1$ to the soliton of the limiting Szegő equation given by (1.7):

$$\lim_{\beta\uparrow 1} \|Q_\beta - Q^+\|_{H^{\frac{1}{2}}} = 0.$$

[3]Note that this phenomenon does not exist for the mass-critical focusing nonlinear Schrödinger equation on $\mathbb{R}$ due to the degeneracy induced by the Galilean symmetry $u_\beta(t,x) = Q_\beta(x-\beta t)e^{i\gamma_\beta(t)}$ with $Q_\beta(x) = Q(x)e^{i\beta x}$ and hence $\|Q_\beta\|_{L^2} = \|Q\|_{L^2}$ for all $\beta \in \mathbb{R}$, and indeed solutions with mass below that of the ground state scatter [9].

See Section 2. Note from (1.9) that this is fundamentally a singular elliptic limit, and the associated almost relativistic traveling waves are arbitrarily small in the critical space:

$$\lim_{\beta\uparrow 1} \|u_\beta(t,\cdot)\|_{L^2} = 0.$$

Hence, another link is made between the half-wave problem and its totally resonant limit given by the Szegő equation through the sole consideration of the full family of *nonlinear* traveling waves.

1.5. **Statement of the result.** In Theorem 1.1, the turbulent solution of (1.1) was constructed as a long time approximation of the turbulent two-soliton of the Szegő equation. The approximation theorem used is valid for any solution of the Szegő equation (respectively of the half-wave equation) with small regular data, not only for two-solitons. In this paper, we take a more efficient approach. Instead of approximating a large class of solutions of (1.1) by their Szegő counterparts, we concentrate on constructing a single solution of (1.1) that mimics the growth mechanism of the turbulent two-soliton of the Szegő equation. Of course, complete integrability is lost, but the analysis initiated by Martel in [31] and revisited in [27] for the nonlocal Hartree problem paves the way to the construction of *compact* two-bubble elements. More precisely, one can in principle extract from the equation the approximate dynamical system driving each solitary wave of an asymptotic two-soliton, at least in a regime where the waves are separated in space, and the robust energy method developed in [27] allows one to follow the flow all the way to $+\infty$.

Theorem 1.2 (Solution with transient turbulent regime and saturated growth)**.** *There exists a universal constant $0 < \delta^* \ll 1$ and, for all $\delta \in (0,\delta^*)$, there exists $0 < \eta^*(\delta) \ll 1$ such that the following holds. For every $\eta \in (0,\eta^*)$, let the times*

$$T_{\text{in}} = \frac{1}{\eta^{2\delta}}, \quad T^- = \frac{\delta}{\eta},$$

then there exists a solution $u \in \mathcal{C}([T_{\text{in}},+\infty),H^1)$ to (1.1) which is $H^{\frac{1}{2}}$-compact as $t \to +\infty$ with the following behavior:
1. Initial data: the initial data at time T_{in} has size

$$\|u(T_{\text{in}})\|_{L^2}^2 \sim \eta, \quad \||D|^{\frac{1}{2}}u(T_{\text{in}})\|_{L^2}^2 \sim 1, \quad \|Du(T_{\text{in}})\|_{L^2}^2 \sim \frac{1}{\eta^{1+2\delta}}.$$

2. Turbulent regime: on $[T_{\text{in}},T^-]$, the solution experiences a turbulent interaction with an explicit monotone growth of the H^1-norm

$$\|u(t)\|_{H^1}^2 = \frac{t^2}{\eta}(1+O(\sqrt{\delta})). \tag{1.10}$$

3. Saturation: the interaction ceases after T^- and there holds the saturation

$$\|u(t)\|_{H^1}^2 = \frac{1}{\eta^3}e^{O(\frac{1}{\delta})} \quad \text{for} \quad t \geqslant T^-.$$

The turbulent interaction behind (1.10) is an explicit energy transfer along the singular branch of traveling waves Q_β, and the solution can more explicitly be described as follows. For all times $t \in [T_{\text{in}},+\infty)$, the solution admits a two solitary

wave decomposition

$$u(t,x) = \sum_{j=1}^{2} \frac{1}{\lambda_j^{\frac{1}{2}}(t)} Q_{\beta_j(t)} \left(\frac{x - x_j(t)}{\lambda_j(t)(1-\beta_j(t))} \right) e^{-i\gamma_j(t)} + \varepsilon(t,x)$$

with the following properties:

1. Structure of the first soliton: the first soliton remains nearly unchanged, i.e. for all $t \geqslant T_{\text{in}}$,

$$\lambda_1(t) \sim 1, \quad 1 - \beta_1(t) \sim \eta, \quad x_1(t) \sim (1-\eta)t, \quad \gamma_1(t) \sim t.$$

2. Concentration of the second soliton: the second soliton behaves like a solitary wave

$$\lambda_2(t) \sim 1, \quad x_2(t) \sim \beta_2 t, \quad \gamma_2(t) \sim t$$

with a concentration of size in the transient turbulent regime:

$$1 - \beta_2(t) = \frac{\eta(1 + O(\sqrt{\delta})}{t^2} \quad \text{for} \quad t \in [T_{\text{in}}, T^-],$$

which saturates after the interaction time T^-:

$$1 - \beta_2(t) = \eta^3 e^{O(\frac{1}{\delta})} \quad \text{for} \quad t \geq T^-.$$

3. Asymptotic compact behaviour: this solution is minimal near $+\infty$, i.e.

$$\lim_{t \to +\infty} \|\varepsilon(t)\|_{H^1} = 0.$$

1.6. **Comments on the result.** Theorem 1.2 exhibits, for a canonical dispersive model, an *explicit mechanism* of growth of high Sobolev norms. To the best of the authors' knowledge, this is one of the first results in which one can control *for all times* a turbulent solution of a nonlinear Hamiltonian PDE.

1. *The two regimes.* The key element behind Theorem 1.2 is the derivation of the leading order ODEs driving the geometrical parameters as in [26]. There are two main new pieces of information. First, we can compute explicitly the rate of concentration which is given by the t-growth as in [43]. This rate is very sensitive to the *phase shift* between the waves in the transient regime, and another phase shift would generate another speed. Note that the growth can be computed for any H^s-Sobolev norm above the energy, i.e. $s > \frac{1}{2}$, and the data can also be taken arbitrarily small in H^1 by a fixed rescaling. Secondly and unlike in the case of the Szegő equation, there is no infinite growth of the H^1-norm for the solution we construct. Here we encountered an essential feature in the structure of the Q_β solitary wave. The limiting solitary wave of the Szegő equation has according to (1.7) a far out decay

$$Q^+(x) \sim \frac{1}{\langle x \rangle},$$

while for Q_β there is a transition regime

$$Q_\beta(x) \sim \frac{1}{\langle x \rangle (1 + (1-\beta)\langle x \rangle)}, \quad \beta < 1. \tag{1.11}$$

In particular, when the waves forming the two-soliton separate and their relative distance becomes large

$$|x_2 - x_1| \gg \frac{1}{1-\beta},$$

their interaction weakens from $\frac{1}{\langle x\rangle}$ to $\frac{1}{\langle x\rangle^2}$, and this explains why the concentration mechanism stops in the far out two-soliton dynamics.

2. *Compact bubbles with energy transfer.* Theorem 1.2 lies within the construction of compact elements which has attracted a considerable attention for the past ten years both for global problems since the pioneering breakthrough work [31] and [32, 27, 36] and blow up problems [39, 46, 37, 26]. It is in particular shown in [26] how the presence of polynomially decaying interactions can lead to dramatic deformations of the soliton dynamics, for example from the straight line motion for each wave to the hyperbolic two body problem of gravitation for the two-soliton of the gravitational Hartree model on $\mathbb{R}^3$. The energy transfer mechanism between KdV waves [41, 35] or the recent multibubble infinite time blow up mechanism of [38] are deeply connected to Theorem 1.2. This is the first instance, however, when modulation analysis used in all the above cited works, is employed to find solutions that exhibit growth of high Sobolev norms. Let us insist that the growth (1.10) does not excite the L^2-scaling instability of the problem as in [26], but the β-instability which according to (1.9) is $\dot{H}^{\frac{1}{2}}$-critical and hence compatible with the small data coercive conservation laws. More generally, there is little understanding of the long time asymptotics of wave equations in small dimensions due to the lack of dispersion, see for example [28], and it is essential for the construction to consider *compact* nondispersive flows.

3. *Specificity of the analysis.* The following two problems are simpler than the result in Theorem 1.2: (i) the construction of an asymptotic two-soliton *without* turbulent interaction in the continuation of [27], and (ii) exhibiting a growth mechanism of the H^1-norm on some sufficiently large time interval as in [44], using the limiting singular Szegő regime (see Theorem 1.1 above). The aim of Theorem 1.2 is to perform both the above in the same time and, in particular, to capture the associated saturation of the H^1-norm which we expect displays some universality, and hence describes the long time dynamical bifurcation of (1.1) from the Szegő singular regime (1.8) *beyond usual Ehrenfest-like times.* We then face two essential difficulties. First, the nonlocal nature of the problem in the presence of slowly decaying solitary waves makes interactions very large and hard to decouple as in [24, 36]. In particular, we need to control the logarithmic instability of the phase shift between the waves, which is central for the derivation of the growth mechanism. This forces us to develop both the complete description of the bifurcation $Q^+ \to Q_\beta$ and a new strategy for the derivation of *sharp* modulation equations for geometrical parameters, see Proposition 4.12. Secondly, the need for high order approximations of the solution required to capture the leading order mechanism is reminiscent of the pioneering two-soliton interaction computations in [34, 35]. But the main difficulty here is the fact that the traveling wave equation (1.9) is a singular elliptic problem which degenerates as $\beta \to 1$. Hence one looses the control of natural energy norms in the concentration process, which a priori should ruin the approach developed in [24]. The wave-like structure of the equation is essential to overcome this difficulty. We also need to develop various new estimates involving the Π^+ projection operator onto positive frequencies since in the concentration process, this projection and the Szegő-like regimes are essential for the analysis.

4. *Regularity shift in the growth of Sobolev norms.* Compared to previous results on the growth of high Sobolev norms for nonlinear Schrödinger equations, see [7, 19,

21, 22, 23, 20], it is interesting to notice that Theorem 1.2 implies the existence of small data in H^1 such that the H^s-norm of the solution becomes large, not only for $s = 1$, but also for $s < 1$ close to 1. Notice that this regularity shift also holds — with unbounded solutions at infinity — for the cubic Szegő equation, see [44, 17], where in [17] this phenomenon is established to be generic.

Having completed this work, let us mention a number of related open problems.

- The main one is probably the existence of a solution of (1.1) such that $\limsup_{t\to\infty} \|u(t)\|_{H^1} = +\infty$.
- What are the possible growth rates ? From the recent paper [50], we know that this rate cannot be bigger than $e^{O(t^2)}$, how optimal is it ?
- Are unbounded solutions in H^1 generic ? Is the behavior $\|u(t)\|_{H^1} \underset{t\to\infty}{\longrightarrow} \infty$ generic, or rather is it generic to have infinitely many forward and backward energy cascades, as in the case of the cubic Szegő equation on the circle ?

To conclude, we hope that Theorem 1.2 is an important step towards a better understanding of the role played by interactions of solitons in turbulent transfers of energy.

1.7. **Strategy of the proof.** We outline in this subsection the main steps and difficulties in the proof of Theorem 1.2.

Step 1: *Description of the bifurcation* $Q^+ \to Q_\beta$. Our first task is to completely describe the solutions to the singular elliptic traveling wave equation

$$\frac{|D| - \beta D}{1-\beta} Q_\beta + Q_\beta - Q_\beta |Q_\beta|^2 = 0$$

in the limit $\beta \to 1$. The local existence and uniqueness of the profile Q_β for β close to 1 in Proposition 2.2 relies on a classical Lyapunov-Schmidt argument, which itself relies on the non degeneracy of the linearized operator close to Q^+ for the Szegő problem proved in [42]. The Lyapunov-Schmidt argument yields the non degeneracy of the linearized operator close to Q_β in Proposition 2.4. We then completely describe the profile in space of Q_β and, in particular, its long range asymptotics which displays a nontrivial boundary layer at $x \sim \frac{1}{1-\beta}$, see Section 3. Here we aimed at avoiding logarithmic losses which would be dramatic for the forthcoming analysis, and this requires the consideration of suitable norms and Fourier multipliers.

Step 2: *Two-soliton ansatz.* We now implement the strategy developed in [24] and construct an approximate solution of the form

$$u = u_1 + u_2$$

after reduction to the slow variables

$$u_j(t,x) = \frac{1}{\lambda_j^{\frac{1}{2}}} v_j(s_j, y_j) e^{i\gamma_j}, \quad \frac{ds_j}{dt} = \frac{1}{\lambda_j(t)}, \quad y_j := \frac{x - x_j(t)}{\lambda_j(t)(1-\beta_j(t))}, \quad j = 1, 2.$$

Here we proceed to an expansion of the profiles v_j after separation of variables

$$v_j(s_j, y_j) = Q_{\beta_j(s_j)}(y_j) + \sum_{n=1}^{N} T_{j,n}(y_j, \mathcal{P}(s_j)),$$

where $\mathcal{P}$ encodes the geometrical parameters of the problem

$$\mathcal{P} = (\lambda_1, \lambda_2, \beta_2, \beta_2, \Gamma, R)$$

and (Γ, R) denote the phase shift and relative distance between the waves *after renormalization*

$$\Gamma = \gamma_2 - \gamma_1, \quad R = \frac{x_2 - x_1}{\lambda_1(1-\beta_1)}$$

which is always large $R \gg 1$. The laws for the parameters are adjusted

$$\frac{(\lambda)_{s_j}}{\lambda_j} = M_j(\mathcal{P}), \quad \frac{(\beta_j)_{s_j}}{1-\beta_j} = B_j(\mathcal{P}) \tag{1.12}$$

in order to ensure the solvability of the elliptic system defining $T_{j,n}$; see Proposition 4.6. In order to keep control of the various terms produced by this procedure, we need to define a notion of admissible function, see Definition 4.1, which is compatible with the properties of Q_β and stable for this nonlinear procedure of construction of the approximate solution. The strategy is conceptually similar to [27], but the functional framework is considerably more challenging due to the slow decay of the solitary wave Q_β and to the singular nature of the bifurcation $Q^+ \to Q_\beta$.

Step 3: *Leading order dynamics.* We now extract the leading order dynamics for the ODEs predicted by (1.12). This step is more delicate than one would expect, in particular because we need to keep track of a logarithmic instability of the phase shift Γ which is essential for the derivation of the turbulent growth. We observe in Proposition 4.12 that *mimicking the conservation laws* of mass and kinetic momentum for the approximate solution provides nonlinear cancellations and a high order approximation of the dynamical system for $\mathcal{P}$. Roughly speaking, this reads

$$\frac{(\beta_1)_t}{1-\beta_1} \sim 0, \quad \frac{(\beta_2)_t}{1-\beta_2} \sim \frac{2\cos\Gamma}{R(1+(1-\beta_1)R)}, \quad R \sim t$$

which reflects the decay (1.11). Hence, $1-\beta_1 \sim \eta$ and as long as $\Gamma \sim 0$ and $t \leqslant \frac{1}{\eta} \sim T^-$, we have the decay

$$1-\beta_2(t) \sim \frac{1}{t^2},$$

which saturates for $t \geqslant T^-$. Keeping the phase under control requires a high order approximation of the modulation equations (Proposition 4.12) and a careful integration of the associated modulation equations; see Subsection 4.8.

Step 4: *Backwards integration and energy bounds.* We now solve the problem from $+\infty$ following the backward integration scheme designed in [39, 31, 32, 27]. In the setting of a suitable bootstrap (Proposition 5.2), the solution decomposes into two bubbles and radiation

$$u(t,x) = \sum_{j=1}^{2} u_j + \varepsilon(t,x), \quad u_j(t,x) = \frac{1}{\lambda_j^{\frac{1}{2}}} v_j(s_j, y_j) e^{i\gamma_j},$$

where the profiles v_j have been constructed above. We pick a sequence $T_n \to +\infty$ and look for uniform backwards estimates for the solution to (1.1) with Cauchy data at T_n given by

$$\varepsilon(T_n) = 0. \tag{1.13}$$

The heart of the analysis is to design an energy estimate to control ε. Following [32, 27], the energy functional is a localization in space of the total conserved energy,

with cut-off functions which are adapted to the dramatic change of size of the second bubble. The outcome is an energy bound of the type

$$\left|\frac{d}{dt}\mathcal{G}(\varepsilon(t))\right| \lesssim \frac{\mathcal{G}(\varepsilon)}{t} + \frac{C_N}{t^N} \tag{1.14}$$

where N is the order of accuracy of the approximate solution and can be made arbitrarily large, and $\mathcal{G}$ is a suitable energy functional with roughly

$$\mathcal{G}(\varepsilon) \sim \|\varepsilon\|^2_{H^{\frac{1}{2}}},$$

see Proposition 5.1. Bootstrapping the bound $\mathcal{G}(\varepsilon(t)) \leqslant \frac{1}{t^{\frac{N}{2}}}$ and integrating in time using the boundary condition (1.13) yields

$$\mathcal{G}(\varepsilon(t)) \lesssim \frac{1}{Nt^{\frac{N}{2}}},$$

which is an improved bound for N universal sufficiently large. The critical point in this argument is the $\frac{1}{t}$ loss only in the RHS of (1.14). In general, the terms induced by the necessary localization procedure may be difficult to control, and sometimes the only known way out is a symmetry assumption on the behaviour of the bubbles as in [27, 38]. This is not an option here since the turbulent regime is in essence asymmetric. Furthermore, a fundamental difficulty here is that the linearized operator close to Q_β depends on β and degenerates as $\beta \to 1$, see (5.16). We show in Section 5 that the above strategy can be implemented with a sharp loss of $\frac{1}{t}$ only, using two new ingredients: a favorable algebra for the localization terms, which seems specific to wave-like problems and is reminiscent of a related algebra in [36], see the proof of (E.14), and the splitting of the motion along positive and negative frequencies which move in space differently. Hence the full energy method relies very strongly on the localization *both in space and frequency* of the infinite dimensional part of the solution.

This paper is organized as follows. In Section 2, we construct the bifurcation $Q^+ \to Q_\beta$ à la Lyapunov-Schmidt, and we study in detail the Q_β profile in Section 3. In Section 4, we produce the two-bubble approximate solution (Proposition 4.6) and derive and study the associated dynamical system for the geometrical parameters (Proposition 4.12 and Subsection 4.8). In Section 5, we close the control of the infinite dimensional remainder by setting up the bootstrap argument (Proposition 5.2), and by using in particular the key energetic control given in Proposition 5.4. The proof of Theorem 1.2 easily follows from Proposition 5.2 as detailed in Subsection 5.8. Appendix A is devoted to simple algebraic formulae involving Q^+. Appendix B revisits the two-soliton dynamics for the Szegő equation on the line studied by Pocovnicu [42]. Appendix C establishes some non degeneracy lemma allowing to implement the modulation theory in this context. Appendix D is devoted to basic commutator estimates. Appendix E contains estimates on some cut-off functions which are crucial in our energy method. Finally, Appendix F is devoted to the coercivity of our energy functional.

Notations. On $L^2(\mathbb{R})$, we adopt the real scalar product

$$(u, v) = \mathrm{Re}\left(\int_{\mathbb{R}} u\overline{v}dx\right). \tag{1.15}$$

For $x \in \mathbb{R}$, we set

$$\langle x\rangle := \sqrt{1+x^2}.$$

If $s > 0$ and f is a tempered distribution such that $\hat{f}$ is locally integrable near $\xi = 0$, we define the tempered distribution $|D|^s f$ by

$$\widehat{|D|^s f}(\xi) = |\xi|^s \widehat{f}(\xi) \ .$$

We define the differential operators

$$\Lambda_x f := x\partial_x f, \quad \Lambda f := \frac{1}{2} f + \Lambda_x f, \quad \tilde{\Lambda}_\beta := (1-\beta)\partial_\beta$$

and the function

$$\Phi_\beta := y\partial_y Q_\beta + (1-\beta)\frac{\partial Q_\beta}{\partial \beta}.$$

We use the Sobolev norm

$$\|f\|_{W^{k,\infty}} = \Sigma_{j=0}^k \|\partial_x^k f\|_{L^\infty}, \quad k \in \mathbb{N}.$$

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2. Existence and uniqueness of traveling waves

2.1. **The limiting Szegő profile.** We consider

$$H^{\frac{1}{2}}_+(\mathbb{R}) := \{u \in H^{\frac{1}{2}}(\mathbb{R}) : supp(\hat{u}) \subset \mathbb{R}_+\} \ ,$$

and, for every $u \in H^{\frac{1}{2}}_+(\mathbb{R}) \setminus \{0\}$,

$$J^+(u) := \frac{(Du,u)\|u\|_{L^2}^2}{\|u\|_{L^4}^4} \ , \ I^+ := \inf_{u \in H^{\frac{1}{2}}_+(\mathbb{R})\setminus\{0\}} J^+(u) \ .$$

It is known ([42]) that I^+ is a minimum and that its minimizers are exactly

$$Q(x) = \frac{C}{x+p} \ , \ \operatorname{Im} p > 0 \ .$$

Moreover, those minimizers which satisfy the following Euler–Lagrange equation

$$DQ + Q - \Pi_+(|Q|^2 Q) = 0 \ ,$$

are given by

$$Q(x) = \mathrm{e}^{i\gamma} Q^+(x + x_0) \ , \ Q^+(x) := \frac{2}{2x+i} \ , \ (\gamma, x_0) \in \mathbb{T} \times \mathbb{R}. \tag{2.1}$$

2.2. **Existence of traveling waves.** To show the existence of nontrivial traveling waves Q_β satisfying (1.9), we consider the minimization problem

$$J_\beta(u) := \frac{((|D| - \beta D)u, u)\|u\|_{L^2}^2}{\|u\|_{L^4}^4} \ , \quad I_\beta := \inf_{u \in H^{\frac{1}{2}}(\mathbb{R}) \setminus \{0\}} J_\beta(u) \ .$$

From [26] and a simple scaling argument, we have the following result:

Proposition 2.1 (Small traveling waves)**.** *For all $0 \leqslant \beta < 1$, the infimum I_β is attained. Moreover, any minimizer Q_β for $J_\beta(u)$ such that*

$$\|Q_\beta\|_{L^2}^2 = \frac{1}{2}\|Q_\beta\|_{L^4}^4 = \frac{((|D| - \beta D)Q_\beta, Q_\beta)}{1-\beta} = \frac{2I_\beta}{1-\beta} \tag{2.2}$$

satisfies the following equation:

$$\frac{|D| - \beta D}{1-\beta} Q_\beta + Q_\beta = |Q_\beta|^2 Q_\beta .$$

In what follows, let $\mathcal{Q}_\beta$ denote the set of minimizers Q_β of $J_\beta(u)$ such that (2.2) holds.

Proposition 2.2 (Profile of Q_β)**.** *If $Q_\beta \in \mathcal{Q}_\beta$ and $\beta \to 1, \beta < 1$, there exist $x(\beta) \in \mathbb{R}$ and $\gamma \in \mathbb{T}$ such that, up to a subsequence,*

$$Q_\beta(x - x(\beta)) \to \mathrm{e}^{i\gamma} Q^+(x) \ ,$$

strongly in $H^{\frac{1}{2}}(\mathbb{R})$. More precisely, for β sufficiently close to 1, we have

$$\left\| Q_\beta(x - x(\beta)) - \mathrm{e}^{i\gamma} Q^+(x) \right\|_{H^{\frac{1}{2}}} \leqslant C(1-\beta)^{1/2} |\log(1-\beta)|^{\frac{1}{2}} . \tag{2.3}$$

Proof. First observe that, since $|D| - \beta D \geq (1-\beta)|D|$,

$$I_\beta \geq (1-\beta) I_0 \ ,$$

and, by plugging $u = Q^+$ in J_β,

$$I_\beta \leqslant (1-\beta) I^+ \ .$$

We claim that indeed,

$$\frac{I_\beta}{1-\beta} \to I^+ \ .$$

Decompose

$$Q_\beta = Q_\beta^+ + Q_\beta^- \ , \ Q_\beta^\pm := \Pi_\pm(Q_\beta) \ .$$

Then identities (2.2) read

$$\|Q_\beta^+\|_{L^2}^2 + \|Q_\beta^-\|_{L^2}^2 = \frac{1}{2}\|Q_\beta^+ + Q_\beta^-\|_{L^4}^4 = (DQ_\beta^+, Q_\beta^+) + \frac{1+\beta}{1-\beta}(|D|Q_\beta^-, Q_\beta^-) = \frac{2I_\beta}{1-\beta} \ .$$

This implies in particular

$$\|Q_\beta^-\|_{L^2}^2 \leqslant 2I^+ \ , \ (|D|Q_\beta^-, Q_\beta^-) \leq 2I_+(1-\beta) \ , \ \|Q_\beta^-\|_{L^4}^4 \leqslant \frac{4I_+^2}{I_0}(1-\beta) \to 0 \ .$$

We are going to improve these estimates on Q_β^-, using the following identity on Fourier transforms, which is an immediate consequence of the equation for Q_β in Proposition 2.2,

$$\widehat{Q_\beta}(\xi) = \frac{1}{1 + \frac{|\xi| - \beta\xi}{1-\beta}} \widehat{|Q_\beta|^2 Q_\beta}(\xi) \ .$$

In particular,

$$\widehat{Q_\beta^-}(\xi) = \frac{\mathbf{1}_{\{\xi<0\}}}{1+\frac{1+\beta}{1-\beta}|\xi|}\widehat{|Q_\beta|^2Q_\beta}(\xi)\ . \tag{2.4}$$

From (2.4) and the Plancherel formula, we immediately get

$$\|Q_\beta^-\|_{L^2}^2 = \frac{1}{2\pi}\int_{-\infty}^0 \frac{1}{\left(1+\frac{1+\beta}{1-\beta}|\xi|\right)^2}|\widehat{|Q_\beta|^2Q_\beta}(\xi)|^2\,d\xi \ \leq C(1-\beta)\ , \tag{2.5}$$

where we used a bound on Q_β in L^3, which is a consequence of identities (2.2) and of the estimate $I_\beta \leqslant (1-\beta)I^+$. Similarly, we have

$$(DQ_\beta^-, Q_\beta^-) = \frac{1}{2\pi}\int_{-\infty}^0 \frac{|\xi|}{\left(1+\frac{1+\beta}{1-\beta}|\xi|\right)^2}|\widehat{|Q_\beta|^2Q_\beta}(\xi)|^2\,d\xi \ \leq C(1-\beta)^2|\log(1-\beta)|\ , \tag{2.6}$$

because of the logarithmic divergence of the integral at $\xi = 0$. This already implies

$$\|Q_\beta^-\|_{L^4}^4 \leqslant C(1-\beta)^3|\log(1-\beta)|\ .$$

Finally, using the bound on Q_β in all the L^p-norms with p finite, we have

$$\begin{aligned}\|Q_\beta^+\|_{L^4}^4 &= \|Q_\beta - Q_\beta^-\|_{L^4}^4 = \|Q_\beta\|_{L^4}^4 - 4\mathrm{Re}\left(\int_{\mathbb{R}}|Q_\beta|^2\overline{Q}_\beta Q_\beta^-\,dx\right) + O(\|Q_\beta^-\|_{L^4}^2)\\ &= \|Q_\beta\|_{L^4}^4 - 4\mathrm{Re}\left(\frac{1}{2\pi}\int_{-\infty}^0 \frac{1}{1+\frac{1+\beta}{1-\beta}|\xi|}|\widehat{|Q_\beta|^2Q_\beta}(\xi)|^2\,d\xi\right) + O((1-\beta)^{3/2}|\log(1-\beta)|^{1/2})\\ &= \|Q_\beta\|_{L^4}^4 - O((1-\beta)|\log(1-\beta)|).\end{aligned}$$

Therefore

$$\begin{aligned}I^+ \leq J^+(Q_\beta^+) &= \frac{(DQ_\beta^+, Q_\beta^+)\|Q_\beta^+\|_{L^2}^2}{\|Q_\beta^+\|_{L^4}^4} = \frac{\left(\frac{2I_\beta}{1-\beta}\right)^2}{\|Q_\beta\|_{L^4}^4 - O((1-\beta)|\log(1-\beta)|)}\\ &= \frac{I_\beta}{1-\beta} - O((1-\beta)\log(1-\beta)) \leqslant I^+ + O((1-\beta)|\log(1-\beta)|).\end{aligned}$$

Summing up, we have proved

$$\begin{aligned}0 \leq I_+ - \frac{I_\beta}{1-\beta} &\lesssim (1-\beta)|\log(1-\beta)|\ ,\\ \left|\|Q_\beta^+\|_{L^2}^2 - 2I^+\right| + \left|\|Q_\beta^+\|_{L^4}^4 - 4I^+\right| + \left|(DQ_\beta^+, Q_\beta^+) - 2I^+\right| &\\ \lesssim (1-\beta)|\log(1-\beta)| &\qquad (2.7)\\ \|Q_\beta^-\|_{\dot H^{1/2}} \lesssim (1-\beta)|\log(1-\beta)|^{\frac12} &\qquad (2.8)\\ \|Q_\beta^-\|_{H^{\frac12}} \lesssim (1-\beta)^{\frac12}.&\end{aligned}$$

By a concentration-compactness argument on the space $H_+^{\frac12}$ (see e.g. [42], Prop. 5.1), this yields (2.3). □

By a straightforward argument, we upgrade the convergence of Q_β to any H^s.

Proposition 2.3. *Let $\beta_n \to 1$, $\beta_n < 1$, and suppose that $Q_{\beta_n} \in \mathcal{Q}_{\beta_n}$ satisfies $Q_{\beta_n} \to Q^+$ in $H^{\frac12}(\mathbb{R})$. Then, for any $s \geqslant 0$, we have*

$$\|Q_{\beta_n}\|_{H^s} \leqslant C_s.$$

In particular, $\|Q_{\beta_n}\|_{L^\infty} \leqslant C$ *and it holds that*

$$Q_{\beta_n} \to Q^+ \quad \text{in} \quad H^s(\mathbb{R}) \quad \text{for all } s \geqslant 0.$$

Proof. It suffices to prove the claim for integer $s \in \mathbb{N}$. By applying ∇^s to the equation satisfied by $Q_n := Q_{\beta_n}$, we obtain that

$$\nabla^s Q_n = \frac{\nabla}{\frac{|D|-\beta_n D}{1-\beta_n} + 1} \nabla^{s-1}(|Q_n|^2 Q_n) =: A_{\beta_n} \nabla^{s-1}(|Q_n|^2 Q_n). \tag{2.9}$$

Using the simple fact that $|\xi| - \beta\xi \geqslant (1-\beta)|\xi|$, we see that $\|A_{\beta_n}\|_{L^2 \to L^2} \leqslant C$ holds. Thus, by choosing $s = 1$, we obtain the uniform bound

$$\|\nabla Q_n\|_{L^2} \leqslant C \|Q_n\|_{L^6}^3 \leqslant C,$$

since $\|Q_n\|_{L^6} \leqslant C$ because of $Q_n \to Q^+$ in $H^{\frac{1}{2}}$. Hence we obtain the uniform bounds $\|Q_n\|_{H^1} \leqslant C$ and $\|Q_n\|_{L^\infty} \leqslant C$ (by Sobolev embedding). Now, by induction over $s \in \mathbb{N}$, Leibniz' rule, and the uniform bounds $\|Q_n\|_{L^\infty} \leqslant C$, we find

$$\|Q_n\|_{H^k} \leqslant C_k$$

for any $k \in \mathbb{N}$. By interpolation, this bound implies that $Q_n \to Q^+$ in H^s for any $s \geqslant 0$, since $Q_n \to Q^+$ in $H^{\frac{1}{2}}$ by assumption. □

2.3. **Invertibility of the linearized operator.** In this section, we fix a solitary wave $Q_\beta \in \mathcal{Q}_\beta$. Let the linearized operator close to this solitary wave be

$$\mathcal{L}_\beta \varepsilon = \frac{|D| - \beta D}{1-\beta}\varepsilon + \varepsilon - 2|Q_\beta|^2 \varepsilon - Q_\beta^2 \overline{\varepsilon}. \tag{2.10}$$

We may now invert $\mathcal{L}_\beta$ and prove the continuity of the inverse in suitable weighted norms.

Proposition 2.4 (Invertibility of $\mathcal{L}_\beta$)**.** *There exist* $\beta_* \in (0,1)$ *such that for all* $\beta \in (\beta_*, 1)$ *and for all* $Q_\beta \in \mathcal{Q}_\beta$*, the following holds. There exists* $C > 0$ *such that for all* $f \in H^{\frac{1}{2}}$ *we have*

$$\|f\|_{H^{\frac{1}{2}}} \leq C \left(\|\mathcal{L}_\beta f\|_{H^{-\frac{1}{2}}} + |(f, iQ_\beta)| + |(f, \partial_x Q_\beta)| \right) . \tag{2.11}$$

Let $g \in H^{-\frac{1}{2}}$ *with*

$$(g, iQ_\beta) = (g, \partial_x Q_\beta) = 0. \tag{2.12}$$

Then, there exists a unique solution to

$$\mathcal{L}_\beta f = g, \quad (f, iQ_\beta) = (f, \partial_x Q_\beta) = 0, \quad f \in H^{\frac{1}{2}} \tag{2.13}$$

and

$$\|f\|_{H^{\frac{1}{2}}} \lesssim \|g\|_{H^{-\frac{1}{2}}}. \tag{2.14}$$

Proof of Proposition 2.4. The invertibility claim follows easily once one proves (2.11). Indeed, denote by P_β the orthogonal projection onto $V_\beta := \mathrm{span}_{\mathbb{R}}(iQ_\beta, \partial_x Q_\beta)$. Since $V_\beta \subset \ker \mathcal{L}_\beta$ from the invariance of the equation on Q_β by translation and phase shift, we have

$$f \in \ker \mathcal{L}_\beta \Rightarrow f - P_\beta f \in \ker \mathcal{L}_\beta .$$

Applying estimate (2.11) to $f - P_\beta f$, we conclude that $f - P_\beta f = 0$, namely $f \in V_\beta$. Therefore, $\ker \mathcal{L}_\beta = V_\beta$. The rest of the statement is just Fredholm alternative applied to the self-adjoint Fredholm operator $\mathcal{L}_\beta$.

In the remaining we will prove (2.11).

Step 1: We first claim that

$$\forall f \in H^{\frac{1}{2}}_+ \ , \ \|f\|_{H^{\frac{1}{2}}} \leq C \left(\|\mathcal{L} f\|_{H^{-\frac{1}{2}}} + |(f, iQ^+)| + |(f, \partial_x Q^+)| \right) \ , \tag{2.15}$$

where $\mathcal{L}$ denotes the linearized operator for the equation on Q^+,

$$\mathcal{L}\varepsilon := D\varepsilon + \varepsilon - \Pi_+(2|Q^+|^2\varepsilon + (Q^+)^2\overline{\varepsilon}) \ , \varepsilon \in H^{\frac{1}{2}}_+ \ . \tag{2.16}$$

To prove this estimate, we closely follow Section 5 of [45]. More precisely, we decompose $f \in H^{\frac{1}{2}}_+$ according to the orthogonal decomposition

$$L^2_+ = (V \oplus iV)^\perp \oplus iV \oplus V, \ V := \mathrm{span}_{\mathbb{R}}(iQ^+, \partial_x Q^+) \ ,$$

which reads

$$f = f' + f_1'' + f_2'' \ .$$

By translation invariance and phase shift invariance, $\mathcal{L} = 0$ on V. Moreover, an exact computation yields

$$\mathcal{L}(Q^+) = -2(DQ^+ + Q^+) \ , \ \mathcal{L}(DQ_+) = -2DQ^+ - 4Q^+ \ .$$

Consequently, $\mathcal{L} : iV \to iV$ is one to one. Finally, $\mathcal{L} : (V \oplus iV)^\perp \to (V \oplus iV)^\perp$ and is coercive (as shown in [45]),

$$(\mathcal{L} f', f') \geq c\|f'\|^2_{H^{\frac{1}{2}}} \ , \tag{2.17}$$

and consequently,

$$\forall f' \in H^{\frac{1}{2}}_+ \cap (V \oplus iV)^\perp \ , \ \|f'\|_{H^{\frac{1}{2}}} \leq C\|\mathcal{L} f'\|_{H^{-\frac{1}{2}}} \ .$$

We now proceed by contradiction. Assume (2.15) fails. Then there exists a sequence (f_n) of $H^{\frac{1}{2}}_+$ such that

$$\|f_n\|_{H^{\frac{1}{2}}} = 1 \ , \ \|\mathcal{L} f_n\|_{H^{-\frac{1}{2}}} \to 0 \ , \ |(f_n, iQ^+)| + |(f_n, \partial_x Q^+)| \to 0 \ .$$

Decomposing $f_n = f_n' + f_{n1}'' + f_{n2}''$, we notice that the last condition exactly means $f_{n2}'' \to 0$ in the plane V. Moreover, since $\|f_{n1}''\|_{L^2} \leq \|f_n\|_{L^2}$, we may assume that $f_{n1}'' \to f_1''$ in the plane iV. Since

$$\mathcal{L} f_n = \mathcal{L} f_n' + \mathcal{L} f_{n1}'' \ ,$$

we have, for every $g \in iV$,

$$(\mathcal{L} f_{n1}'', g) = (\mathcal{L} f_n, g) \to 0 \ ,$$

whence $(\mathcal{L} f_1'', g) = 0$, or $\mathcal{L} f_1'' = 0$, which implies $f_1'' = 0$ since $\mathcal{L} : iV \to iV$ is one to one. Finally, we conclude that $\mathcal{L} f_n' \to 0$ in $H^{-\frac{1}{2}}$, which implies $f_n' \to 0$ in $H^{\frac{1}{2}}$, and finally $f_n \to 0$ in $H^{\frac{1}{2}}$, a contradiction.

Step 2: Proof of (2.11). This now follows from a standard perturbation argument. Indeed, since (2.14) is translation and phase-shift invariant, it is enough to prove it for $Q_\beta = Q_{\beta_n} \to Q^+$, $\beta_n \to 1$, $n \geq N$ sufficiently large. In the following, we write

$$Q_n = Q_{\beta_n}.$$

For $f \in H^{\frac{1}{2}}$, we observe that

$$\|\mathcal{L}_{\beta_n} f\|^2_{H^{-\frac{1}{2}}} = \|\Pi_+ \mathcal{L}_{\beta_n} f\|^2_{H^{-\frac{1}{2}}} + \|\Pi_- \mathcal{L}_{\beta_n} f\|^2_{H^{-\frac{1}{2}}} \ .$$

Write $f^{\pm} := \Pi_{\pm}(f)$. We have

$$\Pi_-(\mathcal{L}_{\beta_n} f) = \frac{1+\beta_n}{1-\beta_n}|D|f^- + f^- - \Pi_-(2|Q_n|^2 f + Q_n^2 \overline{f})$$

hence, using the L^4 bound for Q_n,

$$(\Pi_-(\mathcal{L}_{\beta_n} f), f^-) \geq \frac{1+\beta_n}{1-\beta_n}(|D|f^-, f^-) + \|f^-\|_{L^2}^2 - \mathcal{O}(1)\|f\|_{L^4}\|f^-\|_{L^4} \ .$$

Using the Gagliardo-Nirenberg inequality for f^- and β_n close to 1, we can absorb $\|f^-\|_{L^4}^2$ with a large factor and get

$$(\Pi_-(\mathcal{L}_{\beta_n} f), f^-) \geq \frac{1}{1-\beta_n}(|D|f^-, f^-) + \|f^-\|_{L^2}^2 - o(1)\|f^+\|_{L^4}^2 \ ,$$

and finally

$$\|\Pi_-(\mathcal{L}_{\beta_n} f)\|_{H^{-\frac{1}{2}}}^2 \geq c\left(\frac{1}{1-\beta_n}(|D|f^-, f^-) + \|f^-\|_{L^2}^2\right) - o(1)\|f^+\|_{L^4}^2 \ .$$

On the other hand,

$$\Pi_+(\mathcal{L}_{\beta_n} f) = \Pi_+(\mathcal{L}_{\beta_n} f^+) + \Pi_+(\mathcal{L}_{\beta_n} f^-) = \mathcal{L} f^+ + r^+ + r^- \ ,$$

with

$$\begin{aligned} r^- &= -\Pi_-(2|Q_n|^2 f^- + Q_n^2 \overline{f^-}) \ , \ \|r^-\|_{H^{-\frac{1}{2}}} \leq \|r^-\|_{L^2} \leq \mathcal{O}(1)\|f^-\|_{L^4} \ , \\ r^+ &= -\Pi_+(2(|Q_n|^2 - |Q^+|^2)f^+ + (Q_n^2 - (Q^+)^2)\overline{f^+}) \ , \ \|r^+\|_{H^{-\frac{1}{2}}} \leq \|r^+\|_{L^2} \leq \ o(1)\|f^+\|_{L^4} \ , \end{aligned}$$

where we have used uniform estimates on Q_n and the fact that $Q_n \to Q^+$ in L^p for every p. Finally,

$$\|\Pi_+(\mathcal{L}_{\beta_n} f)\|_{H^{-\frac{1}{2}}}^2 \geq \|\mathcal{L} f^+\|_{H^{\frac{1}{2}}}^2 - o(1)\|f^+\|_{L^4}^2 - \mathcal{O}(1)\|f^-\|_{L^4}^2 \ . \tag{2.18}$$

Summing up, we get, using again the absorption of $\|f^-\|_{L^4}$,

$$\|\mathcal{L}_{\beta_n} f\|_{H^{-\frac{1}{2}}}^2 \geq c\left(\frac{1}{1-\beta_n}(|D|f^-, f^-) + \|f^-\|_{L^2}^2\right) + \|\mathcal{L} f^+\|_{H^{\frac{1}{2}}}^2 - o(1)\|f^+\|_{L^4}^2 \ .$$

On the other hand,

$$|(f, \partial_x Q_n)|^2 + |(f, iQ_n)|^2 \geqslant |(f^+, \partial_x Q^+)|^2 + |(f^+, iQ^+)|^2 - o(1)\|f\|_{L^2}^2 \ .$$

Summing the last two inequalities and using estimate (2.15) for f^+, we absorb the term $o(1)(\|f^+\|_{L^4}^2 + \|f\|_{L^2}^2)$ and obtain the desired estimate. □

Remark 2.5. We also have the estimate

$$\|f\|_{H^{\frac{1}{2}}} \leq C\left(\|\mathcal{L}_\beta f\|_{H^{-\frac{1}{2}}} + |(f, iQ^+)| + |(f, \partial_x Q^+)|\right) \ , \tag{2.19}$$

if β is close enough to 1 and Q_β is close enough to Q^+. This will be useful in the next subsection for defining a smooth branch of Q_β.

2.4. Uniqueness of traveling waves for $\beta \in (\beta_*, 1)$ close to 1.

Proposition 2.6. *There exists $\beta_* \in (0,1)$ such that the following holds.*

- *For every $\beta \in (\beta_*, 1)$, for every $Q_\beta, \tilde{Q}_\beta$ in $\mathcal{Q}_\beta$, there exists $(\gamma, y) \in \mathbb{T} \times \mathbb{R}$ such that*

$$\tilde{Q}_\beta(x) = \mathrm{e}^{i\gamma} Q_\beta(x - y) \ .$$

- *There exists a neighborhood U of Q^+ in $H^{\frac12}$ such that, for every $\beta \in (\beta_*, 1)$, $\mathcal{Q}_\beta \cap U$ contains a unique point Q_β satisfying*

$$(Q_\beta, iQ^+) = (Q_\beta, \partial_x Q^+) = 0 \ .$$

Moreover, we have

$$\|Q_\beta - Q^+\|_{H^1} = O\left(|1-\beta|^{\frac12} |\log(1-\beta)|^{\frac12}\right) \ . \tag{2.20}$$

The map $\beta \in (\beta_*, 1) \mapsto Q_\beta \in H^{\frac12}$ is smooth, tends to Q^+ as β tends to 1, and its derivative is uniquely determined by

$$\begin{cases} \mathcal{L}_\beta(\partial_\beta Q_\beta) = \frac{2}{1-\beta^2}(Q_\beta^- - \Pi_-(|Q_\beta|^2 Q_\beta)) \\ (\partial_\beta Q_\beta, iQ^+) = (\partial_\beta Q_\beta, \partial_x Q^+) = 0 \end{cases} \tag{2.21}$$

Proof. Let us prove the first item. We may assume that Q_β and $\tilde{Q}_\beta$ tend to Q^+ as β tends to 1. For $(\gamma, y) \in \mathbb{T} \times \mathbb{R}$, we then define

$$\varepsilon(x, \gamma, y, \beta) := \tilde{Q}_\beta(x) - \mathrm{e}^{i\gamma} Q_\beta(x - y) \ ,$$

and

$$f(\gamma, y, \beta) := (\varepsilon(., \gamma, y, \beta), i\tilde{Q}_\beta) \ , \ g(\gamma, y, \beta) := (\varepsilon(., \gamma, y, \beta), \partial_x \tilde{Q}_\beta) \ .$$

These two functions are smooth in (γ, y) and their Jacobian matrix at $(\gamma, y) = (0, 0)$ is close to

$$\begin{pmatrix} (-iQ^+, iQ^+) & (\partial_x Q^+, iQ^+) \\ (-iQ^+, \partial_x Q^+) & (\partial_x Q_+, \partial_x Q^+) \end{pmatrix} = \begin{pmatrix} -2\pi & 2\pi \\ -2\pi & 4\pi \end{pmatrix}$$

therefore it is uniformly invertible. Moreover, as β goes to 1, $f(0, 0, \beta)$ and $g(0, 0, \beta)$ tend to 0. By the implicit function theorem, we conclude that there exist functions $\gamma(\beta), y(\beta)$ with values near $(0, 0)$ such that

$$f(\gamma(\beta), y(\beta), \beta) = g(\gamma(\beta), y(\beta), \beta) = 0 \ .$$

Then, coming back to the equations satisfied by Q_β and $\tilde{Q}_\beta$, we infer that $\varepsilon(x, \beta) := \varepsilon(x, \gamma(\beta), y(\beta), \beta)$ satisfies

$$\|\mathcal{L}_{\tilde{Q}_\beta} \varepsilon(\,.\,, \beta)\|_{H^{-\frac12}} \leqslant Co(1) \|\varepsilon(\,.\,, \beta)\|_{H^{\frac12}} \ ,$$

and, using estimate (2.14), we conclude that $\varepsilon(x, \beta) = 0$.

Let us come to the second item. Select a family (Q_β^0), with $Q_\beta^0 \in \mathcal{Q}_\beta$, which tends to Q^+ as β tends to 1. Applying the implicit function theorem as before to the functions

$$\tilde{f}(\gamma, y, \beta) := (\mathrm{e}^{i\gamma} Q_\beta^0(. - y), iQ^+) \ , \ \tilde{g}(\gamma, y, \beta) := (\mathrm{e}^{i\gamma} Q_\beta^0(. - y), \partial_x Q^+) \ ,$$

we find functions $\tilde{\gamma}(\beta), \tilde{y}(\beta)$ valued near $(0, 0)$ which cancel $\tilde{f}, \tilde{g}$. This provides the existence of Q_β. The uniqueness comes from Remark 2.5. Furthermore, as a consequence of (2.3), we get

$$\|Q_\beta - Q^+\|_{H^{\frac12}} = O\left(|1-\beta|^{\frac12} |\log(1-\beta)|^{\frac12}\right) \ .$$

Coming back to the equation satisfied by Q_β,

$$Q_\beta = \left(\frac{|D| - \beta D}{1-\beta} + 1\right)^{-1} (|Q_\beta|^2 Q_\beta) \ ,$$

and expanding in the L^2-norm

$$|Q_\beta|^2 Q_\beta = |Q^+|^2 Q^+ + O((1-\beta)^{\frac12} |\log(1-\beta)|^{\frac12}) \ ,$$

we infer, in the L^2 norm,

$$DQ_\beta = D(D+1)^{-1}\Pi^+(|Q^+|^2Q^+)+D\left(\frac{(1+\beta)|D|}{1-\beta}+1\right)^{-1}\Pi^-(|Q^+|^2Q^+)+O((1-\beta)^{\frac{1}{2}}|\log(1-\beta)|^{\frac{1}{2}}) ,$$

and finally

$$DQ_\beta = DQ^+ + O((1-\beta)^{\frac{1}{2}}|\log(1-\beta)|^{\frac{1}{2}}) ,$$

in the L^2 norm, which completes the proof of (2.20).

Using again the equation satisfied by Q_β and the estimate from Remark 2.5, it is then straightforward to prove that the map $\beta \mapsto Q_\beta$ is smooth on $(\beta_*, 1)$ and that its derivative satisfies

$$\mathcal{L}_\beta(\partial_\beta Q_\beta) + \frac{(|D|-D)Q_\beta}{(1-\beta)^2} = 0 \ , \ (\partial_\beta Q_\beta, iQ^+) = (\partial_\beta Q_\beta, \partial_x Q^+) = 0.$$

Notice that $(|D|-D)Q_\beta = -2DQ_\beta^-$. Projecting the equation for Q_β onto negative Fourier modes, we get

$$\frac{2DQ_\beta^-}{1-\beta} = \frac{2}{1+\beta}(Q_\beta^- - \Pi_-(|Q_\beta|^2Q_\beta)) ,$$

which, plugged into the equation on $\partial_\beta Q_\beta$, leads to (2.21). □

3. Properties of Q_β

We collect in this section information on Q_β which will be essential for the construction of the two-bubble approximate solutions.

3.1. **Weighted norms and Fourier multipliers.** For every function f on $\mathbb{R}$ and $\beta \in (\beta_*, 1)$, we define the following weighted norm,

$$\|f\|_\beta := \sup_{x\in\mathbb{R}} \langle x\rangle (1+(1-\beta)|x|)|f(x)| .$$

The next lemma will be crucial in all our estimates.

Lemma 3.1. *Let $\{m_\beta\}_{\beta_*<\beta<1}$ be a family of functions on $\mathbb{R}$ such that*

$$\sup_\beta \|m_\beta\|_{L^2} \le M_0 , \tag{3.1}$$

$$|xm_\beta(x)| \le \frac{M_0}{1+(1-\beta)|x|} , \tag{3.2}$$

for some $M_0 > 0$. Assume $\{a_\beta, b_\beta\}_{\beta_*<\beta<1}$ is bounded in L^∞ and is tight in L^2, namely

$$\sup_{\beta_*<\beta<1} \int_{|x|>R} [|a_\beta(x)|^2 + |b_\beta(x)|^2]\,dx \underset{R\to\infty}{\longrightarrow} 0 .$$

Then there exists a constant $A > 0$ independent of β such that, if $f, h \in L^2$ satisfy

$$f = m_\beta * (a_\beta f + b_\beta \overline{f}) + h ,$$

the following estimate holds,

$$\|f\|_\beta \le A[(\|a_\beta\|_{L^\infty} + \|a_\beta\|_{L^2} + \|b_\beta\|_{L^\infty} + \|b_\beta\|_{L^2})\|f\|_{L^2} + \|h\|_\beta] .$$

Proof. First of all, we have trivially

$$\|f\|_{L^\infty} \leq \|m_\beta\|_{L^2} \left(\|a_\beta\|_{L^\infty} + \|b_\beta\|_{L^\infty}\right) \|f\|_{L^2} + \|h\|_{L^\infty} \ ,$$

hence it is enough to estimate $|f(x)|$ for x large enough. Let $R_0 > 0$ such that

$$\sup_\beta \|m_\beta\|_{L^2} \left[\left(\int_{|y|\geq R_0/2} |a_\beta(y)|^2 \, dy \right)^{1/2} + \left(\int_{|y|\geq R_0/2} |b_\beta(y)|^2) \, dy \right)^{1/2} \right] \leq \frac{1}{8} \ .$$

For every $R > 0$, we set

$$M(R) := \sup_{|x|\geq R} |f(x)| \ .$$

For $|x| \geq R$, and $R \geq R_0$, we write

$$\begin{aligned} |m_\beta * (a_\beta f + b_\beta \overline{f})(x)| &\leq \left| \int_{|y|\leq \frac{R}{2}} m_\beta(x-y)(a_\beta(y) f(y) + b_\beta(y)\overline{f(y)}) \, dy \right| \\ &+ \left| \int_{|y|\geq \frac{R}{2}} m_\beta(x-y)(a_\beta(y) f(y) + b_\beta(y)\overline{f(y)}) dy \right| \\ &\leq \frac{C}{R(1+(1-\beta)R)} (\|a_\beta\|_{L^2} + \|b_\beta\|_{L^2}) \|f\|_{L^2} + \frac{1}{8} M\left(\frac{R}{2}\right) \ . \end{aligned}$$

This implies, for every $R \geq R_0$,

$$M(R) \leq \frac{C(\|a_\beta\|_{L^2} + \|b_\beta\|_{L^2}) \|f\|_{L^2} + \|h\|_\beta}{R(1+(1-\beta)R)} + \frac{1}{8} M\left(\frac{R}{2}\right) \ .$$

Applying this to $R = 2^n$ for $n \geq n_0$, we obtain

$$M(2^n) \leq K 2^{-n} (1+(1-\beta)2^n)^{-1} + \frac{1}{8} M(2^{n-1}) \ , \ K := C(\|a_\beta\|_{L^2} + \|b_\beta\|_{L^2}) \|f\|_{L^2} + \|h\|_\beta \ .$$

Iterating, we get

$$\begin{aligned} M(2^n) &\leq K \sum_{p=0}^{n-n_0} 2^{-(n-p)} (1+(1-\beta)2^{n-p})^{-1} \left(\frac{1}{8}\right)^p + \left(\frac{1}{8}\right)^{n-n_0+1} M(2^{n_0-1}) \\ &\leq K 2^{-n} (1+(1-\beta)2^n)^{-1} \sum_{p=0}^{n-n_0} 2^{-p} + \left(\frac{1}{8}\right)^{n-n_0+1} M(2^{n_0-1}) \\ &\leq (2K + 4^{n_0} M(2^{n_0-1})) \, 2^{-n} (1+(1-\beta)2^n)^{-1} \ . \end{aligned}$$

Since $|x| \sim 2^n$ for $2^n \leq |x| \leq 2^{n+1}$, this completes the proof of the lemma. □

We now introduce an important class of families $\{m_\beta\}_{\beta_*<\beta<1}$ satisfying estimates (3.1), (3.2). Denote by $\mathcal{M}$ the class of families $\{\mu_\beta\}_{\beta_*<\beta<1}$ such that the Fourier transform is given by

$$\hat{\mu}_\beta(\xi) = A_\beta \left(f_+(\xi) \, \mathbf{1}_{\xi>0} + f_- \left(-\frac{1+\beta}{1-\beta} \xi \right) \mathbf{1}_{\xi<0} \right) \ , \tag{3.3}$$

where $f_\pm \in C^\infty([0,+\infty))$ satisfy the following requirements,

$$\forall j \geq 0, \forall \zeta \in (0,+\infty), |f_\pm^{(j)}(\zeta)| \leq C_j (1+\zeta)^{-j-1} \ , \ f_+(0) = f_-(0) \ ,$$

and where $\beta \mapsto A_\beta$ is smooth on $(\beta_*, 1)$ and is bounded with bounded derivatives of any order. Indeed, the L^2-estimate (3.1) on μ_β is provided by

$$|f_\pm(\zeta)| \leq C_0 (1+\zeta)^{-1} \ ,$$

while (3.2) comes from

$$x\mu_\beta(x) = A_\beta \left(F_+(x) - F_- \left(-\frac{1-\beta}{1+\beta} x \right) \right) \ , \ F_\pm(y) := \int_0^{+\infty} i f'_\pm(\zeta)\, e^{iy\zeta}\, \frac{d\zeta}{2\pi} = O\left(\frac{1}{1+|y|} \right) .$$

The advantage of the class $\mathcal{M}$ is that it is stable through various important operations. The first one is of course the product of convolution, which corresponds to the product of functions $\beta \mapsto A_\beta$ and $\zeta \mapsto f_\pm(\zeta)$. The second one is the operator $x\partial_x + 1$, which corresponds to replacing $f_\pm$ by $-\zeta f'_\pm$. Finally, if $\{\mu_\beta\}_{\beta_*<\beta<1}$ belongs to class $\mathcal{M}$, then

$$(1-\beta)\partial_\beta \hat{\mu}_\beta(\xi) = (1-\beta) A'_\beta \left(f_+(\xi)\, \mathbf{1}_{\xi>0} + f_- \left(-\frac{1+\beta}{1-\beta}\xi \right) \mathbf{1}_{\xi<0} \right) \quad (3.4)$$

$$+ \frac{2A_\beta}{1+\beta}\, g_- \left(-\frac{1+\beta}{1-\beta}\xi \right) \mathbf{1}_{\xi<0} \ , \quad (3.5)$$

where $g_-(\zeta) := \zeta f'_-(\zeta)$. Hence the family

$$\{(1-\beta)\partial_\beta \mu_\beta\}_{\beta_*<\beta<1}$$

is a sum of elements of class $\mathcal{M}$.

A typical example of a family in class $\mathcal{M}$ is

$$m_\beta = \mathcal{F}^{-1} \left(\frac{1}{1 + \frac{|\xi| - \beta\xi}{1-\beta}} \right) \ ,$$

which corresponds to

$$A_\beta = 1 \ , \ f_+(\zeta) = f_-(\zeta) = (1+\zeta)^{-1} \ .$$

The above considerations lead to the following result, which will be of constant use in the sequel.

Lemma 3.2. *All the multipliers*

$$m_{\beta,p,q} := (x\partial_x)^p ((1-\beta)\partial_\beta)^q m_\beta, \ p,q \geq 0 \ ,$$

and any convolution products between them satisfy properties (3.1) and (3.2).

We complete this subsection with three auxiliary results. The first one is the crucial estimate for $\mathcal{L}_\beta$ regarding the weighted norm $\| \ \|_\beta$.

Proposition 3.3 (Continuity of $\mathcal{L}_\beta^{-1}$ in weighted norms)**.** *Let $\beta \in (\beta_*, 1)$ and $g \in H^{-\frac{1}{2}}$ with*

$$(g, iQ_\beta) = (g, \partial_x Q_\beta) = 0.$$

Then any solution f to

$$\mathcal{L}_\beta f = g, \ f \in H^{\frac{1}{2}}$$

satisfies:

$$\|f\|_\beta \leq C(\|g\|_{H^{-\frac{1}{2}}} + |(f, iQ_\beta)| + |(f, \partial_x Q_\beta)| + \|m_\beta * g\|_\beta) \quad (3.6)$$

where

$$m_\beta = \mathcal{F}^{-1} \left(\frac{1}{1 + \frac{|\xi| - \beta\xi}{1-\beta}} \right) \ .$$

Proof. The equation reads

$$f = m_\beta * g + m_\beta * (2|Q_\beta|^2 f + Q_\beta^2 \overline{f}) \ ,$$

so we are in position to apply Lemma 3.1 with $a_\beta = 2|Q_\beta|^2$, $b_\beta = Q_\beta^2$, $h = m_\beta * g$. In view of the L^∞-estimates and the tightness property for the family Q_β obtained from Proposition 2.6, we infer

$$\|f\|_\beta \leq B(\|f\|_{L^2} + \|m_\beta * g\|_\beta) \ .$$

On the other hand, by Proposition 2.4,

$$\|f\|_{L^2} \leq \|f\|_{H^{\frac{1}{2}}} \lesssim \|g\|_{H^{-\frac{1}{2}}} + |(f, iQ_\beta)| + |(f, \partial_x Q_\beta)|.$$

This completes the proof. □

Remark 3.4. In view of Remark 2.5, one can replace

$$|(f, iQ_\beta)| + |(f, \partial_x Q_\beta)|$$

by

$$|(f, iQ_+)| + |(f, \partial_x Q_+)|$$

in the right hand side of the estimate (3.6).

The second result is the following lemma.

Lemma 3.5. *Assume μ_β satisfies (3.1) and (3.2). Then*

$$\|\mu_\beta * (h_1 h_2)\|_\beta \lesssim \|h_1\|_\beta \, \|h_2\|_\beta \ .$$

Proof. First of all, the L^∞-bound is an easy consequence of $L^2 * L^2 \subset L^\infty$, so we may assume $|x| \geq 1$. Then we split

$$\begin{aligned}
\mu_\beta * (h_1 h_2)(x) &= \int_{|y|<\frac{|x|}{2}} \mu_\beta(x-y) h_1(y) h_2(y)\, dy + \\
&\quad \int_{|y|\geq\frac{|x|}{2}} \mu_\beta(x-y) h_1(y) h_2(y)\, dy \\
&= O(|x|^{-1}(1+(1-\beta)|x|)^{-1})\|h_1 h_2\|_{L^1} \\
&\quad + \|\mu_\beta\|_{L^2} \, \|h_1 h_2\|_{L^2(|y|>|x|/2)} \\
&\leq O(|x|^{-1}(1+(1-\beta)|x|)^{-1})\|h_1\|_{L^2}\|h_2\|_{L^2} \\
&\quad + O(|x|^{-3/2}(1+(1-\beta)|x|)^{-2})\|h_1\|_\beta \|h_2\|_\beta \ ,
\end{aligned}$$

and the lemma follows. □

The third result concerns the L^p norm of elements of class $\mathcal{M}$.

Lemma 3.6. *If $\{\mu_\beta\}_{\beta_*<\beta<1}$ belongs to class $\mathcal{M}$, then there exists $C>0$ such that, for every $p \in (1,\infty)$, for every $\beta \in (\beta_*, 1)$,*

$$\|\mu_\beta\|_{L^p} \leq C \, \max\left(\frac{1}{p-1}, p\right) \ .$$

Proof. From (3.3), the following holds,

$$\mu_\beta(x) = A_\beta \left(\mu_+(x) + \frac{1-\beta}{1+\beta}\mu_-\left(-\frac{1-\beta}{1+\beta}x\right)\right) \ , \ \mu_\pm := \mathcal{F}^{-1}(f_\pm) \ .$$

It is therefore sufficient to prove that, for every $f \in C^1(\mathbb{R}_+)$ such that

$$|f(\xi)| \leq \frac{C}{1+\xi} \ , \ |f'(\xi)| \leq \frac{C}{(1+\xi)^2} \ ,$$

the inverse Fourier transform $\mu = \mathcal{F}^{-1}(f)$ satisfies

$$\forall p \in (1,\infty) \ , \ \|\mu\|_{L^p(\mathbb{R})} \leq \tilde{C} \max\left(\frac{1}{p-1}, p\right) \ .$$

First, an integration by part leads to

$$x\mu(x) = \frac{if(0)}{2\pi} + i\int_0^\infty e^{ix\xi} f'(\xi)\,\frac{d\xi}{2\pi} \ ,$$

which provides the bound

$$|\mu(x)| \lesssim \frac{1}{|x|} \ .$$

Secondly, if x is close to 0, introducing a cut-off function φ such that $\varphi = 1$ near 0, and writing

$$\mu(x) = \int_0^\infty e^{ix\xi}\varphi(x\xi) f(\xi)\,\frac{d\xi}{2\pi} + \int_0^\infty e^{ix\xi}(1-\varphi(x\xi)) f(\xi)\,\frac{d\xi}{2\pi} \ := \mu_<(x) + \mu_>(x) \ ,$$

we observe that

$$|\mu_<(x)| \lesssim \log\left(\frac{1}{|x|}\right) \ ,$$

while

$$|x\mu_>(x)| \lesssim \int_{\mathbb{R}} \left|\frac{d}{d\xi}[\varphi(x\xi) f(\xi)]\right| d\xi \lesssim |x| \ .$$

We infer that, near $x = 0$,

$$|\mu(x)| \lesssim \log\left(\frac{1}{|x|}\right) \ .$$

Consequently,

$$\begin{aligned} \|\mu\|_{L^p}^p &\lesssim \int_{|x|\leq 1} \left(\log\left(\frac{1}{|x|}\right)\right)^p dx + \int_{|x|\geq 1} \frac{dx}{|x|^p} \\ &\lesssim p^p + \frac{1}{p-1} \ . \end{aligned}$$

This completes the proof. □

3.2. **Weighted estimates on Q_β.**

Proposition 3.7. *For every $p, q \in \mathbb{N}$, there exists $C_{p,q}$ such that*

$$\forall \beta \in (\beta_*, 1) \ , \ \|(x\partial_x)^p((1-\beta)\partial_\beta)^q Q_\beta\|_\beta \leqslant C_{p,q} \ .$$

Proof. First assume $p = q = 0$. We use the identity

$$Q_\beta = m_\beta * (Q_\beta |Q_\beta|^2) \ ,$$

and Lemma 3.1 with

$$m_\beta = \mathcal{F}^{-1}\left(\frac{1}{1+\frac{|\xi|-\beta\xi}{1-\beta}}\right) \ , \ a_\beta = |Q_\beta|^2 \ , \ b_\beta = 0 \ , \ h = 0 \ ,$$

and we easily obtain

$$\|Q_\beta\|_\beta \leq C_{0,0} \ .$$

Now let us prove the estimate for $p = 0$ and every q. Set $\tilde{\Lambda}_\beta := (1-\beta)\partial_\beta$. From equation (2.21), we have

$$\begin{cases} \mathcal{L}_\beta(\tilde{\Lambda}_\beta Q_\beta) = \frac{2}{1+\beta}(Q_\beta^- - \Pi_-(|Q_\beta|^2 Q_\beta)) \\ (\tilde{\Lambda}_\beta Q_\beta, iQ^+) = (\tilde{\Lambda}_\beta Q_\beta, \partial_x Q^+) = 0 \end{cases}$$

From a priori H^s estimates on Q_β and inequality (3.6) — in fact Remark 3.4— we infer

$$\left\|\tilde{\Lambda}_\beta Q_\beta\right\|_\beta \leq C\left(1+\left\|m_\beta * (Q_\beta^- - \Pi_-(|Q_\beta|^2 Q_\beta))\right\|_\beta\right) .$$

From the equation (1.9) of Q_β, we have

$$Q_\beta = m_\beta * (|Q_\beta|^2 Q_\beta) ,$$

so that, with $m_\beta^- := \Pi_- m_\beta$,

$$m_\beta * (Q_\beta^- - \Pi_-(|Q_\beta|^2 Q_\beta)) = (m_\beta^- * m_\beta^- - m_\beta^-) * (|Q_\beta|^2 Q_\beta) .$$

Notice that

$$\mathcal{F}(m_\beta^- * m_\beta^- - m_\beta^-)(\xi) = \mathbf{1}_{\xi<0} \frac{\frac{1+\beta}{1-\beta}\xi}{(1-\frac{1+\beta}{1-\beta}\xi)^2} ,$$

so that $\{m_\beta^- * m_\beta^- - m_\beta^-\}_{\beta_*<\beta<1}$ belongs to class $\mathcal{M}$, and therefore Lemma 3.5 yields

$$\left\|\tilde{\Lambda}_\beta Q_\beta\right\|_\beta \leqslant C_{0,1} .$$

For further reference, we are going to estimate $\left\|D\partial_\beta Q_\beta^-\right\|_{L^2}$. Projecting the equation of $\tilde{\Lambda}_\beta Q_\beta$ onto negative Fourier modes, we get

$$(1+\beta)D\partial_\beta Q_\beta^- = \tilde{\Lambda}_\beta Q_\beta^- - \Pi_-(2|Q_\beta|^2\tilde{\Lambda}_\beta Q_\beta + Q_\beta^2\overline{\tilde{\Lambda}_\beta Q_\beta}) - \frac{2}{1+\beta}(Q_\beta^- - \Pi_-(|Q_\beta|^2 Q_\beta)) .$$

From the estimate on $\tilde{\Lambda}_\beta Q_\beta$ we just established, we infer

$$\left\|D\partial_\beta Q_\beta^-\right\|_{L^2} \leq C_1' .$$

Let us prove by induction on $q \geq 1$ that

$$\left\|\tilde{\Lambda}_\beta^q Q_\beta\right\|_\beta \leq C_{0,q} , \ \left\|D\partial_\beta \tilde{\Lambda}_\beta^{q-1} Q_\beta\right\|_{L^2} \leq C_q' , \tag{3.7}$$

where $C_{0,q}$ and C_q' are independent of β. Notice that we just proved the case $q=1$. In order to deal with higher orders, we observe that, for every function f_β depending smoothly on β,

$$\mathcal{L}_\beta(\tilde{\Lambda}_\beta f_\beta) = \tilde{\Lambda}_\beta(\mathcal{L}_\beta f_\beta) + \frac{2Df_\beta^-}{1-\beta} + 4\mathrm{Re}(\overline{Q}_\beta\tilde{\Lambda}_\beta Q_\beta) f_\beta + 2Q_\beta\tilde{\Lambda}_\beta Q_\beta\overline{f}_\beta .$$

From this identity and the formula for $\tilde{\Lambda}_\beta Q_\beta$, we infer that $\mathcal{L}_\beta((\tilde{\Lambda}_\beta)^{q+1}Q_\beta)$ is a linear combination of terms of the following form.

- $D\partial_\beta(\tilde{\Lambda}_\beta)^r Q_\beta$, with $r \leq q-1$.
- $A_\beta(\tilde{\Lambda}_\beta)^r Q_\beta^-$ for $r \leq q$ and A_β depends smoothly on β, is bounded as well as its derivatives.
- $B_\beta\Pi_-\left((\tilde{\Lambda}_\beta)^a Q_\beta(\tilde{\Lambda}_\beta)^b Q_\beta\overline{(\tilde{\Lambda}_\beta)^c Q_\beta}\right)$, where $a+b+c \leq q$, and B_β depends smoothly on β, is bounded as well as its derivatives.
- $C_\beta(\tilde{\Lambda}_\beta)^a Q_\beta(\tilde{\Lambda}_\beta)^b Q_\beta\overline{(\tilde{\Lambda}_\beta)^c Q_\beta}$, where $a+b+c \leq q+1$, $a,b,c \leqslant q$, and C_β depends smoothly on β, is bounded as well as its derivatives.

Since all these terms are bounded in L^2 by the induction assumption, and since $((\tilde{\Lambda}_\beta)^{q+1}Q_\beta, iQ^+) = ((\tilde{\Lambda}_\beta)^{q+1}Q_\beta, \partial_x Q^+) = 0$, we infer from inequality (3.6) — in

fact Remark 3.4— that $\left\|(\tilde\Lambda_\beta)^{q+1}Q_\beta\right\|_{L^2}$ is bounded independently of β.

Now let us prove (3.7) at step $q+1$. Applying $(\tilde\Lambda_\beta)^{q+1}$ to

$$Q_\beta = m_\beta * (|Q_\beta|^2 Q_\beta) \ ,$$

we obtain

$$(\tilde\Lambda_\beta)^{q+1}Q_\beta = m_\beta * \left(2|Q_\beta|^2(\tilde\Lambda_\beta)^{q+1}Q_\beta + Q_\beta^2\overline{(\tilde\Lambda_\beta)^{q+1}Q_\beta}\right) + R_{\beta,q} \ ,$$

where $R_{\beta,q}$ is a finite sum of terms of the form

$$(\tilde\Lambda_\beta)^a m_\beta * \left[(\tilde\Lambda_\beta)^b Q_\beta (\tilde\Lambda_\beta)^c Q_\beta \overline{(\tilde\Lambda_\beta)^d Q_\beta}\right] \ , \ a+b+c+d = q+1 \ , \ \max(b,c,d) \le q \ .$$

Using Lemma 3.1, the L^2 estimate on $(\tilde\Lambda_\beta)^{q+1}Q_\beta$, and Lemmas 3.2 and 3.5, as well as the induction assumption, we infer

$$\left\|(\tilde\Lambda_\beta)^{q+1}Q_\beta\right\|_\beta \le C_{0,q+1} \ .$$

Furthermore,

$$D\partial_\beta(\tilde\Lambda_\beta)^q Q_\beta^- = \frac{Dm_\beta}{1-\beta} * \left(2|Q_\beta|^2(\tilde\Lambda_\beta)^{q+1}Q_\beta + Q_\beta^2\overline{(\tilde\Lambda_\beta)^{q+1}Q_\beta}\right) + (1-\beta)^{-1}DR_{\beta,q} \ ,$$

where $(1-\beta)^{-1}DR_{\beta,q}$ is a finite sum of terms of the form

$$(1-\beta)^{-1}D(\tilde\Lambda_\beta)^a m_\beta * \left[(\tilde\Lambda_\beta)^b Q_\beta (\tilde\Lambda_\beta)^c Q_\beta \overline{(\tilde\Lambda_\beta)^d Q_\beta}\right] \ , \ a+b+c+d = q+1 \ , \ \max(b,c,d) \le q \ .$$

It remains to observe that, if $\{\mu_\beta\}$ is an element of class $\mathcal{M}$, then

$$(1-\beta)^{-1}\widehat{D\mu_\beta^-}(\xi) = \mathbf{1}_{\xi<0}\frac{i\xi}{1-\beta}f_-\left(-\frac{1+\beta}{1-\beta}\xi\right)$$

is uniformly bounded in L^∞, therefore the convolution with $(1-\beta)^{-1}D\mu_\beta^-$ is uniformly bounded on L^2. This proves the L^2-estimate on $D\partial_\beta(\tilde\Lambda_\beta)^q Q_\beta^-$, and completes the proof of (3.7) at step $q+1$.

Finally, we prove the estimate for every p,q, by induction on $p+q$. Assume that

$$\left\|\Lambda_x^r(\tilde\Lambda_\beta)^s Q_\beta\right\|_\beta \le C_{r,s} \ , \ r+s \le n \ ,$$

and let us prove the inequality for $r+s = n+1$. Since the case $r=0$ is already known, we may assume $r = p+1, s = q$ with $p+q = n$. Recall that $\Lambda_x := x\partial_x$. We use the identity

$$\Lambda_x(f*g) = \Lambda_x(f)*g + f*\Lambda_x(g) + f*g = (\Lambda_x + I)f*g + f*\Lambda_x(g) \tag{3.8}$$

to obtain

$$\Lambda_x^p(\tilde\Lambda_\beta)^q Q_\beta = m_\beta * \left(2|Q_\beta|^2\Lambda_x^p(\tilde\Lambda_\beta)^q Q_\beta + Q_\beta^2\overline{\Lambda_x^p(\tilde\Lambda_\beta)^q Q_\beta}\right) + R_{\beta,p,q} \ ,$$

where $R_{\beta,p,q}$ is a finite sum of terms of the form

$$(\Lambda_x+I)^{a'}(\tilde\Lambda_\beta)^a m_\beta * \left[\Lambda_x^{b'}(\tilde\Lambda_\beta)^b Q_\beta \Lambda_x^{c'}(\tilde\Lambda_\beta)^c Q_\beta \overline{\Lambda_x^{d'}(\tilde\Lambda_\beta)^d Q_\beta}\right] \ ,$$

$$a+b+c+d = q \ , \ a'+b'+c'+d' = p \ , \ \max(b,c,d) \le q-1 \text{ or } \max(b',c',d') \le p-1 \ .$$

Let us first prove that $\Lambda_x^{p+1}(\tilde\Lambda_\beta)^q Q_\beta$ is uniformly bounded in L^2. We apply Λ_x to the above formula giving $\Lambda_x^p(\tilde\Lambda_\beta)^q Q_\beta$. We expand $\Lambda_x R_{\beta,p,q}$ using again identity (3.8), and we get, by the induction assumption, that $\Lambda_x R_{\beta,p,q}$ is uniformly bounded in L^2. As for the term

$$m_\beta * \left(2|Q_\beta|^2\Lambda_x^p(\tilde\Lambda_\beta)^q Q_\beta + Q_\beta^2\overline{\Lambda_x^p(\tilde\Lambda_\beta)^q Q_\beta}\right) \ ,$$

we write

$$x\partial_x[m_\beta * f] = x\partial_x m_\beta * f + \partial_x m_\beta * (xf) \ .$$

From the induction assumption, we easily get that

$$\left\| x\partial_x m_\beta * \left(2|Q_\beta|^2 \Lambda_x^p(\tilde{\Lambda}_\beta)^q Q_\beta + Q_\beta^2 \overline{\Lambda_x^p(\tilde{\Lambda}_\beta)^q Q_\beta} \right) \right\|_{L^2} \leq A_{p,q} \ .$$

On the other hand, since

$$\widehat{\partial_x m_\beta}(\xi) = \frac{i\xi}{1 + \frac{|\xi| - \beta\xi}{1-\beta}}$$

is uniformly bounded, the uniform bounds on

$$\|xQ_\beta\|_{L^\infty} \, , \|Q_\beta\|_{L^\infty} \ , \ \left\| \Lambda_x^p(\tilde{\Lambda}_\beta)^q Q_\beta \right\|_{L^2}$$

imply

$$\left\| \partial_x m_\beta * \left(2x|Q_\beta|^2 \Lambda_x^p(\tilde{\Lambda}_\beta)^q Q_\beta + xQ_\beta^2 \overline{\Lambda_x^p(\tilde{\Lambda}_\beta)^q Q_\beta} \right) \right\|_{L^2} \leq B_{p,q} \ .$$

Summing up, we have proved that $\Lambda_x^{p+1}(\tilde{\Lambda}_\beta)^q Q_\beta$ is uniformly bounded in L^2. It remains to prove a uniform bound of the weighted norm. But this is now a consequence of the formula

$$\Lambda_x^{p+1}(\tilde{\Lambda}_\beta)^q Q_\beta = m_\beta * \left(2|Q_\beta|^2 \Lambda_x^{p+1}(\tilde{\Lambda}_\beta)^q Q_\beta + Q_\beta^2 \overline{\Lambda_x^{p+1}(\tilde{\Lambda}_\beta)^q Q_\beta} \right) + R_{\beta,p+1,q} \ ,$$

of Lemmas 3.1, 3.2, 3.5 and of the induction assumption. The proof is complete. □

3.3. **Inverting $\mathcal{L}_\beta$ with a special right hand side.** In this section, we consider the equation

$$\mathcal{L}_\beta(i\rho_\beta) = i\partial_y Q_\beta \ , \ (i\rho_\beta, iQ_\beta) = (i\rho_\beta, \partial_y Q_\beta) = 0 \ . \tag{3.9}$$

Since $i\partial_y Q_\beta$ is orthogonal to iQ_β and $\partial_y Q_\beta$, this equation has a unique solution given by Proposition 3.3. The next lemma describes this solution as β tends to 1.

Lemma 3.8. *Let $i\rho_\beta$ be defined by (3.9). Then,*

$$i\rho_\beta = Q_\beta + \frac{i}{2}\partial_y Q_\beta + O((1-\beta)^{\frac{1}{2}} |\log(1-\beta)|^{\frac{1}{2}}) \text{ in } H^{\frac{1}{2}}(\mathbb{R}). \tag{3.10}$$

Proof. A computation based on the equation satisfied by Q_β shows that

$$\mathcal{L}_\beta\left(Q_\beta + \frac{i}{2}\partial_y Q_\beta \right) = -2|Q_\beta|^2 Q_\beta + iQ_\beta^2 \overline{\partial_y Q_\beta}.$$

On the other hand, we have

$$\mathcal{L}_\beta(\overline{Q_\beta}) = \frac{\overline{2\beta D Q_\beta}}{1-\beta} - |Q_\beta|^2 \overline{Q_\beta} - Q_\beta^3.$$

From the last two equations, we conclude that

$$\begin{aligned} \mathcal{L}_\beta\left(Q_\beta + \frac{i}{2}\partial_y Q_\beta + \frac{1}{2}(1-\beta)\overline{Q_\beta} \right) &= -2|Q_\beta|^2 Q_\beta + iQ_\beta^2 \overline{\partial_y Q_\beta} + i\beta\partial_y \overline{Q_\beta} \\ &\quad - \frac{1}{2}(1-\beta)|Q_\beta|^2 \overline{Q_\beta} - \frac{1}{2}(1-\beta)Q_\beta^3 =: RHS \end{aligned}$$

Using (2.20) and Proposition 2.3, we then notice that

$$\begin{aligned} RHS &= -2|Q^+|^2 Q^+ - i|Q^+|^4 - i\overline{Q^+}^2 + O((1-\beta)^{1/2} |\log(1-\beta)|^{1/2}) \text{ in } H^{-1/2} \\ &= i\partial_y Q^+ + O((1-\beta)^{1/2} |\log(1-\beta)|^{1/2}) \text{ in } H^{-1/2} \\ &= i\partial_y Q_\beta + O((1-\beta)^{1/2} |\log(1-\beta)|^{1/2}) \text{ in } H^{-1/2} \end{aligned}$$

Thus, denoting

$$g_\beta := Q_\beta + \frac{i}{2}\partial_y Q_\beta + \frac{1}{2}(1-\beta)\overline{Q_\beta},$$

we have that

$$\mathcal{L}_\beta(g_\beta) = i\partial_y Q_\beta + O((1-\beta)^{1/2}|\log(1-\beta)|^{1/2}) \text{ in } H^{-1/2}.$$

Notice that

$$\Big(Q_\beta + \frac{i}{2}\partial_y Q_\beta, iQ_\beta\Big) = \Big(Q_\beta + \frac{i}{2}\partial_y Q_\beta, \partial_y Q_\beta\Big) = 0.$$

Then, considering

$$\tilde{g}_\beta := g_\beta - \frac{1}{2}(1-\beta)\mathrm{Proj}_{(iQ_\beta, \partial_y Q_\beta)}\overline{Q_\beta}\ , \tag{3.11}$$

we have that $(\tilde{g}_\beta, iQ_\beta) = (\tilde{g}_\beta, \partial_y Q_\beta) = 0$ and

$$\mathcal{L}_\beta(\tilde{g}_\beta) = \mathcal{L}_\beta(g_\beta) = i\partial_y Q_\beta + O((1-\beta)^{1/2}|\log(1-\beta)|^{1/2}) \text{ in } H^{-1/2}\ .$$

Since $\mathcal{L}_\beta(i\rho_\beta) = i\partial_y Q_\beta$, it follows that $(i\rho_\beta - \tilde{g}_\beta, iQ_\beta) = (i\rho_\beta - \tilde{g}_\beta, \partial_y Q_\beta) = 0$ and

$$\mathcal{L}_\beta(i\rho_\beta - \tilde{g}_\beta) = O((1-\beta)^{1/2}|\log(1-\beta)|^{1/2}) \text{ in } H^{-1/2}.$$

Then, by Proposition 2.4, we have that

$$i\rho_\beta - \tilde{g}_\beta = O((1-\beta)^{1/2}|\log(1-\beta)|^{1/2}) \text{ in } H^{1/2}$$

In view of (3.11), we have $\tilde{g}_\beta = Q_\beta + \frac{i}{2}\partial_y Q_\beta + O(1-\beta)$ in $H^{\frac{1}{2}}(\mathbb{R})$, thus (3.10) is proved. □

3.4. **The profiles of $Q_\beta(x)$ and of $\partial_x Q_\beta(x)$ at infinity.**

Proposition 3.9. *Consider the following function,*

$$F(x) = \int_0^\infty \frac{\alpha\, e^{-\alpha}}{\alpha - ix}\, d\alpha\ ,\ x \in \mathbb{R}\ , \tag{3.12}$$

and the quantity

$$c_\beta := \frac{i}{2\pi}\int_{\mathbb{R}} |Q_\beta(x)|^2 Q_\beta(x)\, dx\ . \tag{3.13}$$

Then, as $\beta \to 1$ and $|x| \to \infty$, we have

$$Q_\beta(x) = \frac{c_\beta}{x} F\left(-\frac{1-\beta}{1+\beta}x\right) + O\left(\frac{1}{x^2}\right) \tag{3.14}$$

$$\partial_x Q_\beta(x) = \frac{ic_\beta}{x}\frac{1-\beta}{1+\beta}F\left(-\frac{1-\beta}{1+\beta}x\right) - \frac{c_\beta}{x^2} + O\left(\frac{1-\beta}{x^2} + \frac{\log|x|}{|x|^3}\right). \tag{3.15}$$

Remark 3.10. (1) From the previous section and by Lemma A.1, we know that c_β tends to 1 as β tends to 1. In the next subsection — see (3.29)— we will prove that

$$c_\beta = 1 + O((1-\beta)|\log(1-\beta)|)\ . \tag{3.16}$$

(2) Notice that $F(x) = 1 + O(|x||\log|x||)$ as $x \to 0$ and $|F(x)| \lesssim \frac{1}{|x|}$ for all $|x| > 0$. Therefore, as $\beta \to 1$ and $|x| \to \infty$, we infer from (3.14) that

$$|Q_\beta(x)| \lesssim \frac{1}{|1-\beta|x^2}, \quad \forall |x| > 0. \tag{3.17}$$

Furthermore, if $0 < 1-\beta \ll 1$, $|x| \gg 1$, and $(1-\beta)|x| \ll 1$, the following asymptotics follows from (3.14) and (3.16):

$$Q_\beta(x) = \frac{1 + O((1-\beta)|\log(1-\beta)|)}{x} \left[1 + O\left((1-\beta)|x|\,|\log((1-\beta)|x|)|\right)\right] + O\left(\frac{1}{x^2}\right). \tag{3.18}$$

(3) In view of the identity

$$F'(x) = \left(\frac{1}{x} - i\right) F(x) - \frac{1}{x}\,, \tag{3.19}$$

the main term in the asymptotics (3.15) for $\partial_x Q_\beta(x)$ is indeed obtained by deriving the main term in the asymptotics (3.14) of $Q_\beta(x)$.

Proof. The starting point is again the formula

$$Q_\beta = m_\beta * (|Q_\beta|^2 Q_\beta)\,,$$

where

$$m_\beta = \mathcal{F}^{-1}\left(\frac{1}{1 + \frac{|\xi| - \beta\xi}{1-\beta}}\right)\,.$$

We notice that, for $x \neq 0$,

$$m_\beta(x) = \frac{1}{2\pi}\left(G(x) + \frac{1-\beta}{1+\beta} G\left(-\frac{1-\beta}{1+\beta}x\right)\right)\,,$$

where

$$G(x) = \int_0^\infty \frac{e^{ix\xi}}{1+\xi}\,d\xi = \int_0^\infty \frac{e^{-\alpha}}{\alpha - ix}\,d\alpha,$$

the second integral being obtained from the former by writing

$$\frac{1}{1+\xi} = \int_0^\infty e^{-\alpha(1+\xi)}\,d\alpha\,.$$

It is easy to check that G is smooth outside $x = 0$, $G(x) = ix^{-1} + x^{-2} + O(x^{-3})$ as $x \to \infty$, and $G(x) \sim \log|x|$ as $x \to 0$. In particular, $G \in L^p(\mathbb{R})$ for every $p \in (1, \infty)$, with

$$\|G\|_{L^p} \leq C \max\left(\frac{1}{p-1}, p\right)\,. \tag{3.20}$$

Next we split

$$xQ_\beta(x) = \int_{\mathbb{R}} (x-y) m_\beta(x-y) |Q_\beta(y)|^2 Q_\beta(y)\,dy + \int_{\mathbb{R}} m_\beta(x-y) y |Q_\beta(y)|^2 Q_\beta(y)\,dy\,.$$

Let us estimate the second integral in the right hand side, writing

$$\begin{aligned}\int_{\mathbb{R}} m_\beta(x-y) y |Q_\beta(y)|^2 Q_\beta(y)\,dy \;&=\; \int_{|y|\leq|x|/2} m_\beta(x-y) y |Q_\beta(y)|^2 Q_\beta(y)\,dy + \\ &\int_{|y|>|x|/2} m_\beta(x-y) y |Q_\beta(y)|^2 Q_\beta(y)\,dy\,.\end{aligned}$$

From Hölder's inequality and the uniform bound $|Q_\beta(x)|\langle x\rangle$ from Proposition 3.7, we have, for every $p > 1$, close to 1,

$$\left|\int_{|y|>|x|/2} m_\beta(x-y) y |Q_\beta(y)|^2 Q_\beta(y)\,dy\right| \lesssim \frac{1}{p-1}|x|^{-1-\frac{1}{p}}\,,$$

which, by choosing

$$p = 1 + \frac{1}{\log|x|} \ ,$$

yields

$$\left| \int_{|y|>|x|/2} m_\beta(x-y) y |Q_\beta(y)|^2 Q_\beta(y)\, dy \right| \lesssim \frac{\log|x|}{x^{2-\frac{1}{\log|x|}}} \lesssim \frac{\log|x|}{|x|^2} \ .$$

On the other hand, because of the bounds on G, we have

$$|m_\beta(x)| \lesssim |x|^{-1} \ , \ |x| \to \infty \ .$$

Indeed, the only non trivial case is $(1-\beta)|x| \leq 1$, so that

$$\left| \frac{1-\beta}{1+\beta} G\left(-\frac{1-\beta}{1+\beta} x \right) \right| \lesssim (1-\beta)|\log[(1-\beta)|x|] \lesssim \frac{1}{|x|} \ .$$

We conclude that

$$\left| \int_{|y|\leq|x|/2} m_\beta(x-y) y |Q_\beta(y)|^2 Q_\beta(y)\, dy \right| \lesssim \frac{1}{|x|} \ ,$$

so that

$$Q_\beta(x) = \frac{1}{x} \int_{\mathbb{R}} (x-y) m_\beta(x-y) |Q_\beta(y)|^2 Q_\beta(y)\, dy + O\left(\frac{1}{x^2}\right) \ .$$

We come to the first integral. We observe that

$$\widehat{xm_\beta}(\xi) = i\partial_\xi \left(\frac{1}{1+\frac{|\xi|-\beta\xi}{1-\beta}} \right) = i\left(\frac{1+\beta}{1-\beta}\right) \mathbf{1}_{\xi<0} \left(1 + \frac{1+\beta}{1-\beta}|\xi|\right)^{-2} - i\frac{\mathbf{1}_{\xi>0}}{(1+\xi)^2} \ ,$$

so that

$$xm_\beta(x) = \frac{i}{2\pi}\left(F\left(-\frac{1-\beta}{1+\beta} x \right) - F(x) \right) \ , \tag{3.21}$$

$$F(x) := \int_0^\infty \frac{e^{ix\xi}}{(1+\xi)^2}\, d\xi = \int_0^\infty \frac{\alpha\, e^{-\alpha}}{\alpha - ix}\, d\alpha = 1 + ixG(x) \ . \tag{3.22}$$

This leads to

$$\begin{aligned} \int_{\mathbb{R}} (x-y) m_\beta(x-y) |Q_\beta(y)|^2 Q_\beta(y)\, dy &= -\int_{\mathbb{R}} F(x-y) g_\beta(y)\, dy \\ &+ \int_{\mathbb{R}} F\left(-\frac{1-\beta}{1+\beta}(x-y) \right) g_\beta(y)\, dy \ , \\ g_\beta &:= \frac{i}{2\pi} |Q_\beta|^2 Q_\beta \ . \end{aligned}$$

Again we are going to estimate the above two integrals by using the properties of F, namely that F is smooth outside the origin, it is bounded near 0, $F(x) = O(x^{-1})$ at infinity, while $|F'(x)| = O(|\log|x||)$ near 0 and $F'(x) = O(x^{-2})$ at infinity. Furthermore, let us recall from Proposition 3.7 that

$$g_\beta(y) = O(\langle y \rangle^{-3}) \ .$$

We infer the following estimates,

$$\begin{aligned}
\int_{\mathbb{R}} F(x-y)g_\beta(y)\,dy &= \int_{|y|\le\frac{|x|}{2}} F(x-y)g_\beta(y)\,dy + \int_{|y|>\frac{|x|}{2}} F(x-y)g_\beta(y)\,dy \\
&= O(|x|^{-1}) + O(x^{-2})\ , \\
\int_{\mathbb{R}} F\left(-\frac{1-\beta}{1+\beta}(x-y)\right) g_\beta(y)\,dy &= F\left(-\frac{1-\beta}{1+\beta}x\right)\int_{|y|\le\frac{|x|}{2}} g_\beta(y)\,dy \\
&+ \int_{|y|>\frac{|x|}{2}} F\left(-\frac{1-\beta}{1+\beta}(x-y)\right) g_\beta(y)\,dy \\
&+ \int_{|y|\le\frac{|x|}{2}} \left(F\left(-\frac{1-\beta}{1+\beta}(x-y)\right) - F\left(-\frac{1-\beta}{1+\beta}x\right)\right) g_\beta(y)\,dy \\
&= c_\beta F\left(-\frac{1-\beta}{1+\beta}x\right) + O(x^{-2}) + O(|x|^{-1}\omega((1-\beta)|x|))\ , \\
\omega(s) &:= \begin{cases} s|\log s| & \text{if } 0<s\le\frac12 \\ \frac1s & \text{if } \frac12\le s \end{cases}\ .
\end{aligned}$$

This completes the proof of (3.14). Let us come to the proof of (3.15). Notice that

$$\begin{aligned}
\widehat{\partial_x m_\beta}(\xi) &= \frac{i\xi}{1+\frac{|\xi|-\beta\xi}{1-\beta}} \\
&= i\left(\mathbf{1}_{\xi>0} - \frac{1-\beta}{1+\beta}\mathbf{1}_{\xi<0} - \frac{\mathbf{1}_{\xi>0}}{1+\xi} + \frac{1-\beta}{1+\beta}\frac{\mathbf{1}_{\xi<0}}{\left(1+\frac{|\xi|-\beta\xi}{1-\beta}\right)}\right)\ ,
\end{aligned}$$

so that, using the formulae $\mathcal{F}^{-1}(\mathbf{1}_{\pm\xi>0}) = \mp\frac{1}{2\pi i}pv\left(\frac1x\right) + \frac12\delta_0$,

$$\partial_x m_\beta(x) = \frac{-1}{\pi(1+\beta)}pv\left(\frac1x\right) + \frac{i\beta}{2(1+\beta)}\delta_0 - \frac{i}{2\pi}G(x) + \frac{i}{2\pi}\left(\frac{1-\beta}{1+\beta}\right)^2 G\left(-\frac{1-\beta}{1+\beta}x\right)\ ,$$

and

$$\begin{aligned}
\partial_x Q_\beta(x) &= \frac{2i}{1+\beta}pv\left(\frac1x\right) * g_\beta + \frac{i\beta}{2(1+\beta)}|Q_\beta(x)|^2 Q_\beta(x) \\
&+ \int_{\mathbb{R}} \left(\left(\frac{1-\beta}{1+\beta}\right)^2 G\left(-\frac{1-\beta}{1+\beta}(x-y)\right) - G(x-y)\right) g_\beta(y)\,dy\ .
\end{aligned}$$

Using, similarly as above, the estimates on G, and Proposition 3.7 for g_β , we have

$$\begin{aligned}
\int_{|y|>|x|/2} G(x-y)g_\beta(y)\,dy &= O\left(\frac{\log|x|}{|x|^3}\right)\ , \\
\int_{|y|\le|x|/2} G(x-y)g_\beta(y)\,dy &= \frac{i}{x}\int_{\mathbb{R}} g_\beta + \frac{1}{x^2}\left(i\int_{\mathbb{R}} y g_\beta + \int_{\mathbb{R}} g_\beta\right) + O\left(\frac{\log|x|}{|x|^3}\right)\ .
\end{aligned}$$

On the other hand,

$$pv\left(\frac1x\right) * g_\beta = \frac1x\int_{\mathbb{R}} g_\beta + \frac{1}{x^2}\int_{\mathbb{R}} y g_\beta + \frac{1}{x^2}\,pv\left(\frac1x\right) * (y^2 g_\beta)\ .$$

From Proposition 3.7, $h_\beta(y) := y^2 g_\beta(y)$ satisfies $h_\beta(y) = O(\langle y\rangle^{-1})$ and $h'_\beta(y) = O(\langle y\rangle^{-2})$. We infer

$$\begin{aligned} pv\left(\frac{1}{x}\right) * h_\beta &= \int_0^\infty \frac{h_\beta(x-z)-h_\beta(x+z)}{z}\,dz \\ &= \int_{||x|-z|>|x|/2} \frac{h_\beta(x-z)-h_\beta(x+z)}{z}\,dz \\ &\quad + \int_{||x|-z|\le|x|/2} \frac{h_\beta(x-z)-h_\beta(x+z)}{z}\,dz \\ &\lesssim \int_{||x|-z|>|x|/2} \frac{dz}{\langle |x|-z\rangle^2} + \int_{||x|-z|\le|x|/2} \frac{dz}{|x|\langle |x|-z\rangle} \\ &= O(|x|^{-1}) + O(|x|^{-1}\log|x|)\ . \end{aligned}$$

Summing up, we have proved that, as $x \to \infty$,

$$\frac{2i}{1+\beta} pv\left(\frac{1}{x}\right) * g_\beta - G * g_\beta = \frac{i(1-\beta)}{(1+\beta)x}\int_{\mathbb{R}} g_\beta - \frac{1}{x^2}\int_{\mathbb{R}} g_\beta + O\left(\frac{1-\beta}{x^2} + \frac{\log|x|}{|x|^3}\right)\ .$$

It remains to study the last integral, namely

$$\int_{\mathbb{R}} \left(\frac{1-\beta}{1+\beta}\right)^2 G\left(-\frac{1-\beta}{1+\beta}(x-y)\right) g_\beta(y)\,dy = \int_{|y|\le|x|/2} ... + \int_{|y|>|x|/2} ...\ .$$

Using again Hölder's inequality and optimizing on the power, we get

$$\left|\int_{|y|>|x|/2} \left(\frac{1-\beta}{1+\beta}\right)^2 G\left(-\frac{1-\beta}{1+\beta}(x-y)\right) g_\beta(y)\,dy\right| \lesssim \frac{(1-\beta)\log|x|}{|x|^3}\ .$$

On the other hand, because of the estimates on G', we have

$$\begin{aligned} &\int_{|y|\le|x|/2} \left(\frac{1-\beta}{1+\beta}\right)^2 G\left(-\frac{1-\beta}{1+\beta}(x-y)\right) g_\beta(y)\,dy = \\ &\left(\frac{1-\beta}{1+\beta}\right)^2 G\left(-\frac{1-\beta}{1+\beta}x\right)\int_{\mathbb{R}} g_\beta(y)\,dy + O\left(\frac{1-\beta}{x^2}\right)\ . \end{aligned}$$

In view of the identity

$$G(x) = \frac{F(x)-1}{ix}\ ,$$

this completes the proof of (3.15). □

3.5. **Further estimates on $\partial_\beta Q_\beta$.** In this subsection, we improve some the estimates on $\dot{Q}_\beta := \partial_\beta Q_\beta$ deduced in Proposition 3.7.

Proposition 3.11. *The following estimates hold as β tends to 1.*

$$\|\dot{Q}^+_\beta\|_{H^{\frac12}} \lesssim |\log(1-\beta)|\ , \tag{3.23}$$

$$|\widehat{\dot{Q}^-_\beta}(\xi)| \le \frac{C}{1-\beta+(1+\beta)|\xi|}\ . \tag{3.24}$$

Furthermore, if $H_\beta = (1-\beta)\partial_\beta^2 Q_\beta$ or $H_\beta = \partial_\beta y \partial_y Q_\beta$, we have similarly

$$\|H^+_\beta\|_{H^{\frac12}} \lesssim |\log(1-\beta)|\ , \tag{3.25}$$

$$|\widehat{H^-_\beta}(\xi)| \le \frac{C}{1-\beta+(1+\beta)|\xi|}\ . \tag{3.26}$$

In particular,

$$\frac{d}{d\beta}\|Q_\beta\|_{L^2}^2 = O(|\log(1-\beta)|)\ , \tag{3.27}$$

$$\frac{d}{d\beta}(DQ_\beta, Q_\beta) = O(|\log(1-\beta)|)\ , \tag{3.28}$$

$$\frac{d}{d\beta}\int_{\mathbb{R}} |Q_\beta|^2 Q_\beta = O(|\log(1-\beta)|)\ , \tag{3.29}$$

and, if H_β is as above, and ρ_β is defined by (3.9), we have

$$|(H_\beta, Q_\beta)| + |(H_\beta, DQ_\beta)| + |(H_\beta, i\rho_\beta)| = O(|\log(1-\beta)|). \tag{3.30}$$

Proof. We project the equation (2.21) for $\dot{Q}_\beta$ onto the negative and positive modes. This gives

$$\begin{aligned}
&(1+\beta)|D|\dot{Q}_\beta^- + (1-\beta)\dot{Q}_\beta^- = \\
&\frac{2}{1+\beta}[Q_\beta^- - \Pi_-(|Q_\beta|^2 Q_\beta)] + \Pi_-[2|Q_\beta|^2(1-\beta)\dot{Q}_\beta + Q_\beta^2(1-\beta)\overline{\dot{Q}_\beta}]\ , \\
&D\dot{Q}_\beta^+ + \dot{Q}_\beta^+ - \Pi_+[2|Q_\beta|^2\dot{Q}_\beta^+ + Q_\beta^2\overline{\dot{Q}_\beta^+}] = \Pi_+[2|Q_\beta|^2\dot{Q}_\beta^- + Q_\beta^2\overline{\dot{Q}_\beta^-}]\ , \\
&(\dot{Q}_\beta^+, iQ^+) = (\dot{Q}_\beta^+, \partial_x Q^+) = 0\ .
\end{aligned}$$

Using the last equation, the invertibility (2.15) of $\mathcal{L}$ defined in (2.16), and a perturbation argument as in Proposition 2.4, we can estimate $\dot{Q}_\beta^+$ by means of $\dot{Q}_\beta^-$ as follows,

$$\|\dot{Q}_\beta^+\|_{H^{\frac12}} \lesssim \||Q_\beta|^2\dot{Q}_\beta^- + Q_\beta^2\overline{\dot{Q}_\beta^-}\|_{H^{-\frac12}} \lesssim \|Q_\beta^2\dot{Q}_\beta^-\|_{L^2}\ . \tag{3.31}$$

On the other hand, the first equation leads to

$$\begin{aligned}
\widehat{\dot{Q}_\beta^-}(\xi) &= \frac{\hat{\ell}_\beta(\xi)}{1-\beta+(1+\beta)|\xi|}\ , \\
\ell_\beta &:= \frac{2}{1+\beta}\Pi_-[m_\beta * (|Q_\beta|^2 Q_\beta) - |Q_\beta|^2 Q_\beta] + \Pi_-[2|Q_\beta|^2(1-\beta)\dot{Q}_\beta + Q_\beta^2(1-\beta)\overline{\dot{Q}_\beta}]\ .
\end{aligned}$$

Using the L^2 bound on $(1-\beta)\dot{Q}_\beta$ from Proposition 3.7, the above expression of ℓ_β implies

$$\|\hat{\ell}_\beta\|_{L^\infty} \le C\ ,$$

which proves (3.24). Coming back to (3.31), we infer, using the L^1 and the L^2 bound on $\widehat{Q_\beta^2}$, and from Young's $L^1 * L^2 \subset L^2$ inequality,

$$\begin{aligned}
\|\dot{Q}_\beta^+\|_{H^{\frac12}} &\lesssim \left[\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\frac{|\widehat{Q_\beta^2}(\xi-\eta)|}{1-\beta+(1+\beta)|\eta|}\,d\eta\right|^2 d\xi\right]^{\frac12} \\
&\lesssim \left(\int_{|\eta|>1}\frac{d\eta}{(1-\beta+(1+\beta)|\eta|)^2}\right)^{\frac12} + \int_{|\eta|\le 1}\frac{d\eta}{1-\beta+(1+\beta)|\eta|} \\
&\lesssim |\log(1-\beta)|.
\end{aligned}$$

This proves (3.23).

Next we prove (3.25) and (3.26). We apply $(1-\beta)\partial_\beta$ to the above equations on Q_β^+ and Q_β^-. With $H_\beta := (1-\beta)\partial_\beta^2 Q_\beta$, we infer

$$
\begin{aligned}
&(1+\beta)|D|H_\beta^- + (1-\beta)H_\beta^- + (|D|-1)(1-\beta)\dot{Q}_\beta^- = \\
&(1-\beta)\partial_\beta\left(\frac{2}{1+\beta}[Q_\beta^- - \Pi_-(|Q_\beta|^2 Q_\beta)] + \Pi_-[2|Q_\beta|^2(1-\beta)\dot{Q}_\beta + Q_\beta^2(1-\beta)\overline{\dot{Q}_\beta}]\right) , \\
&DH_\beta^+ + H_\beta^+ - \Pi_+[2|Q_\beta|^2 H_\beta^+ + Q_\beta^2\overline{H_\beta^+}] = \\
&\Pi_+[2|Q_\beta|^2 H_\beta^- + Q_\beta^2\overline{H_\beta^-} + 2(1-\beta)\partial_\beta(|Q_\beta|^2)\dot{Q}_\beta + (1-\beta)\partial_\beta(Q_\beta^2)\overline{\dot{Q}_\beta}] , \\
&(H_\beta^+, iQ^+) = (H_\beta^+, \partial_x Q^+) = 0 .
\end{aligned}
$$

In view of (3.24), the Fourier transform of $(|D|-1)(1-\beta)\dot{Q}_\beta^-$ is uniformly bounded. Furthermore, using again (3.24) and the L^2 bound on $[(1-\beta)\partial_\beta]^k Q_\beta$ from Proposition 3.7, the Fourier transform of the right hand side of the equation on H_β^- is uniformly bounded. This provides estimate (3.26). In order to obtain (3.25), we use the equation on H_β^+. Notice that, again by (3.23) and (3.24) combined with the Hausdorff–Young inequality,

$$
\begin{aligned}
&\|2(1-\beta)\partial_\beta(|Q_\beta|^2)\dot{Q}_\beta + (1-\beta)\partial_\beta(Q_\beta^2)\overline{\dot{Q}_\beta}\|_{L^2} \lesssim (1-\beta)\|\dot{Q}_\beta\|_{L^4}^2 \\
&\lesssim (1-\beta)\left(\int_{\mathbb{R}} \frac{d\xi}{((1-\beta)+(1+\beta)|\xi|)^{4/3}}\right)^{3/2} + (1-\beta)|\log(1-\beta)|^2 \\
&\lesssim (1-\beta)^{1/2} .
\end{aligned}
$$

By the perturbation argument of Proposition 2.4, we infer

$$
\|H_\beta^+\|_{H^{\frac{1}{2}}} \lesssim \|Q_\beta^2 H_\beta^-\|_{L^2} + O((1-\beta)^{1/2}) ,
$$

and we obtain (3.25) exactly as we obtained (3.23) above.

Next we deal with the case of $H_\beta := y\partial_y\dot{Q}_\beta$. Applying $y\partial_y$ to the equation on Q_β, we get

$$
\mathcal{L}_\beta(y\partial_y Q_\beta) = |Q_\beta|^2 Q_\beta - Q_\beta ,
$$

and, taking the derivative with respect to β and projecting on the negative and positive modes, we obtain

$$
\begin{aligned}
&(1+\beta)|D|H_\beta^- + (1-\beta)H_\beta^- = \\
&\Pi_-[2|Q_\beta|^2(1-\beta)H_\beta + Q_\beta^2(1-\beta)\overline{H_\beta}] + \frac{2}{1+\beta}\Pi_-(y\partial_y Q_\beta) \\
&+ \frac{2}{1+\beta}[Q_\beta^- - \Pi_-(|Q_\beta|^2 Q_\beta)] + \Pi_-[2|Q_\beta|^2(1-\beta)\dot{Q}_\beta + Q_\beta^2(1-\beta)\overline{\dot{Q}_\beta}] - (1-\beta)\dot{Q}_\beta^- \\
&- \frac{2}{1+\beta}\Pi_-[2|Q_\beta|^2 y\partial_y Q_\beta + Q_\beta^2\overline{y\partial_y Q_\beta}] \\
&+ 2\Pi_-[(1-\beta)\overline{\dot{Q}_\beta}Q_\beta y\partial_y Q_\beta + (1-\beta)\dot{Q}_\beta\overline{Q_\beta}y\partial_y Q_\beta + (1-\beta)\dot{Q}_\beta Q_\beta\overline{y\partial_y Q_\beta}], \\
&DH_\beta^+ + H_\beta^+ - \Pi_+[2|Q_\beta|^2 H_\beta^+ + Q_\beta^2\overline{H_\beta^+}] = \\
&\Pi_+[2|Q_\beta|^2 H_\beta^- + Q_\beta^2\overline{H_\beta^-} + 2|Q_\beta|^2\dot{Q}_\beta + Q_\beta^2\overline{\dot{Q}_\beta}] - \dot{Q}_\beta^+ \\
&+ 2\Pi_+[\overline{\dot{Q}_\beta}Q_\beta y\partial_y Q_\beta + \dot{Q}_\beta\overline{Q_\beta}y\partial_y Q_\beta + \dot{Q}_\beta Q_\beta\overline{y\partial_y Q_\beta}], \\
&(H_\beta^+, iQ^+) = (H_\beta^+, \partial_x Q^+) = 0 .
\end{aligned}
$$

Again, from Proposition 3.7, we notice that the Fourier transform of the right hand side of the equation on H^-_β is bounded. This provides (3.26). Using again the perturbation argument of Proposition 2.4, we infer

$$\|H^+_\beta\|_{H^{\frac12}} \lesssim \|Q^2_\beta H^-_\beta\|_{L^2} + \|Q^2_\beta \dot Q_\beta\|_{L^2} + \|\dot Q^+_\beta\|_{L^2} + \|\dot Q_\beta Q_\beta y\partial_y Q_\beta\|_{L^2}$$

and (3.25) again follows from (3.26), (3.24), (3.23), and the L^1- and L^2- bounds on $\widehat{Q_\beta y\partial_y Q_\beta}$.

Let us come to the proof of (3.27). We have

$$\frac{d}{d\beta}\|Q_\beta\|^2_{L^2} = 2(Q_\beta, \dot Q_\beta) = 2(Q^+_\beta, \dot Q^+_\beta) + 2(Q^-_\beta, \dot Q^-_\beta)\ .$$

From (3.23) and the L^2 bound on Q_β, we infer

$$|(Q^+_\beta, \dot Q^+_\beta)| \lesssim |\log(1-\beta)|\ .$$

From (3.24) and the representation of Q^-_β, we infer

$$|(Q^-_\beta, \dot Q^-_\beta)| \lesssim \int_{\mathbb{R}} \frac{(1-\beta)|\widehat{|Q_\beta|^2 Q_\beta}(\xi)|}{(1-\beta+(1+\beta)|\xi|)^2}\, d\xi = O(1)\ .$$

This completes the proof of (3.27). The proof of (3.28) is similar. As for (3.29), we write

$$\frac{d}{d\beta}\int_{\mathbb{R}} |Q_\beta|^2 Q_\beta = 2\int_{\mathbb{R}} |Q_\beta|^2 \dot Q_\beta + \int_{\mathbb{R}} Q^2_\beta \overline{\dot Q_\beta}\ .$$

Write $\dot Q_\beta = \dot Q^+_\beta + \dot Q^-_\beta$ in the two integrals of the above right hand side. The contribution of $\dot Q^+_\beta$ is $O(|\log(1-\beta)|)$ because of (3.23). As for the contribution of $\dot Q^-_\beta$, we evaluate it by means of the Plancherel theorem. In view of (3.24), it is $O(|\log(1-\beta)|)$. This completes the proof of (3.29).
The proof of the first two estimates of (3.30) follows exactly the same lines as (3.27). As for the last estimate, we recall from (3.10) that

$$\|i\rho_\beta - Q_\beta - \frac12 DQ_\beta\|_{L^2} \lesssim (1-\beta)^{1/2}|\log(1-\beta)|^{1/2}\ ,$$

so that

$$|(H_\beta, i\rho_\beta)| \lesssim |\log(1-\beta)| + (\|H^+_\beta\|_{L^2} + \|H^-_\beta\|_{L^2})(1-\beta)^{1/2}|\log(1-\beta)|^{1/2}\ ,$$

and the proof is completed by using (3.25) and (3.26). □

4. The two-bubble approximate solution

This section is devoted to the construction of the two-bubble approximate solution. The general strategy follows the lines of [27] for the Hartree problem with the additional difficulties of keeping very carefully track of the leading order terms generated by the critically slow decay of the solitary wave and getting estimates which are uniform in the singular limit $\beta \to 1$.

4.1. **Renormalization and slow variables.** Let

$$u_j(t,x) = \frac{1}{\lambda_j^{\frac{1}{2}}} v_j(s_j, y_j) e^{i\gamma_j}, \quad \frac{ds_j}{dt} = \frac{1}{\lambda_j(t)}, \quad y_j := \frac{x - x_j(t)}{\lambda_j(t)(1-\beta_j(t))},$$

for $j = 1, 2$. We have

$$\begin{aligned} & i\partial_t u_j - |D| u_j + u_j |u_j|^2 \\ = \; & \frac{1}{\lambda_j^{\frac{3}{2}}} \left[i\partial_{s_j} v_j - \frac{(|D| - \beta_j D) v_j}{1-\beta_j} - i \frac{(\lambda_j)_{s_j}}{\lambda_j} \Lambda v_j - \frac{i}{1-\beta_j} \left(\frac{(x_j)_{s_j}}{\lambda_j} - \beta_j \right) \partial_{y_j} v_j \right. \\ + \; & \left. \frac{i(\beta_j)_{s_j}}{1-\beta_j} y_j \partial_{y_j} v_j - (\gamma_j)_{s_j} v_j + v_j |v_j|^2 \right] e^{i\gamma_j}(s_j, y_j). \end{aligned}$$

Let us define the relative numbers

$$X = x_2 - x_1, \quad \mu = \frac{\lambda_2}{\lambda_1}, \quad \Gamma = \gamma_2 - \gamma_1,$$

and

$$b = \frac{1-\beta_2}{1-\beta_1}, \quad R = \frac{X}{\lambda_1(1-\beta_1)}. \tag{4.1}$$

We observe the relation

$$y_1 = R + \mu b y_2. \tag{4.2}$$

We then decompose $u(t,x) = u_1(t,x) + u_2(t,x)$, expand the nonlinearity

$$u|u|^2 = u_1(|u_1|^2 + 2|u_2|^2 + u_1\bar{u_2}) + u_2(|u_2|^2 + 2|u_1|^2 + u_2\bar{u_1})$$

and split the contributions of crossed terms using a cut off function

$$\chi_R(x) = \chi\left(\frac{y_1}{R}\right) = \chi\left(1 + \frac{\mu b}{R} y_2\right) \tag{4.3}$$

to obtain:

$$i\partial_t u - |D|u + u|u|^2 = \frac{1}{\lambda_1^{\frac{3}{2}}} \mathcal{E}_1(s_1, y_1) e^{i\gamma_1} + \frac{1}{\lambda_2^{\frac{3}{2}}} \mathcal{E}_2(s_2, y_2) e^{i\gamma_2}$$

with

$$\begin{aligned} \mathcal{E}_1 = \; & i\partial_{s_1} v_1 - \frac{(|D| - \beta_1 D) v_1}{1-\beta_1} - v_1 + v_1 |v_1|^2 \\ - \; & i \frac{(\lambda_1)_{s_1}}{\lambda_1} \Lambda v_1 - \frac{i}{1-\beta_1} \left(\frac{(x_1)_{s_1}}{\lambda_1} - \beta_1 \right) \partial_{y_1} v_1 + \frac{i(\beta_1)_{s_1}}{1-\beta_1} y_1 \partial_{y_1} v_1 - [(\gamma_1)_{s_1} - 1] v_1 \\ + \; & \chi_R \left[\frac{2}{\mu} v_1 |v_2|^2 + \frac{e^{-i\Gamma}}{\sqrt{\mu}} v_1^2 \overline{v_2} + 2 \frac{e^{i\Gamma}}{\sqrt{\mu}} |v_1|^2 v_2 + \frac{e^{2i\Gamma}}{\mu} \overline{v_1} v_2^2 \right], \end{aligned}$$

$$\begin{aligned} \mathcal{E}_2 = \; & i\partial_{s_2} v_2 - \frac{(|D| - \beta_2 D) v_2}{1-\beta_2} - v_2 + v_2 |v_2|^2 \\ - \; & i \frac{(\lambda_2)_{s_2}}{\lambda_2} \Lambda v_2 - \frac{i}{1-\beta_2} \left(\frac{(x_2)_{s_2}}{\lambda_2} - \beta_2 \right) \partial_{y_2} v_2 + \frac{i(\beta_2)_{s_2}}{1-\beta_2} y_2 \partial_{y_2} v_2 - [(\gamma_2)_{s_2} - 1] v_2 \\ + \; & (1 - \chi_R) \left[2\mu v |v_1|^2 v_2 + 2\sqrt{\mu} e^{-i\Gamma} v_1 |v_2|^2 + \sqrt{\mu} e^{i\Gamma} \overline{v_1} v_2^2 + \mu e^{-2i\Gamma} v_1^2 \overline{v_2} \right]. \end{aligned}$$

The full vector of parameters is denoted by

$$\mathcal{P} = (\lambda_1, \lambda_2, \beta_1, \beta_2, \Gamma, R). \tag{4.4}$$

Following [27], we now look for a solution to

$$\mathcal{E}_1 = \mathcal{E}_2 = 0$$

in the form of a slowly modulated two-bubble, i.e.

$$v_j(s_j, y_j) = V_j(y_j, \mathcal{P}(s_j))$$

where the time dependence of the parameters is frozen for translation and phase invariances:

$$\frac{(x_j)_{s_j}}{\lambda_j} = \beta_j, \quad (\gamma_j)_{s_j} = 1, \tag{4.5}$$

the dependence of scaling and speed is computed iteratively according to a dynamical system

$$\frac{(\lambda_j)_{s_j}}{\lambda_j} = M_j(\mathcal{P}), \quad \frac{(\beta_j)_{s_j}}{1-\beta_j} = B_j(\mathcal{P}), \tag{4.6}$$

$$\Gamma_{s_1} = \frac{1}{\mu} - 1, \quad \Gamma_{s_2} = 1 - \mu, \quad X_t = \beta_2 - \beta_1 \tag{4.7}$$

and the remaining time derivatives for (b, R) are modeled after (4.1), (4.5), (4.6):

$$R_{s_1} = 1 - b + (B_1 - M_1)R, \quad R_{s_2} = \mu(1 - b + (B_1 - M_1)R). \tag{4.8}$$

Hence

$$\begin{aligned}
\mathcal{E}_1 &= -\frac{(|D| - \beta_1 D)V_1}{1-\beta_1} - V_1 + V_1|V_1|^2 - iM_1\Lambda V_1 + iB_1\left[y_1\partial_{y_1}V_1 + (1-\beta_1)\frac{\partial V_1}{\partial \beta_1}\right] \\
&+ i\lambda_1 M_1 \frac{\partial V_1}{\partial \lambda_1} + i\lambda_1 M_2 \frac{\partial V_1}{\partial \lambda_2} + i\frac{(1-\beta_2)B_2}{\mu}\frac{\partial V_1}{\partial \beta_2} \\
&+ i\frac{1-\mu}{\mu}\frac{\partial V_1}{\partial \Gamma} + i(1 - b + (B_1 - M_1)R)\frac{\partial V_1}{\partial R} \\
&+ \chi_R\left[\frac{2}{\mu}V_1|V_2|^2 + \frac{e^{-i\Gamma}}{\sqrt{\mu}}V_1^2\overline{V_2} + 2\frac{e^{i\Gamma}}{\sqrt{\mu}}|V_1|^2V_2 + \frac{e^{2i\Gamma}}{\mu}\overline{V_1}V_2^2\right],
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
\mathcal{E}_2 &= -\frac{(|D| - \beta_2 D)V_2}{1-\beta_2} - V_2 + V_2|V_2|^2 - iM_2\Lambda V_2 + iB_2\left[y_2\partial_{y_2}V_2 + (1-\beta_2)\frac{\partial V_2}{\partial \beta_2}\right] \\
&+ i\lambda_2 M_2 \frac{\partial V_2}{\partial \lambda_2} + i\lambda_2 M_1 \frac{\partial V_2}{\partial \lambda_1} + i\mu(1-\beta_1)B_1\frac{\partial V_2}{\partial \beta_1} \\
&+ i(1-\mu)\frac{\partial V_2}{\partial \Gamma} + i\mu(1 - b + (B_1 - M_1)R)\frac{\partial V_2}{\partial R} \\
&+ (1-\chi_R)\left[2\mu|V_1|^2V_2 + 2\sqrt{\mu}e^{-i\Gamma}V_1|V_2|^2 + \sqrt{\mu}e^{i\Gamma}\overline{V_1}V_2^2 + \mu e^{-2i\Gamma}V_1^2\overline{V_2}\right].
\end{aligned} \tag{4.10}$$

and we need to solve the system of *nonlinear elliptic equations* in V_1, V_2,

$$\begin{cases} \mathcal{E}_1(y_1) = 0 \quad \text{with} \quad y_2 = \frac{y_1 - R}{b\mu}, \\ \mathcal{E}_2(y_2) = 0 \quad \text{with} \quad y_1 = R + b\mu y_2. \end{cases} \tag{4.11}$$

in a suitable range of parameters $\mathcal{P}$.

4.2. **Definition of admissible functions.** We define the open set of parameters:

$$\mathcal{P} \in \mathcal{O} \equiv \begin{cases} \beta_j \in (\beta^*, 1), \quad j = 1, 2 \\ R > R_* \\ |1 - \lambda_1| + |1 - \lambda_2| < \eta^\delta \\ \frac{\eta}{2} < 1 - \beta_1 < 2\eta, \quad 1 - \beta_2 \geq e^{-R}, \quad 0 < b < \delta \end{cases} \tag{4.12}$$

for some universal constants $R_* \gg 1$, $0 < \eta, \delta \ll 1$ to be chosen later.

We now define a suitable topology:

Definition 4.1 (Admissible function)**.** *We consider functions $g = g(y, \mathcal{P}) : \mathbb{R} \times \mathcal{O} \to \mathbb{C}$.*
(i) (*L^∞-admissibility*). *We say that g is L^∞-admissible if $\forall \alpha \in \mathbb{N}^7$, $\exists A_\alpha$, $\forall \mathcal{P} \in \mathcal{O}$,*

$$\left\| \Lambda_y^{\alpha_1} \Lambda_R^{\alpha_2} \partial_{\lambda_1}^{\alpha_3} \partial_{\lambda_2}^{\alpha_4} \partial_\Gamma^{\alpha_5} \tilde{\Lambda}_{\beta_1}^{\alpha_6} \tilde{\Lambda}_{\beta_2}^{\alpha_7} g(\cdot, \mathcal{P}) \right\|_\infty \leqslant A_\alpha. \tag{4.13}$$

(ii) (*Admissibility with respect to a bubble*). *Let $j \in \{1, 2\}$. We say that g is admissible with respect to the bubble j — or j–admissible — if $\forall \alpha \in \mathbb{N}^7$, $\exists A_\alpha$, $\forall \mathcal{P} \in \mathcal{O}$,*

$$\left\| \Lambda_y^{\alpha_1} \Lambda_R^{\alpha_2} \partial_{\lambda_1}^{\alpha_3} \partial_{\lambda_2}^{\alpha_4} \partial_\Gamma^{\alpha_5} \tilde{\Lambda}_{\beta_1}^{\alpha_6} \tilde{\Lambda}_{\beta_2}^{\alpha_7} g(\cdot, \mathcal{P}) \right\|_{\beta_j} \leqslant A_\alpha. \tag{4.14}$$

(iii) (*Strong admissibility with respect to a bubble*). *Let $j \in \{1, 2\}$. We say that g is strongly admissible with respect to the bubble j — or j–strongly admissible — if it is j–admissible and if, for every family $\{\mu_\beta\}_{\beta \in (\beta*, 1)}$ of multipliers in the class $\mathcal{M}$, the convolution product*

$$\mu_{\beta_j} * g(., \mathcal{P})$$

is j–admissible.

Notice that admissibility with respect to the bubble j implies L^∞-admissibility. Furthermore, we have the following fundamental property.

Lemma 4.2 (Admissibility of Q_β)**.** *For $j = 1, 2$, Q_{β_j} is strongly j–admissible.*

Proof. Admissibility of Q_{β_j} with respect to the bubble j is a straightforward consequence of Proposition 3.7. Given $\{\mu_\beta\}_{\beta \in (\beta*, 1)}$ a family of multipliers in the class $\mathcal{M}$, let us come to the j–admissibility of $\mu_{\beta_j} * Q_{\beta_j}$. From the identity

$$Q_\beta = m_\beta * (|Q_\beta|^2 Q_\beta) \ ,$$

and the invariance of $\mathcal{M}$ by convolution, we infer that

$$\mu_\beta * Q_\beta = \tilde{\mu}_\beta * (|Q_\beta|^2 Q_\beta) \ ,$$

where $\{\tilde{\mu}_\beta\}_{\beta \in (\beta*, 1)}$ belongs to $\mathcal{M}$. Then, applying $\tilde{\Lambda}_\beta^p \Lambda_y^q$ to this identity, and using the stability properties of class $\mathcal{M}$ through these operations, the j–admissibility of $\mu_{\beta_j} * Q_{\beta_j}$ follows from the j–admissibility of Q_{β_j} and from Lemma 3.5. □

4.3. **Stability properties of admissible functions.** We now prove some elementary stability properties of admissible functions.

Lemma 4.3 (Stability properties of admissible functions)**.** *The following stability properties hold.*
(i) (*Stability by derivation*). *Assume g is j–admissible (resp. strongly j–admissible). Then*

$$\Lambda_y g, \Lambda_R g, \partial_{\lambda_j} g, \partial_\Gamma g, \tilde{\Lambda}_{\beta_j} g \tag{4.15}$$

are j–admissible (resp. strongly j–admissible).

(ii) (*Stability by multiplication*). *If g is j–admissible, h is L^∞-admissible, then gh is j–admissible. Furthermore, if g and h are j–admissible, then gh is strongly j–admissible.*
(iii) (*Exchange of variables*). *Given a function $g = g(y)$, we define*

$$g^\sharp(y_1) := g\left(\frac{y_1 - R}{b\mu}\right) \tag{4.16}$$

and

$$g^\flat(y_2) := g(R + b\mu y_2) \ . \tag{4.17}$$

If g_2 is 2–admissible, then $R(1+(1-\beta_1)R)b^{-1}\chi_R g_2^\sharp$ is L^∞-admissible, and $b^{-1}\chi_R g_2^\sharp$ is 1–admissible. If g_1 is 1–admissible, then $R(1+(1-\beta_1)R)((1-\chi_R)g_1)^\flat$ is L^∞-admissible.
(iv) (*Stability by scalar product*). *If g is j–admissible, then (g, iQ_{β_j}) and $(g, \partial_y Q_{\beta_j})$ are L^∞-admissible.*
(v) (*Stability by convolution*). *If g is strongly j–admissible and if $\{\mu_\beta\}_{\beta\in(\beta*,1)}$ belongs to class $\mathcal{M}$, then $\mu_{\beta_j} * g$ is strongly j–admissible.*
(vi) (*Mixed cubic nonlinearity and convolution*). *Assume g_1, h_1 are 1–admissible, and g_2, h_2 are 2–admissible. Then*

$$R(1+(1-\beta_1)R)b^{-1}\chi_R g_1 g_2^\sharp h_2^\sharp \ , \ R(1+(1-\beta_1)R)b^{-1}\chi_R g_1 h_1 g_2^\sharp$$

are strongly 1–admissible, and

$$R(1+(1-\beta_1)R)((1-\chi_R)g_1)^\flat g_2 h_2 \ , \ R(1+(1-\beta_1)R)((1-\chi_R)g_1 h_1)^\flat g_2$$

are strongly 2–admissible.

Proof of Lemma 4.3. The first two properties are almost immediate — notice that the strong admissibility of gh is a consequence of Lemma 3.5.
Property (iii) is established by first observing that $|y_1| \le \frac{R}{2}$ on the support of $\chi_R\, g_2^\sharp$, so that

$$(1-\beta_2)\frac{|y_1 - R|}{b\mu} \ge (1-\beta_1)\frac{R}{2\mu} \ .$$

Similarly, $R + b\mu y_2 \ge R/4$ on the support of $((1-\chi_R)g_1)^\flat$, so that

$$(1-\beta_1)|R + b\mu y_2| \ge (1-\beta_1)\frac{R}{4} \ .$$

In the first case, we also have, on the support of $\chi_R\, g_2^\sharp$,

$$|y_1 - R| \ge \frac{1}{4}(|y_1| + R) \ ,$$

so that

$$\|\chi_R g_2^\sharp\|_{\beta_1} \lesssim b\|g_2\|_{\beta_2} \text{ and } R(1+(1-\beta_1)R)\|\chi_R g_2^\sharp\|_{L^\infty} \lesssim b\|g_2\|_\beta.$$

We argue similarly for $((1-\chi_R)g_1)^\flat$. Furthermore,

$$\Lambda_{y_1}(g_2^\sharp) = (\Lambda_{y_2} g_2)^\sharp + \frac{R}{b\mu}(\partial_{y_2} g_2)^\sharp \ , \ \Lambda_{y_2} g_1^\flat = (\Lambda_{y_1} g_1)^\flat - R(\partial_{y_1} g_1)^\flat \ ,$$

with similar formulae for derivatives $\tilde\Lambda_{\beta_j}, \Lambda_R, \partial_{\lambda_j}$. Since

$$\partial_{y_j}^k g_j(y_j) = O(\langle y_j\rangle^{-k-1}) \ ,$$

this provides the correct decay of derivatives of $\chi_R\, g_2^\sharp$ and of $((1-\chi_R)g_1)^\flat$.
Let us prove property (iv). The L^∞-admissibility of (g, iQ_{β_j}) is a consequence of the Cauchy–Schwarz inequality and of the j–admissibility of g and Q_{β_j}. As for the L^∞-admissibility of $(g, \partial_y Q_{\beta_j})$, it is a consequence of the j–admissibility of g and

of the boundedness in L^2 of $\tilde{\Lambda}^a_\beta Q_\beta \tilde{\Lambda}^b_\beta Q_\beta \tilde{\Lambda}^c_\beta Q_\beta$. The latter fact follows from the identity

$$\partial_y Q_\beta = \partial_y m_\beta * (|Q_\beta|^2 Q_\beta) \ ,$$

and of the boundedness of the Fourier transforms of $\partial_y \tilde{\Lambda}^q_\beta m_\beta$.

Property (v) is an immediate consequence of the invariance of class $\mathcal{M}$ by convolution.

Finally, let us prove property (vi). By properties (iii) and (ii), we immediately get that

$$R(1+(1-\beta_1)R)b^{-1}\chi_R g_1 g_2^\sharp h_2^\sharp \ , \ R(1+(1-\beta_1)R)b^{-1}\chi_R g_1 h_1 g_2^\sharp$$

are strongly 1–admissible, and

$$R(1+(1-\beta_1)R)((1-\chi_R)g_1)^\flat g_2 h_2$$

is strongly 2–admissible. Furthermore, $R^2((1-\chi_R)g_1h_1)^\flat g_2$ is 2–admissible for the same reasons.

The strong admissibility of $R(1+(1-\beta_1)R)((1-\chi_R)g_1h_1)^\flat g_2$ requires a specific proof, as follows. We proceed as in the proof of Lemma 3.5. First of all, the L^∞-bound of $\mu_{\beta_2} * R^2((1-\chi_R)g_1h_1)^\flat g_2)$ is a consequence of $L^2 * L^2 \subset L^\infty$. Then we consider the case $|y_1| \geq 1$. We split

$$\begin{aligned}
&\mu_{\beta_2} * ((1-\chi_R)^\flat g_1^\flat h_1^\flat g_2)(y_2) = \\
&\int_{|y_2'|<\frac{|y_2|}{2}} \mu_{\beta_2}(y_2-y_2')(1-\chi_R)(R+\mu b y_2')g_1(R+\mu b y_2')h_1(R+\mu b y_2')g_2(y_2')\,dy_2' \\
+ &\int_{|y_2'|\geq\frac{|y_2|}{2}} \mu_{\beta_2}(y_2-y_2')(1-\chi_R)(R+\mu b y_2')g_1(R+\mu b y_2')h_1(R+\mu b y_2')g_2(y_2')\,dy_2' \ .
\end{aligned}$$

In view of decaying properties of μ_β and of the L^∞-bound on $(1-\chi_R)g_1h_1$, the first term in the right hand side is bounded by

$$\begin{aligned}
&\frac{\|g_2\|_{\beta_2}}{|y_2|(1+(1-\beta_2)|y_2|)R^2(1+(1-\beta_1)R)^2}\int_{\mathbb{R}}\frac{dy_2'}{(1+|y_2'|+(1-\beta_2)|y_2'|^2)}\,dy_2' \\
\lesssim \ &\frac{|\log(1-\beta_2)|}{|y_2|(1+(1-\beta_2)|y_2|)R^2(1+(1-\beta_1)R)^2} \ .
\end{aligned}$$

For the second term, we need the following L^p bound on μ_β, proved in Lemma 3.6,

$$\|\mu_\beta\|_{L^p(\mathbb{R})} \leq \frac{C}{p-1} \ , \ 1<p\leq 2.$$

Using this bound and Hölder's inequality, we infer that, for $2 \leq q < \infty$, the second term is bounded by

$$\begin{aligned}
&\frac{Cq}{R^2(1+(1-\beta_1)R)(1+|y_2|)(1+(1-\beta_2)|y_2|)}\left(\int_{\mathbb{R}}\frac{dy_2'}{(1+(1-\beta_2)|y_2'|)^q}\right)^{\frac{1}{q}} \\
\lesssim &\frac{Cq(1-\beta_2)^{-1/q}}{R^2(1+(1-\beta_1)R)(1+|y_2|)(1+(1-\beta_2)|y_2|)} \ .
\end{aligned}$$

Optimizing on q, we get the bound

$$\frac{|\log(1-\beta_2)|}{(1+|y_2|)(1+(1-\beta_2)|y_2|)R^2(1+(1-\beta_1)R)}.$$

We conclude that

$$\|\mu_{\beta_2} * ((1-\chi_R)^\flat g_1^\flat h_1^\flat g_2)\|_{\beta_2} \lesssim \frac{|\log(1-\beta_2)|}{R^2(1+(1-\beta_1)R)} \leq \frac{1}{R(1+(1-\beta_1)R)}$$

because of the assumption

$$1 - \beta_2 \geq e^{-R} \tag{4.18}$$

from (4.12). Similar estimates hold for the derivatives. This completes the proof. □

Remark 4.4. Because b is bounded but can be small in the set of parameters $\mathcal{O}$, there is some asymmetry between bubble 1 and bubble 2, which is reflected by the specificity of the last case in property (vi), for which we had to introduce assumption (4.18).

4.4. **Continuity of $\mathcal{L}_\beta^{-1}$ on admissible functions.** We claim a uniform continuity property of $\mathcal{L}_\beta^{-1}$ with respect to Schwartz-like norms which will be essential to control the error in the construction of the approximate 2-bubble. Recall that

$$\Phi_\beta := y\partial_y Q_\beta + (1-\beta)\partial_\beta Q_\beta \ .$$

Lemma 4.5 (Generalized invertibility)**.** *Let $j = 1$ or $j = 2$, let d be a nonnegative integer, and $\alpha \in \mathbb{R}$ such that $|\alpha| < \alpha_*(d)$. If $\eta < \eta_*(d)$ and if g is of the form*

$$g(y, \mathcal{P}) = \sum_{r=-d}^{d} g_r(y, \mathcal{P}^*)\, e^{ir\Gamma} \ ,$$

where $\mathcal{P}^* := (\lambda_1, \lambda_2, \beta_1, \beta_2, R)$, and each g_r , $r = -d, \dots, d$, is strongly j–admissible, then the problem

$$\mathcal{L}_{\beta_j} f - i\alpha\partial_\Gamma f = g - iM(\mathcal{P})\Lambda Q_{\beta_j} + iB(\mathcal{P})\Phi_{\beta_j} \ , \ (f, iQ_{\beta_j}) = (f, \partial_y Q_{\beta_j}) = 0 \ ,$$

admits a unique solution (f, M, B), where $M(\mathcal{P}), B(\mathcal{P})$ are real valued, and

$$f(y, \mathcal{P}) = \sum_{r=-d}^{d} f_r(y, \mathcal{P}^*)\, e^{ir\Gamma} \ ,$$

where each f_r , $r = -d, \dots, d$, is in $H^{\frac{1}{2}}$ in the variable y. Furthermore, M, B are L^∞-admissible, and f is strongly j–admissible.

Proof. Since $\mathcal{L}_\beta$ is not $\mathbb{C}$–linear, it is preferable to use the Fourier expansion in cosines and sines, so we write

$$g(y, \mathcal{P}^*) = g_0(y, \mathcal{P}^*) + \sum_{r=1}^{d} [g_r^+(y, \mathcal{P}^*)\cos(r\Gamma) + g_r^-(y, \mathcal{P}^*)\sin(r\Gamma)] \ ,$$

$$f(y, \mathcal{P}^*) = f_0(y, \mathcal{P}^*) + \sum_{r=1}^{d} [f_r^+(y, \mathcal{P}^*)\cos(r\Gamma) + f_r^-(y, \mathcal{P}^*)\sin(r\Gamma)] \ ,$$

$$M(\mathcal{P}) = M_0(\mathcal{P}^*) + \sum_{r=1}^{d} [M_r^+(\mathcal{P}^*)\cos(r\Gamma) + M_r^-(\mathcal{P}^*)\sin(r\Gamma)] \ ,$$

$$B(\mathcal{P}) = B_0(\mathcal{P}^*) + \sum_{r=1}^{d} [B_r^+(\mathcal{P}^*)\cos(r\Gamma) + B_r^-(\mathcal{P}^*)\sin(r\Gamma)] \ .$$

The problem on f, M, B is therefore equivalent to the following family of problems

$$\mathcal{L}_{\beta_j} f_0 = iM_0\Lambda Q_{\beta_j} - iB_0\Phi_{\beta_j} + g_0 \ , \ (f_0, iQ_{\beta_j}) = (f_0, \partial_y Q_{\beta_j}) = 0 \ , \tag{4.19}$$

$$\begin{cases} \mathcal{L}_{\beta_j} f_r^+ - i\alpha r f_r^- = iM_r^+\Lambda Q_{\beta_j} - iB_r^+\Phi_{\beta_j} + g_r^+ \ , \ (f_r^+, iQ_{\beta_j}) = (f_r^+, \partial_y Q_{\beta_j}) = 0 \ , \\ \mathcal{L}_{\beta_j} f_r^- + i\alpha r f_r^+ = iM_r^-\Lambda Q_{\beta_j} - iB_r^-\Phi_{\beta_j} + g_r^- \ , \ (f_r^-, iQ_{\beta_j}) = (f_r^-, \partial_y Q_{\beta_j}) = 0 \ , \end{cases} \tag{4.20}$$

Let us first deal with (4.19). Recall from Proposition 2.4 that

$$\ker \mathcal{L}_{\beta_j} = span_{\mathbb{R}}\{iQ_{\beta_j}, \partial_{y_j} Q_{\beta_j}\},$$

and that the range of $\mathcal{L}_{\beta_j}$ coincides with the orthogonal of $span_{\mathbb{R}}\{iQ_{\beta_j}, \partial_{y_j} Q_{\beta_j}\}$.

Consequently, the real numbers M_0, B_0 must satisfy the orthogonality conditions

$$(g_0 - iM_0 \Lambda Q_{\beta_j} + iB_0 \Phi_{\beta_j}, iQ_{\beta_j}) = (g_0 - iM_0 \Lambda Q_{\beta_j} + iB_0 \Phi_{\beta_j}, \partial_{y_j} Q_{\beta_j}) = 0 \ .$$

Notice that, in view of (3.27), (3.28),

$$\begin{aligned}
(i\Lambda Q_\beta, iQ_\beta) &= (\Lambda Q_\beta, Q_\beta) = 0 \ , \\
(i\Phi_\beta, iQ_\beta) &= (\Phi_\beta, Q_\beta) = \frac{1-\beta}{2}\frac{d}{d\beta}\|Q_\beta\|_{L^2}^2 - \frac{1}{2}\|Q_\beta\|_{L^2}^2 = -\pi + O((1-\beta)|\log(1-\beta)|) \ , \\
(i\Lambda Q_\beta, \partial_y Q_\beta) &= \frac{1}{2}(Q_\beta, DQ_\beta) = \pi + O((1-\beta)|\log(1-\beta)|) \ , \\
(i\Phi_\beta, \partial_y Q_\beta) &= \frac{1-\beta}{2}\frac{d}{d\beta}(Q_\beta, DQ_\beta) = O((1-\beta)|\log(1-\beta)|) \ .
\end{aligned}$$

In view of these identities, we infer that M_0, B_0 are characterized for β_j close enough to 1 — hence for η small enough —, given by the following formulae

$$B_0 = \frac{2(g_0, iQ_{\beta_j})}{\|Q_{\beta_j}\|_{L^2}^2 - \tilde{\Lambda}_{\beta_j}\|Q_{\beta_j}\|_{L^2}^2} \ , \tag{4.21}$$

$$M_0 = \frac{2(g_0, \partial_y Q_{\beta_j})}{(Q_{\beta_j}, DQ_{\beta_j})} + \frac{2(g_0, iQ_{\beta_j})\tilde{\Lambda}_{\beta_j}\|Q_{\beta_j}\|_{L^2}^2}{(Q_{\beta_j}, DQ_{\beta_j})(\|Q_{\beta_j}\|_{L^2}^2 - \tilde{\Lambda}_{\beta_j}\|Q_{\beta_j}\|_{L^2}^2)} \ . \tag{4.22}$$

In view of these formulae and of property (v) in Lemma 4.3, we conclude that M_0 and B_0 are L^∞-admissible.

Then Proposition 3.3 provides existence and uniqueness of function f_0, as well as the estimate

$$\|f_0\|_{\beta_j} \lesssim \|g_0\|_{L^2} + \|m_{\beta_j} * g_0\|_{\beta_j} \ .$$

Applying inductively $\Lambda_y^p \tilde{\Lambda}_{\beta_j}^q$ to the identity

$$f_0 = m_{\beta_j} * (iM_0 \Lambda Q_{\beta_j} - iB_0 \Phi_{\beta_j} + g_0) + m_{\beta_j} * (2|Q_{\beta_j}|^2 f_0 + Q_{\beta_j}^2 \overline{f}_0) \ ,$$

and using that $\Lambda Q_{\beta_j}, \Phi_{\beta_j}$ and g_0 are strongly j–admissible, we conclude from Lemma 3.1 that f_0 is strongly j–admissible.

Let us come to the systems (4.20). Given $g \in H^{-\frac{1}{2}}$, define

$$\begin{aligned}
B[g] &:= \frac{2(g, iQ_{\beta_j})}{\|Q_{\beta_j}\|_{L^2}^2 - \tilde{\Lambda}_{\beta_j}\|Q_{\beta_j}\|_{L^2}^2} \ , \\
M[g] &:= \frac{2(g, \partial_y Q_{\beta_j})}{(Q_{\beta_j}, DQ_{\beta_j})} + \frac{2(g, iQ_{\beta_j})\tilde{\Lambda}_{\beta_j}\|Q_{\beta_j}\|_{L^2}^2}{(Q_{\beta_j}, DQ_{\beta_j})(\|Q_{\beta_j}\|_{L^2}^2 - \tilde{\Lambda}_{\beta_j}\|Q_{\beta_j}\|_{L^2}^2)} \ .
\end{aligned}$$

and let

$$\mathcal{L}_\beta^{-1} : H^{-\frac{1}{2}} \cap (\ker \mathcal{L}_\beta)^\perp \to H^{\frac{1}{2}} \cap (\ker \mathcal{L}_\beta)^\perp$$

be the $\mathbb{R}$–linear isomorphism provided by Proposition 2.4 . Then the system (4.20) is equivalent to

$$\begin{cases}
f_r^+ = \mathcal{L}_{\beta_j}^{-1}(g_r^+ + i\alpha r f_r^- + iM[g_r^+ + i\alpha r f_r^-]\Lambda Q_{\beta_j} - iB[g_r^+ + i\alpha r f_r^-]\Phi_{\beta_j}), \\
f_r^- = \mathcal{L}_{\beta_j}^{-1}(g_r^- - i\alpha r f_r^+ + iM[g_r^- - i\alpha r f_r^+]\Lambda Q_{\beta_j} - iB[g_r^- - i\alpha r f_r^+]\Phi_{\beta_j}) \ .
\end{cases}$$

The right hand side in the above side defines a mapping of $(f_r^+, f_r^-) \in H^{1/2} \times H^{1/2}$ which is contracting if αr is small enough. This provides existence and uniqueness of (f_r^+, f_r^-) as well as uniform bounds in $H^{1/2}$, and the formulae

$$M_r^+ = M[g_r^+ + i\alpha r f_r^-], B_r^+ = B[g_r^+ + i\alpha r f_r^-], M_r^- = M[g_r^- - i\alpha r f_r^+]\,,\ B_r^- = B[g_r^- - i\alpha r f_r^+]\,.$$

The strong j–admissibility of f_r^+ and f_r^- and the L^∞-admissibility of $M_r^\pm, B_r^\pm$ are then obtained from the system

$$\begin{cases} f_r^+ = m_{\beta_j} * (iM_r^+ \Lambda Q_{\beta_j} - iB_r^+ \Phi_{\beta_j} + g_r^+ + i\alpha r f_r^-) + m_{\beta_j} * (2|Q_{\beta_j}|^2 f_r^+ + Q_{\beta_j}^2 \overline{f_r^+})\,, \\ f_r^- = m_{\beta_j} * (iM_r^- \Lambda Q_{\beta_j} - iB_r^- \Phi_{\beta_j} + g_r^- - i\alpha r f_r^+) + m_{\beta_j} * (2|Q_{\beta_j}|^2 f_r^- + Q_{\beta_j}^2 \overline{f_r^-})\,, \end{cases}$$

applying again Lemma 3.1. □

4.5. **Construction of the approximate solution.** We are now in position to construct the approximate two-bubble solution.

Proposition 4.6 (Construction of the two-bubble)**.** *Let N be a positive integer, $0 < \eta \ll \eta_*(N)$. We can find an expansion of the slowly modulated two-bubble for $j = 1, 2$:*

$$\begin{aligned} V_j^{(N)}(y_j, \mathcal{P}) &= \sum_{n=0}^{N} T_{j,n}(y_j, \mathcal{P}), \\ M_j^{(N)}(\mathcal{P}) &= \sum_{n=0}^{N} M_{j,n}(\mathcal{P}), \\ B_j^{(N)}(\mathcal{P}) &= \sum_{n=0}^{N} B_{j,n}(\mathcal{P}) \end{aligned}$$

such that the following holds:

(1) (*Initialization*). *For $j = 1, 2$, $T_{j,0} = Q_{\beta_j}(y_j)$, $M_{j,0} = B_{j,0} = 0$.*

(2) (*Control of the error*). *Let $0 \leqslant n \leqslant N$ and $(\mathcal{E}_{j,n})_{j=1,2}$ be given by (4.9), (4.10) with $V_j = V_j^{(n)}$. Then $b^{-1}(1+(1-\beta_1)R)R^{n+1}\mathcal{E}_{1,n}$ is strongly 1–admissible, and $(1+(1-\beta_1)R)R^{n+1}\mathcal{E}_{2,n}$ is strongly 2–admissible.*

(3) (*Control of the profile*). *For all $0 \leqslant n \leqslant N$, $j = 1, 2$, $b^{-1}(1+(1-\beta_1)R)R^n T_{1,n}$ is strongly 1–admissible, and $(1+(1-\beta_1)R)R^n T_{2,n}$ is strongly 2–admissible.*

(4) (*Orthogonality*). *For $j = 1, 2$, $n \geqslant 1$, $(T_{j,n}, iQ_{\beta_j}) = (T_{j,n}, \partial_{y_j} Q_{\beta_j}) = 0$.*

(5) (*Control of the modulation equations*). *For all $0 \leq n \leq N$, $b^{-1}(1+(1-\beta_1)R)R^n B_{1,n}$, $b^{-1}(1+(1-\beta_1)R)R^n M_{1,n}$, $(1+(1-\beta_1)R)R^n B_{2,n}$, and $(1 + (1 - \beta_1)R)R^n M_{2,n}$ are L^∞-admissible.*

Proof of Proposition 4.6. We argue by induction on N. In order to deal with the dependence on the phase Γ, we need a more refined description of the error and claim inductively:

$$T_{j,n} = \sum_{r=-d_n}^{d_n} T_{j,n,r}\, e^{ir\Gamma} \tag{4.23}$$

where d_n is an integer, $b^{-1}(1 + (1 - \beta_1)R)R^n T_{1,n,r}$ is strongly 1–admissible, $(1 + (1 - \beta_1)R)R^n T_{2,n,r}$ is strongly 2–admissible, and they do not depend on Γ.

Moreover,

$$\mathcal{E}_{j,n} = \sum_{r=-d_{n+1}}^{d_{n+1}} \mathcal{E}_{j,n,r}\, e^{ir\Gamma} \tag{4.24}$$

where $b^{-1}(1+(1-\beta_1)R)R^{n+1}\mathcal{E}_{1,n,r}$ is strongly 1–admissible, $(1+(1-\beta_1)R)R^{n+1}\mathcal{E}_{2,n,r}$ is strongly 2–admissible, and they do not depend on Γ. Finally,

$$M_{j,n} = \sum_{r=-d_n}^{d_n} M_{j,n,r}\, e^{ir\Gamma}$$

$$B_{j,n} = \sum_{r=-d_n}^{d_n} B_{j,n,r}\, e^{ir\Gamma}$$

where $b^{-1}(1+(1-\beta_1)R)R^n M_{1,n,r}, b^{-1}(1+(1-\beta_1)R)R^n B_{1,n,r}, (1+(1-\beta_1)R)R^n M_{2,n,r}$, $(1+(1-\beta_1)R)R^n B_{2,n,r}$ are L^∞–admissible and do not depend on Γ nor y.

Step 1: Initialization $N = 0$. We inject the decomposition

$$V_j = V_j^{(0)} = Q_{\beta_j}(y_j), \quad M_{j,0} = B_{j,0} = 0$$

$j = 1, 2$, into the definitions (4.9) and (4.10) of the errors and compute from the equation of Q_{β_j}:

$$\begin{aligned}
\mathcal{E}_{1,0} &= \chi_R \left[\frac{2}{\mu} Q_{\beta_1} |Q_{\beta_2}|^2 + \frac{e^{-i\Gamma}}{\sqrt{\mu}} Q_{\beta_1}^2 \overline{Q_{\beta_2}} + 2\frac{e^{i\Gamma}}{\sqrt{\mu}} |Q_{\beta_1}|^2 Q_{\beta_2} + \frac{e^{2i\Gamma}}{\mu} \overline{Q_{\beta_1}} Q_{\beta_2}^2 \right] , \\
\mathcal{E}_{2,0} &= (1-\chi_R) \left[2\mu |Q_{\beta_1}|^2 Q_{\beta_2} + 2\sqrt{\mu} e^{-i\Gamma} Q_{\beta_1} |Q_{\beta_2}|^2 + \sqrt{\mu} e^{i\Gamma} \overline{Q_{\beta_1}} Q_{\beta_2}^2 + \mu e^{-2i\Gamma} Q_{\beta_1}^2 \overline{Q_{\beta_2}} \right] .
\end{aligned}$$

We now recall from that Q_{β_j} is strongly j–admissible. Therefore, a direct application of Lemma 4.3, property (vi), ensures that $b^{-1}(1+(1-\beta_1)R)R\mathcal{E}_{1,0}$ is strongly 1–admissible, and $(1+(1-\beta_1)R)R\mathcal{E}_{2,0}$ is strongly 2–admissible. Notice that we have (4.24) with $n = 0$, $d_1 = 2$, and that the admissibility properties transfer to the Fourier coefficients by integration in the Γ variable.

Step 2: Induction. We assume the claim for $N = n$ and prove it for $N = n+1$. We expand

$$V_j^{(n+1)} = V_j^{(n)} + T_{j,n+1}, \quad j = 1, 2 \tag{4.25}$$

and show how to choose $(T_{j,n+1}, M_{j,n+1}, B_{j,n+1})$ so that the corresponding errors $\mathcal{E}_{j,n+1}$ are such that $b^{-1}(1+(1-\beta_1)R)R^{n+2}\mathcal{E}_{1,n+1}$ is strongly 1–admissible, and $(1+(1-\beta_1)R)R^{n+2}\mathcal{E}_{2,n+1}$ is strongly 1–admissible. We focus onto the first bubble, the computations for the second bubble are completely analogous, except that there is no gain of a b factor.

In general, we split the error term $\mathcal{E}_1$ into four contributions: the nonlinear term,

$$\mathrm{NL}_1 = -\frac{(|D| - \beta_1 D)V_1}{1-\beta_1} - V_1 + V_1|V_1|^2, \tag{4.26}$$

the interaction term,

$$\mathrm{Int}_1 = \chi_R \left[\frac{2}{\mu} V_1 |V_2|^2 + \frac{e^{-i\Gamma}}{\sqrt{\mu}} (V_1)^2 \overline{V_2} + 2\frac{e^{i\Gamma}}{\sqrt{\mu}} |V_1|^2 V_2 + \frac{e^{2i\Gamma}}{\mu} \overline{V_1} (V_2)^2 \right] , \tag{4.27}$$

the leading order term for modulation equations,

$$\mathrm{Mod}_1 = -iM_1 \Lambda V_1 + iB_1[\Lambda_{y_1} V_1 + \tilde{\Lambda}_{\beta_1} V_1] + i\frac{1-\mu}{\mu}\frac{\partial V_1}{\partial \Gamma} \tag{4.28}$$

and the lower order term for modulation equations,

$$\begin{aligned} \mathrm{Modlow}_1 &= i\lambda_1(M_1\partial_{\lambda_1}V_1 + M_2\partial_{\lambda_2}V_1) + i\frac{B_2}{\mu}\tilde{\Lambda}_{\beta_2}V_1 + \\ &+ i\left(\frac{1-b}{R} + B_1 - M_1\right)\Lambda_R V_1 \ . \end{aligned} \tag{4.29}$$

Notice that we dropped the notation $V^\sharp$ and $V^\flat$ in these formulae, since the indices $1,2$ unambiguously suggest the arguments y_1, y_2.

Step 3: Choice of $T_{1,n+1}, M_{1,n+1}, B_{1,n+1}$. We inject the decomposition (4.25) into (4.26) - (4.29) and define $\mathcal{E}^{(k)}_{1,n+1}$, $k = 1,\dots,4$ by

$$\begin{aligned} &\mathrm{NL}_{1,n+1} = \mathrm{NL}_{1,n} - \mathcal{L}_{\beta_1}T_{1,n+1} + \mathcal{E}^{(1)}_{1,n+1} \\ &\mathrm{Int}_{1,n+1} = \mathrm{Int}_{1,n} + \mathcal{E}^{(2)}_{1,n+1} \\ &\mathrm{Mod}_{1,n+1} = \mathrm{Mod}_{1,n} + \{-iM_{1,n+1}\Lambda Q_{\beta_1} + iB_{1,n+1}\Phi_{\beta_1}\} + \frac{i(1-\mu)}{\mu}\frac{\partial T_{1,n+1}}{\partial\Gamma} + \mathcal{E}^{(3)}_{1,n+1} \\ &\mathrm{Modlow}_{1,n+1} = \mathrm{Modlow}_{1,n} + \mathcal{E}^{(4)}_{1,n+1}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{E}_{1,n+1} &= \mathcal{E}_{1,n} - \mathcal{L}_{\beta_1}T_{1,n+1} + \frac{i(1-\mu)}{\mu}\frac{\partial T_{1,n+1}}{\partial\Gamma} - iM_{1,n+1}\Lambda Q_{\beta_1} + iB_{1,n+1}\Phi_{\beta_1} \\ &+ \Sigma_{k=1}^4\mathcal{E}^{(k)}_{1,n+1}. \end{aligned}$$

The smallness assumption on η and the definition of $\mathcal{O}$ imply that $1-\mu$ is small enough with respect to n, and we may therefore use Lemma 4.5 to solve the equation

$$\mathcal{L}_{\beta_1}T_{1,n+1} + iM_{1,n+1}\Lambda Q_{\beta_1} - iB_{1,n+1}\Phi_{\beta_1} - i\frac{1-\mu}{\mu}\partial_\Gamma T_{1,n+1} = \mathcal{E}_{1,n} \ .$$

From the inductive assumption on $\mathcal{E}_{1,n}$ and Lemma 4.5, we infer that $b^{-1}(1+(1-\beta_1)R)R^{n+1}T_{1,n+1}$ is strongly 1–admissible, and that $b^{-1}(1+(1-\beta_1)R)R^{n+1}M_{1,n+1}$, $b^{-1}(1+(1-\beta_1)R)R^{n+1}B_{1,n+1}$ are L^∞-admissible. Furthermore, $T_{1,n+1}, M_{1,n+1}, B_{j,n+1}$ are trigonometric polynomials of degree d_{n+1}.

Step 4: Estimating $\mathcal{E}^{(1)}_{1,n+1}$. Explicitly:

$$\begin{aligned} \mathcal{E}^{(1)}_{1,n+1} &= 2\left[|V_1^{(n)}|^2 - |Q_{\beta_1}|^2\right]T_{1,n+1} + \left[(V_1^{(n)})^2 - Q_{\beta_1}^2\right]\overline{T_{1,n+1}} \\ &+ 2V_1^{(n)}|T_{1,n+1}|^2 + \overline{V_1^{(n)}}T_{1,n+1}^2 + T_{1,n+1}|T_{1,n+1}|^2. \end{aligned} \tag{4.30}$$

First of all, we observe that $\mathcal{E}^{(1)}_{1,n+1}$ is a trigonometric polynomial in Γ, with a degree $d^{(1)}_{n+2}$ depending only on n. Secondly, using Lemma 4.3, the 1–admissibility of $b^{-1}(1+(1\beta_1)R)R^kT_{1,k}$, and the 2–admissibility of $(1+(1\beta_1)R)R^kT_{2,k}$ for $k \le n+1$, we conclude that $b^{-1}(1+(1-\beta_1)R)R^{n+2}\mathcal{E}^{(1)}_{1,n+1}$ is strongly 1–admissible.

Step 5: Estimating $\mathcal{E}^{(2)}_{1,n+1}$. First of all, we observe that $\mathcal{E}^{(2)}_{1,n+1}$ is a trigonometric polynomial in Γ, with a degree $d^{(2)}_{n+2}$ depending only on n. We then expand the interaction term $\mathrm{Int}_{1,n+1}$ (4.27). Notice that each term contains an exchange of variables. Let us consider the term

$$\frac{e^{2i\Gamma}}{\mu}\chi_R\overline{V_1^{(n)}}T_{2,n+1}V_2^{(n)}.$$

Recall that $V_j^{(n)}$ is j–admissible by the induction assumption, and that $(1+(1-\beta_1)R)R^{n+1}T_{2,n+1}$ is 1 admissible by step 3. By Lemma 4.3, (vi), we infer that

$$b^{-1}(1+(1-\beta_1)R)R^{n+2}\frac{e^{2i\Gamma}}{\mu}\chi_R\overline{V_1^{(n)}}T_{2,n+1}V_2^{(n)}$$

is strongly 1-admissible. The other terms can be treated similarly. We therefore conclude that $b^{-1}(1+(1-\beta_1)R)R^{n+2}\mathcal{E}_{1,n+1}^{(2)}$ is strongly 1–admissible.

Step 6: Estimating $\mathcal{E}_{1,n+1}^{(3)}$. Again, $\mathcal{E}_{1,n+1}^{(3)}$ is a trigonometric polynomial in Γ, with a degree $d_{n+2}^{(3)}$ depending only on n.

Let us first observe that the term $i\frac{1-\mu}{\mu}\frac{\partial T_{1,n+1}}{\partial\Gamma}$ is absent in $\mathcal{E}_{1,n+1}^{(3)}$ since it is now a part of the equation of $T_{1,n+1}$. For example, let us deal with the contribution of the term $-iM_1\Lambda V_1$ to $\mathcal{E}_{1,n+1}^{(3)}$. The other contributions can be handled similarly. We have

$$M_1^{(n+1)}\Lambda V_1^{(n+1)}-M_1^{(n)}\Lambda V_1^{(n)}-M_{1,n+1}\Lambda Q_{\beta_1}=M_{1,n+1}\Lambda(V_1^{(n)}-Q_{\beta_1})+M_1^{(n+1)}\Lambda T_{1,n+1}\,.$$

Let us consider the first term $M_{1,n+1}\Lambda(V_1^{(n)}-Q_{\beta_1})$ in the right hand side. By step 3, we know that $b^{-1}(1+(1-\beta_1)R)R^{n+1}M_{1,n+1}$ is L^∞-admissible, and independent on y_1. On the other hand, $R\Lambda(V_1^{(n)}-Q_{\beta_1})$ is strongly 1–admissible. Hence $b^{-1}(1+(1-\beta_1)R)R^{n+2}M_{1,n+1}\Lambda(V_1^{(n)}-Q_{\beta_1})$ is strongly 1–admissible.

Let us come to the second term $M_1^{(n+1)}\Lambda T_{1,n+1}$ in the right hand side. From step 3, $b^{-1}(1+(1-\beta_1)R)R^{n+1}T_{1,n+1}$ is strongly 1–admissible, while, from step 3 and the induction hypothesis

$$b^{-1}(1+(1-\beta_1)R)RM_1^{(n+1)}=b^{-1}(1+(1-\beta_1)R)R\sum_{k=1}^{n+1}M_{1,k}$$

is L^∞-admissible and independent on y_1. We infer that $b^{-1}(1+(1-\beta_1)R)R^{n+2}M_1^{n+1}\Lambda T_{1,n+1}$ is strongly 1–admissible.

Summing up, $b^{-1}(1+(1-\beta_1)R)R^{n+2}\mathcal{E}_{1,n+1}^{(3)}$ is strongly 1–admissible.

Step 7: Estimating $\mathcal{E}_{1,n+1}^{(4)}$. Finally, we deal with $b^{-1}(1+(1-\beta_1)R)R^{n+2}\mathcal{E}_{1,n+1}^{(4)}$ via the lower order term for modulation equations (4.29). In fact, the worst behavior occurs in this part, and comes from the term

$$i\frac{1-b}{R}\Lambda_R T_{1,n+1}\,.$$

Indeed, this one only provides a gain of R, so we get exactly that

$$b^{-1}(1+(1-\beta_1)R)R^{n+2}i\frac{1-b}{R}\Lambda_R T_{1,n+1}$$

is strongly 1–admissible. The other terms are easier and left to the reader.

Defining $d_{n+2}:=\max\{d_{n+2}^{(k)},k=1,\dots,4\}$, this completes the proof.

□

As a consequence of Proposition 4.6, we establish some additional estimates which will be useful in Section 5.

Corollary 4.7. *If $V_j=V_j^{(N)}$ as in Proposition 4.6, and if*

$$\partial'\in\{\partial_\Gamma,\Lambda_R,\partial_{\lambda_{j+1}},(1-\beta_{j+1})\partial_{\beta_{j+1}}\}$$

with $\{j, j+1\} = \{1, 2\}$, *we have*

$$\|D\Pi^- \partial' V_j\|_{L^2} \lesssim \frac{1-\beta_j}{R} \ .$$

Proof. From Proposition 4.6, we know that V_j is j–admissible, and that $R(V_j - Q_{\beta_j})$ is j–admissible. Moreover, $R^{N+1}\mathcal{E}_j$ is j–admissible, and RM_j, RB_j are L^∞-admissible. Consequently, in view of the expressions (4.9), (4.10) of $\mathcal{E}_j$ and of Lemma 4.3, we conclude that

$$\left(\frac{|D| - \beta_j D}{1-\beta_j} + 1\right) V_j - |V_j|^2 V_j = F_j \ ,$$

where RF_j is j–admissible. Furthermore, since $\partial' Q_{\beta_j} = 0$, $R\partial' V_j$ is j–admissible, and so is $R\partial'(|V_j|^2 V_j)$. This implies in particular

$$\left\| \left(\frac{|D| - \beta_j D}{1-\beta_j} + 1\right) V_j \right\|_{L^2} \lesssim \frac{1}{R} \ .$$

The proof is completed by observing that the operator

$$D\Pi^- \left(\frac{|D| - \beta_j D}{1-\beta_j} + 1\right)^{-1}$$

has a norm $O(1-\beta_j)$ on L^2. □

Corollary 4.8. *If* $M_2 = M_2^{(N)}$ *as in Proposition 4.6, we have*

$$|\partial_\Gamma M_2| + |R\partial_R M_2| + \sum_{k=1}^{2} (1-\beta_k)|\partial_{\beta_k} M_2| \lesssim \frac{|1-\mu| + (1-\beta_2)|\log(1-\beta_2)| + R^{-1}}{R(1+(1-\beta_1)R)} \ .$$

Proof. Since $R^2(1+(1-\beta_1)R)(M_2 - M_{2,1})$ is L^∞-admissible from Proposition 4.6, we just have to prove the estimate for $M_{2,1}$. From the construction of Proposition 4.6 — see also the proof of Lemma 4.5, we have

$$M_{2,1} = \frac{2(\mathcal{E}_{2,0} + i(1-\mu)\partial_\Gamma T_{2,1}, \partial_{y_2} Q_{\beta_2})}{(Q_{\beta_2}, DQ_{\beta_2})} + \frac{2(\mathcal{E}_{2,0} + i(1-\mu)\partial_\Gamma T_{2,1}, iQ_{\beta_2})\tilde{\Lambda}_{\beta_2}\|Q_{\beta_2}\|_{L^2}^2}{(Q_{\beta_2}, DQ_{\beta_2})(\|Q_{\beta_2}\|_{L^2}^2 - \tilde{\Lambda}_{\beta_2}\|Q_{\beta_2}\|_{L^2}^2)} \ .$$

Since Q_{β_2}, and $R(1+(1-\beta_1)R)T_{2,1}$ are 2–admissible, and since $(Q_{\beta_2}, DQ_{\beta_2})^{-1}$, $(\|Q_{\beta_2}\|_{L^2}^2 - \tilde{\Lambda}_{\beta_2}\|Q_{\beta_2}\|_{L^2}^2)^{-1}$ are L^∞-admissible, the only terms to be estimated are

$$(\mathcal{E}_{2,0}, \partial_{y_2} Q_{\beta_2}), (\mathcal{E}_{2,0}, iQ_{\beta_2})\tilde{\Lambda}_{\beta_2}\|Q_{\beta_2}\|_{L^2}^2 \ ,$$

with

$$\mathcal{E}_{2,0} = (1-\chi_R)\left(2\mu|Q_{\beta_1}|^2 Q_{\beta_2} + 2\sqrt{\mu}e^{-i\Gamma}Q_{\beta_1}|Q_{\beta_2}|^2 + \sqrt{\mu}e^{i\Gamma}\overline{Q_{\beta_1}}Q_{\beta_2}^2 + \mu e^{-2i\Gamma}Q_{\beta_1}^2\overline{Q_{\beta_2}}\right) \ .$$

We already know that $R(1+(1-\beta_1)R)\mathcal{E}_{2,0}$ is 2–admissible. Furthermore, from Proposition 3.11, we have

$$|\tilde{\Lambda}_{\beta_2}\|Q_{\beta_2}\|_{L^2}^2| + |\tilde{\Lambda}_{\beta_2}^2\|Q_{\beta_2}\|_{L^2}^2| \lesssim (1-\beta_2)|\log(1-\beta_2)| \ .$$

This implies the claimed estimate for $(\mathcal{E}_{2,0}, iQ_{\beta_2})\tilde{\Lambda}_{\beta_2}\|Q_{\beta_2}\|_{L^2}^2$. As for $(\mathcal{E}_{2,0}, \partial_{y_2}Q_{\beta_2})$, since $R(1-\chi_R)Q_{\beta_1}$ is L^∞-admissible, we just have to study the contribution of the terms with only one factor Q_{β_1}, namely

$$2\sqrt{\mu}((1-\chi_R)e^{-i\Gamma}Q_{\beta_1}|Q_{\beta_2}|^2, \partial_{y_2}Q_{\beta_2}) + \sqrt{\mu}((1-\chi_R)e^{i\Gamma}\overline{Q_{\beta_1}}Q_{\beta_2}^2, \partial_{y_2}Q_{\beta_2}) \ .$$

After integrating by parts, this quantity is equal to

$$-\sqrt{\mu}\,\mathrm{Re}\left(e^{-i\Gamma}\int_{\mathbb{R}} \partial_{y_2}((1-\chi_R)Q_{\beta_1})|Q_{\beta_2}|^2\overline{Q_{\beta_2}}\,dy_2\right) \ .$$

Since $R^2(1+(1-\beta_1)R)\partial_{y_2}((1-\chi_R)Q_{\beta_1})$ is L^∞-admissible, this completes the proof. □

4.6. **Improved decay for $T_{2,1}$.** In this subsection, we improve some estimates of the first correction $T_{2,1}$ to Q_{β_2} in the approximate solution we have constructed in the previous paragraph.

Lemma 4.9. *We have*

$$\Big(i\frac{\partial T_{2,1}}{\partial\Gamma},\partial_{y_2}Q_{\beta_2}\Big)=-2\pi\mathrm{Re}\,(e^{i\Gamma}\overline{Q_{\beta_1}(R)})+O\Big(\frac{|1-\mu|+(1-\beta_2)^{1/2}|\log(1-\beta_2)|^{1/2}+R^{-1}}{R(1+(1-\beta_1)R)}\Big).$$

Proof. Writing $i\partial_{y_2}Q_{\beta_2}=\mathcal{L}_{\beta_2}(i\rho_{\beta_2})$, we have

$$(i\partial_\Gamma T_{2,1},\partial_{y_2}Q_{\beta_2})=-(\partial_\Gamma T_{2,1},i\partial_{y_2}Q_{\beta_2})=-(\partial_\Gamma T_{2,1},\mathcal{L}_{\beta_2}i\rho_{\beta_2})=-(\partial_\Gamma\mathcal{L}_{\beta_2}(T_{2,1}),i\rho_{\beta_2})$$

$$=-\Big(\partial_\Gamma\mathcal{E}_{2,0}-i\partial_\Gamma M_{2,1}\Lambda Q_{\beta_2}+i\partial_\Gamma B_{2,1}(y_2\partial_{y_2}Q_{\beta_2}+(1-\beta_2)\partial_{\beta_2}Q_{\beta_2})+i\frac{1-\mu}{\mu}\partial_\Gamma^2T_{2,1},i\rho_{\beta_2}\Big)$$

$$=I+II+III+IV \tag{4.31}$$

For IV, we have by Proposition 4.6 that

$$|IV|\lesssim\frac{|1-\mu|}{R(1+(1-\beta_1)R)}\ . \tag{4.32}$$

For III, we have by Proposition 4.6 that $|\partial_\Gamma B_{2,1}|\lesssim\frac{1}{R(1+(1-\beta_1)R)}$. Then,

$$|III|=\Big|\Big(i\partial_\Gamma B_{2,1}(y_2\partial_{y_2}Q_{\beta_2}+(1-\beta_2)\partial_{\beta_2}Q_{\beta_2}),i\rho_{\beta_2}\Big)\Big|$$

$$\lesssim\Big(\big|(iy_2\partial_{y_2}Q_{\beta_2},i\rho_{\beta_2})\big|+(1-\beta_2)\big|(i\partial_{\beta_2}Q_{\beta_2},i\rho_{\beta_2})\big|\Big)\frac{1}{R(1+(1-\beta_1)R)}$$

Using Proposition 3.11 and (3.10),

$$(1-\beta_2)\big|(i\partial_{\beta_2}Q_{\beta_2},i\rho_{\beta_2})\big|\lesssim(1-\beta_2)|\log(1-\beta_2)|. \tag{4.33}$$

Then, by (3.10), (2.20) and the identity

$$y\partial_yQ^+=Q^++\frac{i}{2}\partial_yQ^+$$

we have

$$(iy_2\partial_{y_2}Q_{\beta_2},i\rho_{\beta_2})=(iy_2\partial_{y_2}Q_{\beta_2},Q_{\beta_2}+\frac{i}{2}\partial_{y_2}Q_{\beta_2})+O((1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12})$$

$$=(iy\partial_yQ^+,Q^++\frac{i}{2}\partial_yQ^+)+O((1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12})$$

$$=O((1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12})\ . \tag{4.34}$$

Thus, we conclude that

$$|III|\lesssim\frac{(1-\beta_2)^{1/2}|\log(1-\beta_2)|^{1/2}}{R(1+(1-\beta_1)R)}. \tag{4.35}$$

For II, we have by Proposition 4.6 that $|\partial_\Gamma M_{2,1}|\lesssim\frac{1}{R(1+(1-\beta_1)R)}$. Then, by (4.34) and (3.10) :

$$(i\Lambda Q_{\beta_2},i\rho_{\beta_2})=\frac12(iQ_{\beta_2},i\rho_{\beta_2})+(iy_2\partial_{y_2}Q_{\beta_2},i\rho_{\beta_2}) \tag{4.36}$$

$$=\frac12(iQ_{\beta_2},Q_{\beta_2})+\frac14(iQ_{\beta_2},i\partial_{y_2}Q_{\beta_2})+O((1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12}) \tag{4.37}$$

$$=O((1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12}).$$

Therefore,

$$|II| \lesssim \frac{(1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}}}{R(1+(1-\beta_1)R)}. \tag{4.38}$$

Finally, for I, we have that

$$\begin{aligned}
I &= -(\partial_\Gamma \mathcal{E}_{2,0}, i\rho_{\beta_2}) \\
&= -\Big((1-\chi_R)[-2i\sqrt{\mu}e^{-i\Gamma}Q_{\beta_1}|Q_{\beta_2}|^2 + i\sqrt{\mu}e^{i\Gamma}\overline{Q_{\beta_1}}Q_{\beta_2}^2 - 2i\mu e^{-2i\Gamma}Q_{\beta_1}^2\overline{Q_{\beta_2}}], i\rho_{\beta_2}\Big) \\
&= -\sqrt{\mu}\mathrm{Re}\left(ie^{i\Gamma}\int (1-\chi_R)\overline{Q_{\beta_1}(y_1)}[2|Q_{\beta_2}|^2 i\rho_{\beta_2} + Q_{\beta_2}^2\overline{i\rho_{\beta_2}}]dy_2\right) + O\Big(\frac{1}{R^2(1+(1-\beta_1)R)^2}\Big) \\
&= \sqrt{\mu}\mathrm{Im}\left(e^{i\Gamma}\int_{|y_2|\leqslant\frac{R}{2b\mu}} \overline{Q_{\beta_1}(y_1)}[2|Q_{\beta_2}|^2 i\rho_{\beta_2} + Q_{\beta_2}^2\overline{i\rho_{\beta_2}}]dy_2\right) + O\Big(\frac{1}{R^2(1+(1-\beta_1)R)^2}\Big).
\end{aligned}$$

Let $z_2 := \frac{b\mu y_2}{R}$. We then Taylor expand for $|z_2| \leqslant \frac{1}{2}$, or equivalently $|y_2| \leqslant \frac{R}{2b\mu}$, and obtain by Proposition 3.7:

$$\begin{aligned}
Q_{\beta_1}(y_1) &= Q_{\beta_1}\left(R(1+z_2)\right) = Q_{\beta_1}(R) - \int_0^1 Rz_2\partial_{y_1}Q_{\beta_1}\left(R(1+tz_2)\right)dt \\
&= Q_{\beta_1}(R) + O\left(\frac{R|z_2|}{R^2(1+(1-\beta_1)R)}\right) = Q_{\beta_1}(R) + O\left(\frac{b|y_2|}{R^2(1+(1-\beta_1)R)}\right)
\end{aligned}$$

Therefore,

$$I = \sqrt{\mu}\mathrm{Im}\left(e^{i\Gamma}\overline{Q_{\beta_1}(R)}\int [2|Q_{\beta_2}|^2 i\rho_{\beta_2} + Q_{\beta_2}^2\overline{i\rho_{\beta_2}}](y_2)dy_2\right) + O\Big(\frac{1}{R^2(1+(1-\beta_1)R)}\Big). \tag{4.39}$$

Using (3.10) and Lemma A.1, we have that

$$\begin{aligned}
\int [2|Q_{\beta_2}|^2 i\rho_{\beta_2} + Q_{\beta_2}^2\overline{i\rho_{\beta_2}}](y_2)dy_2 &= 3\int |Q_{\beta_2}|^2 Q_{\beta_2}dy_2 + i\int |Q_{\beta_2}|^2\partial_{y_2}Q_{\beta_2}dy_2 \\
&\quad - \frac{i}{2}\int Q_{\beta_2}^2\overline{\partial_{y_2}Q_{\beta_2}}dy_2 + O((1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}}) \\
&= 3\int |Q^+|^2 Q^+dy + i\int |Q^+|^2\partial_y Q^+dy - \frac{i}{2}\int (Q^+)^2\overline{\partial_y Q^+}dy + O((1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}}) \\
&= -6\pi i + 2\pi i + 2\pi i + O((1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}}) \\
&= -2\pi i + O((1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}}).
\end{aligned} \tag{4.40}$$

Then,

$$I = -2\pi\mathrm{Re}\left(e^{i\Gamma}\overline{Q_{\beta_1}(R)}\right) + O\Big(\frac{|1-\mu| + (1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}}}{R(1+(1-\beta_1)R)}\Big) + O\Big(\frac{1}{R^2(1+(1-\beta_1)R)}\Big).$$

Combining this with (4.32), (4.31), (4.35) and (4.38), the conclusion of the lemma follows.

□

Lemma 4.10. *We have*

$$\begin{aligned}
\Big(i\frac{\partial T_{2,1}}{\partial R}, \partial_{y_2}Q_{\beta_2}\Big) &= -2\pi\mathrm{Im}\left(e^{i\Gamma}\overline{\partial_{y_1}Q_{\beta_1}(R)}\right) + O\Big(\frac{|1-\mu| + (1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}}}{R^2(1+(1-\beta_1)R)}\Big) \\
&\quad + O\Big(\frac{1}{R^3(1+(1-\beta_1)R)}\Big).
\end{aligned}$$

Proof. The proof follows the same lines as the above one. With the same notation as above, we have

$$\begin{aligned}
&(i\partial_R T_{2,1}, \partial_{y_2} Q_{\beta_2}) \\
&= -\Big(\partial_R \mathcal{E}_{2,0} - i\partial_R M_{2,1}\Lambda Q_{\beta_2} + i\partial_R B_{2,1}(y_2\partial_{y_2}Q_{\beta_2} + (1-\beta_2)\partial_{\beta_2}Q_{\beta_2}), i\rho_{\beta_2}\Big) \\
&- \Big(i\frac{1-\mu}{\mu}\partial_\Gamma \partial_R T_{2,1}, i\rho_{\beta_2}\Big) \\
&= V + VI + VII + VIII. \qquad (4.41)
\end{aligned}$$

By Proposition 4.6, we have that

$$|VIII| \lesssim \frac{|1-\mu|}{R^2(1+(1-\beta_1)R)} \qquad (4.42)$$

Using Proposition 4.6, we have that $|\partial_R B_{2,1}| \lesssim \frac{1}{R^2(1+(1-\beta_1)R)}$. Then, it follows by (4.33) and (4.34) that

$$|VII| \lesssim \frac{(1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}}}{R^2(1+(1-\beta_1)R)}.$$

Since $|\partial_R M_{2,1}| \lesssim \frac{1}{R^2(1+(1-\beta_1)R)}$ by Proposition 4.6, we have according to (4.36) that

$$|VI| \lesssim \frac{(1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}}}{R^2(1+(1-\beta_1)R)} \qquad (4.43)$$

Lastly, by (4.40) we have that $\int (2|Q_{\beta_2}|^2 i\rho_{\beta_2} + Q_{\beta_2}^2\overline{i\rho_{\beta_2}})dy_2 = -2\pi i + O((1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}})$, and thus

$$\begin{aligned}
V = -(\partial_R \mathcal{E}_{2,0}, i\rho_{\beta_2}) &= \sqrt{\mu}\mathrm{Re}\, e^{i\Gamma}\int (1-\chi_R)\overline{\partial_{y_1}Q_{\beta_1}}(2|Q_{\beta_2}|^2 i\rho_{\beta_2} + Q_{\beta_2}^2\overline{i\rho_{\beta_2}})dy_2 \\
&+ O\left(\int \chi'\left(1+\mu\frac{by_2}{R}\right)\frac{\mu b|y_2|}{R^2}(|Q_{\beta_1}|^2 + |Q_{\beta_1}||Q_{\beta_2}|)|Q_{\beta_2} i\rho_{\beta_2}|dy_2\right) \\
&+ O\Big(\int (1-\chi_R)|Q_{\beta_1}||\partial_{y_1}Q_{\beta_1}||Q_{\beta_2} i\rho_{\beta_2}|dy_2\Big) \\
&= -2\pi \mathrm{Im}\,(e^{i\Gamma}\overline{\partial_{y_1}Q_{\beta_1}(R)}) + O\left(\frac{(1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}}}{R^2(1+(1-\beta_1)R)}\right) \\
&+ O\left(\frac{1}{R^3(1+(1-\beta_1)R)}\right).
\end{aligned}$$

□

Lemma 4.11. *We have*

$$\Big|(1-\beta_2)\Big(i\frac{\partial T_{2,1}}{\partial\beta_2}, \partial_{y_2}Q_{\beta_2}\Big)\Big| \lesssim \frac{(1-\beta_2)^{\frac{1}{2}}|\log(1-\beta_2)|^{\frac{1}{2}} + |1-\mu|}{R(1+(1-\beta_1)R)} + \frac{\sqrt{b}}{R^2(1+(1-\beta_1)R)}.$$

Proof. Using the symmetry of $\mathcal{L}_{\beta_2}$ with respect to the real scalar product, we write

$$
\begin{aligned}
(1-\beta_2)\Big(i\frac{\partial T_{2,1}}{\partial\beta_2},\partial_{y_2}Q_{\beta_2}\Big) &= -(1-\beta_2)\Big(\frac{\partial T_{2,1}}{\partial\beta_2},i\partial_{y_2}Q_{\beta_2}\Big) \qquad (4.44)\\
&= -\Big((1-\beta_2)\frac{\partial T_{2,1}}{\partial\beta_2},\mathcal{L}_{\beta_2}(i\rho_{\beta_2})\Big) = -\Big((1-\beta_2)\mathcal{L}_{\beta_2}\Big(\frac{\partial T_{2,1}}{\partial\beta_2}\Big),i\rho_{\beta_2}\Big)\\
&= -(1-\beta_2)\Big(\partial_{\beta_2}(\mathcal{L}_{\beta_2}T_{2,1})+2\partial_{\beta_2}(|Q_{\beta_2}|^2)T_{2,1}+\partial_{\beta_2}(Q_{\beta_2}^2)\overline{T_{2,1}},i\rho_{\beta_2}\Big)\\
&\quad-\Big(\frac{2DT_{2,1}^-}{1-\beta_2},i\rho_{\beta_2}\Big).
\end{aligned}
$$

We start by estimating the last term. Firstly,

$$
\Big|\Big(\frac{2DT_{2,1}^-}{1-\beta_2},i\rho_{\beta_2}\Big)\Big| \leqslant \Big\|\frac{2DT_{2,1}^-}{1-\beta_2}\Big\|_{L^2}\|i\rho_{\beta_2}^-\|_{L^2}.
$$

Projecting the equation satisfied by $T_{2,1}$ onto negative frequencies, we obtain:

$$
\begin{aligned}
-\frac{1+\beta_2}{1-\beta_2}DT_{2,1}^- + T_{2,1}^- - 2\Pi^-(|Q_{\beta_2}|^2T_{2,1}) - \Pi^-(Q_{\beta_2}^2\overline{T_{2,1}})&\\
= \Pi^-(\mathcal{E}_{2,0}) - iM_{2,1}\Pi^-(\Lambda Q_{\beta_2}) + iB_{2,1}\Pi^-(y_2\partial_{y_2}Q_{\beta_2}+(1-\beta_2)\partial_{\beta_2}Q_{\beta_2})&\\
+ i(1-\mu)\partial_\Gamma T_{2,1}^-&
\end{aligned}
$$

and therefore, using the 2–admissibility of $R(1+(1-\beta_1)R)T_{2,1}$ and $R(1+(1-\beta_1)R)\mathcal{E}_{2,0}$, as well as the L^∞-admissibility $R(1+(1-\beta_1)R)B_{2,1}$ and $R(1+(1-\beta_1)R)M_{2,1}$, we infer

$$
\Big\|\frac{2DT_{2,1}^-}{1-\beta_2}\Big\|_{L^2} \lesssim \frac{1}{R(1+(1-\beta_1)R)}.
$$

On the other hand, by (3.10), we have

$$
\|i\rho_{\beta_2}^-\|_{L^2} = \Big\|Q_{\beta_2}^- + \frac{i}{2}\partial_{y_2}Q_{\beta_2}^-\Big\|_{L^2} + O((1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{1/2}) \lesssim (1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{1/2}.
$$

This shows that

$$
\Big|\Big(\frac{2DT_{2,1}^-}{1-\beta_2},i\rho_{\beta_2}\Big)\Big| \lesssim \frac{(1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{1/2}}{R(1+(1-\beta_1)R)}. \qquad (4.45)
$$

Then, by (3.10), we easily notice that, for every $p\in(2,\infty)$,

$$
\begin{aligned}
\Big|\Big(2\partial_{\beta_2}(|Q_{\beta_2}|^2)T_{2,1}+\partial_{\beta_2}(Q_{\beta_2}^2)\overline{T_{2,1}},i\rho_{\beta_2}\Big)\Big| &\lesssim \frac{\|\dot Q_{\beta_2}^+\|_{L^2}+\|\dot Q_{\beta_2}^-\|_{L^p}+\|\dot Q_{\beta_2}^-\|_{L^2}(1-\beta_2)^{1/2}|\log(1-\beta_2)|^{1/2}}{R(1+(1-\beta_1)R)}\\
&\lesssim \frac{|\log(1-\beta_2)|+p(1-\beta_2)^{-1/p}+|\log(1-\beta_2)|^{1/2}}{R(1+(1-\beta_1)R)},
\end{aligned}
$$

where we have used (3.23) and (3.24) combined to the Hausdorff–Young inequality. Choosing $p=|\log(1-\beta_2)|$, we conclude

$$
(1-\beta_2)\Big|\Big(2\partial_{\beta_2}(|Q_{\beta_2}|^2)T_{2,1}+\partial_{\beta_2}(Q_{\beta_2}^2)\overline{T_{2,1}},i\rho_{\beta_2}\Big)\Big| \lesssim \frac{(1-\beta_2)|\log(1-\beta_2)|}{R(1+(1-\beta_1)R)}. \qquad (4.46)
$$

Finally, we deal with the term

$$
(1-\beta_2)\Big(\partial_{\beta_2}(\mathcal{L}_{\beta_2}T_{2,1}),i\rho_{\beta_2}\Big).
$$

Recalling the equation of $T_{2,1}$, we have:

$$\begin{aligned}\partial_{\beta_2}\big(\mathcal{L}_{\beta_2}T_{2,1}\big) &= \partial_{\beta_2}\mathcal{E}_{2,0} - i\partial_{\beta_2}M_{2,1}\Lambda Q_{\beta_2} - iM_{2,1}\Lambda\partial_{\beta_2}Q_{\beta_2} \\ &+ i\partial_{\beta_2}B_{2,1}\left[y_2\partial_{y_2}Q_{\beta_2} + (1-\beta_2)\frac{\partial Q_{\beta_2}}{\partial\beta_2}\right] \\ &+ iB_{2,1}\left[y_2\partial_{y_2}\partial_{\beta_2}Q_{\beta_2} - \partial_{\beta_2}Q_{\beta_2} + (1-\beta_2)\partial_{\beta_2}^2Q_{\beta_2}\right] \\ &+ i(1-\mu)\partial_\Gamma\partial_{\beta_2}T_{2,1}\end{aligned} \tag{4.47}$$

with

$$\mathcal{E}_{2,0} = (1-\chi_R)\left[2\sqrt{\mu}e^{-i\Gamma}Q_{\beta_1}|Q_{\beta_2}|^2 + \mu e^{-2i\Gamma}Q_{\beta_1}^2\overline{Q_{\beta_2}} + 2\mu Q_{\beta_2}|Q_{\beta_1}|^2 + \sqrt{\mu}e^{i\Gamma}\overline{Q_{\beta_1}}Q_{\beta_2}^2\right].$$

Because of Proposition 4.6, we have the pointwise bound on the Fourier coefficients of $\mathcal{E}_{2,0}$:

$$\sum_{\pm}\sum_{r=1}^{2}|\mathcal{E}_{2,0,r}^{\pm}(y_2)| + |\mathcal{E}_{2,0,0}(y_2)| \lesssim \frac{1}{R(1+(1-\beta_1)R)\langle y_2\rangle(1+(1-\beta_2)|y_2|)}. \tag{4.48}$$

Using the fact that $\frac{\partial y_1}{\partial\beta_2} = -\frac{\mu y_2}{1-\beta_1}$, we also have the pointwise bound

$$\begin{aligned}&\sum_{\pm}\sum_{r=1}^{2}|\partial_{\beta_2}\mathcal{E}_{2,0,r}^{p}m(y_2)| + |\partial_{\beta_2}\mathcal{E}_{2,0,0}(y_2)| \\ &\lesssim \frac{|y_2|}{(1-\beta_1)R}\mathbf{1}_{|y_1|\sim R,|y_2|\sim\frac{R}{b}}\Big(|Q_{\beta_1}||Q_{\beta_2}|^2 + |Q_{\beta_1}|^2|Q_{\beta_2}|\Big) \\ &+ \frac{|y_2|}{1-\beta_1}\mathbf{1}_{|y_1|\geqslant\frac{R}{4}}|\partial_{y_1}Q_{\beta_1}|\Big(|Q_{\beta_1}||Q_{\beta_2}| + |Q_{\beta_2}|^2\Big) \\ &+ \mathbf{1}_{|y_1|\geqslant\frac{R}{4}}|\partial_{\beta_2}Q_{\beta_2}|\Big(|Q_{\beta_1}|^2 + |Q_{\beta_1}||Q_{\beta_2}|\Big) \\ &= IX + X + XI\ .\end{aligned} \tag{4.49}$$

Using the bounds (3.7) on Q_β and (3.23), (3.24) combined with Hausdorff–Young yield

$$\|IX\|_{L^2} + \|X\|_{L^2} \lesssim \frac{1}{(1-\beta_1)\sqrt{b}R^2(1+(1-\beta_1)R)}\ , \tag{4.50}$$

$$\|XI\|_{L^2+L^p} \lesssim \frac{|\log(1-\beta_2)| + p(1-\beta_2)^{-1/p}}{R(1+(1-\beta_1)R)}\ ,\ 2\leq p<\infty\ . \tag{4.51}$$

We are going to use this to estimate $\partial_{\beta_2}B_{2,1}$ and $\partial_{\beta_2}M_{2,1}$. Recall that

$$B_{2,1,r}^{\pm} = -\frac{(\mathcal{E}_{2,0,r}^{\pm} + i(1-\mu)T_{2,1,r}^{\pm}, iQ_{\beta_2})}{\big(iy_2\partial_{y_2}Q_{\beta_2} + i(1-\beta_2)\frac{\partial Q_{\beta_2}}{\partial\beta_2}, iQ_{\beta_2}\big)},$$

and a similar identity for $B_{2,1,0}$. Taking the derivative with respect to β_2 and using (4.48), we have

$$\begin{aligned}|\partial_{\beta_2}B_{2,1,r}^{\pm}| &\lesssim \Big|\Big(\partial_{\beta_2}\mathcal{E}_{2,0,r}^{\pm} + i(1-\mu)\partial_{\beta_2}T_{2,1,r}^{\pm}, iQ_{\beta_2}\Big)\Big| \\ &+ \Big|(\mathcal{E}_{2,0,r}^{\pm} + i(1-\mu)T_{2,1,r}^{\pm}, i\partial_{\beta_2}Q_{\beta_2})\Big| \\ &+ \frac{1}{R(1+(1-\beta_1)R)}\Big|\big(iy_2\partial_{y_2}\partial_{\beta_2}Q_{\beta_2} - i\partial_{\beta_2}Q_{\beta_2} + i(1-\beta_2)\partial_{\beta_2}^2Q_{\beta_2}, iQ_{\beta_2}\big)\Big| \\ &+ \frac{1}{R(1+(1-\beta_1)R)}\Big|\big(iy_2\partial_{y_2}Q_{\beta_2} + i(1-\beta_2)\partial_{\beta_2}Q_{\beta_2}, i\partial_{\beta_2}Q_{\beta_2}\big)\Big|\ .\end{aligned}$$

We estimate the inner products in the right hand side of the above inequality as follows. Notice that, from the admissibility properties and (4.48), for every $q \in (1,2]$,

$$\|Q_{\beta_2}\|_{L^q} \lesssim \frac{1}{q-1} \ , \ \|T^{\pm}_{2,1,r}\|_{L^q} + \|\mathcal{E}^{\pm}_{2,1,r}\|_{L^q} \lesssim \frac{1}{(q-1)R(1+(1-\beta_1)R)} \ .$$

Given $p \in [2,\infty)$, using (4.50), (4.51), (3.23), (3.24), Hölder's inequality leads to

$$\begin{aligned}
|(\partial_{\beta_2}\mathcal{E}^{\pm}_{2,0,r}, iQ_{\beta_2})| &\lesssim \frac{1}{(1-\beta_1)\sqrt{b}R^2(1+(1-\beta_1)R)} + \frac{p|\log(1-\beta_2)| + p^2(1-\beta_2)^{-1/p}}{R(1+(1-\beta_1)R)} \\
|(\mathcal{E}^{\pm}_{2,0,r}, i\partial_{\beta_2}Q_{\beta_2})| &\lesssim \frac{p|\log(1-\beta_2)| + p^2(1-\beta_2)^{-1/p}}{R(1+(1-\beta_1)R)} \\
|(T^{\pm}_{2,1,r}, i\partial_{\beta_2}Q_{\beta_2})| &\lesssim \frac{p|\log(1-\beta_2)| + p^2(1-\beta_2)^{-1/p}}{R(1+(1-\beta_1)R)} \\
|(\partial_{\beta_2}T^{\pm}_{2,1,r}, iQ_{\beta_2})| &\lesssim \frac{1}{(1-\beta_2)R(1+(1-\beta_1)R)} \ .
\end{aligned}$$

The other inner products are estimated thanks to (3.30). Choosing $p = |\log(1-\beta_2)|$ in the above inequalities, we infer

$$(1-\beta_2)|\partial_{\beta_2}B^{\pm}_{2,1,r}| \lesssim \frac{\sqrt{b}}{R^2(1+(1-\beta_1)R)} + \frac{(1-\beta_2)(\log(1-\beta_2))^2}{R(1+(1-\beta_1)R)} + \frac{|1-\mu|}{R(1+(1-\beta_1)R)} \ .$$

We obtain the same estimate for $(1-\beta_2)\partial_{\beta_2}B_{2,1,0}$.
Arguing analogously, we obtain

$$(1-\beta_2)|\partial_{\beta_2}M^{\pm}_{2,1,r}| \lesssim \frac{\sqrt{b}}{R^2(1+(1-\beta_1)R)} + \frac{(1-\beta_2)(\log(1-\beta_2))^2}{R(1+(1-\beta_1)R)} + \frac{|1-\mu|}{R(1+(1-\beta_1)R)} \ .$$

Putting together the above estimates and using the fact that $|B_{2,1}| + |M_{2,1}| \lesssim \frac{1}{R(1+(1-\beta_1)R)}$, we obtain from (4.47):

$$\Big|(1-\beta_2)\Big(\partial_{\beta_2}(\mathcal{L}_{\beta_2}T_{2,1}), i\rho_{\beta_2}\Big)\Big| \lesssim \frac{\sqrt{b}}{R^2(1+(1-\beta_1)R)} + \frac{(1-\beta_2)(\log(1-\beta_2))^2 + |1-\mu|}{R(1+(1-\beta_1)R)} \ .$$

This together with (4.44), (4.45), and (4.46) show that

$$\Big|(1-\beta_2)\Big(i\frac{\partial T_{2,1}}{\partial\beta_2}, \partial_{y_2}Q_{\beta_2}\Big)\Big| \lesssim \frac{(1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12} + |1-\mu|}{R(1+(1-\beta_1)R)} + \frac{\sqrt{b}}{R^2(1+(1-\beta_1)R)} \ ,$$

which proves (4.44).

□

4.7. **Sharp modulation equations.** We now compute explicitly the leading order modulation equations. We need to exhibit some fine cancellations which could be computed to the expense of lengthy computations[4] which can be avoided using the following nonlinear algebra.

Before stating the result, let us define some more notation. We set

$$N_\beta := \frac{1}{2\pi}\|Q_\beta\|^2_{L^2} \ , \ P_\beta := \frac{1}{2\pi}(DQ_\beta, Q_\beta) \ . \tag{4.52}$$

and we recall that

$$c_\beta := \frac{i}{2\pi}\int_{\mathbb{R}} |Q_\beta(y)|^2 Q_\beta(y)\,dy \ .$$

[4]because we need the cancellation to the order 2 in the scaling law.

and the asymptotics from Proposition 3.11,

$$\begin{aligned} N_\beta &= 1 + O((1-\beta)\log(1-\beta)) \ , \tilde{\Lambda}_\beta N_\beta = O((1-\beta)\log(1-\beta)) \ , \\ P_\beta &= 1 + O((1-\beta)\log(1-\beta)) \ , \ c_\beta = 1 + O((1-\beta)\log(1-\beta)) \ . \end{aligned}$$

Proposition 4.12 (Sharp modulation equations). *Let $B_j^{(N)}, M_j^{(N)}$ be defined by Proposition 4.6. The following estimates hold for $\mathcal{P} \in \mathcal{O}$.*

$$B_1^{(N)} = 2\frac{\mathrm{Re}\left(Q_{\beta_2}\left(-\frac{R}{b\mu}\right)\overline{c_{\beta_1}}\, e^{i\Gamma}\right)}{N_{\beta_1} - \tilde{\Lambda}_{\beta_1} N_{\beta_1}} + O\left(\frac{b(|1-\mu| + R^{-1})}{R(1+(1-\beta_1)R)}\right) \ , \tag{4.53}$$

$$B_2^{(N)} = 2\frac{\mathrm{Re}\left(Q_{\beta_1}(R)\,\overline{c_{\beta_2}}\, e^{-i\Gamma}\right)}{N_{\beta_2} - \tilde{\Lambda}_{\beta_2} N_{\beta_2}} + O\left(\frac{|1-\mu| + R^{-1}}{R(1+(1-\beta_1)R)}\right) \ . \tag{4.54}$$

$$M_1^{(N)} - \frac{\tilde{\Lambda}_{\beta_1} P_{\beta_1}}{P_{\beta_1}} B_1^{(N)} = O\left(\frac{b(|1-\mu| + R^{-1})}{R(1+(1-\beta_1)R)}\right) \ , \tag{4.55}$$

$$\begin{aligned} &M_2^{(N)} - \frac{\tilde{\Lambda}_{\beta_2} P_{\beta_2}}{P_{\beta_2}} B_2^{(N)} + 2(1-\mu)\mathrm{Re}\left(e^{i\Gamma}\overline{Q_{\beta_1}(R)}\right) + 2\mathrm{Im}\left(e^{i\Gamma}\overline{\partial_{y_1} Q_{\beta_1}(R)}\right) \\ &= O\Big(\frac{(|1-\mu| + R^{-1})(|1-\mu| + b + (1-\beta_2)^{1/2}|\log(1-\beta_2)|^{1/2}) + R^{-2}}{R(1+(1-\beta_1)R)}\Big) \ . \end{aligned} \tag{4.56}$$

Proof. We recall the system of *nonlinear elliptic equations* solved in Proposition 4.6.

$$\|\mathcal{E}_{1,N}\|_{\beta_1} = O(bR^{-N-1}) \ , \ \|\mathcal{E}_{2,N}\|_{\beta_2} = O(R^{-N-1}) \ .$$

To simplify the notation, we will use v_j instead of $V_j^{(N)}$ all along this proof. We will also drop the indices (N) from B_j, M_j for $j = 1, 2$.

Let us recall the expressions of $\mathcal{E}_1, \mathcal{E}_2$.

$$\begin{aligned} \mathcal{E}_1 &= -\frac{(|D| - \beta_1 D)v_1}{1-\beta_1} - v_1 + v_1|v_1|^2 - iM_1\Lambda v_1 + iB_1\left[y_1\partial_{y_1}v_1 + (1-\beta_1)\frac{\partial v_1}{\partial \beta_1}\right] \\ &+ i\lambda_1 M_1\frac{\partial v_1}{\partial \lambda_1} + i\lambda_1 M_2\frac{\partial v_1}{\partial \lambda_2} + i\frac{(1-\beta_2)B_2}{\mu}\frac{\partial v_1}{\partial \beta_2} \\ &+ i\frac{1-\mu}{\mu}\frac{\partial v_1}{\partial \Gamma} + i(1 - b + (B_1 - M_1)R)\frac{\partial v_1}{\partial R} \\ &+ \chi_R\left[\frac{2}{\mu}v_1|v_2|^2 + \frac{e^{-i\Gamma}}{\sqrt{\mu}}v_1^2\overline{v_2} + 2\frac{e^{i\Gamma}}{\sqrt{\mu}}|v_1|^2 v_2 + \frac{e^{2i\Gamma}}{\mu}\overline{v_1}v_2^2\right] , \end{aligned} \tag{4.57}$$

$$\begin{aligned} \mathcal{E}_2 &= -\frac{(|D| - \beta_2 D)v_2}{1-\beta_2} - v_2 + v_2|v_2|^2 - iM_2\Lambda v_2 + iB_2\left[y_2\partial_{y_2}v_2 + (1-\beta_2)\frac{\partial v_2}{\partial \beta_2}\right] \\ &+ i\lambda_2 M_2\frac{\partial v_2}{\partial \lambda_2} + i\lambda_2 M_1\frac{\partial v_2}{\partial \lambda_1} + i\mu(1-\beta_1)B_1\frac{\partial v_2}{\partial \beta_1} \\ &+ i(1-\mu)\frac{\partial v_2}{\partial \Gamma} + i\mu(1 - b + (B_1 - M_1)R)\frac{\partial v_2}{\partial R} \\ &+ (1-\chi_R)\left[2\sqrt{\mu}e^{-i\Gamma}v_1|v_2|^2 + \mu e^{-2i\Gamma}v_1^2\overline{v_2} + 2\mu v_2|v_1|^2 + \sqrt{\mu}e^{i\Gamma}\bar{v_1}v_2^2\right] . \end{aligned} \tag{4.58}$$

Our strategy is to extract information on B_j, M_j from (4.11), (4.57), (4.58) and the admissibility properties of v_1, v_2.

Step 1: Speed for the first bubble and estimate on B_1. We take the scalar product of (4.57) with iv_1. We observe the cancellations

$$\Big(-\frac{(|D|-\beta_1 D)v_1}{1-\beta_1} - v_1 + v_1|v_1|^2, iv_1\Big) = 0 \ , \ (i\Lambda v_1, iv_1) = 0 \ .$$

Recall from Proposition 4.6 that

$$|B_1| + |M_1| \lesssim \frac{b}{R(1+(1-\beta_1)R)} \ ,$$

and that $b^{-1}(1+(1-\beta_1)R)R^j T_{1,j}$ is 1–admissible. We obtain

$$B_1[(\Lambda_{y_1} Q_{\beta_1} + \tilde{\Lambda}_{\beta_1} Q_{\beta_1}, Q_{\beta_1}) + O(R^{-1})] = -\frac{1}{\sqrt{\mu}} \mathrm{Im}\left(e^{i\Gamma} \int_{\mathbb{R}} \chi_R |v_1|^2 \overline{v}_1 v_2 \, dy_1 \right)$$
$$+O\left(\int_{\mathbb{R}} \chi_R |v_1|^2 |v_2|^2 \, dy_1 + \frac{b}{R(1+(1-\beta_1)R)} \left[|1-\mu| + \frac{1}{R} \right] \right) \ .$$

From the 2–admissibility of v_2, we have

$$\chi_R(y_1) \left| v_2\left(\frac{y_1 - R}{b\mu} \right) \right|^2 \leq \frac{b^2}{R^2(1+(1-\beta_1)R)^2} \ .$$

This allows to neglect the integral

$$\int_{\mathbb{R}} \chi_R |v_1|^2 |v_2|^2 \, dy_1 \ .$$

On the other hand,

$$\left| \chi_R(y_1) \left(v_2\left(\frac{y_1 - R}{b\mu} \right) - v_2\left(\frac{-R}{b\mu} \right) \right) \right| \lesssim \frac{b^2}{R^2(1+(1-\beta_1)R)} \ ,$$

and more precisely, since $R^j(1+(1-\beta_1)R)T_{2,j}$ is 2–admissible,

$$\left| v_2\left(\frac{-R}{b\mu} \right) - Q_{\beta_2}\left(-\frac{R}{b\mu} \right) \right| \lesssim \frac{b}{R^2(1+(1-\beta_1)R)^2} \ .$$

Therefore we can replace v_2 by $Q_{\beta_2}(-R/(b\mu))$ in the integral

$$\int_{\mathbb{R}} \chi_R |v_1|^2 \overline{v}_1 v_2 \, dy_1 \ .$$

Similarly, because of the estimates on $T_{1,j}$, one can replace v_1 by Q_{β_1} in the above integral, and finally drop the factor χ_R, since the tale of $|Q_{\beta_1}|^3$ at infinity is small enough. Identifying the coefficient of B_1, we infer

$$\begin{aligned} -\pi B_1(N_{\beta_1} - \tilde{\Lambda}_{\beta_1} N_{\beta_1}) \ &= \ -\frac{1}{\sqrt{\mu}} \mathrm{Im}\left(Q_{\beta_2}\left(\frac{-R}{b\mu} \right) \int_{\mathbb{R}} |Q_{\beta_1}|^2 \overline{Q}_{\beta_1} \, e^{i\Gamma} \right) \\ &\quad +O\left(\frac{b}{R(1+(1-\beta_1)R)} \left[|1-\mu| + \frac{1}{R} \right] \right) \ , \end{aligned}$$

which, using the notation for c_β, provides (4.53). Notice that the factor $1/\sqrt{\mu}$ has been replaced by 1 up to an error

$$O\left(\frac{|1-\mu| b}{R(1+(1-\beta_1)R)} \right) \ .$$

Step 2: Speed for the second bubble and estimate on B_2. We proceed for the second bubble exactly as in Step 1. This leads to (4.54), as can be checked easily by the reader. Notice that the absence of the factor b in the remainder term is due to the slightly different estimate for $T_{2,1}$ in Proposition 4.6.

Step 3: Scaling for the first bubble and estimate on M_1. We take the scalar product of (4.57) with $\partial_{y_1} v_1$. We observe the cancellation

$$\Big(- \frac{(|D| - \beta_1 D) v_1}{1-\beta_1} - v_1 + v_1 |v_1|^2, \partial_{y_1} v_1 \Big) = 0.$$

We now compute the leading order non linear term. First, by integration by parts,

$$\begin{aligned} &\left(\chi_R \left[\frac{2}{\mu} v_1 |v_2|^2 + \frac{e^{2i\Gamma}}{\mu} \overline{v_1} v_2^2 \right], \partial_{y_1} v_1 \right) \\ &= -\frac{1}{\mu} \int |v_1|^2 \partial_{y_1} (\chi_R |v_2|^2) dy_1 - \frac{1}{2\mu} Re \left(\int e^{2i\Gamma} \overline{v_1}^2 \partial_{y_1} (\chi_R v_2^2) dy_1 \right). \end{aligned} \tag{4.59}$$

From Proposition 4.6, we have the rough bound

$$|v_j| + |y_j \partial_{y_j} v_j| \lesssim \frac{1}{\langle y_j \rangle (1 + (1-\beta_j)|y_j|)}. \tag{4.60}$$

Combining this with the fact that on the support of χ_R we have $\frac{R}{2\mu b} \leqslant |y_2| \leqslant \frac{3R}{2\mu b}$, we estimate

$$|\partial_{y_1} (\chi_R |v_2|^2)| \lesssim \frac{\mathbf{1}_{|y_2| \geqslant \frac{R}{2\mu b}}}{R} |v_2|^2 + \frac{\mathbf{1}_{|y_2| \geqslant \frac{R}{2\mu b}}}{b\mu} \partial_{y_2} (|v_2|^2) \lesssim \frac{b^2}{R^3 (1 + (1-\beta_1) R)^2}$$

Then, by (4.59) and (4.60), we have

$$\left| \left(\chi_R \Big[\frac{2}{\mu} v_1 |v_2|^2 + \frac{e^{2i\Gamma}}{\mu} \overline{v_1} v_2^2 \Big], \partial_{y_1} v_1 \right) \right| \lesssim \frac{b^2}{R^3 (1 + (1-\beta_1) R)^2}. \tag{4.61}$$

For the remaining nonlinear term, we integrate by parts and obtain

$$\begin{aligned} &\left(\chi_R \left[\frac{e^{-i\Gamma}}{\sqrt{\mu}} v_1^2 \overline{v_2} + 2 \frac{e^{i\Gamma}}{\sqrt{\mu}} |v_1|^2 v_2 \right], \partial_{y_1} v_1 \right) \\ &= \mathrm{Re} \left(\int \frac{\chi_R}{\sqrt{\mu}} \left[e^{-i\Gamma} v_1^2 \overline{v_2} \overline{\partial_{y_1} v_1} + 2 e^{i\Gamma} v_1 \overline{v_1} v_2 \overline{\partial_{y_1} v_1} \right] dy_1 \right) \\ &= \mathrm{Re} \left(\int \frac{\chi_R}{\sqrt{\mu}} \left[e^{-i\Gamma} \overline{v_2} \left[\partial_{y_1} (v_1^2 \overline{v_1}) - 2 v_1 \partial_{y_1} v_1 \overline{v_1} \right] + 2 e^{i\Gamma} v_1 \overline{v_1} v_2 \overline{\partial_{y_1} v_1} \right] dy_1 \right) \\ &= -\mathrm{Re} \left(\int \frac{e^{-i\Gamma}}{\sqrt{\mu}} v_1 |v_1|^2 \partial_{y_1} \left[\chi_R \overline{v_2} \right] dy_1 \right) \end{aligned} \tag{4.62}$$

We extract the leading order term using the following pointwise bound which is a consequence of the 1–admissibility of $b^{-1} R (1 + (1-\beta_1) R)(v_1 - Q_{\beta_1})$, and of the 2–admissibility of $R(1 + (1-\beta_1)R)(v_2 - Q_{\beta_2})$,

$$\begin{aligned} &\Big| v_1 |v_1|^2 \partial_{y_1} \left[\chi_R \overline{v_2} \right] - Q_{\beta_1} |Q_{\beta_1}|^2 \partial_{y_1} \left[\chi_R \overline{Q_{\beta_2}} \right] \Big| \\ &\qquad \lesssim \frac{b}{R^3 (1 + (1-\beta_1) R)^2 \langle y_1 \rangle^3} \end{aligned}$$

and thus:

$$\begin{aligned} -\mathrm{Re} \left(\int \frac{e^{-i\Gamma}}{\sqrt{\mu}} v_1 |v_1|^2 \partial_{y_1} \left[\chi_R \overline{v_2} \right] dy_1 \right) &= -\mathrm{Re} \left(\int \frac{e^{-i\Gamma}}{\sqrt{\mu}} Q_{\beta_1} |Q_{\beta_1}|^2 \partial_{y_1} \left[\chi_R \overline{Q_{\beta_2}} \right] dy_1 \right) \\ &\quad + O \left(\frac{b}{R^3 (1 + (1-\beta_1) R)^2} \right) \end{aligned} \tag{4.63}$$

We now compute the leading order term. Let

$$z_1 = \frac{y_1}{R}$$

Then, using $|\partial_{y_2}^2 Q_{\beta_2}| \lesssim \frac{1}{\langle y_2\rangle^3}$, we have for $|z_1| \leqslant \frac{1}{2}$ that

$$\begin{aligned}\partial_{y_2} Q_{\beta_2}(y_2) &= \partial_{y_2} Q_{\beta_2}\left(\frac{-R}{b\mu}(1-z_1)\right)\\ &= \partial_{y_2} Q_{\beta_2}\left(\frac{-R}{b\mu}\right) + \int_0^1 \frac{Rz_1}{b\mu}\partial_{y_2}^2 Q_{\beta_2}\left(\frac{-R}{b\mu}(1-tz_1)\right) dt\\ &= \partial_{y_2} Q_{\beta_2}\left(\frac{-R}{b\mu}\right) + O\left(\frac{R|z_1|}{b}\Big(\frac{b}{R}\Big)^3\right) = \partial_{y_2} Q_{\beta_2}\left(\frac{-R}{b\mu}\right) + O\left(\frac{b^2|y_1|}{R^3}\right).\end{aligned}$$

Thus,

$$\begin{aligned}&-\operatorname{Re}\left(\int \frac{e^{-i\Gamma}}{\sqrt{\mu}} Q_{\beta_1}|Q_{\beta_1}|^2 \partial_{y_1}\left[\chi_R \overline{Q_{\beta_2}}\right] dy_1\right)\\ &= -\operatorname{Re}\left(\int_{|y_1|\leqslant \frac{R}{2}} \frac{e^{-i\Gamma}}{b\mu\sqrt{\mu}} Q_{\beta_1}|Q_{\beta_1}|^2 \chi_R \overline{\partial_{y_2} Q_{\beta_2}} dy_1\right)\\ &-\operatorname{Re}\left(\int_{\frac{R}{4}\leqslant|y_1|\leqslant\frac{R}{2}} \frac{e^{-i\Gamma}}{\sqrt{\mu}} Q_{\beta_1}|Q_{\beta_1}|^2 \left[\partial_{y_1}\chi_R\right] \overline{Q_{\beta_2}} dy_1\right)\\ &= -\operatorname{Re}\left(\int_{|y_1|\leqslant \frac{R}{2}} \frac{e^{-i\Gamma}}{b\mu\sqrt{\mu}} Q_{\beta_1}|Q_{\beta_1}|^2 \partial_{y_2}\overline{Q_{\beta_2}} dy_1\right)\\ &-\operatorname{Re}\left(\int_{|y_1|\leqslant \frac{R}{2}} \frac{e^{-i\Gamma}}{b\mu\sqrt{\mu}} (\chi_R - 1) Q_{\beta_1}|Q_{\beta_1}|^2 \partial_{y_2}\overline{Q_{\beta_2}} dy_1\right) + O\left(\frac{b}{R^4}\right)\\ &= -\operatorname{Re}\left(\overline{\partial_{y_2} Q_{\beta_2}\left(\frac{-R}{b\mu}\right)} \int_{|y_1|\leqslant \frac{R}{2}} \frac{e^{-i\Gamma}}{b\mu\sqrt{\mu}} Q_{\beta_1}|Q_{\beta_1}|^2 dy_1\right) + O\left(\frac{b}{R^3}\right)\\ &= -\frac{1}{b\mu}\operatorname{Re}\left(\partial_{y_2} Q_{\beta_2}\left(\frac{-R}{b\mu}\right) \frac{e^{i\Gamma}}{\sqrt{\mu}} \int \overline{Q}_{\beta_1}|Q_{\beta_1}|^2 dy_1\right) + O\left(\frac{b}{R^3}\right)\\ &= \frac{2\pi}{b\mu\sqrt{\mu}}\operatorname{Im}\left(e^{i\Gamma}\overline{c}_{\beta_1}\partial_{y_2} Q_{\beta_2}\left(\frac{-R}{b\mu}\right)\right) + O\left(\frac{b}{R^3}\right). \end{aligned} \tag{4.64}$$

Finally we use again the following bound,

$$\frac{1}{b\mu}\left|\partial_{y_2} Q_{\beta_2}\left(\frac{-R}{b\mu}\right)\right| \lesssim \frac{b}{R^2(1+(1-\beta_1)R)}.$$

This together with (4.62), (4.63), and (4.64), yields

$$\left|\left(\chi_R\left[\frac{e^{-i\Gamma}}{\sqrt{\mu}} v_1^2\overline{v_2} + 2\frac{e^{i\Gamma}}{\sqrt{\mu}}|v_1|^2 v_2\right], \partial_{y_1} v_1\right)\right| \lesssim \frac{b}{R^2(1+(1-\beta_1)R)}.$$

Combining this with (4.61), we get that the contribution of the nonlinearity is

$$\left|\left(\chi_R\Big[\frac{2}{\mu}v_1|v_2|^2 + \frac{e^{2i\Gamma}}{\mu}\overline{v_1}v_2^2 + \frac{e^{-i\Gamma}}{\sqrt{\mu}}v_1^2\overline{v_2} + 2\frac{e^{i\Gamma}}{\sqrt{\mu}}|v_1|^2 v_2\Big], \partial_{y_1} v_1\right)\right| \lesssim \frac{b}{R^2(1+(1-\beta_1)R)}. \tag{4.65}$$

In view of the expression (4.57) of $\mathcal{E}_1$, we infer —assuming N large enough—,

$$\begin{aligned} & M_1(-i\Lambda v_1, \partial_{y_1} v_1) + B_1 \left(i y_1 \partial_{y_1} v_1 + i(1-\beta_1)\frac{\partial v_1}{\partial \beta_1}, \partial_{y_1} v_1 \right) \\ + \quad & \lambda_1 M_1 \left(i\frac{\partial v_1}{\partial \lambda_1}, \partial_{y_1} v_1 \right) + \lambda_1 M_2 \left(i\frac{\partial v_1}{\partial \lambda_2}, \partial_{y_1} v_1 \right) + \frac{(1-\beta_2)B_2}{\mu} \left(i\frac{\partial v_1}{\partial \beta_2}, \partial_{y_1} v_1 \right) \\ + \quad & \frac{1-\mu}{\mu} \left(i\frac{\partial v_1}{\partial \Gamma}, \partial_{y_1} v_1 \right) + (1 - b + (B_1 - M_1)R) \left(i\frac{\partial v_1}{\partial R}, \partial_{y_1} v_1 \right) = O(bR^{-2}(1+(1-\beta_1)R)^{-1}). \end{aligned}$$

We now compute the terms involving the modulation equations. First, by Proposition 4.6, we have that

$$\begin{aligned} (-i\Lambda v_1, \partial_{y_1} v_1) &= (-i\Lambda Q_{\beta_1}, \partial_{y_1} Q_{\beta_1}) + O\left(\frac{b}{R(1+(1-\beta_1)R)} \right) \\ &= -\pi P_{\beta_1} + O\left(\frac{b}{R(1+(1-\beta_1)R)} \right). \end{aligned} \tag{4.66}$$

On the other hand,

$$\begin{aligned} \left(i y_1 \partial_{y_1} v_1 + i(1-\beta_1)\frac{\partial v_1}{\partial \beta_1}, \partial_{y_1} v_1 \right) &= \left(i(1-\beta_1)\frac{\partial Q_{\beta_1}}{\partial \beta_1}, \partial_{y_1} Q_{\beta_1} \right) + O\left(\frac{b}{R(1+(1-\beta_1)R)} \right), \\ &= \pi \tilde{\Lambda}_{\beta_1} P_{\beta_1} + O\left(\frac{b}{R(1+(1-\beta_1)R)} \right). \end{aligned}$$

Then, by Proposition 4.6,

$$\begin{aligned} & \left| \left(i\frac{\partial v_1}{\partial \lambda_1}, \partial_{y_1} v_1 \right) \right| + \left| \left(i\frac{\partial v_1}{\partial \lambda_2}, \partial_{y_1} v_1 \right) \right| + \left| \left(i(1-\beta_2)\frac{\partial v_1}{\partial_{\beta_2}}, \partial_{y_1} v_1 \right) \right| \\ & \lesssim \frac{b}{R(1+(1-\beta_1)R}, \end{aligned}$$

and

$$\begin{aligned} & \left| \frac{1-\mu}{\mu} \left(i\frac{\partial v_1}{\partial \Gamma}, \partial_{y_1} v_1 \right) \right| + \left| \left(i\frac{\partial v_1}{\partial R}, \partial_{y_1} v_1 \right) \right| \\ & \lesssim \frac{b(|1-\mu| + R^{-1})}{R(1+(1-\beta_1)R)}. \end{aligned}$$

The collection of above bounds yields the identity:

$$-\pi P_{\beta_1} M_1 + \tilde{\Lambda}_{\beta_1} P_{\beta_1} B_1 = O\left(\frac{b(|1-\mu| + R^{-1})}{R(1+(1-\beta_1)R)} \right),$$

which leads to the bound

$$\left| M_1 - \frac{\tilde{\Lambda}_{\beta_1} P_{\beta_1}}{P_{\beta_1}} B_1 \right| \lesssim \frac{b(|1-\mu| + R^{-1})}{R(1+(1-\beta_1)R)}. \tag{4.67}$$

Step 4: Scaling for the second bubble and estimate on M_2. We take the scalar product of (4.10) with $\partial_{y_2} v_2$. We observe the cancellation

$$\left(-\frac{(|D| - \beta_2 D)v_2}{1-\beta_2} - v_2 + v_2 |v_2|^2, \partial_{y_2} v_2 \right) = 0.$$

We now compute the contribution of the non linear term. Firstly, by integration by parts,

$$\begin{aligned}&\left((1-\chi_R)\left[2\mu v_2|v_1|^2+\mu e^{-2i\Gamma}\overline{v_2}v_1^2\right],\partial_{y_2}v_2\right) \qquad (4.68)\\ &\quad=-\mu\int|v_2|^2\partial_{y_2}((1-\chi_R)|v_1|^2)dy_2-\frac{\mu}{2}Re\left(\int e^{-2i\Gamma}\overline{v_2}^2\partial_{y_2}((1-\chi_R)v_1^2)dy_2\right).\end{aligned}$$

By the rough bound (4.60), we have

$$\begin{aligned}\left|\partial_{y_2}\left((1-\chi_R)|v_1|^2\right)\right| &\lesssim \frac{b\mu}{R}\mathbf{1}_{\frac{R}{4}\leqslant|y_1|\leqslant\frac{R}{2}}|v_1|^2+b\mu\mathbf{1}_{|y_1|\geqslant\frac{R}{4}}\partial_{y_1}(|v_1|^2)\\ &\lesssim \frac{b\mathbf{1}_{\frac{R}{4}\leqslant|y_1|\leqslant\frac{R}{2}}}{R\langle y_1\rangle^2}+b\frac{\mathbf{1}_{|y_1|\geqslant\frac{R}{4}}}{\langle y_1\rangle^3}\lesssim\frac{b}{R^3(1+(1-\beta_1)R)^2}.\end{aligned}$$

Then, by (4.68), we have

$$\left|\left((1-\chi_R)\left[2\mu v_2|v_1|^2+\mu e^{-2i\Gamma}\overline{v_2}v_1^2\right],\partial_{y_2}v_2\right)\right|\lesssim\frac{b}{R^3(1+(1-\beta_1)R)^2}. \qquad (4.69)$$

For the remaining nonlinear term, we integrate by parts and obtain

$$\begin{aligned}&\left((1-\chi_R)\left[\sqrt{\mu}e^{i\Gamma}v_2^2\overline{v_1}+2\sqrt{\mu}e^{-i\Gamma}|v_2|^2v_1\right],\partial_{y_2}v_2\right)\\ &=\mathrm{Re}\left(\int\sqrt{\mu}(1-\chi_R)\left[e^{i\Gamma}v_2^2\overline{v_1}\overline{\partial_{y_2}v_2}+2e^{-i\Gamma}v_2\overline{v_2}v_1\overline{\partial_{y_2}v_2}\right]dy_2\right)\\ &=\mathrm{Re}\left(\int\sqrt{\mu}(1-\chi_R)\left[e^{i\Gamma}\overline{v_1}\left[\partial_{y_2}(v_2^2\overline{v_2})-2v_2\partial_{y_2}v_2\overline{v_2}\right]+2e^{-i\Gamma}v_2\overline{v_2}v_1\overline{\partial_{y_2}v_2}\right]dy_2\right)\\ &=-\mathrm{Re}\left(\int\sqrt{\mu}e^{i\Gamma}v_2|v_2|^2\partial_{y_2}\left[(1-\chi_R)\overline{v_1}\right]dy_2\right) \qquad (4.70)\end{aligned}$$

We extract the leading order term using the pointwise bound:

$$\begin{aligned}&\left|v_2|v_2|^2\partial_{y_2}\left[(1-\chi_R)\overline{v_1}\right]-Q_{\beta_2}|Q_{\beta_2}|^2\partial_{y_2}\left[(1-\chi_R)\overline{Q_{\beta_1}}\right]\right|\\ \lesssim\ &\frac{1}{R\langle y_2\rangle^3}\left[\frac{b\mu\mathbf{1}_{\frac{R}{4}\leqslant|y_1|\leqslant\frac{R}{2}}}{R\langle y_1\rangle(1+(1-\beta_1)|y_1|)}+\frac{b\mu\mathbf{1}_{|y_1|\geqslant\frac{R}{4}}}{\langle y_1\rangle^2(1+(1-\beta_1)|y_1|)}\right]\lesssim\frac{b}{R^3\langle y_2\rangle^3(1+(1-\beta_1)R)}.\end{aligned}$$

Thus,

$$\begin{aligned}&-\mathrm{Re}\left(\int\sqrt{\mu}e^{i\Gamma}v_2|v_2|^2\partial_{y_2}\left[(1-\chi_R)\overline{v_1}\right]dy_2\right)\\ &=-\mathrm{Re}\left(\int\sqrt{\mu}e^{i\Gamma}Q_{\beta_2}|Q_{\beta_2}|^2\partial_{y_2}\left[(1-\chi_R)\overline{Q_{\beta_1}}\right]dy_2\right)+O\left(\frac{b}{R^3(1+(1-\beta_1)R)}\right)\end{aligned} \qquad (4.71)$$

We now compute the leading order term. Let $z_2=\frac{b\mu y_2}{R}$, then for $|z_2|\leqslant\frac{1}{2}$:

$$\begin{aligned}\partial_{y_1}Q_{\beta_1}(y_1) &= \partial_{y_1}Q_{\beta_1}\left(R(1+z_2)\right)=\partial_{y_1}Q_{\beta_1}(R)+\int_0^1Rz_2\partial_{y_1}^2Q_{\beta_1}\left(R(1+tz_2)\right)dt\\ &= \partial_{y_1}Q_{\beta_1}(R)+O\left(\frac{R|z_2|}{R^3}\right)=\partial_{y_1}Q_{\beta_1}(R)+O\left(\frac{b\mu|y_2|}{R^3}\right)\end{aligned}$$

and thus:

$$\begin{aligned}
& -\mathrm{Re}\left(\int \sqrt{\mu}e^{i\Gamma}Q_{\beta_2}|Q_{\beta_2}|^2\partial_{y_2}\left[(1-\chi_R)\overline{Q_{\beta_1}}\right]dy_2\right)\\
= & -\mathrm{Re}\left(\int_{|y_2|\geqslant\frac{R}{2b\mu}} b\mu\sqrt{\mu}e^{i\Gamma}Q_{\beta_2}|Q_{\beta_2}|^2\partial_{y_1}\left[(1-\chi_R)\overline{Q_{\beta_1}}\right]dy_2\right)\\
- & \ \mathrm{Re}\left(\int_{|y_2|\leqslant\frac{R}{2b\mu}} b\mu\sqrt{\mu}e^{i\Gamma}Q_{\beta_2}|Q_{\beta_2}|^2\partial_{y_1}\left[(1-\chi_R)\overline{Q_{\beta_1}}\right]dy_2\right)\\
= & -\mathrm{Re}\left(\int_{|y_2|\leqslant\frac{R}{2b\mu}} b\mu\sqrt{\mu}e^{i\Gamma}Q_{\beta_2}|Q_{\beta_2}|^2\partial_{y_1}\overline{Q_{\beta_1}}dy_2\right)+O\left(\frac{b^3}{R^4}\right)\\
= & -\mathrm{Re}\left(b\mu\sqrt{\mu}e^{i\Gamma}\overline{\partial_{y_1}Q_{\beta_1}(R)}\int_{|y_2|\leqslant\frac{R}{2b\mu}} Q_{\beta_2}|Q_{\beta_2}|^2dy_2\right)\\
+ & \ O\left(\frac{b^3}{R^4}+b\int_{|y_2|\leqslant\frac{R}{2b\mu}}\frac{b\mu}{R^3}\frac{|y_2|}{\langle y_2\rangle^3}dy_2\right)\\
= & -\mathrm{Re}\left(b\mu\sqrt{\mu}e^{i\Gamma}\overline{\partial_{y_1}Q_{\beta_1}(R)}\int Q_{\beta_2}|Q_{\beta_2}|^2dy_2\right)+O\left(\frac{b}{R^3}\right)\\
= & \ O\left(\frac{b}{R^2(1+(1-\beta_1)R)}\right)
\end{aligned}$$

where we used (3.15) in the last step. Combining this with (4.70) and (4.71), we obtain that

$$\left|\left((1-\chi_R)\left[\sqrt{\mu}e^{i\Gamma}v_2^2\overline{v_1}+2\sqrt{\mu}e^{-i\Gamma}|v_2|^2v_1\right],\partial_{y_2}v_2\right)\right|\lesssim\frac{b}{R^2(1+(1-\beta_1)R)}\,.$$

This, together with (4.69) yields

$$\begin{aligned}
&\left|\left((1-\chi_R)\left[2\mu v_2|v_1|^2+\mu e^{-2i\Gamma}\overline{v_2}v_1^2+\sqrt{\mu}e^{i\Gamma}v_2^2\overline{v_1}+2\sqrt{\mu}e^{-i\Gamma}|v_2|^2v_1\right],\partial_{y_2}v_2\right)\right|\\
&\lesssim\frac{b}{R^2(1+(1-\beta_1)R)}\,.
\end{aligned}$$

In view of the expression (4.58) of $\mathcal{E}_2$, we infer

$$\begin{aligned}
& M_2(-i\Lambda v_2,\partial_{y_2}v_2)+B_2\left(iy_2\partial_{y_2}v_2+i(1-\beta_2)\frac{\partial v_2}{\partial\beta_2},\partial_{y_2}v_2\right)\\
+ & \ \lambda_2M_2\left(i\frac{\partial v_2}{\partial\lambda_2},\partial_{y_2}v_2\right)+\lambda_2M_1\left(i\frac{\partial v_2}{\partial\lambda_1},\partial_{y_2}v_2\right)+\mu B_1\left(i(1-\beta_1)\frac{\partial v_2}{\partial\beta_1},\partial_{y_2}v_2\right)\\
+ & \ (1-\mu)\left(i\frac{\partial v_2}{\partial\Gamma},\partial_{y_2}v_2\right)+(1-b+(B_1-M_1)R)\left(i\frac{\partial v_2}{\partial R},\partial_{y_2}v_2\right)\\
= & \ O\left(\frac{b}{R^2(1+(1-\beta_1)R)}+\frac{1}{R^{N+1}}\right).
\end{aligned}$$

Next we compute the terms involving the modulation equations. On the one hand,

$$(i\Lambda v_2,\partial_{y_2}v_2)=(i\Lambda Q_{\beta_2},\partial_{y_2}Q_{\beta_2})+O\left(\frac{1}{R(1+(1-\beta_1)R)}\right)=\pi P_{\beta_2}+O\left(\frac{1}{R(1+(1-\beta_1)R)}\right).$$

On the other hand, taking into account Lemma 4.11 and (3.23), (3.24),

$$\begin{aligned}\Big(iy_2\partial_{y_2}v_2+i(1-\beta_2)\frac{\partial v_2}{\partial\beta_2},\partial_{y_2}v_2\Big) &= \Big(i(1-\beta_2)\frac{\partial Q_{\beta_2}}{\partial\beta_2},\partial_{y_2}Q_{\beta_2}\Big)\\ &+ O\left(\frac{|1-\mu|+(1-\beta_2)^{1/2}|\log(1-\beta_2)|^{1/2}+R^{-1}}{R(1+(1-\beta_1)R)}\right),\\ &= \pi\tilde\Lambda_{\beta_2}P_{\beta_2}+O\left(\frac{|1-\mu|+(1-\beta_2)^{1/2}|\log(1-\beta_2)|^{1/2}+R^{-1}}{R(1+(1-\beta_1)R)}\right).\end{aligned}$$

Then, by construction,

$$\Big|\Big(i\frac{\partial v_2}{\partial\lambda_1},\partial_{y_2}v_2\Big)\Big|+\Big|\Big(i\frac{\partial v_2}{\partial\lambda_2},\partial_{y_2}v_2\Big)\Big|+\Big|\Big(i(1-\beta_1)\frac{\partial v_2}{\partial\beta_1},\partial_{y_2}v_2\Big)\Big|\lesssim\frac{1}{R(1+(1-\beta_1)R)}.$$

Moreover, by Lemmas 4.9 and 4.10, we have

$$\begin{aligned}&(1-\mu)\Big(i\frac{\partial v_2}{\partial\Gamma},\partial_{y_2}v_2\Big)+\mu(1-b)\Big(i\frac{\partial v_2}{\partial R},\partial_{y_2}v_2\Big)\\ &=(1-\mu)\Big[-2\pi\mathrm{Re}\,(e^{i\Gamma}\overline{Q_{\beta_1}(R)})\Big]+\mu(1-b)\Big[-2\pi\mathrm{Im}\,(e^{i\Gamma}\overline{\partial_{y_1}Q_{\beta_1}(R)})\Big]\\ &+O\Big(\frac{(1-\mu)^2+|1-\mu|((1-\beta_2)^{1/2}|\log(1-\beta_2)|^{1/2}+R^{-1})+R^{-2}}{R(1+(1-\beta_1)R)}\Big).\end{aligned}$$

Notice that, in view of (3.15), the factor $\mu(1-b)$ in the above right hand side can be replaced by 1 up to the expense of the additional error

$$O\left(\frac{b(|1-\mu|+R^{-1})}{R(1+(1-\beta_1)R}\right).$$

Summing up, we obtain

$$\begin{aligned}&M_2-\frac{\tilde\Lambda_{\beta_2}P_{\beta_2}}{P_{\beta_2}}B_2+2(1-\mu)\mathrm{Re}\,(e^{i\Gamma}\overline{Q_{\beta_1}(R)})+2\mathrm{Im}\,(e^{i\Gamma}\overline{\partial_{y_1}Q_{\beta_1}(R)})=\\ &O\Big(\frac{(|1-\mu|+R^{-1})(|1-\mu|+b+(1-\beta_2)^{1/2}|\log(1-\beta_2)|^{1/2})+R^{-2}}{R(1+(1-\beta_1)R)}\Big).\end{aligned}$$

This completes the proof. □

4.8. **Solving the reduced dynamical system.** Our aim in this section is to exhibit a suitable exact solution to the idealized dynamical system

$$(S)^\infty\quad\begin{cases}(x_j)_t=\beta_j,\quad(\gamma_j)_t=\frac{1}{\lambda_j},\\ (\lambda_j)_t=M_j(\mathcal{P}),\quad\frac{(\beta_j)_t}{1-\beta_j}=\frac{B_j(\mathcal{P})}{\lambda_j},\qquad j=1,2,\\ \Gamma=\gamma_2-\gamma_1,\quad R=\frac{x_2-x_1}{\lambda_1(1-\beta_1)}\end{cases}\tag{4.72}$$

with $\mathcal{P}=(\lambda_1,\lambda_2,\beta_1,\beta_2,\Gamma,R)$, which will correspond to the leading order two-soliton motion, and where from now on and for the rest of this paper we omit the subscript N for the sake of simplicity.

Let $0<\eta,\delta\ll1$. Define the times

$$T_{\mathrm{in}}=\frac{1}{\eta^{2\delta}}<T^-=\frac{\delta}{\eta}\tag{4.73}$$

and consider explicitly the solution

$$\tilde{\mathcal{P}}^\infty=(\lambda_1^\infty,\lambda_2^\infty,\beta_1^\infty,\beta_2^\infty,\gamma_1^\infty,\gamma_2^\infty,x_1^\infty,x_2^\infty)$$

to (4.72) with data at $t = T^-$:

$$\begin{cases} \lambda_1^\infty = 1, \quad \lambda_2^\infty = 1, \\ \gamma_1^\infty = \gamma_2^\infty = 0, \\ 1 - \beta_1^\infty = \eta, \quad b^\infty = \frac{1}{(T^-)^2} \quad \text{ie} \quad 1 - \beta_2^\infty = \frac{\eta}{(T^-)^2}, \\ x_1^\infty = 0, \quad R^\infty = T^- \quad \text{ie} \quad x_2^\infty = T^- \eta = \delta. \end{cases} \tag{4.74}$$

The fact that the system (4.72) with data (4.74) admits a unique maximal solution is a simple consequence of the Cauchy–Lipschitz theorem.

We first claim the backwards control of this solution in the following perturbative form.

Lemma 4.13 (Control of the solution in the perturbative turbulent regime)**.** *Let $\delta > 0$ small enough and $0 < \eta < \eta^*(\delta, N)$ small enough. Let $\tilde{\mathcal{P}}$ be the solution to the approximate system*

$$\begin{cases} (x_j)_t = \beta_j + O\left(\frac{1}{t^3}\right), \quad (\gamma_j)_t = \frac{1}{\lambda_j} + O\left(\frac{1}{t^3}\right), \\ (\lambda_j)_t = M_j(\mathcal{P}) + O\left(\frac{1}{t^3}\right), \quad \frac{(\beta_j)_t}{1-\beta_j} = \frac{B_j(\mathcal{P})}{\lambda_j} + O\left(\frac{1}{t^3}\right), \quad j = 1, 2, \\ \Gamma = \gamma_2 - \gamma_1, \quad R = \frac{x_2 - x_1}{\lambda_1(1-\beta_1)} \end{cases} \tag{4.75}$$

with initial data at T^- satisfying :

$$|\tilde{\mathcal{P}}(T^-) - \tilde{\mathcal{P}}^\infty(T^-)| \leqslant \eta^{10}, \tag{4.76}$$

then the parameters satisfy in $t \in [T_{\text{in}}, T^-]$ the bounds:

$$\begin{cases} \lambda_1(t) = 1 + O\left(\frac{\eta^\delta}{t}\right), \quad \lambda_2(t) = 1 + O\left(\frac{\eta^\delta + \eta t |\log \eta t|}{t}\right) \\ 1 - \beta_1(t) = \eta(1 + O(\eta^\delta)), \quad b(t) = \frac{1 + O(\sqrt{\delta})}{t^2} \\ \Gamma(t) = O(\eta t |\log \eta t|) \\ R = t(1 + O(\eta^\delta)). \end{cases} \tag{4.77}$$

Remark 4.14. Notice that the small quantity $\eta t |\log \eta t|$ grows on $[T_{\text{in}}, T^-]$ from $(1-\delta)\eta^{1-\delta}|\log \eta|$ to $\delta |\log \delta|$. Therefore, if δ is small and if $\eta < \eta^*(\delta)$, this quantity is first smaller that η^δ, then it becomes bigger than η^δ. This explains why we have to keep both quantities in the remainder terms.

Proof of Lemma 4.13. From (4.74) and (4.76), we may assume the following bounds:

$$\begin{cases} |\lambda_1(t) - 1| \leqslant \frac{\eta^\delta}{t}, \quad j = 1, 2 \\ |\lambda_2(t) - 1| \leqslant K \frac{\eta^\delta + \eta t |\log(\eta t)|}{t} \\ |1 - \beta_1(t) - \eta| \leqslant \eta^{1+\delta}, \\ \frac{\eta}{2t^2} \leqslant 1 - \beta_2(t) \leqslant \frac{2\eta}{t^2} \\ \frac{|R(t) - t|}{t} \leqslant \eta^\delta \\ |\Gamma(t)| \leqslant K(\eta^\delta + \eta t |\log(\eta t)|) \end{cases} \tag{4.78}$$

and aim at improving them for some large enough universal constant K, and for $0 < \delta < \delta^*(K)$, $0 < \eta < \eta^*(K, \delta)$, which proves (4.77) through a standard continuity argument. The difficulty is that the growth of Sobolev norms in (4.77) relies on an uniform control of the phase which is not allowed to move, and this requires two integrations in time in the presence of $O(\frac{1}{t^2})$ decay only and hence some suitable cancellation in the modulation equations.

Step 1: Leading order modulation equations. We extract the leading order modulation equations of Proposition 4.12 in the regime (4.77) using the sharp description of the asymptotic structure of Q_β given by Proposition 3.9. We estimate from (4.78)

$$R \sim t \leqslant \frac{\delta}{\eta}$$

and hence

$$0 < (1-\beta_1)R \lesssim \eta t \lesssim \delta \ll 1.$$

Now we appeal to the precise description of Q_β given by (3.18):

$$\begin{aligned} Q_{\beta_1}(R) &= \frac{1+O((1-\beta_1)|\log(1-\beta_1)|)}{R}\left[1+O((1-\beta_1)R\log((1-\beta_1)R))\right]+O\left(\frac{1}{R^2}\right) \\ &= \frac{1}{t}+O\left(\frac{\eta^\delta}{t}+\eta|\log\eta t|\right) \end{aligned} \tag{4.79}$$

where we used the localization of R given by (4.78) in the last step. Similarly, using (3.15), it follows that

$$\begin{aligned} &\partial_{y_1} Q_{\beta_1}(R) \\ &= \frac{1+O(\eta|\log\eta|)}{R^2}\left\{-1+\frac{i}{2}(1-\beta_1)R\left[1+O((1-\beta_1)R|\log(1-\beta_1)R|)\right]\right\}+O\left(\frac{\eta^\delta}{R^2}\right) \\ &= \frac{1}{t^2}\left[-1+\frac{i}{2}\eta t\right]+O\left(\frac{\eta^\delta}{t^2}+\eta^2|\log(\eta t)|\right). \end{aligned} \tag{4.80}$$

We also have

$$(1-\beta_2)\frac{R}{b\mu} = \frac{(1-\beta_1)R}{\mu} \lesssim (1-\beta_1)R \lesssim \delta$$

and thus,

$$\begin{aligned} &Q_{\beta_2}\left(-\frac{R}{b\mu}\right) \\ &= -\frac{1+O((1-\beta_2)|\log(1-\beta_2)|)}{\frac{R}{b\mu}}\left[1+O((1-\beta_2)\frac{R}{b}\log((1-\beta_2)\frac{R}{b}))\right]+O\left(\frac{b^2}{R^2}\right) \\ &= O\left(\frac{b}{t}\right). \end{aligned} \tag{4.81}$$

We now compute the leading order modulation equations of Proposition 4.12. We first have the rough bound

$$B_1 = O\left(\frac{b}{t}\right) \tag{4.82}$$

and the finer control from (4.79):

$$\begin{aligned} B_2 &= 2\left[1+O((1-\beta_2)|\log(1-\beta_2)|)\right]\mathrm{Re}\left\{(\cos\Gamma - i\sin\Gamma)\left[\frac{1}{t}+O\left(\frac{\eta^\delta}{t}+\eta|\log\eta t|\right)\right]\right\} \\ &+ O\left(\frac{1-\mu}{t}+\frac{1}{t^2}\right) \\ &= \frac{2\cos\Gamma}{t}+O\left(\frac{\eta^\delta}{t}+\eta|\log(\eta t)|+\frac{|1-\mu|}{t}\right) \qquad (4.83) \\ &= \frac{2}{t}+O\left(\frac{\eta^\delta}{t}+\eta|\log(\eta t)|\right) \qquad (4.84) \end{aligned}$$

where in the last step we used from (4.78):

$$\begin{aligned}\frac{\Gamma^2}{t} &\lesssim \frac{K^2(\eta^{2\delta}+\eta^2t^2|\log(\eta t)|^2)}{t} \lesssim \frac{\eta^\delta}{t}+\eta|\log(\eta t)|K^2\eta t|\log(\eta t)| \\ &\lesssim \frac{\eta^\delta}{t}+\eta|\log(\eta t)|K^2\delta|\log\delta| \lesssim \frac{\eta^\delta}{t}+\eta|\log(\eta t)| \end{aligned} \tag{4.85}$$

for $\delta<\delta^*(K)$ small enough. We similarly derive the rough bound

$$|M_1| \lesssim |1-\beta_1||\log(1-\beta_1)||B_1| + \frac{|b(1-\mu)|}{t}+\frac{|b|}{t^2} \lesssim \frac{\eta^{2\delta}}{t^2}. \tag{4.86}$$

We now estimate M_2. First we compute from (4.79), (4.85):

$$\begin{aligned} 2(1-\mu)\mathrm{Re}\left(e^{i\Gamma}\overline{Q_{\beta_1}(R)}\right) &= 2(1-\mu)\mathrm{Re}\left\{(\cos\Gamma+i\sin\Gamma)\left(\frac{1}{t}+O\left(\frac{\eta^\delta}{t}+\eta|\log\eta t|\right)\right)\right\} \\ &= \frac{2(1-\mu)\cos\Gamma}{t}+|1-\mu|O\left(\frac{\eta^\delta}{t}+\eta|\log\eta t|\right) \\ &= \frac{2(1-\mu)}{t}+O\left(\frac{\eta^\delta}{t^2}+K\eta^2|\log\eta t|^2\right) \end{aligned}$$

where we used in the last step from (4.78):

$$|1-\mu| \lesssim |\lambda_2-1|+|\lambda_1-1| \lesssim K\frac{\eta^\delta+\eta t|\log(\eta t)|}{t} \tag{4.87}$$

and hence

$$|1-\mu|\left(\frac{\eta^\delta}{t}+\eta|\log\eta t|\right) \lesssim K\frac{(\eta^\delta+\eta t|\log\eta t|)^2}{t^2} \lesssim \frac{\eta^\delta}{t^2}+K\eta^2|\log\eta t|^2$$

for $\eta<\eta^*(K,\delta)$ small enough. similarly from (4.80):

$$\begin{aligned} 2\mathrm{Im}\left\{e^{i\Gamma}\overline{\partial_{y_1}Q_{\beta_1}(R)}\right\} &= 2\mathrm{Im}\left\{(\cos\Gamma+i\sin\Gamma)\left[\frac{1}{t^2}\left(-1+\frac{i}{2}\eta t\right)+O\left(\frac{\eta^\delta}{t^2}+\eta^2|\log(\eta t)|\right)\right]\right\} \\ &= -\frac{2\sin\Gamma}{t^2}+\frac{\eta}{t}\cos\Gamma+O\left(\frac{\eta^\delta}{t^2}+\eta^2|\log(\eta t)|\right) \\ &= -\frac{2\Gamma}{t^2}+\frac{\eta}{t}+O\left(\frac{\eta^\delta}{t^2}+\eta^2|\log(\eta t)|\right), \end{aligned}$$

where we used in the last step the development of $\cos\Gamma,\sin\Gamma$ with the bounds:

$$\begin{aligned}\frac{|\Gamma|^3}{t^2}+\frac{\eta\Gamma^2}{t} &\lesssim \frac{K^3(\eta^{3\delta}+\eta^3t^3|\log\eta t|^3)}{t^2}+\frac{\eta K^2(\eta^{2\delta}+\eta^2t^2|\log\eta t|^2)}{t} \\ &\lesssim \frac{\eta^\delta}{t^2}+\eta^2|\log\eta t|\left[K^3\eta t|\log\eta t|^2+K^2\eta t|\log\eta t|\right] \\ &\lesssim \frac{\eta^\delta}{t^2}+\eta^2|\log\eta t|K^3\delta|\log\delta|^2 \leqslant \frac{\eta^\delta}{t^2}+\eta^2|\log\eta t| \end{aligned}$$

for $\delta<\delta^*(K)$ small enough. Using from (4.83) the rough bound $|B_2|\lesssim\frac{1}{t}$ ensures the finer bound from (4.56):

$$\begin{aligned}&\left|M_2+\frac{2(1-\mu)}{t}-\frac{2\Gamma}{t^2}+\frac{\eta}{t}\right| \lesssim \frac{\eta^\delta}{t^2}+K^2\eta^2|\log(\eta t)|^2+\frac{|1-\beta_2||\log(1-\beta_2)|}{t} \\ \lesssim\ &\frac{\eta^\delta}{t^2}+K^2\eta^2|\log(\eta t)|^2 \end{aligned} \tag{4.88}$$

where we used (4.78) in the last step to estimate $1-\beta_2$.

Step 2: Control of the speeds. We first integrate the law for β_2 from (4.83):

$$\frac{(\beta_2)_t}{1-\beta_2} = \frac{B_2}{\lambda_2} = \frac{1}{\lambda_2}\left[\frac{2}{t} + O\left(\frac{\eta^\delta}{t} + \eta|\log(\eta t)|\right)\right] = \frac{2}{t} + O\left(\frac{\eta^\delta}{t} + \eta|\log(\eta t)|\right).$$

We integrate on $[t, T^-]$ and use

$$\int_t^{T^-} \eta|\log(\eta\tau)|d\tau \lesssim \int_0^\delta |\log\sigma|d\sigma \leqslant \sqrt{\delta}$$
$$\int_t^{T^-} \frac{\eta^\delta}{\tau}d\tau = O\left(\eta^\delta|\log\eta|\right) \leqslant \sqrt{\delta},$$

for $\eta < \eta^*(K,\delta)$, to estimate

$$-\log\left(\frac{1-\beta_2(T^-)}{1-\beta_2(t)}\right) = 2\log\left(\frac{T^-}{t}\right) + O(\sqrt{\delta})$$

from which using the initialization (4.74), (4.76):

$$1-\beta_2(t) = \frac{(T^-)^2(1-\beta_2(T^-))}{t^2}e^{O(\sqrt{\delta})} = \left[1+O(\sqrt{\delta})\right]\frac{\eta}{t^2}. \tag{4.89}$$

We now compute for β_1 from (4.82):

$$\left|\frac{(\beta_1)_t}{1-\beta_1}\right| = \left|\frac{B_1}{\lambda_1}\right| \lesssim \frac{b}{t} \lesssim \frac{1}{t^3}$$

which time integration using (4.74), (4.76) yields

$$1-\beta_1(t) = (1-\beta_1(T^-))e^{O(\frac{1}{t^2})} = \eta\left(1+O\left(\frac{1}{t^2}\right)\right). \tag{4.90}$$

Since $t \geq T_{\text{in}} = \eta^{-\delta}$, this improves the estimate on $1-\beta_1-\eta$. This yields with (4.89):

$$b(t) = \frac{1+O(\sqrt{\delta})}{t^2}. \tag{4.91}$$

Step 3: Control of the scaling and the phase shift. We need to be extra careful to reintegrate the law for Γ which requires two integrations in time in the presence of $\frac{1}{t^2}$ decay only, and hence the possibility of logarithmic losses which would be dramatic to control the smallness of the phase and hence the growth of the Sobolev norm. We first integrate λ_1 from (4.86):

$$|(\lambda_1)_t| \lesssim |M_1| \lesssim \frac{\eta^{2\delta}}{t^2}$$

and hence from (4.74), (4.76):

$$\lambda_1(t) = 1 + O\left(\frac{\eta^{2\delta}}{t}\right). \tag{4.92}$$

Now consider

$$v = 1-\mu$$

. Using (4.92), we have

$$\begin{aligned}\Gamma_t &= \frac{1}{\lambda_2} - \frac{1}{\lambda_1} = \frac{1-\mu}{\lambda_2} = \frac{1-\mu}{\lambda_1(1-(1-\mu))} = v\left[1+O\left(\frac{\eta^{2\delta}}{t}\right)\right][1+O(v)] \\ &= v + O\left(\frac{\eta^\delta}{t^2}\right) + O(v^2)\ .\end{aligned}$$

and we now estimate from (4.87):

$$v^2 \lesssim K^2 \frac{(\eta^\delta + \eta t|\log \eta t|)^2}{t^2} \lesssim \frac{\eta^\delta}{t^2} + K^2\eta^2|\log \eta t|^2,$$

whence the first equation,

$$\Gamma_t = v + O\left(\frac{\eta^\delta}{t^2} + K^2\eta^2|\log \eta t|^2\right) .$$

Hence from (4.86), (4.88):

$$\begin{aligned} v_t &= -\mu_t = -\mu\left[\frac{(\lambda_2)_t}{\lambda_2} - \frac{(\lambda_1)_t}{\lambda_1}\right] = \mu\frac{M_1}{\lambda_1} - \frac{M_2}{\lambda_1} \\ &= -M_2\left[1 + O\left(\frac{\eta^{2\delta}}{t}\right)\right] + O\left(\frac{\eta^{2\delta}}{t^2}\right) = -M_2 + O\left(\frac{\eta^{2\delta}}{t^2}\right) . \end{aligned}$$

and hence from (4.88):

$$v_t = \frac{2v}{t} - \frac{2\Gamma}{t^2} + \frac{\eta}{t} + O\left(\frac{\eta^\delta}{t^2} + K^2\eta^2|\log(\eta t)|^2\right).$$

We therefore obtain the following system,

$$\left\{ \begin{array}{l} \Gamma_t = v + R_\Gamma(t), \\ v_t = \frac{2v}{t} - \frac{2\Gamma}{t^2} + \frac{\eta}{t} + R_v(t) \end{array} \right. \tag{4.93}$$

with

$$|R_\Gamma(t)| + |R_v(t)| \lesssim \frac{\eta^\delta}{t^2} + K^2\eta^2|\log(\eta t)|^2,$$

and with the initial data

$$\Gamma(T^-) = O(\eta^{10}) \, , v(T^-) = O(\eta^{10}).$$

A basis of solutions to the linear homogeneous system

$$\left\{ \begin{array}{l} \Gamma_t = v \\ v_t = \frac{2v}{t} - \frac{2\Gamma}{t^2} \end{array} \right. \tag{4.94}$$

is given by $\{(\Gamma_1(t), v_1(t)) = (t, 1), (\Gamma_2(t), v_2(t)) = (t^2, 2t)\}$, with Wronskian

$$W = v_2\Gamma_1 - \Gamma_2 v_1 = t^2$$

and hence the explicit solution with data (4.74) is given by:

$$\begin{aligned} \Gamma(t) &= \Gamma_0(t) - \Gamma_1(t)\int_t^{T^-} \frac{R_\Gamma v_2 - R_v\Gamma_2}{W}d\tau - \Gamma_2(t)\int_t^{T^-} \frac{R_v\Gamma_1 - R_\Gamma v_1}{W}d\tau, \\ v(t) &= v_0(t) - v_1(t)\int_t^{T^-} \frac{R_\Gamma v_2 - R_v\Gamma_2}{W}d\tau - v_2(t)\int_t^{T^-} \frac{R_v\Gamma_1 - R_\Gamma v_1}{W}d\tau, \end{aligned}$$

where (Γ_0, v_0) is the explicit homogeneous solution given by

$$\Gamma_0(t) = \Gamma_1(t)\left(O(\eta^{10}) + \int_t^{T^-} \frac{\eta}{\tau}\psi_2\frac{d\tau}{W}\right) - \Gamma_2(t)\left(O(\eta^{10}) + \int_t^{T^-} \frac{\eta}{\tau}\Gamma_1\frac{d\tau}{W}\right) = O\left(\eta t(|\log \eta t|)\right),$$

and

$$\begin{aligned} v_0(t) &= v_1(t)\left(O(\eta^{10}) + \int_t^{T^-} \frac{\eta}{\tau}\psi_2 \frac{d\tau}{W}\right) - v_2(t)\left(O(\eta^{10}) + \int_t^{T^-} \frac{\eta}{\tau}\Gamma_1 \frac{d\tau}{W}\right) \\ &= \int_t^{T^-} \frac{\eta}{\tau} d\tau - 2t\int_t^{T^-} \frac{\eta}{\tau^2} d\tau = \eta \log\left(\frac{T^-}{t}\right) - 2\eta t\left(\frac{1}{t} - \frac{1}{T^-}\right) \\ &= O\left(\frac{\eta t(|\log \eta t|)}{t}\right). \end{aligned}$$

We now estimate the error:

$$\begin{aligned} &\left| v_1(t)\int_t^{T^-} \frac{R_\Gamma v_2 - R_v\Gamma_2}{W} d\tau - v_2(t)\int_t^{T^-} \frac{R_v\Gamma_1 - R_\Gamma v_1}{W}\right| \\ \lesssim &\int_t^{T^-} \left[\frac{\eta^\delta}{\tau^2} + K^2\eta^2 |\log(\eta\tau)|^2\right] d\tau \lesssim \frac{\eta^\delta}{t} + K^2\eta\int_0^\delta |\log\tau|^2 d\tau \lesssim \frac{\eta^\delta}{t} + K^2\eta\delta|\log\delta|^2 \\ \lesssim &\frac{\eta^\delta}{t} + \frac{K^2\delta|\log\delta|^2\eta t|\log\eta t|}{t|\log\eta t|} \lesssim \frac{\eta^\delta}{t} + \frac{K^2\delta|\log\delta|^2}{|\log\delta|}\frac{\eta t|\log\eta t|}{t} \lesssim \frac{\eta^\delta + \eta t|\log\eta t|}{t}, \end{aligned}$$

for $\delta < \delta^*(K)$ small enough, and similarly:

$$\left|\Gamma_1(t)\int_t^{T^-} \frac{R_\Gamma v_2 - R_v\Gamma_2}{W} d\tau - \Gamma_2(t)\int_t^{T^-} \frac{R_v\Gamma_1 - R_\Gamma v_1}{W}\right| \lesssim \eta^\delta + \eta t|\log\eta t|.$$

The collection of above bounds using the modified initial data easily ensures

$$|v(t)| \lesssim \frac{\eta^\delta + \eta t|\log\eta t|}{t}, \quad |\Gamma(t)| \lesssim \eta^\delta + \eta t|\log\eta t|$$

which closes the bootstrap (4.77) for λ_2, Γ on $[T_{\text{in}}, T^-]$ for K universal large enough.

Step 4: Control of the centers and the relative distance.

We compute from (4.90), (4.91):

$$\begin{aligned} (x_2)_t - (x_1)_t &= \beta_2 - \beta_1 = 1 - \beta_1 - (1 - \beta_2) = (1 - \beta_1)(1 - b(t)) \\ &= \eta\left(1 + O\left(\frac{1}{t^2}\right)\right)\left[1 - \frac{1 + O(\sqrt{\delta})}{t^2}\right] = \eta\left(1 + O\left(\frac{1}{t^2}\right)\right). \end{aligned}$$

Hence using $(x_2 - x_1)(T^-) = \eta T^- + O(\eta^9)$ from (4.74), we obtain by integration in time:

$$(x_2 - x_1)(t) = (x_2 - x_1)(T^-) + \eta(t - T^-) + O\left(\frac{\eta}{t}\right) = \eta t + O\left(\frac{\eta}{t}\right),$$

and hence, using (4.90), (4.92):

$$\begin{aligned} \frac{R(t) - t}{t} &= \frac{x_2 - x_1}{t\lambda_1(1 - \beta_1)} - 1 = \frac{x_2 - x_1}{\eta t}(1 + O(\eta^{2\delta})) - 1 \\ &= O(\eta^{2\delta}) \leqslant \frac{1}{2}\eta^\delta \end{aligned}$$

which closes the R bound in (4.77).

□

We now come back the exact solution $\tilde{\mathcal{P}}^\infty$ of (4.72) with data (4.74) and claim that the corresponding dynamics is frozen for $t \geq T^-$.

Lemma 4.15 (Post interaction dynamics)**.** *For δ sufficiently small and $\eta < \eta^*(\delta)$, there holds on $[T^-, +\infty)$:*

$$\begin{cases} \lambda_1^\infty(t) = 1 + O(\eta), \quad \lambda_2^\infty(t) = 1 + O(\eta) \\ 1 - \beta_1^\infty(t) = \eta(1 + O(\eta^\delta)), \quad 1 - \beta_2^\infty(t) = \eta^3 e^{O(\frac{1}{\delta})}, \\ \Gamma^\infty(t) = O(t) \\ R^\infty = t(1 + O(\eta^\delta)). \end{cases} \tag{4.95}$$

Proof. We bootstrap the following bounds on $[T^-, +\infty)$,

$$\begin{cases} |1 - \lambda_1(t)| + |1 - \lambda_2| \leqslant K\eta, \\ |1 - \beta_1 - \eta| \leqslant K\eta^\delta, \quad |1 - \beta_2| \leqslant \eta^2 \\ R(t) \geqslant \frac{t}{2} \end{cases} \tag{4.96}$$

for some large enough universal constant $K = K(\delta)$, and where we omit the ∞ subscript for the sake of clarity. Notice that the notation $A \lesssim B$ in this context means $A \leq C\,B$ with a constant C independent of δ, assuming $\eta < \eta^*(\delta)$.

By (4.96) we have

$$|b| \lesssim \eta \tag{4.97}$$

and using (3.17) and (3.15), it follows for $R(1 - \beta_1) \gtrsim \delta$ that

$$\begin{aligned} |Q_{\beta_1}(R)| &\lesssim \frac{1}{\eta t^2}, \\ |Q'_{\beta_1}(R)| &\lesssim \frac{1}{t^2}, \\ \left|Q_{\beta_2}\left(-\frac{R}{b\mu}\right)\right| &\lesssim \frac{b}{(1-\beta_1)t^2} \lesssim \frac{1}{t^2}. \end{aligned}$$

We may therefore estimate in brute force the parameters using Proposition 4.12:

$$\begin{aligned} |B_1| &\lesssim \frac{\eta}{t^2} + \frac{K\eta^2}{\eta t^2} \lesssim \frac{1}{t^2} \\ |B_2| &\lesssim \frac{1}{\eta t^2} + \frac{K\eta}{\eta t^2} \lesssim \frac{1}{\eta t^2} \\ |M_1| &\lesssim \frac{(1-\beta_1)|\log(1-\beta_1)|}{t^2} + \frac{K\eta^2}{\eta t^2} + \frac{K\eta}{t^2} \lesssim \frac{\eta|\log\eta|}{t^2} \\ |M_2| &\lesssim \frac{1}{t^2} + \frac{|1-\mu|}{\eta t^2}\ . \end{aligned}$$

We therefore control the speeds on $[T^-, +\infty)$ using (4.77):

$$\begin{aligned} \left|\frac{(\beta_1)_t}{1-\beta_1}\right| \lesssim |B_1| \lesssim \frac{1}{t^2}, \quad \text{i.e.} \quad 1 - \beta_1(t) = \eta e^{O\left(\frac{1}{T^-}\right)} = \eta(1 + O(\eta)) \\ \left|\frac{(\beta_2)_t}{1-\beta_2}\right| \lesssim |B_2| \lesssim \frac{1}{\eta t^2}, \quad \text{i.e.} \quad 1 - \beta_2(t) = \frac{\eta}{(T^-)^2} e^{O\left(\frac{1}{\eta T^-}\right)} = \eta^3 e^{O(\frac{1}{\delta})} \end{aligned}$$

and similarly for the first size,

$$\left|\frac{(\lambda_1)_t}{\lambda_1}\right| \lesssim |M_1| \lesssim \frac{\sqrt{\eta}}{t^2}, \quad \text{i.e.} \quad \lambda_1(t) = 1 + O(\eta).$$

Hence:

$$|\mu_t| \lesssim |M_2| + |M_1| \lesssim \frac{|1-\mu|}{\eta t^2} + \frac{1}{t^2}$$

from which we infer, using $\mu(T^-)=1$,

$$|1-\mu(t)| \lesssim \frac{1}{T^-} + \int_{T^-}^{t} \frac{|1-\mu(\tau)|}{\eta\tau^2} d\tau.$$

By Gronwall's lemma, we conclude

$$|1-\mu(t)| \lesssim \frac{1}{T^-} e^{O\left(\frac{1}{\eta T^-}\right)} = \eta\, e^{O(\frac{1}{\delta})}\ .$$

Hence the control of scalings and speeds is closed for $K=K(\delta)$ large enough in (4.96). We now integrate the position.

$$(x_2)_t - (x_1)_t = \beta_2 - \beta_1 = 1-\beta_1-(1-\beta_2) = \eta(1+O(\eta))$$

from which we get

$$x_2(t)-x_1(t) = \eta(1+O(\eta))(t-T^-) + \eta T^- = \eta t + O(\eta^2 t)$$

and

$$R(t) = \frac{x_2-x_1}{t\lambda_1(1-\beta_1)} \geqslant \frac{2}{3},$$

which concludes the proof of Lemma 4.15. □

5. Energy estimates

This section is devoted to the construction of an exact solution to (1.1) with two-soliton asymptotic behavior and transient turbulent regime. The strategy is based as in [27, 40] on an energy method near the explicit approximate solution which can be closed thanks to the arbitrary high order expansion of the approximate solution, and the $R(t)\sim t$ distance between the two waves.

5.1. Backwards integration and parametrization of the flow.

Given parameters

$$\mathcal{P} = (\lambda_1,\lambda_2,\beta_1,\beta_2,\Gamma,R), \quad \widetilde{\mathcal{P}} = (\mathcal{P},x_1,x_2,\gamma_1,\gamma_2),$$

we let

$$\Phi^{(N)}_{\widetilde{\mathcal{P}}}(x) = \Phi^{(N,1)}_{\widetilde{\mathcal{P}}}(x) + \Phi^{(N,2)}_{\widetilde{\mathcal{P}}}(x)$$

with

$$\Phi^{(N,j)}_{\widetilde{\mathcal{P}}}(x) = \frac{1}{\lambda_j^{\frac12}} V_j^{(N)}(y_j,\mathcal{P}) e^{i\gamma_j}, \quad y_j = \frac{x-x_j}{\lambda_j(1-\beta_j)}, \quad j=1,2,$$

constructed in Proposition 4.6. We now fix one and for all a large enough number $N\gg 1$, and for the rest of the paper, we omit the subscript N in order to ease notations. We then pick a small enough universal constant $\delta>0$ and, for $0<\eta<\eta^*(\delta)$, we consider

$$\widetilde{\mathcal{P}}^\infty = (\lambda_1^\infty,\lambda_2^\infty,\gamma_1^\infty,\gamma_2^\infty,x_1^\infty,x_2^\infty)$$

to be the exact solution to (4.72) with data (4.74) which is well defined on $[T^-,+\infty)$ from Lemma 4.15.

We now build an exact solution to the full system (1.1) by integrating backwards in time from $+\infty$: we let a sequence $T_n\to+\infty$ and consider $u_n(t)$ the solution to

$$\begin{cases} i\partial_t u_n = |D|u_n - |u_n|^2 u_n, \\ u_n(T_n) = \Phi_{\widetilde{\mathcal{P}}^\infty(T_n)}(x). \end{cases} \tag{5.1}$$

We will very precisely study the properties of $u_n(t)$. Here and in the sequel, we omit as much as possible the subscript n to ease notations.

From standard modulation argument, as the solution remains close in $H^{\frac12}$ to a

modulated tube around the decoupled two solitary waves , we may consider a decomposition of the flow

$$u(t,x) = \Phi_{\widetilde{\mathcal{P}}(t)}(x) + \varepsilon(t,x) \tag{5.2}$$

where the parameters

$$\widetilde{\mathcal{P}}(t) = (\lambda_1(t), \lambda_2(t), \beta_1(t), \beta_2(t), x_1(t), x_2(t), \Gamma(t), R(t)) \ ,$$

with the explicit dependence

$$\Gamma = \gamma_2 - \gamma_1, \quad R = \frac{x_2 - x_1}{\lambda_1(1-\beta_1)} \ , \tag{5.3}$$

are chosen for each fixed t in order to manufacture suitable orthogonality conditions on the remainders

$$\varepsilon_j(t, y_j) := \lambda_j^{\frac{1}{2}}(t)\varepsilon\big(t, \lambda_j(t)(1-\beta_j(t))y_j + x_j(t)\big)e^{-i\gamma_j(t)}, \quad j = 1, 2. \tag{5.4}$$

Observe that

$$\|\varepsilon\|_{L^2}^2 = (1-\beta_j)\|\varepsilon_j\|_{L^2}^2, \quad j = 1, 2. \tag{5.5}$$

Let ω be the symplectic form

$$\omega(f,g) = Im \int f\bar{g}dx = (f, ig),$$

and consider the generalized null space of the operator $i\mathcal{L}_\beta$ formed of functions $f \in H^{1/2}$ such that $(i\mathcal{L}_\beta)^2 f = 0$. This generalized null subspace consists of iQ_β, $\partial_y Q_\beta$, ΛQ_β, and $i\rho_\beta$, where ρ_β is the unique $H^{\frac{1}{2}}$ solution to the problem (3.9). Indeed, one can directly check that $i\mathcal{L}_\beta(iQ_\beta) = i\mathcal{L}_\beta(\partial_y Q_\beta) = 0$ and

$$(i\mathcal{L}_\beta)^2(\Lambda Q_\beta) = (i\mathcal{L}_\beta)^2(i\rho_\beta) = 0.$$

We then impose the set of symplectic orthogonality conditions:

$$\omega(\varepsilon_j, iQ_{\beta_j}) = \omega(\varepsilon_j, \partial_{y_j} Q_{\beta_j}) = \omega(\varepsilon_j, \Lambda Q_{\beta_j}) = \omega(\varepsilon_j, i\rho_j) = 0, \quad j = 1, 2,$$

or equivalently,

$$(\varepsilon_j, Q_{\beta_j}) = (\varepsilon_j, i\partial_{y_j} Q_{\beta_j}) = (\varepsilon_j, i\Lambda Q_{\beta_j}) = (\varepsilon_j, \rho_j) = 0, \quad j = 1, 2. \tag{5.6}$$

Let $\sigma_j := (\lambda_j, x_j, \gamma_j, \beta_j)$, $j = 1, 2$ and Σ be a compact subset of

$$\big(\mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R} \times (1-\beta_*, 1)\big)^2.$$

For $(\sigma_1, \sigma_2) \in \Sigma$ and $f \in H^{1/2}$, we define

$$\mathcal{S}_{\sigma_j} f(x) = \frac{1}{\lambda_j^{1/2}} f\Big(\frac{x - x_j}{\lambda_j(1-\beta_j)}\Big)e^{i\gamma_j}.$$

The existence and uniqueness for each t of $\widetilde{\mathcal{P}}(t)$ ensuring the decomposition (5.2), (5.6) is now a standard consequence of the implicit function theorem applied to the function $G : H^{1/2} \times \Sigma \to \mathbb{R}^8$, $G(\psi, \sigma) = 0$, where G is defined by

$$G(\psi,\sigma) = \begin{pmatrix} (\psi - \mathcal{S}_{\sigma_1}V_1(\mathcal{P}) - \mathcal{S}_{\sigma_2}V_2(\mathcal{P}), \mathcal{S}_{\sigma_1}Q_{\beta_1}) \\ (\psi - \mathcal{S}_{\sigma_1}V_1(\mathcal{P}) - \mathcal{S}_{\sigma_2}V_2(\mathcal{P}), \mathcal{S}_{\sigma_1}i\partial_x Q_{\beta_1}) \\ (\psi - \mathcal{S}_{\sigma_1}V_1(\mathcal{P}) - \mathcal{S}_{\sigma_2}V_2(\mathcal{P}), \mathcal{S}_{\sigma_1}i\Lambda Q_{\beta_1}) \\ (\psi - \mathcal{S}_{\sigma_1}V_1(\mathcal{P}) - \mathcal{S}_{\sigma_2}V_2(\mathcal{P}), \mathcal{S}_{\sigma_1}\rho_{\beta_1}) \\ (\psi - \mathcal{S}_{\sigma_1}V_1(\mathcal{P}) - \mathcal{S}_{\sigma_2}V_2(\mathcal{P}), \mathcal{S}_{\sigma_2}Q_{\beta_2}) \\ (\psi - \mathcal{S}_{\sigma_1}V_1(\mathcal{P}) - \mathcal{S}_{\sigma_2}V_2(\mathcal{P}), \mathcal{S}_{\sigma_2}i\partial_x Q_{\beta_2}) \\ (\psi - \mathcal{S}_{\sigma_1}V_1(\mathcal{P}) - \mathcal{S}_{\sigma_2}V_2(\mathcal{P}), \mathcal{S}_{\sigma_2}i\Lambda Q_{\beta_2}) \\ (\psi - \mathcal{S}_{\sigma_1}V_1(\mathcal{P}) - \mathcal{S}_{\sigma_2}V_2(\mathcal{P}), \mathcal{S}_{\sigma_2}\rho_{\beta_2}) \end{pmatrix},$$

where $\sigma = (\sigma_1, \sigma_2)$ and $\mathcal{P} = (\lambda_1, \lambda_2, \beta_1, \beta_2, \Gamma, R)$. The key ingredient here is that, for any $(\sigma_1^0, \sigma_2^0) \in \Sigma$, the Jacobian matrix

$$\partial_\sigma G\big(\mathcal{S}_{\sigma_1^0} V_1^{(N)} + \mathcal{S}_{\sigma_2^0} V_2^{(N)}, \sigma\big)\Big|_{\sigma=(\sigma_1^0,\sigma_2^0)}$$

is invertible, which follows from the fact that the matrix

$$A_j = \begin{pmatrix} (\Lambda Q_{\beta_j}, Q_{\beta_j}) & (\Lambda Q_{\beta_j}, i\partial_{y_j} Q_{\beta_j}) & (\Lambda Q_{\beta_j}, i\Lambda Q_{\beta_j}) & (\Lambda Q_{\beta_j}, \rho_j) \\ (iQ_{\beta_j}, Q_{\beta_j}) & (iQ_{\beta_j}, i\partial_{y_j} Q_{\beta_j}) & (iQ_{\beta_j}, i\Lambda Q_{\beta_j}) & (iQ_{\beta_j}, \rho_j) \\ (\partial_{y_j} Q_{\beta_j}, Q_{\beta_j}) & (\partial_{y_j} Q_{\beta_j}, i\partial_{y_j} Q_{\beta_j}) & (\partial_{y_j} Q_{\beta_j}, i\Lambda Q_{\beta_j}) & (\partial_{y_j} Q_{\beta_j}, \rho_j) \\ (\Sigma_j, Q_{\beta_j}) & (\Sigma_j, i\partial_{y_j} Q_{\beta_j}) & (\Sigma_j, i\Lambda Q_{\beta_j}) & (\Sigma_j, \rho_j) \end{pmatrix}$$

with

$$\Sigma_j := y\partial_y Q_{\beta_j} + (1-\beta_j)\partial_{\beta_j} Q_{\beta_j} \tag{5.7}$$

is non degenerate

$$\lim_{\beta_j \to 1} |\det A_j| \neq 0, \quad j = 1, 2, \tag{5.8}$$

see Appendix C.

5.2. **Localized $H^{\frac{1}{2}}$-energy.** The heart of our analysis is the derivation of a suitable monotonicity formula for a *suitable localized $H^{\frac{1}{2}}$ energy identity*. The localization procedure is mandatory in order to dynamically adapt the functional to the dramatically changing size of the bubble, but this will lead to serious difficulties due to nonlocal nature of the problem and the slow decay of the solitary wave. The limiting Szegő problem will arise in the form of various different estimates for $\Pi^{\pm}\varepsilon$ which will be essential to close the estimates.

Let us start by introducing suitable cut-off functions which adapt the energy functional to the dramatic change of size of the second solitary wave.

Space localization. We pick explicitly a sufficiently smooth non increasing function

$$\Psi_1(z_1) = \begin{cases} 1 \quad \text{for} \quad z_1 \leqslant \frac{1}{4} \\ (1-z_1)^{10} \quad \text{for} \quad \frac{1}{2} \leqslant z_1 \leqslant 1 \\ 0 \quad \text{for} \quad z_1 \geqslant 1. \end{cases} . \tag{5.9}$$

and let

$$\Phi_1(t, z_1) = \Psi_1 + b(t)(1-\Psi_1) = \begin{cases} 1 \quad \text{for} \quad z_1 \leqslant \frac{1}{4} \\ b(t) \quad \text{for} \quad z_1 \geqslant 1. \end{cases} \tag{5.10}$$

From this function of (t, z_1) we deduce a function of (t, y_1) and (t, x) via the following change of variables,

$$\phi(t, x) = \phi_1(t, y_1) = \Phi_1(t, z_1), \quad z_1 = \frac{y_1}{R(t)(1-b(t))}.$$

We then define the localization associated to kinetic momentum

$$\zeta(t, x) = \beta_1(t) + (1-\beta_1(t))(1-\phi(t, x)), \tag{5.11}$$

so that

$$\zeta(t, x) = \zeta_1(t, y_1) = \begin{cases} \beta_1(t) \quad \text{for} \quad y_1 \leqslant \frac{(1-b(t))R(t)}{4} \\ \beta_2(t) \quad \text{for} \quad y_1 \geqslant (1-b(t))R(t). \end{cases} . \tag{5.12}$$

similarly, let

$$\widetilde{\Phi_1}(t, z_1) = \mu(t)\Psi_1(z_1) + (1-\Psi_1(z_1)) = \begin{cases} \mu(t) \quad \text{for} \quad z_1 \leqslant \frac{1}{4} \\ 1 \quad \text{for} \quad z_1 \geqslant 1, \end{cases} \tag{5.13}$$

with the same change of variables as before,

$$\tilde{\phi}(t,x) = \tilde{\phi}_1(t,y_1) = \widetilde{\Phi_1}(t,z_1), \quad z_1 = \frac{y_1}{R(t)(1-b(t))} .$$

We define the localization attached to the localization of mass,

$$\theta(t,x) = \frac{1}{\lambda_2(t)}\tilde{\phi}(t,x) = \theta_1(t,y_1), \tag{5.14}$$

so that

$$\theta(t,y_1) = \begin{cases} \frac{1}{\lambda_1(t)} & \text{for} \quad y_1 \leqslant \frac{(1-b(t))R(t)}{4} \\ \frac{1}{\lambda_2(t)} & \text{for} \quad y_1 \geqslant (1-b(t))R(t) \end{cases} .$$

Explicit estimates used throughout the proof involving functions ζ, θ are stated in Appendix E.

Localized energy. We now introduce the localized energy functional:

$$\begin{aligned} \mathcal{G}(\varepsilon) : &= \frac{1}{2}\left[(|D|\varepsilon - \zeta D\varepsilon, \varepsilon) + (\theta\varepsilon, \varepsilon)\right] \\ &- \frac{1}{4}\left[\int_{\mathbb{R}} (|\varepsilon+\Phi|^4 - |\Phi|^4)dx - 4(\varepsilon, \Phi|\Phi|^2)\right] \end{aligned} \tag{5.15}$$

Notice that the inner products are taken in the x variable, and that Φ denotes the approximate solution $\Phi_{\widetilde{\mathcal{P}}(t)}$. This functional will be used as our main energy functional. We indeed first claim that $\mathcal{G}$ is a coercive functional.

Proposition 5.1 (Coercivity of the localized energy)**.** *There holds*[5] *:*

$$\mathcal{G}(\varepsilon) \gtrsim (1-\beta_1)\left[\int |\varepsilon_1|^2 dy_1 + \int \phi_1 ||D|^{\frac{1}{2}}\varepsilon_1^+|^2 dy_1\right] + \int ||D|^{\frac{1}{2}}\varepsilon_1^-|^2 dy_1 \tag{5.16}$$

where ε_1 *was defined in* (5.4)*.*

The proof adapts the argument in [33] and relies on a careful localization of the kinetic energy and the coercivity of the limiting Szegő quadratic form. A key fact is that the relative distance R between the solitary waves is always large. The presence of the localization ϕ_1 in (5.16) is an essential difficulty of the analysis and shows that one looses control of $\|D^{\frac{1}{2}}\varepsilon^+\|_{L^2}$ as $\beta_1 \to 1$ (through the factor $1-\beta_1$), which reflects the singular nature of the bifurcation $Q^+ \to Q_\beta$. This will be a fundamental issue for the forthcoming analysis. The proof of Proposition 5.1 is detailed in Appendix F.

5.3. **Bootstrap argument.** Since $\varepsilon(T_n) = 0$ and $\mathcal{P}(T_n) = \mathcal{P}^\infty(T_n)$, we run a bootstrap argument in the following form. Let

$$\tilde{\beta}_j := \log(1-\beta_j) \tag{5.17}$$

and

$$|\Delta\lambda_j|(t) := \sup_{\tau\in[t,T_n]} |\lambda_j - \lambda_j^\infty|(\tau), \quad |\Delta\tilde{\beta}_j|(t) := \sup_{\tau\in[t,T_n]} |\tilde{\beta}_j - \tilde{\beta}_j^\infty|(\tau), \tag{5.18}$$

$$|\Delta R|(t) := \sup_{\tau\in[t,T_n]} |R - R^\infty|(\tau), \quad |\Delta\Gamma|(t) := \sup_{\tau\in[t,T_n]} |\Gamma - \Gamma^\infty|(\tau), \tag{5.19}$$

[5] for some universal coercivity constant which is related to the coercivity of the limiting Szegő functional (2.17).

we assume on some interval $[T, T_n]$, with $T_{\rm in} \le T \le T_n$, the H^1-bounds:

$$\forall t \in [T_{\rm in}, T_n], \quad \begin{cases} \mathcal{G}(\varepsilon(t)) \leqslant \frac{1}{t^{\frac{N}{2}}} \\ \|\varepsilon(t)\|_{H^1}^2 \leqslant \frac{1}{t^{\frac{N}{4}}} \end{cases} \tag{5.20}$$

and the bounds on the parameters:
1. post interaction estimates: for $t \in [T^-, T_n] \cap [T, T_n]$,

$$\begin{cases} |\Delta R| \leqslant \frac{1}{t^{\frac{N}{8}-1}} \\ |\Delta\tilde{\beta}_j| + |\Delta\Gamma| \leqslant \frac{1}{t^{\frac{N}{8}}}, \\ \sum_{j=1,2} |\Delta\lambda_j| \leqslant \frac{1}{t^{\frac{N}{8}+1}}; \end{cases} \tag{5.21}$$

2. rough turbulent bounds: for $t \in [T_{\rm in}, T^-] \cap [T, T_n]$,

$$\begin{cases} |\lambda_1 - 1| + |\lambda_2 - 1| \leqslant \frac{1}{t} \\ \frac{\eta}{2} \leqslant 1 - \beta_1(t) \leqslant 2\eta, \quad \frac{1}{2} \leqslant t^2 b(t) \leqslant 2 \\ |\Gamma(t)| \leqslant \sqrt{\delta} \\ \frac{t}{2} \leqslant R \leqslant 2t. \end{cases} \tag{5.22}$$

The heart of our analysis is that all these bounds can be improved.

Proposition 5.2 (Bootstrap)**.** *For $N \geqslant N^*$ large enough and $0 < \eta < \eta^*(N)$ small enough, the following holds:*

$$\forall t \in [T, T_n], \quad \begin{cases} \mathcal{G}(\varepsilon(t)) \lesssim \frac{1}{Nt^{\frac{N}{2}}} \\ \|\varepsilon(t)\|_{H^1}^2 \lesssim \frac{1}{Nt^{\frac{N}{4}}} \end{cases} \tag{5.23}$$

and the bounds on the parameters:
1. post interaction estimates: for $t \in [T^-, T_n] \cap [T, T_n]$,

$$\begin{cases} |\Delta R| \lesssim \frac{1}{Nt^{\frac{N}{8}-1}} \\ |\Delta\tilde{\beta}_j| + |\Delta\Gamma| \lesssim \frac{1}{Nt^{\frac{N}{8}}}, \\ \sum_{j=1,2} |\Delta\lambda_j| \lesssim \frac{1}{Nt^{\frac{N}{8}+1}}; \end{cases} \tag{5.24}$$

2. rough turbulent bounds: for $t \in [T_{\rm in}, T^-] \cap [T, T_n]$, $\mathcal{P}$ satisfies (4.77).

Of course, the bounds (5.23), (5.24), (4.77) improve on (5.20), (5.21), (5.22) for N universal large enough, so that we can finally set $T = T_{\rm in}$. Proposition 5.2 is the heart of the analysis and implies Theorem 1.2 through a now classical argument which we detail in Subsection 5.8 for the convenience of the reader.

From now until Subsection 5.8, we assume the bounds (5.20), (5.21), (5.22) and aim at improving them. Since $t \geqslant T_{in} = \frac{1}{\eta^{2\delta}}$, we will systematically use the bound

$$\frac{1}{\eta^C t^{\sqrt{N}}} \leqslant 1 \quad \text{for} \quad N \geqslant N(\delta), \ \eta < \eta^*(N).$$

Let us also observe from (5.21), (5.22), (4.95) injected into Proposition 4.12 the bounds: $\forall t \in [T_{\rm in}, T_n]$,

$$|B_1| + |M_1| \lesssim \frac{b}{t}, \quad |B_2| \lesssim \frac{1}{t}, \quad |M_2| \lesssim \frac{1}{t^2}. \tag{5.25}$$

5.4. **Equation for ε.** Let us start by writing the equation for ε. Using $\frac{ds_j}{dt} = \frac{1}{\lambda_j}$, we compute from (5.3) the generalized modulation equations:

$$\Gamma_{s_1} = \frac{(\gamma_2)_{s_2}}{\mu} - (\gamma_1)_{s_1} = \frac{1}{\mu} - 1 + \frac{(\gamma_2)_{s_2} - 1}{\mu} - ((\gamma_1)_{s_1} - 1), \quad \Gamma_{s_2} = \mu \Gamma_{s_1} \tag{5.26}$$

and

$$\begin{aligned} R_{s_1} &= 1 - b + (B_1 - M_1)R + \frac{1}{1-\beta_1}\left(\frac{(x_2)_{s_2}}{\lambda_2} - \beta_2\right) - \frac{1}{1-\beta_1}\left(\frac{(x_1)_{s_1}}{\lambda_1} - \beta_1\right) \\ &- R\left(\frac{(\lambda_1)_{s_1}}{\lambda_1} - M_1\right) + R\left(\frac{(\beta_1)_{s_1}}{1-\beta_1} - B_1\right) \end{aligned} \tag{5.27}$$

We compute by construction:

$$i\partial_t \Phi_{\tilde{P}} - |D|\Phi_{\tilde{P}} + \Phi_{\tilde{P}}|\Phi_{\tilde{P}}|^2 = \Psi + \sum_{j=1}^{2} \frac{1}{\lambda_j^{\frac{3}{2}}} S_j\left(t, y_j\right) e^{i\gamma_j}, \quad j = 1, 2,$$

where

$$\begin{aligned} S_j(t, y_j) &:= -i\left[\frac{(\lambda_j)_{s_j}}{\lambda_j} - M_j\right]\Lambda V_j - \frac{1}{1-\beta_j}\left[\frac{(x_j)_{s_j}}{\lambda_j} - \beta_j\right] i\partial_{y_j} V_j \\ &+ \left[\frac{(\beta_j)_{s_j}}{1-\beta_j} - B_j\right] i[y_j \partial_{y_j} V_j + (1-\beta_j)\partial_{\beta_j} V_j] - [(\gamma_j)_{s_1} - 1]V_j \\ &+ \tilde{S}_j \end{aligned} \tag{5.28}$$

encodes the deviation of modulation equations from the idealized dynamical system (4.72) with the lower order error computed from (5.26):

$$\begin{aligned} \tilde{S}_1 &:= i\left[\frac{\gamma_{s_2} - 1}{\mu} - (\gamma_{s_1} - 1)\right]\frac{\partial V_1}{\partial \Gamma} \\ &+ i\Bigg\{\frac{1}{1-\beta_1}\left(\frac{x_{s_2}}{\lambda_2} - \beta_2\right) - \frac{1}{1-\beta_1}\left(\frac{x_{s_1}}{\lambda_1} - \beta_1\right) - R\left(\frac{(\lambda_1)_{s_1}}{\lambda_1} - M_1\right) \\ &+ R\left(\frac{(\beta_1)_{s_1}}{1-\beta_1} - B_1\right)\Bigg\}\frac{\partial V_1}{\partial R} \\ &+ i\lambda_1\left[\frac{(\lambda_1)_{s_1}}{\lambda_1} - M_1\right]\frac{\partial V_1}{\partial \lambda_1} + i\lambda_1\left[\frac{(\lambda_2)_{s_2}}{\lambda_2} - M_2\right]\frac{\partial V_1}{\partial \lambda_2} \\ &+ i\frac{(1-\beta_2)}{\mu}\left[\frac{(\beta_2)_{s_2}}{1-\beta_2} - B_2\right]\frac{\partial V_1}{\partial \beta_2} \end{aligned} \tag{5.29}$$

$$\begin{aligned} \tilde{S}_2 &:= i\left[\gamma_{s_2} - 1 - \mu(\gamma_{s_1} - 1)\right]\frac{\partial V_2}{\partial \Gamma} \\ &+ i\mu\Bigg\{\frac{1}{1-\beta_1}\left(\frac{x_{s_2}}{\lambda_2} - \beta_2\right) - \frac{1}{1-\beta_1}\left(\frac{x_{s_1}}{\lambda_1} - \beta_1\right) - R\left(\frac{(\lambda_1)_{s_1}}{\lambda_1} - M_1\right) \\ &+ R\left(\frac{(\beta_1)_{s_1}}{1-\beta_1} - B_1\right)\Bigg\}\frac{\partial V_2}{\partial R} \\ &+ i\lambda_2\left[\frac{(\lambda_1)_{s_1}}{\lambda_1} - M_1\right]\frac{\partial V_2}{\partial \lambda_1} + i\lambda_2\left[\frac{(\lambda_2)_{s_2}}{\lambda_2} - M_2\right]\frac{\partial V_2}{\partial \lambda_2} \\ &+ i\mu(1-\beta_1)\left[\frac{(\beta_1)_{s_1}}{1-\beta_1} - B_1\right]\frac{\partial V_2}{\partial \beta_1}. \end{aligned} \tag{5.30}$$

The error term

$$\Psi(t,x)=\sum_{j=1}^{2}\frac{1}{\lambda_j^{\frac{3}{2}}}\mathcal{E}_{j,N}\left(y_j,\mathcal{P}(t)\right)e^{i\gamma_j} \tag{5.31}$$

encodes the error in the construction of V_j and satisfies by construction

$$\|\Psi\|_{H^2}\leqslant\frac{C_N}{\eta^{C_0}R^{N+1}}\leqslant\frac{1}{\eta^{C}t^{N+1}}, \tag{5.32}$$

where we recall that N will be fixed later and $\eta<\eta^*(N)$. We write the equation for ε,

$$i\partial_t\varepsilon-|D|\varepsilon+2|\Phi_{\tilde{P}}|^2\varepsilon+(\Phi_{\tilde{P}})^2\overline{\varepsilon}=-N(\varepsilon)-\Psi-\sum_{j=1}^{2}\frac{1}{\lambda_j^{\frac{3}{2}}}S_j\left(t,y_j\right)e^{i\gamma_j}, \tag{5.33}$$

where

$$N(\varepsilon)=(\Phi_{\tilde{P}}+\varepsilon)|\Phi_{\tilde{P}}+\varepsilon|^2-\Phi_{\tilde{P}}|\Phi_{\tilde{P}}|^2-2|\Phi_{\tilde{P}}|^2\varepsilon-(\Phi_{\tilde{P}})^2\overline{\varepsilon}.$$

In the sequel, we use the notation

$$j+1=1\quad\text{for}\quad j=2.$$

5.5. **Modulation equations.** At this stage we can evaluate the right hand side of the modulation system applied to the parameters $\mathcal{P}(t)$ given by the modulation argument.

Lemma 5.3 (Modulation equations)**.** *Let*

$$\mathrm{Mod}_j(t):=\left|\frac{(\lambda_j)_{s_j}}{\lambda_j}-M_j\right|+\frac{1}{1-\beta_j}\left|\frac{(x_j)_{s_j}}{\lambda_j}-\beta_j\right|+\left|\frac{(\beta_j)_{s_j}}{1-\beta_j}-B_j\right|+\left|(\gamma_j)_{s_j}-1\right|,$$

then

$$\mathrm{Mod}_j(t)\lesssim\frac{1}{\eta^{C}t^{N+1}}+\frac{\|\varepsilon_j\|_{L^2}}{t}. \tag{5.34}$$

Proof of Lemma 5.3. Let $j=1$ or $j=2$ and consider a generic multiplier

$$\Theta(t,x)=\frac{1}{\lambda_j^{\frac{1}{2}}}\Theta_j(y_j,\beta_j)e^{i\gamma_j}, \tag{5.35}$$

with Θ_j strongly j-admissible. We compute from (5.33):

$$\begin{aligned}\frac{d}{dt}(\varepsilon,\Theta)&=(\varepsilon,\partial_t\Theta)+(i\partial_t\varepsilon,i\Theta)=(\varepsilon,-i\partial_t(i\Theta)+|D|(i\Theta)-2|\Phi_{\tilde{\mathcal{P}}}|^2(i\Theta)-(\Phi_{\tilde{\mathcal{P}}})^2\overline{i\Theta}))\\&\quad-\left(N(\varepsilon)+\Psi+\Sigma_{k=1}^{2}\frac{1}{\lambda_k^{\frac{3}{2}}}S_k\left(y_k\right)e^{i\gamma_k},i\Theta\right)\end{aligned} \tag{5.36}$$

and estimate all terms in this identity.

The linear terms. Using the fact that M_j, B_j are L^∞-admissible, we estimate:

$$\begin{aligned}
i\partial_t\Theta - |D|\Theta &= \frac{1}{\lambda_j^{\frac{3}{2}}}\Big[-\frac{(|D|-\beta_j D)\Theta_j}{1-\beta_j} - \Theta_j - i\frac{(\lambda_j)_{s_j}}{\lambda_j}\Lambda\Theta_j - \frac{i}{1-\beta_j}\left(\frac{(x_j)_{s_j}}{\lambda_j} - \beta_j\right)\partial_{y_j}\Theta_j \\
&\quad + i\frac{(\beta_j)_{s_j}}{1-\beta_j}[y_j\partial_{y_j}\Theta_j + (1-\beta_j)\partial_{\beta_j}\Theta_j] - ((\gamma_j)_{s_j}-1)\Theta_j\Big]e^{i\gamma_j}(y_j) \\
&= -\frac{1}{\lambda_j^{\frac{3}{2}}}\left[\frac{(|D|-\beta_j D)\Theta_j}{1-\beta_j} + \Theta_j\right]e^{i\gamma_j}(y_j) \\
&\quad + O\left(\mathrm{Mod}_j(t)\big(|\Lambda\Theta_j(y_j)| + |\partial_{y_j}\Theta(y_j)| + |y_j\partial_{y_j}\Theta_j| + |(1-\beta_j)\partial_{\beta_j}\Theta_j| + |\Theta_j(y_j)|\big)\right) \\
&\quad + O\Big(|M_j||\Lambda\Theta_j| + |B_j||y_j\partial_{y_j}\Theta_j + (1-\beta_j)\partial_{\beta_j}\Theta_j| \\
&= -\frac{1}{\lambda_j^{\frac{3}{2}}}\left[\frac{(|D|-\beta_j D)\Theta_j}{1-\beta_j} + \Theta_j\right]e^{i\gamma_j}(y_j) \\
&\quad + \Big(\mathrm{Mod}_j(t) + \frac{1}{t}\Big)O\Big(|\Theta_j| + |\partial_{y_j}\Theta_j| + |\Lambda\Theta_j| + |(1-\beta_j)\partial_{\beta_j}\Theta_j|\Big).
\end{aligned}$$

Then, changing to the y_j variable, using the definition of ε_j in (5.4), and Cauchy–Schwarz, we have:

$$\begin{aligned}
(\varepsilon, -i\partial_t(i\Theta) + |D|i\Theta - 2|\Phi_{\tilde{\mathcal{P}}}|^2(i\Theta) - (\Phi_{\tilde{\mathcal{P}}})^2\overline{(i\Theta)}) &= \Big(\varepsilon, \frac{1}{\lambda_j^{\frac{3}{2}}}\left[\mathcal{L}_{\beta_j}(i\Theta_j)\right]e^{i\gamma_j}(y_j)\Big) \\
&+ (1-\beta_j)\Big(\mathrm{Mod}_j(t)\|\varepsilon_j\|_{L^2} + \frac{\|\varepsilon_j\|_{L^2}}{t}\Big) \\
&\times O\Big(\|\Theta_j\|_{L^2} + \|\partial_{y_j}\Theta_j\|_{L^2} + \|\Lambda\Theta_j\|_{L^2} + \|(1-\beta_j)\partial_{\beta_j}\Theta_j\|_{L^2}\Big) \\
&+ (1-\beta_j)O\Big(\|(|V_j|^2 - |Q_{\beta_j}|^2)\Theta_j\|_{L^2}\|\varepsilon_j\|_{L^2}\Big) \\
&+ (1-\beta_j)O\Big(\||V_{j+1}|^2\Theta_j\|_{L^2}\|\varepsilon_j\|_{L^2}\Big) \\
&+ (1-\beta_j)O\Big(\|V_jV_{j+1}\Theta_j\|_{L^2}\|\varepsilon_j\|_{L^2}\Big)
\end{aligned}$$

with the convention $y_{j+1} = y_1$ for $j = 2$. To estimate the remainder, we estimate using that $R(V_j - Q_{\beta_j})$ is j-admissible:

$$\|(|V_j|^2 - |Q_{\beta_j}|^2)\Theta_j\|_{L^2} \lesssim \|\frac{1}{R\langle y_j\rangle}\|_{L^2} \lesssim \frac{1}{t}.$$

We now use

$$y_1 = R + b\mu y_2 \tag{5.37}$$

so that $|y_1| \leqslant \frac{R}{2}$ implies $|y_2| \geqslant \frac{R}{2\mu b}$ and hence the bounds

$$\begin{aligned}
\int \frac{dy_1}{\langle y_1\rangle^2\langle y_2\rangle^4} &= \int_{|y_1|\leqslant\frac{R}{2}} \frac{dy_1}{\langle y_1\rangle^2\langle y_2\rangle^4} + \int_{|y_1|\geqslant\frac{R}{2}} \frac{dy_1}{\langle y_1\rangle^2\langle y_2\rangle^4} \\
&\lesssim \frac{b^4}{R^4}\int_{|y_1|\leqslant\frac{R}{2}} \frac{dy_1}{\langle y_1\rangle^2} + \frac{1}{R^2}\int \frac{bdy_2}{\langle y_2\rangle^4} \lesssim \frac{b}{R^2} \lesssim \frac{b}{t^2},
\end{aligned}$$

$$\begin{aligned}\int \frac{dy_1}{\langle y_2\rangle^2\langle y_1\rangle^4} &= \int_{|y_1|\leqslant \frac{R}{2}} \frac{dy_1}{\langle y_2\rangle^2\langle y_1\rangle^4} + \int_{|y_1|\geqslant \frac{R}{2}} \frac{dy_1}{\langle y_2\rangle^2\langle y_1\rangle^4} \\ &\lesssim \frac{b^2}{R^2}\int_{|y_1|\leqslant \frac{R}{2}} \frac{dy_1}{\langle y_1\rangle^4} + \frac{1}{R^4}\int \frac{b\,dy_2}{\langle y_2\rangle^2} \lesssim \frac{1}{t^2},\end{aligned}$$

which implies

$$\||V_{j+1}|^2\Theta_j\|_{L^2} + \|V_jV_{j+1}\Theta_j\|_{L^2} \lesssim \frac{1}{t}.$$

The above collection of bounds yields

$$\begin{aligned}(\varepsilon, -i\partial_t(i\Theta) + |D|\Theta - 2|\Phi_{\tilde P}|^2(i\Theta) - (\Phi_{\tilde P})^2\overline{(i\Theta)}) &= \frac{1-\beta_j}{\lambda_j}\big(\varepsilon_j, \mathcal{L}_{\beta_j}(i\Theta_j)\big) \\ &+ (1-\beta_j)O\Big[\frac{\|\varepsilon_j\|_{L^2}}{t} + \mathrm{Mod}_j(t)\|\varepsilon_j\|_{L^2}\Big] \qquad (5.38)\end{aligned}$$

The nonlinear term. We estimate using (5.20):

$$\begin{aligned}|(N(\varepsilon), i\Theta_j)| &\lesssim (1-\beta_j)\int \frac{|\varepsilon_j|^2|\Phi_{\tilde P}| + |\varepsilon_j|^3}{\langle y_j\rangle}dy_j \lesssim (1-\beta_j)\big(\|\varepsilon_j\|_{L^2}^2 + \|\varepsilon_j\|_{L^2}^2\|\varepsilon_j\|_{H^1}\big) \\ &\lesssim (1-\beta_j)\|\varepsilon_j\|_{L^2}^2 \leqslant (1-\beta_j)\frac{\|\varepsilon_j\|_{L^2}}{t} \qquad (5.39)\end{aligned}$$

The Ψ term. From (5.32),

$$(\Psi, i\Theta) \lesssim \frac{1}{\eta^C t^{N+1}}. \qquad (5.40)$$

The S-terms and conclusion. We now pick

$$\Theta_j \in A_j := \{Q_{\beta_j}, i\partial_{y_j}Q_{\beta_j}, \Lambda Q_{\beta_j}, \rho_j\}$$

which are strongly j-admissible, and estimate all terms in (5.36) using (5.38), (5.39), (5.40). The derivative in time of (ε, Θ) drops using the orthogonality conditions (5.6). Moreover, the same orthogonality conditions (5.6) imply that $(\varepsilon_j, \mathcal{L}_{\beta_j}(i\Theta_j)) = 0$. We now use Appendix C to compute all the scalar products and conclude:

$$\Big|\Big((S_j - \tilde S_j)e^{i\gamma_j}, i\Theta\Big)\Big| \sim (1-\beta_j)\mathrm{Mod}_j.$$

Thus, in order to estimate Mod_j, we are left with computing the crossed terms and the error $\tilde S_j$ terms given by (5.30), (5.29). The detailed estimates are given below.

Case $j = 1$. We rescale to the y_1 variable and use the 1–admissibility of $R(V_1 - Q_{\beta_1})$ to estimate:

$$|(\tilde S_1 e^{i\gamma_1}, i\Theta)| \lesssim (1-\beta_1)\frac{|\mathrm{Mod}_1| + |\mathrm{Mod}_2|}{t}.$$

We now recall (5.37) to estimate:

$$\begin{aligned}
&\int \frac{dy_1}{\langle y_1\rangle(1+(1-\beta_1)\langle y_1\rangle)\langle y_2\rangle(1+(1-\beta_2)\langle y_2\rangle)}\\
\lesssim\ &\int_{|y_1|\leqslant\frac{R}{2}} \frac{dy_1}{\langle y_1\rangle(1+(1-\beta_1)\langle y_1\rangle)\langle y_2\rangle}\\
+\ &\int_{|y_1|\geqslant\frac{R}{2}} \frac{dy_1}{\langle y_1\rangle(1+(1-\beta_1)\langle y_1\rangle)\langle y_2\rangle(1+(1-\beta_2)\langle y_2\rangle)}\\
\lesssim\ &\frac{b}{R}\int \frac{dy_1}{\langle y_1\rangle(1+(1-\beta_1)\langle y_1\rangle)} + \frac{1}{R(1+\eta R)}\int \frac{bdy_2}{\langle y_2\rangle(1+(1-\beta_2)\langle y_2\rangle)}\\
\lesssim\ &\frac{b}{t}\frac{|\log\eta|+\log t}{1+\eta t} \lesssim \frac{b|\log\eta|}{t},
\end{aligned}$$

and hence the estimate of the crossed term:

$$|(S_2e^{i\gamma_2},\Theta_1)| \lesssim (1-\beta_1)\left[\frac{b|\log\eta|}{t}(\mathrm{Mod}_2+\mathrm{Mod}_1)\right].$$

This yields the first bound,

$$\mathrm{Mod}_1 \lesssim \frac{\mathrm{Mod}_1+\mathrm{Mod}_2}{t} + \frac{\|\varepsilon_1\|_{L^2}}{t} + \frac{1}{\eta^C t^{N+1}}, \tag{5.41}$$

Case $j=2$. We estimate similarly

$$|(\tilde{S}_2e^{i\gamma_2},\Theta)| \lesssim (1-\beta_2)\frac{\mathrm{Mod}_1+\mathrm{Mod}_2}{t}$$

and

$$\begin{aligned}
&\int \frac{dy_2}{\langle y_1\rangle(1+(1-\beta_1)\langle y_1\rangle)(\langle y_2\rangle(1+(1-\beta_2)\langle y_2\rangle)}\\
\lesssim\ &\frac{1}{b}\int \frac{dy_1}{\langle y_1\rangle(1+(1-\beta_1)\langle y_1\rangle)(\langle y_2\rangle(1+(1-\beta_2)\langle y_2\rangle)} \lesssim \frac{|\log\eta|}{t}
\end{aligned}$$

from which

$$\begin{aligned}
\mathrm{Mod}_2 &\lesssim \frac{\mathrm{Mod}_1+\mathrm{Mod}_2}{t} + \frac{|\log\eta|}{t}(\mathrm{Mod}_1+\mathrm{Mod}_2) + \frac{\|\varepsilon_2\|_{L^2}}{t}\\
&\lesssim \frac{|\log\eta|}{t}(\mathrm{Mod}_1+\mathrm{Mod}_2) + \frac{\|\varepsilon_2\|_{L^2}}{t} + \frac{1}{\eta^C t^{N+1}}.
\end{aligned}$$

Conclusion. Combined with (5.41), since $t \gg |\log\eta|$, this yields

$$\mathrm{Mod}_1+\mathrm{Mod}_2 \lesssim \frac{\|\varepsilon_1\|_{L^2}+\|\varepsilon_2\|_{L^2}}{t} + \frac{1}{\eta^C t^{N+1}}$$

and hence using $\|\varepsilon_1\|_{L^2} = \sqrt{b}\|\varepsilon_2\|_{L^2}$:

$$\begin{aligned}
\mathrm{Mod}_2 &\lesssim \mathrm{Mod}_1+\mathrm{Mod}_2 \lesssim \frac{\|\varepsilon_1\|_{L^2}+\|\varepsilon_2\|_{L^2}}{t} + \frac{1}{\eta^C t^{N+1}} \lesssim \frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_2\|_{L^2}(1+\sqrt{b})}{t}\\
&\lesssim \frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_2\|_{L^2}}{t}
\end{aligned}$$

and from (5.41):

$$\begin{aligned}
\mathrm{Mod}_1 &\lesssim \frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_1\|_{L^2}}{t} + \frac{\|\varepsilon_2\|_{L^2}}{t^2} \lesssim \frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_1\|_{L^2}}{t}\left[1+\frac{1}{t\sqrt{b}}\right]\\
&\lesssim \frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_1\|_{L^2}}{t}
\end{aligned}$$

where we used

$$t\sqrt{b} \gtrsim 1 \tag{5.42}$$

□

5.6. **Energy estimate.** We are now in position to derive the key monotonicity formula for the linearized energy $\mathcal{G}$ which is the second crucial element of our analysis.

Proposition 5.4 (Energy estimate for $\mathcal{G}$). *There holds the improved pointwise bound on* $[T_{\text{in}}, T_n]$*:*

$$\mathcal{G}(\varepsilon(t)) \leqslant \frac{C}{Nt^{\frac{N}{2}}} \tag{5.43}$$

for some universal constant C *independent of* N, η, t.

Proof of Proposition 5.4. The proof relies on the careful treatment of all terms induced by the localization of mass and energy when computing the time variation of the energy $\mathcal{G}$. The main difficulty is the loss of control of the kinetic energy and mass as $\beta \to 1$ for ε_1^+ as reflected by (5.16), which forces different set of estimates for $\varepsilon^{\pm}$.
We rewrite (5.33) as:

$$i\partial_t \varepsilon - |D|\varepsilon + (\Phi + \varepsilon)|\Phi + \varepsilon|^2 - \Phi|\Phi|^2 = F, \tag{5.44}$$

$$F := -\Psi - S, \quad S = \sum_{j=1}^{2} \frac{1}{\lambda_j^{\frac{3}{2}}} S_j\left(y_j\right) e^{i\gamma_j},$$

or equivalently

$$\left|\begin{array}{l} i\partial_t \varepsilon - |D|\varepsilon + 2|\Phi|^2\varepsilon + \Phi^2\overline{\varepsilon} = G \\ N(\varepsilon) := (\Phi + \varepsilon)|\Phi + \varepsilon|^2 - \Phi|\Phi|^2 - 2|\Phi|^2\varepsilon - \Phi^2\overline{\varepsilon} \\ G := F - N(\varepsilon) = -\Psi - S - N(\varepsilon). \end{array}\right. \tag{5.45}$$

Step 1: Localization of mass. We compute the localized mass conservation law and claim

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}(\theta\varepsilon, \varepsilon) &= \frac{1}{2}((\partial_t\theta)\varepsilon, \varepsilon) + (-i|D|\varepsilon, \theta\varepsilon) + (i\Phi^2, \theta\varepsilon^2) + (iS, \theta\varepsilon) \\ &+ O\left(\frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}\right). \end{aligned} \tag{5.46}$$

Indeed, from (5.45):

$$\begin{aligned} \frac{d}{dt}\frac{1}{2}(\theta\varepsilon, \varepsilon) &= (\theta\partial_t\varepsilon, \varepsilon) + \frac{1}{2}((\partial_t\theta)\varepsilon, \varepsilon) \\ &= (-i|D|\varepsilon + i(2|\Phi|^2\varepsilon + \Phi^2\bar{\varepsilon}) - iG, \theta\varepsilon) + \frac{1}{2}((\partial_t\theta)\varepsilon, \varepsilon) \\ &= (i\Phi^2, \theta\varepsilon^2) - (iG, \theta\varepsilon) + \frac{1}{2}((\partial_t\theta)\varepsilon, \varepsilon) \\ &= \frac{1}{2}((\partial_t\theta)\varepsilon, \varepsilon) + (-i|D|\varepsilon, \theta\varepsilon) + (i\Phi^2, \theta\varepsilon^2) + (iN(\varepsilon), \theta\varepsilon) + (i\Psi, \theta\varepsilon) + (iS, \theta\varepsilon). \end{aligned} \tag{5.47}$$

We estimate from (5.32), (5.20):

$$|(\Psi, \theta\varepsilon)| \lesssim \frac{\|\varepsilon\|_{L^2}}{\eta^C t^{N+1}} \lesssim \frac{1}{t^{N+1}}.$$

For the nonlinear term, we estimate from (5.20) and (5.16),

$$|(N(\varepsilon), \theta\varepsilon)| \lesssim \int (|\varepsilon|^4 + |\varepsilon|^3) \lesssim \|\varepsilon\|_{L^\infty}\|\varepsilon\|_{L^2}^2 \lesssim \frac{\mathcal{G}}{t}$$

and (5.46) is proved.

Step 2: Localization of kinetic momentum. We compute the localized kinetic momentum conservation law and claim

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\zeta D\varepsilon,\varepsilon) &= (2\Phi|\varepsilon|^2+\overline{\Phi}\varepsilon^2,\zeta\partial_x\Phi)+O\left(\frac{\mathcal{G}}{t}+\frac{1}{t^{N+1}}\right) \\ &+ \frac{1}{2}(\partial_t\zeta D\varepsilon,\varepsilon)+(-i|D|\varepsilon,\zeta D\varepsilon+\frac{1}{2}\varepsilon D\zeta)+(iS,\zeta D\varepsilon+\frac{1}{2}\varepsilon D\zeta).\end{aligned} \tag{5.48}$$

Indeed, we compute from (5.44):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\zeta D\varepsilon,\varepsilon) &= \frac{1}{2}(\partial_t\zeta D\varepsilon,\varepsilon)+\frac{1}{2}(\zeta D\partial_t\varepsilon,\varepsilon)+\frac{1}{2}(\zeta D\varepsilon,\partial_t\varepsilon) \\ &= \frac{1}{2}(\partial_t\zeta D\varepsilon,\varepsilon)+(\partial_t\varepsilon,\zeta D\varepsilon+\frac{1}{2}\varepsilon D\zeta) \\ &= \frac{1}{2}(\partial_t\zeta D\varepsilon,\varepsilon)+(-i|D|\varepsilon+i(2|\Phi|^2\varepsilon+\Phi^2\bar{\varepsilon})-iG,\zeta D\varepsilon+\frac{1}{2}\varepsilon D\zeta) \\ &= \frac{1}{2}(\partial_t\zeta D\varepsilon,\varepsilon)+(-i|D|\varepsilon,\zeta D\varepsilon+\frac{1}{2}\varepsilon D\zeta) \\ &+ (i(2|\Phi|^2\varepsilon+\Phi^2\bar{\varepsilon}),\zeta D\varepsilon+\frac{1}{2}\varepsilon D\zeta)+(-iG,\zeta D\varepsilon+\frac{1}{2}\varepsilon D\zeta).\end{aligned}$$

We integrate by parts the quadratic term using the pointwise bound (E.2):

$$(i(2|\Phi|^2\varepsilon+\Phi^2\bar{\varepsilon}),\zeta D\varepsilon)=(2\Phi|\varepsilon|^2+\overline{\Phi}\varepsilon^2,\zeta\partial_x\Phi)+O\left(\frac{\|\varepsilon\|_{L^2}^2}{t}\right).$$

We estimate from (5.32) after integrating by parts:

$$|(i\Psi,\zeta D\varepsilon+\frac{1}{2}\varepsilon D\zeta)|\lesssim\|\Psi\|_{H^1}\|\varepsilon\|_{L^2}\lesssim\frac{1}{t^{N+1}}.$$

For the nonlinear term:

$$|(iN(\varepsilon),\zeta D\varepsilon+\frac{1}{2}\varepsilon D\zeta)|\lesssim\|\varepsilon\|_{H^1}\|\varepsilon\|_{L^2}^2\lesssim\frac{\mathcal{G}}{t}$$

and (5.48) is proved.

Step 3: Localized energy identity. We now compute the variation of the linearized energy:

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}(|D|\varepsilon,\varepsilon)-\frac{1}{4}\left[\int(|\varepsilon+\Phi|^4-|\Phi|^4)-(4\varepsilon,\Phi|\Phi|^2)\right]\right\} \\ &\quad=(\partial_t\varepsilon,|D|\varepsilon)-\big((\varepsilon+\Phi)|\varepsilon+\Phi|^2,\partial_t\varepsilon+\partial_t\Phi\big)+(\Phi|\Phi|^2,\partial_t\Phi) \\ &\quad+(\partial_t\varepsilon,\Phi|\Phi|^2)+(\varepsilon,\partial_t(\Phi|\Phi|^2)) \\ &\quad=(\partial_t\varepsilon,|D|\varepsilon-(\varepsilon+\Phi)|\varepsilon+\Phi|^2+\Phi|\Phi|^2)-(\partial_t\Phi,N(\varepsilon)) \\ &\quad=(i\Psi+iS,|D|\varepsilon-(\varepsilon+\Phi)|\varepsilon+\Phi|^2+\Phi|\Phi|^2)-(\partial_t\Phi,N(\varepsilon))\end{aligned} \tag{5.49}$$

We estimate all terms in (5.49) and in particular first extract the quadratic terms. From (5.32), Sobolev, $\|\Phi\|_{L^\infty}\lesssim 1$ and (5.20):

$$\left|(i\Psi,|D|\varepsilon-(\Phi+\varepsilon)|\Phi+\varepsilon|^2+\Phi|\Phi|^2)\right|\lesssim\|\Psi\|_{H^1}\|\varepsilon\|_{L^2}\lesssim\frac{1}{t^{N+1}}. \tag{5.50}$$

Let us estimate the term $(\partial_t\Phi, N(\varepsilon))$. Since V_j, $R(V_j - Q_{\beta_j})$ are j–admissible, and RM_j, RB_j are L^∞-admissible, we compute

$$\partial_{s_j} V_j = \sum_{k=1}^{2} \Big[\frac{\partial V_j}{\partial \lambda_k}(\lambda_k)_{s_j} + (1-\beta_k)\frac{\partial V_j}{\partial \beta_k}\cdot\frac{(\beta_k)_{s_j}}{1-\beta_k}\Big] + \frac{\partial V_j}{\partial \Gamma}\Gamma_{s_j} + \frac{\partial V_j}{\partial R}R_{s_j}$$

and hence, using (5.34) and the bootstrap assumption, we infer

$$|\partial_s V_j| \lesssim \frac{1}{t\langle y_j\rangle}\,. \tag{5.51}$$

Consequently, the admissibility of V_j, (5.34), and the bounds $1-\beta_1 \sim \eta$ and $1-\beta_2 \gtrsim \eta^3$ ensure

$$\begin{aligned}\partial_t\Phi = \sum_{j=1}^{2}\frac{1}{\lambda_j^{\frac{3}{2}}}\Big[\partial_{s_j}V_j - \frac{(\lambda_j)_{s_j}}{\lambda_j}\Lambda V_j - \frac{1}{1-\beta_j}\Big[\frac{(x_j)_{s_j}}{\lambda_j} - \beta_j\Big]\partial_{y_j}V_j - \frac{\beta_j}{1-\beta_j}\partial_{y_j}V_j \\ + \frac{(\beta_j)_{s_j}}{1-\beta_j}y_j\partial_{y_j}V_j + i(\gamma_j)_{s_j}V_j\Big]e^{i\gamma_j}(y_j) = O\left(\sum_{j=1}^{2}\frac{1}{\eta^C\langle y_j\rangle}\right)\end{aligned} \tag{5.52}$$

We use this with (5.20) to estimate:

$$\begin{aligned}&-(\partial_t\Phi, N(\varepsilon)) = -(\partial_t\Phi, (\Phi+\varepsilon)|\Phi+\varepsilon|^2 - \Phi|\Phi|^2 - 2|\Phi|^2\varepsilon - \Phi^2\overline{\varepsilon})\\ =\ & -\left(\partial_t\Phi, 2\Phi|\varepsilon|^2 + \overline{\Phi}\varepsilon^2\right) + O\left(\frac{\|\varepsilon\|_{H^1}\|\varepsilon\|_{L^2}^2}{\eta^C}\right) = -\left(\partial_t\Phi, 2\Phi|\varepsilon|^2 + \overline{\Phi}\varepsilon^2\right) + O\left(\frac{\mathcal{G}}{t}\right).\end{aligned}$$

similarly, using (5.34) and (5.20):

$$\begin{aligned}&(iS, |D|\varepsilon - (\Phi+\varepsilon)|\Phi+\varepsilon|^2 + \Phi|\Phi|^2)\\ =\ & (iS, |D|\varepsilon - 2|\Phi|^2\varepsilon - \Phi^2\overline{\varepsilon}) + O\left(\frac{\mathrm{Mod}_1 + \mathrm{Mod}_2}{\eta^C}\|\varepsilon\|_{L^2}^2\right)\\ =\ & (iS, |D|\varepsilon - 2|\Phi|^2\varepsilon - \Phi^2\overline{\varepsilon}) + O\left(\frac{\mathcal{G}}{t}\right).\end{aligned}$$

The collection of above bounds yields

$$\begin{aligned}&\frac{d}{dt}\left\{\frac{1}{2}(|D|\varepsilon,\varepsilon) - \frac{1}{4}\left[\int(|\varepsilon+\Phi|^4 - |\Phi|^4) - (4\varepsilon, \Phi|\Phi|^2)\right]\right\}\\ =\ & (\varepsilon, |D|(iS) - 2|\Phi|^2(iS) - \Phi^2\overline{iS}) - \left(\partial_t\Phi, 2\Phi|\varepsilon|^2 + \overline{\Phi}\varepsilon^2\right)\\ +\ & O\left(\frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}\right).\end{aligned} \tag{5.53}$$

We now treat the remaining quadratic terms more carefully and combine them with the leading order quadratic terms in (5.46), (5.48). Indeed, we rewrite (5.52) using (5.34), (5.51), (5.25) and the j-admissibility of V_j:

$$\begin{aligned}&\partial_t\Phi\\ =\ & \sum_{j=1}^{2}\frac{1}{\lambda_j^{\frac{3}{2}}}\left[\partial_{s_j}V_j - \frac{(\lambda_j)_{s_j}}{\lambda_j}\Lambda V_j - \frac{1}{1-\beta_j}\left[\frac{(x_j)_{s_j}}{\lambda_j}\right]\partial_{y_j}V_j + \frac{(\beta_j)_{s_j}}{1-\beta_j}y_j\partial_{y_j}V_j + i(\gamma_j)_{s_j}V_j\right]e^{i\gamma_j}(y_j)\\ =\ & \sum_{j=1}^{2}\frac{i}{\lambda_j}\Phi^{(j)} - \beta_j\partial_x\Phi^{(j)} + O\left(\sum_{j=1}^{2}\frac{1}{t}\frac{1}{\langle y_j\rangle}\right)\end{aligned}$$

where we have set

$$\Phi^{(j)}(t,x) := \frac{1}{\lambda_j^{\frac{1}{2}}} V_j\left(\frac{x-x_j}{\lambda_j(1-\beta_j)}\right) e^{i\gamma_j} \ .$$

We infer the bound

$$\begin{aligned}
& -\left(\partial_t\Phi, 2\Phi|\varepsilon|^2+\overline{\Phi}\varepsilon^2\right)+(\varepsilon^2, i\theta\Phi^2)-(2\Phi|\varepsilon|^2+\overline{\Phi}\varepsilon^2, \zeta\partial_x\Phi) \\
= & \left(\beta_1\partial_x\Phi^{(1)}+\beta_2\partial_x\Phi^{(2)}+O\left(\Sigma_{j=1}^2\frac{1}{t}\frac{1}{\langle y_j\rangle}\right), 2\Phi|\varepsilon|^2+\overline{\Phi}\varepsilon^2\right)-(2\Phi|\varepsilon|^2+\overline{\Phi}\varepsilon^2, \zeta\partial_x\Phi) \\
+ & (\varepsilon^2, i\Phi\left[\theta\Phi-\Sigma_{j=1}^2\frac{1}{\lambda_j}\Phi^{(j)}\right])-2(i\left(\frac{\Phi^{(1)}}{\lambda_1}+\frac{\Phi^{(2)}}{\lambda_2}\right), (\Phi^{(1)}+\Phi^{(2)})|\varepsilon|^2) \\
= & -(2\Phi|\varepsilon|^2+\overline{\Phi}\varepsilon^2, (\zeta-\beta_1)\partial_x\Phi^{(1)}+(\zeta-\beta_2)\partial_x\Phi^{(2)})+(\varepsilon^2, i\Phi\Sigma_{j=1}^2\left(\theta-\frac{1}{\lambda_j}\right)\Phi^{(j)}) \\
- & \frac{2}{\lambda_1}(|\varepsilon|^2, i\Phi^{(1)}\overline{\Phi^{(2)}})-\frac{2}{\lambda_2}(|\varepsilon|^2, i\Phi^{(2)}\overline{\Phi^{(1)}})+O\left(\frac{\|\varepsilon\|_{L^2}^2}{t}\right)
\end{aligned}$$

We recall (5.37), and hence $|y_1|\leqslant\frac{R}{2}$ implies $|y_2|\geqslant\frac{R}{2b\mu}$ from which

$$\||\Phi^{(1)}\Phi^{(2)}\|_{L^\infty}\lesssim\|\frac{1}{\langle y_1\rangle\langle y_2\rangle}\|_{L^\infty}\lesssim\frac{1}{t}$$

and hence

$$|(|\varepsilon|^2, i\Phi^{(1)}\overline{\Phi^{(2)}})|+|(|\varepsilon|^2, i\Phi^{(2)}\overline{\Phi^{(1)}})|\lesssim\frac{\|\varepsilon\|_{L^2}^2}{t}\lesssim\frac{\mathcal{G}}{t}.$$

We then use $\langle y_1\rangle\gtrsim R$ on $\mathrm{Supp}(1-\phi_1)$ and $\mathrm{Supp}(\frac{1}{\lambda_1}-\theta)$ and the explicit formula (5.11) to estimate:

$$\begin{aligned}
\left|(\zeta-\beta_1)\partial_x\Phi^{(1)}\right| &\lesssim\frac{|1-\phi_1|}{\langle y_1\rangle^2}\lesssim\frac{1}{t} \\
\left|\left(\theta-\frac{1}{\lambda_1}\right)\Phi^{(1)}\right| &\lesssim\frac{1}{t} \ .
\end{aligned}$$

Similarly, we use $\langle y_2\rangle\gtrsim R$ on $\mathrm{Supp}(b-\phi_1)$ and $\mathrm{Supp}(\frac{1}{\lambda_2}-\theta)$, and the relation $\beta_2-\zeta=(1-\beta_1)(\phi_1-b)$ to get

$$\begin{aligned}
\left|(\zeta-\beta_2)\partial_x\Phi^{(2)}\right| &\lesssim\frac{|b-\phi_1|}{b\langle y_2\rangle^2}\lesssim\frac{1}{t} \\
\left|\left(\theta-\frac{1}{\lambda_2}\right)\Phi^{(2)}\right| &\lesssim\frac{1}{t}.
\end{aligned}$$

The second estimate above is straightforward. Let us explain how to obtain the first estimate. Recall that $b-\phi_1=(b-1)\Psi_1$, and $0\leq\Psi(z_1)\leq 1$, with $\Psi_1(z_1)=1$ for $z_1\leq 1/4$, $\Psi_1(z_1)=(1-z_1)^{10}$ for $1/2\leq z_1\leq 1$, and $\Psi_1(z_1)=0$ for $z_1\geq 1$, so we may assume $z_1\geq 1$. Moreover, recall that

$$1-z_1=1-\frac{y_1}{R(1-b)}=1-\frac{R+\mu by_2}{R(1-b)}=\frac{-b}{1-b}\left(1+\frac{\mu y_2}{R}\right)\geq 0 \ .$$

If

$$-1\geq\frac{\mu y_2}{R}\geq-\frac{1}{\sqrt{b}},$$

then $|1-z_1| \lesssim \sqrt{b}$, and

$$\frac{\Psi_1}{b\langle y_2\rangle^2} \lesssim \frac{b^4}{\langle y_2\rangle^2} \leq \frac{b^4}{R^2} \lesssim \frac{1}{t}\,.$$

On the other hand, if

$$\frac{\mu y_2}{R} \leq -\frac{1}{\sqrt{b}},$$

then $\langle y_2\rangle \gtrsim R/\sqrt{b}$, and

$$\frac{\Psi_1}{b\langle y_2\rangle^2} \leq \frac{1}{b\langle y_2\rangle^2} \lesssim \frac{1}{R^2} \lesssim \frac{1}{t}\,.$$

We conclude using $\|\Phi\|_{L^\infty} \lesssim 1$:

$$-\left(\partial_t\Phi, 2\Phi|\varepsilon|^2 + \overline{\Phi}\varepsilon^2\right) + (\varepsilon^2, i\theta\Phi^2) - (2\Phi|\varepsilon|^2 + \overline{\Phi}\varepsilon^2, \zeta\partial_x\Phi) = O\left(\frac{\mathcal{G}}{t}\right).$$

Injecting this estimate into (5.46), (5.48) and (5.53) yields the full localized energy identity:

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{1}{2}(|D|\varepsilon+\theta\varepsilon,\varepsilon) - \frac{1}{2}(\zeta D\varepsilon,\varepsilon) - \frac{1}{4}\left[\int(|\varepsilon+\Phi|^4 - |\Phi|^4) - (4\varepsilon,\Phi|\Phi|^2)\right]\right\}\\
=\;& \frac{1}{2}((\partial_t\theta)\varepsilon,\varepsilon) + (-i|D|\varepsilon,\theta\varepsilon) + (\varepsilon,|D|(iS) + i\theta S - 2|\Phi|^2(iS) - \Phi^2\overline{iS})\\
-\;& \frac{1}{2}(\partial_t\zeta D\varepsilon,\varepsilon) + (i|D|\varepsilon,\zeta D\varepsilon + \frac{1}{2}\varepsilon D\zeta) - (iS,\zeta D\varepsilon + \frac{1}{2}\varepsilon D\zeta)\\
+\;& O\left(\frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}\right)\\
=\;& \frac{1}{2}((\partial_t\theta)\varepsilon,\varepsilon) + (-i|D|\varepsilon,\theta\varepsilon) - \frac{1}{2}(\partial_t\zeta D\varepsilon,\varepsilon) + (i|D|\varepsilon,\zeta D\varepsilon + \frac{1}{2}\varepsilon D\zeta)\\
+\;& (\varepsilon,(|D|-\zeta D)(iS) + i\theta S - 2|\Phi|^2(iS) - \Phi^2\overline{iS}) + \frac{1}{2}(\varepsilon, iSD\zeta)\\
+\;& O\left(\frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}\right) \qquad (5.54)
\end{aligned}$$

where we integrated by parts the term $(iS,\zeta D\varepsilon + \frac{1}{2}\varepsilon D\zeta)$ in the last step. We now estimate all remaining terms in (5.54). The linear terms in (5.54) induced by the localization of the mass and kinetic momentum[6] are particularly critical for our analysis.

Step 4: Modulation equations terms. We estimate the remaining modulation equations terms in (5.54) and claim

$$\left|(\varepsilon,(|D|-\zeta D)(iS) + i\theta S - 2|\Phi|^2(iS) - \Phi^2\overline{iS})\right| + |(\varepsilon, iSD\zeta)| \lesssim \frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}. \quad (5.55)$$

Indeed, we first estimate the S terms in the y_1 variable. From (5.28), (5.29) and (5.34) with $\|\varepsilon_2\|_{L^2} = \frac{\|\varepsilon_1\|_{L^2}}{\sqrt{b}}$:

$$\begin{aligned}
\|S_1\|_{H^1_{y_1}} &\lesssim \mathrm{Mod}_1 + \frac{\mathrm{Mod}_2}{t} \lesssim \frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_1\|_{L^2}}{t}\left[1 + \frac{1}{\sqrt{b}t}\right]\\
&\lesssim \frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_1\|_{L^2}}{t} \qquad (5.56)
\end{aligned}$$

[6] which is necessary due to the dramatic change of size of each bubble.

where we used (5.42) in the last step, and

$$\|S_2\|_{H^1_{y_2}} \lesssim \mathrm{Mod}_2 + \frac{\mathrm{Mod}_1}{t} \lesssim \frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_2\|_{L^2}}{t}. \tag{5.57}$$

We also have similarly the pointwise bound using the admissibility of V_j:

$$|\partial^k_{y_2} S_2| \lesssim \frac{1}{\langle y_2\rangle^k}\left[\frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_2\|_{L^2}}{t}\right]. \tag{5.58}$$

In particular,

$$\begin{aligned} \|S\|_{L^2_{y_1}} &\lesssim \|S_1\|_{L^2_{y_1}} + \|S_2\|_{L^2_{y_1}} \lesssim \|S_1\|_{L^2_{y_1}} + \sqrt{b}\|S_2\|_{L^2_{y_2}} \\ &\lesssim \frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_1\|_{L^2}}{t}. \end{aligned} \tag{5.59}$$

We therefore renormalize to the y_1 variable and estimate from (5.59), (5.11):

$$|(\varepsilon, iSD\zeta)| \lesssim (1-\beta_1)|(\varepsilon_1, S\partial_{y_1}\phi_1)| \lesssim \frac{1-\beta_1}{t}\|\varepsilon_1\|^2_{L^2} + \frac{1}{t^{N+1}},$$

and similarly using $\|\Phi\|_{L^\infty} \lesssim 1$:

$$\left|\left|(\varepsilon, i\theta S - 2|\Phi|^2(iS) - \Phi^2\overline{iS})\right| \lesssim (1-\beta_1)\|\varepsilon_1\|_{L^2}\|S\|_{L^2_{y_1}} \lesssim \frac{1-\beta_1}{t}\|\varepsilon_1\|^2_{L^2} + \frac{1}{t^{N+1}}.$$

We now use $\zeta_1 = \beta_1 + (1-\beta_1)(1-\phi_1) = 1 - (1-\beta_1)\phi_1$ to compute:

$$\begin{aligned} |(\varepsilon, (|D| - \zeta D)(iS))| &\lesssim |((|D| - \zeta_1 D)\varepsilon_1, iS)| \\ &\lesssim (1-\beta_1)|(\phi_1 D\varepsilon_1, iS)| + |(i\varepsilon_1^-, D\Pi^- S)| := I + II. \end{aligned}$$

We claim:

$$I + II \lesssim \frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}} \tag{5.60}$$

which concludes the proof of (5.55).

Control of I. We split $S = S_1 + S_2$ and first estimate after an integration by parts and using (5.56):

$$|(1-\beta_1)|(\phi_1 D\varepsilon_1, iS_1)| \lesssim (1-\beta_1)\|\varepsilon_1\|_{L^2}\|S_1\|_{H^1_{y_1}} \lesssim \frac{1}{t^{N+1}} + \frac{1-\beta_1}{t}\|\varepsilon_1\|^2_{L^2} \lesssim \frac{1}{t^{N+1}} + \frac{\mathcal{G}}{t}.$$

Next,

$$\begin{aligned} &|(1-\beta_1)|(\phi_1 D\varepsilon_1, iS_2) \lesssim (1-\beta_1)|(\phi_2 D\varepsilon_2, iS_2)| \\ \lesssim\ &(1-\beta_2)|(\varepsilon_2, iDS_2)| + (1-\beta_1)|(\varepsilon_2, D((\phi_2 - b)iS_2)|. \end{aligned}$$

The first term is estimated from (5.57):

$$(1-\beta_2)|(\varepsilon_2, iDS_2)| \lesssim (1-\beta_2)\|\varepsilon_2\|_{L^2}\left[\frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_2\|_{L^2}}{t}\right] \lesssim \frac{1}{t^{N+1}} + \frac{\mathcal{G}}{t}.$$

The second term is estimated using (5.58), (5.37), $\langle y_2\rangle \gtrsim \frac{b}{R}$ on $\mathrm{Supp}(b-\phi_2)$ and $\|\partial_{y_2}\phi_2\|_{L^\infty} \lesssim b\|\partial_{y_1}\phi_1\|_{L^\infty} \lesssim \frac{b}{R}$ so that:

$$\begin{aligned} (1-\beta_1)|(\varepsilon_2, D((\phi_2 - b)iS_2)| &\lesssim (1-\beta_1)\frac{b}{R}\left[\frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_2\|_{L^2}}{t}\right]\int \frac{|\varepsilon_2|}{\langle y_2\rangle}dy_2 \\ &\lesssim \frac{1-\beta_2}{t}\|\varepsilon_2\|^2_{L^2} + \frac{1}{t^{N+1}} \lesssim \frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}} \end{aligned}$$

which concludes the proof of (5.60) for I.

Control of II. Consider $S_j - \tilde{S}_j$. Then by commuting the null space relations

$$\mathcal{L}_\beta(\Lambda Q_\beta) = -Q_\beta, \quad \mathcal{L}_\beta(iQ_\beta) = 0, \quad \mathcal{L}_\beta(\partial_y Q_\beta) = 0$$

and (2.21) with Π^-, we estimate:

$$\|D\Pi^-(\Lambda Q_\beta)\|_{L^2} + \|D\Pi^- Q_\beta\|_{L^2} + \|D\Pi^-\tilde{\Lambda} Q_\beta\|_{L^2} + \|D\Pi^-\partial_y Q_\beta\|_{L^2} \lesssim 1-\beta.$$

Hence from (5.28):

$$\|D\Pi^-(S_j - \tilde{S}_j)\|_{L^2_{y_j}} \lesssim (1-\beta_j)\mathrm{Mod}_j \lesssim (1-\beta_j)\left[\frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_j\|_{L^2}}{t}\right]$$

from which:

$$|(i\varepsilon_1, D\Pi^-(S_1 - \tilde{S}_1))| \lesssim (1-\beta_1)\|\varepsilon_1\|_{L^2}\left[\frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_1\|_{L^2}}{t}\right] \lesssim \frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}$$

and renormalizing to the y_2 variable:

$$\begin{aligned} |(i\varepsilon_1, D\Pi^-(S_2 - \tilde{S}_2))| &= |(i\varepsilon_2, D\Pi^-(S_2 - \tilde{S}_2))| \lesssim (1-\beta_2)\|\varepsilon_2\|_{L^2}\left[\frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_2\|_{L^2}}{t}\right] \\ &\lesssim \frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}. \end{aligned}$$

We now argue similarly for the $\tilde{S}_j$ terms. Indeed, from Corollary 4.7, we have

$$\|D\Pi^-\partial_\Gamma V_j\|_{L^2} + \|D\Pi^-\Lambda_R V_j\|_{L^2} + \|D\Pi^-\partial_{\lambda_{j+1}} V_j\|_{L^2} + \|D\Pi^-(1-\beta_{j+1})\partial_{\beta_{j+1}} V_j\|_{L^2} \lesssim \frac{1-\beta_j}{R}.$$

Hence, arguing like for (5.56):

$$\|\|D\Pi^-\tilde{S}_1\|_{L^2_{y_1}} \lesssim \frac{1-\beta_1}{t}\left[\mathrm{Mod}_1 + \mathrm{Mod}_2\right] \lesssim (1-\beta_1)\left[\frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_1\|_{L^2}}{t}\right]$$

which implies

$$|(i\varepsilon_1, D\Pi^-\tilde{S}_1)| \lesssim (1-\beta_1)\|\varepsilon_1\|_{L^2}\left[\frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_1\|_{L^2}}{t}\right] \lesssim \frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}.$$

similarly:

$$\|\|D\Pi^-\tilde{S}_2\|_{L^2_{y_2}} \lesssim \frac{1-\beta_2}{t}\left[\mathrm{Mod}_1 + \mathrm{Mod}_2\right] \lesssim (1-\beta_2)\left[\frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_2\|_{L^2}}{t}\right]$$

and

$$|(i\varepsilon_2, D\Pi^-\tilde{S}_2)| \lesssim (1-\beta_2)\|\varepsilon_2\|_{L^2}\left[\frac{1}{\eta^C t^{N+1}} + \frac{\|\varepsilon_2\|_{L^2}}{t}\right] \lesssim \frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}.$$

This concludes the proof of (5.60).

Step 5: Linear momentum terms. Let

$$\widetilde{\varepsilon_1} = \frac{\varepsilon_1^-}{\langle z_1\rangle^{\frac{1+\alpha}{2}}}, \quad z_1 = \frac{y_1}{R}, \tag{5.61}$$

we claim:

$$\begin{aligned} &-\frac{1}{2}(\partial_t\zeta D\varepsilon, \varepsilon) + (i|D|\varepsilon, \zeta D\varepsilon + \frac{1}{2}\varepsilon D\zeta) \\ &= \frac{d}{dt}\left\{o_{\eta\to 0}(1)\mathcal{G}\right\} + O\left(\frac{1}{t^{N+1}} + \frac{1}{t}\left[\mathcal{G}(t) + \frac{\|\widetilde{\varepsilon_1}\|^2_{L^2}}{t}\right]\right). \end{aligned} \tag{5.62}$$

We first compute:

$$\begin{aligned}
&(\partial_t\zeta D\varepsilon,\varepsilon)+(-i|D|\varepsilon,\varepsilon D\zeta)=(\partial_t\zeta(D\varepsilon^+ + D\varepsilon^-),\varepsilon^+ +\varepsilon^-)+(D\varepsilon^+ - D\varepsilon^-,(\varepsilon^+ +\varepsilon^-)\partial_x\zeta)\\
=\;&((\partial_t\zeta+\partial_x\zeta)D\varepsilon^+,\varepsilon^+)+((\partial_t\zeta-\partial_x\zeta)D\varepsilon^-,\varepsilon^-)+((\partial_t\zeta-\partial_x\zeta)D\varepsilon^-,\varepsilon^+)+((\partial_t\zeta+\partial_x\zeta)D\varepsilon^+,\varepsilon^-)\\
=\;&((\partial_t\zeta+\partial_x\zeta)D\varepsilon^+,\varepsilon^+)+((\partial_t\zeta+\partial_x\zeta)D\varepsilon^-,\varepsilon^-)+((\partial_t\zeta+\partial_x\zeta)D\varepsilon^-,\varepsilon^+)+((\partial_t\zeta+\partial_x\zeta)D\varepsilon^+,\varepsilon^-)\\
-\;&2(\partial_x\zeta D\varepsilon^-,\varepsilon^-)-2(\partial_x\zeta D\varepsilon^-,\varepsilon^+)
\end{aligned}$$

and now estimate the various contributions.

Term $|(\partial_x\zeta D\varepsilon^-,\varepsilon^-)|$. We claim:

$$|(\partial_x\zeta D\varepsilon^-,\varepsilon^-)|\lesssim\frac{1}{t}\left[\mathcal{G}(t)+\frac{\|\widetilde{\varepsilon_1}\|_{L^2}^2}{t}\right]. \tag{5.63}$$

Indeed, recall (5.11) and renormalize to the y_1 variable to compute:

$$|(\partial_x\zeta D\varepsilon^-,\varepsilon^-)|\lesssim|(D\varepsilon_1^-,\partial_{y_1}\phi_1\varepsilon_1^-)|.$$

We then commute:

$$\begin{aligned}
&|(D\varepsilon_1^-,\partial_{y_1}\phi_1\varepsilon_1^-)|=\frac{1}{R(1-b)}|(\frac{1}{\langle z_1\rangle^{\frac{1+\alpha}{2}}}\chi_R D\varepsilon_1^-,\varepsilon_1^-)|\\
=\;&\frac{1}{R(1-b)}\left|(\frac{1}{\langle z_1\rangle^{\frac{1+\alpha}{2}}}[-|D|^{\frac12}\chi_R|D|^{\frac12}\varepsilon_1^- +[|D|^{\frac12},\chi_R]|D|^{\frac12}\varepsilon_1^-],\varepsilon_1^-)\right|\\
\lesssim\;&\frac{1}{R}\left\{|(\chi_R|D|^{\frac12}\varepsilon_1^-,|D|^{\frac12}\widetilde{\varepsilon_1})|+|(|D|^{\frac12}\varepsilon_1^-,[|D|^{\frac12},\chi_R]\widetilde{\varepsilon_1})|\right\}\\
\lesssim\;&\frac{1}{R}\||D|^{\frac12}\varepsilon_1^-\|_{L^2}\left[\|\chi_R\langle z_1\rangle^{\frac{1+\alpha}{2}}\|_{L^\infty}\|\frac{1}{\langle z_1\rangle^{\frac{1+\alpha}{2}}}|D|^{\frac12}\widetilde{\varepsilon_1}\|_{L^2}+\|[|D|^{\frac12},\chi_R]\widetilde{\varepsilon_1}\|_{L^2}\right].
\end{aligned}$$

We estimate from (D.1):

$$\|[|D|^{\frac12},\chi_R]\widetilde{\varepsilon_1}\|_{L^2}\lesssim\frac{1}{\sqrt{R}}\|\widetilde{\varepsilon_1}\|_{L^2}$$

and from (D.2) applied to $\frac{1}{\langle z_1\rangle^{\frac{1+\alpha}{2}}}$:

$$\begin{aligned}
\|\frac{1}{\langle z_1\rangle^{\frac{1+\alpha}{2}}}|D|^{\frac12}\widetilde{\varepsilon_1}\|_{L^2}\;&\lesssim\;\||D|^{\frac12}\varepsilon_1^-\|_{L^2}+\|\frac{1}{\langle z_1\rangle^{\frac{1+\alpha}{2}}}[|D|^{\frac12},\frac{1}{\langle z_1\rangle^{\frac{1+\alpha}{2}}}]\varepsilon_1^-\|_{L^2}\\
&\lesssim\;\||D|^{\frac12}\varepsilon_1^-\|_{L^2}+\frac{1}{\sqrt{R}}\|\widetilde{\varepsilon_1}\|_{L^2}
\end{aligned} \tag{5.64}$$

and hence the bound:

$$|(D\varepsilon_1^-,\partial_{y_1}\phi_1\varepsilon_1^-)|\lesssim\frac{1}{R}\left[\||D|^{\frac12}\varepsilon_1^-\|_{L^2}^2+\frac{\|\widetilde{\varepsilon_1}\|_{L^2}^2}{R}\right]\lesssim\frac{1}{t}\left[\mathcal{G}+\frac{\|\widetilde{\varepsilon_1}\|_{L^2}^2}{t}\right],$$

this is (5.63).

Term $(\partial_x\zeta D\varepsilon^-,\varepsilon^+)$. This term cannot be treated directly due to the η loss in $\|\varepsilon_1^\pm\|_{L^2}\lesssim\frac{\mathcal{G}}{\eta}$. We claim that

$$(\partial_x\zeta D\varepsilon^-,\varepsilon^+)=\frac{d}{dt}\left\{o_{\eta\to0}(\mathcal{G})\right\}+O\left(\frac{1}{t^{N+1}}+\frac{\mathcal{G}(t)}{t}\right). \tag{5.65}$$

Indeed, first we renormalize to the y_1 variable,

$$(\partial_x\zeta D\varepsilon^-,\varepsilon^+)=\frac{1}{\lambda_1^2}(D\varepsilon_1^-,\partial_{y_1}\phi_1\varepsilon_1^+)$$

and now we need to use the equation. We rewrite (5.44) as

$$\begin{aligned} & i\partial_t\varepsilon - |D|\varepsilon = \tilde F \\ & \tilde F(t,x) = -\Psi - S - ((\Phi+\varepsilon)|\Phi+\varepsilon|^2 - \Phi|\Phi|^2), \quad \tilde F(t,x) = \tilde F_1(s_1,y_1) \end{aligned}$$

and renormalize to the $y_1 = \frac{x-x_1}{\lambda_1(1-\beta_1)}$ variable so that

$$\begin{aligned} & i\partial_{s_1}\varepsilon_1 - \frac{|D|-\beta_1 D}{1-\beta_1}\varepsilon_1 \\ = \;& \lambda_1^{1+\frac12}\tilde F_1 + i\frac{(\lambda_1)_{s_1}}{\lambda_1}(\frac{\varepsilon_1}{2} + y_1\partial_{y_1}\varepsilon_1) - i\frac{(\beta_1)_{s_1}}{1-\beta_1}y_1\partial_{y_1}\varepsilon_1 + i\frac{(x_1)_t-\beta_1}{1-\beta_1}\partial_{y_1}\varepsilon_1 + \gamma_{s_1}\varepsilon_1 \end{aligned}$$

and thus after projecting with Π^- and using $[\Pi^\pm, \partial_y] = [\Pi^\pm, y\partial_y] = 0$:

$$\begin{aligned} & i\partial_{s_1}\varepsilon_1^- + \frac{1+\beta_1}{1-\beta_1}D\varepsilon_1^- \qquad (5.66) \\ = \;& \lambda_1^{1+\frac12}\Pi^-\tilde F_1 + i\frac{(\lambda_1)_{s_1}}{\lambda_1}(\frac{\varepsilon_1^-}{2} + y_1\partial_{y_1}\varepsilon_1^-) - i\frac{(\beta_1)_{s_1}}{1-\beta_1}y_1\partial_{y_1}\varepsilon_1^- + i\frac{(x_1)_t-\beta_1}{1-\beta_1}\partial_{y_1}\varepsilon_1^- + (\gamma_1)_{s_1}\varepsilon_1^-, \end{aligned}$$

and

$$\begin{aligned} & i\partial_{s_1}\varepsilon_1^+ - D\varepsilon_1^+ \qquad (5.67) \\ = \;& \lambda_1^{1+\frac12}\Pi^+\tilde F_1 + i\frac{(\lambda_1)_{s_1}}{\lambda_1}(\frac{\varepsilon_1^+}{2} + y_1\partial_{y_1}\varepsilon_1^+) - i\frac{(\beta_1)_{s_1}}{1-\beta_1}y_1\partial_{y_1}\varepsilon_1^+ + i\frac{(x_1)_t-\beta_1}{1-\beta_1}\partial_{y_1}\varepsilon_1^+ + (\gamma_1)_{s_1}\varepsilon_1^+. \end{aligned}$$

Using (5.66), we have

$$\begin{aligned} & \frac{1}{\lambda_1^2}(D\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+) = \frac{1-\beta_1}{\lambda_1(1+\beta_1)}(-i\partial_t\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+) \\ + \;& \frac{1-\beta_1}{\lambda_1^2(1+\beta_1)}\left(\lambda_1^{1+\frac12}\Pi^-\tilde F_1 + i\frac{(\lambda_1)_{s_1}}{\lambda_1}(\frac{\varepsilon_1^-}{2} + y_1\partial_{y_1}\varepsilon_1^-) - i\frac{(\beta_1)_{s_1}}{1-\beta_1}y_1\partial_{y_1}\varepsilon_1^-\right. \\ + \;& \left. i\frac{(x_1)_t-\beta_1}{1-\beta_1}\partial_{y_1}\varepsilon_1^- + (\gamma_1)_{s_1}\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+\right). \end{aligned}$$

We use $\mathrm{Supp}(\partial_{y_1}\phi_1) \subset \{\frac{t}{4} \leqslant y_1 \leqslant t\}$, $\|\partial_{y_1}\phi_1\|_{L^\infty} \lesssim \frac1t$ and the rough bound

$$\left|\frac{(\lambda_1)_{s_1}}{\lambda_1}\right| + \left|\frac{(\beta_1)_{s_1}}{1-\beta_1}\right| + \left|\frac{(x_1)_t-\beta_1}{1-\beta_1}\right| \lesssim \frac1t, \quad |(\gamma_1)_{s_1}| \lesssim 1 \qquad (5.68)$$

to estimate

$$(1-\beta_1)\left|\left(i\frac{(\lambda_1)_{s_1}}{\lambda_1}\frac{\varepsilon_1^-}{2} + (\gamma_1)_{s_1}\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+\right)\right| \lesssim \frac{1-\beta_1}{t}\|\varepsilon_1\|_{L^2}^2 \lesssim \frac{\mathcal G}{t},$$

and

$$\begin{aligned} & (1-\beta_1)\left|\left(i\frac{(\lambda_1)_{s_1}}{\lambda_1}y_1\partial_{y_1}\varepsilon_1^- - i\frac{(\beta_1)_{s_1}}{1-\beta_1}y_1\partial_{y_1}\varepsilon_1^- + i\frac{(x_1)_t-\beta_1}{1-\beta_1}\partial_{y_1}\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+\right)\right| \\ \lesssim \;& \frac{1-\beta_1}{t}\|\varepsilon_1\|_{L^2}^2 \lesssim \frac{\mathcal G}{t}. \end{aligned}$$

Indeed, in order to absorb the derivative in the second estimate, we make use of the commutator estimate (D.9). For instance,

$$|(y_1\partial_{y_1}\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+)| = |(\partial_{y_1}[\Pi^+, y_1\partial_{y_1}\phi_1]\varepsilon_1^-, \varepsilon_1^+)| + O(\|\varepsilon_1\|_{L^2}^2) \lesssim \|\varepsilon_1\|_{L^2}^2 \ ,$$

ad the two other terms are treated similarly.

The rough L^∞-bound $\|\varepsilon_1\|_{L^\infty} \leq 1$, (5.32) and (5.59) ensure

$$\|\tilde F_1\|_{L^2} \lesssim \|\varepsilon_1\|_{L^2} + \frac{1}{\eta^C t^{N+1}} \qquad (5.69)$$

and hence:

$$(1-\beta_1)|(\Pi^-\tilde{F}_1, \partial_{y_1}\phi_1\varepsilon_1^+)| \lesssim \frac{1-\beta_1}{t}\|\varepsilon_1\|_{L^2}^2 + \frac{1}{t^{N+1}} \lesssim \frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}.$$

We now integrate by parts in time:

$$\begin{aligned} \frac{1}{\lambda_1^2}(D\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+) &= -\frac{1-\beta_1}{\lambda_1(1+\beta_1)}(i\partial_t\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+) + O\left(\frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}\right) \\ = \quad -\frac{d}{dt}\left\{\frac{1-\beta_1}{\lambda_1(1+\beta_1)}(i\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+)\right\} &- \frac{1-\beta_1}{\lambda_1(1+\beta_1)}\left(\varepsilon_1^-, \partial_{y_1}\phi_1 i\partial_t\varepsilon_1^+\right) + O\left(\frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}\right) \end{aligned}$$

where we used (5.68) and the rough bound

$$|\partial_t\partial_{y_1}\phi_1| \lesssim \frac{1}{t}$$

in the last step. We now inject (5.67) and conclude using a similar chain of estimates as above:

$$\begin{aligned} \frac{1}{\lambda_1^2}(D\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+) &= -\frac{d}{dt}\left\{\frac{1-\beta_1}{\lambda_1(1+\beta_1)}(i\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+)\right\} + O\left(\frac{\mathcal{G}}{t} + \frac{1}{t^{N+1}}\right) \\ &- \frac{1-\beta_1}{\lambda_1(1+\beta_1)}(\varepsilon_1^-, \partial_{y_1}\phi_1 D\varepsilon_1^+). \end{aligned}$$

The last term is handled using again the commutator estimate (D.9):

$$|(\varepsilon_1^-, \partial_{y_1}\phi_1 D\varepsilon_1^+)| = |(\varepsilon_1^-, [\partial_{y_1}\phi_1, \Pi^+]D\varepsilon_1^+)| \lesssim \|\varepsilon_1^+\|_{L^2}\|D[\partial_{y_1}\phi_1, \Pi^+]\varepsilon_1^-\|_{L^2} \lesssim \frac{\|\varepsilon_1\|_{L^2}^2}{t^2}$$

and the boundary term in time is estimated using $\|\partial_{y_1}\phi_1\|_{L^\infty} \lesssim \frac{1}{t}$:

$$\left|\frac{1-\beta_1}{\lambda_1(1+\beta_1)}(i\varepsilon_1^-, \partial_{y_1}\phi_1\varepsilon_1^+)\right| \lesssim \frac{1-\beta_1}{t}\|\varepsilon_1\|_{L^2}^2 \lesssim \frac{\mathcal{G}}{t}.$$

The collection of above bounds yields (5.65).

Term $(-i|D|\varepsilon, \zeta D\varepsilon)$. We claim similarly

$$(-i|D|\varepsilon, \zeta D\varepsilon) = \frac{d}{dt}\left\{o_{\eta\to 0}(\mathcal{G})\right\} + O\left(\frac{1}{t^{N+1}} + \frac{\mathcal{G}(t)}{t}\right). \tag{5.70}$$

Indeed, we compute:

$$\begin{aligned} (-i|D|\varepsilon, \zeta D\varepsilon) &= \frac{1}{\lambda_1^2(1-\beta_1)}(-i(D\varepsilon_1^+ - D\varepsilon_1^-), \zeta_1(D\varepsilon_1^+ + D\varepsilon_1^-)) \\ &= \frac{1}{\lambda_1^2(1-\beta_1)}\left[(-iD\varepsilon_1^+, \zeta_1 D\varepsilon_1^-) + (iD\varepsilon_1^-, \zeta_1 D\varepsilon_1^+)\right] \\ &= \frac{2}{\lambda_1^2(1-\beta_1)}(iD\varepsilon_1^-, \zeta_1 D\varepsilon_1^+) = -\frac{2}{\lambda_1^2}(iD\varepsilon_1^-, \phi_1 D\varepsilon_1^+). \end{aligned}$$

We compute from (5.66):

$$\begin{aligned} &\frac{1}{\lambda_1^2}(iD\varepsilon_1^-, \phi_1 D\varepsilon_1^+) = \frac{1-\beta_1}{\lambda_1(1+\beta_1)}(i\partial_t\varepsilon_1^-, i\phi_1 D\varepsilon_1^+) \\ - \quad &\frac{1-\beta_1}{\lambda_1^2(1+\beta_1)}\left(\lambda_1^{1+\frac{1}{2}}\Pi^-\tilde{F}_1 + i\frac{(\lambda_1)_{s_1}}{\lambda_1}(\frac{\varepsilon_1^-}{2} + y_1\partial_{y_1}\varepsilon_1^-) - i\frac{(\beta_1)_{s_1}}{1-\beta_1}y_1\partial_{y_1}\varepsilon_1^-\right. \\ + \quad &\left. i\frac{(x_1)_t - \beta_1}{1-\beta_1}\partial_{y_1}\varepsilon_1^- + \gamma_{s_1}\varepsilon_1^-, i\phi_1 D\varepsilon_1^+\right). \end{aligned}$$

We estimate from (5.69), (D.9):

$$\begin{aligned}&(1-\beta_1)|(\Pi^-\tilde F_1, i\phi_1 D\varepsilon_1^+)| = (1-\beta_1)|(\Pi^-\tilde F_1, [\Pi^+,\phi_1]D\varepsilon_1^+)|\\ \lesssim\ & (1-\beta_1)\|D[\Pi^+,\phi_1]\Pi^-\tilde F_1\|_{L^2}\|\varepsilon_1^+\|_{L^2} \lesssim \frac{1-\beta_1}{t}\|\tilde F_1\|_{L^2}\|\varepsilon_1\|_{L^2} \lesssim \frac{1-\beta_1}{t}\|\varepsilon_1\|_{L^2}^2 + \frac{1}{t^{N+1}}\\ \lesssim\ & \frac{\mathcal G}{t} + \frac{1}{t^{N+1}}.\end{aligned}$$

We integrate by parts,

$$\begin{aligned}|(iy_1\partial_{y_1}\varepsilon_1^-, i\phi_1 D\varepsilon_1^+)| &= |(i\varepsilon_1^-, \partial_{y_1}(y_1\phi_1\partial_{y_1}\varepsilon_1^+)|\\ &\lesssim |(i\varepsilon_1^-, (\phi_1 + y_1\partial_{y_1}\phi_1)\partial_{y_1}\varepsilon_1^+)| + (i\varepsilon_1^-, y_1\phi_1\partial_{y_1}^2\varepsilon_1^+)|\ .\end{aligned}$$

For the first term, we estimate from (D.9):

$$|(i\varepsilon_1^-, (\phi_1 + y_1\partial_{y_1}\phi_1)\partial_{y_1}\varepsilon_1^+)| \lesssim \|\varepsilon_1^+\|_{L^2}\|\partial_{y_1}[\Pi^+, \phi_1 + y_1\partial_{y_1}\phi_1]\varepsilon_1^-\|_{L^2} \lesssim \frac{1}{t}\|\varepsilon_1\|_{L^2}^2.$$

For the second term, we use $[\Pi^+, y_1]\partial_{y_1}\varepsilon_1 = 0$ and (D.10) to estimate

$$\begin{aligned}|(i\varepsilon_1^-, y_1\phi_1\partial_{y_1}^2\varepsilon_1^+)| &= |(i\varepsilon_1^-, \phi_1\Pi^+(y_1\partial_{y_1}^2\varepsilon_1^+))| = (i\varepsilon_1^-, [\phi_1, \Pi^+](y_1\partial_{y_1}^2\varepsilon_1^+))|\\ &\lesssim \|\varepsilon_1^+\|_{L^2}\|\langle y_1\rangle\partial_{y_1}^2[\Pi^+,\phi_1]\varepsilon_1^-\|_{L^2} \lesssim \frac{\|\varepsilon_1\|_{L^2}^2}{t}.\end{aligned}$$

Similarly,

$$|(i\partial_{y_1}\varepsilon_1^-, \phi_1 D\varepsilon_1^+)| + |(\varepsilon_1^-, \phi_1 D\varepsilon_1^+)| \lesssim \frac{1}{t}\|\varepsilon_1\|_{L^2}^2.$$

We therefore integrate by parts and using (5.68)

$$\begin{aligned}&\frac{1}{\lambda_1^2}(iD\varepsilon_1^-, \phi_1 D\varepsilon_1^+) = \frac{1-\beta_1}{\lambda_1(1+\beta_1)}(i\partial_t\varepsilon_1^-, i\phi_1 D\varepsilon_1^+) + O\left(\frac{\mathcal G}{t} + \frac{1}{t^{N+1}}\right)\\ =\ & \frac{d}{dt}\left\{\frac{1-\beta_1}{\lambda_1(1+\beta_1)}(\varepsilon_1^-, \phi_1 D\varepsilon_1^+)\right\} - \frac{1-\beta_1}{\lambda_1^2(1+\beta_1)}(i\phi_1\varepsilon_1^-, Di\partial_{s_1}\varepsilon_1^+) + O\left(\frac{\mathcal G}{t} + \frac{1}{t^{N+1}}\right).\end{aligned}$$

We now reinject (5.67) and estimate all terms similarly as above using (D.9), (D.10), and (5.70) follows through a completely similar chain of estimates.

$\underline{(\partial_t+\partial_x)\zeta \text{ terms}}$. These terms gain an extra $1-\beta_1$ which is essential to treat the degeneracy of the kinetic energy and the L^2 mass for ε_1^+ in the lower bound (5.16), and we claim:

$$\begin{aligned}&|((\partial_t\zeta + \partial_x\zeta)D\varepsilon^+, \varepsilon^+)| + |((\partial_t\zeta + \partial_x\zeta)D\varepsilon^-, \varepsilon^-)| + |((\partial_t\zeta + \partial_x\zeta)D\varepsilon^-, \varepsilon^+)|\\ +\ & |((\partial_t\zeta + \partial_x\zeta)D\varepsilon^+, \varepsilon^-)| \lesssim \frac{\mathcal G}{t}.\end{aligned} \tag{5.71}$$

Indeed, let

$$\psi(t,x) = \frac{\partial_t\zeta + \partial_x\zeta}{\sqrt{\phi}}, \quad \psi(x) = \psi_1(y_1). \tag{5.72}$$

We estimate, after renormalization to the y_1 variable, using (5.11), (E.3), (E.12), (E.14), (E.15),

$$\begin{aligned}
& |(D\varepsilon^+, (\partial_t\zeta + \partial_x\zeta)\varepsilon^\pm)| \lesssim \left|(\sqrt{\phi_1} D\varepsilon_1^+, \psi_1\varepsilon_1^\pm)\right| \\
\lesssim\ & \left|(D(\sqrt{\phi_1}\varepsilon_1^+), \psi_1\varepsilon_1^\pm)\right| + \int \frac{|\partial_{y_1}\phi_1|}{\sqrt{\phi_1}} |\psi_1||\varepsilon_1|^2 dy_1 \\
\lesssim\ & \||D|^{\frac12}(\sqrt{\phi_1}\varepsilon_1^+)\|_{L^2} \left[\|[|D|^{\frac12}, \psi_1]\varepsilon_1^\pm\|_{L^2} + \|\psi_1|D|^{\frac12}\varepsilon_1^\pm\|_{L^2}\right] + \frac{1-\beta_1}{R^2}\|\varepsilon_1\|_{L^2}^2 \\
\lesssim\ & \left[\|\sqrt{\phi_1}|D|^{\frac12}\varepsilon_1^+\|_{L^2} + \frac{1}{\sqrt{R}}\|\varepsilon_1\|_{L^2}\right] \times \left[\frac{1-\beta_1}{t^{\frac32}}\|\varepsilon_1\|_{L^2} + \frac{1-\beta_1}{t}\|\sqrt{\phi_1}|D|^{\frac12}\varepsilon_1^\pm\|_{L^2}\right] \\
+\ & \frac{1-\beta_1}{R^2}\int |\varepsilon_1|^2 dy_1 \lesssim \frac{1}{t}\mathcal{G}(t)\ .
\end{aligned}$$

Finally, we infer

$$\begin{aligned}
& |((\partial_t\zeta + \partial_x\zeta)D\varepsilon^-, \varepsilon^\pm)| = |(\psi\sqrt{\phi} D\varepsilon^-, \varepsilon^\pm)| = |(\sqrt{\phi_1} D\varepsilon_1^-, \psi_1\varepsilon_1^\pm)| \\
=\ & \left|([\sqrt{\phi_1}, |D|^{\frac12}]\varepsilon_1^- + |D|^{\frac12}\sqrt{\phi_1}|D|^{\frac12}\varepsilon_1^-, \psi_1\varepsilon_1^\pm)\right| \\
\lesssim\ & \|[\sqrt{\phi_1}, |D|^{\frac12}]\varepsilon_1^-\|_{L^2}\|\psi_1\varepsilon_1^\pm\|_{L^2} + \|\sqrt{\phi_1}|D|^{\frac12}\varepsilon_1^-\|_{L^2}\left(\|\psi_1|D|^{\frac12}\varepsilon_1^\pm\|_{L^2} + \|[|D|^{\frac12}, \psi_1]\varepsilon_1\|_{L^2}\right)\ ,
\end{aligned}$$

and hence, using (E.14), (E.15), (E.12),

$$\begin{aligned}
& |((\partial_t\zeta + \partial_x\zeta)D\varepsilon^-, \varepsilon^\pm)| \lesssim \frac{1}{\sqrt{t}}\frac{1-\beta_1}{t}\|\varepsilon_1\|_{L^2}^2 \\
+\ & \||D|^{\frac12}\varepsilon_1^-\|_{L^2}\left(\frac{1-\beta_1}{t}\|\sqrt{\phi_1}|D|^{\frac12}\varepsilon_1^\pm\|_{L^2} + \frac{1-\beta_1}{t^{\frac32}}\|\varepsilon_1\|_{L^2}\right) \lesssim \frac{\mathcal{G}}{t},
\end{aligned}$$

and (5.71) is proved.

Step 6: Control of mass terms. We claim:

$$\begin{aligned}
& \frac12((\partial_t\theta)\varepsilon, \varepsilon) + (-i|D|\varepsilon, \theta\varepsilon) = \frac{d}{dt}\left\{o_{\eta\to 0}(\mathcal{G})\right\} \qquad (5.73) \\
+\ & O\left(\frac{1}{t^{N+1}} + \frac1t\left[\mathcal{G}(t) + \frac{\|\widetilde{\varepsilon}_1\|_{L^2}^2}{t}\right]\right).
\end{aligned}$$

Indeed, we split $\varepsilon = \varepsilon^+ + \varepsilon^-$ and compute:

$$\begin{aligned}
& \frac12((\partial_t\theta)\varepsilon, \varepsilon) + (-i|D|\varepsilon, \theta\varepsilon) = \frac12((\partial_t\theta)(\varepsilon^+ + \varepsilon^-), \varepsilon^+ + \varepsilon^-) + (-i(D\varepsilon^+ - D\varepsilon^-), \theta(\varepsilon^+ + \varepsilon^-)) \\
=\ & \frac12((\partial_t\theta)(\varepsilon^+ + \varepsilon^-), \varepsilon^+ + \varepsilon^-) - (\partial_x\varepsilon^+ - \partial_x\varepsilon^-, \theta(\varepsilon^+ + \varepsilon^-)) \\
=\ & \frac12((\partial_t\theta + \partial_x\theta)\varepsilon^+, \varepsilon^+) + \frac12((\partial_t\theta - \partial_x\theta)\varepsilon^-, \varepsilon^-) + (\partial_t\theta\varepsilon^+, \varepsilon^-) + (\partial_x\varepsilon^-, \theta\varepsilon^+) - (\partial_x\varepsilon^+, \theta\varepsilon^-) \\
=\ & \frac12((\partial_t\theta + \partial_x\theta)\varepsilon^+, \varepsilon^+) + \frac12((\partial_t\theta - \partial_x\theta)\varepsilon^-, \varepsilon^-) + ((\partial_t\theta + \partial_x\theta)\varepsilon^+, \varepsilon^-) \\
-\ & (\partial_x\theta\varepsilon^+, \varepsilon^-) + (\partial_x\varepsilon^-, \theta\varepsilon^+) + (\varepsilon^+, \partial_x\theta\varepsilon^- + \theta\partial_x\varepsilon^-) \\
=\ & \frac12((\partial_t\theta + \partial_x\theta)\varepsilon^+, \varepsilon^+) + \frac12((\partial_t\theta - \partial_x\theta)\varepsilon^-, \varepsilon^-) + ((\partial_t\theta + \partial_x\theta)\varepsilon^+, \varepsilon^-) \\
+\ & 2(\theta\varepsilon^+, \partial_x\varepsilon^-)\ ,
\end{aligned}$$

and estimate all terms.

$(\partial_t+\partial_x)\theta$ terms. We estimate from (E.19),

$$|((\partial_x\theta+\partial_t\theta)\varepsilon^{\pm},\varepsilon^{\pm})|\lesssim\frac{\|\varepsilon\|_{L^2}^2}{t}\lesssim\frac{\mathcal{G}}{t}.$$

Term $((\partial_t-\partial_x)\theta\varepsilon^-,\varepsilon^-)$. For $t\geqslant T^-$, we use

$$|\lambda_2-\lambda_1|\lesssim\eta$$

and (E.17):

$$|((\partial_x\theta)\varepsilon^-,\varepsilon^-)|\lesssim\frac{|\lambda_2-\lambda_1|}{(1-\beta_1)R}\|\varepsilon\|_{L^2}^2\lesssim\frac{\mathcal{G}(t)}{t}.$$

For $t\leqslant T^-$, we use the bound

$$|\lambda_2-\lambda_1|\lesssim\frac{1}{t}$$

and the space localization of $\partial_{y_1}\theta_1$ to estimate from (E.16):

$$|((\partial_x\theta)\varepsilon^-,\varepsilon^-)|\lesssim|((\langle z_1\rangle^{1+\alpha}\partial_{y_1}\theta_1)\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_1)|\lesssim\frac{|\lambda_2-\lambda_1|}{t}\|\widetilde{\varepsilon}_1\|_{L^2}^2\lesssim\frac{\|\widetilde{\varepsilon}_1\|_{L^2}^2}{t^2}.$$

Term $(\theta\varepsilon^+,\partial_x\varepsilon^-)$. For the last term, we renormalize to the y_1 variable

$$(\partial_x\varepsilon^-,\theta\varepsilon^+)=\frac{1}{\lambda_1}(D\varepsilon_1^-,-i\theta_1\varepsilon_1^+)$$

and hence, using (5.66),

$$\begin{aligned}(\partial_x\varepsilon^-,\theta\varepsilon^+)&=\frac{1-\beta_1}{1+\beta_1}(i\partial_t\varepsilon_1^-,i\theta_1\varepsilon_1^+)\\&+\frac{1-\beta_1}{\lambda_1(1+\beta_1)}\left(\lambda_1^{1+\frac12}\Pi^-\tilde{F}_1+i\frac{(\lambda_1)_{s_1}}{\lambda_1}(\frac{\varepsilon_1^-}{2}+y_1\partial_{y_1}\varepsilon_1^-)\right.\\&\left.-i\frac{(\beta_1)_{s_1}}{1-\beta_1}y_1\partial_{y_1}\varepsilon_1^-+i\frac{(x_1)_t-\beta_1}{1-\beta_1}\partial_{y_1}\varepsilon_1^-+(\gamma_1)_{s_1}\varepsilon_1^-,\theta_1\varepsilon_1^+\right)\\&=\frac{1-\beta_1}{1+\beta_1}(i\partial_t\varepsilon_1^-,i\theta_1\varepsilon_1^+)+O\left(\frac{1-\beta_1}{t}\|\varepsilon_1\|_{L^2}^2\right)\end{aligned}$$

and hence, integrating by parts in time and using (5.67), (5.68), (5.69),

$$\begin{aligned}(\partial_x\varepsilon^-,\theta\varepsilon^+)&=\frac{d}{dt}\left\{\frac{1-\beta_1}{1+\beta_1}(\varepsilon_1^-,\theta_1\varepsilon_1^+)\right\}\\&-\frac{1-\beta_1}{\lambda_1(1+\beta_1)}(\varepsilon_1^-,\theta_1D\varepsilon_1^+)+O\left(\frac{1-\beta_1}{t}\|\varepsilon_1\|_{L^2}^2+\frac{1}{t^{N+1}}.\right)\ .\end{aligned}$$

We estimate from (D.9),

$$|(\varepsilon_1^-,\theta_1D\varepsilon_1^+)|=|(D[\Pi^+,\theta_1]\varepsilon_1^-,\varepsilon_1^+)|\lesssim\frac{1}{t}\|\varepsilon_1\|_{L^2}^2$$

and, for the boundary term in time, we use

$$\theta_1=\frac{1}{\lambda_2}\left[\mu\Psi_1+1-\Psi_1\right]\ ,$$

to compute

$$\frac{1-\beta_1}{1+\beta_1}(\varepsilon_1^-,\theta_1\varepsilon_1^+)=\frac{1-\beta_1}{\lambda_2(1+\beta_1)}(\varepsilon_1^-,(\mu\Psi_1+1-\Psi_1)\varepsilon_1^+)=\frac{1-\beta_1}{\lambda_2(1+\beta_1)}(\mu-1)(\varepsilon_1^-,\Psi_1\varepsilon_1^+)\,.$$

Hence

$$|\frac{1-\beta_1}{1+\beta_1}(\varepsilon_1^-,\theta_1\varepsilon_1^+)|\lesssim|\lambda_2-\lambda_1|(1-\beta_1)\|\varepsilon_1\|_{L^2}^2\lesssim\eta^{\delta}\mathcal{G}$$

which concludes the proof of (5.73).

Step 7: Small time improved bound for $\|\widetilde{\varepsilon}_1\|_{L^2}$. The collection of above estimates yields the differential control:

$$\left|\frac{d}{dt}\left\{\mathcal{G}(t)(1+o_{\eta\to 0}(1))\right\}\right| \lesssim \frac{1}{t}\left[\mathcal{G}(t)+\frac{\|\widetilde{\varepsilon}_1\|_{L^2}^2}{t}\right]+\frac{1}{t^{N+1}}. \tag{5.74}$$

We now estimate the $\widetilde{\varepsilon}_1$ term first through the following space time bound:

$$\int_t^{T_n}\frac{\|\widetilde{\varepsilon}_1(\tau)\|_{L^2}^2}{\tau}d\tau \lesssim \int_t^{T_n}\left[\mathcal{G}(\tau)+\frac{1}{\tau^{N+1}}\right]d\tau. \tag{5.75}$$

which improves on the trivial bound $\|\widetilde{\varepsilon}_1(t)\|_{L^2}^2 \lesssim \|\varepsilon_1^-\|_{L^2}^2 \lesssim \frac{\mathcal{G}(t)}{\eta}$ for $t\leqslant T^-$. Indeed, let

$$h(s_1,y_1)=H\left(\frac{y_1}{s_1}\right),\quad H(z_1)=\int_{z_1}^{+\infty}\frac{dz}{1+\langle z\rangle^{1+\alpha}}.$$

We estimate from (5.66):

$$\begin{aligned}
&\frac{1}{2}\frac{d}{ds_1}\int h|\varepsilon_1^-|^2=\frac{1}{2}\int\left(\partial_{s_1}-\frac{1+\beta_1}{1-\beta_1}\partial_{y_1}\right)h|\varepsilon_1^-|^2\\
+\;&\left(ih\varepsilon_1^-,\lambda_1^{1+\frac{1}{2}}\Pi^-\tilde{F}+i\frac{(\lambda_1)_{s_1}}{\lambda_1}(\frac{\varepsilon_1^-}{2}+y_1\partial_{y_1}\varepsilon_1^-)-i\frac{(\beta_1)_{s_1}}{1-\beta_1}y_1\partial_{y_1}\varepsilon_1^-+i\frac{(x_1)_t-\beta_1}{1-\beta_1}\partial_{y_1}\varepsilon_1^-+(\gamma_1)_{s_1}\varepsilon_1^-\right)\\
=\;&\frac{1}{2}\int\left(\partial_{s_1}-\frac{1+\beta_1}{1-\beta_1}\partial_{y_1}\right)h|\varepsilon_1^-|^2+O\left(\|\varepsilon_1\|_{L^2}^2+\frac{1}{t^{N+1}}\right)
\end{aligned}$$

where we integrated by parts and use (5.68), (5.69) in the last step. Moreover,

$$\left(\partial_{s_1}-\frac{1+\beta_1}{1-\beta_1}\partial_{y_1}\right)h=\frac{1}{s_1}\left(-z_1-\frac{1+\beta_1}{1-\beta_1}\right)\partial_{z_1}H=\frac{1}{s_1}\left(z_1+\frac{1+\beta_1}{1-\beta_1}\right)\frac{1}{1+\langle z_1\rangle^{1+\alpha}}$$

and hence the bound using $\frac{|z_1|}{\langle z_1\rangle^{1+\alpha}}\leqslant 1$:

$$\frac{1}{s_1}\int\left(\frac{1+\beta_1}{1-\beta_1}\right)\frac{|\varepsilon_1^-|^2}{1+\langle z_1\rangle^{1+\alpha}}\leqslant C\frac{\mathcal{G}}{1-\beta_1}+\frac{1}{t^{N+1}}+\frac{1}{2}\frac{d}{ds_1}\int h|\varepsilon_1^-|^2.$$

We integrate this on $[s_1(t),s_1(T_n)]$ with $\varepsilon_1(s_1(T_n))=0$ and (5.75) follows from $s_1\sim t\sim R$, $\frac{\eta}{2}\leqslant 1-\beta_1\leqslant 2\eta$.

Step 8: Conclusion. We integrate (5.74) in time on $[t,T_n]$ using $\varepsilon(T_n)=0$ so that

$$\mathcal{G}(t)\lesssim\int_t^{T_n}\frac{\mathcal{G}(\tau)}{\tau}d\tau+\int_t^{T_n}\frac{\|\widetilde{\varepsilon}_1\|_{L^2}^2}{\tau^2}d\tau+\frac{1}{t^N}.$$

The first term is estimated using the bootstrap bound (5.20):

$$\int_t^{T_n}\frac{\mathcal{G}(\tau)}{\tau}d\tau\lesssim\int_t^{T_n}\frac{1}{\tau^{1+\frac{N}{2}}}d\tau\lesssim\frac{1}{N}\frac{1}{t^{\frac{N}{2}}}.$$

For the second term, we estimate from (5.75):

$$\begin{aligned}
\int_t^{T_n}\frac{\|\widetilde{\varepsilon}_1\|_{L^2}^2}{\tau^2}d\tau\;&\lesssim\;\frac{1}{t}\int_t^{T_n}\frac{\|\widetilde{\varepsilon}_1\|_{L^2}^2}{\tau}d\tau\lesssim\frac{1}{t}\int_t^{T_n}\left[\mathcal{G}(\tau)+\frac{1}{\tau^{N+1}}\right]d\tau\\
&\lesssim\;\frac{1}{t^{N+1}}+\frac{1}{t}\int_t^{T_n}\frac{d\tau}{\tau^{\frac{N}{2}}}\lesssim\frac{1}{N}\frac{1}{t^{\frac{N}{2}}}
\end{aligned}$$

which concludes the proof of (5.43). □

5.7. **Proof of the bootstrap Proposition 5.2.** We are now in position to conclude the control of the geometrical parameters and the H^1 bound.

Proof of Proposition 5.2. First observe that (5.43) yields the improved $H^{\frac{1}{2}}$ bound in (5.23). Moreover, the bounds (5.21) at T^- and (5.34), (5.16) (5.43) allow us to apply the perturbative Lemma 4.13 and conclude that $\mathcal{P}$ satisfies (4.77). We therefore need to prove (5.24) and the improved H^1 bound in (5.23).

Step 1: Proof of (5.24). Recall (5.17) so that

$$(1-\beta_j)\partial_{\beta_j} = \partial_{\tilde{\beta}_j}.$$

Since RM_j, RB_j are L^∞-admissible, we have

$$\sum_{j,k=1}^{2} \left|\partial_{\lambda_j} M_k\right| + |\partial_{\tilde{\beta}_k} M_k| + \sum_{j=1}^{2} |R\partial_R M_j| + |\partial_\Gamma M_j| \lesssim \frac{1}{t} \tag{5.76}$$

$$\sum_{j,k=1}^{2} \left|\partial_{\lambda_j} B_k\right| + |\partial_{\tilde{\beta}_k} B_k| + \sum_{j=1}^{2} |R\partial_R B_j| + |\partial_\Gamma B_j| \lesssim \frac{1}{t}. \tag{5.77}$$

For $t \geq T^-$, the same chain of estimates like for the proof of Proposition 4.12 using

$$|1-\mu| \lesssim \eta \quad \text{for} \quad t \geq T^-$$

ensures the more precise control:

$$\sum_{j=1}^{2} |\partial_\Gamma M_j| + |R\partial_R M_j| + \sum_{j,k=1}^{2} |\partial_{\tilde{\beta}_k} M_j| \lesssim \frac{1}{t^2}. \tag{5.78}$$

Indeed, if $j=1$, we know that $b^{-1}R(1+(1-\beta_1)R)M_1$ is L^∞-admissible, so that

$$|\partial_\Gamma M_1| + |R\partial_R M_1| + \sum_{k=1}^{2} |\partial_{\tilde{\beta}_k} M_1| \lesssim \frac{b}{R(1+(1-\beta_1)R)} \lesssim \frac{1}{t^2}$$

since, for $t \geq T^-$, $b \simeq \eta^2$, $1-\beta_1 \sim \eta$ and $R \sim t$. If $j=2$, Corollary 4.8 leads to

$$|\partial_\Gamma M_2| + |R\partial_R M_2| + \sum_{k=1}^{2} (1-\beta_k)|\partial_{\beta_k} M_2| \lesssim \frac{|1-\mu| + (1-\beta_2)|\log(1-\beta_2)| + R^{-1}}{R(1+(1-\beta_1)R)}.$$

Since, for $t \geq T^-$, $|1-\mu| \lesssim \eta, 1-\beta_2 \simeq \eta^3, 1-\beta_1 \sim \eta, R \sim t$, we infer (5.78). Recalling (5.18), (5.19), then (5.76), (5.77), (5.78) ensure:

$$|B_j - B_j^\infty| \lesssim \frac{1}{t}\left[\sum_{j=1}^{2}(|\Delta\lambda_j| + |\Delta\tilde{\beta}_j|) + |\Delta\Gamma|\right] + \frac{1}{t^2}|\Delta R| \lesssim \frac{1}{t^{\frac{N}{8}+1}}$$

$$|M_j - M_j^\infty| \lesssim \frac{1}{t}\sum_{j=1}^{2}|\Delta\lambda_j| + \frac{1}{t^2}\left(\sum_{j=1,2}|\Delta\tilde{\beta}_j| + |\Delta\Gamma|\right) + \frac{|\Delta R|}{t^3} \lesssim \frac{1}{t^{\frac{N}{8}+2}}.$$

Moreover, from (4.72), (5.34), (5.43):

$$|(\lambda_j)_t - (\lambda_j^\infty)_t| = \left|\frac{(\lambda_j)_{s_j}}{\lambda_j} - \frac{(\lambda_j^\infty)_{s_j^\infty}}{\lambda_j^\infty}\right| \lesssim |M_j - M_j^\infty| + \mathrm{Mod}_j \lesssim \frac{1}{t^{\frac{N}{8}+2}}$$

which time integration using (5.82) ensures:

$$|\Delta\lambda_j| \lesssim \frac{1}{Nt^{\frac{N}{8}+1}}.$$

We now compute similarly:

$$|(\tilde{\beta}_j)_t-(\tilde{\beta}_j^\infty)_t| = \left|\frac{1}{\lambda_j}\left[B_j + O(\mathrm{Mod}_j)\right] - \frac{1}{\lambda_j^\infty}B_j^\infty\right| \lesssim \frac{1}{t^{\frac{N}{4}}}+|B_j-B_j^\infty|+|B_j^\infty||\lambda_j-\lambda_j^\infty| \lesssim \frac{1}{t^{\frac{N}{8}+1}}$$

and hence by integration in time:

$$|\Delta\tilde{\beta}_j| \lesssim \frac{1}{Nt^{\frac{N}{8}}}.$$

We now compute the phase shift:

$$|\Gamma_t - \Gamma_t^\infty| = \left|\frac{1}{\lambda_2} - \frac{1}{\lambda_2^\infty} - \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_2^\infty}\right) + O(\mathrm{Mod}_1 + \mathrm{Mod}_2)\right| \lesssim \frac{1}{Nt^{\frac{N}{8}+1}}$$

and hence

$$|\Delta\Gamma| \lesssim \frac{1}{N^2t^{\frac{N}{8}}}.$$

We now estimate from (5.27):

$$|R_t - R_t^\infty| \lesssim \sum_{j=1,2} |\Delta\tilde{\beta}_j| + |\Delta\lambda_j| + \mathrm{Mod}_j + |\Delta\Gamma| + \frac{1}{t}|\Delta R| \lesssim \frac{1}{t^{\frac{N}{8}}}$$

which time integration concludes the proof of (5.24).

Step 2: Proof of the H^1 bound in (5.23). Since we have closed the $H^{\frac{1}{2}}$ bound at the linear level, closing the H^1 bound or any higher Sobolev norm is now elementary. Recall (5.45)

$$i\partial_t\varepsilon - |D|\varepsilon + 2|\Phi|^2\varepsilon + \Phi^2\overline{\varepsilon} = G.$$

Let

$$z = |D|^{\frac{1}{2}}\varepsilon,$$

then:

$$\begin{cases} i\partial_t z - |D|z + 2|\Phi|^2 z + \Phi^2\overline{z} = \tilde{G} \\ \tilde{G} = |D|^{\frac{1}{2}}G - 2[|D|^{\frac{1}{2}}, |\Phi|^2]\varepsilon - [|D|^{\frac{1}{2}}, \Phi^2]\overline{\varepsilon} \end{cases}. \tag{5.79}$$

We now run an energy identity on (5.79). We consider

$$\mathcal{G}_0(z) := \frac{1}{2}\Big[(|D|z, z) + (z, z) - (2|\Phi|^2 z + \Phi^2\overline{z}, z)\Big]$$

then from (5.20):

$$\|z\|_{H^{\frac{1}{2}}}^2 \lesssim \mathcal{G}_0(z) + \|z\|_{L^2}^2 \lesssim \mathcal{G}_0(z) + \frac{1}{t^{\frac{N}{2}}}. \tag{5.80}$$

We compute the associated energy identity:

$$\begin{aligned}
\frac{d}{dt}\mathcal{G}_0 &= (\partial_t z, |D|z + z - 2|\Phi|^2 z - \Phi^2\overline{z}) - (\partial_t(|\Phi|^2)z + \frac{1}{2}\partial_t\Phi^2\overline{z}, z) \\
&= (-i\tilde{G}, |D|z + z - 2|\Phi|^2 z - \Phi^2\overline{z}) + (z^2, i\Phi^2) - (\partial_t\Phi, 2\Phi|z|^2 + \bar{\Phi}z^2) \\
&= (z^2, i\Phi^2) + (i|D|^{\frac{1}{2}}(\Psi + S), |D|z + z - 2|\Phi|^2 z - \Phi^2\overline{z}) \\
&+ \left(i\left[2[|D|^{\frac{1}{2}}, |\Phi|^2]\varepsilon - [|D|^{\frac{1}{2}}, \Phi^2]\overline{\varepsilon}\right], |D|z + z - 2|\Phi|^2 z - \Phi^2\overline{z}\right) \\
&+ \left(i|D|^{\frac{1}{2}}N(\varepsilon), |D|z + z - 2|\Phi|^2 z - \Phi^2\overline{z}\right) \\
&- (\partial_t\Phi, 2\Phi|z|^2 + \bar{\Phi}z^2) \\
&= I + II + III + IV + V.
\end{aligned} \tag{5.81}$$

We now estimate all terms in (5.81). From (5.20) and $\|\Phi\|_{L^\infty} \lesssim 1$:

$$|I| = |(z^2, i\Phi^2)| \lesssim \|\varepsilon\|^2_{H^{\frac12}} \leqslant \frac{1}{t^{\frac N4}}.$$

For II, we use (5.32) and an integration by parts and (5.20) to estimate:

$$|(i|D|^{\frac12}\Psi, |D|z + z - 2|\Phi|^2 z - \Phi^2\overline{z})| \lesssim \|\Psi\|_{H^{\frac32}} \|z\|_{L^2} \lesssim \frac{1}{\eta^C t^{N+1}} \frac{1}{t^{\frac N4}} \leqslant \frac{1}{t^{\frac N4}}.$$

For the modulation equation term, we estimate in brute force using the admissibility of V_j, (5.34) and (5.20):

$$\|S\|_{H^{\frac32}} \lesssim \frac{1}{\eta^C}(\mathrm{Mod}_1 + \mathrm{Mod}_2) \lesssim \frac{1}{\eta^C}\frac{1}{t^{\frac N4}}$$

and hence:

$$|(i|D|^{\frac12}S, |D|z + z - 2|\Phi|^2 z - \Phi^2\overline{z})| \lesssim \|S\|_{H^{\frac32}} \|z\|_{L^2} \lesssim \frac{1}{\eta^C t^{\frac N4}} \frac{1}{t^{\frac N4}} \leqslant \frac{1}{t^{\frac N4}}.$$

For III, we use that for any function χ:

$$|D|^{\frac12}[|D|^{\frac12}, \chi] = [|D|, \chi] - [|D|^{\frac12}, \chi]|D|^{\frac12}$$

and hence using (5.20), (D.1) with $R = 1$ and the admissibility of V_j:

$$\begin{aligned}
&\left|\left(2i[|D|^{\frac12}, |\Phi|^2]\varepsilon, |D|z\right)\right| \lesssim \||D|^{\frac12}[|D|^{\frac12}, |\Phi|^2]\varepsilon\|_{L^2}\|z\|_{H^{\frac12}} \\
\lesssim\ & \left(\|[|D|, |\Phi|^2]\varepsilon\|_{L^2} + \|[|D|^{\frac12}, |\Phi|^2]|D|^{\frac12}\varepsilon\|_{L^2}\right)\|z\|_{H^{\frac12}} \lesssim \frac{1}{\eta^C}\|\varepsilon\|_{H^{\frac12}}\|z\|_{H^{\frac12}} \\
\lesssim\ & \frac{1}{\eta^C t^{\frac N4}}\|z\|_{H^{\frac12}} \lesssim \frac{1}{t^{\frac N4+1}}.
\end{aligned}$$

The term $(2i[|D|^{\frac12}, |\Phi|^2]\varepsilon, z - 2|\Phi|^2 z - \Phi^2\overline{z})$ being easier to handle and proceeding analogously for the terms containing $i[|D|^{\frac12}, \Phi^2]\bar\varepsilon$, we conclude that

$$|III| \lesssim \frac{1}{t^{\frac N4+1}}$$

For IV, we develop the cubic non linear term. The most dangerous nonlinear term is the following which we estimate in brute force by Sobolev and (5.20):

$$\begin{aligned}
\left|\left(i|D|^{\frac12}(\varepsilon|\varepsilon|^2), |D|z\right)\right| = \left|\left(i|D|(|\varepsilon|^2\varepsilon), |D|^{\frac12}z\right)\right| &\lesssim \|D(|\varepsilon|^2\varepsilon)\|_{L^2}\|z\|_{H^{\frac12}} \\
&\lesssim \|D\varepsilon\|_{L^2}\|\varepsilon\|^2_{L^\infty}\|z\|_{H^{\frac12}} \lesssim (\|\varepsilon\|^2_{L^2} + \|D\varepsilon\|^2_{L^2})\|z\|^2_{H^{\frac12}} \\
&\lesssim \frac{1}{t^{\frac N4+1}}.
\end{aligned}$$

Then, by the fractional Leibniz rule and (5.20), we also have

$$\begin{aligned}
\left|\left(i|D|^{\frac12}(\varepsilon|\varepsilon|^2), z - 2|\Phi|^2 z - \Phi^2\overline{z}\right)\right| &\lesssim \||D|^{\frac12}\varepsilon\|_{L^4}\|\varepsilon^2\|_{L^4}\|z\|_{L^2} \lesssim \|z\|_{H^{\frac12}}\|\varepsilon\|^2_{H^{\frac12}}\|z\|_{L^2} \\
&\lesssim \frac{1}{t^{\frac N4+1}}.
\end{aligned}$$

We argue similarly for the quadratic terms and obtain:

$$\begin{aligned}
\left|\left(i|D|^{\frac12}(2|\varepsilon|^2\Phi + \varepsilon^2\overline{\Phi}), |D|z + z - 2|\Phi|^2 z - \Phi^2\overline{z}\right)\right| &\lesssim \frac{1}{\eta^{\frac{1+4\delta}{2}}}(\|\varepsilon\|^2_{L^2} + \|\varepsilon\|^2_{\dot H^1})\|z\|_{H^{\frac12}} \\
&\lesssim \frac{1}{t^{\frac N4+1}}
\end{aligned}$$

Finally, to estimate V, we use from (5.52) the rough bound $\|\partial_t \Phi\|_{L^\infty} \lesssim \frac{1}{\eta^C}$ to estimate:

$$|V| = |(\partial_t \Phi, 2\Phi|z|^2 + \bar{\Phi} z^2)| \lesssim \|\partial_t \Phi\|_{L^\infty} \|\Phi\|_{L^\infty} \|z\|_{L^2}^2 \lesssim \frac{1}{\eta^C t^{\frac{N}{2}}} \lesssim \frac{1}{t^{\frac{N}{4}+1}}.$$

The collection of above bounds yields

$$\left| \frac{d}{dt} \mathcal{G}_0 \right| \lesssim \frac{1}{t^{\frac{N}{4}+1}}$$

which time integration using $\varepsilon(T_n) = z(T_n) = 0$ with (5.80) yields

$$\|z\|_{H^{\frac{1}{2}}}^2 \lesssim \frac{1}{N t^{\frac{N}{4}}}.$$

This concludes the proof of (5.23) and of Proposition 5.2. □

5.8. **Proof of Theorem 1.2.** We are now in position to conclude the proof of Theorem 1.2 as a simple consequence of Proposition 5.2. The argument is now classical [31], we recall it for the convenience of the reader.

Proof of Theorem 1.2. First observe that Proposition 5.2 implies that $u_n(t)$ solution to (5.1) satisfies:

$$\forall n \geqslant 1, \quad \forall t \in [T_{in}, T_n], \quad \|u_n(t) - \Phi_{\widetilde{\mathcal{P}}^\infty}(t)\|_{H^1} \leqslant \frac{1}{t^{\frac{N}{10}}}. \tag{5.82}$$

We now let $n \to +\infty$ and extract a non trivial limit to produce the dynamics described by Theorem 1.2.

Step 1: $H^{\frac{1}{2}}$-compactness. We claim that that the sequence $u_n(T_{in})$ is up to a subsequence $H^{\frac{1}{2}}$ compact. Indeed it is H^1 bounded from (5.82). We now claim that it is $H^{\frac{1}{2}}$ tight: $\forall \varepsilon_0 > 0$, $\exists R(\varepsilon_0)$ such that:

$$\int_{|x| \geqslant R(\varepsilon_0)} |u_n(T_{in})|^2 + \int_{|x| \geqslant R(\varepsilon_0)} ||D|^{\frac{1}{2}} u_n(T_{in})|^2 < \varepsilon_0. \tag{5.83}$$

Indeed, pick $\varepsilon_0 > 0$, then from (5.82), we may find a time $T(\varepsilon_0)$ such that

$$\|u_n(T(\varepsilon_0)) - \Phi_{\widetilde{\mathcal{P}}^\infty}(T(\varepsilon_0))\|_{H^1} < \varepsilon_0$$

and then by construction of $\Phi_{\widetilde{\mathcal{P}}^\infty}$, we may find $R = R(\varepsilon_0)$ such that

$$\forall R \geqslant R(\varepsilon_0), \quad \int (1 - \chi_R) |\Phi_{\widetilde{\mathcal{P}}^\infty}(T(\varepsilon_0))|^2 + \int (1 - \chi_R) ||D|^{\frac{1}{2}} \Phi_{\widetilde{\mathcal{P}}^\infty}(T(\varepsilon_0))|^2 < \varepsilon_0$$

from which

$$\int (1 - \chi_R) |u_n(T(\varepsilon_0))|^2 + \int (1 - \chi_R) ||D|^{\frac{1}{2}} u_n(T(\varepsilon_0))|^2 \lesssim \varepsilon_0.$$

We now propagate this information backwards at T_{in} by localizing the mass and energy conservation laws. Indeed, a brute force computation and (D.4) ensure

$$\left| \frac{d}{dt} \int (1 - \chi_R) |u_n|^2 \right| \lesssim \frac{\|u_n\|_{L^2}^2}{R} \lesssim \frac{1}{R}$$

and hence

$$\int (1 - \chi_R) |u_n(T_{in})|^2 \lesssim \varepsilon_0 + \frac{T(\varepsilon_0) - T_{in}}{R(\varepsilon_0)} \lesssim \varepsilon_0$$

by possibly raising the value of $R(\varepsilon_0)$. We similarly localize the conservation of energy with $\zeta_R = 1 - \chi_R$ and estimate using (D.3):

$$\left|\frac{d}{dt}\left\{\frac{1}{2}\int \zeta_R||D|^{\frac{1}{2}}u_n|^2 + \frac{1}{4}\int \zeta_R|u_n|^4\right\}\right| = \left|(\partial_t u_n, [|D|^{\frac{1}{2}}, \zeta_R]|D|^{\frac{1}{2}}u_n)\right|$$
$$\lesssim \frac{\|u_n\|_{H^1}^2 + \|u_n\|_{H^1}^4}{\sqrt{R}} \lesssim \frac{1}{\eta^C\sqrt{R}}$$

from which

$$\int (1-\chi_R)||D|^{\frac{1}{2}}u_n(T_{in})|^2 \lesssim \varepsilon_0 + \frac{T(\varepsilon_0) - T_{in}}{\eta^C R(\varepsilon_0)} \lesssim \varepsilon_0$$

by possibly raising the value of $R(\varepsilon_0)$, and (5.83) is proved.

Step 2: Conclusion. The H^1 global bound and the tightness (5.83) ensure using the compactness of the Sobolev embedding $H^1 \hookrightarrow H^{\frac{1}{2}}_{\text{loc}}$ the strong convergence up to a subsequence

$$u_n(T_{in}) \to u(T_{in}) \text{ in } H^{\frac{1}{2}} \text{ as } n \to +\infty.$$

Let u be the solution to (1.1) with data $u(T_{in})$, then the continuity of the flow in $H^{\frac{1}{2}}$ now yield the convergence of the whole sequence

$$\forall t \geq T_{in}, \quad u_n(t) \to u(t) \text{ in } H^{\frac{1}{2}} \text{ as } n \to +\infty$$

and hence from (5.82) and lower semi continuity of the norm:

$$\forall t \geq T_{in}, \quad \|u(t) - \Phi_{\tilde{\mathcal{P}}^\infty}(t)\|_{H^1} \leqslant \frac{1}{t^{\frac{N}{10}}}.$$

Moreover, since the modulation equation are computed from local in space scalar products, we have[7]

$$\forall t \geq T_{in}, \quad \tilde{\mathcal{P}}_{u_n}(t) \to \tilde{\mathcal{P}}_u(t) \text{ as } n \to +\infty,$$

and hence passing to the limit in the estimates (5.24), (4.77) ensures that u satisfies the expected dynamics of Theorem 1.2. □

Appendix A. Algebra for the Szegő profile

Lemma A.1 (Algebraic relations). *There holds:*

$$\int |Q^+|^2 = 2\pi, \quad \int \partial_y Q^+ \overline{Q^+} = 2i\pi, \tag{A.1}$$

$$\int |Q^+|^2\overline{\partial_y Q^+} = 2\pi, \quad \int (Q^+)^2\overline{\partial_y Q^+} = -4\pi, \tag{A.2}$$

$$\int |Q^+|^2\overline{Q^+} = 2i\pi, \quad \int (Q^+)^2\overline{Q^+} = -2i\pi. \tag{A.3}$$

$$(y\partial_y Q^+, iQ^+) = 0 \tag{A.4}$$

$$(y\partial_y Q^+, \partial_y Q^+) = 0. \tag{A.5}$$

Proof. Since

$$Q^+(y) = \frac{1}{y + \frac{i}{2}},$$

these formulas are for instance easy consequences of the residue theorem. □

[7] see for example [33] for a detailed proof in a similar functional setting.

Appendix B. The resonant two-soliton Szegő dynamics

This appendix revisits the result of Pocovnicu [43] about two-soliton solutions for the cubic Szegő equation on the line, by putting emphasis on the ODE system on modulation parameters. For ease of notation, in this appendix we set

$$Q(x) := Q^+(x) = \frac{1}{x+\frac{i}{2}} ,$$

and we look for a solution $u = u(t,x)$ of the cubic Szegő equation on the line

$$i\partial_t u - Du + \Pi(u^2\overline{u}) = 0$$

of the form

$$u(t,x) = \alpha_1(t)Q\left(\frac{x-x_1(t)}{\kappa_1(t)}\right) + \alpha_2(t)Q\left(\frac{x-x_2(t)}{\kappa_2(t)}\right) =: \alpha_1 Q_1 + \alpha_2 Q_2 .$$

B.1. **Derivation of the system.** Notice that

$$Q' = -Q^2 \ , \ xQ'(x) = -Q(x) + \frac{i}{2}Q(x)^2 ,$$

so that

$$\begin{aligned} Du - i\partial_t u &= i\frac{\alpha_1}{\kappa_1}Q_1^2 - \left(i\dot\alpha_1 + i\alpha_1\frac{\dot\kappa_1}{\kappa_1}\right)Q_1 - \alpha_1\left(i\frac{\dot x_1}{\kappa_1} + \frac{1}{2}\frac{\dot\kappa_1}{\kappa_1}\right)Q_1^2 + \\ &\quad i\frac{\alpha_2}{\kappa_2}Q_2^2 - \left(i\dot\alpha_2 + i\alpha_2\frac{\dot\kappa_2}{\kappa_2}\right)Q_2 - \alpha_2\left(i\frac{\dot x_2}{\kappa_2} + \frac{1}{2}\frac{\dot\kappa_2}{\kappa_2}\right)Q_2^2 \end{aligned}$$

On the other hand, using partial fraction decompositions, it is easy to check the following identities, for $j,k = 1,2$,

$$\begin{aligned} \Pi(Q_j^2\overline{Q}_k) &= -\frac{\kappa_j\kappa_k}{\left(x_j - x_k - i\frac{\kappa_j+\kappa_k}{2}\right)^2}Q_j + \frac{\kappa_k}{x_j - x_k - i\frac{\kappa_j+\kappa_k}{2}}Q_j^2 , \\ \Pi(Q_1Q_2\overline{Q}_j) &= \frac{\kappa_2\kappa_j\,Q_1}{\left(x_1 - x_2 + i\frac{\kappa_2-\kappa_1}{2}\right)\left(x_1 - x_j - i\frac{\kappa_1+\kappa_j}{2}\right)} + \frac{\kappa_1\kappa_j\,Q_2}{\left(x_2 - x_1 + i\frac{\kappa_1-\kappa_2}{2}\right)\left(x_2 - x_j - i\frac{\kappa_2+\kappa_j}{2}\right)} . \end{aligned}$$

This leads to

$$\begin{aligned} \Pi(u^2\overline{u}) &= \Pi(Q_1^2\overline{Q}_1) + \Pi(Q_1^2\overline{Q}_2) + 2\Pi(Q_1Q_2\overline{Q}_1) + 2\Pi(Q_1Q_2\overline{Q}_2) + \Pi(Q_2^2\overline{Q}_1) + \Pi(Q_2^2\overline{Q}_2) \\ &= \beta_1 Q_1^2 + \gamma_1 Q_1 + \beta_2 Q_2^2 + \gamma_2 Q_2 , \end{aligned}$$

with

$$\begin{aligned} \beta_1 &= i\alpha_1^2\overline{\alpha}_1 + \frac{\kappa_2}{\left(x_1 - x_2 - i\frac{\kappa_1+\kappa_2}{2}\right)}\alpha_1^2\overline{\alpha}_2 \\ \gamma_1 &= \alpha_1^2\overline{\alpha}_1 - \frac{\kappa_1\kappa_2\alpha_1^2\overline{\alpha}_2}{\left(x_1 - x_2 - i\frac{\kappa_1+\kappa_2}{2}\right)^2} + \frac{2i\kappa_2\alpha_1\alpha_2\overline{\alpha}_1}{x_1 - x_2 + i\frac{\kappa_2-\kappa_1}{2}} + \frac{2\kappa_2^2\alpha_1\alpha_2\overline{\alpha}_2}{\left(x_1 - x_2 + i\frac{\kappa_2-\kappa_1}{2}\right)\left(x_1 - x_2 - i\frac{\kappa_1+\kappa_2}{2}\right)} \\ \beta_2 &= i\alpha_2^2\overline{\alpha}_2 + \frac{\kappa_1}{\left(x_2 - x_1 - i\frac{\kappa_2+\kappa_1}{2}\right)}\alpha_2^2\overline{\alpha}_1 \\ \gamma_2 &= \alpha_2^2\overline{\alpha}_2 - \frac{\kappa_2\kappa_1\alpha_2^2\overline{\alpha}_1}{\left(x_2 - x_1 - i\frac{\kappa_2+\kappa_1}{2}\right)^2} + \frac{2i\kappa_1\alpha_2\alpha_1\overline{\alpha}_2}{x_2 - x_1 + i\frac{\kappa_1-\kappa_2}{2}} + \frac{2\kappa_1^2\alpha_2\alpha_1\overline{\alpha}_1}{\left(x_2 - x_1 + i\frac{\kappa_1-\kappa_2}{2}\right)\left(x_2 - x_1 - i\frac{\kappa_2+\kappa_1}{2}\right)} \end{aligned}$$

Identifying $i\partial_t u$ and $\Pi(u^2\overline{u})$, we obtain the following system,

$$\begin{aligned}
i\frac{1-\dot{x}_1}{\kappa_1}-\frac{1}{2}\frac{\dot{\kappa}_1}{\kappa_1} &= i|\alpha_1|^2+\frac{\kappa_2}{\left(x_1-x_2-i\frac{\kappa_1+\kappa_2}{2}\right)}\alpha_1\overline{\alpha}_2\\
-i\left(\frac{\dot{\alpha}_1}{\alpha_1}+\frac{\dot{\kappa}_1}{\kappa_1}\right) &= |\alpha_1|^2-\frac{\kappa_1\kappa_2\alpha_1\overline{\alpha}_2}{\left(x_1-x_2-i\frac{\kappa_1+\kappa_2}{2}\right)^2}+\frac{2i\kappa_2\alpha_2\overline{\alpha}_1}{x_1-x_2+i\frac{\kappa_2-\kappa_1}{2}}+\frac{2\kappa_2^2|\alpha_2|^2\left(x_1-x_2-i\frac{\kappa_1+\kappa_2}{2}\right)^{-1}}{\left(x_1-x_2+i\frac{\kappa_2-\kappa_1}{2}\right)}\\
i\frac{1-\dot{x}_2}{\kappa_2}-\frac{1}{2}\frac{\dot{\kappa}_2}{\kappa_2} &= i|\alpha_2|^2+\frac{\kappa_1}{\left(x_2-x_1-i\frac{\kappa_2+\kappa_1}{2}\right)}\alpha_2\overline{\alpha}_1\\
-i\left(\frac{\dot{\alpha}_2}{\alpha_2}+\frac{\dot{\kappa}_2}{\kappa_2}\right) &= |\alpha_2|^2-\frac{\kappa_2\kappa_1\alpha_2\overline{\alpha}_1}{\left(x_2-x_1-i\frac{\kappa_2+\kappa_1}{2}\right)^2}+\frac{2i\kappa_1\alpha_1\overline{\alpha}_2}{x_2-x_1+i\frac{\kappa_1-\kappa_2}{2}}+\frac{2\kappa_1^2|\alpha_1|^2\left(x_2-x_1-i\frac{\kappa_2+\kappa_1}{2}\right)^{-1}}{\left(x_2-x_1+i\frac{\kappa_1-\kappa_2}{2}\right)}
\end{aligned}$$

B.2. **Conservation laws.** Taking the real part of the combination of the first and of the third equation with coefficients κ_1 and κ_2, we derive the first conservation law,

$$\frac{\kappa_1+\kappa_2}{2}=K\ . \tag{B.1}$$

The other conservation laws are not so easy to figure out. The first one corresponds to the mass conservation,

$$\|u\|_{L^2}^2=|\alpha_1|^2\|Q_1\|_{L^2}^2+|\alpha_2|^2\|Q_2\|_{L^2}^2+2\mathrm{Re}[\alpha_1\overline{\alpha}_2(Q_1|Q_2)]\ .$$

An elementary computation leads to

$$(2\pi)^{-1}\|u\|_{L^2}^2=|\alpha_1|^2\kappa_1+|\alpha_2|^2\kappa_2+2\kappa_1\kappa_2\mathrm{Im}\left(\frac{\alpha_1\overline{\alpha}_2}{x_1-x_2-i\frac{\kappa_1+\kappa_2}{2}}\right)=:C\ . \tag{B.2}$$

For the other conservation laws, we use the Lax pair property for the Hankel operators H_u, ensuring that the eigenvalues of H_u^2 are conservation laws. Recalling that $H_u(h):=\Pi(u\overline{h})$, the matrix of H_u in the basis (Q_1,Q_2) is

$$\mathscr{M}=\begin{pmatrix} i\alpha_1 & \frac{\alpha_1\kappa_2}{x_1-x_2-i\frac{\kappa_1+\kappa_2}{2}}\\ \frac{\alpha_2\kappa_1}{x_2-x_1-i\frac{\kappa_1+\kappa_2}{2}} & i\alpha_2\end{pmatrix}\ .$$

Since H_u is antilinear the trace of H_u^2 is

$$\mathrm{tr}\left(\mathscr{M}\overline{\mathscr{M}}\right)=|\alpha_1|^2+|\alpha_2|^2-2\kappa_1\kappa_2\mathrm{Re}\left(\frac{\alpha_1\overline{\alpha}_2}{\left(x_1-x_2-i\frac{\kappa_1+\kappa_2}{2}\right)^2}\right)=M\ , \tag{B.3}$$

which is also the momentum of u, divided by 2π. The determinant of H_u^2 is

$$|\det\mathscr{M}|^2=|\alpha_1|^2|\alpha_2|^2\left(1-\frac{\kappa_1\kappa_2}{(x_1-x_2)^2+\left(\frac{\kappa_1+\kappa_2}{2}\right)^2}\right)^2=D\ . \tag{B.4}$$

Let us specify the link of D with the conservation laws K,M and

$$H:=\frac{1}{2\pi}\|u\|_{L^4}^4.$$

We claim that

$$4KD=2MC-H\ .$$

Proof. Let us check this identity by calculating H. We set $X := x_1 - x_2$.

$$\begin{aligned} H &= |\alpha_1|^4 \frac{\|Q_1\|_{L^2}^4}{2\pi} + |\alpha_2|^4 \frac{\|Q_2\|_{L^2}^4}{2\pi} + 4|\alpha_1|^2|\alpha_2|^2 \frac{\|Q_1Q_2\|_{L^2}^2}{2\pi} \\ &+ 2\mathrm{Re}\left(\alpha_1^2\overline{\alpha}_2^2 \frac{(Q_1^2|Q_2^2)}{2\pi}\right) + 4\mathrm{Re}\left(\alpha_1^2\overline{\alpha}_1\overline{\alpha}_2 \frac{(Q_1^2|Q_1Q_2)}{2\pi}\right) + 4\mathrm{Re}\left(\alpha_1\alpha_2\overline{\alpha}_2^2 \frac{(Q_1Q_2|Q_2^2)}{2\pi}\right) . \end{aligned}$$

Using

$$\begin{aligned} \frac{\|Q_1\|_{L^2}^4}{2\pi} &= 2\kappa_1 \ , \ \frac{\|Q_2\|_{L^2}^4}{2\pi} = 2\kappa_2 \ , \ \frac{\|Q_1Q_2\|_{L^2}^2}{2\pi} = \frac{2K\kappa_1\kappa_2}{X^2+K^2} \ , \ \frac{(Q_1^2|Q_2^2)}{2\pi} = \frac{2i\kappa_1^2\kappa_2^2}{(X-iK)^3} \ , \\ \frac{(Q_1^2|Q_1Q_2)}{2\pi} &= \frac{-i\kappa_1\kappa_2}{X-iK} - \frac{\kappa_1^2\kappa_2}{(X-iK)^2} \ , \ \frac{(Q_1Q_2|Q_2^2)}{2\pi} = \frac{-i\kappa_1\kappa_2}{X-iK} - \frac{\kappa_1\kappa_2^2}{(X-iK)^2} \ , \end{aligned}$$

we infer

$$\begin{aligned} H &= 2\kappa_1|\alpha_1|^4 + 2\kappa_2|\alpha_2|^4 + |\alpha_1|^2|\alpha_2|^2 \frac{8K\kappa_1\kappa_2}{X^2+K^2} + 4\mathrm{Re}\left(\alpha_1^2\overline{\alpha}_2^2 \frac{i\kappa_1^2\kappa_2^2}{(X-iK)^3}\right) \\ &+ 4\mathrm{Re}\left(\alpha_1^2\overline{\alpha}_1\overline{\alpha}_2\left(\frac{-i\kappa_1\kappa_2}{X-iK} - \frac{\kappa_1^2\kappa_2}{(X-iK)^2}\right)\right) + 4\mathrm{Re}\left(\alpha_1\alpha_2\overline{\alpha}_2^2\left(\frac{-i\kappa_1\kappa_2}{X-iK} - \frac{\kappa_1\kappa_2^2}{(X-iK)^2}\right)\right) \\ &= 2\kappa_1|\alpha_1|^4 + 2\kappa_2|\alpha_2|^4 + |\alpha_1|^2|\alpha_2|^2 \frac{8K\kappa_1\kappa_2}{X^2+K^2} + 4\mathrm{Re}\left(\alpha_1^2\overline{\alpha}_2^2 \frac{i\kappa_1^2\kappa_2^2}{(X-iK)^3}\right) \\ &+ 4\kappa_1\kappa_2(|\alpha_1|^2+|\alpha_2|^2)\mathrm{Im}\left(\frac{\alpha_1\overline{\alpha}_2}{X-iK}\right) - 4(\kappa_1|\alpha_1|^2+\kappa_2|\alpha_2|^2)\kappa_1\kappa_2 K\mathrm{Re}\left(\frac{\alpha_1\overline{\alpha}_2}{(X-iK)^2}\right) . \end{aligned}$$

On the other hand,

$$\begin{aligned} M &= |\alpha_1|^2 + |\alpha_2|^2 - 2\kappa_1\kappa_2\mathrm{Re}\left(\frac{\alpha_1\overline{\alpha}_2}{(X-iK)^2}\right) \ , \\ C &= \kappa_1|\alpha_1|^2 + \kappa_2|\alpha_2|^2 + 2\kappa_1\kappa_2\mathrm{Im}\left(\frac{\alpha_1\overline{\alpha}_2}{X-iK}\right) \ , \end{aligned}$$

hence

$$\begin{aligned} 2MC &= 2\kappa_1|\alpha_1|^4 + 2\kappa_2|\alpha_2|^4 + 4K|\alpha_1|^2|\alpha_2|^2 - 4(\kappa_1|\alpha_1|^2+\kappa_2|\alpha_2|^2)\kappa_1\kappa_2\mathrm{Re}\left(\frac{\alpha_1\overline{\alpha}_2}{(X-iK)^2}\right) \\ &+ 4(|\alpha_1|^2+|\alpha_2|^2)\kappa_1\kappa_2\mathrm{Im}\left(\frac{\alpha_1\overline{\alpha}_2}{X-iK}\right) - 8\kappa_1^2\kappa_2^2\mathrm{Re}\left(\frac{\alpha_1\overline{\alpha}_2}{(X-iK)^2}\right)\mathrm{Im}\left(\frac{\alpha_1\overline{\alpha}_2}{X-iK}\right) , \end{aligned}$$

and

$$\begin{aligned} 2MC - H &= 4K|\alpha_1|^2|\alpha_2|^2\left(1 - \frac{2\kappa_1\kappa_2}{X^2+K^2}\right) \\ &- 8\kappa_1^2\kappa_2^2\mathrm{Re}\left(\frac{\alpha_1\overline{\alpha}_2}{(X-iK)^2}\right)\mathrm{Im}\left(\frac{\alpha_1\overline{\alpha}_2}{X-iK}\right) - 4\mathrm{Re}\left(\alpha_1^2\overline{\alpha}_2^2\frac{i\kappa_1^2\kappa_2^2}{(X-iK)^3}\right) . \end{aligned}$$

Now just observe that, for every complex numbers a, b,

$$-8\mathrm{Re}(a)\mathrm{Im}(b) + 4\mathrm{Im}(ab) = 4\mathrm{Im}(a)\mathrm{Re}(b) - 4\mathrm{Im}(b)\mathrm{Re}(a) = 4\mathrm{Im}(a\overline{b}) .$$

Applying this identity to

$$a = \frac{\alpha_1\overline{\alpha}_2}{(X-iK)^2} \ , \ b = \frac{\alpha_1\overline{\alpha}_2}{X-iK} \ ,$$

we infer

$$-8\mathrm{Re}\left(\frac{\alpha_1\overline{\alpha}_2}{(X-iK)^2}\right)\mathrm{Im}\left(\frac{\alpha_1\overline{\alpha}_2}{X-iK}\right) - 4\mathrm{Re}\left(\frac{i\alpha_1^2\overline{\alpha}_2^2}{(X-iK)^3}\right) = \frac{4K}{(X^2+K^2)^2}$$

and finally

$$2MC - H = 4K|\alpha_1|^2|\alpha_2|^2 \left(1 - \frac{2\kappa_1\kappa_2}{X^2+K^2} + \frac{\kappa_1^2\kappa_2^2}{(X^2+K^2)^2}\right) = 4KD \ .$$

□

B.3. **The reduced variables.** Notice that

$$\dot{x}_1 + \dot{x}_2 = 2 - C \ . \tag{B.5}$$

Therefore, it is natural to introduce

$$X := x_1 - x_2 \ , \ \nu := \frac{\kappa_1 - \kappa_2}{2} \ .$$

Setting

$$\zeta_1 := \frac{\alpha_1}{X - iK} \ , \ \zeta_2 := \frac{\alpha_2}{X + iK} \ ,$$

the system reads

$$\begin{aligned} \dot{X} &= (X^2+K^2)[(K-\nu)|\zeta_2|^2 - (K+\nu)|\zeta_1|^2] \ , \\ \dot{\nu} &= -2(K^2-\nu^2)\mathrm{Re}[\zeta_1\overline{\zeta}_2(X - iK)] \ . \end{aligned}$$

Furthermore, the last three conservation laws read

$$\begin{aligned} C &= (X^2+K^2)[(K+\nu)|\zeta_1|^2 + (K-\nu)|\zeta_2|^2] + 2(K^2-\nu^2)\mathrm{Im}[\zeta_1\overline{\zeta}_2(X-iK)] \\ M &= (X^2+K^2)(|\zeta_1|^2 + |\zeta_2|^2) - 2(K^2-\nu^2)\mathrm{Re}(\zeta_1\overline{\zeta}_2) \\ &= (K^2-\nu^2)|\zeta_1 - \zeta_2|^2 + (X^2+\nu^2)(|\zeta_1|^2 + |\zeta_2|^2) \\ D &= |\zeta_1|^2|\zeta_2|^2(X^2+\nu^2)^2 \end{aligned}$$

B.4. **The resonance condition.** Notice that

$$\begin{aligned} M^2 - 4D &= (K^2-\nu^2)^2|\zeta_1 - \zeta_2|^4 + 2(K^2-\nu^2)(X^2+\nu^2)|\zeta_1-\zeta_2|^2(|\zeta_1|^2+|\zeta_2|^2) \\ &+ (X^2+\nu^2)^2(|\zeta_1|^2 - |\zeta_2|^2)^2 \ . \end{aligned}$$

Therefore, this conservation law cancels if and only if

$$\zeta_1 = \zeta_2 =: \zeta \ .$$

In this case, the above three conservation laws degenerate as

$$|\zeta|^2(X^2+\nu^2) = \frac{M}{2} = \sqrt{D} \ , \ C = KM \ .$$

Using the laws M, C, H, K and the identity

$$2MC - H = 4KD \ ,$$

we observe that the condition $M^2 = 4D$ is therefore equivalent to the set of two conditions,

$$MC = H \quad \text{and} \quad C = KM \ .$$

Indeed, on the one hand, $M^2 = 4D$ implies $C = KM$ as we have already observed, and therefore,

$$4KD = 2M^2K - H = 8KD - H$$

so that $H = 4KD = 2MC - H$, hence $H = MC$. On the other hand, if $MC = H$ and $C = KM$, then $MC = 4KD$ and $C = KM$, hence $KM^2 = 4KD$, so $M^2 = 4D$.

Under the resonance condition, the system in the reduced variables can be written

$$\begin{aligned} \dot{X} &= -M\,\nu\,\frac{X^2+K^2}{X^2+\nu^2}\,, \\ \dot{\nu} &= -M\,X\,\frac{K^2-\nu^2}{X^2+\nu^2}\,. \end{aligned}$$

In particular,

$$\frac{d}{dt}(X\nu) = -M\,K^2\,.$$

This means that $X\nu$ cancels exactly once, so either X cancels and ν keeps the same sign, or ν cancels and X keeps the same sign. In both cases, $|X(t)|$ tends to infinity like $KM|t|$, and $|\nu|$ tends to K. Furthermore, in this case, we have

$$|\alpha_1| = |\alpha_2|,$$

and the phase shift is given by

$$\frac{\alpha_1}{\alpha_2} = \mathrm{e}^{i\Gamma} = \frac{X-iK}{X+iK}\,,$$

so the phase shift cancels at infinity. More precisely,

$$i\dot{\Gamma} = \frac{\dot{X}}{X-iK} - \frac{\dot{X}}{X+iK} = -M\,\frac{\nu}{X^2+\nu^2}\,[X+iK-(X-iK)] = \frac{-2iKM\,\nu}{X^2+\nu^2}\,.$$

Since $|X(t)|$ tends to infinity like $KM|t|$, we conclude that $|\dot{\Gamma}(t)|$ cancels as fast as t^{-2}.

Appendix C. Proof of the non degeneracy (5.8)

The non degeneracy (5.8) follows from an explicit computation on the limiting Szegő profile Q^+. However, before proceeding with the limiting process, we need more precise information on $i\rho_\beta$ and $(1-\beta)\partial_\beta Q_\beta$.

By (3.10) and Lemma 3.8, we have

$$\rho = -iQ_\beta + \frac{1}{2}\partial_y Q_\beta + o_{\beta\to 1}(1), \qquad \Sigma = y\partial_y Q_\beta + o_{\beta\to 1}(1).$$

which together with Lemma A.1 ensures:

$$\begin{aligned} \det A_j &= \det \begin{pmatrix} (\Lambda Q_{\beta_j}, Q_{\beta_j}) & (\Lambda Q_{\beta_j}, i\partial_{y_j} Q_{\beta_j}) & (\Lambda Q_{\beta_j}, i\Lambda Q_{\beta_j}) & (\Lambda Q_{\beta_j}, \rho_j) \\ (iQ_{\beta_j}, Q_{\beta_j}) & (iQ_{\beta_j}, i\partial_{y_j} Q_{\beta_j}) & (iQ_{\beta_j}, i\Lambda Q_{\beta_j}) & (iQ_{\beta_j}, \rho_j) \\ (\partial_{y_j} Q_{\beta_j}, Q_{\beta_j}) & (\partial_{y_j} Q_{\beta_j}, i\partial_{y_j} Q_{\beta_j}) & (\partial_{y_j} Q_{\beta_j}, i\Lambda Q_{\beta_j}) & (\partial_{y_j} Q_{\beta_j}, \rho_j) \\ (\Sigma_j, Q_{\beta_j}) & (\Sigma_j, i\partial_{y_j} Q_{\beta_j}) & (\Sigma_j, i\Lambda Q_{\beta_j}) & (\Sigma_j, \rho_j) \end{pmatrix} \\ &\to \begin{pmatrix} 0 & -\pi & 0 & 0 \\ 0 & 0 & 0 & -\pi \\ 0 & 0 & \pi & 0 \\ -\pi & 0 & 0 & 0 \end{pmatrix} = -\pi^4 \quad \text{as} \quad \beta_j \uparrow 1 \end{aligned}$$

and (5.8) is proved.

Appendix D. Commutator estimates

This Appendix is devoted to the derivation of commutator estimates used all along Section 5. All proofs are more or less standard but the involved norms and associated decay are critical for the proof of Proposition 5.4, so we display all estimates in detail.

We let in this section χ denote a bounded Lipschitz continuous function and let

$$\chi_R(x) = \chi\left(\frac{x}{R}\right), \quad R \geqslant 1.$$

Lemma D.1 ($|D|^{\frac12}$ commutator)**.** *There holds the global bound*

$$\|[|D|^{\frac12}, \chi_R]g\|_{L^2} \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{\sqrt{R}}\|g\|_{L^2}, \tag{D.1}$$

and the weighted bound for $0 < \alpha < 1$*:*

$$\|\frac{1}{\langle z\rangle^{\frac{1+\alpha}{2}}}[|D|^{\frac12}, \chi_R]g\|_{L^2} \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{\sqrt{R}}\|\frac{g}{\langle z\rangle^{\frac{1+\alpha}{2}}}\|_{L^2} \quad \textit{with} \quad z = \frac{x}{R}. \tag{D.2}$$

Proof. **Step 1:** Kernel representation. First we provide a description of the operator $|D|^{\frac12}$ in the space variables. This operator is the convolution operator with the tempered distribution

$$k := \mathcal{F}^{-1}(|\xi|^{\frac12}) .$$

From the properties of the Fourier transform we know that k is homogeneous of degree $-3/2$, and is even. As a consequence, it is characterized up to a multiplicative constant. For every function φ in the Schwartz space, we therefore have

$$\langle k, \varphi\rangle = c\int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0)}{|x|^{\frac32}}\, dx \ , \ c := (2\pi)^{-\frac12}\frac{\int_{\mathbb{R}} |\xi|^{\frac12} e^{-\frac{\xi^2}{2}}\, d\xi}{\int_{\mathbb{R}} \frac{e^{-\frac{x^2}{2}}-1}{|x|^{\frac32}}\, dx} \ ,$$

and

$$k * \varphi(x) = c\int_{\mathbb{R}} \frac{\varphi(y) - \varphi(x)}{|x-y|^{\frac32}}\, dy \ .$$

Consequently, we can write

$$[|D|^{\frac12}, \chi_R]\, g(x) = c\int_{\mathbb{R}} \frac{\chi_R(y) - \chi_R(x)}{|x-y|^{\frac32}}\, g(y)\, dy \ .$$

Step 2: Proof of (D.1). We split the kernel in two parts,

$$\begin{aligned}
[|D|^{\frac12}, \chi_R]\, g(x) &= c(T^{med}g(x) + T^{off}g(x)) \ , \\
T^{med}g(x) &:= \int_{|x-y|\leq 5R} \frac{\chi_R(y) - \chi_R(x)}{|x-y|^{\frac32}}\, g(y)\, dy \ , \\
T^{off}g(x) &:= \int_{|x-y|> 5R} \frac{\chi_R(y) - \chi_R(x)}{|x-y|^{\frac32}}\, g(y)\, dy \ .
\end{aligned}$$

We have

$$\frac{|\chi_R(x) - \chi_R(y)|}{|x-y|^{\frac32}} \lesssim \frac{\|\chi_R'\|_{L^\infty}}{|x-y|^{\frac12}} \lesssim \frac{1}{R|x-y|^{\frac12}}$$

and hence, by Young's inequality,

$$\|T^{med}\|_{L^2} \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{R}\|\frac{\mathbf{1}_{|x|\leqslant 5R}}{|x|^{\frac12}} \star g\|_{L^2} \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{R}\|\frac{\mathbf{1}_{|x|\leqslant 5R}}{|x|^{\frac12}}\|_{L^1}\|g\|_{L^2} \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{\sqrt{R}}\|g\|_{L^2}.$$

Similarly,

$$\|T^{off}\|_{L^2} \lesssim \|\chi\|_{L^\infty} \Big\| \frac{\mathbf{1}_{|x|\geqslant 5R}}{|x|^{\frac{3}{2}}} \star g \Big\|_{L^2} \lesssim \|\chi\|_{W^{1,\infty}} \Big\| \frac{\mathbf{1}_{|x|\geqslant 5R}}{|x|^{\frac{3}{2}}} \Big\|_{L^1} \|g\|_{L^2} \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{\sqrt{R}} \|g\|_{L^2}$$

and (D.1) is proved.

Step 3: Proof of (D.2). For $|x-y| \leqslant 5R$, we have $\langle \frac{y}{R} \rangle \lesssim \langle \frac{x}{R} \rangle$ and we infer

$$\Big| \frac{1}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} T^{med} g \Big| \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{R} \frac{\mathbf{1}_{|x|\leqslant 5R}}{|x|^{\frac{1}{2}}} \star \frac{|g|}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}}$$

from which, as above, from Young's inequality,

$$\Big\| \frac{1}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} T^{med} g \Big\|_{L^2} \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{\sqrt{R}} \Big\| \frac{|g|}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} \Big\|_{L^2}.$$

For $|x-y| \geqslant 5R$, we distinguish

$$T_1^{off} = \int_{|x-y|>5R, \quad |y|\leqslant 2|x|} \frac{\chi_R(y) - \chi_R(x)}{|x-y|^{\frac{3}{2}}} g(y)\, dy$$
$$T_2^{off} = \int_{|x-y|>5R, \quad |y|\geqslant 2|x|} \frac{\chi_R(y) - \chi_R(x)}{|x-y|^{\frac{3}{2}}} g(y)\, dy$$

For the first kernel, $\langle \frac{y}{R} \rangle \lesssim \langle \frac{x}{R} \rangle$ and thus

$$\Big| \frac{1}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} T_1^{off} g \Big| \lesssim \|\chi\|_{L^\infty} \frac{\mathbf{1}_{|x|\geqslant 5R}}{|x|^{\frac{3}{2}}} \star \frac{|g|}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}}$$

from which, as above,

$$\Big\| \frac{1}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} T_1^{off} g \Big\|_{L^2} \lesssim \|\chi\|_{L^\infty} \Big\| \frac{\mathbf{1}_{|x|\geqslant 5R}}{|x|^{\frac{3}{2}}} \Big\|_{L^1} \Big\| \frac{g}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} \Big\|_{L^2} \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{\sqrt{R}} \Big\| \frac{g}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} \Big\|_{L^2}.$$

For the second kernel, $|y| \geqslant 2|x|$ and $|x-y| \geqslant 5R$, we have $|y| \gtrsim R$ and $|x-y| \gtrsim |y|$. Therefore, from Cauchy–Schwarz' inequality,

$$\begin{aligned} |T_2^{off} g| &\lesssim \|\chi\|_{L^\infty} \int_{|y|\gtrsim R} \frac{|g(y)|}{\langle \frac{y}{R} \rangle^{\frac{1+\alpha}{2}}} \Big(\frac{|y|}{R} \Big)^{\frac{1+\alpha}{2}} \frac{dy}{|y|^{\frac{3}{2}}} \\ &\lesssim \frac{\|\chi\|_{L^\infty}}{R^{\frac{1+\alpha}{2}}} \Big\| \frac{g}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} \Big\|_{L^2} \Big(\int_{|y|\gtrsim R} \frac{dy}{|y|^{2-\alpha}} \Big)^{\frac{1}{2}} \\ &\lesssim \frac{\|\chi\|_{L^\infty}}{R} \Big\| \frac{g}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} \Big\|_{L^2} \end{aligned}$$

where we used $\alpha < 1$, from which

$$\Big\| \frac{T^{off} g}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} \Big\|_{L^2} \lesssim \frac{\|\chi\|_{L^\infty}}{R} \Big\| \frac{g}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} \Big\|_{L^2} \Big\| \frac{1}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} \Big\|_{L^2} \lesssim \frac{\|\chi\|_{W^{1,\infty}}}{\sqrt{R}} \Big\| \frac{g}{\langle \frac{x}{R} \rangle^{\frac{1+\alpha}{2}}} \Big\|_{L^2}$$

where we simply changed variables and used $\alpha > 0$ in the last step. This concludes the proof of (D.2). □

We shall also use the following slightly different version.

Lemma D.2 (Commutator estimate in L^2)**.** *For a general function χ such that $\partial_x\chi \in L^1$, there holds the following bounds.*

$$\|[|D|^{\frac12},\chi]g\|_{L^2} \lesssim \||\xi|^{\frac12}\widehat{\chi}\|_{L^1}\|g\|_{L^2} \lesssim (\|\partial_x\chi\|_{L^1}\|\partial_{xx}\chi\|_{L^1})^{\frac12}\,\|g\|_{L^2}, \tag{D.3}$$

$$\|[|D|,\chi]g\|_{L^2} + \|[\Pi^+|D|,\chi]\|_{L^2} \lesssim \left(\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1}\right)^{\frac12}\|g\|_{L^2}, \tag{D.4}$$

$$\begin{aligned} &\||D|^{\frac12}[|D|^{\frac12},\chi]g\|_{L^2} \\ \lesssim\ & (\|\partial_x\chi\|_{L^1}\|\partial_{xx}\chi\|_{L^1})^{\frac12}\,\||D|^{\frac12}g\|_{L^2} + \left(\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1}\right)^{\frac12}\|g\|_{L^2}, \end{aligned} \tag{D.5}$$

Proof. **Step 1:** Proof of (D.3). Since $\partial_x\chi \in L^1$, $\hat\chi(\xi)$ is discontinuous only at $\xi = 0$, with a mild singularity justifying the calculations below for every g in the Schwartz space. We have

$$\widehat{[|D|^{\frac12},\chi]g} = \widehat{|D|^{\frac12}(\chi g)}(\xi) - \widehat{\chi(|D|^{\frac12}g)}(\xi) = \int_{\mathbb{R}}(|\xi|^{\frac12}-|\eta|^{\frac12})\hat\chi(\xi-\eta)\hat g(\eta)\,d\eta.$$

We use

$$\left||\xi|^{\frac12}-|\eta|^{\frac12}\right| \leqslant |\xi-\eta|^{\frac12}. \tag{D.6}$$

to estimate pointwise

$$\left|\widehat{[|D|^{\frac12},\chi]g}(\xi)\right| \lesssim \int_{\mathbb{R}}|\xi-\eta|^{\frac12}|\hat\chi|(\xi-\eta)|\hat g|(\eta)\,d\eta = |\xi|^{\frac12}|\hat\chi| \star |\hat g|.$$

We conclude, from Young's inequality and the Plancherel formula,

$$\|[|D|^{\frac12},\chi]g\|_{L^2} \lesssim \|(|\xi|^{\frac12}|\hat\chi|)\star|\hat g|\|_{L^2} \lesssim \||\xi|^{\frac12}\hat\chi\|_{L^1}\|g\|_{L^2}.$$

Finally, we estimate

$$\begin{aligned} \int |\xi|^{\frac12}|\hat\chi|d\xi &\leqslant \int_{|\xi|\leqslant A}\frac{\|\widehat{\partial_x\chi}\|_{L^\infty}}{|\xi|^{\frac12}}d\xi + \int_{|\xi|\geqslant A}\frac{\|\widehat{\partial_{xx}\chi}\|_{L^\infty}}{|\xi|^{\frac32}}d\xi \lesssim \sqrt{A}\|\partial_x\chi\|_{L^1} + \frac{\|\partial_{xx}\chi\|_{L^1}}{\sqrt{A}} \\ &\lesssim (\|\partial_x\chi\|_{L^1}\|\partial_{xx}\chi\|_{L^1})^{\frac12}. \end{aligned} \tag{D.7}$$

by optimizing in A.

Step 2: Proof of (D.4). We compute

$$\begin{aligned} |\widehat{[|D|,\chi]g}(\xi)| &= \left|\widehat{|D|(\chi g)}(\xi) - \widehat{\chi(|D|g)}(\xi)\right| = \left|\int_{\mathbb{R}}(|\xi|-|\eta|)\hat\chi(\xi-\eta)\hat g(\eta)\,d\eta\right| \\ &\lesssim \int_{\mathbb{R}}|\xi-\eta||\hat\chi|(\xi-\eta)|\hat g|(\eta)\,d\eta = (|\xi||\hat\chi|)\star|\hat g| \end{aligned}$$

and hence

$$\|[|D|,\chi]g\|_{L^2} \lesssim \|(|\xi||\hat\chi|)\star|\hat g|\|_{L^2} \lesssim \||\xi|\hat\chi\|_{L^1}\|g\|_{L^2}.$$

We now estimate

$$\int|\xi||\hat\chi| \lesssim \int_{|\xi|\leqslant A}\|\partial_x\chi\|_{L^1}d\xi + \int_{|\xi|\geqslant A}\frac{\|\partial_x^3\chi\|_{L^1}}{|\xi|^2}d\xi \lesssim \left(\|\partial_x\chi\|_{L^1}\|\partial_x^3\chi\|_{L^1}\right)^{\frac12} \tag{D.8}$$

and the first commutator estimate in (D.4) is proved. Similarly,

$$\begin{aligned} |\widehat{[\Pi^+|D|,\chi]g}| &= \left|\int_{\mathbb{R}}(|\eta|\mathbf{1}_{\eta>0} - |\xi|\mathbf{1}_{\xi>0})\hat\chi(\xi-\eta)\hat g(\eta)\,d\eta\right| \\ &\lesssim \int_{\mathbb{R}}|\xi-\eta||\hat\chi|(\xi-\eta)|\hat g|(\eta)\,d\eta = (|\xi||\hat\chi|)\star|\hat g| \end{aligned}$$

and the conclusion follows as above.

Step 3: Proof of (D.5). We compute, using (D.6),

$$\begin{aligned} ||D|^{\frac{1}{2}}\widehat{[|D|^{\frac{1}{2}},\chi]g}(\xi)| &\lesssim \int_{\mathbb{R}} |\xi|^{\frac{1}{2}} \Big||\xi|^{\frac{1}{2}} - |\eta|^{\frac{1}{2}}\Big| |\hat{\chi}(\xi-\eta)\hat{g}(\eta)|\, d\eta \\ &\lesssim \int_{\mathbb{R}} |\xi-\eta||\hat{\chi}(\xi-\eta)||\hat{g}(\eta)| d\eta + \int_{\mathbb{R}} |\xi-\eta|^{\frac{1}{2}}|\hat{\chi}|(\xi-\eta)|\eta|^{\frac{1}{2}}|\hat{g}|(\eta)\, d\eta \\ &\lesssim |\xi\chi| \star |\hat{g}| + |\xi|^{\frac{1}{2}}|\hat{\chi}| \star |\eta^{\frac{1}{2}}\hat{g}| \end{aligned}$$

and the conclusion follows as in the previous two steps. □

We similarly estimate $\Pi^{\pm}$ commutators.

Lemma D.3 ($\Pi^{\pm}$ commutator)**.** *Assume that the derivative χ' is supported in $[1,2]$. Then there holds*

$$\Big\| D^k[\Pi^{\pm},\chi_R]g\Big\|_{L^2} \lesssim \frac{\|\chi\|_{W^{k+1,\infty}}}{R^k}\|g\|_{L^2}, \quad k=1,2, \tag{D.9}$$

and

$$\big\|\langle x\rangle D^2[\Pi^{\pm},\chi_R]g\big\|_{L^2} \lesssim \frac{\|\chi\|_{W^{3,\infty}}}{R}\|g\|_{L^2}. \tag{D.10}$$

Proof. We recall the standard representation formula

$$[\Pi^+,\chi_R]g(x) = c\int \frac{\chi_R(x)-\chi_R(y)}{x-y} g(y)dy.$$

Step 1: Case $k=1$. We take a derivative,

$$\partial_x[\Pi^+,\chi_R]g(x) = -c\int \frac{\chi_R(x)-\chi_R(y)-(x-y)\chi_R'(x)}{(x-y)^2} g(y)dy.$$

We now split the kernel as

$$\begin{aligned} \partial_x[\Pi^+,\chi_R]g(x) &= -c(T_R^{med}g(x) + T^{off}g(x))\ , \\ T_R^{med}g(x) &:= \int_{|x-y|<R} \frac{\chi_R(x)-\chi_R(y)-(x-y)\chi_R'(x)}{(x-y)^2} g(y)\, dy\ , \\ T_R^{off}g(x) &:= \int_{|x-y|>R} \frac{\chi_R(x)-\chi_R(y)-(x-y)\chi_R'(x)}{(x-y)^2} g(y)\, dy\ . \end{aligned}$$

We estimate

$$\left|\frac{\chi_R(x)-\chi_R(y)-(x-y)\chi_R'(x)}{(x-y)^2}\right| \lesssim \|\chi_R''\|_{L^\infty} \lesssim \frac{\|\chi\|_{W^{2,\infty}}}{R^2}. \tag{D.11}$$

Hence, by (D.11) and Young's inequality,

$$\|T_R^{med}g\|_{L^2} \lesssim \frac{\|\chi\|_{W^{2,\infty}}}{R^2}\|\mathbf{1}_{|x-y|<R}\star g\|_{L^2} \lesssim \frac{1}{R^2}\|\mathbf{1}_{|x-y|<R}\|_{L^1}\|g\|_{L^2} \lesssim \frac{\|\chi\|_{W^{2,\infty}}}{R}\|g\|_{L^2}.$$

Off the diagonal, we use the special structure of χ_R. Firstly, we have

$$\begin{aligned} |T^{off}g| &\lesssim \|\chi\|_{L^\infty}\int_{|x-y|>R} \frac{|g(y)|}{|x-y|^2}dy + \int_{|x-y|>R} \frac{|\chi_R'(x)-\chi_R'(y)|}{|x-y|}|g(y)|dy \\ &+ \int_{|x-y|>R} \frac{1}{|x-y|}|\chi_R'(y)g(y)|dy := I + II + III. \end{aligned}$$

The first term is estimated by Young's inequality,

$$\begin{aligned}\|I\|_{L^2} &\lesssim \|\chi\|_{L^\infty}\|\frac{1}{|x|^2}\mathbf{1}_{|x|>R}\star|g|\|_{L^2}\lesssim\|\chi\|_{L^\infty}\|\frac{1}{|x|^2}\mathbf{1}_{|x|>R}\|_{L^1}\|g\|_{L^2}\\ &\lesssim \frac{\|\chi\|_{W^{2,\infty}}}{R}\|g\|_{L^2}.\end{aligned}$$

For the second term, we use Young's inequality and the fact that $\chi_R'(x)$ is supported in $R\leqslant|x|\leqslant 2R$. We obtain

$$\begin{aligned}\|II\|_{L^2} &\lesssim \|\frac{\chi_R'}{|x|}\mathbf{1}_{|x|>R}\star g\|_{L^2}\lesssim\|\frac{\chi_R'}{|x|}\mathbf{1}_{|x|>R}\|_{L^1}\|g\|_{L^2}\lesssim\frac{\|\chi'\|_{L^\infty}}{R}\|g\|_{L^2}\int_{\frac{R}{\leqslant}|x|\leqslant 2R}\frac{dx}{\langle x\rangle}\\ &\lesssim \frac{\|\chi\|_{W^{2,\infty}}}{R}\|g\|_{L^2}.\end{aligned}$$

The last term is treated with Young's and Cauchy Schwarz's inequalities,

$$\begin{aligned}\|III\|_{L^2} &\lesssim \|\frac{1}{|x|}\mathbf{1}_{|x|>R}\star(\chi_R'g)\|_{L^2}|\lesssim\|\frac{1}{|x|}\mathbf{1}_{|x|>R}\|_{L^2}\|\chi_R'g\|_{L^1}\lesssim\frac{\|\chi'\|_{L^\infty}}{\sqrt{R}}\frac{1}{R}\|g\|_{L^1(R\leqslant|x|\leqslant 2R)}\\ &\lesssim \frac{\|\chi\|_{W^{2,\infty}}}{R}\|g\|_{L^2}.\end{aligned}$$

The collection of above bounds yields (D.9)for $k=1$.

Step 2: Case $k=2$. The proof is similar. We take two derivatives,

$$\begin{aligned}\partial_x^2[\Pi^+,\chi_R]g(x) &= 2c\int\frac{\chi_R(x)-\chi_R(y)-(x-y)\chi_R'(x)+\frac{1}{2}(x-y)^2\chi_R''(x)}{(x-y)^3}g(y)dy\\ &= c(T_R^{med}g(x)+T^{off}g(x)).\end{aligned}$$

We estimate

$$\left|\frac{\chi_R(x)-\chi_R(y)-(x-y)\chi_R'(x)+\frac{1}{2}(x-y)^2\chi_R''(x)}{(x-y)^3}\right|\lesssim\|\chi_R'''\|_{L^\infty}\lesssim\frac{\|\chi\|_{W^{3,\infty}}}{R^3}$$

from which

$$\|T_R^{med}g\|_{L^2}\lesssim\frac{\|\chi\|_{W^{3,\infty}}}{R^3}\|\mathbf{1}_{|x-y|<R}\star g\|_{L^2}\lesssim\frac{1}{R^3}\|\mathbf{1}_{|x-y|<R}\|_{L^1}\|g\|_{L^2}\lesssim\frac{\|\chi\|_{W^{3,\infty}}}{R^2}\|g\|_{L^2}.$$

Off the diagonal, we split

$$\begin{aligned}|T^{off}g| &\lesssim \|\chi\|_{L^\infty}\int_{|x-y|>R}\frac{|g(y)|}{|x-y|^3}dy\\ &+ \int_{|x-y|>R}\frac{|\chi_R'(x)-\chi_R'(y)|}{|x-y|^2}|g(y)|dy+\int_{|x-y|>R}\frac{|\chi_R'(y)|}{|x-y|^2}|g(y)|dy\\ &+ \int_{|x-y|>R}\frac{|\chi_R''(x)-\chi_R''(y)|}{|x-y|}|g(y)|dy+\int_{|x-y|>R}\frac{1}{|x-y|}|\chi_R''(y)g(y)|dy\\ &:= I+II+III.\end{aligned}$$

The first term is estimated by Young's inequality,

$$\begin{aligned}\|I\|_{L^2} &\lesssim \|\chi\|_{L^\infty}\|\frac{1}{|x|^3}\mathbf{1}_{|x|>R}\star|g|\|_{L^2}\lesssim\|\chi\|_{L^\infty}\|\frac{1}{|x|^3}\mathbf{1}_{|x|>R}\|_{L^1}\|g\|_{L^2}\\ &\lesssim \frac{\|\chi\|_{W^{3,\infty}}}{R^2}\|g\|_{L^2}.\end{aligned}$$

For the second term, we use Young's inequality and the fact that $\chi_R'(x)$ is supported in $R \leqslant |x| \leqslant 2R$. We obtain

$$\begin{aligned} \|\frac{\chi_R'}{|x|^2}\mathbf{1}_{|x|>R} \star g\|_{L^2} &\lesssim \|\frac{\chi_R'}{|x|^2}\mathbf{1}_{|x|>R}\|_{L^1}\|g\|_{L^2} \lesssim \frac{\|\chi'\|_{L^\infty}}{R}\|g\|_{L^2}\int_{R\leqslant|x|\leqslant 2R}\frac{dx}{\langle x\rangle^2} \\ &\lesssim \frac{\|\chi\|_{W^{3,\infty}}}{R^2}\|g\|_{L^2} \end{aligned}$$

and

$$\begin{aligned} \|\frac{1}{|x|^2}\mathbf{1}_{|x|>R} \star (\chi_R' g)\|_{L^2}| &\lesssim \|\frac{1}{|x|^2}\mathbf{1}_{|x|>R}\|_{L^2}\|\chi_R' g\|_{L^1} \lesssim \frac{\|\chi\|_{W^{3,\infty}}}{R^{\frac{3}{2}}}\frac{1}{R}\|g\|_{L^1(R\leqslant|x|\leqslant 2R)} \\ &\lesssim \frac{\|\chi\|_{W^{3,\infty}}}{R^2}\|g\|_{L^2} \end{aligned}$$

and hence

$$II \lesssim \frac{\|\chi\|_{W^{3,\infty}}}{R^2}\|g\|_{L^2}.$$

For the last term, we have

$$\begin{aligned} \|\frac{\chi_R''}{|x|}\mathbf{1}_{|x|>R} \star g\|_{L^2} &\lesssim \|\frac{\chi_R''}{|x|}\mathbf{1}_{|x|>R}\|_{L^1}\|g\|_{L^2} \lesssim \frac{\|\chi\|_{W^{3,\infty}}}{R^2}\|g\|_{L^2}\int_{R\leqslant|x|\leqslant 2R}\frac{dx}{\langle x\rangle} \\ &\lesssim \frac{\|\chi\|_{W^{3,\infty}}}{R^2}\|g\|_{L^2} \end{aligned}$$

and

$$\begin{aligned} \|\frac{1}{|x|}\mathbf{1}_{|x|>R} \star (\chi_R'' g)\|_{L^2}| &\lesssim \|\frac{1}{|x|}\mathbf{1}_{|x|>R}\|_{L^2}\|\chi_R'' g\|_{L^1} \lesssim \frac{\|\chi\|_{W^{3,\infty}}}{\sqrt{R}}\frac{1}{R^2}\|g\|_{L^1(R\leqslant|x|\leqslant 2R)} \\ &\lesssim \frac{\|\chi\|_{W^{3,\infty}}}{R^2}\|g\|_{L^2}. \end{aligned}$$

The collection of above bounds yields (D.9) for $k = 2$.

Step 3: Proof of (D.10). We revisit the estimates of step 2 in the presence of the additional $\langle x\rangle$ weight. For $|x| \leqslant 10R$, we estimate directly from (D.9),

$$\Big\|\langle x\rangle D^k[\Pi^\pm, \chi_R]g\Big\|_{L^2(|x|\leqslant 10R)} \lesssim \frac{\|\chi\|_{W^{2,\infty}}}{R}\|g\|_{L^2}.$$

We therefore assume $|x| \geqslant 10R$. Since $\chi' = 0$ outside $[1, 2]$, $|x - y| < R$ implies $\chi_R(x) - \chi_R(y) = 0$ and $\chi_R'(x) = 0$. For $|x - y| > R$, we have $|x - y| > |x|$ if x, y do not have the same sign, and if x, y have the same sign, necessarily $|y| \leqslant R$, for otherwise $\chi_R(x) - \chi_R(y) = 0$ again. In both cases, $|x - y| \gtrsim |x|$, and hence

$$\|T^{off} g\|_{L^\infty(|x|\geqslant 10R)} \lesssim \int_{|x-y|\gtrsim|x|}\frac{1}{|x-y|^3}|g(y)|dy \lesssim \|g\|_{L^2}\|\frac{\mathbf{1}_{|z|\gtrsim|x|}}{|z|^3}\|_{L^2} \lesssim \frac{1}{|x|^{\frac{5}{2}}}\|g\|_{L^2},$$

therefore

$$\|\langle x\rangle T^{off} g\|_{L^2(|x|\geqslant 10R)} \lesssim \|g\|_{L^2}\|\frac{1}{\langle x\rangle^{\frac{3}{2}}}\|_{L^2(|x|\geqslant 10R)} \lesssim \frac{\|g\|_{L^2}}{R},$$

and (D.10) is proved. □

We will need a standard localization formula for the kinetic energy.

Lemma D.4 (Localization of the kinetic energy)**.** *There holds for given functions Z, f,*

$$\begin{aligned}\int |Z|^2||D|^{\frac12}f|^2 &= \int ||D|^{\frac12}(Zf)|^2 \\ &+ O\left(\|[|D|^{\frac12},Z]f\|_{L^2}\left[\||D|^{\frac12}(Zf)\|_{L^2}+\|Z|D|^{\frac12}f\|_{L^2}\right]\right).\end{aligned} \tag{D.12}$$

In particular, for $\chi_R(y) := \chi(\frac{y}{R})$ with χ a smooth function satisfying

$$\chi(y) = \begin{cases} 1, & \text{if } |y| < \frac14 \\ 0, & \text{if } |y| > \frac12, \end{cases}$$

we have

$$\int \chi_R^2||D|^{\frac12}f|^2 = \int ||D|^{\frac12}(\chi_R f)|^2 + O\left(\frac{\|f\|_{L^2}^2+\||D|^{\frac12}(\chi_R f)\|_{L^2}^2}{\sqrt{R}}\right). \tag{D.13}$$

Proof. We expand and estimate

$$\begin{aligned}\int ||D|^{\frac12}(Zf)|^2 &= (|D|^{\frac12}(Zf), |D|^{\frac12}(Zf)) = ([|D|^{\frac12},Z]f + Z|D|^{\frac12}f, |D|^{\frac12}(Zf)) \\ &= O\left(\|[|D|^{\frac12},Z]f\|_{L^2}\||D|^{\frac12}(Zf)\|_{L^2}\right) + (Z|D|^{\frac12}f, [|D|^{\frac12},Z]f + Z|D|^{\frac12}f) \\ &= \int Z^2||D|^{\frac12}f|^2 + O\left(\|[|D|^{\frac12},Z]f\|_{L^2}\left[\||D|^{\frac12}(Zf)\|_{L^2}+\|Z|D|^{\frac12}f\|_{L^2}\right]\right)\end{aligned}$$

and (D.12) follows. We then estimate from (D.1),

$$\|[|D|^{\frac12},\chi_R]\|_{L^2\to L^2} \lesssim \frac{1}{\sqrt{R}}$$

and (D.13) follows. □

Finally, for establishing the coercivity of our energy functional, we need the following — non sharp — estimate.

Lemma D.5. *Let χ be a smooth function satisfying*

$$\chi(y) = \begin{cases} 1, & \text{if } |y| < \frac14 \\ 0, & \text{if } |y| > \frac12, \end{cases}$$

There holds:

$$\left\|\frac{(\chi_R u^+)^-}{\langle y\rangle}\right\|_{L^2} \lesssim \frac{1}{R^{\frac13}}\|u^+\|_{L^2}. \tag{D.14}$$

Proof. Using a standard duality argument, it suffices to show that

$$\left|((\chi_R u^+)^-, v)\right| \lesssim \frac{1}{R^{\frac13}}\|u^+\|_{L^2}\|\langle y\rangle v\|_{L^2} \tag{D.15}$$

for any $v \in L^2(\mathbb{R})$ such that $\langle y\rangle v \in L^2(\mathbb{R})$. Let $0 < \eta < 1$ and consider a cut off function

$$\zeta_\eta(\xi) = \zeta\left(\frac{\xi}{\eta}\right), \quad \zeta(\xi) = \begin{cases} 1 & \text{for } |\xi| \leqslant 1, \\ 0 & \text{for } |\xi| \geqslant 2 \end{cases}$$

and let

$$\hat v(\xi) = \zeta_\eta\hat v(\xi) + (1-\zeta_\eta)\hat v(\xi) =: \hat{v_1}(\xi) + \hat{v_2}(\xi).$$

For the high frequency part, we compute, using Plancherel's identity, and the fact that $|y| \geqslant \frac{R}{4}$ on the support of $1-\chi_R$:

$$|((\chi_R u^+)^-, v_2)| = |(\chi_R u^+, v_2^-)| = |((1-\chi_R)u^+, v_2^-)| \lesssim \frac{1}{R}\|u^+\|_{L^2}\|\langle y\rangle v_2^-\|_{L^2}$$

and, by construction and Plancherel's identity,

$$\|\langle y\rangle v_2^-\|_{L^2}^2 \lesssim \int |\hat{v}_2^-|^2 + |\partial_\xi \widehat{v_2^-}|^2 \lesssim \frac{1}{\eta^2}\left[\int |\hat{v}|^2 + |\partial_\xi \hat{v}|^2\right] \lesssim \frac{\|\langle y\rangle v\|_{L^2}^2}{\eta^2}.$$

We estimate, for the low frequency part,

$$|((\chi_R u^+)^-, v_1)| = \|u^+\|_{L^2}\|v_1\|_{L^2} \lesssim \|u^+\|_{L^2}\|\hat{v}_1\|_{L^2}$$

and

$$\|\hat{v_1}\|_{L^2}^2 \lesssim \int_{|\xi|\leqslant 2\eta} |\hat{v}|^2 \lesssim \eta\|\hat{v}\|_{L^\infty}^2 \lesssim \eta\|v\|_{L^1}^2 \lesssim \eta\|\langle y\rangle v\|_{L^2}^2.$$

The collection of above bounds and the choice $\eta = \frac{1}{R^{\frac{2}{3}}}$ yield

$$\left|((\chi_R u^+)^-, v)\right| \lesssim \left[\frac{1}{\eta R} + \sqrt{\eta}\right]\|u^+\|_{L^2}\|\langle y\rangle v\|_{L^2} \lesssim \frac{1}{R^{\frac{1}{3}}}\|u^+\|_{L^2}\|\langle y\rangle v\|_{L^2},$$

which proves (D.15). □

Appendix E. Estimates on the cut-off functions

This Appendix is devoted to the derivation of various estimates related to the localization of mass and kinetic energy which are used throughout Section 5. Recall (5.9), (5.10).

ζ estimates. We recall the definition of the cut-off functions, see (5.10), (5.13). The function Ψ_1 is smooth enough, non increasing, with

$$\Psi_1(z_1) = \left| \begin{array}{ll} 1 & \text{for } \ z_1 \leqslant \frac{1}{4} \\ (1-z_1)^{10} & \text{for } \ \frac{1}{2} \leqslant z_1 \leqslant 1 \\ 0 & \text{for } \ z_1 \geqslant 1. \end{array} \right. .$$

Furthermore, $\Phi_1 = \Psi_1 + b(1-\Psi_1)$, and

$$\phi_1(y_1) = \Phi_1\left(\frac{y_1}{R(1-b)}\right) \ , \ \phi(x) = \phi_1\left(\frac{x-x_1}{\lambda_1(1-\beta_1)}\right) \ .$$

Then, by construction, $b\partial_b\Phi_1 = \Phi_1 - \Psi_1 \leq \Phi_1$, and there holds the global control

$$|(1-z_1)\partial_{z_1}\Phi_1| \lesssim \Phi_1. \tag{E.1}$$

Then, since, by (5.11), $\zeta = \beta_1 + (1-\beta_1)(1-\phi)$, we have

$$|\partial_x\zeta| \lesssim (1-\beta_1)|\partial_x\phi| \lesssim \frac{1}{R}, \quad |b\partial_{z_1}\Phi_1| \lesssim b \lesssim \Phi_1. \tag{E.2}$$

We estimate

$$|\partial_{y_1}\phi_1| \lesssim \frac{1}{R}, \tag{E.3}$$

and

$$\|\partial_{y_1}^2\phi_1\|_{L^1} \lesssim \frac{1}{R}, \quad \|\partial_{y_1}^3\phi_1\|_{L^1} \lesssim \frac{1}{R^2}$$

from which, using (D.3),

$$\|[|D^{\frac{1}{2}}, \partial_{y_1}\phi_1]\|_{L^2\to L^2} \lesssim \frac{1}{R^{\frac{3}{2}}}. \tag{E.4}$$

More generally,

$$\|\Phi_1\|_{W^{k,\infty}_{z_1}} \lesssim 1, \quad k = 2, 3 \tag{E.5}$$

and hence, from $\phi_1 = \Phi_1\left(\frac{y_1}{R(1-b)}\right)$ and (D.9),

$$\|D^k[\Pi^\pm, \phi_1]g\|_{L^2} \lesssim \frac{\|g\|_{L^2}}{R^k}, \quad k = 1, 2 \tag{E.6}$$

$$\|D[\Pi^\pm, \partial_{y_1}\phi_1]g\|_{L^2} \lesssim \frac{1}{R^2}\|g\|_{L^2} \tag{E.7}$$

Next we compute

$$\partial_t\zeta + \partial_x\zeta = (\beta_1)_t\phi_1 - (1-\beta_1)(\partial_t\phi_1 + \partial_x\phi_1) = (1-\beta_1)W\left(t, \frac{y_1}{R(1-b)}\right) , \tag{E.8}$$

with

$$\begin{aligned}
W(t, z_1) &= \frac{(\beta_1)_t}{1-\beta_1}\Phi_1 - \frac{b_t}{b}(\Phi_1 - \Psi_1) - \frac{1-(x_1)_t}{\lambda_1(1-\beta_1)(1-b)R}\partial_{z_1}\Phi_1 \\
&- \left(-\frac{(\lambda_1)_t}{\lambda_1} + \frac{(\beta_1)_t}{1-\beta_1} + \frac{b_t}{1-b} - \frac{R_t}{R}\right) z_1\partial_{z_1}\Phi_1 \\
&= \frac{(\beta_1)_t}{1-\beta_1}\Phi_1 - \frac{b_t}{b}(\Phi_1 - \Psi_1) + \frac{\lambda_1 R_t z_1 - 1}{\lambda_1 R}\partial_{z_1}\Phi_1 + \frac{b}{\lambda_1(1-b)R}\partial_{z_1}\Phi_1 \\
&+ \frac{(x_1)_t - \beta_1}{\lambda_1(1-\beta_1)(1-b)R}\partial_{z_1}\Phi_1 - \left(-\frac{(\lambda_1)_t}{\lambda_1} + \frac{(\beta_1)_t}{1-\beta_1} + \frac{b_t}{1-b}\right) z_1\partial_{z_1}\Phi_1 \, .
\end{aligned} \tag{E.9}$$

We now use the bounds (5.25), (5.34), (5.20) and $b \lesssim \phi_1$ to derive

$$\left|\frac{(\beta_1)_t}{1-\beta_1}\right| + \left|\frac{(\lambda_1)_t}{\lambda_1}\right| + \left|\frac{(x_1)_t - \beta_1}{1-\beta_1}\right| + |b_t| \lesssim \frac{b}{t} \lesssim \frac{\phi_1}{t}$$

and hence, we obtain

$$|\partial_t\zeta + \partial_x\zeta| \lesssim \frac{1-\beta_1}{t}\phi_1 + (1-\beta_1)\,|1 - \lambda_1 R_t z_1|\,\frac{|\partial_{z_1}\Phi_1|}{R} \, . \tag{E.10}$$

Then we compute

$$\begin{aligned}
R_t &= \frac{(x_2)_t - (x_1)_t}{\lambda_1(1-\beta_1)} + R\left[-\frac{(\lambda_1)_t}{\lambda_1} + \frac{(\beta_1)_t}{1-\beta_1}\right] \\
&= \frac{\beta_2 - \beta_1}{\lambda_1(1-\beta_1)} + O(b)
\end{aligned}$$

and hence

$$1 - \lambda_1 R_t z_1 = 1 - \left[\frac{\beta_2 - \beta_1}{1-\beta_1} + O(b)\right] z_1 = 1 - z_1 + O(bz_1).$$

Injecting this into (E.10) with (E.1) and $b \lesssim \phi_1$, $R \sim t$, finally yields the fundamental estimate,

$$|\partial_t\zeta + \partial_x\zeta| \lesssim \frac{1-\beta_1}{t}\phi_1 \, . \tag{E.11}$$

Next we estimate the first three derivatives of $\sqrt{\phi_1}$ with respect to y_1. Since $\Phi_1 = b + (1-b)\Psi_1$, with Ψ_1 non increasing, we have

$$\Phi_1(z_1) \geq \frac{1}{2^{11}} \ , \ z_1 \leq \frac{1}{2} \ ,$$

hence $\partial_{z_1}^k\sqrt{\Phi_1}(z_1)$ are bounded for $k = 1, 2, 3$ and $z_1 \leq \frac{1}{2}$. As for $\frac{1}{2} \leq z_1 \leq 1$,

$$\sqrt{\Phi_1}(z_1) = \left(b + (1-b)(1-z_1)^{10}\right)^{\frac{1}{2}} \ ,$$

hence again $\partial_{z_1}^k\sqrt{\Phi_1}(z_1)$ are bounded for $k = 1, 2, 3$. Consequently,

$$\|\partial_{y_1}^k\sqrt{\phi_1}\|_{L^1} \lesssim \int_{\frac{(1-b)R}{4} \leqslant |y_1| \leqslant (1-b)R} \frac{1}{R^k} dy_1 \lesssim \frac{1}{R^{k-1}}, \quad k = 1, 2, 3 \ ,$$

and thus, from (D.3), (D.5),

$$\|[|D|^{\frac{1}{2}}, \sqrt{\phi_1}]f\|_{L^2} \lesssim \frac{1}{\sqrt{R}}\|f\|_{L^2}, \quad \|[|D|, \sqrt{\phi_1}]f\|_{L^2} \lesssim \frac{1}{R}\|f\|_{L^2} \tag{E.12}$$

$$\||D|^{\frac{1}{2}}[|D|^{\frac{1}{2}}, \sqrt{\phi_1}]f\|_{L^2} \lesssim \frac{1}{\sqrt{R}}\||D|^{\frac{1}{2}}f\|_{L^2} + \frac{1}{R}\|f\|_{L^2}. \tag{E.13}$$

According to (5.72), consider now

$$\psi = \frac{\partial_t\zeta + \partial_x\zeta}{\sqrt{\phi}}, \quad \psi(x) = \psi_1(y_1)$$

then from (E.11):

$$|\psi| \lesssim \frac{(1-\beta_1)}{t}\sqrt{\phi}. \tag{E.14}$$

In order to estimate the first two derivatives of ψ_1 with respect to y_1, we use (E.8), (E.9). We already noticed that the first three derivatives of $\sqrt{\Phi_1}(z_1)$ are bounded. By a similar argument, the first two derivatives of $\Psi_1/\sqrt{\Phi_1}$ are bounded. Consequently, using again $R \sim t$,

$$\partial_{y_1}\psi_1 = O\left(\frac{1-\beta_1}{t^2}\mathbf{1}_{\frac{R(1-b)}{4} \leqslant y_1 \leqslant R(1-b)}\right) \ , \ \partial_{y_1}^2\psi_1 = O\left(\frac{1-\beta_1}{t^3}\mathbf{1}_{\frac{R(1-b)}{4} \leqslant y_1 \leqslant R(1-b)}\right).$$

Hence

$$\|\partial_{y_1}\psi_1\|_{L^1} \lesssim \frac{1-\beta_1}{t}, \quad \|\partial_{y_1}^2\psi_1\|_{L^1} \lesssim \frac{1-\beta_1}{t^2}.$$

We conclude, from (D.3), that

$$\|[|D|^{\frac{1}{2}}, \psi_1]\|_{L^2 \to L^2} \lesssim \frac{1-\beta_1}{t^{\frac{3}{2}}}. \tag{E.15}$$

θ estimates. Recall from (5.13), (5.14):

$$\theta(t,x) = \theta(t,y_1) = \frac{1}{\lambda_1}\Psi_1(z_1) + \frac{1}{\lambda_2}(1-\Psi_1)(z_1).$$

Hence

$$|\partial_{y_1}\theta_1| \lesssim \frac{|\lambda_2 - \lambda_1|}{R}\mathbf{1}_{\frac{(1-b)R}{2} \leqslant y_1 \leqslant (1-b)R} \ , \tag{E.16}$$

and therefore

$$|\partial_x\theta| \lesssim \frac{|\lambda_1 - \lambda_2|}{(1-\beta_1)R} \ . \tag{E.17}$$

Next

$$[\Pi^{\pm}, \theta_1] = [\Pi^{\pm}, \frac{1}{\lambda_1}\Psi_1 + \frac{1}{\lambda_2}(1-\Psi_1)] = \frac{\lambda_2 - \lambda_1}{\lambda_1\lambda_2}[[\Pi^{\pm}, \Psi_1]$$

and hence from (D.9):

$$\left\|\partial_{y_1}[\Pi^{\pm}, \theta_1]g\right\|_{L^2} \lesssim \frac{|\lambda_2 - \lambda_1|}{R}\|g\|_{L^2} \ . \tag{E.18}$$

We now estimate more carefully:

$$\begin{aligned}(\partial_t + \partial_x)\theta = &-\frac{(\lambda_1)_t}{\lambda_1^2}\Psi_1 - \frac{(\lambda_2)_t}{\lambda_2^2}(1-\Psi_1)\\ + &\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\left[\frac{\beta_1 - (x_1)_t}{(1-\beta_1)\lambda_1 R(1-b)} + \frac{1}{\lambda_1 R(1-b)}\right]\partial_{z_1}\Psi_1\\ + &\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\left[-\frac{(\lambda_1)_t}{\lambda_1} + \frac{(\beta_1)_t}{1-\beta_1} - \frac{R_t}{R} + \frac{b_t}{1-b}\right]z_1\partial_{z_1}\Psi_1\end{aligned}$$

and hence

$$(\partial_t + \partial_x)\theta = O\left(\frac{1}{t}\right). \tag{E.19}$$

Appendix F. Proof of Proposition 5.1

This Appendix is devoted to the proof of Proposition 5.1. We recall the coercivity of the linearized Szegő operator which we will use in the following form: there exists a universal constant $0 < c_0 < 1$ such that for $u \in H^{\frac{1}{2}}_+$,

$$(\mathcal{L}_+ u, u) \geqslant c_0 \|u\|^2_{H^{\frac{1}{2}}_+} - \frac{1}{c_0}\left[(u, \partial_y Q^+)^2 + (u, iQ^+)^2\right]. \tag{F.1}$$

Proof of Proposition 5.1. We define the following functionals:

$$\mathcal{G}_1(\varepsilon) = \int ||D|^{\frac{1}{2}}\varepsilon_1^+|^2 dy_1 + \frac{1}{1-\beta_1}\int ||D|^{\frac{1}{2}}\varepsilon_1^-|^2 dy_1 + \lambda_1\|\varepsilon_1\|^2_{L^2} - (2|\Phi^{(1)}|^2\varepsilon_1 + (\Phi^{(1)})^2\overline{\varepsilon_1}, \varepsilon_1)$$
$$\mathcal{G}_0(\varepsilon) = \beta_1 \int ||D|^{\frac{1}{2}}\varepsilon_1^+|^2 dy_1 + (1-\beta_1)\mathcal{G}_1(\varepsilon)$$

where

$$\Phi^{(1)}(y_1) = V_1(\mathcal{P}, y_1) + \frac{1}{\mu^{\frac{1}{2}}} V_2(\mathcal{P}, y_2) e^{i\Gamma}.$$

Then the full functional $\mathcal{G}$ is exactly given by:

$$\begin{aligned} \mathcal{G}(\varepsilon) &= \frac{1}{2}\left[\frac{1}{\lambda_1}\mathcal{G}_0(\varepsilon,\varepsilon) - (\zeta D\varepsilon, \varepsilon) + ((\theta-1)\varepsilon, \varepsilon)\right] \\ &- \frac{1}{4}\left[\int (|\varepsilon+\Phi|^4 - |\Phi|^4) - 4(\varepsilon, \Phi|\Phi|^2) - 2(2|\Phi|^2\varepsilon + \Phi^2\overline{\varepsilon}, \varepsilon)\right] \end{aligned} \tag{F.2}$$

The heart of the proof is the derivation of a suitable coercivity for $\mathcal{G}_0$.

Step 1: Splitting and coercivity for the first bubble. Let $\chi_\ell(y_1) = \chi^{(0)}(\frac{y_1}{R})$, where $\chi^{(0)}$ is a smooth cut off function satisfying:

$$\chi^{(0)}(y_1) = \begin{cases} 1 & \text{for } y_1 \leqslant \frac{1}{10} \\ 0 & \text{for } y_1 \geqslant \frac{1}{5} \end{cases}.$$

We now split the L^2 norm:

$$\begin{aligned} \int |\varepsilon_1|^2 dy_1 &= \int |\varepsilon_1^+| dy_1 + \int |\varepsilon_1^-|^2 dy_1 = \int |\chi_l\varepsilon_1^+|^2 dy_1 + \int (1-\chi_l^2)|\varepsilon_1^+|^2 dy_1 + \int |\varepsilon_1^-|^2 dy_1 \\ &= \int |(\chi_l\varepsilon_1^+)^+|^2 dy_1 + \int |(\chi_l\varepsilon_1^+)^-|^2 dy_1 + \int (1-\chi_l^2)|\varepsilon_1^+|^2 dy_1 + \int |\varepsilon_1^-|^2 dy_1. \end{aligned}$$

We now split the kinetic energy according to (D.13):

$$\begin{aligned}\int ||D|^{\frac12}\varepsilon_1^+|^2dy_1 &= \int \chi_l^2||D|^{\frac12}\varepsilon_1^+|^2dy_1+\int(1-\chi_l^2)||D|^{\frac12}\varepsilon_1^+|^2dy_1\\ &= \int\left|\left[|D|^{\frac12}(\chi_l\varepsilon_1^+)\right]^+\right|^2dy_1+\int\left|\left[|D|^{\frac12}(\chi_l\varepsilon_1^+)\right]^-\right|^2dy_1\\ &+ \int(1-\chi_l^2)||D|^{\frac12}\varepsilon_1^+|^2dy_1\\ &+ O\left(\frac{\|\varepsilon_1\|_{L^2}^2+\||D|^{\frac12}(\chi_l\varepsilon_1^+)\|_{L^2}^2}{\sqrt{R}}\right)\end{aligned}$$

We now decompose the potential energy. We first estimate:

$$(2|\Phi^{(1)}|^2\varepsilon_1+(\Phi^{(1)})^2\overline{\varepsilon_1},\varepsilon_1)=(2|\Phi^{(1)}|^2\varepsilon_1^++(\Phi^{(1)})^2\overline{\varepsilon_1^+},\varepsilon_1^+)+O\left(\|(\Phi^{(1)})^2\varepsilon_1^-\|_{L^2}\|\varepsilon_1\|_{L^2}\right).$$

We now estimate from $|V_j|\lesssim\frac{1}{\langle y_j\rangle}$ and Sobolev:

$$\int|V_1|^4|\varepsilon_1^-|^2dy_1\lesssim\int\frac{|\varepsilon_1^-|^2}{\langle y_1\rangle^2}dy_1\lesssim\|\varepsilon_1^-\|_{L^4}^2\lesssim\|\varepsilon_1^-\|_{L^2}\|\varepsilon_1^-\|_{\dot H^{\frac12}}\lesssim\|\varepsilon_1\|_{L^2}\|\varepsilon_1^-\|_{\dot H^{\frac12}}$$
$$\int|V_2|^4|\varepsilon_1^-|^2dy_1\lesssim\int\frac{|\varepsilon_1^-|^2}{\langle y_2\rangle^4}dy_1\lesssim\sqrt{b}\|\varepsilon_1\|_{L^2}\|\varepsilon_1^-\|_{\dot H^{\frac12}}$$

We now develop the potential term:

$$\begin{aligned}\int|\Phi^{(1)}|^2|\varepsilon_1^+|^2dy_1 &= \int|\Phi^{(1)}|^2\left[\chi_l^2|\varepsilon_1^+|^2+(1-\chi_l^2)|\varepsilon_1^+|^2\right]dy_1\\ &= \int|Q_{\beta_1}|^2|\chi_l\varepsilon_1^+|^2dy_1+O\left(\frac{|\log\eta|^4\|\varepsilon_1\|_{L^2}^2}{R}\right)\\ &+ \int|\Phi^{(1)}|^2(1-\chi_l^2)|\varepsilon_1^+|^2dy_1\\ &= \int|Q^+|^2|\chi_l\varepsilon_1^+|^2dy_1+O\left(\left[\frac{|\log\eta|^4}{R}+(1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12}\right]\|\varepsilon_1\|_{L^2}^2\right)\\ &+ \int|\Phi^{(1)}|^2(1-\chi_l^2)|\varepsilon_1^+|^2dy_1\end{aligned}$$

by construction of V_1, the support properties of χ_l and the rough bound

$$\|Q_{\beta_1}-Q^+\|_{L^\infty}\lesssim\|Q_{\beta_1}-Q^+\|_{H^1}\lesssim(1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12}.$$

We now use (D.14) and $|Q^+|\lesssim\frac{1}{\langle y_1\rangle}$ which ensure

$$\int\frac{|(\chi_l\varepsilon_1^+)^-|^2}{\langle y_1\rangle^2}dy_1\lesssim\frac{1}{R^{\frac23}}\|\varepsilon_1\|_{L^2}^2 \tag{F.3}$$

to conclude:

$$\begin{aligned}\int|\Phi^{(1)}|^2|\varepsilon_1^+|^2dy_1 &= \int|Q^+|^2\left|\left[(\chi_l\varepsilon_1^+)\right]^+\right|^2dy_1+O\left(\left[\frac{1}{R^{\frac23}}+(1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12}\right]\|\varepsilon_1\|_{L^2}^2\right)\\ &+\int|\Phi^{(1)}|^2(1-\chi_l^2)|\varepsilon_1^+|^2dy_1.\end{aligned}$$

We argue similarly for the second potential term and obtain the first decomposition:

$$\begin{aligned}\mathcal{G}_1(\varepsilon) &= (\mathcal{L}_+(\chi_l\varepsilon_1^+)^+,(\chi_l\varepsilon_1^+)^+) + (\lambda_1-1)\int |(\chi_\ell\varepsilon_1^+)^+|^2dy_1 \\ &+ \frac{1}{1-\beta_1}\int ||D|^{\frac12}\varepsilon_1^-|^2dy_1 + \int ||D|^{\frac12}(\chi_l\varepsilon_1^+)^-|^2dy_1 + \lambda_1\int |(\chi_l\varepsilon_1^+)^-|^2dy_1 + \lambda_1\int |\varepsilon_1^-|^2dy_1 \\ &+ \int (1-\chi_l^2)||D|^{\frac12}\varepsilon_1^+|^2dy_1 + \lambda_1\int (1-\chi_l^2)|\varepsilon_1^+|^2dy_1 \\ &- 2\int |\Phi^{(1)}|^2(1-\chi_l^2)|\varepsilon_1^+|^2 - \mathrm{Re}\int (\Phi^{(1)})^2(1-\chi_l^2)\overline{(\varepsilon_1^+)^2} \\ &+ O\left(\left[\frac{1}{\sqrt{R}} + (1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12}\right]\|\varepsilon_1\|_{L^2}^2 + \|\varepsilon_1\|_{L^2}^{\frac32}\||D|^{\frac12}\varepsilon_1^-\|_{L^2}^{\frac12} + \frac{1}{\sqrt{R}}\||D|^{\frac12}(\chi_l\varepsilon_1^+)\|_{L^2}^2\right).\end{aligned} \tag{F.4}$$

From the choice of orthogonality conditions (5.6) we have:

$$\begin{aligned}0 &= (\varepsilon_1,Q_{\beta_1})^2 = (\varepsilon_1^+,Q^+)^2 + O((1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12}\|\varepsilon_1\|_{L^2}^2) \\ &= (\chi_l\varepsilon_1^+,Q^+)^2 + O\left(\left[(1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12} + \frac{1}{R}\right]\|\varepsilon_1\|_{L^2}^2\right) \\ &= ((\chi_l\varepsilon_1^+)^+,Q^+)^2 + O\left(\left[(1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12} + \frac{1}{R}\right]\|\varepsilon_1\|_{L^2}^2\right),\end{aligned}$$

and similarly:

$$0 = (\varepsilon_1,i\partial_{y_1}Q_{\beta_1})^2 = ((\chi_l\varepsilon_1^+)^+,i\partial_{y_1}Q^+)^2 + O\left(\left[(1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12} + \frac{1}{R}\right]\|\varepsilon_1\|_{L^2}^2\right).$$

We now apply the coercivity estimate (F.1) to $(\chi_l\varepsilon_1^+)^+$ and obtain from (F.4) the control:

$$\begin{aligned}\mathcal{G}_1(\varepsilon) &\geqslant c_0\left[\||D|^{\frac12}(\chi_l\varepsilon_1^+)\|_{L^2}^2 + \|\chi_l\varepsilon_1^+\|_{L^2}^2\right] + \frac{1}{(1-\beta_1)}\int ||D|^{\frac12}\varepsilon_1^-|^2 \\ &+ (\lambda_1-1)\int |(\chi_\ell\varepsilon_1^+)^+|^2dy_1 + \lambda_1\int |\varepsilon_1^-|^2 + O\left(\left[(1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12} + \frac{1}{\sqrt{R}}\right]\|\varepsilon_1\|_{L^2}^2\right) \\ &+ \int (1-\chi_l^2)||D|^{\frac12}\varepsilon_1^+|^2 + \lambda_1\int (1-\chi_l^2)|\varepsilon_1^+|^2 \\ &- 2\int |\Phi^{(1)}|^2(1-\chi_l^2)|\varepsilon_1^+|^2 - \mathrm{Re}\int (\Phi^{(1)})^2(1-\chi_l^2)\overline{(\varepsilon_1^+)^2} \\ &+ O\Big(\|\varepsilon_1\|_{L^2}^{\frac32}\||D|^{\frac12}\varepsilon_1^-\|_{L^2}^{\frac12} + \frac{1}{\sqrt{R}}\||D|^{1/2}(\chi_\ell\varepsilon_1^+)\|_{L^2}^2\Big)\end{aligned} \tag{F.5}$$

Step 2: Coercivity for the second bubble. We now consider $\chi_R(y_2) = \chi^{(1)}(\frac{y_2}{R})$, where $\chi^{(1)}$ is a smooth cut off function satisfying

$$\chi^{(1)}(y_2) = \begin{cases} 0 & \text{for } y_2 \leqslant -3 \\ 1 & \text{for } y_2 \geqslant -2, \end{cases}$$

and let

$$\begin{aligned}\mathcal{G}_2(\varepsilon) :&= b\int \chi_r^2||D|^{\frac12}\varepsilon_1^+|^2dy_1 + \lambda_1\int |\chi_r\varepsilon_1^+|^2 - 2\int |\Phi^{(1)}|^2\chi_r^2|\varepsilon_1^+|^2dy_1 \\ &- \mathrm{Re}\int (\Phi^{(1)})^2\chi_r^2\overline{(\varepsilon_1^+)^2}dy_1.\end{aligned}$$

$\mathcal{G}_2$ will be useful in finding a lower bound for $\mathcal{G}_1$. We observe from the support property of χ_l, χ_r and by construction of V_j the bounds:

$$\left|2\int |\Phi^{(1)}|^2(1-\chi_l^2-\chi_r^2)|\varepsilon_1^+|^2dy_1 - \mathrm{Re}\int (\Phi^{(1)})^2(1-\chi_l^2-\chi_r^2)\overline{(\varepsilon_1^+)^2}dy_1\right| \lesssim \frac{1}{R^2}\int |\varepsilon_1|^2$$

and therefore rewrite (F.5):

$$\begin{aligned}
\mathcal{G}_1(\varepsilon) &\geqslant \mathcal{G}_2(\varepsilon) + c_0\left[\||D|^{\frac12}(\chi_l\varepsilon_1^+)\|_{L^2}^2 + \|\chi_l\varepsilon_1^+\|_{L^2}^2\right] + \frac{1}{(1-\beta_1)}\int ||D|^{\frac12}\varepsilon_1^-|^2dy_1 \qquad \text{(F.6)}\\
&+ (\lambda_1-1)\int |(\chi_\ell\varepsilon_1^+)^+|^2dy_1 + \lambda_1\int |\varepsilon_1^-|^2dy_1 + \int (1-\chi_l^2-b\chi_r^2)||D|^{\frac12}\varepsilon_1^+|^2dy_1\\
&+ \lambda_1\int (1-\chi_l^2-\chi_r^2)|\varepsilon_1^+|^2dy_1\\
&+ O\left(\left[(1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12} + \frac{1}{\sqrt{R}}\right]\|\varepsilon_1\|_{L^2}^2 + \|\varepsilon_1\|_{L^2}^{\frac32}\||D|^{\frac12}\varepsilon_1^-\|_{L^2}^2 + \frac{\||D|^{\frac12}(\chi_\ell\varepsilon_1^+)\|_{L^2}^2}{\sqrt{R}}\right).
\end{aligned}$$

We renormalize to the y_2 variable using the formula

$$\varepsilon_1(y_1) = \frac{e^{i\Gamma}}{\sqrt{\mu}}\varepsilon_2\left(\frac{y_1-R}{b\mu}\right) = \frac{e^{i\Gamma}}{\sqrt{\mu}}\varepsilon_2(y_2)$$

and compute:

$$\begin{aligned}
\mu\mathcal{G}_2(\varepsilon) = b\Big[&\int \chi_r^2||D|^{\frac12}\varepsilon_2^+|^2dy_2 + \lambda_2\int |\chi_r\varepsilon_2^+|^2dy_2 - 2\int |\Phi^{(2)}|^2\chi_r^2|\varepsilon_2^+|^2dy_2\\
&- \mathrm{Re}\int (\Phi^{(2)})^2\chi_r^2\overline{(\varepsilon_2^+)^2}dy_2\Big].
\end{aligned}$$

where

$$\Phi^{(2)}(y_2) = \mu^{\frac12}V_1(\mathcal{P}, y_1)e^{-i\Gamma} + V_2(\mathcal{P}, y_2).$$

We estimate using (D.13):

$$\begin{aligned}
\int (1-\chi_l^2)||D|^{\frac12}\varepsilon_2^+|^2dy_2 = &\int (1-\chi_l^2-\chi_r^2)||D|^{\frac12}\varepsilon_2^+|^2dy_2 + \int ||D|^{\frac12}(\chi_r\varepsilon_2^+)|^2dy_2\\
&+ O\left(\frac{\|\varepsilon_2^+\|_{L^2}^2 + \||D|^{\frac12}(\chi_r\varepsilon_2^+)\|_{L^2}^2}{\sqrt{R}}\right)
\end{aligned}$$

and estimate as for the first bubble the potential energy to obtain:

$$\begin{aligned}
\mu\mathcal{G}_2(\varepsilon) &= b\left[(\mathcal{L}_+(\chi_r\varepsilon_2^+)^+, (\chi_r\varepsilon_2^+)^+) + \int ||D|^{\frac12}(\chi_r\varepsilon_2^+)^-|^2dy_2 + \lambda_2\int |(\chi_r\varepsilon_2^+)^-|^2dy_2\right]\\
&+ b(\lambda_2-1)\int |(\chi_r\varepsilon_2^+)^+|^2dy_2\\
&+ bO\left(\left[\frac{1}{\sqrt{R}} + (1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12}\right]\|\varepsilon_2\|_{L^2}^2 + \frac{\||D|^{\frac12}(\chi_r\varepsilon_2^+)\|_{L^2}^2}{\sqrt{R}}\right).
\end{aligned}$$

We estimate using the orthogonality conditions (5.6):

$$((\chi_r\varepsilon_2^+)^+, Q^+)^2 + ((\chi_r\varepsilon_2^+)^+, i\partial_y Q^+)^2 \lesssim \left[(1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12} + \frac{1}{R}\right]\|\varepsilon_2\|_{L^2}^2$$

and hence conclude using the coercivity (F.1):

$$\begin{aligned}\mathcal{G}_2(\varepsilon) &\geqslant \frac{bc_0}{\mu}\left[\||D|^{\frac12}(\chi_r\varepsilon_2^+)\|_{L^2}^2+\int\chi_r^2|\varepsilon_2^+|^2dy_2\right]+\frac{b(\lambda_2-1)}{\mu}\int|(\chi_r\varepsilon_2^+)^+|^2dy_2\\ &+ \frac{b}{\mu}O\left(\left[\frac{1}{\sqrt{R}}+(1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12}\right]\|\varepsilon_2\|_{L^2}^2+\frac{\||D|^{\frac12}(\chi_r\varepsilon_2^+)\|_{L^2}^2}{\sqrt{R}}\right)\end{aligned}\tag{F.7}$$

Step 3: Coercivity of $\mathcal{G}_0$. We sum (F.6) and (F.7) and conclude:

$$\begin{aligned}\mathcal{G}_1(\varepsilon) &\geqslant c_0\left[\||D|^{\frac12}(\chi_l\varepsilon_1^+)\|_{L^2}^2+\|\chi_l\varepsilon_1\|_{L^2}^2\right]+\frac{1}{(1-\beta_1)}\int||D|^{\frac12}\varepsilon_1^-|^2dy_1\\ &+ (\lambda_1-1)\int|(\chi_\ell\varepsilon_1^+)^+|^2dy_1+\lambda_1\int|\varepsilon_1^-|^2dy_1\\ &+ \int(1-\chi_l^2-b\chi_r^2)||D|^{\frac12}\varepsilon_1^+|^2dy_1+\int(1-\chi_l^2-\chi_r^2)|\varepsilon_1^+|^2dy_1\\ &+ O\left(\left[(1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12}+\frac{1}{\sqrt{R}}\right]\|\varepsilon_1\|_{L^2}^2+\|\varepsilon_1\|_{L^2}^{\frac32}\||D|^{\frac12}\varepsilon_1^-\|_{L^2}^{\frac12}+\frac{\||D|^{\frac12}(\chi_\ell\varepsilon_1^+)\|_{L^2}^2}{\sqrt{R}}\right)\\ &+ bc_0\left[\||D|^{\frac12}(\chi_r\varepsilon_2^+)\|_{L^2}^2+\int\chi_r^2|\varepsilon_2^+|^2dy_2\right]+b(\lambda_2-1)\int|(\chi_r\varepsilon_2^+)^+|^2dy_2\\ &+ bO\left(\left[\frac{1}{\sqrt{R}}+(1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12}\right]\|\varepsilon_2\|_{L^2}^2+\frac{\||D|^{\frac12}(\chi_r\varepsilon_2^+)\|_{L^2}^2}{\sqrt{R}}\right).\end{aligned}$$

which after renormalization to the y_1 variable implies:

$$\begin{aligned}\mathcal{G}_1(\varepsilon) &\geqslant c_0\left[\int(\chi_l^2+b\chi_r^2)||D|^{\frac12}\varepsilon_1^+|^2dy_1+\|\varepsilon_1\|_{L^2}^2\right]+\frac{1}{1-\beta_1}\int||D|^{\frac12}\varepsilon_1^-|^2dy_1\\ &+ \int(1-\chi_l^2-b\chi_r^2)||D|^{\frac12}\varepsilon_1^+|^2dy_1+\mathcal{E}rr(\varepsilon),\end{aligned}\tag{F.8}$$

where

$$\begin{aligned}\mathcal{E}rr(\varepsilon) &= c_0\Big(\frac{1}{\mu}-1\Big)\|\chi_r\varepsilon_1^+\|_{L^2}^2+c_0(\lambda_1-1)\|(\chi_\ell\varepsilon_1^+)^+\|_{L^2}^2+\frac{\lambda_2-1}{\mu}\|(\chi_r\varepsilon_1^+)^+\|_{L^2}^2\\ &+(\lambda_1-1)\|\varepsilon_1^-\|_{L^2}^2+O\Big((1-\beta_1)^{\frac12}|\log(1-\beta_1)|^{\frac12}+(1-\beta_2)^{\frac12}|\log(1-\beta_2)|^{\frac12}+\frac{1}{\sqrt{R}}\Big)\|\varepsilon_1\|_{L^2}^2\\ &+O\left(\frac{\||D|^{\frac12}(\chi_\ell\varepsilon_1^+)\|_{L^2}^2+b\||D|^{\frac12}(\chi_r\varepsilon_1^+)\|_{L^2}^2}{\sqrt{R}}\right)\\ &+O\Big(\|\varepsilon_1\|_{L^2}^{\frac32}\||D|^{\frac12}\varepsilon_1^-\|_{L^2}^{\frac12}\Big).\end{aligned}\tag{F.9}$$

Equivalently, this yields the lower bound:

$$\begin{aligned}\mathcal{G}_0(\varepsilon) &= \beta_1\int||D|^{\frac12}\varepsilon_1^+|^2dy_1+(1-\beta_1)\mathcal{G}_1\\ &\geqslant c_0(1-\beta_1)\|\varepsilon_1\|_{L^2}^2+\int\big[\beta_1+(1-\beta_1)(1-\phi_0+c_0\phi_0)\big]||D|^{\frac12}\varepsilon_1^+|^2dy_1\\ &+ \int||D|^{\frac12}\varepsilon_1^-|^2dy_1+(1-\beta_1)\mathcal{E}rr(\varepsilon),\end{aligned}$$

with

$$\phi_0=\chi_l^2+b\chi_r^2.\tag{F.10}$$

We now observe from the support property of χ_r, χ_l, ϕ_1 that $\phi_1 \geqslant \phi_0$ and since $c_0 < 1$ and $1 - \beta_1 > 0$, we have

$$\beta_1 + (1-\beta_1)(1-\phi_0 + c_0\phi_0) \geqslant \beta_1 + (1-\beta_1)(1-\phi_1 + c_0\phi_1).$$

We therefore have obtained the coercivity:

$$\begin{aligned} \mathcal{G}_0(\varepsilon) &\geqslant c_0(1-\beta_1)\int |\varepsilon_1|^2 \\ &+ \int \left[\beta_1 + (1-\beta_1)(1-\phi_1+c_0\phi_1)\right] ||D|^{\frac{1}{2}}\varepsilon_1^+|^2 \\ &+ \int ||D|^{\frac{1}{2}}\varepsilon_1^-|^2 + (1-\beta_1)\mathcal{E}rr(\varepsilon). \end{aligned} \tag{F.11}$$

Step 4: Control of the kinetic momentum and coercivity of $\mathcal{G}$. We now consider the full functional given by (F.2):

$$\mathcal{G}(\varepsilon) = \frac{1}{2}\left[\frac{1}{\lambda_1}\mathcal{G}_0(\varepsilon,\varepsilon) - (\zeta D\varepsilon, \varepsilon) + ((\theta-1)\varepsilon,\varepsilon)\right] + \mathcal{N}(\varepsilon)$$

$$\mathcal{N}(\varepsilon) = \frac{1}{4}\left[\int (|\varepsilon+\Phi|^4 - |\Phi|^4) - 4(\varepsilon, \Phi|\Phi|^2) - 2(2|\Phi|^2\varepsilon + \Phi^2\overline{\varepsilon}, \varepsilon)\right].$$

The cubic and higher order terms are easily estimated using the rough bound $\|\varepsilon\|_{H^1} \ll 1$:

$$\begin{aligned} \mathcal{N}(\varepsilon) \lesssim \int |\varepsilon|^4 + C|\varepsilon^3||\Phi|dx &\lesssim \|\varepsilon\|_{L^\infty}(\|\varepsilon\|_{L^\infty} + \|\Phi\|_{L^\infty})\|\varepsilon\|_{L^2}^2 \lesssim \|\varepsilon\|_{H^1}(\|\varepsilon\|_{H^1}+1)\|\varepsilon\|_{L^2}^2 \\ &\leqslant \frac{c_0}{10}\|\varepsilon\|_{L^2}^2 = \frac{c_0}{10}(1-\beta_1)\|\varepsilon_1\|_{L^2}^2. \end{aligned}$$

The L^2 error is estimated from $|\mu| \ll 1$:

$$|((\theta-1)\varepsilon,\varepsilon)| \lesssim |\mu|\|\varepsilon\|_{L^2}^2 \leqslant \frac{c_0}{10}(1-\beta_1)\|\varepsilon_1\|_{L^2}^2.$$

We therefore conclude from (F.11):

$$\begin{aligned} 2\mathcal{G}(\varepsilon) &\geqslant \frac{c_0(1-\beta_1)}{\lambda_1}\left[\int |\varepsilon_1|^2 dy_1 + \int \phi_1 ||D|^{\frac{1}{2}}\varepsilon_1^+|^2 dy_1\right] + \frac{1-\beta_1}{\lambda_1}\mathcal{E}rr(\varepsilon) \\ &+ \frac{1}{\lambda_1}\left[\int \zeta_1 ||D|^{\frac{1}{2}}\varepsilon_1^+|^2 + \frac{1}{\lambda_1}\int ||D|^{\frac{1}{2}}\varepsilon_1^-|^2 dy_1 - (\zeta_1 D\varepsilon_1, \varepsilon_1)\right]. \end{aligned}$$

We now estimate the kinetic momentum term. We first compute from (5.12):

$$\begin{aligned} &\int \zeta_1 ||D|^{\frac{1}{2}}\varepsilon_1^+|^2 - (\zeta_1 D\varepsilon_1^+, \varepsilon_1^+) \\ =\ &\int (1-(1-\beta_1)\phi_1)||D|^{\frac{1}{2}}\varepsilon_1^+|^2 - ((1-(1-\beta_1)\phi_1)D\varepsilon_1^+, \varepsilon_1^+) \\ =\ &-(1-\beta_1)\left[\int \phi_1 ||D|^{\frac{1}{2}}\varepsilon_1^+|^2 - (\phi_1 D\varepsilon_1^+, \varepsilon_1^+)\right] \end{aligned}$$

We then estimate using (E.12) and (D.12):

$$\begin{aligned} &(\phi_1 D\varepsilon_1^+, \varepsilon_1^+) = (\phi_1|D|\varepsilon_1^+, \varepsilon_1^+) = (\sqrt{\phi_1}|D|\varepsilon_1^+, \sqrt{\phi_1}\varepsilon_1^+) \\ =\ &([\sqrt{\phi_1}, |D|]\varepsilon_1^+ + |D|(\sqrt{\phi_1}\varepsilon_1^+), \sqrt{\phi_1}\varepsilon_1^+) = \int ||D|^{\frac{1}{2}}\sqrt{\phi_1}\varepsilon_1^+|^2 + O\left(\frac{1}{R}\|\varepsilon_1\|_{L^2}^2\right) \\ =\ &\int \phi_1 ||D|^{\frac{1}{2}}\varepsilon_1^+|^2 + O\left(\frac{1}{R}\|\varepsilon_1\|_{L^2}^2 + \frac{1}{\sqrt{R}}\left[\|\varepsilon_1\|_{L^2}^2 + \|\sqrt{\phi_1}|D|^{\frac{1}{2}}\varepsilon_1^+\|_{L^2}^2\right]\right) \end{aligned}$$

which yield thanks to the smallness of $\frac{1}{\sqrt{R}}$:

$$\int \zeta_1 ||D|^{\frac12}\varepsilon_1^+|^2 dy_1 - (\zeta_1 D\varepsilon_1^+, \varepsilon_1^+) \geqslant -\frac{c_0(1-\beta_1)}{10}\Big[\int \phi_1 ||D|^{\frac12}\varepsilon_1^+|^2 dy_1 + \int |\varepsilon_1|^2 dy_1\Big].$$

similarly using (D.1):

$$\begin{aligned} -(\zeta_1 D\varepsilon_1^-, \varepsilon_1^-) &= (\beta_1 + (1-\beta_1)\phi_1 |D|\varepsilon_1^-, \varepsilon_1^-) \\ &= \beta_1 \||D|^{\frac12}\varepsilon_1^-\|_{L^2}^2 + (1-\beta_1) O\left(\||D|^{\frac12}\varepsilon_1^-\|_{L^2}^2 + \frac{\|\varepsilon_1\|_{L^2}^2}{R}\right) \\ &\geqslant \frac12 \||D|^{\frac12}\varepsilon_1^-\|_{L^2}^2 - \frac{c_0}{10}(1-\beta_1)\|\varepsilon_1\|_{L^2}^2. \end{aligned}$$

For the crossed terms, we estimate from (D.9):

$$\begin{aligned} &|(\zeta_1 D\varepsilon_1^-, \varepsilon_1^+)| + |(\zeta_1, D\varepsilon_1^+, \varepsilon_1^-)| = (1-\beta_1)|(\phi_1 D\varepsilon_1^-, \varepsilon_1^+)| + |(\phi_1, D\varepsilon_1^+, \varepsilon_1^-)| \\ \lesssim\ & (1-\beta_1)\left[|(\varepsilon_1^-, D[\Pi^+, \phi_1]\varepsilon_1^+)| + |(\varepsilon_1^+, D[\Pi^-, \phi_1]\varepsilon_1^-)|\right] \lesssim \frac{1-\beta_1}{R^2}\|\varepsilon_1\|_{L^2}^2 \end{aligned}$$

The collection of above estimates yields the lower bound:

$$\mathcal{G}(\varepsilon) \geqslant \frac{c_0(1-\beta_1)}{2\lambda_1}\left[\int |\varepsilon_1|^2 + \int \phi_1 ||D|^{\frac12}\varepsilon_1^+|^2\right] + \int ||D|^{\frac12}\varepsilon_1^-|^2 + \frac{1-\beta_1}{\lambda_1}\mathcal{E}rr(\varepsilon). \tag{F.12}$$

Finally, we need to treat the error $\mathcal{E}rr(\varepsilon)$ defined in (F.9). Most of the terms can be bounded using the hypothesis

$$|\lambda_1 - 1| + |\lambda_2 - 1| + |\mu - 1| + |1-\beta_1| + |1-\beta_2| + \frac{1}{R} \ll 1.$$

We turn to the last term in (F.9) and by Young's inequality obtain that

$$\begin{aligned} C\|\varepsilon_1\|_{L^2}^{\frac32} \||D|^{\frac12}\varepsilon_1^-\|_{L^2}^{\frac12} &= C\Big(\sqrt{\frac{c_0}{3\lambda_1 C}}\|\varepsilon_1\|_{L^2}\Big)^{\frac32}\Big(\sqrt{\frac{3\lambda_1 C}{c_0}}\||D|^{\frac12}\varepsilon_1^-\|_{L^2}\Big)^{\frac12} \\ &\leqslant \frac{c_0}{4\lambda_1}\|\varepsilon_1\|_{L^2}^2 + \frac{C(3\lambda_1 C)^3}{4c_0^3}\||D|^{\frac12}\varepsilon_1^-\|_{L^2}^2. \end{aligned}$$

Thus, the last term in $\frac{1-\beta_1}{\lambda_1}\mathcal{E}rr(\varepsilon)$ has a lower bound:

$$-\frac{c_0(1-\beta_1)}{4\lambda_1}\|\varepsilon_1\|_{L^2}^2 - \frac{C(3\lambda_1 C)^3(1-\beta_1)}{4c_0^3}\||D|^{\frac12}\varepsilon_1^-\|_{L^2}^2,$$

Since $0 < 1-\beta_1 \ll 1$, it can be absorbed by the main terms in (F.12) to obtain:

$$\mathcal{G}(\varepsilon) \geqslant \frac{c_0(1-\beta_1)}{5\lambda_1}\left[\int |\varepsilon_1|^2 + \int \phi_1 ||D|^{\frac12}\varepsilon_1^+|^2\right] + \frac{1}{\lambda_1}\int ||D|^{\frac12}\varepsilon_1^-|^2$$

which concludes the proof of Proposition 5.1. □

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