Linear time algorithms for finding independent spanning trees on pyramid networks †

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Abstract

The use of independent spanning trees (ISTs) has scientific applications in fault-tolerant requirement in network protocols and secure message distributions. Most of the designs of ISTs are for those interconnection networks with vertex symmetric property, implying that one can find ISTs rooted on a designated vertex, and, by the vertex symmetry property of the given network, hence have solved the ISTs problem on any arbitrary vertex. The existence of asymmetry makes the ISTs problem even harder than its symmetric counterpart. In this paper, we propose linear time algorithms that solved 3-ISTs rooted at an arbitrary vertex of pyramid networks.

Keywords: independent spanning trees; interconnection networks; pyramid networks; graph algorithms.

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1 Introduction

The vertex set and the edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. Let $P = \langle v_1, v_2, \ldots, v_j \rangle$ be a *path* from v_1 to v_j in G. We use $I(P)$ to denote the set $V(P) \setminus \{v_1, v_j\}$. Two paths P and Q are called *disjoint*, denoted by $P \| Q$, if $V(P) \cap V(Q) = \emptyset$. We say that P and Q are *internally disjoint* if $I(P) \cap I(Q) = \emptyset$. A *spanning tree* of a graph G is a tree T such that $V(T) = V(G)$. Given a designated vertex $r \in V$ as a root vertex and a natural number k, spanning trees $T_1, T_2, \ldots, T_k$ of G are called k-independent spanning trees of G rooted at r (refered as k-ISTs for short) if the k paths connecting r and v, for any vertex v, in $T_1, T_2, \ldots, T_k$ are pairwise internally disjoint in G.

The use of independent spanning trees has scientific applications in fault-tolerant requirement in network protocols and secure message distributions [1]. For example, an efficient and reliable broadcasting can be achieved by sending k copies of the message rooted at the source vertex along k-ISTs such that the message can be broadcasted to all the other vertices in a network with at most $k-1$ faulty vertices. Although the ISTs problem is hard for general graphs, several results are known for some special classes of graphs, especially on interconnection networks, such as odd graphs [5], n-dimensional torus [9], product graphs [6], chordal rings [4], recursive circulant graphs [11], parity cubes [10], and hypercubes [8]. Most of the above results are for symmetric graphs implying that one can find ISTs rooted on a designated vertex, and, by the vertex symmetry property of the given graph, hence have solved the ISTs problem on any arbitrary vertex. However, the investigation into the ISTs problem on interconnection networks with no vertex symmetric property, to the best of our knowledge, are poor.

We focus attention on the construction of ISTs of a given pyramid network. Pyramid networks have potentially powerful architecture for many applications such as image processing, visualization, and data mining [3]. The major advantage of pyramids which is important for image processing systems is hierarchical abstracting and transferring the data from different directions and forward them toward the apex of a pyramid [7]. A pyramid network is a 3-connected graph. In [2], Cheriyan and Maheshwari showed that, for any 3-connected graph, 3-ISTs rooted at any vertex can be found in $O(|V||E|)$ time. In this paper, we propose linear time algorithms for finding 3-ISTs of a pyramid network rooted at an arbitrary vertex.

2 Preliminaries

All graphs considered here are finite, undirected, without loops and multiple edges. A *path* in a graph G is a single vertex or a sequence of distinct vertices $\langle v_1, v_2, \ldots, v_n\rangle$ such that (v_1, v_2), $(v_2, v_3), \ldots, (v_{n-1}, v_n)$ are edges of G and the v_i are distinct. A vertex of degree 0 is considered as an *isolated* vertex.

A $2^k \times 2^k$ *square mesh*, denoted by M_k, has the vertex set $V(M_k) = \{(x, y) \,|\, 0 \leq x, y \leq 2^k - 1\}$ where any two vertices (x_1, y_1) and (x_2, y_2) are connected by an edge iff $|x_1 - x_2| + |y_1 - y_2| = 1$. Each $M_k, k \geq 2$, can be partitioned into $2^{k-2} \times 2^{k-2}$ square submeshes M_2 and each M_2 is called a *cluster*. Every cluster can be further decomposed into 2×2 M_1 and each M_1 is said to be a *block*. Let v, v^+, v^{++}, v^- denote the four vertices of a block in a clockwise order starting from v, and B, B^+, B^{++}, B^- denote the four blocks of a cluster in a clockwise order starting from B.

Let $\mathbb{P}_n$ be an *n-dimensional pyramid* with the vertex set $\bigcup_{k=0}^{n} V_k$, where $V_k = \{(k; x, y) \,|\, 0 \leq x, y \leq 2^k - 1\}$. Vertex $(0; 0, 0)$ is the *apex* of $\mathbb{P}_n$ and also denoted as Δ. The subgraph induced by V_k is connected as an M_k and called *layer* k of $\mathbb{P}_n$. Let S_0 and S_1 denote the layers 0 and 1 of $\mathbb{P}_n$ with the vertex set $(0; 0, 0)$ and $\{(1; x, y) \,|\, 0 \leq x, y \leq 3\}$, respectively. Let $S(k; a, b)$ denote the cluster on layer k with the vertex set $\{(k; x, y) \,|\, 4a \leq x \leq 4a + 3 \text{ and } 4b \leq y \leq 4b + 3\}$, where $2 \leq k \leq n$ and $0 \leq a, b \leq 2^{k-2} - 1$. Notice that layer k has 2^{k-2} clusters. Vertex $(k; x, y)$ has exactly four *children* $(k + 1; 2x, 2y), (k + 1; 2x, 2y + 1), (k + 1; 2x + 1, 2y), (k + 1; 2x + 1, 2y + 1)$ in V_{k+1} and a *parent vertex* $(k - 1; \lfloor \frac{x}{2} \rfloor, \lfloor \frac{y}{2} \rfloor)$ in V_{k-1}. For simplicity, we shall use the abbreviated notation S_k for $S(k; a, b)$, whenever no confusion can arise. Let $p(v)$ denote the parent vertex of v and $l(v)$ the layer number of v. The block containing the four children of v, denoted $\mathcal{C}_{\mathcal{B}}(v)$, is called the *child block* of v. In an S_k, the block with the vertex set $\{p(v) | v \in V(S_{k+1})\}$ is called the *parent block* of S_{k+1} and denoted by $\mathcal{P}_{\mathcal{B}}(S_{k+1})$. An edge between v and $p(v)$ is called a *layer edge*, while every edge of a cluster is called a *cluster edge*. Every vertex on the shortest path from v to Δ is called an *ancestor* of v.

In a cluster S, every vertex of degree 4 is called an *inner vertex*, while the remaining vertices of S are said to be *outer vertices*. An outer vertex of degree 2 is also called a *corner vertex*. The subgraph induced by all inner vertices of S is an *i-cycle*, while the cycle induced by all of the outer vertices is an *o-cycle*. For any two vertices u and v in an o-cycle (respectively, i-cycle) O, each path from u to

v is called an $o-path$ (respectively, $i-path$). Vertex v is also labeled as u^{+d} (respectively, u^{-d}) if v can be visited from u via d edges of O in a clockwise order (respectively, a counterclockwise order). In paricular, $u = v$ if $d = 0$. Figure 1 depicts an example of a 2-layered pyramid $\mathbb{P}_2$. The dash lines indicate layer edges, while the solid lines are cluster edges. Each bold 4-cycle is a block. The top-right block of $S(2;0,0)$ has a corner vertex $(2;0,3)$, an inner vertex $(2;1,2)$ and outer vertices $(2;0,3)$, $(2;1,3)$ and $(2;0,2)$. The subgraph induced by the set $X = \{(2;1,1),(2;1,2),(2;2,2),(2;2,1)\}$ is the i-cycle of $S(2;0,0)$, and the o-cycle of $S(2;0,0)$ is induced by $V(S(2;0,0)) \setminus V(X)$.

3 Constructions of 3-ISTs Rooted at Different Layers

For a given root vertex r, we shall construct 3-ISTs of $\mathbb{P}_n$ rooted at r by considering the position of r to fulfill the vertex asymmetry of $\mathbb{P}_n$.

3.1 Rooted at the Apex

Before the construction of 3-ISTs of $\mathbb{P}_n$ rooted at an arbitrary vertex, we first define a vertex coloring on every vertex of $\mathbb{P}_n$ for selecting edges to be tree edges of a spanning tree. Let $\pi_0 : V \to \{0,1,2,3\}$ be a coloring on $\mathbb{P}_n$, where for each vertex $v \in V(\mathbb{P}_n - r)$, $\pi_0(v) \in \{0,1,2,3\}$, and the four vertices of each block are assigned distinct colors. For convenience, let ρ_1, ρ_2 and ρ_3 be three distinct color numbers in $\{1,2,3\}$ in the remaining text. In S_1, we arbitrarily select a vertex v as the corner vertex and then assign colors $0, \rho_1, \rho_2$ and ρ_3 to vertices v, v^+, v^- and v^{++}, respectively. Let $\hat{H}_{1,\rho_i}$ be the subgraph with the vertex set $V(S_1) \cup \{r\}$, where $1 \le i \le 3$. And let $E(\hat{H}_{1,\rho_1}) = E(\langle v^{++}, v^+, v, v^- \rangle) \cup \{(r, v^+)\}$, $E(\hat{H}_{1,\rho_2}) = E(\langle v^{++}, v^-, v, v^+ \rangle) \cup \{(r, v^-)\}$ and $E(\hat{H}_{1,\rho_3}) = E(\langle v^+, v^{++}, v^- \rangle) \cup \{(r, v), (r, v^{++})\}$. The π_0 coloring of a cluster is determined by the coloring of its parent block. For a cluster $S_k, k \ge 2$, let x, x^+, x^- and x^{++} be the four parent vertices of S_k such that $\pi_0(x) = 0$, $\pi_0(x^+) = \rho_1$, $\pi_0(x^-) = \rho_2$ and $\pi_0(x^{++}) = \rho_3$. We assign color $\pi_0(x^+)$ to the corner vertex of $\mathcal{C}_{\mathcal{B}}(x^+)$, to the two outer vertices incident to $\mathcal{C}_{\mathcal{B}}(x^+)$, and to the outer vertex incident to $\mathcal{C}_{\mathcal{B}}(x)$ in $\mathcal{C}_{\mathcal{B}}(x^-)$ (see Statement 2 of `HFinding Algorithm`). Statement 3 assigns color $\pi_0(x^-)$ to the corner vertex of $\mathcal{C}_{\mathcal{B}}(x^-)$, to the two outer vertices incident to $\mathcal{C}_{\mathcal{B}}(x^-)$, and to the outer vertex incident to $\mathcal{C}_{\mathcal{B}}(x^{++})$ in $\mathcal{C}_{\mathcal{B}}(x^+)$. In Statement 4, we assign color $\pi_0(x^{++})$ to each inner vertex of S_k. The remaining uncolored vertices of S_k are assigned color 0 in Statement 5.

We now define three graphs for any cluster, and then each subgraph will be selected to be

Input: Given a coloring on the vertices x, x^+, x^-, x^{++} of $\mathcal{P}_{\mathcal{B}}(S_k)$ as: $\pi_0(x) = 0$, $\pi_0(x^+) = \rho_1$, $\pi_0(x^-) = \rho_2$, $\pi_0(x^{++}) = \rho_3$.
Output: The colorings of blocks $\mathcal{C}_{\mathcal{B}}(x), \mathcal{C}_{\mathcal{B}}(x^+), \mathcal{C}_{\mathcal{B}}(x^-)$ and $\mathcal{C}_{\mathcal{B}}(x^{++})$ in S_k.
begin

Let the inner vertex of $\mathcal{C}_{\mathcal{B}}(x^{++})$ be $\hat{h}_{k,\rho_3}$ and the corner vertices of $\mathcal{C}_{\mathcal{B}}(x^+)$ and $\mathcal{C}_{\mathcal{B}}(x^-)$ be $\hat{h}_{k,\rho_1}$ and $\hat{h}_{k,\rho_2}$, respectively;

Assign color ρ_1 to every vertex of $\{\hat{h}_{k,\rho_1}, \hat{h}^{+2}_{k,\rho_1}, \hat{h}^{-2}_{k,\rho_1}, \hat{h}^{-5}_{k,\rho_1}\}$;

Assign color ρ_2 to every vertex of $\{\hat{h}_{k,\rho_2}, \hat{h}^{+2}_{k,\rho_2}, \hat{h}^{-2}_{k,\rho_2}, \hat{h}^{-5}_{k,\rho_2}\}$;

Assign color ρ_3 to every inner vertex of S_k;

Assign color 0 to every uncolored outer vertex of S_k;

$E(\hat{H}_{k,\rho_1}) = E(P_{k,\rho_1}) \cup \{(\hat{h}_{k,\rho_3}, \hat{h}^{+2}_{k,\rho_1}), (\hat{h}^{-1}_{k,\rho_3}, \hat{h}^{-1}_{k,\rho_1}), (\hat{h}^{+2}_{k,\rho_3}, \hat{h}^{-2}_{k,\rho_1}), (\hat{h}^{+1}_{k,\rho_3}, \hat{h}^{-5}_{k,\rho_1}), (\hat{h}_{k,\rho_1}, p(\hat{h}_{k,\rho_1}))\}$;

$E(\hat{H}_{k,\rho_2}) = E(P_{k,\rho_2}) \cup \{(\hat{h}_{k,\rho_3}, \hat{h}^{-2}_{k,\rho_2}), (\hat{h}^{-1}_{k,\rho_3}, \hat{h}^{-5}_{k,\rho_2}), (\hat{h}^{+2}_{k,\rho_3}, \hat{h}^{+2}_{k,\rho_2}), (\hat{h}^{+1}_{k,\rho_3}, \hat{h}^{-1}_{k,\rho_2}), (\hat{h}_{k,\rho_2}, p(\hat{h}_{k,\rho_2}))\}$;

$E(\hat{H}_{k,\rho_3}) = E(P_{k,\rho_3}) \cup \{(u,v) \in E(S_k) | \pi_0(u) = \rho_3, \pi_0(v) \neq \rho_3\} \cup \{(\hat{h}_{k,\rho_1}, \hat{h}^{-1}_{k,\rho_1}), (\hat{h}_{k,\rho_2}, \hat{h}^{-1}_{k,\rho_2}), (\hat{h}^{+3}_{k,\rho_1}, p(\hat{h}^{+3}_{k,\rho_1})), (\hat{h}^{-3}_{k,\rho_1}, p(\hat{h}^{-3}_{k,\rho_1})), (\hat{h}_{k,\rho_3}, p(\hat{h}_{k,\rho_3}))\}$;

end
`HFinding Algorithm`

a subgraph of one of the 3-ISTs of $\mathbb{P}_n$. Let $\hat{H}_i(k; a, b)$ denote the subgraph with the vertex set $V(S(k; a, b)) + V(\mathcal{P}_{\mathcal{B}}(S(k; a, b)))$, where $1 \leq i \leq 3$, $2 \leq k \leq n$ and $0 \leq a, b \leq 2^{k-2} - 1$. A vertex $v \in V(\hat{H}_i(k; a, b))$, denoted $\hat{h}_i(k; a, b)$, is called a *junction vertex* of $\hat{H}_i(k; a, b)$ if $\pi_0(v) = \pi_0(p(v)) = i$. The vertex of color ρ_i in S_1 is denoted by $\hat{h}_{1,\rho_i}$ for each i and the vertex of color 0 in S_1 is $\hat{h}_{1,0}$. For simplicity, we shall also use the abbreviated notation $\hat{h}_{k,i}$ and $\hat{H}_{k,i}$ for $\hat{h}_i(k; a, b)$ and $\hat{H}_i(k; a, b)$, respectively, and let $\hat{h}_k = \{\hat{h}_{k,1}, \hat{h}_{k,2}, \hat{h}_{k,3}\}$.

By the first statement of `HFinding Algorithm`, there are exactly two junction vertices on the o-cycle of S_k, while one junction vertex belongs to the i-cycle of S_k. In Statements 6-8, we define three subgraphs $\hat{H}_{k,\rho_1}$, $\hat{H}_{k,\rho_2}$ and $\hat{H}_{k,\rho_3}$ on S_k. $\hat{H}_{k,\rho_1}$ has the outer path

$P_{k,\rho_1} = \langle \hat{h}_{k,\rho_2}, \hat{h}^{-5}_{k,\rho_1}, \ldots, \hat{h}^{-1}_{k,\rho_1}, \hat{h}_{k,\rho_1}, \hat{h}^{+1}_{k,\rho_1}, \hat{h}^{+2}_{k,\rho_1}, \ldots, \hat{h}^{+5}_{k,\rho_1} \rangle$, and every inner vertex of S_k has exactly one edge incident to P_{k,ρ_1} in $\hat{H}_{k,\rho_1}$ (See Statement 6). $\hat{H}_{k,\rho_2}$ contains the outer path $P_{k,\rho_2} = \langle \hat{h}_{k,\rho_1}, \hat{h}^{-5}_{k,\rho_2}, \ldots, \hat{h}^{-1}_{k,\rho_2}, \hat{h}_{k,\rho_2}, \hat{h}^{+1}_{k,\rho_2}, \hat{h}^{+2}_{k,\rho_2}, \ldots, \hat{h}^{+5}_{k,\rho_2} \rangle$, and every inner vertex has an edge incident to P_{k,ρ_2} (See Statement 7). $\hat{H}_{k,\rho_3}$ has the inner path $P_{k,\rho_3} = \langle \hat{h}^{-3}_{k,\rho_3}, \hat{h}^{-2}_{k,\rho_3}, \hat{h}^{-1}_{k,\rho_3}, \hat{h}_{k,\rho_3} \rangle$ and every outer vertex $v \in V(S_k)$ has an edge incident to P_{k,ρ_3} if v is not a corner vertex. Among the four corner vertices of S_k, $\hat{h}_{k,\rho_1}$ and $\hat{h}_{k,\rho_2}$ are adjacent to $\hat{h}^{-1}_{k,\rho_1}$ and $\hat{h}^{-1}_{k,\rho_2}$, respectively, in $\hat{H}_{k,\rho_3}$. And the two corner vertices of color 0 have layer edges $(\hat{h}^{+3}_{k,\rho_1}, p(\hat{h}^{+3}_{k,\rho_1}))$ and $(\hat{h}^{-3}_{k,\rho_1}, p(\hat{h}^{-3}_{k,\rho_1}))$ in $\hat{H}_{k,\rho_3}$ (See Statement 8). For the connectivity of layers, the edge $(\hat{h}_{k,\rho_i}, p(\hat{h}_{k,\rho_i}))$ is used to connect $\hat{H}_{k,\rho_i}$ and

$\hat{H}_{k-1,\rho_i}$, where $1 \leq i \leq 3$ (See Statements 6-8). It can be seen that $P_{k,\rho_1}, P_{k,\rho_2}$ and P_{k,ρ_3} are pairwise disjoint. Figure 2 illustrates subgraphs $\hat{H}_{k,\rho_1}, \hat{H}_{k,\rho_2}$ and $\hat{H}_{k,\rho_3}$ of a cluster S_k on the condition that the parent vertex of the top-left block of S_k is of color 0. The bold edges are edges of subgraphs and the grey vertices represent junction vertices. The formal definitions of $\hat{H}_{k,\rho_1}, \hat{H}_{k,\rho_2}$ and $\hat{H}_{k,\rho_3}$ refer to `HFinding Algorithm`.

The following observation gives some properties of $\hat{h}_{k,\rho_1}$ and $\hat{h}_{k,\rho_2}$:

Observation 3.1. *Let $\hat{h}_{k,\rho_1}$ and $\hat{h}_{k,\rho_2}$ be on the o-cycle of S_k, where $2 \leq k \leq n$.*

(1) $\hat{h}^{+d}_{k,\rho_1} = \hat{h}^{-(6-d)}_{k,\rho_2}$ and $\hat{h}^{+d}_{k,\rho_2} = \hat{h}^{-(6-d)}_{k,\rho_1}$.

(2) If $v = \hat{h}^{+d}_{k,\rho_1}$, then v is an end-vertex of $\langle \hat{h}^{+d}_{k,\rho_1}, \hat{h}^{+(d-1)}_{k,\rho_1}, \ldots, \hat{h}_{k,\rho_1} \rangle$ and $\langle \hat{h}^{-(6-d)}_{k,\rho_2}, \hat{h}^{-(6-d-1)}_{k,\rho_2}, \ldots, \hat{h}_{k,\rho_2} \rangle$ in $\hat{H}_{k,\rho_1}$ and $\hat{H}_{k,\rho_2}$, respectively.

(3) If $v = \hat{h}^{-d}_{k,\rho_1}$, then v is an end-vertex of $\langle \hat{h}^{-d}_{k,\rho_1}, \hat{h}^{-(d-1)}_{k,\rho_1}, \ldots, \hat{h}_{k,\rho_1} \rangle$ and $\langle \hat{h}^{+(6-d)}_{k,\rho_2}, \hat{h}^{+(6-d-1)}_{k,\rho_2}, \ldots, \hat{h}_{k,\rho_2} \rangle$ in $\hat{H}_{k,\rho_1}$ and $\hat{H}_{k,\rho_2}$, respectively.

Proof. Let B_1 and B_2 be the block containing $\hat{h}_{k,\rho_1}$ and $\hat{h}_{k,\rho_2}$, respectively. According to the first statement of `HFinding Algorithm`, both $\hat{h}_{k,\rho_1}$ and $\hat{h}_{k,\rho_2}$ are corner vertices and B_1 is in fact B_2^{++}. Thus, $\hat{h}_{k,\rho_1} = \hat{h}^{+6}_{k,\rho_2}$ and $\hat{h}_{k,\rho_2} = \hat{h}^{+6}_{k,\rho_1}$. Besides, both (2) and (3) immediately follow (1).

Q. E. D.

Since the π_0 coloring of a cluster can be decided by its parent block, the coloring of S_2 is determined by the coloring of S_1. After that, the coloring and subgraphs of every cluster in $\mathbb{P}_n$ can then be established. We now continue with the derivation of building spanning subgraphs $T^0_{\rho_i}, i = 1, 2, 3$, on $\mathbb{P}_n$. Each $T^0_{\rho_i}$ is regarded as an n-layered graph. Let $V(T^0_{\rho_i}) = V(\mathbb{P}_n)$ and $E(T^0_{\rho_i}) = E(\hat{H}_{1,\rho_i}) \bigcup_{k=2}^{n} \bigcup_{a,b=0}^{2^{k-2}-1} E(\hat{H}_{\rho_i}(k; a, b))$ for $i = 1, 2, 3$.

To show that there are three pairwise internally disjoint paths from any vertex u_k to the apex r, we first find three pairwise internally disjoint paths from u_k to the block B containing $p(u_k)$.

Lemma 3.1. *If u_k is a vertex of $S_k, 2 \leq k \leq n$, and u'_{k-1} is a dummy vertex with dummy edges incident to every vertex of $\mathcal{P}_{\mathcal{B}}(S_k)$, then paths from u_k to u'_{k-1} in T^0_1, T^0_2 and T^0_3 are pairwise internally disjoint.*

Proof. Let $\hat{h}_{k,\rho_1}$ and $\hat{h}_{k,\rho_2}$ be outer vertices and $\hat{h}_{k,\rho_3}$ be an inner vertex. Since $\hat{h}_{k,\rho_1}, \hat{h}_{k,\rho_2}$ and

$\hat{h}_{k,\rho_3}$ belong to three different blocks, $p(\hat{h}_{k,\rho_1}), p(\hat{h}_{k,\rho_2})$ and $p(\hat{h}_{k,\rho_3})$ are distinct. Let P_{ρ_i} be the path from u_k to u'_{k-1} in $T^0_{\rho_i}$, where $1 \leq i \leq 3$. Each P_{ρ_i} can be regarded as a concatenation of subpaths $\langle u_k \rangle Q_{\rho_i} \langle t_i, u'_{k-1} \rangle$, where $t_i \in V(\mathcal{P}_{\mathcal{B}}(S_k))$. Let v_i be the neighbor of u_k in $\hat{H}_{k,\rho_i}$ for each i. Note that $v_i \neq v_j$ and $t_i \neq t_j$ if $i \neq j$. We shall prove $Q_{\rho_1}, Q_{\rho_2}, Q_{\rho_3}$ pairwise disjoint instead of showing $P_{\rho_1}, P_{\rho_2}, P_{\rho_3}$ pairwise internally disjoint.

Case 1. u_k is a corner vertex and $\pi_0(u_k) = 0$.

By `HFinding Algorithm`, u_k is either $\hat{h}^{+3}_{k,\rho_1}$ or $\hat{h}^{-3}_{k,\rho_1}$. From the definition of $E(\hat{H}_{k,\rho_3})$, $(u_k, p(u_k)) \in E(\hat{H}_{k,\rho_3})$. Then $P_{\rho_3} = \langle u_k, p(u_k), u'_{k-1} \rangle$ and thus Q_{ρ_3} is empty. So, $Q_{\rho_3} \| Q_{\rho_1}$ and $Q_{\rho_3} \| Q_{\rho_2}$. If $u_k = \hat{h}^{+3}_{k,\rho_1}$, by Observation 3.1 (2), u_k is an end-vertex of $\langle \hat{h}^{+3}_{k,\rho_1}, \hat{h}^{+2}_{k,\rho_1}, \hat{h}^{+1}_{k,\rho_1}, \hat{h}_{k,\rho_1} \rangle$ in $\hat{H}_{k,\rho_1}$ and is also an end-vertex of $\langle \hat{h}^{-3}_{k,\rho_2}, \hat{h}^{-2}_{k,\rho_2}, \hat{h}^{-1}_{k,\rho_2}, \hat{h}_{k,\rho_2} \rangle$ in $\hat{H}_{k,\rho_2}$. It means that $Q_{\rho_1} = \langle \hat{h}^{+2}_{k,\rho_1}, \hat{h}^{+1}_{k,\rho_1}, \hat{h}_{k,\rho_1} \rangle$ and $Q_{\rho_2} = \langle \hat{h}^{-2}_{k,\rho_2}, \hat{h}^{-1}_{k,\rho_2}, \hat{h}_{k,\rho_2} \rangle$. For the case $u_k = \hat{h}^{-3}_{k,\rho_1}$, by Observation 3.1 (3), u_k is an end-vertex of $\langle \hat{h}^{-3}_{k,\rho_1}, \hat{h}^{-2}_{k,\rho_1}, \hat{h}^{-1}_{k,\rho_1}, \hat{h}_{k,\rho_1} \rangle$ in $\hat{H}_{k,\rho_1}$ and an end-vertex of $\langle \hat{h}^{+3}_{k,\rho_2}, \hat{h}^{+2}_{k,\rho_2}, \hat{h}^{+1}_{k,\rho_2}, \hat{h}_{k,\rho_2} \rangle$ in $\hat{H}_{k,\rho_2}$. Then $Q_{\rho_1} = \langle \hat{h}^{-2}_{k,\rho_1}, \hat{h}^{-1}_{k,\rho_1}, \hat{h}_{k,\rho_1} \rangle$ and $Q_{\rho_2} = \langle \hat{h}^{+2}_{k,\rho_2}, \hat{h}^{+1}_{k,\rho_2}, \hat{h}_{k,\rho_2} \rangle$. It concludes that $Q_{\rho_1} \| Q_{\rho_2}$ for cases $u_k = \hat{h}^{+3}_{k,\rho_1}$ and $u_k = \hat{h}^{-3}_{k,\rho_1}$.

Case 2. $u_k = \hat{h}_{k,\rho_1}$.

Obviously, $P_{\rho_1} = \langle u_k, p(u_k), u'_{k-1} \rangle$ and thus Q_{ρ_1} is empty. So, $Q_{\rho_1} \| Q_{\rho_2}$ and $Q_{\rho_1} \| Q_{\rho_3}$. Furthermore, since $\hat{h}_{k,\rho_1}$ and $\hat{h}^{+5}_{k,\rho_2}$ are the two end-vertices of the path P_{k,ρ_2} and $\hat{h}_{k,\rho_2} \in V(P_{k,\rho_2})$, Q_{ρ_2} is a subpath of P_{k,ρ_2} and Q_{ρ_2} does not contain the vertex $\hat{h}^{+5}_{k,\rho_2}$. From the definition of $\hat{H}_{k,\rho_3}$, $\hat{h}_{k,\rho_1}$ and $\hat{h}^{-1}_{k,\rho_1}$ are adjacent and $\hat{h}^{-1}_{k,\rho_1}$ is adjacent to an inner vertex v in $E(\hat{H}_{k,\rho_3})$. Since $v \in V(P_{k,\rho_3})$, $Q_{\rho_3} - \hat{h}^{-1}_{k,\rho_1}$ is a subpath of P_{k,ρ_3}. That is, $Q_{\rho_3} - \hat{h}^{+5}_{k,\rho_2}$ is a subpath of P_{k,ρ_3} because $\hat{h}^{-1}_{k,\rho_1} = \hat{h}^{+5}_{k,\rho_2}$. Moreover, since $P_{k,\rho_3} \| P_{k,\rho_2}$ and $\hat{h}^{+5}_{k,\rho_2} \notin V(Q_{\rho_2})$, $Q_{\rho_3} \| Q_{\rho_2}$.

Case 3. $u_k = \hat{h}_{k,\rho_2}$.

The proof is similar to *Case 1* and is left to the reader.

Case 4. u_k is an inner vertex.

Notably, v_1 and v_2 belong to the same block. If $u_k = \hat{h}_{k,\rho_3}$, then Q_{ρ_3} is empty. By the definitions of $\hat{H}_{k,\rho_1}$ and $\hat{H}_{k,\rho_2}$, v_1 and v_2 are $\hat{h}^{+2}_{k,\rho_1}$ and $\hat{h}^{-2}_{k,\rho_2}$, respectively. Then $Q_{\rho_1} = \langle \hat{h}^{+2}_{k,\rho_1}, \hat{h}^{+1}_{k,\rho_1}, \hat{h}_{k,\rho_1} \rangle$ and $Q_{\rho_2} = \langle \hat{h}^{-2}_{k,\rho_2}, \hat{h}^{-1}_{k,\rho_2}, \hat{h}_{k,\rho_2} \rangle$. So, Q_{ρ_1}, Q_{ρ_2} and Q_{ρ_3} are pairwise disjoint. Similarly, we get pairwise disjoint paths Q_{ρ_1}, Q_{ρ_2} and Q_{ρ_3} for cases $u_k = \hat{h}^{-j}_{k,\rho_3}$, where $j = 1, 2, 3$.

Case 5. u_k is adjacent to a corner vertex.

Clearly, $v_i \in V(P_{k,\rho_i})$ for each i. That is, Q_{ρ_3} is an i-path, while both Q_{ρ_1} and Q_{ρ_2} are o-paths. So, $Q_{\rho_3} \| Q_{\rho_1}$ and $Q_{\rho_3} \| Q_{\rho_2}$. Vertex u_k is either $\hat{h}_{k,\rho_1}^{+d_1}$ or $\hat{h}_{k,\rho_1}^{-d_2}$, where $1 \leq d_1, d_2 \leq 5$ and $d_1, d_2 \neq 3$. For the former case, by Observation 3.1 (1), $Q_{\rho_1} = \langle \hat{h}_{k,\rho_1}^{+(d_1-1)}, \ldots, \hat{h}_{k,\rho_1} \rangle$ and $Q_{\rho_2} = \langle \hat{h}_{k,\rho_2}^{-(6-d_1-1)}, \ldots, \hat{h}_{k,\rho_2} \rangle$. Thus, $Q_{\rho_1} \| Q_{\rho_2}$. For the latter case, by Observation 3.1 (2), $Q_{\rho_1} = \langle \hat{h}_{k,\rho_1}^{-(d_2-1)}, \ldots, \hat{h}_{k,\rho_1} \rangle$ and $Q_{\rho_2} = \langle \hat{h}_{k,\rho_2}^{+(6-d_2-1)}, \ldots, \hat{h}_{k,\rho_2} \rangle$. Therefore, $Q_{\rho_1} \| Q_{\rho_2}$.

Q. E. D.

We further present three pairwise internally disjoint paths from a block B on a cluster S_k to $\mathcal{P}_{\mathcal{B}}(S_k)$.

Lemma 3.2. *Let B be a block of S_k, $2 \leq k \leq n$. If u'_k and u'_{k-1} are dummy vertices with dummy edges incident to every vertex of B and $\mathcal{P}_{\mathcal{B}}(S_k)$, respectively, then the paths from u'_k to u'_{k-1} in T_i^0, for $i = 1, 2, 3$, are pairwise internally disjoint.*

Proof. Let $V(B) = \{v_0, v_{\rho_1}, v_{\rho_2}, v_{\rho_3}\}$ and $V(\mathcal{P}_{\mathcal{B}}(S_k)) = \{t_0, t_{\rho_1}, t_{\rho_2}, t_{\rho_3}\}$, where $\pi_0(v_0) = \pi_0(t_0) = 0$ and $\pi_0(v_{\rho_i}) = \pi_0(t_{\rho_i}) = \rho_i$ for each i. Let P_{ρ_i} denote the path from u'_k to u'_{k-1} in $T_{\rho_i}^0$ and Q_{ρ_j} be the subpath of P_{ρ_j} on S_k for each i. Let $\hat{h}_{k,\rho_1}$ and $\hat{h}_{k,\rho_2}$ be outer vertices and $\hat{h}_{k,\rho_3}$ be an inner vertex. By the definition of $E(\hat{H}_{k,\rho_j})$, P_{ρ_j} must visit v_{ρ_j} and t_{ρ_j}, where $j = 1, 2$. We shall prove $Q_{\rho_1}, Q_{\rho_2}, Q_{\rho_3}$ pairwise disjoint instead of showing $P_{\rho_1}, P_{\rho_2}, P_{\rho_3}$ pairwise internally disjoint.

According to the `HFinding Algorithm`, $v_{\rho_1} \in \{\hat{h}_{k,\rho_1}, \hat{h}_{k,\rho_1}^{+2}, \hat{h}_{k,\rho_1}^{-2}, \hat{h}_{k,\rho_1}^{-5}\}$ and $v_{\rho_2} \in \{\hat{h}_{k,\rho_2}, \hat{h}_{k,\rho_2}^{+2}, \hat{h}_{k,\rho_2}^{-2}, \hat{h}_{k,\rho_2}^{-5}\}$. Actually, the ordered pair $(v_{\rho_1}, v_{\rho_2}) \in \{(\hat{h}_{k,\rho_1}, \hat{h}_{k,\rho_2}^{-5}), (\hat{h}_{k,\rho_1}^{+2}, \hat{h}_{k,\rho_2}^{-2}), (\hat{h}_{k,\rho_1}^{-2}, \hat{h}_{k,\rho_2}^{+2}), (\hat{h}_{k,\rho_1}^{-5}, \hat{h}_{k,\rho_2})\}$. Since $p(\hat{h}_{k,\rho_j})$ is t_{ρ_j} for each j, we get the ordered pair (Q_{ρ_1}, Q_{ρ_2}) as follows:

$$(Q_{\rho_1}, Q_{\rho_2}) = \begin{cases} (\langle \hat{h}_{k,\rho_1} \rangle, \langle \hat{h}_{k,\rho_2}^{-5}, \hat{h}_{k,\rho_2}^{-4}, \ldots, \hat{h}_{k,\rho_2} \rangle) & \text{if } (v_{\rho_1}, v_{\rho_2}) = (\hat{h}_{k,\rho_1}, \hat{h}_{k,\rho_2}^{-5}), \\ (\langle \hat{h}_{k,\rho_1}^{+2}, \hat{h}_{k,\rho_1}^{+1}, \hat{h}_{k,\rho_1} \rangle, \langle \hat{h}_{k,\rho_2}^{-2}, \hat{h}_{k,\rho_2}^{-1}, \hat{h}_{k,\rho_2} \rangle) & \text{if } (v_{\rho_1}, v_{\rho_2}) = (\hat{h}_{k,\rho_1}^{+2}, \hat{h}_{k,\rho_2}^{-2}), \\ (\langle \hat{h}_{k,\rho_1}^{-2}, \hat{h}_{k,\rho_1}^{-1}, \hat{h}_{k,\rho_1} \rangle, \langle \hat{h}_{k,\rho_2}^{+2}, \hat{h}_{k,\rho_2}^{+1}, \hat{h}_{k,\rho_2} \rangle) & \text{if } (v_{\rho_1}, v_{\rho_2}) = (\hat{h}_{k,\rho_1}^{-2}, \hat{h}_{k,\rho_2}^{+2}), \\ (\langle \hat{h}_{k,\rho_1}^{-5}, \hat{h}_{k,\rho_1}^{-4}, \ldots, \hat{h}_{k,\rho_1} \rangle, \langle \hat{h}_{k,\rho_2} \rangle) & \text{if } (v_{\rho_1}, v_{\rho_2}) = (\hat{h}_{k,\rho_1}^{-5}, \hat{h}_{k,\rho_2}). \end{cases}$$

It is clear that $Q_{\rho_1} \| Q_{\rho_2}$ for each case. Consider P_{ρ_3}. By the definition of $E(\hat{H}_{k,\rho_3})$, P_{ρ_3} visits either t_0 or t_{ρ_3}.

Case 1. P_{ρ_3} visits t_0.

By the definition of $E(\hat{H}_{k,\rho_3})$, $P_{\rho_3} = \langle u'_k, v_0, t_0, u'_{k-1} \rangle$ and thus $Q_{\rho_3} = \langle v_0 \rangle$. Clearly, $Q_{\rho_3} \| Q_{\rho_1}$ and $Q_{\rho_3} \| Q_{\rho_2}$ for every ordered pair (v_{ρ_1}, v_{ρ_2}).

Case 2. P_{ρ_3} visits both v_{ρ_3} and t_{ρ_3}.

Since every vertex of color ρ_3 is an inner vertex, Q_{ρ_3} is a subpath of P_{k,ρ_3} and is in fact an i-path. However, both Q_{ρ_1} and Q_{ρ_2} are o-paths. So, $Q_{\rho_3} \| Q_{\rho_1}$ and $Q_{\rho_3} \| Q_{\rho_2}$.

Case 3. P_{ρ_3} visits both v_0 and t_{ρ_3}.

If v_0 is a corner vertex, then v_0 must be $\hat{h}^{+3}_{k,\rho_1}$ and $t_{\rho_3} = p(v_0)$. Since, by the definition of $E(\hat{H}_{k,\rho_3})$, $(\hat{h}^{+3}_{k,\rho_1}, p(\hat{h}^{+3}_{k,\rho_1})) \in E(\hat{H}_{k,\rho_3})$, $P_{\rho_3} = \langle u'_k, v_0, t_{\rho_3}, u'_{k-1} \rangle$. That is, $Q_{\rho_3} = \langle v_0 \rangle$. So, $Q_{\rho_3} \| Q_{\rho_1}$ and $Q_{\rho_3} \| Q_{\rho_2}$ for every ordered pair (v_{ρ_1}, v_{ρ_2}). Otherwise, v_0 is not a corner vertex and so v_0 is adjacent to an inner vertex, say v. Since $v \in V(P_{k,\rho_3})$, $Q_{\rho_3} - v_0$ is a subpath of P_{k,ρ_3}. According to `HFinding Algorithm`, $v_0 \in \{\hat{h}^{-1}_{k,\rho_1}, \hat{h}^{-1}_{k,\rho_2}\}$. Since both Q_{ρ_1} and Q_{ρ_2} are o-paths and neither Q_{ρ_1} nor Q_{ρ_2} contains v_0 for every ordered pair (v_{ρ_1}, v_{ρ_2}), $Q_{\rho_3} \| Q_{\rho_1}$ and $Q_{\rho_3} \| Q_{\rho_2}$.

Q. E. D.

Theorem 3.1. *Graphs T^0_1, T^0_2 and T^0_3 are 3-ISTs of $\mathbb{P}_n$ rooted at apex.*

Proof. Let ρ_1, ρ_2 and ρ_3 be color numbers of the vertices v, v^+, v^- and v^{++} in S_1. Since $E(\hat{H}_{1,\rho_1}) = E(\langle v^{++}, v^+, v, v^- \rangle) \cup \{(r, v^+)\}$, $E(\hat{H}_{1,\rho_2}) = E(\langle v^{++}, v^-, v, v^+ \rangle) \cup \{(r, v^-)\}$ and $E(\hat{H}_{1,\rho_3}) = E(\langle v^+, v^{++}, v^- \rangle) \cup \{(r, v), (r, v^{++})\}$, every $\hat{H}_{1,\rho_i}, i = 1, 2, 3$, is a tree. According to `HFinding Algorithm`, every $\hat{H}_{k,i}$ is acyclic, where $2 \leq k \leq n$. For connectivity, each $\hat{h}_{k,i}$ has a layer edge $(\hat{h}_{k,i}, p(\hat{h}_{k,i})) \in E(\hat{H}_{k,i})$ for connecting subgraphs $\hat{H}_{k,i}$ and $\hat{H}_{k-1,i}$. It follows that T^0_i is a tree. Actually, T^0_i is a spanning tree of $\mathbb{P}_n$ due to $V(T^0_i) = V(\mathbb{P}_n)$.

For any vertex $u_k = (k; x, y) \in V(\mathbb{P}_n)$, we now want to find three pairwise internally disjoint paths from u_k to the apex r. The case $u_k = (0; 0, 0)$ is trivial. Consider $k = 1$. If $u_k = \hat{h}_{1,0}$, we get three pairwise internally disjoint paths $\langle u_k, \hat{h}_{1,\rho_1}, r \rangle$, $\langle u_k, \hat{h}_{1,\rho_2}, r \rangle$ and $\langle u_k, r \rangle$ from u_k to r in $T^0_{\rho_1}$, $T^0_{\rho_2}$ and $T^0_{\rho_3}$, respectively. Similarly, we get three pairwise internally disjoint paths $\langle u_k, r \rangle$, $\langle u_k, \hat{h}_{1,0}, \hat{h}_{1,\rho_2}, r \rangle$ and $\langle u_k, \hat{h}_{1,\rho_3}, r \rangle$ for the case $u_k = \hat{h}_{1,\rho_1}$; paths $\langle u_k, \hat{h}_{1,\rho_0}, \hat{h}_{1,\rho_1}, r \rangle$, $\langle u_k, r \rangle$ and $\langle u_k, \hat{h}_{1,\rho_3}, r \rangle$ for $u_k = \hat{h}_{1,\rho_2}$; and paths $\langle u_k, \hat{h}_{1,\rho_1}, r \rangle$, $\langle u_k, \hat{h}_{1,\rho_2}, r \rangle$ and $\langle u_k, r \rangle$ for $u_k = \hat{h}_{1,\rho_3}$.

For $k \geq 2$, let u'_j be the dummy vertex with dummy edges incident to every vertex of $\mathcal{P}_{\mathcal{B}}(S_{j+1})$, where $1 \leq j \leq k-1$. From Lemma 3.1, we get pairwise internally disjoint paths X_{k,ρ_i} from u_k to u'_{k-1} in $T^0_{\rho_i}$ for each $i = 1, 2, 3$. By applying Lemma 3.2, we get pairwise internally disjoint paths X_{j,ρ_i} from u'_j to u'_{j-1} in $T^0_{\rho_i}$ for each i, where $2 \leq j \leq k-1$. Moreover, since $(\hat{h}_{1,\rho_i}, r) \in E(\hat{H}_{1,\rho_i})$,

we get three pairwise internally disjoint paths from u_k to r.

Q. E. D.

Theorem 3.2. *The construction of 3-ISTs of $\mathbb{P}_n$ rooted at the apex can be solved in $O(|V|)$ time, where $|V|$ is the number of vertices of $\mathbb{P}_n$.*

Proof. At first, we assign color numbers to the four vertices of S_1 and build three subgraphs $\hat{H}_{1,i}$, $i = 1, 2, 3$, on S_1 in constant time. By using `HFinding Algorithm`, the coloring of every cluster is determined by the coloring of its parent block and costs constant time. So, the coloring of $\mathbb{P}_n$ costs $O(|V|)$ time. According to π_0 coloring, $\hat{h}_{k,1}$, $\hat{h}_{k,2}$ and $\hat{h}_{k,3}$, $k = 2, 3, \ldots, n$, of S_k can be decided in constant time and then every $\hat{H}_{k,i}$ is established in constant time. Since each edge set $E(T_i^0), i = 1, 2, 3$, has $|V| - 1$ edges and each edge of $E(T_i^0)$ can be checked in constant time, the finding of $E(T_i^0)$ costs $O(|V|)$ time. Therefore, the construction of 3-ISTs of $\mathbb{P}_n$ rooted at the apex is solved in $O(|V|)$ time.

Q. E. D.

3.2 Rooted at a Vertex on Layer 1

In S_1, we let r as the corner vertex and assign colors $0, \rho_1, \rho_2$ and ρ_3 to vertices r, r^+, r^- and r^{++}, respectively. Due to $r \in V(S_1)$, r^+, r^- and r^{++} are actually $\hat{h}_{1,\rho_1}, \hat{h}_{1,\rho_2}$ and $\hat{h}_{1,\rho_3}$, respectively. To color $V(\mathbb{P}_n)$, we also apply `HFinding Algorithm` iteratively on every block of $\mathbb{P}_n$. Since the coloring of a cluster can be decided by its parent block, the π_0 coloring of S_2 is determined by the coloring of S_1. In this way, the coloring and subgraphs of every cluster of $\mathbb{P}_n$ can be established.

Let $\hat{H}^1_{1,\rho_i}$ have the vertex set $V(S_1) \cup \{\Delta\}$. Let $E(\hat{H}^1_{1,\rho_1}) = E(\langle r^-, r^{++}, r^+, r\rangle) \cup \{(r^+, \Delta)\}$, $E(\hat{H}^1_{1,\rho_2}) = E(\langle r^+, r^{++}, r^-, r\rangle) \cup \{(r^-, \Delta)\}$ and $E(\hat{H}^1_{1,\rho_3}) = \{\{(v, \Delta)|v \in V(S_1)\}$. Notations S_k, $\hat{h}_{k,i}$ and $\hat{H}_{k,i}$, where $2 \leq k \leq n$ and $1 \leq i \leq 3$, refer to Section 3.1. Let $V(T_i^1) = V(\mathbb{P}_n)$ and $E(T_i^1) = E(\hat{H}^1_{1,i}) \bigcup_{k=2}^{n} \bigcup_{a,b=0}^{2^{k-2}-1} E(\hat{H}_i(k; a, b))$ for each i.

Theorem 3.3. *Graphs T_1^1, T_2^1 and T_3^1 are 3-ISTs of $\mathbb{P}_n$ rooted at a vertex of S_1.*

Proof. Let $0, \rho_1, \rho_2$ and ρ_3 be color numbers of the vertices r, r^+, r^- and r^{++} in S_1. Since $E(\hat{H}^1_{1,\rho_1}) = E(\langle r^-, r^{++}, r^+, r\rangle) \cup \{(r^+, \Delta)\}$, $E(\hat{H}^1_{1,\rho_2}) = E(\langle r^+, r^{++}, r^-, r\rangle) \cup \{(r^-, \Delta)\}$ and $E(\hat{H}^1_{1,\rho_3}) = \{\{(v, \Delta)|v \in V(S_1)\}$, every $\hat{H}^1_{1,i}, i = 1, 2, 3$, is a tree. According to `HFinding Algorithm`, every $\hat{H}_{k,i}$

is acyclic, where $2 \leq k \leq n$. For connectivity, each $\hat{h}_{k,i}$ has a layer edge $(\hat{h}_{k,i}, p(\hat{h}_{k,i})) \in E(\hat{H}_{k,i})$ for connecting subgraphs $\hat{H}_{k,i}$ and $\hat{H}_{k-1,i}$, where $3 \leq k \leq n$. In addition, each $\hat{h}_{2,i}$ has a layer edge $(\hat{h}_{2,i}, p(\hat{h}_{2,i})) \in E(\hat{H}_{2,i})$ for connecting subgraphs $\hat{H}_{2,i}$ and $\hat{H}^1_{1,i}$. It follows that T^1_i is a tree. Actually, T^1_i is a spanning tree of $\mathbb{P}_n$ due to $V(T^1_i) = V(\mathbb{P}_n)$.

For any vertex $u_k = (k; x, y) \in V(\mathbb{P}_n)$, we now want to find three pairwise internally disjoint paths from u_k to r. If u_k is Δ, by the definition of each $\hat{H}^1_{1,\rho_i}$, we have pairwise internally paths $\langle u_k, r^+, r\rangle$, $\langle u_k, r^-, r\rangle$ and $\langle u_k, r\rangle$ in $T^1_{\rho_1}$, $T^1_{\rho_2}$ and $T^1_{\rho_3}$, respectively, from u_k to r. The case $u_k = r$ is trivial. Consider the case $k = 1$ and $u_k \neq r$. If $u_k = r^{++}$, then paths $\langle u_k, r^+, r\rangle$, $\langle u_k, r^-, r\rangle$ and $\langle u_k, p(r), r\rangle$ are in $T^1_{\rho_1}$, $T^1_{\rho_2}$ and $T^1_{\rho_3}$, respectively, and are pairwise internally disjoint. Similarly we have three pairwise internally disjoint paths for the case u_k is a neighbor of r and leave the proof to the reader.

For $k \geq 2$, let u'_j be the dummy vertex with dummy edges incident to every vertex of $\mathcal{P}_{\mathcal{B}}(S_{j+1})$, where $1 \leq j \leq k-1$. From Lemma 3.1, we get pairwise internally disjoint paths X_{k,ρ_i} from u_k to u'_{k-1} in $T^1_{\rho_i}$ for each $i = 1, 2, 3$. By applying Lemma 3.2, we get pairwise internally disjoint paths X_{j,ρ_i} from u'_j to u'_{j-1} in $T^1_{\rho_i}$ for each i, where $2 \leq j \leq k-1$. Let X_{1,ρ_i} be the path from u'_1 to r in $T^1_{\rho_i}$ for each i. For each u'_j, $1 \leq j \leq k-1$, if v is a neighbor of u'_j and $\pi_0(v) = 0$, then, by the definition of each $E(\hat{H}_{j,\rho_i})$, either v or v^{++} can be on one of the paths in $\{X_{j,\rho_1}, X_{j,\rho_2}, X_{j,\rho_3}\}$. Let v_i be the neighbor of u'_1 in X_{1,ρ_i} for each i. Since $\pi_0(r) = 0$, by the definition of each $E(\hat{H}_{2,\rho_i})$, either r or r^{++} is a vertex in $\{v_1, v_2, v_3\}$. If $r \in \{v_1, v_2, v_3\}$, then we get three pairwise internally disjoint paths $\langle u'_1, r\rangle, \langle u'_1, r^+, r\rangle$ and $\langle u'_1, r^-, r\rangle$. Otherwise, we get three pairwise internally disjoint paths $\langle u'_1, r^{++}, \Delta, r\rangle, \langle u'_1, r^+, r\rangle$ and $\langle u'_1, r^-, r\rangle$. Therefore, we get three pairwise internally disjoint paths from u_k to r.

Q. E. D.

Theorem 3.4. *The construction of 3-ISTs of $\mathbb{P}_n$ rooted at a vertex on layer 1 can be solved in $O(|V|)$ time, where $|V|$ is the number of vertices of $\mathbb{P}_n$.*

Proof. Similar to the proof of Theorem 3.2.

Q. E. D.

3.3 Rooted at Vertices on Higher Layers

The construction of 3-ISTs rooted at r, $2 \leq l(r) \leq n$, has two stages: *Backbone Establishment Stage* and *Branching Stage*. In Backbone Establishment Stage, we shall first define subgraphs on every cluster S_k that contains an ancestor of r. Then dedicated layer edges of $\mathbb{P}_n$ are selected for connecting subgraphs of S_k and S_{k-1}, for each $k = 1, 2, \ldots, l(r)$, to establish tree backbones. We further do an expansion on tree backbones to build 3-ISTs of $\mathbb{P}_n$ rooted at r in Branching Stage. The detailed description of the two stages are given in the following two subsections.

3.3.1 Backbone Establishment Stage

A block is an *a-block* if it contains an ancestor of r (including r). Let A_k denote the a-block on layer k of $\mathbb{P}_n$, where $1 \leq k \leq l(r)$. Let S_k be the cluster containing an a-block on layer k and $\mathcal{BB}_r$ be the subgraph induced by the vertex set $\bigcup_{k=0}^{l(r)} V(S_k)$. A block B of $\mathcal{BB}_r$ is said to be a *b-block* if B is not an a-block. Every vertex of an a-block (respectively, a b-block) is called an *a-vertex* (respectively, a *b-vertex*). The corner vertex and inner vertex of A_k are denoted as $a_{k,0}$ and $a_{k,3}$, respectively, and $a^+_{k,0}$ and $a^-_{k,0}$ are also denoted by $a_{k,1}$ and $a_{k,2}$, respectively. Every $V(S_k)$ can be partitioned into four vertex subsets $U_{k,0}$, $U_{k,1}$, $U_{k,2}$ and $U_{k,3}$. Let $U_{k,0} = \{\, a_{k,0}, a^{+3}_{k,0}, a^{+6}_{k,0}, a^{+9}_{k,0} \,\}$, $U_{k,1} = \{\, a^{+1}_{k,0}, a^{+2}_{k,0}, a^{+5}_{k,0}, a^{+8}_{k,0} \,\}$, $U_{k,2} = \{\, a^{+11}_{k,0}, a^{+10}_{k,0}, a^{+7}_{k,0}, a^{+4}_{k,0} \,\}$ and $U_{k,3} = \{\, a_{k,3}, a^{-1}_{k,3}, a^{-2}_{k,3}, a^{-3}_{k,3} \,\}$. Notice that every $U_{k,i}$ contains exactly one vertex of each block of $\mathcal{BB}_r$.

Let $\pi_1 : V \to \{0, 1, 2, 3\}$ be a coloring on $\mathcal{BB}_r$, where for each vertex $v \in V(\mathcal{BB}_r)$, $\pi_1(v) \in \{0, 1, 2, 3\}$. For an S_k, $2 \leq k \leq l(r) - 1$, if A_k is the top-left block or top-right block of S_k, we assign color i to every vertex of $U_{k,i}$. Otherwise, we assign colors 0, 2, 1 and 3 to every vertex of $U_{k,0}$, $U_{k,1}$, $U_{k,2}$ and $U_{k,3}$, respectively. According to the above coloring rules for π_1, we get colorings Π_0, Π_1, Π_2 and Π_3 for a cluster with respect to an A_k locating at the top-left, top-right, bottom-left and bottom-right block of S_k, respectively. Figure 3 depicts the four possible π_1 colorings Π_0, Π_1, Π_2 and Π_3 of a cluster S_k. The block having grey vertices along with bold edges indicates an a-block.

Let $h_{k,i}, i = 0, 1, 2, 3$, denote the a-vertex of color i in A_k and $h_k = \{h_{k,1}, h_{k,2}, h_{k,3}\}$. The coloring of S_1 is the same as the coloring of A_2. That is, if v is the corner vertex of A_2, then we assign colors $\pi_1(v)$, $\pi_1(v^+)$, $\pi_1(v^-)$ and $\pi_1(v^{++})$ to $p(v)$, $p(v)^+$, $p(v)^-$ and $p(v)^{++}$, respectively. The π_1 coloring of $S_{l(r)}$ is treated as a special case of π_1 and determined by the π_1 coloring of $A_{l(r)-1}$ in

`RootColoring Algorithm`.

Input: π_1 coloring of $A_{l(r)-1}$.
Output: π_1 coloring of $S_{l(r)}$.
begin

Assign color 0 to all vertices of $U_{l(r),0}$;

Case $\pi_1(p(r)) \in \{0, 3\}$.
Assign color 3 to all vertices of $U_{l(r),3}$;
If $\pi_1(p(r)^+) = 1$, then assign color 1 to all vertices of $U_{l(r),1}$ and color 2 to $U_{l(r),2}$
else assign color 2 to every vertex of $U_{l(r),1}$ and color 1 to $U_{l(r),2}$;

Case $\pi_1(p(r)) = 1$.
Assign color 1 to all vertices of $U_{l(r),3}$;
If $\pi_1(p(r)^+) = 0$, then assign color 2 to all vertices of $U_{l(r),1}$ and color 3 to $U_{l(r),2}$
else assign color 3 to every vertex of $U_{l(r),1}$ and color 2 to $U_{l(r),2}$;

Case $\pi_1(p(r)) = 2$.
Assign color 2 to all vertices of $U_{l(r),3}$;
If $\pi_1(p(r)^+) = 0$, then assign color 1 to all vertices of $U_{l(r),1}$ and color 3 to $U_{l(r),2}$
else assign color 3 to every vertex of $U_{l(r),1}$ and color 1 to $U_{l(r),2}$;

Adjust the coloring of $A_{l(r)}$:
Assign color $\pi_1(r^{++})$ to $a_{l(r),0}$, and then assign color $\pi_1(r)$ to r^{++};
Assign color 0 to r;

end
`RootColoring Algorithm`

We are able to establish three tree backbones $TB_{r,i}$ on $\mathcal{BB}_r$ for each $i = 1, 2, 3$. Each $TB_{r,i}$ is regarded as an $l(r)$-layered graph. For a $v \in V(\mathcal{BB}_r)$, let $c_i(v)$ denote the child of v of color i. Let $H_{k,i}$ denote a subgraph of an S_k, where $1 \leq k \leq l(r)$ and $1 \leq i \leq 3$. Let $V(H_{1,i}) = \{h_{1,0}, h_{1,1}, h_{1,2}, h_{1,3}, \Delta\}$ and $E(H_{1,i}) = \{(h_{1,i}, \Delta)\}$. We define two o-paths

$P_{k,\rho_1} = \langle a_{k,1}, a_{k,1}^{+1}, a_{k,1}^{+2}, \ldots, a_{k,1}^{+9}\rangle$, $P_{k,\rho_2} = \langle a_{k,2}, a_{k,2}^{-1}, a_{k,2}^{-2}, \ldots, a_{k,2}^{-9}\rangle$ and an i-path $P_{k,\rho_3} = \langle a_{k,3}, a_{k,3}^{-1}, a_{k,3}^{-2}, a_{k,3}^{-3}\rangle$ for every S_k. For convenience, let a'_1, a'_2 and a'_3 denote as $a_{l(r),1}^{+1}, a_{l(r),2}^{-1}$ and $a_{l(r),3}^{-1}$, respectively, and let $\rho_i = \pi_1(a'_i)$ for each i. The three subgraphs H_{k,ρ_1}, H_{k,ρ_2} and H_{k,ρ_3} of S_k are defined in the following:

$E(H_{k,\rho_j}) = E(P_{k,\rho_j}) \cup \{(u, v) \in E(S_k) | u \notin V(A_k), \pi_1(u) = \rho_3 \text{ and } \pi_1(v) = \rho_j\} \cup \{(v, p(v)) | v \notin V(A_{l(r)}) \text{ and } \pi_1(v) = \rho_j\}$ for $j = 1, 2$, and

$E(H_{k,\rho_3}) = E(P_{k,\rho_3}) \cup \{(u, v) \in E(S_k) | u \notin V(A_k), \pi_1(u) = \rho_3 \text{ and } \pi_1(v) = \rho_1 \text{ or } \rho_2\}$

$\cup\{(v, p(v)) | v \notin V(A_{l(r)}) \text{ and } \pi_1(v) = \rho_3\} \cup \{(v, p(v)) | v \notin V(A_k) \text{ and } \pi_1(v) = 0\}$.

Among the four a-vertices of an $A_k, 2 \leq k \leq l(r) - 1$, h_{k,ρ_i} is adjacent to a b-vertex of H_{k,ρ_i} and its parent, while the other three a-vertices are isolated in H_{k,ρ_i}. For connectivity, every a-vertex of H_{k,ρ_i} is incident to a child of color ρ_i in H_{k+1,ρ_i}. Figure 4 illustrates subgraphs $H_{k,\rho_1}, H_{k,\rho_2}$ and

H_{k,ρ_3} of S_k. Bold edges are edges of subgraphs $H_{k,\rho_i}, 1 \leq i \leq 3$, and grey vertices are a-vertices.

Let $\mathcal{E}_1 = E(\langle r^+, r^{++}, r^-, r\rangle) \cup \{(r^-, p(r))\}$, $\mathcal{E}_2 = E(\langle r^-, r^{++}, r^+, r\rangle) \cup \{(r^+, p(r))\}$ and $\mathcal{E}_3 = \{(v, p(r)) | v \in V(A_{l(r)})\}$. For each $TB_{r,i}$, let R_i denote the subgraph with the vertex set $V(A_{l(r)}) \cup \{p(r)\}$ and an edge set in $\{\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3\}$, where $i = 1, 2, 3$. The formal definition of each $TB_{r,i}$ is in the following:

Definition 3.1.

(1) $E(R_{\pi_1(r^-)}) = \mathcal{E}_1$, $E(R_{\pi_1(r^+)}) = \mathcal{E}_2$ *and* $E(R_{\pi_1(r^{++})}) = \mathcal{E}_3$.

(2) $V(TB_{r,\rho_i}) = V(\mathcal{BB}_r)$,

$E(TB_{r,\rho_i}) = E(H_{1,\rho_i}) \cup E(R_{\rho_i}) \bigcup_{k=2}^{l(r)} \bigcup_{a,b=0}^{2^{k-2}-1} E(H_{\rho_i}(k; a, b))$, *where* $\rho_i = \pi_1(a'_i)$ *and* $i = 1, 2, 3$.

Lemma 3.3. *The graphs* TB_1, TB_2 *and* TB_3 *are spanning trees of* $\mathcal{BB}_r$ *rooted at* r.

Proof. Since $E(H_{1,i}) = \{(h_{1,i}, \Delta)\}, 1 \leq i \leq 3$, every $H_{1,i}$ is clearly acyclic. Since each $H_{k,i}$, $k = 2, 3, \ldots, l(r) - 1$, is constructed as a tree along with three isolated a-vertices on S_k, every $H_{k,i}$ is acyclic. In the $S_{l(r)}$, $A_{l(r)}$ and $p(r)$ form three tree subgraphs R_1, R_2 and R_3, and the remaining three b-blocks form tree subgraphs $H_{l(r),1}, H_{l(r),2}$ and $H_{l(r),3}$. In addition, $H_{l(r),i}$ and R_i share a common a-vertex $u \in \{a_{l(r),1}, a_{l(r),2}, a_{l(r),3}\}$. For connectivity, each vertex $v \notin V(A_{l(r)})$ of color i in $H_{k,i}$ has an edge $(v, p(v)) \in E(H_{k,i})$ for connecting subgraphs $H_{k,i}$ and $H_{k-1,i}$, where $k = 2, 3, \ldots, l(r)$ and each $p(v)$ is in fact an a-vertex of $H_{k-1,i}$. Thus, $TB_{r,i}$ is a tree. Actually, each $TB_{r,i}$ is a spanning tree because $V(TB_{r,i}) = V(\mathcal{BB}_r)$.

Q. E. D.

For convenience, let u'_j be the dummy vertex with dummy edges incident to every vertex of h_j for $j = 1, 2, \ldots, l(r) - 1$ and $u'_{l(r)}$ be the dummy vertex with dummy edges incident to a'_1, a'_2 and a'_3 in the remaining text of this section.

Lemma 3.4. *The paths from* u'_{k-1} *to* u'_k *in each* $TB_{r,i}$ *are pairwise internally disjoint, where* $1 \leq i \leq 3$ *and* $2 \leq k \leq l(r) - 1$.

Proof. Let $a_{k,0}^{+1}$ and $a_{k,0}^{+11}$ be h_{k,ρ_1} and h_{k,ρ_2}, respectively. Let P_{ρ_i} be the path from $c_{\rho_i}(h_{k-1,\rho_i})$ to h_{k,ρ_i} in H_{k,ρ_i} for $i = 1, 2, 3$. Since $c_{\rho_i}(h_{k-1,\rho_i}) \in V(P_{k,\rho_i})$, P_{ρ_i} is a subpath of P_{k,ρ_i}. Then $P_{\rho_3} \| P_{\rho_1}$ and $P_{\rho_3} \| P_{\rho_2}$ because $P_{k,\rho_3} \| P_{k,\rho_1}$ and $P_{k,\rho_3} \| P_{k,\rho_2}$. It remains to prove $P_{\rho_1} \| P_{\rho_2}$.

We first consider the case when $k = 2$. By the definition of the π_1 coloring, the coloring of S_1 is the same as the coloring of A_2. So, $a^{+2}_{k,0} = c_{\rho_1}(h_{1,\rho_1})$ and $a^{+10}_{k,0} = c_{\rho_2}(h_{1,\rho_2})$. Thus, we get disjoint paths $P_{\rho_1} = \langle a^{+2}_{k,0}, a^{+1}_{k,0}\rangle$ and $P_{\rho_2} = \langle a^{+10}_{k,0}, a^{+11}_{k,0}\rangle$. For the case $3 \leq k \leq l(r) - 1$, if $c_{\rho_1}(h_{k-1,\rho_1})$ is an a-vertex, $P_{\rho_1} = \langle h_{k,\rho_1}\rangle$. By the definition of H_{k,ρ_2}, $h_{k,\rho_1} \notin V(H_{k,\rho_2})$. That is, $h_{k,\rho_1} \notin V(P_{\rho_2})$. Thus, $P_{\rho_1} \| P_{\rho_2}$. Due to the same reason, P_{ρ_1} and P_{ρ_2} are disjoint for the case that $c_{\rho_2}(h_{k-1,\rho_2})$ is an a-vertex. Consider the case that both $c_{\rho_1}(h_{k-1,\rho_1})$ and $c_{\rho_1}(h_{k-1,\rho_2})$ are b-vertices. Since h_{k-1,ρ_1} and h_{k-1,ρ_2} are not adjacent, $\mathcal{C}_{\mathcal{B}}(h_{k-1,\rho_1})$ and $\mathcal{C}_{\mathcal{B}}(h_{k-1,\rho_2})$ must be two non-adjacent b-blocks of S_k. So, $\mathcal{C}_{\mathcal{B}}(h_{k-1,\rho_1})$ is either A^-_k or A^+_k.

Case 1. $\mathcal{C}_{\mathcal{B}}(h_{k-1,\rho_1}) = A^-_k$.

We first assume that A_k is the top-left block of S_k. Then $\mathcal{C}_{\mathcal{B}}(h_{k-1,\rho_1})$ is the bottom-left block of S_k and hence h_{k-1,ρ_1} is the bottom-left vertex of A_{k-1}. Based on the coloring rules of π_1, every vertex of $U_{k,i}$ is assigned the color number i when A_k is the top-left block of S_k. So, $\pi_1(h_{k,\rho_1}) = \pi_1(a^{+1}_{k,0}) = 1$. Then $\rho_1 = 1$ and $\pi_1(h_{k-1,\rho_1}) = 1$. However, every bottom-left vertex of an a-block must be of color 0 (in Π_2), 3 (in Π_1) or 2 (in Π_0 and Π_3) which contradicts the fact that h_{k-1,ρ_1} is the bottom-left vertex of A_{k-1}. This implies that A_k can not be the top-left block of S_k. Due to the same reason, contradiction occurs on the condition that A_k is the top-right block, bottom-left block or bottom-right block of S_k.

Case 2. $\mathcal{C}_{\mathcal{B}}(h_{k-1,\rho_1}) = A^+_k$.

Since, by the definition of $U_{k,1}$, $\pi_1(a^{+1}_{k,0}) = \pi_1(a^{+2}_{k,0})$, it follows that $a^{+2}_{k,0}$ is the unique vertex of color ρ_1 in $\mathcal{C}_{\mathcal{B}}(h_{k-1,\rho_1})$ and hence $c_{\rho_1}(h_{k-1,\rho_1})$ must be $a^{+2}_{k,0}$. So, $P_{\rho_1} = \langle a^{+2}_{k,0}, a^{+1}_{k,0}\rangle$. We also get $P_{\rho_2} = \langle a^{+10}_{k,0}, a^{+11}_{k,0}\rangle$ because both $a^{+10}_{k,0}$ and $a^{+11}_{k,0}$ are in $U_{k,2}$ and are of the same color. Therefore, $P_{\rho_1} \| P_{\rho_2}$.

Q. E. D.

For any vertex v of $\mathcal{BB}_r$, we want to show the paths from v to r in TB_1, TB_2 and TB_3 are pairwise internally disjoint. Each path from v to r in a $TB_{r,i}, 1 \leq i \leq 3$, can be regarded as a concatenation of three subpaths, including the subpath from v to its nearest a-vertex $h_{k,i}, l(v) \leq l(k)$ (see Lemma 3.5), the subpath from $h_{k,i}$ to $h_{l(r)-1,i}$ (see Lemma 3.3), the subpath from $h_{l(r)-1,i}$ to a vertex a of $A_{l(r)}$ (see Corollaries 3.1 and 3.2) and the subpath from a to r (see Lemma 3.7). We are ready to give proofs for the above subpaths.

Lemma 3.5. *If u_k is a b-vertex of S_k, $2 \leq k \leq l(r)-1$, then the paths from u_k to u'_k in each $TB_{r,i}, i = 1, 2, 3$, are pairwise internally disjoint.*

Proof. Let $a_{k,0}^{+1}$ and $a_{k,0}^{+11}$ be h_{k,ρ_1} and h_{k,ρ_2}, respectively. Since each $TB_{r,\rho_i}, i = 1, 2, 3$, by Lemma 3.3, is a spanning tree of $\mathcal{BB}_r$, we have the path P_{ρ_i} from u_k to h_{k,ρ_i} in TB_{r,ρ_i} for each i. We first prove $V(P_{\rho_3}) \cap V(P_{\rho_j}) = \{u_k\}$, where $j = 1, 2$. If u_k is a corner vertex, then $\pi_1(u_k) = 0$ and u_k^{++} is an inner vertex of color ρ_3. By the definition of H_{k,ρ_3}, $(u_k, p(u_k)), (u_k^{++}, p(u_k)) \in E(H_{k,\rho_3})$. Thus, $\langle u_k, p(u_k), u_k^{++}\rangle$ is a subpath of P_{ρ_3}. Since $u_k^{++} \in V(P_{k,\rho_3})$, $P_{\rho_3} - u_k - p(u_k)$ is a subpath of P_{k,ρ_3}. In contrast, P_{ρ_j} is a subpath of P_{k,ρ_j} because $u_k \in V(P_{k,\rho_j})$. Moreover, since $p(u_k) \notin V(P_{k,\rho_j})$ and $P_{k,\rho_3} \| P_{k,\rho_j}$, $V(P_{\rho_3}) \cap V(P_{\rho_j}) = \{u_k\}$. Otherwise, u_k is either an inner vertex or adjacent to an inner vertex. Then $P_{\rho_j} - u_k$ and $P_{\rho_3} - u_k$ are subpaths of P_{k,ρ_j} and P_{k,ρ_3}, respectively. Therefore, $V(P_{\rho_3}) \cap V(P_{\rho_j}) = \{u_k\}$. It remains to prove $V(P_{\rho_1}) \cap V(P_{\rho_2}) = \{u_k\}$.

Case 1. u_k is an outer vertex.

Let $u_k = a_{k,0}^{+d}$, where $2 \leq d \leq 10$. Since $h_{k,\rho_1} = a_{k,0}^{+1}$ and $h_{k,\rho_2} = a_{k,0}^{+11}$, $P_{\rho_1} = \langle a_{k,0}^{+d}, a_{k,0}^{+(d-1)}, \cdots, a_{k,0}^{+1}\rangle$ and $P_{\rho_2} = \langle a_{k,0}^{+d}, a_{k,0}^{+(d+1)}, \cdots, a_{k,0}^{+11}\rangle$. Clearly, $V(P_{\rho_1}) \cap V(P_{\rho_2}) = \{u_k\}$.

Case 2. u_k is an inner vertex.

Every inner vertex is of color ρ_3 and has neighbors $a_{k,0}^{+d}$ ($\in U_{k,1}$) and $a_{k,0}^{+(d+2)}$ ($\in U_{k,2}$) of colors ρ_1 and ρ_2, respectively, where $d \in \{2, 5, 8\}$. By the definitions of H_{k,ρ_1} and H_{k,ρ_2}, $(u_k, a_{k,0}^{+d}) \in E(H_{k,\rho_1})$ and $(u_k, a_{k,0}^{+(d+2)}) \in E(H_{k,\rho_2})$. Then paths $P_{\rho_1} = \langle u_k, a_{k,0}^{+d}, a_{k,0}^{+(d-1)}, \cdots, a_{k,0}^{+1}\rangle$ and $P_{\rho_2} = \langle u_k, a_{k,0}^{+(d+2)}, a_{k,0}^{+(d+3)}, \cdots, a_{k,0}^{+11}\rangle$ share the common vertex u_k.

Q. E. D.

The next result follows immediately.

Corollary 3.1. *If $u_{l(r)}$ is a b-vertex of $S_{l(r)}$, then the paths from $u_{l(r)}$ to $u'_{l(r)}$ in each $TB_{r,i}, i = 1, 2, 3$, are pairwise internally disjoint.*

Lemma 3.6. *If u_k is an a-vertex of S_k, $1 \leq k \leq l(r)-2$, then the paths from u_k to u'_{k+1} in each $TB_{r,i}, i = 1, 2, 3$, are pairwise internally disjoint.*

Proof. Let $a_{k+1,0}^{+1}$ and $a_{k+1,0}^{+11}$ be h_{k+1,ρ_1} and h_{k+1,ρ_2}, respectively. Let B be the block containing $c_{\rho_i}(u_k)$ for $i = 1, 2, 3$. By the definition of $E(H_{k+1,\rho_i})$, the edge $(u_k, c_{\rho_i}(u_k)) \in E(H_{k+1,\rho_i})$ for each i. If B is an a-block, paths $\langle u_k, c_{\rho_i}(u_k), u'_{k+1}\rangle$ in TB_{r,ρ_i}, for $i = 1, 2, 3$, are pairwise internally disjoint.

For the case B is a b-block, let P_{ρ_i} be the path from $c_{\rho_i}(u_k)$ to h_{k+1,ρ_i} in H_{k+1,ρ_i}. Since $c_{\rho_i}(u_k) \in V(P_{k+1,\rho_i})$, P_{ρ_i} is a subpath of P_{k+1,ρ_i}. Recall that P_{k+1,ρ_3} is an $i-$path, whereas both P_{k+1,ρ_1} and P_{k+1,ρ_2} are $o-$paths. Thus, $P_{\rho_3} \| P_{\rho_1}$ and $P_{\rho_3} \| P_{\rho_2}$. It remains to prove $P_{\rho_1} \| P_{\rho_2}$. Since $c_{\rho_1}(u_k) \in U_{k+1,1}$ and $c_{\rho_2}(u_k) \in U_{k+1,2}$, we let $a_{k+1,0}^{+d}$ and $a_{k+1,0}^{+(d+2)}$ be $c_{\rho_1}(u_k)$ and $c_{\rho_2}(u_k)$, respectively, where $d \in \{2,5,8\}$. Then paths $P_{\rho_1} = \langle a_{k+1,0}^{+d}, a_{k+1,0}^{+(d-1)}, \cdots, a_{k+1,0}^{+1} \rangle$ and $P_{\rho_2} = \langle a_{k+1,0}^{+(d+2)}, a_{k+1,0}^{+(d+3)}, \cdots, a_{k+1,0}^{+11} \rangle$ are disjoint.

Q. E. D.

Corollary 3.2. *If $u_{l(r)-1}$ is an a-vertex of $S_{l(r)-1}$, then the paths from $u_{l(r)-1}$ to $u'_{l(r)}$ in each $TB_{r,i}, i = 1,2,3,$ are pairwise internally disjoint.*

Lemma 3.7. *The paths from $u'_{l(r)}$ to r in each $TB_{r,i}$, for $i = 1,2,3,$ are pairwise internally disjoint.*

Proof. Let $\pi_1(a'_i) = \rho_i$ for $i = 1,2,3$. Recall that $(a_{l(r),i}, a'_i) \in E(H_{l(r),\rho_i})$. Let $P_{\rho_i} \subseteq R_{\rho_i}$ be the path from $a_{l(r),i}$ to r in TB_{r,ρ_i} for each i.

Case 1. r is $a_{l(r),1}$.

Intuitively, $P_{\rho_1} = \langle r \rangle$. Vertices r^+, r^- and r^{++} are $a_{l(r),3}$, $a_{l(r),0}$ and $a_{l(r),2}$, respectively. Since $\pi_1(r^+) = \rho_3$ and $E(R_{\pi_1(r^+)}) = \mathcal{E}_2$, $E(R_{\rho_3}) = \mathcal{E}_2$. Then $P_{\rho_3} = \langle r^+, r \rangle$ reveals the fact that $P_{\rho_3} \subseteq R_{\rho_3}$. Recall that we assign color $\pi_1(r^{++})$ to $a_{l(r),0}$ in `RootColoring Algorithm`. So $\pi_1(r^-) = \rho_2$. Furthermore, since $E(R_{\pi_1(r^-)}) = \mathcal{E}_1$, it follows that $E(R_{\rho_2}) = \mathcal{E}_1$. Then, $P_{\rho_2} = \langle r^{++}, r^-, r \rangle$ due to $P_{\rho_2} \subseteq R_{\rho_2}$. Clearly, P_{ρ_1}, P_{ρ_2} and P_{ρ_3} are pairwise internally disjoint.

Case 2. $r \in \{a_{l(r),0}, a_{l(r),2}, a_{l(r),3}\}$.

Similar to the proof of *Case 1.*

Q. E. D.

Lemma 3.8. *The paths from u'_k to r in each $TB_{r,i}$ are pairwise internally disjoint, where $1 \leq k \leq l(r) - 1$ and $1 \leq i \leq 3$.*

Proof. Let $\rho_i = \pi_1(a'_i)$ for $i = 1,2,3$. By Lemma 3.4, we have pairwise internally disjoint paths X_{j-1,ρ_i} from u'_{j-1} to u'_j, where $1 \leq i \leq 3$, $k+1 \leq j \leq l(r) - 1$ and $h_{j-1,\rho_i}, h_{j,\rho_i} \in V(X_{j-1,\rho_i})$. For each i, let $X_{l(r)-1,\rho_i}$ be the path from $u'_{l(r)-1}$ to $u'_{l(r)}$ and let $X_{l(r),\rho_i}$ be the path from $u'_{l(r)}$ to r in TB_{r,ρ_i}. Let $Y_{l(r)-1,\rho_i}$ be the subpath of $X_{l(r)-1,\rho_i}$ from $h_{l(r)-1,\rho_i}$ to a'_i in TB_{r,ρ_i} for each i. Notably,

$h_{l(r)-1,\rho_{i_1}}, a'_{i_1} \notin V(X_{l(r)-1,\rho_{i_2}})$ if $i_1 \neq i_2$. We shall list pairwise internally disjoint paths from $u'_{l(r)-1}$ to r in each TB_{r,ρ_i} to complete the proof.

Case 1. $\pi_1(p(r)) = 0$.

From `RootColoring Algorithm`, every vertex of $U_{l(r),1}, U_{l(r),2}$ and $U_{l(r),3}$ is assigned the color $\pi_1(p(r)^+), \pi_1(p(r)^-)$ and $\pi_1(p(r)^{++})$, respectively. So, $\pi_1(a'_1) = \pi_1(p(r)^+)$, $\pi_1(a'_2) = \pi_1(p(r)^-)$ and $\pi_1(a'_3) = \pi_1(p(r)^{++})$, implying $p(r)^+, p(r)^-$ and $p(r)^{++}$ are $h_{l(r)-1,\rho_1}$,$h_{l(r)-1,\rho_2}$ and $h_{l(r)-1,\rho_3}$, respectively. Since $c_{\rho_1}(h_{l(r)-1,\rho_1}) = a'_1$ and $c_{\rho_2}(h_{l(r)-1,\rho_2}) = a'_2$, it follows that $Y_{l(r)-1,\rho_1} = \langle h_{l(r)-1,\rho_1}, a'_1 \rangle$ and $Y_{l(r)-1,\rho_2} = \langle h_{l(r)-1,\rho_2}, a'_2 \rangle$. Since $h_{l(r)-1,\rho_1}, h_{l(r)-1,\rho_2}, a'_1, a'_2 \notin V(Y_{l(r)-1,\rho_3})$, $Y_{l(r)-1,\rho_1}$, $Y_{l(r)-1,\rho_2}$ and $Y_{l(r)-1,\rho_3}$ are pairwise disjoint. Moreover, by Lemma 3.7, we have pairwise internally disjoint paths $X_{l(r),\rho_i}$ from $u'_{l(r)}$ to r, where $1 \leq i \leq 3$ and $a'_i \in V(X_{l(r),\rho_i})$.

Case 2. $\pi_1(p(r)) = 1$.

Vertex $p(r)$ is $h_{l(r)-1,1}$. By the definition of R_1, the path from $p(r)$ to r in $TB_{r,1}$, denoted P_1, is a subgraph of R_1. From `RootColoring Algorithm`, every vertex of $U_{l(r),3}$ is assigned color 1. If $\pi_1(p(r)^+) = 0$, every vertex of $U_{l(r),1}$ and $U_{l(r),2}$ is assigned color 2 and 3, respectively. So, $p(r)^-$ and $p(r)^{++}$ are $h_{l(r)-1,3}$ and $h_{l(r)-1,2}$, respectively. Then $Y_{l(r)-1,3} = \langle p(r)^-, a'_2 \rangle$. Since $p(r)^-, a'_2 \notin V(Y_{l(r)-1,\rho_3})$, $Y_{l(r)-1,2}$ and $Y_{l(r)-1,3}$ are disjoint. By Lemma 3.7, we have internally disjoint paths $X_{l(r),i_2}$ from $u'_{l(r)}$ to r, where $i_2 = 2, 3$ and $a'_{i_2} \in V(X_{l(r),i_2})$. Notably, each $X_{l(r),i_2}$ is a subgraph of R_{i_2} and shares a unique common vertex r with P_1. For the case $\pi_1(p(r)^+) = 3$, due to the same reason, we get pairwise internally disjoint paths from $u'_{l(r)-1}$ to r in each $TB_{r,i}$.

Case 3. $\pi_1(p(r)) \in \{2, 3\}$.

Similar to the proof of *Case 2.*

Q. E. D.

Theorem 3.5. *For a given root vertex r, $2 \leq l(r) \leq n$, in $\mathbb{P}_n$, $\mathcal{BB}_r$ has $3-ISTs$.*

Proof. Let $\rho_i = \pi_1(a'_i)$ for $i = 1, 2, 3$. Let $v \in V(\mathcal{BB}_r)$. By Lemma 3.3, we have paths P_{ρ_i} from v to r in TB_{r,ρ_i} for each i. We want to show P_{ρ_1}, P_{ρ_2} and P_{ρ_3} are pairwise internally disjoint.

Case 1. $l(v) = 0$.

Vertex v is Δ and adjacent to every vertex of h_1. By Lemma 3.4, we have pairwise internally disjoint paths X_{1,ρ_i} from v to u'_2, where $1 \leq i \leq 3$ and $h_{1,\rho_i}, h_{2,\rho_i} \in V(X_{1,\rho_i})$. Furthermore, by Lemma 3.8, we

have pairwise internally disjoint paths X_{2,ρ_i} from u'_2 to r, where $1 \leq i \leq 3$ and $h_{2,\rho_i}, a'_i \in V(X_{2,\rho_i})$. Let $Y_{1,\rho_i} = X_{1,\rho_i} - u'_2$ and $Y_{2,\rho_i} = X_{2,\rho_i} - u'_2$ for each i. Clearly, paths $Y_{1,\rho_i}Y_{2,\rho_i}$ from v to r are pairwise internally disjoint for $i = 1, 2, 3$.

Case 2. v is an a-vertex and $1 \leq l(v) \leq l(r) - 2$.

By Lemma 3.6, we have pairwise internally disjoint paths X_{1,ρ_i} from v to $u'_{l(v)+1}$ in each TB_{r,ρ_i}, where $1 \leq i \leq 3$ and $h_{l(v)+1,\rho_i} \in V(X_{1,\rho_i})$. Furthermore, by Lemma 3.8, we have pairwise internally disjoint paths X_{2,ρ_i} from $u'_{l(v)+1}$ to r, where $1 \leq i \leq 3$ and $h_{l(v)+1,\rho_i} \in V(X_{2,\rho_i})$. Let $Y_{1,\rho_i} = X_{1,\rho_i} - u'_{l(v)+1}$ and $Y_{2,\rho_i} = X_{2,\rho_i} - u'_{l(v)+1}$ for each i. Clearly, paths $Y_{1,\rho_i}Y_{2,\rho_i}$ from v to r are pairwise internally disjoint for $i = 1, 2, 3$.

Case 3. v is an a-vertex and $l(v) = l(r) - 1$.

If $v = p(r)$, then, by the definition of each R_{ρ_i}, we have three pairwise internally disjoint paths $\langle v, r^-, r\rangle$, $\langle v, r^+, r\rangle$ and $\langle v, r\rangle$. Otherwise, $v \neq p(r)$. Since $c_{\rho_i}(v) \in V(P_{k+1,\rho_i})$, P_{ρ_i} is a subpath of P_{k+1,ρ_i}. So, $P_{\rho_3} \| P_{\rho_1}$ and $P_{\rho_3} \| P_{\rho_2}$ because $P_{k+1,\rho_3} \| P_{k+1,\rho_1}$ and $P_{k+1,\rho_3} \| P_{k+1,\rho_2}$. Moreover, since $c_{\rho_1}(v) \in U_{k+1,1}$ and $c_{\rho_2}(v) \in U_{k+1,2}$ belong to the same block, it follows that $P_{\rho_1} \| P_{\rho_2}$.

Case 4. v is a b-vertex and $2 \leq l(v) \leq l(r) - 1$.

By Lemma 3.5, we have pairwise internally disjoint paths X_{1,ρ_i} from v to $u'_{l(v)}$ in TB_{r,ρ_i}, where $1 \leq i \leq 3$ and $h_{l(v),\rho_i} \in V(X_{1,\rho_i})$. Furthermore, by Lemma 3.8, we have pairwise internally disjoint paths X_{2,ρ_i}, from $u'_{l(v)}$ to r, where $1 \leq i \leq 3$ and $h_{l(v),\rho_i}, a'_i \in V(X_{2,\rho_i})$. Let $Y_{1,\rho_i} = X_{1,\rho_i} - u'_{l(v)}$ and $Y_{2,\rho_i} = X_{2,\rho_i} - u'_{l(v)}$ for each i. Clearly, paths $Y_{1,\rho_i}Y_{2,\rho_i}$ from v to r are pairwise internally disjoint for $i = 1, 2, 3$.

Case 5. $v \in V(A_{l(r)})$.

The case $v = r$ is trivial. For cases $v \in \{r^+, r^-, r^{++}\}$, we shall list three pairwise internally disjoint paths for each case, where each path is a subpath of R_{ρ_i}. If $v = r^+$, we have paths $\langle r^+, r^{++}, r^-, r\rangle$, $\langle r^+, r\rangle$ and $\langle r^+, p(r^+), r\rangle$. For the case $v = r^-$, we have $\langle r^-, r\rangle$, $\langle r^-, r^{++}, r^+, r\rangle$ and $\langle r^-, p(r^-), r\rangle$. And we have $\langle r^{++}, r^-, r\rangle$, $\langle r^{++}, r^+, r\rangle$ and $\langle r^{++}, p(r^{++}), r\rangle$ for the case $v = r^{++}$.

Case 6. $v \in V(S_{l(r)}) \setminus V(A_{l(r)})$.

By Corollary 3.1, we get pairwise internally disjoint paths X_{1,ρ_i} from v to $u'_{l(r)}$ in TB_{r,ρ_i}, where $1 \leq i \leq 3$ and $a'_i \in V(X_{1,\rho_i})$. Furthermore, by Lemma 3.7, we have pairwise internally disjoint

paths X_{2,ρ_i} from $u'_{l(r)}$ to r, where $1 \leq i \leq 3$ and $a'_i \in V(X_{2,\rho_i})$. Let $Y_{1,\rho_i} = X_{1,\rho_i} - u'_{l(r)}$ and $Y_{2,\rho_i} = X_{2,\rho_i} - u'_{l(r)}$ for each i. Clearly, paths $Y_{1,\rho_i}Y_{2,\rho_i}$ from v to r are pairwise internally disjoint for $i = 1, 2, 3$.

Q. E. D.

3.3.2 Branching Stage

A *subpyramid* $\mathbb{P}_n(u)$ is a subgraph of $\mathbb{P}_n$ by letting a vertex $u \in V(\mathbb{P}_n)$ be the apex of $\mathbb{P}_n(u)$. A *flat subpyramid* $\overline{\mathbb{P}}_n(u)$ is a subpyramid without an apex u and a layer 1. Let V_b denote the set of b-vertices of $\mathcal{BB}_r$. The subgraph $\mathbb{P}_n - \mathcal{BB}_r$ is regarded as an union of flat subpyramids, and every flat subpyramid is defined as $\overline{\mathbb{P}}_n(p(v))$, where $v \in V(A_{l(r)}) \cup V_b$. In particular, $\overline{\mathbb{P}}_n(p(v))$ is defined to be empty if $v \in S_{l(r)}$ and $l(r) = n$. For easily describing notations in $\overline{\mathbb{P}}_n(p(v))$, we let $\overline{S_k}$ denote the cluster on layer k of $\overline{\mathbb{P}}_n(p(v))$, where $2 \leq k \leq n - l(v) + 1$. That is, $\overline{S_k}$ is actually a cluster on layer $l(v)+k-1$ of $\mathbb{P}_n$. For a $\overline{\mathbb{P}}_n(p(v))$, $\mathcal{P}_{\mathcal{B}}(\overline{S_2})$ is either $A_{l(r)}$ or a b-block of $\mathcal{BB}_r$, and $\mathcal{P}_{\mathcal{B}}(\overline{S_2})$ is also denoted by $\overline{S_1}$. Since $\overline{S_1}$ is in fact a block of $\mathcal{BB}_r$, $\overline{S_1}$ has a π_1 coloring in Backbone Establishment Stage. Henceforth, every cluster of $\overline{\mathbb{P}}_n(p(v))$ can have a π_0 coloring by applying `HFinding Algorithm` on its parent block. Notations $\hat{h}_{k,i}$ and $\hat{H}_{k,i}$, $1 \leq i \leq 3$, refer to Section 3.1. Let $T_i(v)$, $1 \leq i \leq 3$, denote the spanning subgraph of $\overline{\mathbb{P}}_n(p(v))$ and have the edge set $\bigcup_{k=2}^{n-l(v)+1} \bigcup_{a,b=0}^{2^{k-2}-1} E(\hat{H}_i(k; a, b))$. Let $T^2_{r,i}$, $1 \leq i \leq 3$, denote the spanning subgraph of $\mathbb{P}_n$ with the edge set $E(TB_{r,i}) \bigcup_{v \in V(A_{l(r)}) \cup V_b} E(T_i(v))$.

Lemma 3.9. *Let $v \in V(A_{l(r)}) \cup V_b$. If u_k is a vertex of $\overline{S_k}$ in $\overline{\mathbb{P}}_n(p(v))$, $2 \leq k \leq n - l(v) + 1$, then the paths from u_k to r in each $T^2_{r,i}$, for $i = 1, 2, 3$, are pairwise internally disjoint.*

Proof. $\overline{S_1}$ is either $A_{l(r)}$ or a b-block of $\mathcal{BB}_r$. If $\overline{S_1} = A_{l(r)}$, then, by Theorem 3.3, we have pairwise internally disjoint paths from u_k to r in each $T^2_{r,i}$. Otherwise, $\overline{S_1}$ is a b-block of $\mathcal{BB}_r$. Let $\overline{u'_j}$ be the dummy vertex with dummy edges incident to every vertex of $\mathcal{P}_{\mathcal{B}}(\overline{S_{j+1}})$ for $j = 1, 2, \ldots, k-1$. Let $u'_{l(r)}$ be the dummy vertex with dummy edges incident to every vertex of $\{a'_1, a'_2, a'_3\}$. Let $\rho_i = \pi_1(a'_i)$ for each i. From Lemma 3.1, we get pairwise internally disjoint paths X_{k,ρ_i} from u_k to $\overline{u'_{k-1}}$ in $T_{\rho_i}(v)$ for each $i = 1, 2, 3$. By applying Lemma 3.2, we get pairwise internally disjoint paths X_{j,ρ_i} from $\overline{u'_j}$ to $\overline{u'_{j-1}}$ in $T_{\rho_i}(v)$ for each i, where $2 \leq j \leq k-1$. Let $Y_{k,\rho_i} = X_{k,\rho_i} - \overline{u'_{k-1}}$ and $Y_{j,\rho_i} = X_{j,\rho_i} - \overline{u'_j} - \overline{u'_{j-1}}$ for each i, j. We now list the three pairwise internally disjoint paths Y_{1,ρ_i} from $\overline{u'_1}$ to r in each TB_{r,ρ_i} to complete the proof.

Let $V(\overline{S_1}) = \{v_0, v_{\rho_1}, v_{\rho_2}, v_{\rho_3}\}$ such that $\pi_1(v_0) = 0$ and $\pi_1(v_{\rho_i}) = \rho_i$ for $i = 1, 2, 3$. Recall that $v_{\rho_3} = v_0^{++}$. By the definition of each $E(\hat{H}_{2,\rho_i})$, either v_{ρ_3} or v_0 can be on the path X_{2,ρ_3}.

Case 1. $\overline{S_1}$ is a b-block of $S_{l(r)}$.

By Lemma 3.7, we have pairwise internally disjoint paths Z_{1,ρ_i} from $u'_{l(r)}$ to r in TB_{r,ρ_i}, where $1 \leq i \leq 3$ and $a'_i \in V(Z_{1,\rho_i})$. Furthermore, from Corollary 3.2, we get pairwise internally disjoint paths P_{ρ_i} from $p(v)$ to $u'_{l(r)}$ in TB_{r,ρ_i}, where $1 \leq i \leq 3$ and $a'_i \in V(P_{\rho_i})$. Since $(v_{\rho_i}, p(v)) \in E(TB_{r,\rho_i})$, $P_{\rho_i} - p(r)$ is the path from v_{ρ_i} to $u'_{l(r)}$ in TB_{r,ρ_i}. If $v_{\rho_3} \in V(X_{2,\rho_3})$, we get paths $Y_{1,\rho_i} = Z_{2,\rho_i} Z_{1,\rho_i} - u'_{l(r)}$ for each i. Otherwise $v_0 \in V(X_{2,\rho_3})$. Since $\langle v_0, p(v_0), v_{\rho_3}\rangle$ is a subgraph of $E(H_{l(r),\rho_3})$, we get the path $Y_{1,\rho_3} = \langle v_0, p(v_0), v_{\rho_3}\rangle Z_{2,\rho_3} Z_{1,\rho_3} - u'_{l(r)}$. In addition, $Y_{1,\rho_1} = Z_{2,\rho_1} Z_{1,\rho_1} - u'_{l(r)}$ and $Y_{1,\rho_2} = Z_{2,\rho_2} Z_{1,\rho_2} - u'_{l(r)}$.

Case 2. $\overline{S_1}$ is not a block of $S_{l(r)}$.

Notice that $\overline{S_1}$ is a b-block and $l(v) < l(r)$. By Lemma 3.8, we have pairwise internally disjoint paths Z_{1,ρ_i} from $u'_{l(v)}$ to r in TB_{r,ρ_i}, where $1 \leq i \leq 3$ and $h_{l(v),\rho_i} \in V(Z_{1,\rho_i})$. Furthermore, from Lemma 3.6, we get pairwise internally disjoint paths P_{ρ_i} from $p(v)$ to $u'_{l(v)}$ in TB_{r,ρ_i}, where $1 \leq i \leq 3$ and $h_{l(v),\rho_i} \in V(P_{\rho_i})$. Since $(v_{\rho_i}, p(v)) \in E(TB_{r,\rho_i})$, $P_{\rho_i} - p(r)$ is the path from v_{ρ_i} to $u'_{l(v)}$ in TB_{r,ρ_i}. If $v_{\rho_3} \in V(X_{2,\rho_3})$, we get paths $Y_{1,\rho_i} = Z_{2,\rho_i} Z_{1,\rho_i} - u'_{l(v)}$ for each i. Otherwise $v_0 \in V(X_{2,\rho_3})$. Since $\langle v_0, p(v_0), v_{\rho_3}\rangle$ is a subgraph of $E(H_{l(v),\rho_3})$, we get the path $Y_{1,\rho_3} = \langle v_0, p(v_0), v_{\rho_3}\rangle Z_{2,\rho_3} Z_{1,\rho_3} - u'_{l(v)}$. In addition, $Y_{1,\rho_1} = Z_{2,\rho_1} Z_{1,\rho_1} - u'_{l(v)}$ and $Y_{1,\rho_2} = Z_{2,\rho_2} Z_{1,\rho_2} - u'_{l(v)}$.

It concludes that paths $Y_{k,\rho_i} Y_{k-1,\rho_i} \dots Y_{1,\rho_i}$ from u_k to r in T^2_{r,ρ_i}, for $i = 1, 2, 3$, are pairwise internally disjoint.

Q. E. D.

Theorem 3.6. *The construction of 3-ISTs of $\mathbb{P}_n$ rooted at $r, 2 \leq l(r) \leq n$, can be solved in $O(|V|)$ time, where $|V|$ is the number of vertices of $\mathbb{P}_n$.*

Proof. The construction of 3-ISTs rooted at r, $2 \leq l(r) \leq n$, contains the Backbone Establishment Stage and the Branching Stage. Let $r = (k; x, y)$. In the Backbone Establishment Stage, every vertex $(k - j; \lfloor \frac{x}{2^j} \rfloor, \lfloor \frac{y}{2^j} \rfloor)$, for $j = 0, 1, \dots, k-1$, is an ancestor of r and can be decided in $O(1)$ time, implying every a-block can be found in $O(1)$ time. Since every cluster of $\mathcal{BB}_r$ can be colored by constant time, π_1 coloring costs $O(|V(\mathcal{BB}_r)|)$ time. We then need additional constant

time to have an `RootColoring Algorithm`. Vertices of h_k of a cluster are decided in constant time and hence every $H_{k,i}$ is established in constant time. Since each edge set $E(H_{x,i}), i = 1, 2, 3$ has $|V(\mathcal{BB}_r)| - 1$ edges and each edge of $E(H_{x,i})$ can be checked in constant time, the finding of $E(H_{x,i})$ costs $O(|V(\mathcal{BB}_r)|)$ time. Therefore, Backbone Establishment Stage costs $O(|V(\mathcal{BB}_r)|)$ time.

In Branching Stage, every flat subpyramid $\overline{\mathbb{P}}_n(p(v))$ needs $O(|V(\overline{\mathbb{P}}_n(p(v)))|)$ time π_0 coloring. By Theorem 3.2, the construction of 3-ISTs of every $\overline{\mathbb{P}}_n(p(v))$ rooted at $p(v)$ can be solved in $O(|V(\overline{\mathbb{P}}_n(p(v)))|)$ time, where $v \in V(A_{l(r)}) \cup V_b$. The overall time complexity of Branching Stage is $O(|V(\mathbb{P}_n - \mathcal{BB}_r)|)$. That is, the construction of 3-ISTs of $\mathbb{P}_n$ rooted at r, can be solved in $O(|V(\mathbb{P}_n)|)$ time.

Q. E. D.

4 Conclusion Remarks

In this paper, we present linear time algorithms to construct 3-ISTs in an n-dimensional pyramid $\mathbb{P}_n$. To the best of our knowledge, this is the first result for constructing ISTs in $\mathbb{P}_n$. The tree height and the average distance of an IST are considered as two performance metrics in our algorithms. The *height* of a tree T is the the number of vertices in a longest path from the root to a leaf. The *average distance* of T is the average of all distances from any vertex to the root. The resulting values are shown in Table 1. In Table 1, the column "height" provides an upper bound to the values of the average distance of a tree, where n is the dimension of the given network. Columns 4-10 demonstrate average distances for different network scales. The data are calculated on different scenarios to see the tree heights and the average distances for different scales of pyramid networks. Although the values of average distances are associated with a proportional increase in the dimension of the networks, the resulting values of $T^0_{\rho_i}, T^1_{\rho_i}$ and T^2_{r,ρ_i} are still bounded by $2.81n, 2.81n$ and $6.54n$, respectively, even when the given network is of order over 1 million vertices.

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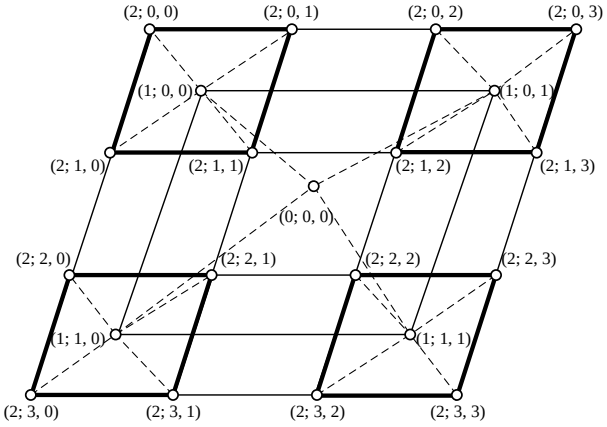


Figure 1: A top view of the pyramid $\mathbb{P}_2$.

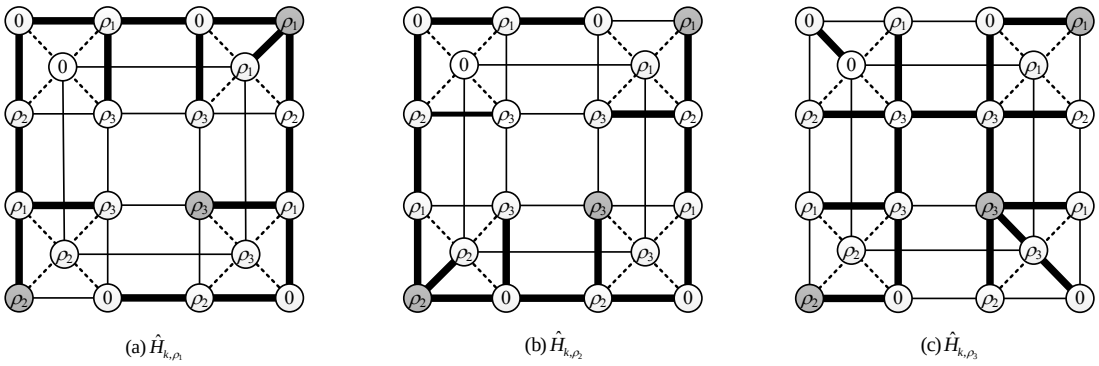


Figure 2: The bold cluster edges establish three subgraphs (a)$\hat{H}_{k,\rho_1}$, (b)$\hat{H}_{k,\rho_2}$, (c)$\hat{H}_{k,\rho_3}$ of a cluster S_k, where the parent vertex of the top-left block is of color 0. The dash lines indicate layer edges.

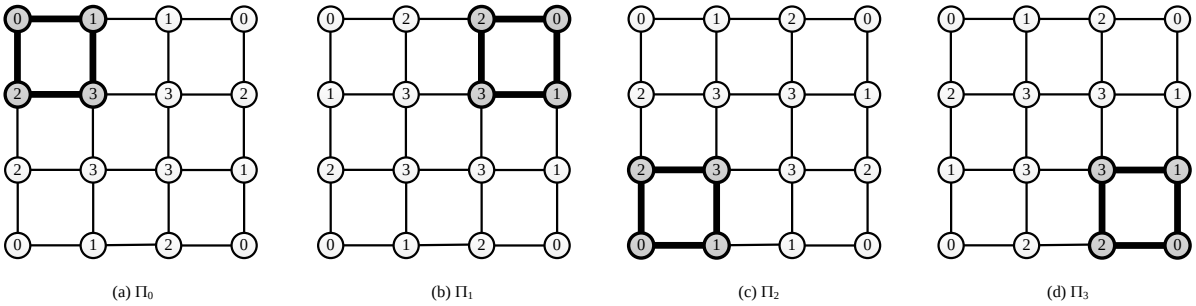
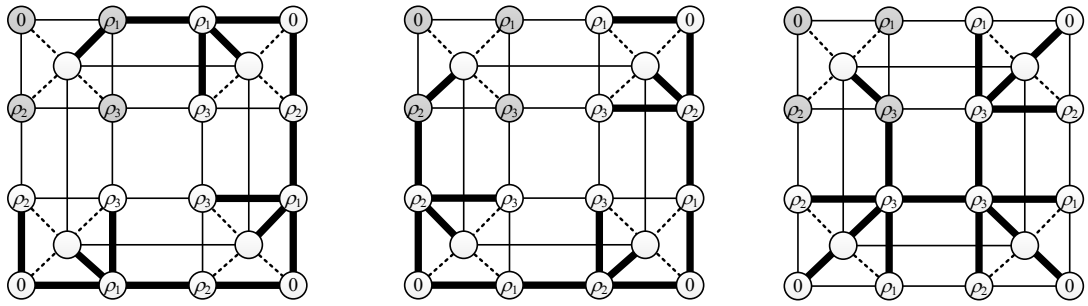


Figure 3: The four possible π_1 colorings Π_0, Π_1, Π_2 and Π_3 of a cluster S_k when A_k is (a) the top-left block; (b) the top-right block; (c) the bottom-left block; (d) the bottom-right block of S_k.



(a) H_{k,ρ_1} (b) H_{k,ρ_2} (c) H_{k,ρ_3}

Figure 4: The three subgraphs (a)H_{k,ρ_1}, (b)H_{k,ρ_2}, (c)H_{k,ρ_3} of a cluster S_k, where the top-left block of S_k is an a-block.

Table 1: The table of tree heights and average distances

			n (dimension)						
	$l(r)$	height	2	3	4	5	6	7	10
$\lvert V(RP_n)\rvert$			21	85	341	1,365	5,461	21,845	>1 million
$T^0_{\rho_1}, T^0_{\rho_2}$	0	$5n-2$	$2.14n$	$2.39n$	$2.53n$	$2.62n$	$2.69n$	$2.73n$	$2.81n$
$T^0_{\rho_3}$	0	$5n-3$	$1.71n$	$2.06n$	$2.28n$	$2.42n$	$2.52n$	$2.59n$	$2.71n$
$T^1_{\rho_1}, T^1_{\rho_2}$	1	$5n-2$	$2.14n$	$2.39n$	$2.53n$	$2.62n$	$2.69n$	$2.73n$	$2.81n$
$T^1_{\rho_3}$	1	$5n-2$	$2.10n$	$2.37n$	$2.53n$	$2.62n$	$2.68n$	$2.73n$	$2.81n$
$T^2_{r,\rho_1}, T^2_{r,\rho_2}$	≥ 2	$\leq 5n-11$ $+5l(r)+\varepsilon$	$\leq 2.14n$	$\leq 3.45n$	$\leq 4.43n$	$\leq 5.11n$	$\leq 5.58n$	$\leq 5.92n$	$\leq 6.54n$
T^2_{r,ρ_3}	≥ 2	$\leq 9n-19$ $+4l(r)+\varepsilon$	$\leq 2.50n$	$\leq 2.60n$	$\leq 3.35n$	$\leq 3.99n$	$\leq 4.47n$	$\leq 4.83n$	$\leq 5.48n$
ε: the distance from an a-vertex of $S_{l(r)}$ to r.									