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Charles Favre

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DEGENERATION OF ENDOMORPHISMS OF THE COMPLEX PROJECTIVE SPACE IN THE HYBRID SPACE

CHARLES FAVRE

ABSTRACT. Consider a meromorphic family of endomorphisms of degree at least 2 of a complex projective space that is parameterized by the unit disk. We prove that the measure of maximal entropy of these endomorphisms converges to the equilibrium measure of the associated non-Archimedean dynamical system when the system degenerates. The convergence holds in the hybrid space constructed by Boucksom and Jonsson. We also infer from our analysis an estimate for the blow-up of the Lyapunov exponent near a pole in one-dimensional families of endomorphisms.

CONTENTS

Introduction	2
1. Degeneration of complex projective manifolds	6
1.1. The general setup	6
1.2. The hybrid circle	6
1.3. The hybrid space	7
2. Model functions	9
2.1. Admissible data	9
2.2. Model functions on degenerations	11
2.3. Model functions on non-Archimedean analytic spaces	13
2.4. Model functions on the hybrid space	15
2.5. Density of model functions	16
3. Degeneration of Monge-Ampère measures of model functions on the hybrid space	18
3.1. Monge-Ampère measures associated to an admissible datum	18
3.2. The Monge-Ampère measure on $X_{\mathbb{C}((t)),r}$	19
3.3. The Chambert-Loir measure associated to an endomorphism of $\mathbb{P}_{\mathbb{C}((t)),r}^k$	21
3.4. Degeneration of measures	22
4. Monge-Ampère of uniform limits of model functions	25
4.1. Uniform limits of model functions	25
4.2. Example of uniform functions	27
4.3. Degeneration of measures of maximal entropy	29
4.4. Lyapunov exponents of endomorphisms	31
5. Questions	33
5.1. Characterization of uniform functions	33
5.2. Effective aspects	33
5.3. Degeneration of volume forms	33
References	34

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INTRODUCTION

This paper is the first of a series in which we analyze degenerations of complex dynamical systems. Inspired by the recent work of S. Boucksom and M. Jonsson [BJ16] we aim here at describing the limit of the equilibrium measures of a meromorphic family of endomorphisms of the projective space.

More specifically we consider a holomorphic family $\{R_t\}_{t \in \mathbb{D}^*}$ of endomorphisms of the complex projective space $\mathbb{P}_{\mathbb{C}}^k$ of degree $d \geq 2$ parameterized by the punctured unit disk, and assume it extends to a meromorphic family over \mathbb{D} . For any $t \neq 0$ small enough, one can attach to R_t its unique measure of maximal entropy μ_t which is obtained as the limit $\frac{1}{d^{kn}}(R_t^n)^*\omega_{\text{FS}}^k$ as $n \rightarrow \infty$, where ω_{FS} is the usual Fubini-Study Kähler form on $\mathbb{P}_{\mathbb{C}}^k$.

The family $\{R_t\}$ also induces an endomorphism R of degree d on the Berkovich analytification of the projective space $\mathbb{P}_{\mathbb{C}((t)),r}^k$ defined over the valued field of formal Laurent series endowed with the t -adic norm normalized by $|t|_r = r \in (0, 1)$. In a similar way as in the complex case, Chambert-Loir [CL06] proved that the sequence of probability measures $\frac{1}{d^{kn}}(R^n)^*\delta_{x_g}$ converges to a measure μ_R where x_g is the Gauß point in $\mathbb{P}_{\mathbb{C}((t)),r}^k$. The entropy properties of μ_R are much more delicate to control in this case, and μ_R is no longer the measure of maximal entropy in general, see [FRL10].

We shall show that μ_t converges towards μ_R as $t \rightarrow 0$. This convergence statement is parallel to the results of L. DeMarco and X. Faber [DMF14] that imply the convergence of μ_t to the residual measure¹ of μ_R in the analytic space $\mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}$ when $k = 1$. Our main result is somehow less precise than the one by DeMarco and Faber, and we postpone to a subsequent article the proof of a generalization of their theorem.

To make sense out of our convergence statement we face the difficulty that our measures live in spaces of very different natures: complex analytic for μ_t and analytic over a non-Archimedean field for μ_R . Constructing spaces mixing complex analytic spaces and non-Archimedean analytic spaces have appeared though in the literature several times, most notably in the work of Morgan-Shalen in the 80's on character varieties, see e.g. [MS85]; and in a paper by Berkovich [Be09]. Such spaces are also implicit in the works of DeMarco and McMullen [DMM08] and Kiwi [Ki06].

We use here a construction by Boucksom and Jonsson [BJ16] of a hybrid analytic space that projects onto the unit disk \mathbb{D} such that the preimage of \mathbb{D}^* is naturally isomorphic to $\mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}^*$ whereas the fiber over 0 is homeomorphic to the non-Archimedean analytic space $\mathbb{P}_{\mathbb{C}((t)),r}^k$.

The construction of this hybrid space can be done as follows. Let A_r be the Banach space of complex power series $f = \sum_{n \in \mathbb{Z}} a_n T^n$ such that $|f|_{\text{hyb},r} := \sum_{n \in \mathbb{Z}} |a_n|_{\text{hyb}} r^n < +\infty$ where $|c|_{\text{hyb}} = \max\{1, |c|\}$ for any $c \in \mathbb{C}^*$. It was proved by Poineau [Po10] that the Berkovich spectrum of A_r is naturally isomorphic to the circle $\{|T| = r\}$ inside the affine line over the Banach ring $(\mathbb{C}, |\cdot|_{\text{hyb}})$, see also §1.2 below. For that reason, we denote by $\mathcal{C}_{\text{hyb}}(r)$ this spectrum.

Now one considers the projective space \mathbb{P}^k over this spectrum. To ease notation we shall write $\mathbb{P}_{\text{hyb},r}^k$ instead of $\mathbb{P}_{\mathcal{C}_{\text{hyb}}(r)}^k$, and denote by $\pi : \mathbb{P}_{\text{hyb},r}^k \rightarrow \mathcal{C}_{\text{hyb}}(r)$ the natural structure map. Recall the following statement due to Boucksom and Jonsson.

¹this is a purely atomic measure as soon as R_t diverges in the parameter space of rational maps of degree d

Theorem A. [BJ16, Appendix A] *There are three natural homeomorphisms*

$$\tau : \bar{\mathbb{D}}_r \rightarrow \mathcal{C}_{\text{hyb}}(r), \psi_{\text{NA}} : \mathbb{P}_{\mathbb{C}((t)),r}^k \rightarrow \pi^{-1}(\tau(0)), \text{ and } \psi : \mathbb{P}_{\mathbb{C}}^k \times \bar{\mathbb{D}}_r^* \rightarrow \pi^{-1}(\tau(\bar{\mathbb{D}}_r^*))$$

such that $\tau^{-1} \circ \pi \circ \psi$ is the projection onto the second factor.

The construction of the hybrid is canonical enough so that any meromorphic family of endomorphisms $\{R_t\}_{t \in \mathbb{D}^*}$ as above induces a continuous map on $\mathbb{P}_{\text{hyb},r}^k$ whose action on $\tau^{-1}(0)$ is conjugated to \mathbb{R} by ψ_{NA} . One can now state our main result.

Theorem B. *Fix $r \in (0, 1)$. Let $\{R_t\}_{t \in \mathbb{D}}$ be any meromorphic family of endomorphisms of degree $d \geq 2$ of $\mathbb{P}_{\mathbb{C}}^k$ that is parameterized by the unit disk, and let \mathbb{R} be the endomorphism induced by this family on the Berkovich space $\mathbb{P}_{\mathbb{C}((t)),r}^k$. For any $t \neq 0$, denote by μ_t the measure of maximal entropy of R_t , and let $\mu_{\mathbb{R}}$ be the Chambert-Loir measure associated to \mathbb{R} .*

Then one has the weak convergence of measures

$$\lim_{t \rightarrow 0} \psi(\cdot, t)_* \mu_t \rightarrow (\psi_{\text{NA}})_* \mu_{\mathbb{R}}$$

in $\mathbb{P}_{\text{hyb},r}^k$.

The above convergence is equivalent to the convergence of integrals

$$(1) \quad \int \Phi d(\psi(\cdot, t)_* \mu_t) \longrightarrow \int \Phi d((\psi_{\text{NA}})_* \mu_{\mathbb{R}}) \text{ as } t \rightarrow 0,$$

for any continuous function Φ on the hybrid space. The bulk of our proof is to prove this convergence for only special functions that we call *model* functions and which are defined as follows. Let \mathcal{L} be the pull-back of $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^k}(1)$ on the product space $\mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}$ by the first projection, and let us fix a reference metrization $|\cdot|_*$ on \mathcal{L} that we assume to be smooth and positive. A (regular) admissible datum \mathcal{F} is the choice of a finite set τ_1, \dots, τ_l of meromorphic sections of a fixed power of \mathcal{L} that are holomorphic over $\mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}^*$ and have no common zeroes (see §2.1 for a precise definition). Any regular admissible datum gives naturally rise to a continuous function Φ on the hybrid space whose restriction to $\mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}^*$ is equal to $\log \max\{|\tau_1|_*, \dots, |\tau_l|_*\}$ (see Theorem 2.10). A model function is any function obtained in this way.

The key observation is that the set of model functions associated to regular admissible data forms a dense set in the space of continuous functions, see Theorem 2.12. For a model function the convergence (1) follows from direct estimates and basic facts about the definition of the complex Monge-Ampère operator as defined originally by E. Bedford and A. Taylor in [BT76]. Using refined Chern-Levine-Nirenberg estimates due to Demailly (see also [FS95]), we prove that our estimates imply the convergence (1) for more general functions than model functions. As an illustration of these ideas we analyze the behaviour of the Lyapunov exponents of R_t as $t \rightarrow 0$.

Recall that the sum of all Lyapunov exponents of the complex endomorphism R_t with respect to the measure μ_t is defined by the integral

$$\text{Lyap}(R_t) = \int \log \|\det(dR_t)\| d\mu_t$$

where $\|\det(dR_t)\|(x)$ is the norm of the determinant of the differential of R_t at a point x computed in terms of e.g. the standard Fubini-Study Kähler metric² on $\mathbb{P}_{\mathbb{C}}^k$. Observe

²Since μ_t is invariant this quantity does not depend on the choice of the metric.

that the integral defining the Lyapunov exponent actually converges, since μ_t is locally the Monge-Ampère measure of a continuous plurisubharmonic (psh) function. Briend and Duval have been able to bound from below each individual Lyapunov exponent of μ_t by $\frac{1}{2} \log d$ so that $\text{Lyap}(R_t) \geq \frac{k}{2} \log d$, see [BD99]. It is also known that the function $t \mapsto \text{Lyap}(f_t)$ is Hölder continuous and subharmonic on \mathbb{D}^* , see e.g. [BB, Corollary 3.4].

In a non-Archimedean context one can define the quantity $\|\det(dR)\|$ using the projective/spherical metric on the projective space. Using the theory developed in [BFJ15] one can argue just as in the complex case that the integral $\text{Lyap}(\mathbf{R}) := \int \log \|\det(dR)\| d\mu_{\mathbf{R}}$ is finite³.

To the author's knowledge, the Lyapunov exponent has been considered only in a few papers over a non-Archimedean field and just in dimension $k = 1$. In this case, Okuyama [Ok15] has proved that the Lyapunov exponent can be computed as the limit of the average of the multiplier of periodic orbits of increasing periods. Jacobs [J16] has given some estimates of $\text{Lyap}(\mathbf{R})$ in terms of the Lipschitz constant of \mathbf{R} w.r.t. the spherical metric. Finally, the author and Rivera-Letelier [FRL16] have recently given a characterization of those rational maps of \mathbb{P}_K^1 having zero Lyapunov exponent, under the assumption that K is a complete metrized field of residual characteristic zero. This applies in particular to the case $K = \mathbb{C}((t))$ endowed with the t -adic norm.

Theorem C. *Under the same assumptions as in the previous theorem, we have*

$$(2) \quad \text{Lyap}(R_t) = \frac{\text{Lyap}(\mathbf{R})}{\log r} \log |t|^{-1} + o(\log |t|^{-1}).$$

In particular we have $\text{Lyap}(\mathbf{R}) \geq 0$.

In dimension 1, the theorem is a consequence from works by DeMarco. More precisely, it follows from a combination of [DeM03, Theorem 1.4] and [DeM16, Proposition 3.1]. Let us mention the following

Conjecture 1. *The function $\text{Lyap}(R_t) - \frac{\text{Lyap}(\mathbf{R})}{\log r} \log |t|^{-1}$ extends continuously at $t = 0$, and $\frac{\text{Lyap}(\mathbf{R})}{\log r}$ is a non-negative rational number.*

A proof of this fact has been given by Ghioca and Ye when $k = 1$ and R_t is a family of cubic polynomials, see [GY16]. But even in dimension 1 this conjecture is widely open for an arbitrary family of rational maps.

Theorems B and C are consequences of results that we present in a higher degree of generality, replacing the product space $\mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}^*$ by any holomorphic family of projective varieties $X \rightarrow \mathbb{D}^*$; and our dynamical setup by a more geometric one working with various metrics on a fixed relatively ample line bundle $L \rightarrow X$. Our setup is described in details in §1.1, and our main results are then Theorems 1.2, 3.5 and 4.2.

We have not tried to prove our results in maximal generality: it is very likely that they extend to degenerations of compact Kähler manifolds for instance (in which case the hybrid space has to be replaced by the construction given in [BJ16, §4]). We have collected a series of questions in §5 that we feel are of some interest for further researchs.

Notation. $|\cdot|_0$ is the trivial norm (on any field); $|\cdot|_{\infty}$ is the standard euclidean norm on the field of complex numbers; and $|\cdot|_{\text{hyb}} = \max\{|\cdot|_{\infty}, |\cdot|_0\}$ is the hybrid norm of Berkovich.

³observe that this quantity depends on the choice of norm on $\mathbb{C}((t))$ hence on our given $r \in (0, 1)$.

For any $r \in (0, 1)$, we set $\mathbb{D}_r = \{z \in \mathbb{C}, |z| < r\}$, $\mathbb{D}_r^* = \mathbb{D} \setminus \{0\}$, $\bar{\mathbb{D}}_r = \{z \in \mathbb{C}, |z| \leq r\}$, and $\bar{\mathbb{D}}_r^* = \bar{\mathbb{D}}_r \setminus \{0\}$. We let $|\cdot|_r$ be the t -adic norm on $\mathbb{C}((t))$ normalized by $|t|_r = r$. We write $\mathcal{O}(\mathbb{D})$ for the ring of holomorphic functions on \mathbb{D} .

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1. DEGENERATION OF COMPLEX PROJECTIVE MANIFOLDS

In this section, we explain the construction of the hybrid space mentioned in the introduction following Boucksom and Jonsson. We shall work in a more general setting than strictly necessary to prove Theorems A and B so as to allow more flexibility in our arguments.

1.1. The general setup. We consider a proper submersion $\pi : X \rightarrow \mathbb{D}^*$ having connected fibers where X is a smooth connected complex manifold of dimension $k + 1$. We fix an *snc model* \mathcal{X} of X that is a smooth connected complex manifold endowed with a proper map⁴ $\pi_{\mathcal{X}} : \mathcal{X} \rightarrow \mathbb{D}$, together with an isomorphism $\pi_{\mathcal{X}}^{-1}(\mathbb{D}^*) \simeq X$ sending $\pi_{\mathcal{X}}$ to π such that the central fiber $X_0 := \pi_{\mathcal{X}}^{-1}(0)$ is a divisor with simple normal crossing singularities. We get a natural embedding $\iota : X \rightarrow \mathcal{X}$.

We suppose given a line bundle $\mathcal{L} \rightarrow \mathcal{X}$ that is relatively very ample over \mathbb{D} . This means that the restriction of \mathcal{L} to the fiber $X_t = \pi_{\mathcal{X}}^{-1}(t)$ is ample for any $t \in \mathbb{D}$. One can show that up to restricting the family to a smaller disk this is equivalent to the existence of an embedding of \mathcal{X} into $\mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}$ compatible with $\pi_{\mathcal{X}}$ such that \mathcal{L} is the restriction to \mathcal{X} of the pull-back of $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^N}(1)$ by the first projection (see [Nak87, §1.4], or [Co06, Theorem 3.2.7]). It follows also from [Nak04, Lemma 1.11] that one may find a finite set of homogeneous polynomials $Q_1, \dots, Q_M \in \mathcal{O}(\mathbb{D})[z_0, \dots, z_N]$ in $N + 1$ variables whose coefficients are holomorphic functions over \mathbb{D} and such that $\mathcal{X} = \{([z], t), Q_1(t, z) = \dots = Q_M(t, z) = 0\}$.

We shall denote by t the holomorphic function π on \mathcal{X} (with values in the unit disk). For any $0 < r < 1$ we write $\mathcal{X}_r = \pi^{-1}(\mathbb{D}_r)$, $X_r = \mathcal{X}_r \cap X$, and $\bar{\mathcal{X}}_r = \pi^{-1}(\bar{\mathbb{D}}_r)$.

1.2. The hybrid circle. Fix $r \in (0, 1)$. Recall from the introduction that one defines the ring

$$A_r := \left\{ f = \sum_{n \in \mathbb{Z}} a_n t^n, \|f\|_{\text{hyb}, r} := \sum_{n \in \mathbb{Z}} |a_n|_{\text{hyb}} r^n < +\infty \right\}.$$

With the norm $\|\cdot\|_{\text{hyb}, r}$ it is a Banach ring, and we let $\mathcal{C}_{\text{hyb}}(r)$ be its Berkovich spectrum, i.e. the set of all multiplicative semi-norms on A_r that are bounded by $\|\cdot\|_{\text{hyb}, r}$ endowed with the topology of the pointwise convergence.

Observe that for any $f \in A_r$ the set of negative integers n for which $a_n \neq 0$ is finite. It follows that any map $f \in A_r$ induces a continuous map on $\bar{\mathbb{D}}_r^*$ that is holomorphic on \mathbb{D}_r , and meromorphic at 0. One can thus define a canonical map τ from the closed disk of radius r to $\mathcal{C}_{\text{hyb}}(r)$ by the formulas:

$$(3) \quad \begin{cases} |f(\tau(0))| = r^{\text{ord}_0(f)}; \\ |f(\tau(z))| = |f(z)|_{\infty}^{\frac{\log r}{\log |z|_{\infty}}} \quad \text{if } 0 < |z| \leq r. \end{cases}$$

for any $f \in A_r$.

This map is injective since $|(t - w)(\tau(z))| = 0$ iff $z = w \neq 0$. It is also continuous since one can write $f(t) = t^{\text{ord}_0(f)}(a + o(1))$ with $a \in \mathbb{C}^*$, and

$$\log |f(\tau(z))| = \log r \frac{\log |f(z)|_{\infty}}{\log |z|_{\infty}} = \log r \frac{\text{ord}_0(f) \log |z|_{\infty} + \log |a + o(1)|}{\log |z|_{\infty}} \rightarrow \log(r^{\text{ord}_0(f)}),$$

when $z \rightarrow 0$.

Proposition 1.1 ([Po10]). *The map $\tau : \bar{\mathbb{D}}_r \rightarrow \mathcal{C}_{\text{hyb}}(r)$ is a homeomorphism.*

⁴since \mathcal{X} and \mathbb{D} are smooth and the latter is a curve, the map $\pi_{\mathcal{X}}$ is automatically flat.

Proof. Since $\mathcal{C}_{\text{hyb}}(r)$ is the analytic spectrum of a Banach ring it is compact. It is therefore sufficient to prove that τ is surjective. Pick any multiplicative semi-norm $f \mapsto |f|$ on A_r bounded by $\|\cdot\|_{\text{hyb},r}$. Observe that $|t|^n = |t^n| \leq r^n$ for all $n \in \mathbb{Z}$ which implies $|t| = r$.

Suppose first that the restriction of $|\cdot|$ to \mathbb{C} is the trivial norm. Then $|\cdot|$ is non-Archimedean since

$$|f + g| = |(f + g)^n|^{1/n} = \left| \sum_{i=0}^n \binom{n}{i} f^i g^{n-i} \right|^{1/n} \leq \left(\sum_{i=0}^n |f|^i |g|^{n-i} \right)^{1/n} \leq (n+1)^{1/n} \max\{|f|, |g|\},$$

and letting $n \rightarrow \infty$. Pick $f = t^{\text{ord}_0(f)}(a + \sum_{n \geq 1} a_n t^n) \in A_r$ with $a \neq 0$. Then we have

$$|f| = r^{\text{ord}_0(f)} \left| a + \sum_{n \geq 1} a_n t^n \right|.$$

Since $\sum_{n \geq 0} |a_n|_{\text{hyb}} r^n < +\infty$, we get $\lim_{N \rightarrow \infty} \sum_{n \geq N} |a_n|_{\text{hyb}} r^n = 0$, so that

$$\left| \sum_{n \geq 1} a_n t^n \right| \leq \left| \sum_{n \geq 1} a_n t^n \right|_{\text{hyb}} \leq \max \left\{ |t|, \left| \sum_{n \geq N} a_n t^n \right|_{\text{hyb}} \right\} < 1$$

because $|\cdot|$ is non-Archimedean. Since $|a| = 1$, it follows that $|f| = r^{\text{ord}_0(f)}$.

Suppose now the restriction of $|\cdot|$ to \mathbb{C} is non-trivial. Then there exists a positive real number $\epsilon \leq 1$ such that $|2| = |2|_\infty^\epsilon$, and this implies $|c| = |c|_\infty^\epsilon$ for any $c \in \mathbb{C}$. Look at the restriction of $|\cdot|$ to the sub-algebra $\mathbb{C}[t]$ of A_r . By Gelfand-Mazur theorem, this restriction has a non-trivial kernel hence $|P(t)| = |P(z)|_\infty^\epsilon$ for some $z \in \mathbb{C}$ and any $P \in \mathbb{C}[t]$. Since $|t| = r$, we have $|z|_\infty = r^{1/\epsilon}$. Now pick any $f \in A_r$, and expand it into power series $f(t) = t^{\text{ord}_0(f)}(a + \sum_{n \geq 1} a_n t^n)$. As above we have

$$\left| \sum_{n \geq N} a_n t^n \right| \leq \sum_{n \geq N} |a_n|_{\text{hyb}} r^n \xrightarrow{N \rightarrow \infty} 0,$$

and we get $|f| = \lim_{N \rightarrow \infty} \left| t^{\text{ord}_0(f)}(a + \sum_{n \leq N} a_n t^n) \right| = |f(z)|_\infty^\epsilon$. \square

1.3. The hybrid space. Any holomorphic function f on the punctured unit disk that is meromorphic at 0 can be expanded as a series $\sum_{n \geq n_0} a_n t^n$ for some $n_0 \in \mathbb{Z}$ with $\sum_{n \geq n_0} |a_n| \rho^n < \infty$ for all $\rho < 1$, hence belongs to A_r . Since X is defined by an homogeneous ideal of polynomials with coefficients in the space of holomorphic functions over \mathbb{D} , one can make a base change and look at the projective A_r -scheme X_{A_r} induced by X . In the sequel, we fix a finite union of affine charts $U_i = \text{Spec } B_i$ with B_i an A_r -algebra of finite type for X_{A_r} . If an embedding \mathcal{X} into $\mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}$ is fixed, then one may choose an index $i \in \{0, \dots, N\}$ and look at

$$U_i = \mathcal{X} \cap \{([z], t) \in \mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}, z_i \neq 0\}$$

so that

$$B_i = A_r[w_1, \dots, w_N] / \langle Q_j(t, w_1, \dots, w_{i-1}, 1, w_{i+1}, \dots, w_N) \rangle_{j=1, \dots, M}.$$

One defines the hybrid space $X_{\text{hyb},r}$ as the analytification (in the sense of Berkovich) of the projective scheme X_{A_r} . As a topological space it is obtained as follows. Set $(U_i)_{\text{hyb},r}$ to

be the space of those multiplicative semi-norms on B_i whose restriction to A_r is bounded by $\|\cdot\|_{\text{hyb},r}$, and endow this space with the topology of the pointwise convergence. Define $X_{\text{hyb},r}$ as the union of $(U_i)_{\text{hyb},r}$ patched together in a natural way using the patching maps defining X_{A_r} .

It is a compact space (any embedding of X into $\mathbb{P}_{A_r}^N$ realizes $X_{\text{hyb},r}$ as a closed subset of $\mathbb{P}_{\text{hyb},r}^N$, and it is easy to check that the latter space is compact). We get a continuous structure map $\pi_{\text{hyb}} : X_{\text{hyb},r} \rightarrow \mathcal{C}_{\text{hyb}}(r)$ sending a semi-norm on B_i to its restriction to A_r .

Observe that A_r is a subring of the field of formal Laurent series $\mathbb{C}((t))$. Endowed with the t -adic norm $|\cdot|_r$ such that $|t|_r = r$, the field $\mathbb{C}((t))$ becomes complete with valuation ring $\mathbb{C}[[t]]$. We may thus consider the projective variety $X_{\mathbb{C}((t))}$ obtained by base change $A_r \rightarrow \mathbb{C}((t))$, and the Berkovich analytification $X_{\mathbb{C}((t)),r}$ of $X_{\mathbb{C}((t))}$ with respect to $|\cdot|_r$.

The latter space can be defined just like the hybrid space above using affine charts, or more intrinsically as follows (see e.g. [Nic16, §2.1]). A point in $X_{\mathbb{C}((t)),r}$ is a pair $(x, |\cdot|)$ where x is a scheme-theoretic point in $X_{\mathbb{C}((t))}$ with residue field $\kappa(x)$ and $|\cdot| : \kappa(x) \rightarrow \mathbb{R}_+$ is a norm whose restriction to $\mathbb{C}((t))$ is $|\cdot|_r$.

The topology on $X_{\mathbb{C}((t)),r}$ is the coarsest one such that the canonical map $\mathfrak{s} : X_{\mathbb{C}((t)),r} \rightarrow X_{\mathbb{C}((t))}$ sending $(x, |\cdot|)$ to x is continuous; and for any affine open set $U \subset X_{\mathbb{C}((t))}$, the map $f \mapsto |f(x)|$ is continuous on $\mathfrak{s}^{-1}(U)$.

Since $X_{\mathbb{C}((t))}$ is projective and connected, it follows from [Ber90, §3] that $X_{\mathbb{C}((t)),r}$ is a compact locally connected and connected space. Note that it is however not a second countable space but is sequentially compact by [Po12].

The next result summarizes the main properties of the map π_{hyb} . Together with Proposition 1.1 it also completes the proof of Theorem A from the introduction.

Theorem 1.2.

- (1) *The natural map $\pi_{\text{hyb}} : X_{\text{hyb},r} \rightarrow \mathcal{C}_{\text{hyb}}(r)$ is continuous and proper;*
- (2) *there exists a natural homeomorphism $\psi_{\text{NA}} : X_{\mathbb{C}((t)),r} \rightarrow \pi_{\text{hyb}}^{-1}(\tau(0))$;*
- (3) *there exists a natural homeomorphism $\psi : \pi^{-1}(\mathbb{D}_r^*) \rightarrow \pi_{\text{hyb}}^{-1}(\tau(\mathbb{D}_r^*))$ such that $\pi_{\text{hyb}} \circ \psi = \tau \circ \pi$.*

In other words the hybrid space gives a way to *see* the complex manifold $X_t = \pi^{-1}(t)$ degenerating to the non-Archimedean analytic variety $X_{\mathbb{C}((t)),r}$.

Proof. For the purpose of the proof, we fix a finite open cover by affine open sets $X_{A_r} = \cup U_i$ with $U_i = \text{Spec } B_i$ where B_i are A_r -algebras of finite type. Recall the definition of $(U_i)_{\text{hyb},r}$ which is a natural subset of $X_{\text{hyb},r}$, and let $(U_i)_{\mathbb{C}((t)),r}$ be the analogous open subset of $X_{\mathbb{C}((t)),r}$ associated to U_i .

The continuity of π_{hyb} follows from the definition and the first statement is clear since $X_{\text{hyb},r}$ is compact.

For the proof of (2), consider a point $x \in \pi_{\text{hyb}}^{-1}(\tau(0))$. By definition this is a multiplicative semi-norm on some B_i whose restriction to A_r is equal to the t -adic norm $|\cdot|_r$. It follows that x naturally induces a semi-norm on the complete tensor product $B_i \hat{\otimes} \mathbb{C}((t))$ (with $\mathbb{C}((t))$ endowed with the norm $|\cdot|_r$). This semi-norm is still multiplicative since the inclusion $A_r \rightarrow \mathbb{C}((t))$ is dense. We get a continuous map from $(U_i)_{\text{hyb},r} \cap \pi_{\text{hyb}}^{-1}(\tau(0))$ to $(U_i)_{\mathbb{C}((t)),r}$. This map is clearly continuous, and its inverse is given by restricting a semi-norm on $B_i \hat{\otimes} \mathbb{C}((t))$ to B_i . These maps respect the patching defining $X_{\text{hyb},r}$ and induces a homeomorphism between $\pi_{\text{hyb}}^{-1}(\tau(0))$ and $X_{\mathbb{C}((t)),r}$ whose inverse is the map ψ_{NA} by definition.

To construct the map ψ in (3), recall that we realized \mathcal{X} as the locus $\{([z], t) \in \mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}, Q_j(t, z) = 0, j = 1, \dots, M\}$, so that we may suppose

$$(4) \quad U_i = \mathcal{X} \cap \{([z], t) \in \mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}, z_i \neq 0\}$$

and $B_i = A_r[w_1, \dots, w_N] / \langle Q_j(t, w_1, \dots, w_{i-1}, 1, w_{i+1}, \dots, w_N) \rangle_{j=1, \dots, M}$.

Pick any point $([z], t_0) \in \mathbb{P}_{\mathbb{C}}^N \times \bar{\mathbb{D}}_r^*$ such that $Q_j(t_0, z) = 0$ for all $j = 1, \dots, M$. Suppose that $z_i \neq 0$, i.e. $([z], t_0) \in U_i$. We may then consider the multiplicative semi-norm

$$f \in B_i \mapsto \left| f \left(t_0, \frac{z_1}{z_i}, \dots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \dots, \frac{z_M}{z_i} \right) \right|_{\infty}^{\frac{\log r}{\log |t_0|_{\infty}}},$$

which defines a point $\psi([z], t_0) \in (U_i)_{\text{hyb}, r}$. This map defines a natural continuous injective map from $\pi^{-1}(\bar{\mathbb{D}}_r^*)$ to $\pi_{\text{hyb}}^{-1}(\tau(\bar{\mathbb{D}}_r^*))$ such that $\pi_{\text{hyb}} \circ \psi = \tau \circ \pi$.

Conversely pick any point $x \in \pi_{\text{hyb}}^{-1}(\tau(\bar{\mathbb{D}}_r^*))$. It is a multiplicative semi-norm on some B_i whose restriction to A_r is equal to $\tau(t_0)$ for some $t_0 \in \bar{\mathbb{D}}_r^*$ (hence to $|c|_{\infty}^{\frac{\log r}{\log |t_0|_{\infty}}}$ for $c \in \mathbb{C}$). The semi-norm x induces a multiplicative norm on the quotient of B_i by the kernel $\mathfrak{P} = \{f \in B_i, |f(x)| = 0\}$. This quotient is isomorphic to \mathbb{C} by Gelfand-Mazur theorem. In particular, we infer the existence of a point $([z], t_0) \in X$ such that $|f(x)| = |f(\psi([z], t_0))|$ for all $f \in B_i$.

This proves ψ is surjective and concludes the proof. \square

Observe that any semi-norm on $\mathcal{C}_{\text{hyb}}(r)$ induces a norm on \mathbb{C} bounded by $|\cdot|_{\text{hyb}}$ which is therefore equal to $|\cdot|_{\infty}^{\epsilon}$ for some $\epsilon \in [0, 1]$. We shall denote by $\mathfrak{n} : X_{\text{hyb}} \rightarrow [0, 1]$ the composition of this map with π_{hyb} . It is a continuous surjective and proper map. When $x \in X$ and $\pi(x) = t \in \mathbb{D}$, then it follows from (3) that

$$\mathfrak{n}(\psi(x)) = \frac{\log r}{\log |t|^{-1}}.$$

2. MODEL FUNCTIONS

We use the same setup as in the previous section. Our aim is to construct natural continuous functions (called model functions) on the hybrid space $X_{\text{hyb}, r}$, and on the Berkovich analytic space $X_{\mathbb{C}((t)), r}$ that are of algebraic origin and form a dense subspace of the space of all continuous functions. These functions will play a key role in the next section to analyze degeneration of measures in the hybrid space.

2.1. Admissible data. Let \mathcal{X} and \mathcal{X}' be two snc models of X . One shall say that \mathcal{X}' dominates \mathcal{X} if there is a proper bimeromorphic morphism $p : \mathcal{X}' \rightarrow \mathcal{X}$ compatible with the natural inclusion maps $\iota : X \rightarrow \mathcal{X}$ and $\iota' : X \rightarrow \mathcal{X}'$, i.e. satisfying $\iota = p \circ \iota'$.

We say an analytic subvariety Z of an snc model \mathcal{X} is *horizontal* when Z equals the closure of $Z \cap X$ in \mathcal{X} . It is *vertical* when it is included in the central fiber \mathcal{X}_0 .

Definition 2.1. A singular admissible datum $\mathcal{F} = \{\mathcal{X}', d, D, \sigma_0, \dots, \sigma_l\}$ is a collection of elements of the following form:

- an snc model $p : \mathcal{X}' \rightarrow \mathcal{X}$ of X dominating \mathcal{X} ;
- a positive integer $d \in \mathbb{N}^*$;
- a vertical divisor D ;
- a finite set of holomorphic sections $\sigma_0, \dots, \sigma_l$ of the line bundle $p^*(\mathcal{L}^{\otimes d}) \otimes \mathcal{O}_{\mathcal{X}'}(D)$ defined in a neighborhood of $p^{-1}(\bar{\mathcal{X}}_r)$ whose common zero locus does not contain any irreducible component of \mathcal{X}'_0 .

When the set of sections has no common zeroes then we say that \mathcal{F} is regular.

For convenience, we shall call the integer d the *degree* of the admissible datum, and refer to D as the *vertical divisor* associated to \mathcal{F} .

There is an equivalent way of thinking about admissible data that we now explain. Recall that a (coherent) fractional ideal sheaf \mathfrak{A} in \mathcal{X} is a (coherent) $\mathcal{O}_{\mathcal{X}}$ -module such that locally $f \cdot \mathfrak{A} \subset \mathcal{O}_{\mathcal{X}}$ for some $f \in \mathcal{O}_{\mathcal{X}}$. We shall say that a fractional ideal sheaf is *vertical* when its co-support is vertical. A log-resolution of a vertical fractional ideal \mathfrak{A} is an scn model $p : \mathcal{X}' \rightarrow \mathcal{X}$ such that $\mathfrak{A} \cdot \mathcal{O}_{\mathcal{X}'}$ is isomorphic to $\mathcal{O}_{\mathcal{X}'}(D)$ for some vertical divisor D . Any scn model \mathcal{X}' is dominated by some log-resolution of \mathfrak{A} by the theorem of Hironaka on the resolution of complex analytic spaces.

We denote by $\text{div}(\sigma)$ the divisor of poles and zeroes of a meromorphic section σ of $\mathcal{L}^{\otimes d}$. Given any finite set of meromorphic sections σ_i of $\mathcal{L}^{\otimes d}$, we also let $\langle \sigma_i \rangle$ be the fractional ideal sheaf locally generated by the meromorphic functions given by σ_i in a trivialization chart of $\mathcal{L}^{\otimes d}$.

Proposition 2.2. *An admissible datum is completely determined by:*

- (1) a positive integer $d \in \mathbb{N}^*$;
- (2) a fractional ideal sheaf \mathfrak{A} in \mathcal{X} such that $t^N \mathfrak{A} \subset \mathcal{O}_{\mathcal{X}}$ for some integer N ;
- (3) a finite set of meromorphic sections τ_0, \dots, τ_l of $\mathcal{L}^{\otimes d}$ defined in a neighborhood of $p^{-1}(\mathcal{X}_r)$ such that $\langle \tau_0, \dots, \tau_l \rangle = \mathfrak{A}$;
- (4) an scn model $p : \mathcal{X}' \rightarrow \mathcal{X}$ of \mathcal{X} dominating \mathcal{X} .

A datum is regular iff its associated fractional ideal sheaf is vertical, and $p : \mathcal{X}' \rightarrow \mathcal{X}$ is a log-resolution of \mathfrak{A} .

Proof. Take any admissible datum $\mathcal{F} = \{\mathcal{X}', d, D, \sigma_0, \dots, \sigma_l\}$, and let σ_{-D} be the canonical meromorphic section of $\mathcal{O}_{\mathcal{X}'}(-D)$ with $\text{div}(\sigma_{-D}) = -D$. A holomorphic section σ of $p^* \mathcal{L}^{\otimes d} \otimes \mathcal{O}_{\mathcal{X}'}(D)$ gives rise to a meromorphic section $\tau' = \sigma \sigma_{-D}$ of $p^* \mathcal{L}^{\otimes d}$ which is the lift by p of a meromorphic section τ of $\mathcal{L}^{\otimes d}$ that is holomorphic off \mathcal{X}_0 .

Let τ_0, \dots, τ_l (resp τ'_0, \dots, τ'_l) be the meromorphic sections of $\mathcal{L}^{\otimes d}$ (resp. of $p^* \mathcal{L}^{\otimes d}$) associated to $\sigma_0, \dots, \sigma_l$ as above, and set $\mathfrak{A} = \langle \tau_i \rangle$. We have

$$\langle \tau_i \rangle \cdot \mathcal{O}_{\mathcal{X}'} = \langle \tau'_i \rangle = \langle \sigma_i \rangle \cdot \mathcal{O}_{\mathcal{X}'}(-D).$$

Since D is a vertical divisor, there exists an integer N such that $t^N \mathcal{O}_{\mathcal{X}'}(-D) \subset \mathcal{O}_{\mathcal{X}'}$ which implies $t^N \langle \tau_i \rangle \cdot \mathcal{O}_{\mathcal{X}'} \subset \mathcal{O}_{\mathcal{X}'}$. Since any coherent ideal sheaf defined on the complement of a subvariety of codimension at least 2 extends to a coherent ideal sheaf of the ambient variety, we get $t^N \mathfrak{A} \subset \mathcal{O}_{\mathcal{X}}$.

When \mathcal{F} is regular, observe that $\langle \tau_i \rangle \cdot \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(-D)$ implies \mathfrak{A} to be vertical, and p to be a log-resolution of \mathfrak{A} .

Conversely let τ_0, \dots, τ_l be meromorphic sections of $\mathcal{L}^{\otimes d}$ such that $t^N \mathfrak{A} \subset \mathcal{O}_{\mathcal{X}}$ where $\mathfrak{A} = \langle \tau_i \rangle$ and $N \in \mathbb{N}$, and pick any scn model $p : \mathcal{X}' \rightarrow \mathcal{X}$. Introduce the vertical divisor D whose order of vanishing along an irreducible component E of the central fiber is equal to $\text{ord}_E(\mathfrak{A}) = -\min\{\text{ord}_E(f), f \in \mathfrak{A}(U)\}$ where U is an affine chart intersecting E . We conclude as before that $\mathfrak{A} \cdot \mathcal{O}_{\mathcal{X}'}(D)$ is a coherent ideal sheaf whose co-support does not contain any irreducible component of \mathcal{X}_0 . Any meromorphic section τ_i lifts to a meromorphic section of $p^* \mathcal{L}^{\otimes d}$ whose divisor of poles and zeroes is greater or equal to $\text{div}(\mathfrak{A} \cdot \mathcal{O}_{\mathcal{X}'}) = -D$. In other words, the lift of τ_i to \mathcal{X}' is a meromorphic section τ'_i of $p^* \mathcal{L}^{\otimes d}$ with $\text{div}(\tau'_i) \geq -D$. Since $\langle \tau_i \rangle = \mathfrak{A}$, then $\langle \tau'_i \rangle \cdot \mathcal{O}_{\mathcal{X}'}(D)$ is a coherent ideal sheaf having horizontal co-support. Let σ_D be the canonical meromorphic section of $\mathcal{O}_{\mathcal{X}'}(D)$ with

$\operatorname{div}(\sigma_D) = D$, and define $\sigma_i = \tau'_i \sigma_D$: these are holomorphic sections of $p^* \mathcal{L}^{\otimes d} \otimes \mathcal{O}_{\mathcal{X}'}(D)$ whose common zero locus does not contain any irreducible vertical component.

When \mathfrak{A} is vertical, and $p : \mathcal{X}' \rightarrow \mathcal{X}$ is a log-resolution of \mathfrak{A} , then $\mathfrak{A} \cdot \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'}(-D)$. It follows that $\langle \tau'_i \rangle \cdot \mathcal{O}_{\mathcal{X}'}(D) = \mathcal{O}_{\mathcal{X}'}$ hence the sections σ_i have no common zeroes. \square

Notation. Given any singular admissible datum \mathcal{F} , we let $\mathfrak{A}_{\mathcal{F}}$ be its associated vertical fractional ideal by the previous proposition.

2.2. Model functions on degenerations. From now on, we fix a smooth positively curved reference metric $|\cdot|_{\star}$ on \mathcal{L} . One can for instance proceed as follows. Recall that \mathcal{X} is embedded into $\mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}$, and \mathcal{L} is the restriction of the pull-back of $\mathcal{O}(1)_{\mathbb{P}_{\mathbb{C}}^N}$ by the first projection. Endow the ample line bundle $\mathcal{O}(1)_{\mathbb{P}_{\mathbb{C}}^N}$ with a metric (unique up to a scalar factor) whose curvature form is the Fubini-Study $(1, 1)$ form on $\mathbb{P}_{\mathbb{C}}^N$. Pull-back this metric to the product space $\mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}$ and restrict it to \mathcal{X} . Recall that $\bar{X}_r = \pi^{-1}(\bar{\mathbb{D}}_r^*) \subset X$.

Let $\mathcal{F} = \{\mathcal{X}', d, D, \sigma_0, \dots, \sigma_l\}$ be any singular admissible datum. Recall from the previous section that we associated to it meromorphic sections τ_0, \dots, τ_l of $\mathcal{L}^{\otimes d}$ that are holomorphic in restriction to X_r . We may thus define a function $\varphi_{\mathcal{F}} : \bar{X}_r \rightarrow \mathbb{R} \cup \{-\infty\}$ given by

$$\varphi_{\mathcal{F}}(x) := \log \max\{|\tau_0|_{\star}, \dots, |\tau_l|_{\star}\}.$$

This is a continuous function on X_r with values in $\mathbb{R} \cup \{-\infty\}$.

Definition 2.3. A model function on X is a function of the form $\varphi_{\mathcal{F}}$ associated to a regular admissible datum \mathcal{F} as above. This is a continuous real-valued function $\varphi_{\mathcal{F}} : \bar{X}_r \rightarrow \mathbb{R}$.

Let \mathcal{F} be a (singular) admissible datum. Let \mathcal{X}'' be any snc model dominating \mathcal{X}' so that the natural map $q : \mathcal{X}'' \rightarrow \mathcal{X}'$ is regular. One may define a (singular) admissible datum $\mathcal{F}_{\mathcal{X}''}$ by choosing the line bundle $\tilde{\mathcal{L}} := q^* \hat{\mathcal{L}}$ on \mathcal{X}'' , and considering the lift of the sections σ_i of $\hat{\mathcal{L}}$ to $\tilde{\mathcal{L}}$. This new admissible datum has the same degree as \mathcal{F} , admits $q^* D$ as its vertical divisor, and we have $\varphi_{\mathcal{F}_{\mathcal{X}''}} = \varphi_{\mathcal{F}}$.

Remark. A function of the form $\varphi_{\mathcal{F}}$ (e.g. any model functions) is completely determined by the data (1)–(3) of Proposition 2.2. The snc model \mathcal{X}' (or equivalently the log-resolution of the fractional ideal) is included in the definition of an admissible datum only for convenience since we shall work in such resolutions most of the time.

Theorem 2.4. Denote by ω the curvature of the metrization induced by $|\cdot|_{\star}$ on \mathcal{L} . It is a positive closed smooth $(1, 1)$ form on \mathcal{X} such that for any singular admissible datum $\mathcal{F} = \{\mathcal{X}', d, D, \sigma_0, \dots, \sigma_l\}$, the following properties hold.

- In any local coordinates (w_0, \dots, w_k) such that the vertical divisor D of \mathcal{F} is defined by the equation $\{\prod_{i=0}^k w_i^{d_i} = 0\}$ with $d_i \in \mathbb{N}$, then we may write

$$(5) \quad \varphi_{\mathcal{F}} = \sum_{0 \leq i \leq k} d_i \log |w_i| + v$$

where v is the sum of a smooth function and a psh function with analytic singularities, so that $\varphi_{\mathcal{F}}$ extends as an L_{loc}^1 -function in a neighborhood of the central fiber in \mathcal{X}' .

- We have the equality of positive closed $(1, 1)$ -currents in \mathcal{X}'_r :

$$(6) \quad dd^c \varphi_{\mathcal{F}} + dp^* \omega = \Omega_{\mathcal{F}} + [D]$$

where $\Omega_{\mathcal{F}}$ is a positive closed $(1, 1)$ -current with analytic singularities.

- When \mathcal{F} is regular, then v is continuous, and $\Omega_{\mathcal{F}}$ admits Lipschitz continuous potentials.

Terminology. We say that a positive closed $(1, 1)$ current on a complex manifold has a continuous (resp. Lipschitz continuous) potential when it can be written locally near any point as the dd^c of a continuous (resp. Lipschitz continuous) psh function u . A psh function having analytic singularities is a psh function u such that one can find holomorphic maps h_0, \dots, h_l and $c > 0$ for which $u - c \log \max\{|h_0|, \dots, |h_l|\}$ is bounded. Observe that any such function is continuous with values in $[-\infty, +\infty)$.

Proof. Let \mathcal{F} be any singular admissible datum.

Choose first any point x outside the central fiber, local coordinates (w_0, w_1, \dots, w_k) near x , and a local trivialization of \mathcal{L} in that chart. In this trivialization a section σ of $p^*(\mathcal{L}^{\otimes d}) \otimes \mathcal{O}_{\mathcal{X}'}(D)$ can be identified to a holomorphic function in the variables w , and $|\sigma|_{\star} = |\sigma(w)|_{\infty} e^{-u}$ where u is a smooth psh function. It follows that

$$\varphi_{\mathcal{F}} = \log \max\{|h_0|, \dots, |h_l|\} - u ,$$

where h_i are holomorphic functions. Recall that $dd^c u$ is the curvature form of the metric $|\cdot|_{\star}$ on $\mathcal{L}^{\otimes d}$ hence is equal to $d\omega$, and $\log \max\{|h_0|, \dots, |h_l|\}$ is a psh function with analytic singularities, so that (5) and (6) hold near x .

Now choose a point $x \in \mathcal{X}'_0$, and choose local coordinates (w_0, w_1, \dots, w_k) such that the central fiber \mathcal{X}'_0 is included in $\prod_{i=0}^k w_i = 0$. More precisely introduce the integers $a_i \in \mathbb{N}$, $d_i \in \mathbb{Z}$ such that we have the equality of divisors $\pi^*[0] = \sum_i a_i [w_i = 0]$; and $D = \sum_i d_i [w_i = 0]$.

Choose a local trivialization of $p^*(\mathcal{L}^{\otimes d})$. In this trivialization a section σ of $p^*(\mathcal{L}^{\otimes d})$ is a holomorphic function in the variables w and its norm can be written as $|\sigma|_{\star} = |\sigma(w)|_{\infty} e^{-u}$ with u psh and smooth.

A section σ of $p^*(\mathcal{L}^{\otimes d}) \otimes \mathcal{O}_{\mathcal{X}'}(D)$ can then be identified to a meromorphic function in the variables w whose divisor of poles and zeroes $\text{div}(\sigma)$ satisfies $\text{div}(\sigma) \geq D$. In other words one can write $\sigma = \prod_i w_i^{d_i} \times h$ where h is holomorphic. Since h is a local section of $p^*(\mathcal{L}^{\otimes d})$, it follows that

$$\log |\sigma|_{\star} = \sum_i d_i \log |w_i| + \log |h| - u ,$$

so that we may write as above

$$\varphi_{\mathcal{F}} = \sum_i d_i \log |w_i| + \log \max\{|h_0|, \dots, |h_l|\} - u ,$$

where u is a smooth psh function. Equations (5) and (6) follow as before.

When \mathcal{F} is regular, then the holomorphic functions h_0, \dots, h_l have no common zeroes, hence the function $\log \max\{|h_0|, \dots, |h_l|\}$ is Lipschitz continuous. \square

Theorem 2.5. *The space of model functions on X_r is stable by sum, and by addition by any real number. Moreover if \mathcal{F} and \mathcal{F}' are admissible data of degree d and d' respectively, then $\frac{\max\{d' \varphi_{\mathcal{F}}, d \varphi_{\mathcal{F}'}\}}{\text{gcd}(d, d')}$ is also a model function.*

Proof. Let $\mathcal{F} = \{\mathcal{X}', d, D, \sigma_0, \dots, \sigma_l\}$ be any regular admissible datum.

Multiplying each section by a constant $\lambda \in \mathbb{C}^*$ modifies the model function by adding $\log |\lambda|$ to $\varphi_{\mathcal{F}}$ which proves the stability by addition by a real number.

Now pick another regular admissible datum \mathcal{F}' . By the previous observation, and replacing \mathcal{X}' by a suitable snc model dominating it we may suppose that both \mathcal{F} and \mathcal{F}' are defined over the same snc model \mathcal{X}' . Let $\hat{\mathcal{L}}' := p^* \mathcal{L}^{\otimes d'} \otimes \mathcal{O}_{\mathcal{X}'}(D')$ be the line bundle and σ'_j the sections of $\hat{\mathcal{L}}'$ associated to \mathcal{F}' .

One can then build a natural regular admissible datum $\mathcal{F} \otimes \mathcal{F}'$ associated to $\hat{\mathcal{L}} \otimes \hat{\mathcal{L}}'$, and to the sections $\sigma_i \otimes \sigma'_j$. This new admissible datum has degree $d + d'$ and vertical divisor $D + D'$. Moreover we have

$$\varphi_{\mathcal{F} \otimes \mathcal{F}'} = \varphi_{\mathcal{F}} + \varphi_{\mathcal{F}'}$$

which implies the stability by sum of model functions.

To see the stability under taking maxima, it is easier to view the regular admissible data \mathcal{F} and \mathcal{F}' in \mathcal{X} given by their degrees $d, d' \in \mathbb{N}^*$, vertical fractional ideals $\mathfrak{A}, \mathfrak{A}'$ and meromorphic sections τ_i and τ'_j of $\mathcal{L}^{\otimes d}$ and $\mathcal{L}^{\otimes d'}$ respectively. The log-resolutions of \mathfrak{A} and \mathfrak{A}' will not play any role in the next argument.

Introduce the integer $\delta = dd' / \gcd(d, d')$, and we consider the set of meromorphic sections $\{\sigma_i^{\delta/d}\} \cup \{(\sigma'_j)^{\delta/d'}\}$ of $\mathcal{L}^{\otimes \delta}$. The fractional ideal sheaf $\langle \sigma_i^{\delta/d}, (\sigma'_j)^{\delta/d'} \rangle$ is then equal to $\mathfrak{A} + \mathfrak{A}'$ which is vertical. We may thus build an admissible datum \mathcal{F}'' by choosing a log-resolution of $\mathfrak{A} + \mathfrak{A}'$, and the associated model function is given by $\varphi_{\mathcal{F}''} = \max\{\frac{\delta}{d}\varphi_{\mathcal{F}}, \frac{\delta}{d'}\varphi_{\mathcal{F}'}\}$ as required. \square

2.3. Model functions on non-Archimedean analytic spaces. We now explain how an admissible datum \mathcal{F} also induces a natural continuous function on the Berkovich analytic space $X_{\mathbb{C}((t)),r}$ following the discussion of [BFJ16].

Let $p : \mathcal{X}' \rightarrow \mathcal{X}$ be any snc model of X dominating \mathcal{X} . To any irreducible component E of the central fiber \mathcal{X}'_0 we may attach a point $x_E \in X_{\mathbb{C}((t)),r}$ corresponding to the generic point on the projective $\mathbb{C}((t))$ -variety $X_{\mathbb{C}((t))}$ and a norm on its field of rational functions in the following way. Pick any rational function f on $X_{\mathbb{C}((t))}$. It defines a rational function on the $\text{Spec } \mathbb{C}[[t]]$ -scheme obtained by base change $\mathcal{X}'_{\mathbb{C}[[t]]}$ whose generic fiber is isomorphic to $X_{\mathbb{C}((t))}$. We then set $|f(x_E)| = r^{\frac{\text{ord}_E(f)}{b_E}}$ where $\text{ord}_E(f)$ is the order of vanishing of f at the generic point of E and $b_E = \text{ord}_E(t) \in \mathbb{N}^*$.

Any such point x_E is called a divisorial point. It is possible to show that the set of divisorial points is dense in $X_{\mathbb{C}((t)),r}$, see e.g. [BFJ16, Corollary 2.4].

To any fractional ideal sheaf \mathfrak{A} defined in a neighborhood of $\bar{\mathcal{X}}_r$, we can attach a function $\log |\mathfrak{A}| : X_{\mathbb{C}((t)),r} \rightarrow \mathbb{R} \cup \{-\infty\}$. When \mathfrak{A} is a vertical ideal sheaf of the $\text{Spec } \mathbb{C}[[t]]$ -scheme $\mathcal{X}_{\mathbb{C}[[t]]}$ this was done e.g. in [BFJ16, § 2.5]. Since we work here with coherent sheaves in the analytic category, we explain this construction in some details using explicit coordinates.

Recall that we realized \mathcal{X} as the locus $\bigcap_{j=1}^M \{([z], t) \in \mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}, Q_j(t, z) = 0\}$, where Q_j are homogeneous polynomials in z with coefficients that are holomorphic functions on $t \in \mathbb{D}$. Recall the definition of the open sets

$$U_i = \mathcal{X} \cap \{([z], t) \in \mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}, z_i \neq 0\}, \quad i = 0, \dots, N.$$

Lemma 2.6. *For any i , one can find finitely many meromorphic functions $f^{(i)}, g_1^{(i)}, \dots, g_l^{(i)}$ defined in a neighborhood of $\bar{\mathcal{X}}_r$ and holomorphic on U_i such that $f^{(i)} \cdot \mathfrak{A}(U_i)$ is an ideal of the ring of holomorphic functions on U_i that is generated by $g_1^{(i)}, \dots, g_l^{(i)}$.*

Proof. Let H_i be the hyperplane section $\{z_i = 0\}$ in \mathcal{X} . Recall that a section of the line bundle $\mathcal{O}(dH_i)$ in a neighborhood of $\bar{\mathcal{X}}_r$ for some d defines a meromorphic function in a neighborhood of $\bar{\mathcal{X}}_r$ which is holomorphic on U_i .

It follows from a theorem of Grauert and Remmert, see [GR58] or [No59] that for a sufficiently large integer d the sheaf $\mathcal{O}(dH_i) \otimes \mathfrak{A}$ is globally generated over a neighborhood of $\bar{\mathcal{X}}_r$. This implies our claim. \square

Recall that the collection of sets $\{(U_i)_{\text{hyb},r}\}_{i=0,\dots,N}$ which consists of all multiplicative semi-norms on the A_r -algebra

$$B_i = A_r[w_1, \dots, w_N] / \langle Q_j(t, w_1, \dots, w_{i-1}, 1, w_{i+1}, \dots, w_N) \rangle_{j=1, \dots, M}$$

forms an open cover of $X_{\mathbb{C}((t)),r}$. For any $x \in (U_i)_{\text{hyb},r}$ we may thus set

$$\log |\mathfrak{A}|(x) := \inf \{ \log |g_j^{(i)}(x)|, j = 1, \dots, l \} - \log |f^{(i)}(x)|.$$

It is easy to check that this definition does not depend on the choice of generators, so that $\log |\mathfrak{A}|$ actually defines a continuous function on $X_{\mathbb{C}((t)),r}$ with values in $[-\infty, +\infty)$. When \mathfrak{A} is vertical, then the function $\log |\mathfrak{A}|$ is a real-valued continuous function.

For any admissible datum \mathcal{F} with associated fractional ideal sheaf $\mathfrak{A}_{\mathcal{F}}$, we set $g_{\mathcal{F}} := \log |\mathfrak{A}_{\mathcal{F}}|$. This defines a continuous function $g_{\mathcal{F}} : X_{\mathbb{C}((t)),r} \rightarrow [-\infty, +\infty)$ (with values in \mathbb{R} when \mathcal{F} is regular).

Lemma 2.7. *For any admissible datum \mathcal{F} , any snc model $p : \mathcal{X}' \rightarrow \mathcal{X}$ and for any component E of the central fiber, we have*

$$g_{\mathcal{F}}(x_E) = \log r \frac{\text{ord}_E(D)}{b_E}$$

where D is the unique vertical divisor such that $\mathfrak{A}_{\mathcal{F}} \cdot \mathcal{O}_{\mathcal{X}'}(D)$ is an ideal subsheaf of $\mathcal{O}_{\mathcal{X}'}$ whose co-support does not contain any vertical component.

Proof. We may suppose $\mathcal{X}' = \mathcal{X}$ and pick a generic point on E which is not included in the co-support of the ideal sheaf $\mathfrak{A}_{\mathcal{F}} \cdot \mathcal{O}_{\mathcal{X}'}(D)$. In a local analytic chart $w = (w_0, \dots, w_k)$ near that point we can write $E = \{w_0 = 0\}$. The ideal \mathfrak{A} is generated by w_0^l for some $l \in \mathbb{Z}$ so that D is the divisor associated to w_0^l too (a section of $\mathcal{O}_{\mathcal{X}'}(D)$ is a meromorphic function whose divisor of poles and zeroes is $\geq -D = -l[w_0 = 0]$).

By definition of x_E we have $\log |\mathfrak{A}|(x_E) = \frac{l}{b_E} \log r$ which implies our claim. \square

Definition 2.8. *A model function on the Berkovich analytic space $X_{\mathbb{C}((t)),r}$ is a function of the form $g_{\mathcal{F}} : X_{\mathbb{C}((t)),r} \rightarrow \mathbb{R}$ for some regular admissible datum \mathcal{F} .*

In [BFJ15], model functions are defined as the difference of two model functions in the sense of our paper. The notion of model functions appears at several places in the literature under various names, see [BFJ15, Table 1].

Proposition 2.9. *Any continuous function on $X_{\mathbb{C}((t)),r}$ is the uniform limit of a sequence of functions of the form $\lambda(g_{\mathcal{F}} - g_{\mathcal{F}'})$ where \mathcal{F} and \mathcal{F}' are admissible data of the same degree, and $\lambda \in \mathbb{Q}_+^*$.*

Proof. Let us introduce the following three spaces of continuous functions on $X_{\mathbb{C}((t)),r}$:

- (1) $\mathcal{F}_1 = \{\lambda(g_{\mathcal{F}} - g_{\mathcal{F}'})\}$ where \mathcal{F} and \mathcal{F}' are admissible data of the same degree, and $\lambda \in \mathbb{Q}_+^*$;
- (2) $\mathcal{F}_2 = \{\lambda(\log |\mathfrak{A}| - \log |\mathfrak{B}|)\}$ where $\mathfrak{A}, \mathfrak{B}$ are two vertical fractional ideal sheaves defined in a neighborhood of $\bar{\mathcal{X}}_r$, and $\lambda \in \mathbb{Q}_+^*$;

- (3) $\mathcal{F}_3 = \{\lambda(\log |\hat{\mathfrak{A}}| - \log |\hat{\mathfrak{B}}|)\}$ where $\hat{\mathfrak{A}}, \hat{\mathfrak{B}}$ are two vertical fractional ideal sheaves of $\mathcal{X}_{\mathbb{C}[[t]]}$, and $\lambda \in \mathbb{Q}_+^*$.

By [BFJ16, Proposition 2.2] we know that \mathcal{F}_3 is dense in $\mathcal{C}^0(X_{\mathbb{C}((t)),r})$. On the other hand Grauert and Remmert's theorem implies that for any fractional ideal sheaf \mathfrak{A} there exist an integer $d \in \mathbb{N}^*$ and sections τ_0, \dots, τ_l of $\mathcal{L}^{\otimes d}$ such that $\langle \tau_i \rangle = \mathfrak{A}$ over a neighborhood of $\bar{\mathcal{X}}_r$. In particular we have $\mathcal{F}_1 = \mathcal{F}_2$.

To conclude it is therefore sufficient to check that for any vertical fractional ideal sheaf $\hat{\mathfrak{A}}$ of $\mathcal{X}_{\mathbb{C}[[t]]}$ there exists a vertical fractional (analytic) sheaf \mathfrak{A} defined in a neighborhood of $\bar{\mathcal{X}}_r$ such that $\log |\hat{\mathfrak{A}}| = \log |\mathfrak{A}|$ on $X_{\mathbb{C}((t)),r}$.

Replacing $\hat{\mathfrak{A}}$ by $t^N \cdot \hat{\mathfrak{A}}$ if necessary we may suppose that $\hat{\mathfrak{A}}$ is an coherent sheaf of ideals of $\mathcal{O}_{\mathcal{X}_{\mathbb{C}[[t]]}}$. Since the ideal sheaf is vertical, there exists an integer l sufficiently large such that $t^l \in \hat{\mathfrak{A}}$. Recall the definition of the open cover as in the proof of Lemma 2.6. It follows that on

$$(U_i)_{\mathbb{C}[[t]]} = \text{Spec } \mathbb{C}[[t]][w_1, \dots, w_N] / \langle Q_j(t, w_1, \dots, w_{i-1}, 1, w_{i+1}, \dots, w_N) \rangle_{j=1, \dots, M}$$

$\hat{\mathfrak{A}}$ is actually generated by elements of the ring

$$\mathcal{O}(\mathbb{D})[w_1, \dots, w_N] / \langle Q_j(t, w_1, \dots, w_{i-1}, 1, w_{i+1}, \dots, w_N) \rangle_{j=1, \dots, M}$$

hence by meromorphic functions on \mathcal{X} that are holomorphic in U_i . It thus defines a vertical ideal sheaf \mathfrak{A} whose values at any point in $(U_i)_{\text{hyb},r}$ coincides with the ones of $\hat{\mathfrak{A}}$.

This concludes the proof. \square

2.4. Model functions on the hybrid space. Recall that any point $x \in X_{\text{hyb},r}$ induces a norm of the field of complex numbers equal to $|\cdot|_{\infty}^{\mathfrak{n}(x)}$ for some $\mathfrak{n}(x) \in [0, 1]$, and we have

$$\mathfrak{n}(\psi(x)) = \frac{\log r}{\log |\pi(x)|^{-1}}$$

for any $x \in X$.

Theorem 2.10. *For any singular admissible datum \mathcal{F} , the function $\Phi_{\mathcal{F}}$ given by*

$$\mathfrak{n} \cdot \varphi_{\mathcal{F}} \circ \psi^{-1} \text{ on } \pi_{\text{hyb}}^{-1}(\tau(\bar{\mathbb{D}}_r^*)), \text{ and by } g_{\mathcal{F}} \circ \psi_{\text{NA}}^{-1} \text{ on } \pi_{\text{hyb}}^{-1}(\tau(0))$$

is continuous on $X_{\text{hyb},r}$ with values in $\mathbb{R} \cup \{-\infty\}$.

Observe that when \mathcal{F} is regular, then the previous result claims that $\Phi_{\mathcal{F}}$ is a real-valued continuous function.

Definition 2.11. *A model function on the hybrid space $X_{\text{hyb},r}$ is a continuous function of the form $\Phi_{\mathcal{F}}$ for some regular admissible datum \mathcal{F} .*

Proof. Let \mathcal{F} be an admissible datum and let \mathfrak{A} be its associated fractional ideal sheaf. The continuity of $\Phi_{\mathcal{F}}$ in restriction to $\pi_{\text{hyb}}^{-1}(\tau(\bar{\mathbb{D}}_r^*))$ (resp. to $\pi_{\text{hyb}}^{-1}(\tau(0))$) follows from the continuity of $\varphi_{\mathcal{F}}$ (resp of $g_{\mathcal{F}}$).

Since $\pi_{\text{hyb}}^{-1}(\tau(0))$ is a closed subset of $X_{\text{hyb},r}$ it is sufficient to prove the following. For any net of points x_n in $\pi_{\text{hyb}}^{-1}(\tau(\bar{\mathbb{D}}_r^*))$ indexed by a set \mathcal{N} and converging to a point $x \in \pi_{\text{hyb}}^{-1}(\tau(0))$, then we have $\Phi_{\mathcal{F}}(x_n) \rightarrow \Phi_{\mathcal{F}}(x)$.

Pick any snc model $p : \mathcal{X}' \rightarrow \mathcal{X}$ obtained from \mathcal{X} by a sequence of blow-ups of smooth centers and write $\mathfrak{A} = \mathfrak{B} \cdot \mathcal{O}_{\mathcal{X}'}(-D)$ where D is a vertical divisor, and \mathfrak{B} is an ideal sheaf whose co-support does not contain any vertical components. Observe that there exists

a relatively ample line bundle $\mathcal{L}' \rightarrow \mathcal{X}'$ so that $(\mathcal{L}')^{\otimes N} \otimes \mathcal{O}_{\mathcal{X}'}(D)$, and $\mathfrak{B} \otimes (\mathcal{L}')^{\otimes N}$ are globally generated for a sufficiently large integer N over \mathbb{D}_r .

We may thus find a finite family of meromorphic functions $w_i^{(j)}, h_\alpha^{(j)}$ on $p^{-1}(\pi^{-1}(\mathbb{D}_r))$ such that $(w_0^{(j)}, \dots, w_k^{(j)})$ form a family of charts $U^{(j)}$ covering \mathcal{X}'_0 ; $h_0^{(j)}, \dots, h_l^{(j)}$ are holomorphic in $U^{(j)}$ and generate $\mathfrak{B}(U^{(j)})$. In each chart, we get

$$(7) \quad \varphi_{\mathcal{F}} = \sum_i d_i^{(j)} \log |w_i^{(j)}| + \log \max_{\alpha} \{|h_{\alpha}^{(j)}|\} + \varphi^{(j)}$$

where $\varphi^{(j)}$ is continuous, and D is defined by the equation $\left\{ \prod_{i=1}^k (w_i^{(j)})^{d_i^{(j)}} = 0 \right\}$.

For each j let $\mathcal{N}^{(j)}$ be the subset of \mathcal{N} of those indices such that x_n belongs to the j -th chart. Write $x_n = (w_{0,n}^{(j)}, \dots, w_{k,n}^{(j)})$ when $n \in \mathcal{N}^{(j)}$. The convergence $x_n \rightarrow x$ then implies $\mathfrak{n}(x_n) \times \log |h| \rightarrow \log |h(x)|$ when n tends to infinity in $\mathcal{N}^{(j)}$, and for all meromorphic function h on $p^{-1}(\pi^{-1}(\mathbb{D}_r))$ that is holomorphic in $U^{(j)}$. We thus get

$$\begin{aligned} \Phi_{\mathcal{F}}(x_n) &= \mathfrak{n}(x_n) \times \varphi_{\mathcal{F}} \circ \psi^{-1}(x_n) \\ &= \mathfrak{n}(x_n) \times \left(\sum_i d_i^{(j)} \log |w_{i,n}^{(j)}| + \log \max_{\alpha} \{|h_{\alpha}^{(j)}(x_n)|\} + \varphi^{(j)}(x_n) \right) \end{aligned}$$

which implies

$$\lim_{n \rightarrow \infty} \Phi_{\mathcal{F}}(x_n) = \sum_i d_i^{(j)} \log |w_i^{(j)}(x)| + \log \max_{\alpha} \{|h_{\alpha}^{(j)}(x)|\} = \log |\mathfrak{A}|(x) = g_{\mathcal{F}}(x),$$

as required. \square

2.5. Density of model functions.

Theorem 2.12. *Let $\mathcal{D}(X_{\text{hyb},r})$ be the space of all functions of the form $q\Phi_{\mathcal{F}} - q'\Phi_{\mathcal{F}'}$ where $\mathcal{F}, \mathcal{F}'$ are regular admissible data and q, q' are positive rational numbers such that $q \deg(\mathcal{F}) = q' \deg(\mathcal{F}')$.*

Then $\mathcal{D}(X_{\text{hyb},r})$ is dense in the space of all continuous functions on $X_{\text{hyb},r}$ endowed with the topology of the uniform convergence.

Proof. We claim that $\mathcal{D}(X_{\text{hyb},r})$ is stable under taking maximum (hence by minimum since it is a \mathbb{Q} -vector space). Pick two functions in $q_1\Phi_{\mathcal{F}_1} - q'_1\Phi_{\mathcal{F}'_1}$ and $q_2\Phi_{\mathcal{F}_2} - q'_2\Phi_{\mathcal{F}'_2}$ in $\mathcal{D}(X_{\text{hyb},r})$ such that $q_1 \deg(\mathcal{F}_1) = q'_1 \deg(\mathcal{F}'_1)$, and $q_2 \deg(\mathcal{F}_2) = q'_2 \deg(\mathcal{F}'_2)$. One can first multiply both functions by a suitably large integer such that q_1, q'_1, q_2 and q'_2 are all integers. One then writes

$$\max\{q_1\Phi_{\mathcal{F}_1} - q'_1\Phi_{\mathcal{F}'_1}, q_2\Phi_{\mathcal{F}_2} - q'_2\Phi_{\mathcal{F}'_2}\} = \max\{q_1\Phi_{\mathcal{F}_1} + q'_2\Phi_{\mathcal{F}'_2}, q_2\Phi_{\mathcal{F}_2} + q'_1\Phi_{\mathcal{F}'_1}\} - q'_1\Phi_{\mathcal{F}'_1} - q'_2\Phi_{\mathcal{F}'_2}$$

and apply Theorem 2.5. This proves the claim.

We then conclude by applying Stone-Weierstrass theorem and the next lemma. \square

Lemma 2.13. *For any two points $x \neq x' \in X_{\text{hyb},r}$, any rational numbers ρ, ρ' , and any positive real number $\epsilon > 0$, there exists $\Phi \in \mathcal{D}(X_{\text{hyb},r})$ such that $|\Phi(x) - \rho| \leq \epsilon$ and $|\Phi(x') - \rho'| \leq \epsilon$.*

Proof. For $\lambda \in \mathbb{C}^*$ and $q \in \mathbb{Z}$ denote by $\lambda t^q \mathcal{F}$ the admissible datum obtained by multiplying all sections by λt^q over X_t . This does not change the degree of \mathcal{F} but its associated vertical

divisor is modified by adding q times the vertical divisor associated to \mathcal{F} . Observe that $\varphi_{\lambda t^q \mathcal{F}} = \varphi_{\mathcal{F}} + \log |\lambda t^q|$ on X , so that

$$\Phi_{\lambda t^q \mathcal{F}} - \Phi_{\mathcal{F}} = \mathfrak{n} (\varphi_{\lambda t^q \mathcal{F}} \circ \psi^{-1} - \varphi_{\mathcal{F}} \circ \psi^{-1}) = \log |\lambda| \mathfrak{n} - q \log r$$

belongs to $\mathcal{D}(X_{\text{hyb},r})$ for any $\lambda \in \mathbb{C}^*$ and for any $q \in \mathbb{Z}$ (and thus for any $q \in \mathbb{Q}$). This implies the lemma when $\pi_{\text{hyb}}(x) = \tau(0)$ and $\pi_{\text{hyb}}(x') \neq \tau(0)$ since $\mathfrak{n}(x) = 0$ and $\mathfrak{n}(x') \neq 0$ in this case.

In the case $\pi_{\text{hyb}}(x) = \pi_{\text{hyb}}(x') = 0$, i.e. both points x, x' belongs to $X_{\mathbb{C}((t)),r}$, then the lemma follows from Proposition 2.9.

To treat the case $x, x' \in \psi^{-1}(X)$, we first recall a few facts. We assumed that \mathcal{X} is embedded in $\mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}$, and \mathcal{L} is the restriction to \mathcal{X} of the pull-back by the first projection of $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^N}(1)$. Any section σ of $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^N}(d)$ is determined by a homogeneous polynomial $P_{\sigma}(z_0, \dots, z_N)$ of degree d in $(N+1)$ -variables with complex coefficients, and we have

$$|\sigma([z])|_{\star} = \frac{|P_{\sigma}(z_0, \dots, z_N)|}{(|z_0|^2 + \dots + |z_N|^2)^{d/2}},$$

for a point $[z] = [z_0 : \dots : z_N] \in \mathbb{P}_{\mathbb{C}}^N$. A meromorphic section σ of $\mathcal{L}^{\otimes d}$ is therefore given by a homogeneous polynomials $P_{\sigma}(z_0, \dots, z_N, t)$ of degree d in z_0, \dots, z_N with coefficients depending meromorphically on $t \in \mathbb{D}$, and we have

$$|\sigma(x)|_{\star} = \frac{|P_{\sigma}(z_0, \dots, z_N, t)|}{(|z_0|^2 + \dots + |z_N|^2)^{d/2}},$$

for any $x = ([z_0 : \dots : z_N], t) \in \mathcal{X} \subset \mathbb{P}_{\mathbb{C}}^N \times \mathbb{D}$.

Pick $\lambda_0, \dots, \lambda_N \in \mathbb{C}^*$ and integers $m_0, \dots, m_N \in \mathbb{Z}$. Then

$$([z_0 : \dots : z_N], t) \mapsto \log \left(\frac{\max\{|\lambda_0 t^{m_0}| |z_0|, \dots, |\lambda_N t^{m_N}| |z_N|\}}{(|z_0|^2 + \dots + |z_N|^2)^{1/2}} \right)$$

is a model function on X associated to a regular admissible datum of degree 1 (in the snc model \mathcal{X} , and with a non-zero vertical divisor that depends on the choices of the integers m_0, \dots, m_N). It follows that the function $\Phi : X_{\text{hyb},r} \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \Phi(\psi([z_0 : \dots : z_N], t)) := & \frac{\log r}{\log |t|^{-1}} (\log \max\{|\lambda_0 t^{m_0}| |z_0|, \dots, |\lambda_N t^{m_N}| |z_N|\} - \\ & \log \max\{|z_0|, \dots, |z_N|\}) , \end{aligned}$$

for all $([z_0 : \dots : z_N], t) \in X \subset \mathcal{X}$ belongs to $\mathcal{D}(X_{\text{hyb},r})$.

Now pick $x \neq x' \in X \subset \mathcal{X}$, and suppose first that $\pi(x) \neq \pi(x')$. Since the group of projective transformations preserving our chosen metrization on $\mathcal{O}(1)_{\mathbb{P}_{\mathbb{C}}^N}$ is transitive, we may assume that $x = ([1 : 0 : \dots : 0], t)$, and $x' = ([0 : 1 : \dots : 0], t')$ with $t' \neq t$. Evaluating the previous function at $\psi(x)$ and $\psi(x')$ respectively, we obtain

$$\Phi(\psi(x)) = -m_0 \log r \log |\lambda_0| \quad \text{and} \quad \Phi(\psi(x')) = -m_1 \log r \log |\lambda_1|$$

and the claim follows easily.

Finally suppose that $x \neq x' \in \mathcal{X}$ belongs to the same fiber X_t with $t \neq 0$. This time, we may assume that $x = ([1 : 0 : \dots : 0], t)$, and $x' = ([w_0 : 1 : \dots : w_N], t)$ so that

$\Phi(\psi(x)) = -m_0 \log r \log |\lambda_0|$ and

$$\Phi(\psi(x')) = \frac{\log r}{\log |t|^{-1}} (\log \max\{|\lambda_0 t^{m_0}| |w_0|, |\lambda_1 t^{m_1}|, \dots, |\lambda_N t^{m_N}| |w_N|\} - \log \max\{|w_0|, 1, \dots, |w_N|\}) .$$

By adjusting m_0 and λ_0 one can achieve at $\Phi(\psi(x))$ taking its values in a fixed open interval, and choosing m_1 negative enough and $\lambda_1 = 1$, $\lambda_i = 0$ for all $i \geq 2$, we can make $|\Phi(\psi(x'))|$ as large as we want. Multiplying Φ by a suitable (small) rational number we get an element $\Phi_1 \in \mathcal{D}(X_{\text{hyb},r})$ for which $\Phi_1(\psi(x))$ and $|\Phi_1(\psi(x')) - \rho'|$ are both as small as we want. In the same manner, we construct $\Phi_2 \in \mathcal{D}(X_{\text{hyb},r})$ for which $|\Phi_2(\psi(x'))| \ll 1$ and $|\Phi_2(\psi(x)) - \rho| \ll 1$, and we conclude the proof of the lemma by taking $\Phi_1 + \Phi_2$. \square

3. DEGENERATION OF MONGE-AMPÈRE MEASURES OF MODEL FUNCTIONS ON THE HYBRID SPACE

We now explain how a regular admissible datum \mathcal{F} gives rise in a natural way to a continuous family of positive measures $\mu_{t,\mathcal{F}}$ on the hybrid space $X_{\text{hyb},r}$. In §3.1, we explain how to associate a continuous family of positive measures to \mathcal{F} on a suitable scn model. In §3.2, we review briefly the definition of the Monge-Ampère operator in a non-Archimedean context following [BFJ15], and define a measure $\mu_{\mathcal{F},\text{NA}}$ on $X_{\mathbb{C}((t)),r}$. In §3.4 we prove the main result of this Section, namely Theorem 3.4 on the convergence of $\mu_{t,\mathcal{F}}$ towards $\mu_{\mathcal{F},\text{NA}}$ in the hybrid space.

3.1. Monge-Ampère measures associated to an admissible datum. We refer to the survey [De93] for the basic theory of intersection of positive closed currents on a complex manifold. Observe that we only need the very first steps of this theory and the definition of the Monge-Ampère measure of a *continuous* psh function, which is due to Bedford and Taylor [BT76].

Let $\mathcal{F} = \{\mathcal{X}', d, D, \sigma_0, \dots, \sigma_l\}$ be any regular admissible datum. Recall from Theorem 2.4 that one can find a positive closed $(1,1)$ -current $\Omega_{\mathcal{F}}$ with Lipschitz continuous potential on the scn model $p : \mathcal{X}' \rightarrow \mathcal{X}$, such that

$$\Omega_{\mathcal{F}} = dd^c \varphi_{\mathcal{F}} - [D] + d \cdot p^* \omega ,$$

where D is the vertical divisor associated to \mathcal{F} . Since $\Omega_{\mathcal{F}}$ has continuous potentials, its k -th power $\Omega_{\mathcal{F}}^{\wedge k}$ is a well-defined positive closed (k,k) -current on \mathcal{X}' . For any $t \in \mathbb{D}$, write $[X_t] = dd^c \log |\pi \circ p - t|$. When t is non-zero, then $[X_t]$ is the current of integration over the fiber $\pi^{-1}(t)$, and $[X_0] = \sum b_E [E]$ where E ranges over all irreducible components of \mathcal{X}'_0 and $b_E = \text{ord}_E(\pi^* t)$.

The positive measure $\mu_{t,\mathcal{F}} = (\Omega_{\mathcal{F}})^{\wedge k} \wedge [X_t]$ is well-defined for any $t \in \mathbb{D}$, and the family of measures $t \mapsto \mu_{t,\mathcal{F}}$ is continuous, see e.g. [De93, Corollary 1.6].

Observe that D being supported on \mathcal{X}'_0 , the measure $\mu_{t,\mathcal{F}}$ for $t \in \mathbb{D}^*$ can be obtained alternatively by restricting $\Omega_{\mathcal{F}}$ to the fiber X_t and consider its Monge-Ampère measure:

$$\mu_{t,\mathcal{F}} = (\Omega_{\mathcal{F}|_{X_t}})^{\wedge k} = (d \cdot \omega_t + dd^c g_{\mathcal{F}|_{X_t}})^{\wedge k}$$

where $\omega_t = \omega|_{X_t}$. The total mass of $\mu_{t,\mathcal{F}}$ is thus equal to $d^k \times \int_{X_t} \omega_t^k$ which can be computed purely in cohomological terms. Indeed the class determined by ω_t in the DeRham cohomology group of X_t is equal to the integral class $c_1(\mathcal{L}|_{X_t}) \in H_{dR}^2(X_t, \mathbb{Z})$, see [GH78, p.139]. It follows that

$$\text{Mass}(\mu_{t,\mathcal{F}}) = d^k c_1(\mathcal{L}|_{X_t})^{\wedge k} \in \mathbb{N}^* .$$

Since $\mu_{t,\mathcal{F}}$ varies continuously this mass is a constant.

The next computation is the key to understand the degeneration of $\mu_{t,\mathcal{F}}$ as $t \rightarrow 0$.

Proposition 3.1. *For any irreducible component E of the central fiber \mathcal{X}'_0 , one has*

$$(8) \quad c_1 \left(p^* \mathcal{L}^{\otimes d} \otimes \mathcal{O}_{\mathcal{X}'(D)}|_E \right)^{\wedge k} = \int_E \Omega_{\mathcal{F}}^k \geq 0 .$$

The left hand side is computed in the DeRham (or singular) cohomology as follows: one restricts the line bundle $p^* \mathcal{L}^{\otimes d} \otimes \mathcal{O}_{\mathcal{X}'(D)}$ to E , take its first Chern class, and consider the degree of its k -th power. The right hand side is computed analytically, as the total mass of the measure $(\Omega_{\mathcal{F}}|_{X_t})^{\wedge k}$ on E .

Proof. Consider the line bundle $\hat{\mathcal{L}} := p^* \mathcal{L}^{\otimes d} \otimes \mathcal{O}_{\mathcal{X}'(D)}$ on \mathcal{X}' . A local section σ of $\hat{\mathcal{L}}$ is the same as a local section of $p^* \mathcal{L}^{\otimes d}$ whose divisor of poles and zeroes satisfies $\text{div}(\sigma) \geq -D$. Endow $\hat{\mathcal{L}}$ with the metric $|\cdot|_{\mathcal{F}} := |\cdot|_{\star} e^{-\varphi_{\mathcal{F}}}$. Choose coordinates w in a trivializing chart such that D is given by the equation $\{\prod_i w_i^{d_i} = 0\}$. By $|\sigma|_{\mathcal{F}} = |\sigma(w)|_{\infty} e^{-u} e^{-g_{\mathcal{F}}}$ with u smooth and $dd^c u = dp^* \omega$. Since $\varphi_{\mathcal{F}} = \sum d_i \log |w_i| + v$ with v smooth we see that $w \mapsto |\sigma(w)|_{\mathcal{F}} = e^{v-u} |\sigma(w)|_{\infty} \prod_i |w_i|^{-d_i}$ is continuous. It follows that $|\cdot|_{\mathcal{F}}$ is a continuous metric on $\hat{\mathcal{L}}$ whose curvature form is equal to $\Omega_{\mathcal{F}}$ by (6). Therefore $c_1(\hat{\mathcal{L}}|_E)$ is represented by the positive closed $(1,1)$ -current $\Omega_{\mathcal{F}}|_E$ and the formula follows from [De93, Corollary 9.3]. \square

3.2. The Monge-Ampère measure on $X_{\mathbb{C}((t)),r}$. We briefly review the definition of the Monge-Ampère operator following A. Chambert-loir [CL06, CL11]. The theory has been expanded and made more precise in [BFJ15, BFJ16-2], [GM16], and we shall extract from the first reference the key Theorem 3.2 below.

Recall that $X_{\mathbb{C}((t))}$ is the projective variety over the field $\mathbb{C}((t))$ obtained from X by base change $A_r \rightarrow \mathbb{C}((t))$. We shall also consider the $\text{Spec}(\mathbb{C}[[t]])$ -scheme $\mathcal{X}_{\mathbb{C}[[t]]}$ obtained by base change $A_r \rightarrow \mathbb{C}[[t]]$. It is a formal scheme whose generic fiber is $X_{\mathbb{C}((t))}$. We also denote by $X_{\mathbb{C}((t)),r}$ the Berkovich analytification of $X_{\mathbb{C}((t))}$ when $\mathbb{C}((t))$ is endowed with the t -adic norm with $|t|_r = r$.

The line bundle $\mathcal{L} \rightarrow \mathcal{X}$ induces natural line bundles $L_{\mathbb{C}((t))} \rightarrow X_{\mathbb{C}((t))}$, $L_{\mathbb{C}[[t]]} \rightarrow \mathcal{X}_{\mathbb{C}[[t]]}$, and $L_{\mathbb{C}((t)),r} \rightarrow X_{\mathbb{C}((t)),r}$. Recall that $L_{\mathbb{C}[[t]]}$ determines a natural metrization $|\cdot|_{\mathcal{L}}$ on $L_{\mathbb{C}((t)),r}$, see [CL11, §1.3.2]. Any other continuous metrization $|\cdot|$ on $L_{\mathbb{C}((t)),r}$ can be thus written $|\cdot| = |\cdot|_{\mathcal{L}} e^{-g}$ for some continuous function $g : X_{\mathbb{C}((t)),r} \rightarrow \mathbb{R}$.

We shall say that $|\cdot|$ is a semi-positive model metrization if g is a positive rational multiple of a model function $\log |\mathfrak{A}|$, and for some (or any) log-resolution $p : \mathcal{X}' \rightarrow \mathcal{X}$ of \mathfrak{A} such that $\mathfrak{A} \cdot \mathcal{O}_{\mathcal{X}'} = \mathcal{O}_{\mathcal{X}'(-D)}$, the line bundle $p^* \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}'(D)}$ is relatively nef in the sense that $p^* \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}'(D)}|_E$ is nef for all irreducible component E of \mathcal{X}'_0 .

To any semi-positive model metrics $|\cdot| = |\cdot|_{\mathcal{L}} e^{-g}$ as above, we associate a positive (atomic) measure⁵ $\text{MA}_{\mathcal{L}}(g)$ on $X_{\mathbb{C}((t)),r}$ as follows:

$$(9) \quad \text{MA}_{\mathcal{L}}(g) := \sum_E c_1 \left(p^* \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}'(D)}|_E \right)^{\wedge k} \delta_{x_E}$$

where E ranges over all irreducible components of the central fiber \mathcal{X}'_0 .

⁵Chambert-Loir uses the notation $(\hat{c}_1(\overline{L}_g)^k|_X)$ instead of $\text{MA}_{\mathcal{L}}(g)$. The latter notation is inspired by the notations used in [BFJ15, §4].

The quantity $c_1(p^*\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}'(D)}|_E)^{\wedge k}$ is here understood as follows. We restrict the line bundle $\hat{L} := p^*L_{\mathbb{C}[[t]]} \otimes \mathcal{O}_{\mathcal{X}'(D)}$ to E viewed as a component of the special fiber of the formal scheme $X_{\mathbb{C}[[t]]}$, and compute the top intersection degree of its first Chern class $c_1(\hat{L}|_E)$ (in E viewed as a projective \mathbb{C} -scheme). Since the (complex) analytification of $\hat{L}|_E$ is isomorphic to $p^*\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}'(D)}|_E$, we see that $c_1(p^*\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}'(D)}|_E)^{\wedge k}$ is identical to the left hand side of (8) by the compatibility results of [Ful88, Example 19.1.1 & Corollary 19.2 (b)].

A general continuous semi-positive metric $|\cdot| = |\cdot|_{\mathcal{L}e^{-g}}$ is by definition a continuous metric on $L_{\mathbb{C}((t)),r}$ such that there exists a sequence of semi-positive model metrics $|\cdot|_n = |\cdot|_{\mathcal{L}e^{-g_n}}$ for which $g_n \rightarrow g$. One associates to any such metric a positive Borel measure on $X_{\mathbb{C}((t)),r}$ by setting $\text{MA}_{\mathcal{L}}(g) = \lim_{n \rightarrow \infty} \text{MA}_{\mathcal{L}}(g_n)$. This measure does not depend on the choice of model metrics converging to $|\cdot|$.

A (singular) semi-positive metric $|\cdot|_{\mathcal{L}e^{-g}}$ is by definition determined by an upper semi-continuous function $g : X_{\mathbb{C}((t)),r} \rightarrow [-\infty, +\infty)$ for which there exists a net of model semi-positive metrics $|\cdot|_{\mathcal{L}e^{-g_n}}$ such that g_n is decreasing pointwise to g , see [BFJ16, Theorem B].

In this terminology, we have

Theorem 3.2. *Let $|\cdot|_{\mathcal{L}e^{-g_n}}$ be a sequence of continuous semi-positive metrics on $L_{\mathbb{C}((t)),r}$ converging uniformly to $|\cdot|_{\mathcal{L}e^{-g}}$. Then the latter metric is again a continuous semi-positive metric and we have*

$$\text{MA}_{\mathcal{L}}(g) = \lim_{n \rightarrow \infty} \text{MA}_{\mathcal{L}}(g_n) .$$

More precisely, given any singular semi-positive metric $|\cdot|_{\mathcal{L}e^{-h}}$, all integrals $\int h d(\text{MA}_{\mathcal{L}}(g_n))$ and $\int h d(\text{MA}_{\mathcal{L}}(g))$ are finite, and we have

$$\int h d(\text{MA}_{\mathcal{L}}(g)) = \lim_{n \rightarrow \infty} \int h d(\text{MA}_{\mathcal{L}}(g_n)) .$$

Proof. The first statement is a consequence of [BFJ15, Theorem 3.1]. The second statement also follows from the arguments of [BFJ15]. The finiteness of the integrals is exactly [BFJ15, Proposition 3.11]. For completeness we give a sktech of the proof of the convergence statement We freely use notation from op. cit.

Let θ be the class in the relative Neron-Severi space $N^1(\mathcal{X}_{\mathbb{C}[[t]]}/S)$ induced by $c_1(\mathcal{L})$ (with $S = \text{Spec } \mathbb{C}[[t]]$). A θ -psh function g is an upper semi-continuous function whose metric $|\cdot|_{\mathcal{L}e^{-g}}$ is semi-positive. For any continuous θ -psh functions g_1, \dots, g_k , [BFJ15, Theorem 3.1] asserts that one can define a Radon measure $(\theta + dd^c g_1) \wedge \dots \wedge (\theta + dd^c g_k)$. This measure has mass $\delta = \int \theta^k = c_1(L)^k$, is symmetric in the entries, and depends continuously on the g_i 's. When all functions are the same $g = g_1 = \dots = g_k$, then we have $\text{MA}_{\mathcal{L}}(g) = (\theta + dd^c g)^{\wedge k}$.

The first step is to prove that one can define a (signed) Radon measure $dd^c h \wedge (\theta + dd^c g_1) \wedge \dots \wedge (\theta + dd^c g_{k-1})$ when h is any θ -psh function, and the g_i 's are continuous θ -psh functions. The point is to check that for any model function φ , the quantity

$$\Lambda(\varphi) := \int h (dd^c \varphi) \wedge (\theta + dd^c g_1) \wedge \dots \wedge (\theta + dd^c g_{k-1})$$

is well-defined and satisfies $|L(\varphi)| \leq 2\delta \sup |\varphi|$. To see that, one first assumes that h is bounded and one writes

$$\begin{aligned} \pm \int h(dd^c \varphi) \wedge (\theta + dd^c g_1) \wedge \cdots \wedge (\theta + dd^c g_{k-1}) &= \\ \pm \int \varphi(dd^c h) \wedge (\theta + dd^c g_1) \wedge \cdots \wedge (\theta + dd^c g_{k-1}) &\leq 2 \sup |\varphi| \delta . \end{aligned}$$

For a general h , we apply the very same estimate to the sequence $\max\{h, -n\}$ and let $n \rightarrow \infty$.

Since the linear form $\varphi \mapsto \Lambda(\varphi)$ is continuous, it defines a Radon measure on $X_{\mathbb{C}((t)),r}$ (of total mass $\leq 2\delta$) which we denote by $dd^c \psi \wedge (\theta + dd^c g_1) \wedge \cdots \wedge (\theta + dd^c g_{k-1})$. Then we write:

$$\begin{aligned} \int \psi d(\text{MA}_{\mathcal{L}}(g)) - \int \psi d(\text{MA}_{\mathcal{L}}(g_n)) &= \int \psi (\theta + dd^c g)^{\wedge k} - \int \psi (\theta + dd^c g_n)^{\wedge k} \\ &= \sum_{j=0}^{k-1} \int \psi dd^c(g - g_n) \wedge (\theta + dd^c g)^j \wedge (\theta + dd^c g_n)^{k-j-1} \\ &= \sum_{j=0}^{k-1} \int (g - g_n) dd^c(\psi) \wedge (\theta + dd^c g)^j \wedge (\theta + dd^c g_n)^{k-j-1} \leq 2k \sup |g - g_n| \delta , \end{aligned}$$

which concludes the proof. \square

In the sequel we shall use the following simple computation.

Proposition 3.3. *Let $\mathcal{F} = \{\mathcal{X}', d, D, \sigma_0, \dots, \sigma_l\}$ be any regular admissible datum. Then the metric $|\cdot|_{\mathcal{L}} e^{-g_{\mathcal{F}}}$ is a semi-positive model metric, and for any singular admissible datum \mathcal{G} , we have*

$$(10) \quad \int g_{\mathcal{G}} \text{MA}_{\mathcal{L}}(g_{\mathcal{F}}) = \sum_E g_{\mathcal{G}}(x_E) c_1 \left(p^* \mathcal{L}^{\otimes d} \otimes \mathcal{O}_{\mathcal{X}'}(D)|_E \right)^{\wedge k}$$

where E ranges over all irreducible components of \mathcal{X}'_0 .

Proof. Since the sections $\sigma_0, \dots, \sigma_l$ of the line bundle $\hat{\mathcal{L}} := p^* \mathcal{L}^{\otimes d} \otimes \mathcal{O}_{\mathcal{X}'}(D)$ have no common zeroes over \mathbb{D}_r , for any compact curve $C \subset \mathcal{X}'_0$ there exists at least one section say σ_0 whose restriction to C is non-zero and

$$\deg(\hat{\mathcal{L}}|_C) = \sum_{p \in E} \text{ord}_p(\sigma_0|_C) \geq 0$$

so that $\hat{\mathcal{L}}$ is relatively nef. The identity (10) follows from the definition of $\text{MA}_{\mathcal{L}}(g_{\mathcal{F}})$ when computed in \mathcal{X}' . \square

3.3. The Chambert-Loir measure associated to an endomorphism of $\mathbb{P}_{\mathbb{C}((t)),r}^k$.

This section may be skipped during a first reading. Suppose $\mathcal{X} = \mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}$, and let \mathcal{L} be the pull-back by the second projection of $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^k}(1)$. This line bundle determines a canonical semi-positive metric $|\cdot|_{\text{can}}$ on $\mathcal{O}_{\mathbb{P}_{\mathbb{C}((t)),r}^k}(1)$, and we shall also denote by $|\cdot|_{\text{can}}$ the induced metric on $\mathcal{O}_{\mathbb{P}_{\mathbb{C}((t)),r}^k}(d)$ for all $d \in \mathbb{Z}$. Note that the norm of a section σ of $\mathcal{O}_{\mathbb{P}_{\mathbb{C}((t)),r}^k}(d)$ is given

in homogeneous coordinates by

$$|\sigma([w])|_{\text{can}} = \frac{|P_\sigma(w_0, \dots, w_k)|}{\max\{|w_0|^d, \dots, |w_k|^d\}},$$

where P_σ is the homogeneous polynomial (of degree d and coefficients in $\mathbb{C}((t))$) determined by σ . The Monge-Ampère measure of $|\cdot|_{\text{can}}$ is the Dirac mass at the divisorial point⁶ x_g corresponding to $\mathbb{P}_{\mathbb{C}}^k \times \{0\}$. In the notation of the previous section, we thus have $\text{MA}_{\mathcal{L}}(0) = \delta_{x_g}$.

Now suppose R is an endomorphism of $\mathbb{P}_{\mathbb{C}((t))}^k$ of degree d given in homogeneous coordinates by $k+1$ polynomials $P_0, \dots, P_k \in \mathbb{C}((t))[w_0, \dots, w_k]$ of degree d having no zeroes in common except for the origin.

There is a natural way to pull-back metric by regular maps. Observe that the pull-back metric $R^*|\cdot|_{\text{can}}$ on $R^*\mathcal{O}_{\mathbb{P}_{\mathbb{C}((t)),r}^k}(1) = \mathcal{O}_{\mathbb{P}_{\mathbb{C}((t)),r}^k}(d)$ can be written $R^*|\cdot|_{\text{can}} = |\cdot|_{\text{can}}e^{-g_1}$ where

$$g_1([w]) = \log \left(\frac{\max\{|P_0|, \dots, |P_k|\}}{\max\{|w_0|^d, \dots, |w_k|^d\}} \right).$$

The metric $R^*|\cdot|_{\text{can}}$ is again semi-positive, see e.g. [FG15, Lemma 2.10] for details. Consider now the metric $|\cdot|_n$ on $\mathcal{O}_{\mathbb{P}_{\mathbb{C}((t)),r}^k}(1)$ obtained by taking the d^n -th root of $(R^n)^*|\cdot|_{\text{can}}$. We get $|\cdot|_{n+1} = |\cdot|_n e^{-\frac{1}{d^n}g_1 \circ R^n}$ so that $|\cdot|_{n+1}$ converges uniformly to a continuous semi-positive metric $|\cdot|_R = |\cdot|_{\text{can}}e^{-g_R}$ on $\mathcal{O}_{\mathbb{P}_{\mathbb{C}((t)),r}^k}(1)$ with

$$(11) \quad g_R = \sum_{n \geq 0} \frac{1}{d^n} g_1 \circ R^n.$$

The Chambert-Loir measure associated to R is by definition $\mu_R := \text{MA}_{\mathcal{L}}(g_R)$.

3.4. Degeneration of measures. Let us return to our general setup as described in §1.1.

Fix a regular admissible datum \mathcal{F} , and recall the definition of ψ and ψ_{NA} from Theorem 1.2. We define a family of positive measures $\mu_{\mathcal{F},t,\text{hyb}}$ on $X_{\text{hyb},r}$ parameterized by $t \in \bar{\mathbb{D}}_r$ as follows:

$$\begin{cases} \mu_{\mathcal{F},t,\text{hyb}} := \psi_*(\mu_{\mathcal{F},t}) & \text{if } t \in \bar{\mathbb{D}}_r^*; \\ \mu_{\mathcal{F},0,\text{hyb}} := (\psi_{\text{NA}})_*(\mu_{\mathcal{F},\text{NA}}). \end{cases}$$

Observe that $\mu_{\mathcal{F},t,\text{hyb}}$ is supported on $\pi_{\text{hyb}}^{-1}(\tau(t))$. We have the following continuity statement.

Theorem 3.4. *For any regular admissible datum \mathcal{F} , one has the weak convergence of measures in $X_{\text{hyb},r}$:*

$$\lim_{t \rightarrow 0} \mu_{\mathcal{F},t,\text{hyb}} = \mu_{\mathcal{F},0,\text{hyb}}.$$

By the density Theorem 2.12, this continuity statement is equivalent to prove

$$\lim_{t \rightarrow 0} \int \Phi_{\mathcal{G}} d\mu_{\mathcal{F},t,\text{hyb}} = \int \Phi_{\mathcal{G}} d\mu_{\mathcal{F},0,\text{hyb}}$$

for any regular admissible datum \mathcal{G} . Since we have $\int \Phi_{\mathcal{G}} d\mu_{\mathcal{F},0,\text{hyb}} = \int g_{\mathcal{G}} d\mu_{\mathcal{F},\text{NA}}$ by definition, we see that the continuity is in fact a consequence of the following (more general) statement by (10).

⁶when suitably interpreted as a norm on $\mathbb{C}((t))[z_1, \dots, z_k]$ this point corresponds to the Gauß norm hence the notation, see [CL11, §2.1].

Theorem 3.5. *Let \mathcal{F} and \mathcal{G} be admissible data, with \mathcal{F} regular. Then one has*

$$\lim_{t \rightarrow 0} \int \Phi_{\mathcal{G}} d\mu_{\mathcal{F}, t, \text{hyb}} = \sum_E g_{\mathcal{G}}(x_E) c_1 \left(p^* \mathcal{L}^{\otimes d} \otimes \mathcal{O}_{\mathcal{X}'}(-D)|_E \right)^{\wedge k},$$

where the sum is taken over all irreducible components E of the central fiber of an snc model \mathcal{X}' which is a log-resolution of the fractional ideal sheaf associated to \mathcal{F} .

Proof. Choose any snc model $p : \mathcal{X}' \rightarrow \mathcal{X}$ which is a log-resolution of the vertical fractional ideal sheaf associated to \mathcal{F} . Decompose the fractional ideal sheaf \mathfrak{A} associated to \mathcal{G} by writing $\mathfrak{A} = \mathfrak{B} \cdot \mathcal{O}_{\mathcal{X}'}(-D)$ where D is a vertical divisor, and \mathfrak{B} is an ideal sheaf whose co-support W does not contain any vertical component. We shall denote by Z the union of $\mathcal{X}'_0 \cap W$ and the singular locus of the central fiber \mathcal{X}'_0 : it is a subvariety included in \mathcal{X}'_0 that does not contain any irreducible component of the central fiber.

Cover the central fiber \mathcal{X}'_0 by finitely many charts $U^{(j)}$ and choose coordinates $w^{(j)} = (w_0^{(j)}, \dots, w_k^{(j)})$ in each of these charts. Let $b^{(j)} \in \mathbb{N}^*$ such that one has

$$t = \pi \circ p = \prod_i (w_i^{(j)})^{b_i^{(j)}} \times \text{unit}$$

in $U_j \subset \mathcal{X}'$. By Theorem 2.4, one can also find integers $d_{i, \mathcal{F}}^{(j)}$ and $d_{i, \mathcal{G}}^{(j)}$, and finitely many holomorphic functions $h_{\alpha}^{(j)}$ such that $\mathfrak{B} \cdot \mathcal{O}_{\mathcal{X}'}(U^{(j)}) = \langle h_{\alpha}^{(j)} \rangle$,

$$\varphi_{\mathcal{F}} = \sum_i d_{i, \mathcal{F}}^{(j)} \log |w_i^{(j)}| + \varphi^{(j)}, \text{ and } \varphi_{\mathcal{G}} = \sum_i d_{i, \mathcal{G}}^{(j)} \log |w_i^{(j)}| + \log \max_{\alpha} |h_{\alpha}^{(j)}| + \psi^{(j)}$$

on U_j where $\varphi^{(j)}$ and $\psi^{(j)}$ are continuous. It follows that one can write for all $w^{(j)} \in U^{(j)} \setminus \mathcal{X}'_0$:

$$\begin{aligned} \Phi_{\mathcal{G}} \circ \psi(w^{(j)}) &= \frac{\log r \cdot \varphi_{\mathcal{G}}(w^{(j)})}{\log |\pi \circ p(w^{(j)})|^{-1}} \\ (12) \quad &= \frac{\log r \cdot \left(\sum_i d_{i, \mathcal{G}}^{(j)} \log |w_i^{(j)}| + \log \max_{\alpha} |h_{\alpha}^{(j)}| \right) + O(1)}{\sum_i b_i^{(j)} \log |w_i^{(j)}| + O(1)}. \end{aligned}$$

Let K be any compact neighborhood of Z inside \mathcal{X}' . Observe that since all integers $b_i^{(j)}$ are non-zero, it follows that

$$(13) \quad \Phi_{\mathcal{G}} \circ \psi(w^{(j)}) \rightarrow \frac{d_{i, \mathcal{G}}^{(j)}}{b_i^{(j)}} \log r \text{ when } w_i^{(j)} \rightarrow 0 \text{ and } w^{(j)} \notin K.$$

In more geometric terms, these estimates imply the

Lemma 3.6. *The function $\Phi_{\mathcal{G}} \circ \psi$ extends to a continuous function on $\mathcal{X}' \setminus K$ whose restriction to an irreducible component E of the central fiber is constant equal to $g_{\mathcal{G}}(x_E)$.*

Proof. The equation (13) implies the continuity statement. Let E be an irreducible component of \mathcal{X}'_0 , and suppose $U^{(j)} \cap E$ is non empty and determined by the equation $w_i^{(j)} = 0$. Then by definition we have

$$g_{\mathcal{G}}(x_E) = \log r \frac{\text{ord}_E(D)}{b_E}$$

where D is the vertical divisor associated to \mathcal{G} , and $b_E = \text{ord}_E(\pi \circ p)$. It follows from Theorem 2.4 that D is given by the equation $(w_i^{(j)})^{d_i^{(j)}} = 0$ in $U^{(j)}$ whereas $b_E = b_i^{(j)}$. This concludes the proof. \square

We shall also use the following

Lemma 3.7. *For any $\epsilon > 0$, there exists a compact neighborhood K of Z , such that*

$$(14) \quad \max \left\{ \int_K (\Omega_{\mathcal{F}}|_{X_t})^{\wedge k}, \int_K \Phi_{\mathcal{G}} \circ \psi (\Omega_{\mathcal{F}}|_{X_t})^{\wedge k} \right\} \leq \epsilon$$

for any $t \in \mathbb{D}$.

To simplify notation, write $\mu_E = c_1(p^* \mathcal{L}^{\otimes d} \otimes \mathcal{O}_{\mathcal{X}'}(D)|_E)^{\wedge k}$ for any irreducible component E of \mathcal{X}'_0 . We then obtain

$$\begin{aligned} \Delta_t := \left| \int \Phi_{\mathcal{G}} d\mu_{\mathcal{F}, t, \text{hyb}} - \sum_E g_{\mathcal{G}}(x_E) \mu_E \right| &= \left| \int \Phi_{\mathcal{G}} \circ \psi d\mu_{\mathcal{F}, t} - \sum_E g_{\mathcal{G}}(x_E) \mu_E \right| \leq \\ & \left| \int_K \Phi_{\mathcal{G}} \circ \psi (\Omega_{\mathcal{F}}|_{X_t})^{\wedge k} \right| + \left| \int_{X_t \setminus K} \Phi_{\mathcal{G}} \circ \psi (\Omega_{\mathcal{F}}|_{X_t})^{\wedge k} - \sum_E g_{\mathcal{G}}(x_E) \mu_E \right|. \end{aligned}$$

Applying (14) and Lemma 3.6, we get

$$\overline{\lim}_{t \rightarrow 0} \Delta_t \leq \epsilon + \sum_E g_{\mathcal{G}}(x_E) \left(\int_{E \setminus K} (\Omega_{\mathcal{F}}|_E)^{\wedge k} - \mu_E \right).$$

By Proposition 3.1, we have $\int_E (\Omega_{\mathcal{F}}|_E)^{\wedge k} = \mu_E$ so that

$$\overline{\lim}_{t \rightarrow 0} \Delta_t \leq \epsilon + \sum_E g_{\mathcal{G}}(x_E) \left(\int_{E \cap K} (\Omega_{\mathcal{F}}|_E)^{\wedge k} \right),$$

and we conclude since $\int_{E \cap K} (\Omega_{\mathcal{F}}|_E)^{\wedge k}$ can be made arbitrarily small by Lemma 3.7. \square

Proof of Lemma 3.7. Let us first estimate the integral $\int_K (\Omega_{\mathcal{F}}|_{X_t})^{\wedge k}$. Since $\Omega_{\mathcal{F}}$ is a positive closed $(1, 1)$ -current with continuous potential, it follows from [De93, Proposition 1.11] that for any irreducible component E of \mathcal{X}'_0 we have

$$(\Omega_{\mathcal{F}}|_E)^{\wedge k}(Z) = 0,$$

so that $\mu_0(Z) = 0$ where $\mu_0 = dd^c(\log |\pi \circ p|) \wedge \Omega_{\mathcal{F}}^{\wedge k}$. Since $t \mapsto (\Omega_{\mathcal{F}}|_{X_t})^{\wedge k} = dd^c(\log |\pi \circ p - t|) \wedge \Omega_{\mathcal{F}}^{\wedge k}$ is continuous, for a sufficiently small compact neighborhood K of Z we have $\int_K (\Omega_{\mathcal{F}}|_{X_t})^{\wedge k} \leq \epsilon$ for all $|t| \ll 1$.

To estimate the integral $\int_K \Phi_{\mathcal{G}} \circ \psi (\Omega_{\mathcal{F}}|_{X_t})^{\wedge k}$, we work in a fixed chart $U \ni (w_0, \dots, w_k)$ near a point $x \in Z$ where we have

$$\Phi_{\mathcal{G}} \circ \psi = \frac{\log r \cdot (\sum_i d_i \log |w_i| + \log \max_{\alpha} |h_{\alpha}|) + \theta}{\sum_i b_i \log |w_i| + O(1)}$$

on U_j where θ is continuous, h_{α} are holomorphic, $d_i \in \mathbb{Z}$, and $b_i \in \mathbb{N}^*$, see (12) above. We decompose $\Phi_{\mathcal{G}} \circ \psi$ into the following sum $\Phi_1 + \Phi_2$, with

$$\Phi_1 = \frac{\log r \cdot (\sum_i d_i \log |w_i|) + \theta}{\sum_i b_i \log |w_i| + O(1)} \quad \text{and} \quad \Phi_2 = \frac{\log r \cdot (\log \max_{\alpha} |h_{\alpha}|)}{\sum_i b_i \log |w_i| + O(1)}.$$

Since Φ_1 is bounded, we have $\int_K \Phi_1(\Omega_{\mathcal{F}}|_{X_t})^{\wedge k} \leq \epsilon$ for K and t small enough by the preceding estimate. Let us prove that

$$A_t = \int_U \log \max_{\alpha} |h_{\alpha}| (\Omega_{\mathcal{F}}|_{X_t})^{\wedge k} = O(1) .$$

Since $\int_K \Phi_1(\Omega_{\mathcal{F}}|_{X_t})^{\wedge k} \leq \frac{\log r \cdot A_t}{\log |t|^{-1}} \rightarrow 0$, this will conclude the proof. Let g be a continuous potential of $\Omega_{\mathcal{F}}$ in the open set U . We can then write

$$A_t = \int_U \log \max_{\alpha} |h_{\alpha}| (\Omega_{\mathcal{F}}|_{X_t})^{\wedge k} = \int_U \log \max_{\alpha} |h_{\alpha}| \left([X_t] \wedge (dd^c)^k g \right) .$$

This integral can be now estimated using the improved Chern-Levine-Nirenberg inequalities of [De93, Proposition 2.6] (with $u_1 = \log \max_{\alpha} |h_{\alpha}|$, $u_2 = \dots = u_k = g$ and $T = [X_t]$). Indeed since the ideal sheaf \mathfrak{B} has a co-support which does not contain any vertical component, the psh function $\log \max_{\alpha} |h_{\alpha}|$ is continuous outside a subvariety W of \mathcal{X}' whose intersection with any fiber X_t has codimension at least 2 (in \mathcal{X}'). \square

4. MONGE-AMPÈRE OF UNIFORM LIMITS OF MODEL FUNCTIONS

In this section, we show how to extend Theorem 3.4 to a much larger class of measures. This will imply a stronger form of Theorem B from the introduction.

4.1. Uniform limits of model functions. We aim at proving a generalization of Theorem 3.4 to a more general class of functions than model ones. To that end we introduce the following definition.

Definition 4.1. *A function $\varphi : X \rightarrow \mathbb{R}$ is said to be uniform if there exists a sequence of regular admissible data \mathcal{F}_n of degree $d_n \rightarrow \infty$ such that*

$$(15) \quad \sup_{X_t} \left| \frac{1}{d_n} \varphi_{\mathcal{F}_n} - \varphi_{\mathcal{F}} \right| \leq \epsilon_n \log |t|^{-1}$$

for all $0 < |t| \leq r$ and for a sequence $\epsilon_n \rightarrow 0$.

The condition imposed by (15) is empty outside $\pi^{-1}(\bar{\mathbb{D}}_r^*)$. This causes no harm since we shall only be interested in the behaviour of uniform functions near the central fiber.

Observe that for a regular admissible datum \mathcal{F} of degree d , the model function $\frac{1}{d} \varphi_{\mathcal{F}}$ is uniform since $\frac{1}{d^n} \varphi_{\mathcal{F}^{\otimes n}} = \varphi_{\mathcal{F}}$ for all n . We refer to the next section for more examples.

Remark. Pick any uniform function φ as in the definition, and consider the function $\Phi := n \cdot \varphi \circ \psi^{-1}$ on $\pi_{\text{hyb}}^{-1}(\tau(\bar{\mathbb{D}}_r^*))$ in the hybrid space. Then (15) implies the uniform convergence $\frac{1}{d_n} \Phi_{\mathcal{F}_n} \rightarrow \Phi$ hence Φ extends continuously to $X_{\text{hyb},r}$. Heuristically uniform functions correspond to continuous ω -psh function on the hybrid space, see Question 1 below for a conjectural characterization of uniform function in this vein.

Let us explain now how to associate a sequence of positive Borel measures to a uniform function.

Theorem 4.2. *Let φ be any uniform function on X , and let \mathcal{F}_n be a sequence of regular admissible data of degree $d_n \rightarrow \infty$ such that (15) holds.*

For any $t \in \bar{\mathbb{D}}_r$, the sequence of measures $\frac{1}{d_n^k} \mu_{t, \mathcal{F}_n, \text{hyb}}$ converges to a positive Borel measure $\text{MA}_{t, \text{hyb}}(\varphi)$, and we have the following weak convergence of measures

$$(16) \quad \lim_{t \rightarrow 0} \text{MA}_{t, \text{hyb}}(\varphi) = \text{MA}_{0, \text{hyb}}(\varphi)$$

in the hybrid space $X_{\text{hyb},r}$. More precisely, for any singular admissible datum \mathcal{G} , we have

$$(17) \quad \lim_{t \rightarrow 0} \int \Phi_{\mathcal{G}} d\text{MA}_{t,\text{hyb}}(\varphi) = \int \Phi_{\mathcal{G}} d\text{MA}_{0,\text{hyb}}(\varphi)$$

Proof. Let \mathcal{F}_n be a sequence of admissible data of degree d_n , such that $\frac{1}{d_n}\varphi_{\mathcal{F}_n} \rightarrow \varphi$, and

$$\sup_{X_t} \left| \frac{1}{d_n}\varphi_{\mathcal{F}_n} - \varphi_{\mathcal{F}} \right| \leq \epsilon_n \log |t|^{-1}$$

with $\epsilon_n \rightarrow 0$. Recall that we wrote ω for the curvature form of the smooth positive metrization $|\cdot|_{\star}$ on \mathcal{L} , and $\omega_t = \omega|_{X_t}$.

For any fixed $t \in \bar{\mathbb{D}}_r^*$, the restriction $\varphi|_{X_t}$ is the uniform limit of the sequence of continuous functions $\frac{1}{d_n}\varphi_{\mathcal{F}_n}|_{X_t}$, and $\omega_t + \frac{1}{d_n}dd^c\varphi_{\mathcal{F}_n}|_{X_t}$ is a positive closed (1,1)-current for all $n \in \mathbb{N}$. It follows from [De93, Corollary 1.6] that $\omega_t + dd^c\varphi|_{X_t}$ is also a positive closed (1,1)-current for which its k -th exterior power is well-defined and

$$\begin{aligned} \mu_{n,t} := \frac{1}{d_n^k} \mu_{t,\mathcal{F}_n,\text{hyb}} &= \psi_* \left(\omega_t + \frac{1}{d_n} dd^c \varphi_{\mathcal{F}_n}|_{X_t} \right)^{\wedge k} \\ &\xrightarrow{n \rightarrow \infty} \psi_* (\omega_t + dd^c \varphi|_{X_t})^{\wedge k} =: \text{MA}_{t,\text{hyb}}(\varphi) . \end{aligned}$$

Recall that $n = \frac{\log r}{\log |\pi|^{-1}}$ on \mathcal{X} so that the function $\Phi = \frac{\varphi \circ \psi}{n}$ which is defined on $\psi(X)$ satisfies

$$\left| \Phi - \frac{1}{d_n} \Phi_{\mathcal{F}_n} \right| \leq \epsilon_n \log r \text{ on } \pi^{-1}(\bar{\mathbb{D}}_r^*) .$$

We thus conclude that Φ extends continuously to $X_{\text{hyb},r}$ and is a uniform limit of the sequence of model functions $\frac{1}{d_n}\Phi_{\mathcal{F}_n}$ on $X_{\text{hyb},r}$. In particular, $g := \Phi \circ \psi_{\text{NA}} : X_{\mathbb{C}((t)),r} \rightarrow \mathbb{R}$ is a uniform limit of the sequence of model functions $\frac{1}{d_n}g_{\mathcal{F}_n}$. It follows from Theorem 3.2 that the Monge-Ampère measure $\text{MA}_{\mathcal{L}}(g)$ is well-defined, and we have the weak convergence of measures

$$\mu_n := \frac{1}{d_n^k} \mu_{0,\mathcal{F}_n,\text{hyb}} = (\psi_{\text{NA}})_* \text{MA}_{\mathcal{L}} \left(\frac{1}{d_n} g_{\mathcal{F}_n} \right) \xrightarrow{n \rightarrow \infty} (\psi_{\text{NA}})_* \text{MA}_{\mathcal{L}}(g) =: \text{MA}_{0,\text{hyb}}(\varphi) .$$

It remains to prove (17) (which implies (16)).

We claim that for any singular admissible data \mathcal{G} there exists a constant $C(\mathcal{G}) > 0$ such that

$$(18) \quad \left| \int \Phi_{\mathcal{G}} d\mu_{n,t} - \int \Phi_{\mathcal{G}} d\text{MA}_{t,\text{hyb}}(\varphi) \right| \leq C(\mathcal{G})\epsilon_n$$

for all $t \in \bar{\mathbb{D}}_r^*$ and all n . Indeed, using the positivity of the current $dd^c\Phi_{\mathcal{G}} + \deg(\mathcal{G})\omega$ on X_r by (6), we get

$$\begin{aligned} \int \Phi_{\mathcal{G}} d\mu_{n,t} - \int \Phi_{\mathcal{G}} d\text{MA}_{t,\text{hyb}}(\varphi) &= \\ &= \frac{\log r}{\log |t|^{-1}} \int_{X_t} \varphi_{\mathcal{G}} \left(\omega_t + \frac{1}{d_n} dd^c \varphi_n|_{X_t} \right)^{\wedge k} - \int_{X_t} \varphi_{\mathcal{G}} (\omega_t + dd^c \varphi|_{X_t})^{\wedge k} = \\ &= \frac{\log r}{\log |t|^{-1}} \sum_{j=0}^{k-1} \int_{X_t} \left(\frac{1}{d_n} \varphi_n - \varphi \right) \left(\omega_t + \frac{1}{d_n} dd^c \varphi_n|_{X_t} \right)^{\wedge j} \wedge (\omega_t + dd^c \varphi|_{X_t})^{\wedge (k-j-1)} \wedge dd^c \varphi_{\mathcal{G}} \\ &\leq \frac{\log r}{\log |t|^{-1}} \sup_{X_t} \left| \frac{1}{d_n} \varphi_n - \varphi \right| \times 2k \deg(\mathcal{G}) \end{aligned}$$

which implies (18) with $C(\mathcal{G}) = 2k |\log r| \deg(\mathcal{G})$.

Let us now prove that $\int_{X_t} \Phi_{\mathcal{G}} d\text{MA}_{t,\text{hyb}}(\varphi) \rightarrow \int_{X_{C((t)),r}} \Phi_{\mathcal{G}} d\text{MA}_{0,\text{hyb}}(\varphi)$. To that end we fix $\epsilon > 0$ arbitrarily small, and take n sufficiently large such that $\epsilon_n \leq \epsilon$. Since $\mu_{n,t} \rightarrow \mu_n$ as $t \rightarrow 0$ by Theorem 3.4, there exists $\eta > 0$ such that

$$\left| \int \Phi_{\mathcal{G}} d\mu_{n,t} - \int \Phi_{\mathcal{G}} d\mu_n \right| \leq \epsilon$$

for all $0 < |t| \leq \eta$. By (18), we infer

$$\left| \int \Phi_{\mathcal{G}} d\text{MA}_{t,\text{hyb}}(\varphi) - \int \Phi_{\mathcal{G}} d\mu_n \right| \leq \epsilon(1 + C(\mathcal{G}))$$

and letting $n \rightarrow \infty$ we conclude that

$$\left| \int \Phi_{\mathcal{G}} d\mu_t - \int \Phi_{\mathcal{G}} d\text{MA}_{0,\text{hyb}}(\varphi) \right| \leq \epsilon(1 + C(\mathcal{G}))$$

for all $|t| \leq \eta$ as was to be shown. \square

4.2. Example of uniform functions. This section is logically not necessary for the rest of the paper.

Proposition 4.3. *Let $\varphi : \mathcal{X} \rightarrow \mathbb{R}$ be any continuous function such that $|\cdot|_{\star} e^{-\varphi}$ induces a semi-positive metric on \mathcal{L} . Then one can find a sequence of admissible data \mathcal{F}_n of degree n such that*

$$(19) \quad \sup_{\pi^{-1}(\bar{\mathbb{D}}_r)} \left| \frac{1}{n} \varphi_{\mathcal{F}_n} - \varphi \right| \rightarrow 0 \text{ as } n \rightarrow \infty .$$

In particular, the function φ is uniform.

This result shows that uniform functions form a quite large class. On the other hand, it is quite easy to show that $\text{MA}_{t,\text{hyb}}(\varphi) \rightarrow \text{MA}_{0,\text{hyb}}(0)$ as $t \rightarrow 0$ for any functions as above (without approximating by model functions). In fact one has the following

Remark. Suppose $|\cdot|_{\star} e^{-\varphi}$ is a semi-positive metric on \mathcal{L} that is continuous in restriction to X , and such that $\sup_{X_t} |\varphi| = o(\log |t|^{-1})$. Then one can copy the proof of (18) to obtain

$$\left| \int \Phi_{\mathcal{G}} d\text{MA}_{t,\text{hyb}}(\varphi) - \int \Phi_{\mathcal{G}} d\text{MA}_{t,\text{hyb}}(0) \right| \leq \frac{\log r}{\log |t|^{-1}} \sup_{X_t} |\varphi| 2k \deg(\mathcal{G}) ,$$

for all admissible data \mathcal{G} , which implies $\text{MA}_{t,\text{hyb}}(\varphi) \rightarrow \text{MA}_{0,\text{hyb}}(0)$.

Proof. The proof is a simple adaptation of the approximation result of Demailly, see [De12, Theorem 14.21]. For any integer m write $|\cdot| = |\cdot|_{\star} e^{-\varphi}$, $|\cdot|_m = |\cdot|_{\star} e^{-m\varphi}$ (which is a metric on $\mathcal{L}^{\otimes m}$), and let ω be the curvature form of our reference metric on \mathcal{X} . Recall that we write $\mathcal{X}_r = \pi^{-1}(\mathbb{D}_r)$, and $\bar{\mathcal{X}}_r = \pi^{-1}(\bar{\mathbb{D}}_r)$.

Consider the Hilbert space

$$\mathcal{H}_m = \left\{ \sigma \in H^0(\mathcal{X}_r, \mathcal{L}^{\otimes m}), \int |\sigma|_m^2 d\text{Vol}_{\omega} < \infty \right\}$$

and set $\varphi_m = \sup_{\sigma \in \mathcal{H}_m(1)} \frac{1}{m} \log |\sigma|_{\star}$ where $\mathcal{H}_m(1)$ is the unit ball of \mathcal{H}_m .

We cover $\bar{\mathcal{X}}_r$ by finitely many charts U_i in which both $K_{\mathcal{X}}$ and \mathcal{L} are trivialized. Pick any section σ of $\mathcal{L}^{\otimes m}$ over \mathcal{X}_r . In each trivializing chart U_i , σ gives rise to a holomorphic function σ_i . And we have $|\sigma|_{\star} = |\sigma_i|_{\infty} e^{-mv_i}$ for some smooth psh functions v_i .

For all $x \in \mathcal{X}_r \cap U_i$, and for any ρ sufficiently small, the mean value inequality for $|\sigma_i|^2$ then implies

$$\begin{aligned} |\sigma(x)|_{\star}^2 &= e^{-2mv_i(x)} |\sigma_i(x)|_{\infty}^2 \leq \frac{e^{-2mv_i(x)} (k+1)!}{\pi^{k+1} \rho^{2(k+1)}} \int_{B(x,\rho)} |\sigma_i|_{\infty}^2 \\ &\leq \frac{C e^{-2mv_i(x)}}{\rho^{2(k+1)}} \int_{B(x,\rho)} |\sigma|_{\star}^2 e^{-2m\varphi} \times e^{2m \sup_{B(x,\rho)} \varphi} \times e^{2m \sup_{B(x,\rho)} v_i} \end{aligned}$$

so that

$$(20) \quad \varphi_m(x) \leq \sup_{B(x,\rho)} \varphi + \frac{1}{2m} \log \left(\frac{C'}{\rho^{2(k+1)}} \right) + C'' \rho .$$

For the lower bound, for any $x \in \mathcal{X}_r$ in the chart U_i , one produces using Ohsawa-Takegoshi's theorem a holomorphic function f such that $f(x) = a$ and

$$\int_{U_i} |f|^2 e^{-2m\varphi} \leq C |a|^2 e^{-2m\varphi(x)} .$$

We denote by $|\cdot|_{\star}$ the induced metric on $K_{\mathcal{X}}^{\pm 1} \otimes \mathcal{L}^{\otimes m}$ by our reference metric ω on \mathcal{X} and $|\cdot|_{\star}$ and \mathcal{L} .

Pick $m_0 \in \mathbb{N}^*$ a sufficiently large integer such that $\mathcal{L}^{\otimes m_0} \otimes K_{\mathcal{X}}$ and $\mathcal{L}^{\otimes m_0} \otimes K_{\mathcal{X}}^{-1}$ are globally generated over a neighborhood of \mathcal{X}_r . Choose two sections τ and τ_- respectively of $\mathcal{L}^{\otimes m_0} \otimes K_{\mathcal{X}}$ and $\mathcal{L}^{\otimes m_0} \otimes K_{\mathcal{X}}^{-1}$ such that $|\tau(x)|_{\star} = |\tau_-(x)|_{\star} = 1$. Pick θ a smooth function having compact support in U_i with constant value 1 in a neighborhood of x . Interpret the $(0,1)$ form $\bar{\partial}(\theta f)$ as a section of $\Lambda^{0,1} T^* \mathcal{X}_r \otimes \mathcal{L}^{\otimes m}$ in the trivialization chart U_i , and consider the section $F = \bar{\partial}(\theta f) \wedge \tau$ of the line bundle $\Lambda^{n,1} T^* \mathcal{X}_r \otimes \mathcal{L}^{\otimes(m+m_0)}$.

On the line bundle $\mathcal{L}^{\otimes m} \otimes \mathcal{L}^{\otimes m_0}$ put the product metric $|\cdot|'_m$ induced by $|\cdot|_{\star} \frac{e^{-m\varphi}}{|x-a|^{\theta(x)(k+2)}}$ in the first factor and $|\cdot|_{\star}$ in the second. The curvature form of this metric is equal to

$$(m + m_0)\omega + m dd^c \varphi + dd^c(\theta \log |x - a|) \geq \omega ,$$

for m_0 large enough. We can solve the equation $\bar{\partial}G = F$ where G is a section over \mathcal{X}_r of the line bundle $\Lambda^{n,0} T^* \mathcal{X}_r \otimes \mathcal{L}^{\otimes(m+m_0)}$ with L^2 -norm bounded by the L^2 -norm of F , see e.g. [De94, Corollary 5.3] (observe that \mathcal{X}_r is indeed weakly pseudoconvex).

Then $\sigma = \tau_- \otimes ((\theta f) \tau - G)$ is a holomorphic section of the line bundle $\mathcal{L}^{\otimes(m+2m_0)}$ such that $\sigma(x) = a\tau(x) \otimes \tau_-(x)$, and we have the integral bound

$$\begin{aligned} \int |\sigma|_m^2 d\text{Vol}_\omega &\leq C_1 \int (|\sigma|'_m)^2 e^{-2m_0\varphi} |x-a|^{2\theta(x)(k+2)} d\text{Vol}_\omega \\ &\leq C_2 \int (|\sigma|'_m)^2 d\text{Vol}_\omega \leq C_3 |a|^2 e^{-2m\varphi(x)}. \end{aligned}$$

Choosing a such that the right hand side is equal to 1, we obtain the lower bound

$$\varphi_m \geq \varphi - \frac{C}{2m}.$$

Now fix $\epsilon > 0$, and observe that \mathcal{H}_m is a separable Hilbert space. We can thus find finitely many sections $\sigma_0, \dots, \sigma_l$ of $\mathcal{L}^{\otimes(m+2m_0)}$ such that $|\varphi_m - \frac{1}{m} \log \max\{|\sigma_0|, \dots, |\sigma_l|\}| \leq \epsilon$ on $\bar{\mathcal{X}}_{r'}$ for some fixed $r' < r$.

Since φ is continuous, one may on the other hand find $\rho > 0$ small enough such that $\sup_{B(x,\rho)} \varphi \leq \varphi(x) + C''\rho \leq \epsilon$ for all $x \in \bar{\mathcal{X}}_{r'}$. For m large enough, we then obtain

$$\left| \varphi - \frac{1}{m+2m_0} \log \max\{|\sigma_0|, \dots, |\sigma_l|\} \right| \leq \frac{C'''}{2m} + \epsilon,$$

on $\bar{\mathcal{X}}_{r'}$. This concludes the proof since $\frac{1}{m+2m_0} \log \max\{|\sigma_0|, \dots, |\sigma_l|\}$ is a function associated to an admissible datum of degree $m+2m_0$. \square

4.3. Degeneration of measures of maximal entropy. Let us now explain how the results of Section 4.1 imply Theorem B from the introduction.

Recall the setting. We let R_t be a meromorphic family of endomorphisms of $\mathbb{P}_{\mathbb{C}}^k$ of a fixed degree d parameterized by the unit disk. In other words, we suppose given $k+1$ homogeneous polynomials $P_{0,t}(w_0, \dots, w_k), \dots, P_{k,t}(w_0, \dots, w_k)$ of degree d whose coefficients are meromorphic functions on \mathbb{D} with a single pole at the origin. These polynomials are uniquely determined up to the multiplication by a meromorphic function $h(t)$ in \mathbb{D} .

We also assume that for any $t \in \mathbb{D}^*$ these polynomials have no common zeroes so that the map

$$R_t([w]) = R_t([w_0 : \dots : w_k]) = [P_{0,t}(w) : \dots : P_{k,t}(w)]$$

has no indeterminacy point. For any integer n , we shall write

$$R_t^{\circ n}([w]) = [P_{0,t}^n(w) : \dots : P_{k,t}^n(w)].$$

Recall that each polynomial P_i^n defines a meromorphic section of the line bundle $\mathcal{L}^{\otimes d^n}$ on $\mathcal{X} := \mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}$ where $\mathcal{L} := \pi_1^* \mathcal{O}_{\mathbb{P}_{\mathbb{C}}^k}(1)$ and π_1 denotes the first projection. We endow \mathcal{L} with the pull-back of the metric on $\mathcal{O}_{\mathbb{P}_{\mathbb{C}}^k}(1)$ whose curvature form is the standard Fubini-Study Kähler metric.

Since the polynomials $P_{i,t}^n$ have no common zeroes over the $\mathbb{P}_{\mathbb{C}}^k \times \{t\}$ when $t \in \mathbb{D}^*$, the fractional ideal sheaf $\mathfrak{A}_n := \langle P_{t,0}^n, \dots, P_{t,k}^n \rangle$ is vertical. Let $p_n : \mathcal{X}_n \rightarrow \mathcal{X}$ be any log-resolution of this vertical ideal sheaf. The set of sections $\{P_{t,0}^n, \dots, P_{t,k}^n\}$ and the degeneration \mathcal{X}_n thus defines a regular admissible datum \mathcal{F}_n of degree d^n by Proposition 2.2, whose model function on X is given by

$$\varphi_{\mathcal{F}_n} = \log \left(\frac{\max\{|P_{0,t}^n(w)|, \dots, |P_{k,t}^n(w)|\}}{(|w_0|^2 + \dots + |w_k|^2)^{d^n/2}} \right).$$

The key estimate is given by the following (standard) result:

Proposition 4.4. *There exists a positive constant $C > 0$ such that*

$$(21) \quad \left| \frac{1}{d^{n+1}} \varphi_{\mathcal{F}_{n+1}} - \frac{1}{d^n} \varphi_{\mathcal{F}_n} \right| \leq \frac{C \log |t|^{-1}}{d^n}$$

on $\mathbb{P}_{\mathbb{C}}^k \times \bar{\mathbb{D}}_r$.

Proof of Proposition 4.4. Observe that $P_{i,t}^{n+1} = P_{i,t}(P_{0,t}^n, \dots, P_{k,t}^n)$ for all n so that (21) is a consequence of the bound

$$c|t|^M \leq \frac{\max\{|P_{0,t}(w)|, \dots, |P_{k,t}(w)|\}}{\max\{|w_0|^d, \dots, |w_k|^d\}} \leq C|t|^{-N}.$$

for some $c, C > 0$, and $M, N \in \mathbb{N}^*$. By compactness of $\bar{\mathbb{D}}_r$, it is sufficient to get this bound in a neighborhood of the origin.

The upper bound is easy to obtain since $|P_{i,t}(w)| \leq C|t|^N \max\{|w_0|^d, \dots, |w_k|^d\}$ for any i . The lower bound follows from the Nullstellensatz applied in the algebraic closure $\widehat{\mathcal{M}}$ of the field \mathcal{M} of meromorphic functions on \mathbb{D}^* . Observe that any element $g \in \widehat{\mathcal{M}}$ can be represented (non uniquely) by a Puiseux series converging in some neighborhood of the origin, so that there exists a rational number q and a positive constant such that $|g(t)| \leq C|t|^q$ for all t small enough.

Since the polynomial $P_{i,t}$ have no common factors for all $t \in \mathbb{D}^*$, it follows that the subvariety of $\mathbb{A}_{\widehat{\mathcal{M}}}^{k+1}$ defined by the vanishing of these polynomials is reduced to the origin. We may thus find an integer N and homogeneous polynomials $q_{i,j,t}$ of degree $N - d$ with coefficients in $\widehat{\mathcal{M}}$ such that

$$w_i^N = \sum_j q_{i,j,t} P_{j,t}.$$

Assuming $|q_{i,j,t}(t)| \leq C|t|^q$ near 0 for all i, j and taking norms of both sides, we get

$$\max\{|w_0|, \dots, |w_k|\}^N \leq C|t|^q \times \max\{|w_0|, \dots, |w_k|\}^{N-d} \times \max\{|P_{0,t}(w)|, \dots, |P_{k,t}(w)|\},$$

which implies the lower bound. \square

Proposition 4.4 implies that $\varphi_R := \lim_{n \rightarrow \infty} \frac{1}{d^n} \varphi_{\mathcal{F}_n}$ is a well-defined function on $\mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}^*$ which is uniform in the sense of §4.1.

Proof of Theorem B. By Theorem 4.2, we have the convergence of measures $\text{MA}_{t,\text{hyb}}(\varphi_R) \rightarrow \text{MA}_{0,\text{hyb}}(\varphi_R)$ in the hybrid space associated to $\mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}$.

To conclude the proof it remains to relate $\text{MA}_{t,\text{hyb}}(\varphi_R)$ to the measure of maximal entropy μ_t of the endomorphism R_t , and $\text{MA}_{0,\text{hyb}}(\varphi_R)$ to the Chambert-Loir measure of the dynamical system \mathbf{R} induced by the family $\{R_t\}$ on $\mathbb{P}_{\mathbb{C}((t))}^k$.

Let ω_{FS} be the standard Fubini-Study $(1, 1)$ -form on $\mathbb{P}_{\mathbb{C}}^k$ so that

$$\frac{1}{d} R_t^* \omega_{\text{FS}} - \omega_{\text{FS}} = \frac{1}{2} dd^c \log \left(\frac{|P_{0,t}(w)|^2 + \dots + |P_{k,t}(w)|^2}{|w_0|^2 + \dots + |w_k|^2} \right).$$

It follows from the previous proposition that $\frac{1}{d^n} (R_t^n)^* \omega_{\text{FS}}$ converges to a positive closed $(1, 1)$ -current T with continuous potential, and [Sib99, Théorème 3.3.2] and [BD01] implies that $T^{\wedge k}$ is the unique measure of maximal entropy of R_t hence is equal to μ_t . On the

other hand for each n , we have

$$\begin{aligned} \frac{1}{d^n} (R_t^n)^* \omega_{\text{FS}} - \left(\omega_{\text{FS}} + \frac{1}{d^n} dd^c \varphi_{\mathcal{F}_n} |_{\mathbb{P}_{\mathbb{C}}^k \times \{t\}} \right) = \\ \frac{1}{d^n} dd^c \log \left(\frac{(|P_{0,t}^n(w)|^2 + \cdots + |P_{k,t}^n(w)|^2)^{1/2}}{\max\{|P_{0,t}^n(w)|, \dots, |P_{k,t}^n(w)|\}} \right). \end{aligned}$$

The right hand side is the dd^c of a function with values in $[0, \frac{\log(k+1)}{2d^n}]$, and therefore we conclude that $T = \lim_n (\omega_{\text{FS}} + \frac{1}{d^n} dd^c \varphi_{\mathcal{F}_n} |_{\mathbb{P}_{\mathbb{C}}^k \times \{t\}})$, and $T^{\wedge k} = \lim_n (\omega_{\text{FS}} + \frac{1}{d^n} dd^c \varphi_{\mathcal{F}_n} |_{\mathbb{P}_{\mathbb{C}}^k \times \{t\}})^{\wedge k}$. Unwinding definitions, we see that the latter convergence implies $\psi_*(\mu_t) = \text{MA}_{t, \text{hyb}}(\varphi_R)$ in the hybrid space.

To identify $\text{MA}_{0, \text{hyb}}(\varphi_R)$ with the Chambert-Loir measure of R , we proceed as follows. By definition $\text{MA}_{0, \text{hyb}}(\varphi) = (\psi_{\text{NA}})_* \text{MA}_{\mathcal{L}}(g)$ where $g = \lim_{n \rightarrow \infty} \frac{1}{d^n} g_{\mathcal{F}_n}$. We claim that $g = g_R$ as defined in (11) hence $\text{MA}_{0, \text{hyb}}(\varphi) = (\psi_{\text{NA}})_* \mu_R$ which proves the theorem.

Let P_i be the homogeneous polynomial of degree d and coefficients in $\mathbb{C}((t))$ associated to $P_{i,t}$. Observe first that we have the following identity in $\mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}$:

$$(22) \quad \frac{1}{d^n} \varphi_{\mathcal{F}_n} = \frac{1}{d^n} \varphi_{\mathcal{F}_1} + \sum_{j=1}^{n-1} \left(\frac{1}{d^{j+1}} \varphi_{\mathcal{F}_{j+1}} - \frac{1}{d^j} \varphi_{\mathcal{F}_j} \right) = \frac{1}{d^n} \varphi_{\mathcal{F}_1} + \sum_{j=0}^{n-1} \frac{1}{d^j} \tilde{\varphi} \circ R^j$$

where

$$\tilde{\varphi}([w], t) = \log \left(\frac{\max\{|P_{0,t}|, \dots, |P_{k,t}|\}}{\max\{|w_0|, \dots, |w_k|\}^d} \right).$$

Recall the definition of g_1 in §3.3, and observe that this function equals $g_{\mathcal{F}_1}$ by definition.

Lemma 4.5. *The function $n \cdot \tilde{\varphi} \circ \psi$ extends continuously to the hybrid space and its restriction to $\pi^{-1}(\tau(0))$ is equal to g_1 .*

From this lemma and Theorem 2.10 we get

$$\frac{1}{d^n} g_{\mathcal{F}_n} = \frac{1}{d^n} g_{\mathcal{F}_1} + \sum_{j=0}^{n-1} \frac{1}{d^j} g_1 \circ R^j$$

and letting $n \rightarrow \infty$, we conclude that $g = g_R$ by (11). \square

Proof of Lemma 4.5. One has $n \cdot (\tilde{\varphi} - \varphi_{\mathcal{F}_1}) = n \cdot \log \left(\frac{(|w_0|^2 + \cdots + |w_k|^2)^{d/2}}{\max\{|w_0|, \dots, |w_k|\}^d} \right)$ so that this function extends continuously to the hybrid space with constant value 0 on $\pi^{-1}(\tau(0))$. \square

4.4. Lyapunov exponents of endomorphisms. Let $R = [P_0 : \cdots : P_k]$ be an endomorphism of the projective complex space $\mathbb{P}_{\mathbb{C}}^k$ given in homogeneous coordinates by $k+1$ polynomials of degree d . The norm of the determinant of the differential $\|\det(dR)\|$ computed with respect to Fubini-Study Kähler form ω satisfies $R^*(\omega^{\wedge k}) = \|\det(dR)\|^2 \omega^{\wedge k}$ and a direct computation in homogeneous coordinates shows:

$$\|\det(dR)\| = \frac{1}{d} \left| \det \left[\frac{\partial P_i}{\partial w_j} \right]_{i,j} \right| \times \left(\frac{|w_0|^2 + \cdots + |w_k|^2}{|P_0|^2 + \cdots + |P_k|^2} \right)^{k/2},$$

see [BJ00, Lemma 3.1]. Recall that the sum of the Lyapunov exponents of R is given by the formula:

$$\text{Lyap}(R) = \int \log \|\det(dR)\| d\mu_R,$$

where μ_R is the measure of maximal entropy of R . Observe also that $\log \|\det(dR)\|$ is locally the sum of a psh function and a smooth function so that the integral is converging since μ_R is locally the Monge-Ampère measure of a continuous function. It was proved in [BD99] that $\text{Lyap}(R) \geq \frac{k}{2} \log d$.

For an endomorphism $R = [P_0 : \cdots : P_k]$ defined over $\mathbb{P}_{\mathbb{C}((t))}^k$, then one uses a slightly different formula setting

$$\|\det(dR)\| = \left| \det \left[\frac{\partial P_i}{\partial w_j} \right]_{i,j} \right| \times \left(\frac{\max\{|w_0|, \dots, |w_k|\}}{\max\{|P_0|, \dots, |P_k|\}} \right)^2,$$

compare with [Ok11, (3.1)]. The sum of the Lyapunov exponents⁷ of the Chambert-Loir measure of R is defined analogously to the complex case by the formula:

$$\text{Lyap}(R) = \int \log \|\det(dR)\| d\mu_R.$$

This integral makes sense and is finite by Theorem 3.2.

Proof of Theorem C. Introduce the two (singular) admissible data \mathcal{G}_1 and \mathcal{G}_2 corresponding to the section $\det \left[\frac{\partial P_{i,t}}{\partial w_j} \right]_{i,j}$ and to the family of sections $P_{0,t}, \dots, P_{k,t}$ respectively. They are of degree $(2d-2)$ and d respectively. We have

$$\varphi_{\mathcal{G}_1} = \left| \det \left[\frac{\partial P_{i,t}}{\partial w_j} \right]_{i,j} \right| \times \frac{1}{\max\{|w_0|, \dots, |w_k|\}^{2d-2}}, \text{ and } \varphi_{\mathcal{G}_2} = \frac{\max\{|P_{0,t}|, \dots, |P_{k,t}|\}}{\max\{|w_0|, \dots, |w_k|\}^d},$$

so that

$$\log \|\det(R_t)\| = \int (\varphi_{\mathcal{G}_1} - 2\varphi_{\mathcal{G}_2} + \tilde{\varphi}) d\mu_{R_t}$$

where

$$\tilde{\varphi} = 2 \log \left(\frac{\max\{|P_{0,t}|, \dots, |P_{k,t}|\}}{|P_{0,t}|^2 + \dots + |P_{k,t}|^2} \right) - \log \left(\frac{\max\{|w_0|, \dots, |w_k|\}}{|w_0|^2 + \dots + |w_k|^2} \right),$$

is a bounded function on $\mathbb{P}_{\mathbb{C}}^k \times \mathbb{D}^*$. We now apply Theorem 4.2 to the uniform function φ_R , and we get the series of equalities

$$\begin{aligned} \text{Lyap}(R_t) &= \int (\varphi_{\mathcal{G}_1} - 2\varphi_{\mathcal{G}_2} + \tilde{\varphi}) d\mu_{R_t} \\ &= \frac{\log |t|^{-1}}{\log r} \int (\Phi_{\mathcal{G}_1} - 2\Phi_{\mathcal{G}_2}) d\text{MA}_{t,\text{hyb}}(\varphi_R) + O(1) \\ &= \frac{\log |t|^{-1}}{\log r} \int (g_{\mathcal{G}_1} - 2g_{\mathcal{G}_2}) d\text{MA}_{0,\text{hyb}}(\varphi_R) + o(\log |t|^{-1}) \\ &= \text{Lyap}(R) \frac{\log |t|^{-1}}{\log r} + o(\log |t|^{-1}). \end{aligned}$$

This concludes the proof. □

⁷one can define each individual Lyapunov exponent of R by looking at the limits $\frac{1}{n} \int \log \|\wedge^l(dR^n)\| d\mu_R$ as $n \rightarrow \infty$ for $l \in \{1, \dots, k\}$ which exist by Kingman's theorem.

5. QUESTIONS

5.1. Characterization of uniform functions. The notion of uniform functions depends on the choice of smooth positive metrization on \mathcal{L} . It would be interesting to explore if one can give a more intrinsic definition of uniform function not relying on the existence of an approximating sequence of model functions.

Let T be any positive closed $(1, 1)$ current on \mathcal{X} , and let E be any irreducible component of the central fiber of an scn model $p : \mathcal{X}' \rightarrow \mathcal{X}$. Then we set $g_T(x_E)$ to be the quotient of the Lelong number of T at a general point in E divided by the integer $b_E = \text{ord}(p^*\pi^*t)$.

Question 1. A function $\varphi : X \rightarrow \mathbb{R}$ is uniform iff

- it is continuous in a neighborhood of \bar{X}_r ;
- there exists a positive closed $(1, 1)$ current T on \mathcal{X}_r such that $T|_{X_r} = \omega + dd^c\varphi$;
- the function g_T extends continuously to $X_{\mathbb{C}((t)),r}$.

The forward implication is easy.

It was proved in [BFJ15] (see also [BGJ+16]) that one can solve the Monge-Ampère equation $\text{MA}_{\mathcal{L}}(g) = \mu$ for a suitable class of positive measures μ on $X_{\mathbb{C}((t)),r}$.

Question 2. Let $|\cdot|_{\mathcal{L}} e^{-g}$ be any continuous semi-positive metrization of $L_{\mathbb{C}((t)),r}$. Is it possible to find a uniform function φ such that $(\psi_{\text{NA}})_* \text{MA}_{\mathcal{L}}(g) = \text{MA}_{0,\text{hyb}}(\varphi)$?

5.2. Effective aspects. One can refine the estimates in §3.4 and prove that for any admissible data \mathcal{F} and \mathcal{G} with \mathcal{F} regular, then the function

$$(23) \quad \int \varphi_{\mathcal{G}} d(\mu_{\mathcal{F},t}) - \left(\int g_{\mathcal{G}} d(\mu_{\mathcal{F},\text{NA}}) \right) \frac{\log |t|^{-1}}{\log r}$$

extends continuously at $t = 0$. In other terms, the error terms in the convergence (17) is of the form $\frac{c+o(1)}{\log |t|^{-1}}$ for some $c \geq 0$.

It would be interesting to see whether the same kind of estimates can hold when $\varphi_{\mathcal{F}}$ is replaced by a general uniform function φ . In particular, when φ is the uniform function of a degenerating family of endomorphisms then (23) is expected to hold true. Observe that this would imply the first assertion of Conjecture 1 from the introduction.

5.3. Degeneration of volume forms. It would also be interesting to further investigate the relationship between the two convergence theorems of measures in the hybrid space given by Theorem A of the present paper and [BJ16, Theorem A]. Let us recall briefly the setting of the latter paper (we have changed slightly their notation so as to match with ours).

Suppose $X \rightarrow \mathbb{D}^*$ is a smooth and proper submersion, and let $\pi : \mathcal{X} \rightarrow \mathbb{D}$ be an scn model of X . To simplify the discussion we shall assume furthermore that \mathcal{X} is smooth and that there exists a relatively ample line bundle $\mathcal{L} \rightarrow \mathcal{X}$.

Let K_{X/\mathbb{D}^*} be the relative canonical line bundle over the punctured disk: in a trivialization (z_1, \dots, z_k, t) where $\pi(z, t) = t$, then sections of K_{X/\mathbb{D}^*} are k -forms $\alpha(z, t) dz_1 \wedge \dots \wedge dz_k$ with α holomorphic. Suppose that there exists a line bundle $\mathcal{K} \rightarrow \mathcal{X}$ whose restriction to X is equal to K_{X/\mathbb{D}^*} , and pick any smooth metric h on $K_{X/\mathbb{D}}$ that extends continuously to \mathcal{K} .

For any fixed $t \in \mathbb{D}^*$ one may consider the smooth volume form μ_t given locally by $\mu_t = \frac{\Omega \wedge \bar{\Omega}}{|\Omega|_h^2}$ where Ω is any local section of K_{X/\mathbb{D}^*} . The family of measures $\{\mu_t\}_{t \in \mathbb{D}^*}$ is in fact smooth, and S. Boucksom and M. Jonsson gave a precise asymptotic formula for the total

mass $\mu_t(X_t)$ as $t \rightarrow 0$. They also proved that the probability measures $\nu_t = \mu_t / \text{Mass}(\mu_t)$ converge to an explicit measure ν_{NA} in the hybrid space.

Let us fix any smooth positive metric on \mathcal{L} , and denote by ω its curvature form. The restriction $\omega_t := \omega|_{X_t}$ is a Kähler form for any $t \in \mathbb{D}^*$. Recall that $\delta = \int_{X_t} \omega_t^k$ is independent on t . Now for any fixed $t \in \mathbb{D}^*$ we may solve the Monge-Ampère equation $(\omega_t + dd^c \varphi_t)^k = \delta \nu_t$ and g_t is uniquely determined if we normalize it by the condition $\sup_{X_t} \varphi_t = 0$.

Question 3. Is it true that the family of functions φ_t is uniform in the sense of §4.1?

If the answer to the previous question is positive, then we may consider the associated function g on $X_{\mathbb{C}((t)),r}$ which defines a continuous semi-positive metrics on $L_{\mathbb{C}((t)),r}$ and Theorem 4.2 together with the results of [BJ16] imply $\text{MA}_{\mathcal{L}}(g) = \lim_t \nu_t = \delta \nu_{\text{NA}}$.

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