Nonsmooth modal analysis of piecewise-linear impact systems

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NONSMOOTH MODAL ANALYSIS OF PIECEWISE-LINEAR IMPACT SYSTEMS

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Summary Periodic solutions of autonomous and conservative second-order dynamical systems of finite dimension n undergoing a single unilateral contact condition are investigated in continuous time. The unilateral constraint is complemented with a purely elastic impact law conserving total energy. The dynamics is linear away from impacts. It is proven that the phase-space is primarily populated by one-dimensional continua of periodic solutions, generating an invariant manifold which can be understood as a *nonsmooth mode of vibration* in the context of vibration analysis. Additionally, it is shown that nonsmooth modes of vibration can be calculated by solving only $k-1$ equations where k is the number of impacts per period. Results are illustrated on a mass-spring chain whose last mass undergoes a contact condition with an obstacle.

MOTIVATION

Nonlinear modal analysis consists in calculating, in the phase space, two-dimensional manifolds which are continuous families of periodic trajectories [1]. When dealing with nonsmooth systems, *i.e.* systems subjected to impact and friction [2], the induced discontinuities are usually regularized [3]. The novelty of the proposed work here is to incorporate the discontinuities in the formulation to exhibit continuous families of periodic nonsmooth trajectories, thus leading to *nonsmooth modal analysis*. The results hold for all autonomous linear dynamical systems subjected to a single contact condition modeled through a unilateral linear scleronomic constraint complemented with a perfectly elastic impact law.

FORMULATION

The vector of generalized coordinates $\mathbf{x}$ is an element of $\mathbb{R}^n$ where n is the number of dofs of the system. During free flights, that is away from impacts, the dynamics is governed by $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{0}$ where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices. Introducing $\overline{\mathbf{x}} = [\mathbf{x}^\top, \dot{\mathbf{x}}^\top]^\top$, this differential equation can be reformulated in the first-order form as $\dot{\overline{\mathbf{x}}} = \mathbf{A}\overline{\mathbf{x}}$ hence if no impact occurs in time interval $[t, t']$, $\overline{\mathbf{x}}(t) = \mathbf{S}(t-t')\overline{\mathbf{x}}(t')$ with $\mathbf{S}(t) = \exp(t\mathbf{A})$. The signed contact distance (gap function) is $g(\mathbf{x}) = \mathbf{w}^\top\mathbf{x} + g_0$. It is shown that the impact law, written in terms of normal contact velocities $\gamma^+ = -\gamma^-$, reads in terms of as $\overline{\mathbf{x}}^+ = \mathbf{N}\overline{\mathbf{x}}^-$ where $\mathbf{N} = \mathbf{M}^{-1/2}(\mathbf{I} - 2\mathbf{r}\mathbf{r}^\top)\mathbf{M}^{1/2}$ and $\mathbf{r}$ is the unit vector given by $\mathbf{r}^\top = [\mathbf{0}_{1,n}, \mathbf{M}^{1/2}\mathbf{w}(\mathbf{w}^\top\mathbf{M}^{-1}\mathbf{w})^{-1/2}]^\top \in \mathbb{R}^{2n}$. We introduce the times of impact $t_0, \ldots, t_k$ with the convention $t_0 = 0$ and period $T = t_k$ so that k is the number of impacts per period (ipp). The dynamics is a succession of free flights of duration $t_i - t_{i-1}$ and impacts at t_i, $i \in [\![1, k]\!]$, so for $t \in [t_{i-1}, t_i)$:

$$\overline{\mathbf{x}}(t) = \mathbf{S}(t-t_i)\mathbf{N}\mathbf{S}(t_i - t_{i-1})\mathbf{N} \ldots \mathbf{N}\mathbf{S}(t-t_0)\overline{\mathbf{x}}(t_0). \tag{1}$$

This expression shows that there exists a linear mapping from a vector of initial conditions to the current state at t. Let u be the endomorphism of matrix $\overline{\mathbf{N}\mathbf{S}}(t_k - t_{k-1})\overline{\mathbf{N}} \ldots \overline{\mathbf{N}\mathbf{S}}(t_1 - t_0)$ in the canonical basis of $\mathbb{R}^{2n}$. The problem of finding periodic solutions of the dynamics with impact times $t_1, \ldots, t_k$ is formulated in three necessary conditions (NC) to be satisfied by $\overline{\mathbf{x}}_0$: Given $T \in \mathbb{R}^*$ and $k \in \mathbb{N}^*$, find $\overline{\mathbf{x}}_0 \in \mathbb{R}^{2n}$ and $t_1, \ldots, t_k$ such that:

$$\begin{cases} u(\overline{\mathbf{x}}_0) = \overline{\mathbf{x}}_0 & \text{NC1} \quad (2a) \\ \overline{\mathbf{x}} \text{ determined by } \overline{\mathbf{x}}_0 \text{ using (1) is such that:} & \\ \forall i \in [\![1, k]\!],\ \overline{g}(\overline{\mathbf{x}}(t_i)) = 0 & \text{NC2} \quad (2b) \\ \forall t \in [0, T],\ \overline{g}(\overline{\mathbf{x}}(t)) \geq 0 & \text{NC3} \quad (2c) \end{cases}$$

with $\overline{g}(\overline{\mathbf{x}}) := g(\mathbf{x})$. Eq. (2a) enforces periodicity through a sequence of free flights over (t_{i-1}, t_i) punctuated by perfectly elastic impacts at t_i, $i \in [\![1, k]\!]$. Eq. (2b) guarantees that the impact law only applies when the separating gap is closed. Ineq. (2c) is necessary to ensure that the unilateral contact condition is not violated during the free flights.

MATHEMATICAL RESULTS

We can show that there is an isomorphism φ between the invariant (maximal) subspace of u and the kernel of a known skew-symmetric matrix $\mathbf{\Pi}$ of size $k \times k$ and function of the impact times. This implies that NC1 is satisfied if and only if there exists a $\boldsymbol{\lambda} \in \mathbb{R}^k$ such that $\mathbf{\Pi}(t_1, \ldots, t_k)\boldsymbol{\lambda} = \mathbf{0}$, and then $\overline{\mathbf{x}}_0$ can be retrieved using $\overline{\mathbf{x}}_0 = \varphi(\boldsymbol{\lambda})$. We also know that NC2 is satisfied by $\overline{\mathbf{x}}$ if and only if $\mathbf{\Sigma}(t_1, \ldots, t_k)\varphi(\overline{\mathbf{x}}_0) = -g_0\mathbf{j}$ where $\mathbf{\Sigma}$ is a symmetric $k \times k$ matrix and $\mathbf{j} = [1, \ldots, 1]^\top \in \mathbb{R}^k$.

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The expressions of $\mathbf{\Pi}$, $\mathbf{\Sigma}$ and φ are known explicitly but are not provided here; they depend only on $\mathbf{M}$, $\mathbf{K}$, g and $t_1, \dots, t_k$. Satisfying NC1 and NC2 reduces to finding $t_1, \dots, t_k$ and $\boldsymbol{\lambda}$ such that $\mathbf{\Pi}\boldsymbol{\lambda} = \mathbf{0}$ and $\mathbf{\Sigma}\boldsymbol{\lambda} = -g_0\mathbf{j}$. Then, $\mathbf{\Sigma}$ is generically invertible so it suffices to find $t_1, \dots, t_k$ such that $\mathbf{\Pi}\mathbf{\Sigma}^{-1}\mathbf{j} = \mathbf{0}$, and then recover $\overline{\mathbf{x}}_0 = \varphi(-g_0\mathbf{\Sigma}^{-1}\mathbf{j})$. The non-generic cases exhibit very specific properties which are not described here. If $\overline{\mathbf{x}}_0$ satisfies NC1 and NC2, it is a solution iff it satisfies NC3. This last condition has to be tested numerically.

It is then proved that if $\overline{\mathbf{x}}_0$ is a solution for $g_0 \neq 0$ then, generically, it lies on a two-dimensional manifold of the phase space. As opposed to smooth nonlinear modes, here the manifolds feature discontinuities, corresponding to the discontinuities at impact times. The positions and velocities as a function of time emanating from the initial conditions $\overline{\mathbf{x}}_0$ can then be calculated. The results are summarized in Fig. 1.

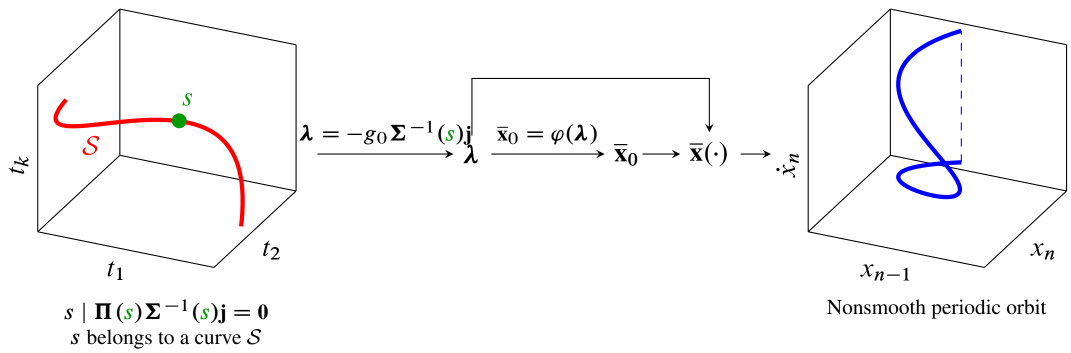

Figure 1: Summary of the mathematical results in the generic case with $g_0 \neq 0$. When $s = (t_1, \dots, t_k)$ travels along the red curve $\mathcal{S}$, the periodic solution describes a nonsmooth mode with k impacts per period.

APPLICATION TO A MECHANICAL SYSTEM

The mathematical developments prove the existence of nonsmooth modes yet also give a constructive way of calculating them by solving $k-1$ equations ($\mathbf{\Pi}\mathbf{\Sigma}^{-1}\mathbf{j} = \mathbf{0}$) independently of the number of dofs. Results are illustrated on a simple spring-mass chain whose free mass is subjected to an elastic Newton impact law, see Fig. 2. A nonsmooth mode with 7-ipp

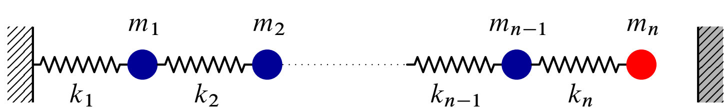

Figure 2: Simple spring-mass chain whose free mass is subjected to impacts.

was calculated following the methodology of Fig. 1 for this system with $n = 5$. The position of the last mass is represented for two initial conditions on the nonsmooth mode in Fig. 3, illustrating the possibilities given by the derivations.

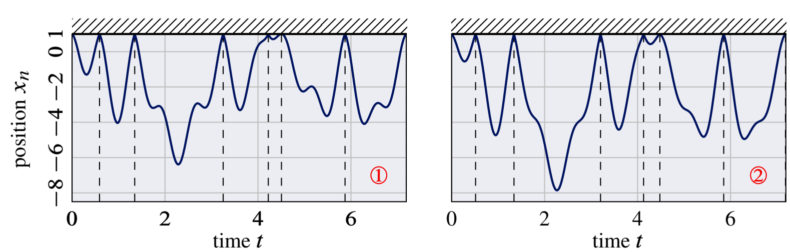

Figure 3: Two periodic trajectories with 7 ipp belonging to the same nonsmooth mode represented in $(\dot{x}_n, t)$. Dashed lines correspond to impact times.

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