Long time behavior in locally activated random walks

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Abstract

We consider a 1-dimensional Brownian motion whose diffusion coefficient varies when it crosses the origin. We study the long time behavior and we establish different regimes, depending on the variations of the diffusion coefficient: emergence of a non-Gaussian multipeaked probability distribution and a dynamical transition to an absorbing static state. We compute the generator and we study the partial differential equation which involves its adjoint. We discuss global existence and blow-up of the solution to this latter equation.

1 Introduction

In this paper we deal with a new class of one dimensional linear diffusion problem in which the diffusivity is modified in a prescribed way upon each crossing of the origin. We study both the system of stochastic differential equations satisfied by the position and the diffusion coefficient of a brownian particle whose diffusion coefficient is modified at each crossing of the origin and the partial differential equation satisfied by the joint distribution of the solution to the stochastic system. In both viewpoints we obtain non-trivial behaviors of the solution: dynamical transition to an absorbing state for the solution to the stochastic system and blow-up of the density of the joint distribution. Global existence versus blow-up has been widely studied for non-linear equations, such as for the Keller-Segel system in two dimensions of space see e.g. [3]. In our case, the partial differential equation is linear and the instability driving the system towards an inhomogeneous state is the diffusion.

Living matter provides a prototypical example of such a problem: the dynamics of a cell or a bacterium in the presence of a localized patch of nutrients, which enhances its ability to move, as for exemple the dynamics of a macrophage that grows by accumulating smaller and spatially localized particles, such as lipids, Figure 1 and [4], or, alternatively, a localized patch of toxins that impairs its mobility. In [1], to describe the movement of such a particle, the following formal system of stochastic differential equations was introduced:

$$\begin{cases} \mathrm{d}X_t &= \sqrt{2A_t}\,\mathrm{d}W_t, \\ \mathrm{d}A_t &= f(A_t)\delta_{X_t=0}\,\mathrm{d}t, \end{cases} \tag{1}$$

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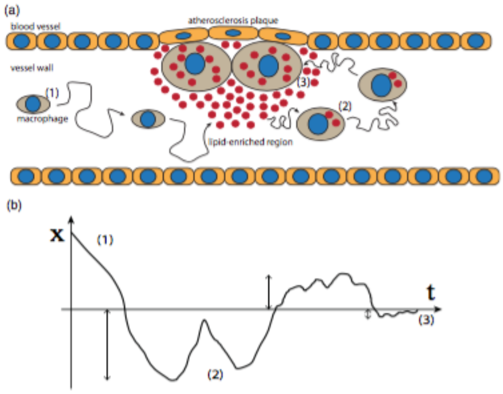

Figure 1: a) Sketch of the different stages of atherosclerosis plaque formation: (1) rapid diffusion of a free macrophage cell; (2) upon entering a localized lipid-enriched region, the macrophage accumulates lipids, and thereby grows and becomes less mobile; and (3) after many crossings of the lipid-enriched region, the macrophage eventually gets trapped, resulting in the formation of an atherosclerotic plaque. (b) Sketch of a one-dimensional particle trajectory of the model of locally decelerated random walk.

where $(W_t)_{t\geq 0}$ is a given standard one-dimensional Brownian motion, X_t and A_t respectively denote the position and the diffusion coefficient of the particle at time t.

In the previous system, the term $\delta_{X_t=0}$ does not make sense. In [1], it was understood in the sense that the generator of the Markov process $(X_t, A_t)_{t\geq 0}$ was

$$\mathcal{L}h(x,a) = a\partial^2_{xx}h(x,a) + f(a)\partial_a h(x,a)\delta_{x=0}, \tag{2}$$

for any smooth function h. Here, we are first interested to give a correct formulation of the term $\delta_{X_t=0}$. Intuitively, the term $\delta_{X_t=0}\,\mathrm{d}t$ should represent a measure on $[0,\infty)$ giving full measure to the set of zeros of the process $(X_t)_{t\geq 0}$. This reminds the notion of local time.

We point out that in the formal system (1) at any time, the diffusion coefficient, A_t, depends on the entire history of the trajectory. Thus the evolution of the particle position, X_t, is intrinsically non-Markovian. Despite these considerations, in the particular case where f is a power function, $f(a) = \pm a^\gamma$, $\gamma \geq 0$, we study the long-time behavior of the process $(X_t)_{t\geq 0}$ solution to the system with the correct formulation of the term $\delta_{X_t=0}$. Our main findings are: (i) The probability distribution of the position has a non-Gaussian tail. (ii) For local acceleration, i.e. f takes nonnegative values, $f(a) = a^\gamma$, a diffusing particle is repelled from the origin, so that the maximum in the probability distribution is at nonzero displacement. (iii) For local deceleration, i.e. f takes negative values, $f(a) = -a^\gamma$, a dynamical transition to an absorbing state occurs: for sufficiently strong deceleration, $\gamma \in (0, 3/2)$, the particle can get trapped at the origin in finite time while if the deceleration process is sufficiently weak, $\gamma \geq 3/2$, the particle never gets trapped.

In a second step, we study the generator of the Markov process $(X_t, A_t)_{t\geq 0}$ solution to the system with the correct formulation of the term $\delta_{X_t=0}$. In order to do so we first prove that the generator of the Markov process $(W_t, L_t^W)_{t\geq 0}$, in a weak sense, is given by

$$\mathcal{L}_0 h(w,l) = \frac{1}{2}\partial^2_{ww} h(w,l) + \partial_l h(w,l)\delta_{l=0},$$

where L_t^W is the local time at 0 of $(W_t)_{t\geq 0}$. We use this result to prove that the density $u(t,x,a)$ of the joint distribution $\mu_t(x,a)$ of $(X_t, A_t)_{t\geq 0}$, defined on $t \geq 0$, $x \in \mathbb{R}$, $a \geq 0$, satisfies, in a weak sense the parabolic equation

$$\partial_t u(t,x,a) = \mathcal{L}^* u(t,x,a) = a\partial^2_{xx} u(t,x,a) - \partial_a \left(f(a)u(t,x,a)\right)\delta_{x=0}, \tag{3}$$

with initial condition $u_0(x,a)$. Here, $\mathcal{L}^*$ is the adjoint of the generator $\mathcal{L}$ defined in (2).

Next we study the partial differential equation (3) and the general questions we are concerned with are the following. Does the joint distribution μ_t at time t, which is a measure, have a density with respect to the Lebesgue measure when considering general initial condition μ_{in}? By studying the regularity of the solution to (3), do we recover the results observed during the probabilistic study? In particular, if $f(a) = -a^\gamma$ with $\gamma \geq 3/2$, can we prove global existence? In the case $\gamma \in (0, 3/2)$ can we prove that the solution to (3) becomes unbounded in finite time in any L^p space (so-called blow-up)?

We describe the plan of the paper. In Section 2 we build and study the correct equation associated with (1). Section 3 is devoted to the computation of the generator of $(X_t, L_t^W)_{t\geq 0}$ from which we deduce the weak formulation satisfied by the joint distribution associated to (X_t, A_t) solution to the correct version of (1). In Section 4 we study equation (3).

2 Mathematical study of a correct version of (1)

In the system (1), the term $\delta_{X_t=0}$ does not make sense. Intuitively, the term $\delta_{X_t=0}\,\mathrm{d}t$ should represent a measure on $[0,\infty)$ to the set of zeros of the process $(X_t)_{t\geq 0}$. This reminds the notion of local time whose definition we recall here for completeness.

For any continuous local martingale $(M_t)_{t\geq 0}$, one can define the *local time at* 0 of $(M_t)_{t\geq 0}$ by:

$$L_t^M := \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\int_0^t \mathbf{1}_{|M_t|\leq\varepsilon}\,\mathrm{d}\left\langle M\right\rangle_t,$$

where the limit holds in probability. The local time satisfies the scaling property

$$L_t^{\lambda M} = \lambda L_t^M \text{ a.s. for any } \lambda > 0.$$

In particular, the process $(\lambda^{-1}L_t^{\lambda M})_{t\geq 0}$ does not actually depend on λ. Since the process $\left(L_t^M\right)_{t\geq 0}$ is continuous and nondecreasing, we can associate to it a measure $\mathrm{d}L_t^M$ without atoms on $\mathbb{R}_+$. This measure is supported by the set $\{t \geq 0 : M_t = 0\}$. For more details on the theory of local times, we refer to [7].

The formal term $\delta_{X_t=0}\,\mathrm{d}t$ should satisfy the scaling invariance property $\delta_{\lambda X_t=0}\,\mathrm{d}t = \delta_{X_t=0}\,\mathrm{d}t$. Comparing this formal property to the properties of local time, it seems natural to replace the term $\delta_{X_t=0}\,\mathrm{d}t$ by the renormalized local time $\mathrm{d}L_t^X/2A_t$. As a consequence, in this work, instead of (1), we will study the system

$$\begin{cases} \mathrm{d}X_t &= \sqrt{2A_t}\,\mathrm{d}W_t, \\ \mathrm{d}A_t &= f(A_t)\dfrac{\mathrm{d}L_t^X}{2A_t}, \end{cases} \tag{4}$$

with a given initial condition (X_0, A_0). In the sequel, f will be a locally Lipschitz continuous function from $(0,\infty)$ to $\mathbb{R}$, and the initial condition will be assumed to satisfy $A_0 > 0$ almost surely. Note that if $(X_t)_{t\geq 0}$ solves (4), then $(X_t)_{t\geq 0}$ is a local martingale.

More precisely, we are interested in proving that Equation (4) defines a Markov process whose generator is given by (2). In order to do so we start by studying a simpler problem

$$\begin{cases} \mathrm{d}X_t &= \sqrt{2A_t}\,\mathrm{d}W_t, \\ \mathrm{d}A_t &= f(A_t)\dfrac{\mathrm{d}L_t^W}{\sqrt{2A_t}}, \end{cases} \tag{5}$$

and we prove that the solutions to systems (4) and (5) coincide when the initial condition satisfies $X_0 = 0$.

Equation (4) may not admit solutions for all positive time, because solutions might blow up in finite time. For example if f is a positive function, the process $(A_t)_{t\geq 0}$ will be nondecreasing, and nothing will a priori prevent it to go to infinity in a finite time τ. In that case, the diffusion coefficient of $(X_t)_{t\geq 0}$ will blow up in finite time, and $(X_t)_{t\geq 0}$ will not admit any extension after time τ.

As a consequence, in the sequel, we will call a *(strong) solution* to Equation (4) (resp. (5)), a triple $(\tau, (X_t)_{0\leq t<\tau}, (A_t)_{0\leq t<\tau})$, where τ is a stopping time of the Brownian motion $(W_t)_{t\geq 0}$ and $(X_t, A_t)_{0\leq t<\tau}$ is a continuous process adapted to $(W_t)_{t\geq 0}$ satisfying Equation (4) (resp. (5)) until time τ.

We will say that such a solution is *maximal*, when the process $(A_t)_{0\leq t<\tau}$ converges either to 0 or ∞ as $t \to \tau$ on the event $\{\tau < \infty\}$. Indeed, in those two cases, the term $f(A_t)$ appearing in the equation becomes ill-defined at time τ, since $f(a)$ is only assumed to make sense for $a \in (0,\infty)$.

2.1 Well-posedness of (5)

The first equation in (5) is explicit in $(W_t, A_t)_{t\geq 0}$: for given $(X_0, (A_t)_{0\leq t<\tau}, (W_t)_{t\geq 0})$, its unique solution is given by

$$\forall t < \tau,\ X_t = X_0 + \int_0^t \sqrt{2A_s}\,\mathrm{d}W_s\,.$$

Moreover, the second equation in (5) is a closed equation on $(A_t)_{t\geq 0}$ and does not depend on $(X_t)_{t\geq 0}$. Thus, studying existence and uniqueness for the equation $\mathrm{d}A_t = f(A_t)\,\mathrm{d}L_t^W/\sqrt{2A_t}$ is enough to obtain existence and uniqueness for system (5).

Recalling that $f : (0,\infty) \to \mathbb{R}$ is assumed to be locally Lipschitz continuous, we obtain the following result.

Proposition 1. *Let* (X_0, A_0) *be a random couple, independent of* $(W_t)_{t\geq 0}$ *and such that* $A_0 > 0$. *Then, there exists a unique maximal strong solution* $(\tau, (X_t)_{0\leq t\leq\tau}, (A_t)_{0\leq t\leq\tau})$ *to Equation* (5) *with initial condition* (X_0, A_0).

Proof. As explained before, we only need to show existence and uniqueness for the equation $\mathrm{d}A_t = f(A_t)\,\mathrm{d}L_t^W/\sqrt{2A_t}$, which is closely related to the ordinary differential equation $y' = f(y)/\sqrt{2y}$.

First, consider the flow Φ associated with $y' = f(y)/\sqrt{2y}$. Namely, $t \mapsto \Phi_x(t)$ is the unique maximal solution to $y' = f(y)/\sqrt{2y}$ satisfying $\Phi_x(0) = x$. This flow is well defined from the local Lipschitz continuity of $y \mapsto f(y)/\sqrt{2y}$. The flow Φ_x is only defined up to a time $T(x)$. Moreover, in the case $T(x) < \infty$, one necessarily has either $\lim_{t\to T(x)} \Phi_x(t) = \infty$ or $\lim_{t\to T(x)} \Phi_x(t) = 0$.

Then, one can check than $A_t = \Phi_{A_0}(L_t^W)$ is defined up to the time $\tau = \sup\{t \geq 0, L_t^W < T(A_0)\}$ and satisfies the equation $\mathrm{d}A_t = f(A_t)\,\mathrm{d}L_t^W/\sqrt{2A_t}$. Indeed, since Φ_x is continuously differentiable and $(L_t^W)_{t\geq 0}$ is a contiuous nondecreasing process, then the usual chain rule holds, namely one has $\mathrm{d}\Phi_x(L_t^W) = \Phi'_x(L_t^W)\,\mathrm{d}L_t^W$. Moreover, on the event $\{\tau < \infty\}$, $\lim_{t\to\tau} A_t$ exists with value 0 or ∞.

For uniqueness of the solution to $\mathrm{d}A_t = f(A_t)\,\mathrm{d}L_t^W/\sqrt{2A_t}$, consider two solutions $(\tau, (A_t)_{0\leq t<\tau})$ and $(\tilde{\tau}, (\tilde{A}_t)_{0\leq t<\tilde{\tau}})$, with $A_0 = \tilde{A}_0$. Then one has the following Grönwall-type inequality: $\forall t < \tau \wedge \tilde{\tau}$

$$|A_t - \tilde{A}_t| = \left| \int_0^t f(A_s)/\sqrt{2A_s} - f(\tilde{A}_s)/\sqrt{2\tilde{A}_s}\,\mathrm{d}L_s^W \right| \leq C \int_0^t |A_s - \tilde{A}_s|\,\mathrm{d}L_s^W\,.$$

Hence, from the expression

$$\begin{aligned} & e^{-CL_t^W} \int_0^t \left| A_s - \tilde{A}_s \right| \,\mathrm{d}L_s^W \\ = & \int_0^t e^{-CL_s^W} \left(|A_s - \tilde{A}_s| - C \int_0^s \left| A_u - \tilde{A}_u \right| \,\mathrm{d}L_u^W \right) \mathrm{d}L_s^W \\ \leq & \ 0\,, \end{aligned}$$

it follows that $|A_t - \tilde{A}_t| = 0$ for $\mathrm{d}L_t^W$-almost all $0 \leq t < \tau \wedge \tilde{\tau}$. □

Remark 2. *As it appears in the proof of Proposition 1, existence for the stochastic differential equation* (5) *still holds true provided the ordinary differential equation* $y' = f(y)/\sqrt{2y}$ *admits a (non necessarily unique) solution.*

2.2 Link with system (4)

In this section, we provide a link between solutions to systems (4) and (5). Consider a solution $(\tau, (X_t)_{0\leq t<\tau}, (A_t)_{0\leq t<\tau})$ to (4) starting from $X_0 = 0$. We prove that the two processes $(X_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ vanish at exactly the same times, until the explosion time τ. Indeed, we show that the measure $\mathrm{d}L_t^X$ has a density with respect to $\mathrm{d}L_t^W$.

Proposition 3. *Let* (X_0, A_0) *be a random variable independent of* $(W_t)_{t\geq 0}$ *with* $A_0 > 0$. *Let* $(\tau, (A_t)_{0\leq t<\tau}, (X_t)_{0\leq t<\tau})$ *be a strong solution to* (4). *Then,*

$$X_t = \sqrt{2A_t}\left(W_t + \frac{X_0}{\sqrt{2A_0}}\right).$$

If in addition we assume that $X_0 = 0$, *then*

$$\forall t < \tau,\ \ \mathrm{d}L_t^X = \sqrt{2A_t}\,\mathrm{d}L_t^W \quad \textit{and} \quad X_t = \sqrt{2A_t}W_t. \tag{6}$$

Proof. Let the stopping time τ_n be the first positive time for which $(A_t)_{0\leq t<\tau}$ reaches $[0, \frac{1}{n}]$. It is defined by

$$\tau_n = \tau \wedge \inf\left\{t \in [0,\tau)\,, A_t \leq \frac{1}{n}\right\}.$$

Almost surely, $\tau_n \to \tau$ as n goes to infinity and $A_t > \frac{1}{n}$ on $[0, \tau_n)$. Then, on the time interval $[0, \tau_n)$, one has

$$\begin{aligned}
\mathrm{d}W_t - \mathrm{d}\left(\frac{X_t}{\sqrt{2A_t}}\right) &= \left(\mathrm{d}W_t - \frac{\mathrm{d}X_t}{\sqrt{2A_t}}\right) - X_t\,\mathrm{d}\left(\frac{1}{\sqrt{2A_t}}\right) \\
&= 0 + (2A_t)^{-3/2}X_t\,\mathrm{d}A_t \\
&= 0\,,
\end{aligned} \tag{7}$$

where the last equality is a consequence of $X_t\,\mathrm{d}L_t^X = 0$. As a consequence, we deduce that $W_t - \frac{X_t}{\sqrt{2A_t}} = W_0 - \frac{X_0}{\sqrt{2A_0}} = -\frac{X_0}{\sqrt{2A_0}}$ up to time τ_n. Letting n go to infinity, we obtain $X_t = \sqrt{2A_t}\left(W_t + \frac{X_0}{\sqrt{2A_0}}\right)$ for all $0 \leq t < \tau$.

According to Tanaka's formula (see for example [7]), the local time of a semimartingale is given by $\mathrm{d}L_t^M = \mathrm{d}|M_t| - \mathrm{sign}(M_t)dM_t$. Hence, it the case $X_0 = 0$, we get

$$\begin{aligned}
\mathrm{d}L_t^X &= \mathrm{d}|X_t| - \mathrm{sign}(X_t)\,\mathrm{d}X_t = \mathrm{d}(\sqrt{2A_t}|W_t|) - \mathrm{sign}(W_t)\sqrt{2A_t}\,\mathrm{d}W_t \\
&= \sqrt{2A_t}\Big(\mathrm{d}|W_t| - \mathrm{sign}(W_t)\,\mathrm{d}W_t\Big) + (|W_t| - \mathrm{sign}(W_t)W_t)\frac{\mathrm{d}A_t}{\sqrt{2A_t}} \\
&= \sqrt{2A_t}\,\mathrm{d}L_t^W.
\end{aligned}$$

□

Proposition 3 is what we needed to establish a link between solutions to (4) and (5).

Corollary 4. *A continuous process* $(X_t, A_t)_{0\leq t<\tau}$ *defined up to time* τ *and satisfying* $X_0 = 0$ *is a strong solution to Equation* (5) *if and only if it is a strong solution to Equation* (4).

Proof. This is a direct consequence of the second equality in (6). □

Corollary 5. *For any initial condition* (X_0, A_0) *independent of* $(W_t)_{t\geq 0}$, *there exists a unique maximal solution to Equation* (4).

Proof. If $X_0 = 0$, there exists a unique maximal solution to (5) from Proposition 1. From Corollary 4, it is also the unique maximale solution to (5).

For a general initial condition, up to the time $\zeta = \inf\{t \geq 0, X_t = 0\}$, system (4) clearly admits a unique solution $X_t = X_0 + \sqrt{2A_0}W_t$, $A_t = A_0$. After ζ, the Markov property allows to apply existence and uniqueness starting from $X_0 = 0$. □

2.3 A discrete time approximation

In this section, we construct an approximation to the process $(X_t, A_t)_{t\geq 0}$. This will give an heuristic justification to equation (6).

The Brownian motion will be discretized by a simple random walk

$$Y_n = \sum_{k=1}^{n} U_k, \quad n \in \mathbb{N},$$

where $(U_k)_{k\in\mathbb{N}^*}$ is a sequence of independant random variables, uniformly distributed on $\{1, -1\}$. We will need the *discrete local time* of $(Y_n)_{n\in\mathbb{N}}$, defined by

$$\Lambda_n = \sum_{k=1}^{n} \mathbf{1}_{Y_k=0}.$$

Let $t > 0$. Then one has the convergence in distribution

$$\left(\sqrt{\frac{t}{n}}Y_n, \sqrt{\frac{t}{n}}\Lambda_n\right) \to \left(W_t, L_t^W\right),$$

as $n \to \infty$.

Two different approximations are natural. Here we modify the step size as in [2].

$$\forall k \in \{1, \ldots, n\} \begin{cases} \hat{X}_k^n &= \hat{X}_{k-1}^n + \sqrt{2\hat{A}_{k-1}^n \frac{t}{n}} U_n, \\ \hat{A}_k^n &= \hat{A}_{k-1}^n + f(\hat{A}_{k-1}^n)\sqrt{\frac{t}{n}} \mathbf{1}_{\hat{X}_k^n=0}. \end{cases}$$

We first state a discrete analog to Equation (6).

Lemma 6. *One has the equalities*

$$X_k^n = \sqrt{2\hat{A}_k^n \frac{t}{n}} Y_k, \text{ and } \mathbf{1}_{\hat{X}_k^n=0} = \mathbf{1}_{Y_k=0}.$$

Consider the Euler scheme associated to $y' = f(y)$, namely

$$y_{n+1}^\delta = y_n^\delta + \delta f(y_n^\delta)$$

where δ is some time step. Then, when $n \to \infty$ and $\delta \to 0$ in the regime $n\delta \to t$, one has

$$y_n^\delta \to \Phi_{y_0}(t).$$

Lemma 7. *$\hat{A}_k^n$ is given by $y_{\Lambda_k}^{\sqrt{t/n}}$.*

Theorem 8. *As n goes to infinity, $(\hat{X}_n^n, \hat{A}_n^n)$ converges in distribution to (X_t, A_t).*

Proof. One has $\hat{A}^n_n = y^{\sqrt{t/n}}_{\Lambda_n}$. Here $\sqrt{\frac{t}{n}}$ and Λ_n respectively converge to 0 and ∞ in the regime $\sqrt{\frac{t}{n}}\Lambda_n \to L^W_t$. As a consequence, $\hat{A}^n_n$ converges to $\Phi_{A_0}(L^W_t) = A_t$.

On the other hand, one has

$$\hat{X}^n_n = \sqrt{2\hat{A}^n_n} \times \sqrt{\frac{t}{n}}U_n \to \sqrt{2A_t} \times W_t = X_t\,.$$

$\square$

Remark 9. *For simplicity, we only proved convergence for a fixed time t. Actually one can prove convergence for the whole trajectory.*

2.4 A particular case: f is a power function

From the second equality in (6), one can expect at least four different long-time behaviors for the process $(X_t)_{t\geq 0}$. Indeed, the process can stop in finite time, in the case where there exists a finite time t such that $A_t = 0$. On the opposite, the process can perform very large oscillations if $(A_t)_{t\geq 0}$ tends to infinity in finite time. Last, when $(A_t)_{t\geq 0}$ takes its values in $(0, \infty)$, we can expect the process $(X_t)_{t\geq 0}$ either to go asymptotically to 0, if $(A_t)_{t\geq 0}$ decreases fast enough, or to be recurrent in $\mathbb{R}$, in the case where $(A_t)_{t\geq 0}$ remains large enough.

Those four behaviors actually do occur in the case of a power function $f(a) = \pm a^\gamma$. The advantage of such a function is that the expression of A_t can explicitely be computed as a function of L^W_t.

When $(A_t)_{t\geq 0}$ remains in $(0, \infty)$ for all positive t, one can derive a polynomial behavior for $(X_t)_{t\geq 0}$. Indeed, up to renormalisation by a power of t, we prove convergence in law to a non-Gaussian distribution for $(X_t)_{t\geq 0}$.

In a first time, we give an explicit expression of A_t, and then we give the asymptotic behavior of X_t, depending on the sign of f.

2.4.1 Explicit of A_t

The following lemma gives the expression of A_t as a function of L^W_t. For simplicity we assume that $X_0 = 1$ and $A_0 = 1$.

Lemma 10. *Let $f(a) = \sigma a^\gamma$, with $\sigma = \pm 1$. The solution $(X_t, A_t)_{0\leq t<\tau}$ to (4) exists up to the time τ defined by*

$$\tau := \begin{cases} +\infty & \text{if } \sigma(3/2-\gamma) \geq 0, \\ \inf\left\{t \geq 0,\ L^W_t = \frac{\sqrt{2}}{\sigma(\gamma-3/2)}\right\} & \text{if } \sigma(3/2-\gamma) < 0\,. \end{cases} \tag{8}$$

For all $t \in [0, \tau)$, one has

$$A_t = \begin{cases} e^{\sigma L^W_t/\sqrt{2}} & \text{if } \gamma = 3/2\,, \\ (1 + \sigma/\sqrt{2}(3/2-\gamma)L^W_t)^{\frac{1}{3/2-\gamma}} & \text{if } \gamma \neq 3/2\,. \end{cases}$$

Proof. As we have seen in the proof of Proposition 1, A_t is given by $A_t = \Phi_{A_0}(L_t^W)$ where Φ_x is the solution to $y'(t) = f(y(t))/\sqrt{2y}$ with initial condition $\Phi_x(0) = x$. Here, Φ_1 is given by

$$\Phi_1(t) = \begin{cases} e^{\sigma t/\sqrt{2}} & \text{if } \gamma = 3/2\,, \\ (1 + \sigma/\sqrt{2}(3/2 - \gamma)t)^{\frac{1}{3/2-\gamma}} & \text{if } \gamma \neq 3/2\,, \end{cases}$$

which lies in $(0, \infty)$ for $0 \leq t < \sqrt{2}(\sigma(\gamma - 3/2))^{-1}$ if $\sigma(3/2 - \gamma) < 0$, and for all $t \geq 0$ otherwise. The expressions of τ and A_t easily follow. $\square$

2.4.2 Local deceleration: $f(a) = -a^\gamma$

In that case the process $(X_t)_{t\geq 0}$ is slowing down and we obtain a dynamical transition to an absorbing state. For sufficiently strong deceleration, the particle might get trapped at the origin in finite time while if the deceleration process is sufficiently weak the particle never gets trapped.

Proposition 11. *Assume that $f(a) = -a^\gamma$, then*

- *if $\gamma < 3/2$, the stopping time τ defined in* (8) *is almost surely finite, and one has $\lim_{t\to\tau}(X_t, A_t) = (0, 0)$;*
- *if $3/2 \leq \gamma < 2$, $\tau = \infty$ and $X_t \to 0$ as t goes to ∞. However, almost surely, for all $t > 0$, there exists $s > t$ such that $X_s \neq 0$;*
- *if $2 \leq \gamma$, then $\tau = \infty$ and the process $(X_t)_{t\geq 0}$ is recurrent in $\mathbb{R}$.*

Proof. The case $\gamma < 3/2$ is a consequence of the fact that $(A_t)_{t\geq 0}$ is absorbed by 0 in finite time.

The case $\gamma \geq 3/2$, $\gamma \neq 2$ follows from the almost sure asymptotic behavior $t^{1/2-\varepsilon} = o(L_t^W)$ and $L_t^W = o(t^{1/2+\varepsilon})$, for any $\varepsilon > 0$, and from the law of iterated logarithm for Brownian motion.

In, the limit case $\gamma = 2$, $(X_t)_{t\geq 0}$ is equivalent in the long time to $(8W_t/L_t^W)_{t\geq 0}$, and it is enough to consider this latter process. From [6], Theorem 4.5, in the case $\beta = 2$, and $f(x) = 1/(x \log x)$, there exist random times $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ such that

$$\forall n \in \mathbb{N}, \ \frac{\sup_{s\leq t_n} |W_s|}{L_{t_n}^W} \geq \log(L_{t_n}^W)\,.$$

Let $(\tilde{t}_n)_{n\in\mathbb{N}}$ be a nondecreasing sequence satisfying $W_{\tilde{t}_n} = \sup_{s\leq t_n} |W_s|$, in particular one has $|W_{\tilde{t}_n}|/L_{\tilde{t}_n}^W \to \infty$. Since $\limsup_{t\to\infty} |W_t| = \infty$, up to a subsequence, one has $\tilde{t}_n \to \infty$. Moreover, one can find another sequence of random times t'_n going to ∞ and satisfying $W_{t'_n} = 0$. As a consequence, one obtains

$$\liminf_{t\to\infty} \frac{|W_t|}{L_t^W} = 0 \text{ and } \limsup_{t\to\infty} \frac{|W_t|}{L_t^W} = \infty\,.$$

Hence, $(|X_t|)_{t\geq 0}$ is recurrent in $[0, \infty)$, and by symmetry this implies that $(X_t)_{t\geq 0}$ is recurrent in $\mathbb{R}$. $\square$

Using the equality in distribution

$$X_t = \sqrt{2A_t}W_t = \sqrt{2}\left(1 - (3/2-\gamma)/\sqrt{2}L_t^W\right)^{\frac{1}{3-2\gamma}} W_t$$
$$\overset{(d)}{=} \sqrt{2}\left(1 + (\gamma - 3/2)\sqrt{t/2}L_1^W\right)^{\frac{1}{3-2\gamma}} \sqrt{t}W_1\,, \tag{9}$$

for $\gamma \neq 3/2$, we deduce that, when $\gamma > 1$, the decreasing rate of the process $(X_t)_{t\geq 0}$ is $t^{\frac{2-\gamma}{3-2\gamma}}$. In this expresson of the rate, the exponent may be nonpositive or nonnegative. More precisely, one has:

Proposition 12. *If $f(a) = -a^\gamma$, with $\gamma > 3/2$, then the convergence in distribution,*

$$t^{\frac{\gamma-2}{3-2\gamma}} X_t \overset{(d)}{\to} C_\gamma (L_1^W)^{\frac{1}{2(3-2\gamma)}} W_1 \quad \textit{as } t \to \infty\,,$$

holds true where $C_\gamma = 2^{\frac{1-\gamma}{3-2\gamma}}(\gamma - 3/2)^{\frac{1}{2(3-2\gamma)}}$.

One can also give the asymptotic behavior as $t \to \tau$ when $\tau < \infty$.

Proposition 13. *If $f(a) = -a^\gamma$, with $\gamma < 3/2$, then, as $t \to 0$,*

$$\mathbf{1}_{t<\tau} t^{\frac{\gamma-2}{3-2\gamma}} X_{\tau-t} \overset{(d)}{\to} C'_\gamma (L_1^W)^{\frac{1}{2(3-2\gamma)}} W_1\,,$$

where $C'_\gamma = 2^{\frac{1-\gamma}{3-2\gamma}}(3/2-\gamma)^{\frac{1}{2(3-2\gamma)}}$.

Proof. We use the reversibility property of the Brownian motion that we recall, for $T > 0$, setting $\zeta = \inf\{t > 0, L_t^W = T\}$, the equality in distribution

$$(W_t, L_t^W)_{0\leq t\leq\zeta} \overset{(d)}{=} (W_{\zeta-t}, T - L_{\zeta-t}^W)_{0\leq t\leq\zeta}\,,$$

holds true. In our situation, we apply this property to the stopping time $\zeta = \tau$, and we then use (9). □

Remark 14. *The fact that $(X_t)_{t\geq 0}$ can be trapped at 0 for $\gamma < 3/2$ was already noticed in [1]. However, the different behavior for $\gamma \in [3/2, 2)$ was not observed.*

Remark 15. *As a consequence of Lemma 10, the survival probability of $(X_t)_{t\geq 0}$ at time t is given for $\gamma < 3/2$ by*

$$S(t) = \mathbb{P}\left(L_t^W \leq \frac{\sqrt{2}}{3/2-\gamma}\right) = \mathbb{P}\left(|W_1| \leq \frac{\sqrt{2}}{\sqrt{t}(3/2-\gamma)}\right) \underset{t\to\infty}{\sim} \frac{2}{(3/2-\gamma)\sqrt{\pi t}}\,,$$

where we used the equalities in distribution $L_t^W = |W_t| = \sqrt{t}|W_1|$. This fact was already observed in [1].

2.4.3 Local acceleration: $f(a) = a^\gamma$

In such a case the diffusion coefficient of $(X_t)_{t\geq 0}$ is nondecreasing. Again, the proof of the following result follows from Lemma 10 and the relation $X_t = \sqrt{2A_t}W_t$.

Proposition 16. *Assume that $f(a) = a^\gamma$, then*

- *if $\gamma \geq 2$, the stopping time τ defined in* (8) *is almost surely finite and* $\lim_{t\to\tau} A_t = \infty$. *Moreover,* $\lim_{t\to\tau} X_t = 0$*;*
- *if* $3/2 < \gamma < 2$, *τ is almost surely finite and* $\lim_{t\to\tau} A_t = \infty$. *Moreover,* $\liminf_{t\to\tau} X_t = -\infty$ *and* $\limsup_{t\to\tau} X_t = \infty$*;*
- *if $\gamma \leq 3/2$, the time τ satisfies $\tau = \infty$, and $A_t \to \infty$ when $t \to \infty$.*

Furthermore, when $\gamma < 3/2$, from equality in distribution (9), one can deduce that the long time behavior of $(X_t)_{t\geq 0}$ is of order $t^{\frac{2-\gamma}{3-2\gamma}}$, where the exponent is positive. More precisely the following result holds true.

Proposition 17. *If $f(a) = a^\gamma$, with $\gamma < 1$, then, as $t \to \infty$, one has the convergence in distribution*

$$t^{\frac{\gamma-2}{3-2\gamma}} X_t \stackrel{(d)}{\to} C'_\gamma (L_1^W)^{\frac{1}{3-2\gamma}} W_1 \,,$$

where $C'_\gamma = 2^{\frac{1-\gamma}{3-2\gamma}} (3/2 - \gamma)^{\frac{1}{2(3-2\gamma)}}$.

We can also describe the rate of explosion of $(X_t, A_t)_{t\geq 0}$ as goes to τ, in the case $\tau < \infty$.

Proposition 18. *If $f(a) = a^\gamma$, with $\gamma > 1$, then, as $t \to 0$, one has the convergence in distribution*

$$\mathbf{1}_{t<\tau} t^{\frac{\gamma-2}{3-2\gamma}} X_{\tau - t} \stackrel{(d)}{\to} C_\gamma (L_1^W)^{\frac{1}{3-2\gamma}} W_1,$$

where $C_\gamma = 2^{\frac{1-\gamma}{3-2\gamma}} (\gamma - 3/2)^{\frac{1}{2(3-2\gamma)}}$.

Remark 19. *The case $\gamma = 0$ was treated by deterministic methods in [1], through an approximation of the Laplace transform of the distribution of X_t. Here, by using the stochastic differential equation* (4)*, we were able to compute the exact asymptotic behavior. We obtain that the growth rate of $(X_t)_{t\geq 0}$ is given by $t^{2/3}$, and that its diffusion coefficient, given by $\left(L_t^W\right)^{3/2}$, behaves as $t^{1/3}$. Those exponents were correctly predicted in [1].*

3 Generator of the process $(X_t, A_t)_{t\geq 0}$

In this section we investigate the generator of the Markov process $(X_t, A_t)_{t\geq 0}$ solution to system (4). The expression of the generator will follow from the generator of the process $(W_t, L_t^W)_{t\geq 0}$, where W_t is a standard Brownian motion and L_t^W is its local time at 0.

Consider the unique maximal solution $(\tau, (X_t)_{0\leq t<\tau}, (A_t)_{0\leq t<\tau})$ of (4), whose existence is ensured by Corollary 5. The first step is to extend its state space in order to define a continuous Markov process for all positive times.

3.1 Extended state space

In the proof of Proposition 1, we mentionned that, when τ is finite, A_t necessarily converges as t goes to τ, either toward 0 or ∞. In the case $A_t \to 0$, one can also determine the behavior of X_t, as stated in the following lemma.

Lemma 20. *On the event $\{\tau < \infty, \lim_{t\to\tau} A_t = 0\}$, X_t converges to 0 as t goes to τ.*

Proof. First, one notices that on the set $\{\tau < \infty, \lim_{t\to\tau} A_t = 0\}$ the limit

$$\lim_{t\to\tau} X_t = X_0 + \int_0^\tau \sqrt{2A_s}\, \mathrm{d}W_s$$

exists almost surely. It is thus enough to prove that the event $E = \{\tau < \infty, \lim_{t\to\tau} A_t = 0, \lim_{t\to\tau} X_t \neq 0\}$ is a null set.

On E, there exists a random variable $h > 0$ such that $X_t \neq 0$ for all $t \in (\tau - h, \tau)$. Hence, on E, one has for all $t \in (\tau - h, \tau)$

$$A_t - A_\tau = \int_t^\tau \mathrm{d}A_s = \int_t^\tau f(A_s)\, \mathrm{d}L_t^X = 0\,.$$

However, on E, one also has $\tau = \inf\{t \geq 0, A_t = 0\}$, which contradicts the fact that A_t is constant on $(\tau - h, \tau)$. As a consequence, E has probability 0, which concludes the proof. $\square$

From Lemma 20, the maximal solution $(\tau, (X_t)_{0\leq t<\tau}, (A_t)_{0\leq t<\tau})$ to Equation (4) can be extended to a process defined for all positive times by setting

$$(X_t, A_t) = \begin{cases} (0,0) & \text{on } \{\tau \leq t, \lim_{s\to\tau} A_s = 0\}\,, \\ (0,\infty) & \text{on } \{\tau \leq t, \lim_{s\to\tau} A_s = \infty\}\,. \end{cases}$$

For notational simpliciy the extended process will still be denoted by $(X_t, A_t)_{t\geq 0}$. This will define a Markov process with state space $\mathcal{E} = (\mathbb{R} \times [0,\infty)) \cup \{(0,\infty)\}$, which is the half plane $\mathbb{R} \times [0,\infty)$ augmented with an additional point $(0,\infty)$. We define the following topology on $\mathcal{E}$: the subset $\mathbb{R} \times [0,\infty)$ is endowed with its usual topology, and we choose the family $(\mathbb{R} \times [\alpha,\infty))_{\alpha>0}$ as a neighborhood basis of $(0,\infty)$. In other words, any sequence $(x_n, a_n)_{n\in\mathbb{N}}$ in $\mathbb{R} \times [0,\infty)$ with $a_n \to \infty$ will satisfy $(x_n, a_n) \to (0,\infty)$ in $\mathcal{E}$.

With these conventions, $(X_t, A_t)_{t\geq 0}$ defines a continuous Markov process with values in $\mathcal{E}$, defined for all positive times. A natural question is then to investigate its generator, and to compute the distribution of (X_t, A_t) for a given $t > 0$. Note that the two points $(0,0)$ and $(0,\infty)$ are absorbing points for the Markov process $(X_t, A_t)_{t\geq 0}$.

3.2 Generator of the process $(W_t, L_t^W)_{t\geq 0}$

Proposition 21. *The generator $\mathcal{L}_0$ of the Markov process $(W_t, L_t^W)_{t\geq 0}$ is given by*

$$\mathcal{L}_0 h(w,l) = \frac{1}{2}\partial_{ww}^2 h(w,l) + \partial_l h(w,l)\delta_{w=0}\,.$$

Remark 22. *Since the coefficient $\delta_{w=0}$ is singular, the previous definition of the generator has to be understood in a weak sense. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function with bounded support, then, for h continuously differentiable in the l-variable and twice continuously differentiable in the w-variable with bounded derivatives, for all $l \in [0, \infty)$, one has*

$$\lim_{t\to 0} \int_{\mathbb{R}} \varphi(w)\mathbb{E}^{w,l}\left[\frac{h(W_t, L_t^W) - h(w,l)}{t}\right] \mathrm{d}w = \int_{\mathbb{R}} \varphi(w)\frac{1}{2}\partial^2_{ww} h(w,l)\,\mathrm{d}w + \varphi(0)\partial_l h(0,l)\,.$$

Here, $\mathbb{E}^{w,l}$ stands for the expectation conditionaly to $\{W_0 = w,\ L_0^W = l\}$. Equivalently, one has, in the distributional sense,

$$\lim_{t\to 0} \mathbb{E}^{w,l}\left[\frac{h(W_t, L_t^W) - h(w,l)}{t}\right] = \frac{1}{2}\partial^2_{ww} h(w,l) + \partial_l h(0,l)\delta_{w=0}\,.$$

Proof. From time invariance of $(L_t^W)_{t\geq 0}$, one can assume that $l = 0$. From the equality $h(W_t, L_t^W) = h(W_t, 0) + L_t^W \int_0^1 \partial_l h(W_t, sL_t^W)\,\mathrm{d}s$, one obtains

$$\mathbb{E}^{w,0}\left[\frac{h(W_t, L_t^W) - h(w,0)}{t}\right] = \mathbb{E}^{w,0}\left[\frac{h(W_t, 0) - h(w,0)}{t}\right] + \mathbb{E}^{w,0}\left[\frac{L_t^W}{t}\int_0^1 \partial_l h(W_t, sL_t^W)\,\mathrm{d}s\right].$$

The first term in the right hand side converges to $\frac{1}{2}\partial^2_{ww} h(w,0)$ as $t \to 0$, since $\frac{1}{2}\partial^2_{ww}$ is the generator of the Brownian motion, using the boundedness of $\partial^2_{ww} h$.

Then, using the fact that the law of (W_t, L_t^W) under $\mathbb{E}^{w,0}$ is the same than the law of $(\sqrt{t}W_1, \sqrt{t}L_1^W)$ under $\mathbb{E}^{w/\sqrt{t},0}$, one gets

$$\begin{aligned}
&\int_{\mathbb{R}} \varphi(w)\mathbb{E}^{w,0}\left[\frac{L_t^W}{t}\int_0^1 \partial_l h(W_t, sL_t^W)\,\mathrm{d}s\right]\mathrm{d}w \\
&= \int_{\mathbb{R}} \varphi(w)\mathbb{E}^{w/\sqrt{t},0}\left[\frac{\sqrt{t}L_1^W}{t}\int_0^1 \partial_l h(\sqrt{t}W_1, s\sqrt{t}L_1^W)\,\mathrm{d}s\right]\mathrm{d}w \\
&= \int_{\mathbb{R}} \varphi(\sqrt{t}w)\mathbb{E}^{w,0}\left[L_1^W\int_0^1 \partial_l h(\sqrt{t}W_1, s\sqrt{t}L_1^W)\,\mathrm{d}s\right]\mathrm{d}w\,.
\end{aligned}$$

From the dominated convergence theorem, this converges as t goes to 0 to

$$\varphi(0)\partial_l h(0,0)\int_{\mathbb{R}} \mathbb{E}^{w,0}[L_1^W]\,\mathrm{d}w\,.$$

Finally, from the occupation time formulation $\int_{\mathbb{R}} L_t^{W+w}\,\mathrm{d}w = t$ (see [7]) one obtains

$$\int_{\mathbb{R}} \mathbb{E}^{w,0}[L_1^W]\,\mathrm{d}w = \int_{\mathbb{R}} \mathbb{E}^{0,0}[L_1^{W+w}]\,\mathrm{d}w = 1\,.$$

□

3.3 Generator of the process defined by system (4)

Since $(X_t, A_t)_{t\geq 0}$ can be obtained as a function of $(W_t, L_t^W)_{t\geq 0}$, we can compute the generator of the former from the generator of the latter.

Proposition 23. *The generator $\mathcal{L}$ of the Markov process $(X_t, A_t)_{t\geq 0}$ solution to system* (4) *is given by*

$$\mathcal{L}h(x,a) = a\partial_{xx}^2 h(x,a) + f(a)\partial_a h(x,a)\delta_{x=0}\,.$$

Remark 24. *Again, the previous definition has to be understood in a weak sense. If $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function with bounded support, then, for h continuously differentiable in the a-variable and twice continuously differentiable in the x-variable with bounded derivatives, for all $a \in [0,\infty)$ one obtains*

$$\begin{aligned} &\lim_{t\to 0}\int_{\mathbb{R}} \varphi(x)\tilde{\mathbb{E}}^{x,a}\left[\frac{h(X_t,A_t)-h(x,a)}{t}\right]\mathrm{d}x \\ = &\int_{\mathbb{R}} \varphi(x)a\partial_{xx}^2 h(x,a)\,\mathrm{d}x + \varphi(0)f(a)\partial_a h(0,a)\,. \end{aligned}$$

Here, $\tilde{\mathbb{E}}^{x,a}$ stands for the expectation conditionaly to $\{X_0 = x,\ A_0 = a\}$.

Remark 25. *Since (X_t, A_t) is a Markov process, the following identity holds for any probability density φ*

$$\int_{\mathbb{R}} \varphi(x)\tilde{\mathbb{E}}^{x,a}[h(X_t,A_t)]\,\mathrm{d}x = \tilde{\mathbb{E}}^{\varphi,a}[h(X_t,A_t)],$$

where $\tilde{E}^{\varphi,a}$ denotes the expectation for an initial condition satisfying $A_0 = a$ and such that X_0 admits φ as density. In other word, when X_0 admits a continuous density v_0, replacing φ by v_t in Remark 24 yields the following time-derivative at $t = 0$:

$$\begin{aligned} &\partial_t\tilde{\mathbb{E}}^{v_0,a}[h(X_t,A_t)] = \partial_t\iint_{\mathbb{R}\times\mathbb{R}^+} h(x,a)\,\mathrm{d}\mu_t(x,a) \\ = &\int_{\mathbb{R}} a\partial_{xx}^2 h(x,a)v_0(x)\,\mathrm{d}x + v_0(0)f(a)\partial_a h(0,a). \end{aligned}$$

This is a weak formulation of Equation (10) *below.*

Proof. Let $(W_t)_{t\geq 0}$ be a Brownian motion started at $X_0/\sqrt{2A_0}$. From the proof of Proposition 1, we know that the process $(X_t, A_t)_{0\leq t<\tau}$ is given by

$$\forall 0 \leq t < \tau,\ \begin{cases} X_t &= \sqrt{2\Phi_{A_0}(L_t^W)}W_t\,, \\ A_t &= \Phi_{A_0}(L_t^W)\,, \end{cases}$$

where $t \to \Phi_x(t)$ is the flow of the differential equation $y' = f(y)/\sqrt{2y}$ with initial condition x. Then, setting $x = \sqrt{2a}w$, one has (if we still denote by $\mathbb{E}^{w,l}$ the expectation conditionally to $\{W_0 = w, L_0^W = l\})$

$$
\begin{aligned}
&\int_{\mathbb{R}} \varphi(x)\tilde{\mathbb{E}}^{x,a}\left[\frac{h(X_t, A_t) - h(x,a)}{t}\right] \mathrm{d}x \\
=&\int_{\mathbb{R}} \varphi(x)\mathbb{E}^{\frac{x}{\sqrt{2a}},0}\left[\frac{h(\sqrt{2\Phi_a(L_t^W)}W_t, \Phi_a(L_t^W)) - h(x,a)}{t}\right] \mathrm{d}x \\
=&\sqrt{2a}\int_{\mathbb{R}} \varphi(\sqrt{2a}w)\mathbb{E}^{w,0}\left[\frac{h(\sqrt{2\Phi_a(L_t^W)}W_t, \Phi_a(L_t^W)) - h(\sqrt{2a}w,a)}{t}\right] \mathrm{d}w\,.
\end{aligned}
$$

Applying Proposition 21 to the function $F(w,l) = h\left(\sqrt{2\Phi_a(l)}w, \Phi_a(l)\right)$ one obtains

$$
\begin{aligned}
&\lim_{t\to 0}\int_{\mathbb{R}} \varphi(x)\tilde{\mathbb{E}}^{x,a}\left[\frac{h(X_t, A_t) - h(x,a)}{t}\right] \mathrm{d}x \\
=&\lim_{t\to 0}\sqrt{2a}\int_{\mathbb{R}} \varphi(\sqrt{2a}w)\mathbb{E}^{w,0}\left[\frac{F(W_t, L_t^W) - F(w,0)}{t}\right] \mathrm{d}w \\
=&\sqrt{\frac{a}{2}}\int_{\mathbb{R}} \varphi(\sqrt{2a}w)\partial^2_{ww}F(w,0)\,\mathrm{d}w + \sqrt{2a}\varphi(0)\partial_l F(0,0) \\
=&\sqrt{\frac{a}{2}}\int_{\mathbb{R}} \varphi(\sqrt{2a}w)(2a)\partial^2_{xx}h(\sqrt{2a}w,a)\,\mathrm{d}w + \Phi_a'(0)\sqrt{2a}\varphi(0)\partial_a h(0,a) \\
=&\int_{\mathbb{R}} \varphi(x)a\partial^2_{xx}h(x,a)\,\mathrm{d}x + f(a)\varphi(0)\partial_a h(0,a)\,.
\end{aligned}
$$

□

4 PDE for the joint distribution

As a consequence of Proposition 23 and Remark 25, the density of the joint distribution $\mu_t(x,a)$ associated to the process $(X_t, A_t)_{t\geq 0}$ solution to (4), satisfies in a weak sense the following equation for $t > 0$, $(x,a) \in \mathbb{R}\times\mathbb{R}_+$:

$$
\partial_t u(t,x,a) = a\partial^2_{xx}u(t,x,a) - \partial_a\left(f(a)\,u(t,x,a)\right)\delta_{x=0}\,, \tag{10}
$$

together with the initial condition:

$$
u(t=0,x,a) = u_0(x,a)\,, \quad (x,a)\in\mathbb{R}\times\mathbb{R}_+\,.
$$

In this Section, we first give the general form of the distribution μ_t at time t. In particular we prove that the measure μ_t has a density with respect to the Lebesgue measure when considering general initial condition. Then, we prove uniqueness of the solution to (10). Furthermore, if $f(a) = -a^\gamma$, by studying the regularity of the solution in a L^p framework, we recover the results observed during the probabilistic study: global existence if $\gamma \geq 3/2$, while, in the case $\gamma < 3/2$, the solution becomes unbounded in finite time (so-called blow-up). Finally, as in [1], using Laplace and Fourier transforms, for a particular initial condition we explicitely compute the solution to (10).

Remark 26. *Since the generator maps continuous function to measure it is possible to write the pde for the density of the joint distribution. In further works it would be interesting to investigate how the pde could be written for joint distribution which involves atoms.*

4.1 Shape of the distribution of (X_t, A_t)

The three lemmas below give the general form of the distribution μ_t at time t of the solution $(X_t, A_t)_{t\geq 0}$ to (4) starting from an initial condition $(x_0, a_0) \in \mathbb{R} \times (0, \infty)$. In particular, one starts from $\mu_0 = \delta_{(x_0,a_0)}$.

Recall that the two points $(0, 0)$ and $(0, \infty)$ are absorbing points for the process $(X_t, A_t)_{t\geq 0}$, so that the distribution starting from those points will be constant, equal to $\delta_{(0,0)}$ or $\delta_{(0,\infty)}$ respectively.

By linearity of Equation (10), the distribution starting from a more general initial condition can be obtained as a mixture of distributions starting at deterministic points. We first consider the case where the initial condition is $\mu_0 = \delta_{(0,a_0)}$, with $a_0 > 0$.

Lemma 27. *Assume that the initial condition (x_0, a_0) satisfies $f(a_0) \neq 0$ and $x_0 = 0$. Then, for all $t > 0$, there exist a measurable function $n_t : \mathbb{R} \times (0, \infty) \to \mathbb{R}$ and two real numbers $p_t \in [0, 1]$ and $q_t \in [0, 1]$ such that*

$$\mu_t(x, a) = n_t(x, a)\,\mathrm{d}x\,\mathrm{d}a + p_t\delta_{(0,0)} + q_t\delta_{(0,\infty)}, \tag{11}$$

where $p_t + q_t + \int_{\mathbb{R}\times[0,\infty)} n_t(x, a)\,\mathrm{d}x\,\mathrm{d}a = 1$. Furthermore, if f is nonnegative, one has $p_t = 0$ for all $t > 0$, while if f is nonpositive, one has $q_t = 0$ for all $t > 0$.

Proof. All we need to prove is that the restriction of μ_t to the set $\mathbb{R} \times (0, \infty)$ admits a density with respect to the Lebesgue measure.

First, since $t^{-1/2}(W_t, L^W_t)$ has the same distribution as (W_1, L^W_1), which has a density, see [5], page 45, it follows that (W_t, L^W_t) with $W_0 = 0$ admits a density γ_t on $\mathbb{R} \times (0, \infty)$. Moreover, as $(X_t)_{0\leq t<\tau}$ starts from the initial condition $x_0 = 0$, Proposition 3 states that $(X_t, A_t) = \left(\sqrt{2\Phi_{a_0}(L^W_t)}W_t, \Phi_{a_0}(L^W_t)\right)$, where Φ_{a_0} is the flow of the differential equation $y' = f(y)$ starting at a_0. For a given a_0, $\Phi_{a_0}(l)$ is defined for all $l \in [0, T(a_0))$ for some $T(a_0) \in [0, \infty]$.

Let us define the mapping $\Psi(w, l) := \left(\sqrt{2\Phi_{a_0}(l)}w, \Phi_{a_0}(l)\right)$, for (w, l) in $\mathbb{R} \times (0, T(a_0))$. To conclude, it is enough to show that Ψ is a local diffeomorphism from $\mathbb{R}\times(0, T(a_0))$ to $\mathbb{R}\times(0, \infty)$. The Jacobian determinant of the $\mathcal{C}^1$ function Ψ is given by

$$J_\Psi(w, l) = \Phi'_{a_0}(l)\sqrt{2\Phi_{a_0}(l)} = f\left(\Phi_{a_0}(l)\right)\,.$$

From uniqueness in the Cauchy-Lipschitz theorem, if $f(\Phi_{a_0}(l)) = 0$ for some l, then $f(\Phi_{a_0}(l)) = 0$ for all $l \in [0, T(a_0))$, but this contradicts the assumption $f(a_0) \neq 0$. As a consequence, J_Ψ does not vanish on $\mathbb{R} \times (0, T(a_0))$, so that Ψ is a local diffeomorphism, and μ_t has a density on $\mathbb{R} \times (0, \infty)$. $\square$

Without the assumption $x_0 = 0$, $(X_t)_{t\geq 0}$ will stay away from 0 for a positive time ζ. In that case $(A_t)_{t\geq 0}$ remains constant on the interval $[0, \zeta]$, and this results in a more complicated expression for μ_t, as stated in the following lemma.

Lemma 28. *Assume that $f(a_0) \neq 0$ and $x_0 \neq 0$. Then, μ_t has the form*

$$\mu_t(x,a) = m_t(x)\,\mathrm{d}x \otimes \delta_{a_0} + n_t(x,a)\,\mathrm{d}x\,\mathrm{d}a + p_t\delta_{(0,0)} + q_t\delta_{(0,\infty)}\,,$$

where m_t, n_t, are measurable functions respectively defined on $\mathbb{R}$ and $\mathbb{R}\times(0,\infty)$.

Proof. This relies on the strong Markov property used at time $\inf\{t>0, X_t = 0\}$ together with Lemma 27. □

The last case to consider is when the process starts from a point where its diffusion coefficients does not change. In that case, $(X_t)_{t\geq 0}$ exists for all positive times, and behaves as a Brownian motion multiplied by some constant.

Lemma 29. *Assume that $f(a_0) = 0$. Then, μ_t is given by*

$$\mu_t(x,a) = \gamma_{2ta_0}^{x_0}(x)\,\mathrm{d}x \otimes \delta_{a_0}\,,$$

where $\gamma_{\sigma^2}^{x_0}$ denotes the Gaussian distribution with mean x_0 and variance σ^2.

Proof. In that case $(X_t, A_t)_{t\geq 0}$ is given by $(X_t, A_t) = (x_0 + \sqrt{2a_0}W_t, a_0)$. □

4.2 Basic facts about weak solution to (10)

This is a linear equation on $u = u(t,x,a)$ defined on $t \geq 0$, $x \in R$, $a \geq 0$. We begin with a proper definition of weak solutions, adapted to our context. We recall that f is assumed to be locally Lipschitz continuous from $(0,\infty)$ to $\mathbb{R}$.

Definition 30. *We say that u is a weak solution to (10) on $(0,T)$ if it satisfies:*

$$u \in L^\infty(0,T;L^1(\mathbb{R}\times\mathbb{R}_+))\,, \quad \partial_x u \in L^1((0,T)\times\mathbb{R}\times\mathbb{R}_+)\,, \tag{12}$$

and $u(t,x,a)$ is a solution to (10) in the sense of distributions in $\mathcal{D}'(\mathbb{R}\times\mathbb{R}_+)$: for any test function φ in $\mathcal{D}(\mathbb{R}\times\mathbb{R}_+)$ and a.e. $t \in (0,T)$,

$$\begin{aligned}&\iint u(t,x,a)\varphi(x,a)\,\mathrm{d}x\,\mathrm{d}a = \iint u_0(x,a)\varphi(x,a)\,\mathrm{d}x\,\mathrm{d}a\\ &- \int_0^t \iint a\partial_x u(s,x,a)\partial_x\varphi(x,a)\,\mathrm{d}x\,\mathrm{d}a\,\mathrm{d}s + \int_0^t \int f(a)u(s,0,a)\partial_a\varphi(0,a)\,\mathrm{d}a\,\mathrm{d}s\,.\end{aligned}$$

Since $\partial_x u(t,x,a)$ belongs to $L^1((0,T)\times\mathbb{R}\times\mathbb{R}_+)$, the solution is well-defined in the distributional sense under assumption (12). In fact we can write $\int_0^T u(t,0,a)\,\mathrm{d}t = -\int_0^T\int_{x>0}\partial_x u(t,x,a)\,\mathrm{d}x\,\mathrm{d}t$.

Weak solutions in the sense of Definition 30 are mass-preserving:

$$M = \iint u_0(x,a)\,\mathrm{d}x\,\mathrm{d}a = \iint u(t,x,a)\,\mathrm{d}x\,\mathrm{d}a\,.$$

Let us first prove that non-negativity is preserved.

Lemma 31. *Assume that u is a weak solution to* (10) *such that $\partial^2_{xx} u \in L^1$. If $|u_0| = u_0$ almost everywhere (initial data non-negative). Then $|u(t,\cdot)| = u(t,\cdot)$ almost everywhere for later times.*

Proof. Observe that if u is solution in L^1 then $|u|$ is subsolution in L^1 since $\mathrm{sgn}(u)\partial^2_{xx} u \le \partial^2_{xx}|u|$ and $\mathrm{sgn}(u)\partial_a(f(a)\,u)\delta_{x=0} = \partial_a(f(a)\,|u|)\delta_{x=0}$. Hence $|u| - u$ is a subsolution, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint (|u| - u)\ \mathrm{d}x\,\mathrm{d}a \le 0.$$

$\square$

Let us next prove that in the case $f(a) \le 0$ (deceleration) and $u_0 \ge 0$, the compact support in a is preserved along time.

Lemma 32. *Assume u is a weak solution to* (10) *with $f(a) \le 0$. Assume in addition that supp$(u_0) \subset \mathbb{R} \times [0, a_0]$ for some $a_0 > 0$. Then supp$(u) \subset \mathbb{R} \times [0, a_0]$ up to the existence time.*

Proof. Consider any non-negative non-decreasing function $\varphi = \varphi(a)$ smooth on $\mathbb{R}_+$ with support included in $(a_0, +\infty)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint u(t,x,a)\varphi(a)\,\mathrm{d}x\,\mathrm{d}a = \int_a f(a)u(0,a)\varphi'(a)\,\mathrm{d}a \le 0\,,$$

which proves that $u\varphi = 0$ for later times. Varying the φ as defined above we conclude that $u = 0$ on $\mathbb{R} \times [a_0, +\infty)$ for later times. $\square$

Similarly, in the case $f(a) \ge 0$ (acceleration) and $u_0 \ge 0$, we can prove that if the support of u_0 is included in $\mathbb{R} \times [a_0, +\infty)$ for some $a_0 > 0$, then the same fact is true for all $t > 0$.

4.3 The different cases for the law of change in the particular case where $f(a) = \pm a^\gamma$

Following the probabilistic study performed in Section 2.4 we consider the three following cases:

(a) acceleration at $x = 0$: $f(a) = a^\gamma$ with $\gamma \ge 0$,

(b) subcritical deceleration at $x = 0$: $f(a) = -a^\gamma$ with $\gamma \ge 3/2$,

(c) supercritical deceleration at $x = 0$: $f(a) = -a^\gamma$ with $\gamma \in [0, 3/2)$.

In this part we will prove the following result:

Theorem 33. *Assume that the initial datum u_0 belongs to L^p, $p \ge 1$, then*

(a) in the acceleration case, there exists a unique weak solution to (10) *that satisfies for all $T > 0$, $\sup_{t\in(0,T)} \iint |u(t,x,a)|^p\,\mathrm{d}x\,\mathrm{d}a < +\infty$,*

(b) in the subcritical deceleration case, for $p = 2$, there exists a unique weak solution to (10) *that satisfies for all $T > 0$,*
$\sup_{t\in(0,T)} \iint |u(t,x,a)|^2\,\mathrm{d}x\,\mathrm{d}a < +\infty$,

(c) in the supercritical deceleration case, any weak solution of (10) *blows-up in finite time.*

Proof of Theorem 33. In the first two cases a) and b), we prove the propagation of L^p bounds, which is the crucial a priori estimate. To prove that solutions blow-up in finite time in the supercritical case c), we show that for an appropriate value of M, the momentum $\int a^M u \, \mathrm{d}a$ becomes infinite in finite time.

Case (a): global existence and uniqueness

Let $p \in [1, +\infty)$, we have the following a priori estimate:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \iint |u(t,x,a)|^p \, \mathrm{d}x \, \mathrm{d}a \leq &- p(p-1) \iint a \, |\partial_x u(t,x,a)|^2 \, |u(t,x,a)|^{p-2} \, \mathrm{d}x \, \mathrm{d}a \\ &- \gamma(p-1) \int a^{\gamma-1} |u(t,0,a)|^p \, \mathrm{d}a \leq 0 \,,\end{aligned}$$

which proves that L^p norms remains finite for all times provided they are finite initially (no finite time appearance of a singularity). By applying the same argument to the modulus of the difference of two solutions one proves similarly uniqueness in L^p.

Case (b): global existence and uniqueness

In this case the a priori estimate writes for $p = 2$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \iint u^2(t,x,a) \, \mathrm{d}x \, \mathrm{d}a &\leq -2 \iint a |\partial_x u(t,x,a)|^2 \, \mathrm{d}x \, \mathrm{d}a + \gamma \int a^{\gamma-1} u^2(t,0,a) \, \mathrm{d}a \\ &\quad - \limsup_{a \to 0} a^\gamma u^2(t,0,a) \\ &\leq -2 \iint a |\partial_x u(t,x,a)|^2 \, \mathrm{d}x \, \mathrm{d}a + \gamma \int a^{\gamma-1} u^2(t,0,a) \, \mathrm{d}a\end{aligned}$$

and we control (with $I_\varepsilon := [-\varepsilon, \varepsilon]$)

$$\begin{aligned}|u(t,0,a)| &\leq \left| u(t,0,a) - \frac{1}{|I_\varepsilon|} \int_{x \in I_\varepsilon} u(t,x,a) \right| + \left| \frac{1}{|I_\varepsilon|} \int_{x \in I_\varepsilon} u(t,x,a) \, \mathrm{d}x \right| \\ &\leq \left| \frac{1}{|I_\varepsilon|} \int_{x \in I_\varepsilon} \Big(u(t,0,a) - u(t,x,a) \Big) \, \mathrm{d}x \right| + \frac{1}{\sqrt{2\varepsilon}} \|u(t,\cdot,a)\|_{L^2_x(\mathbb{R})} \\ &\leq \left| \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \int_0^x \partial_y u(t,y,a) \, \mathrm{d}y \, \mathrm{d}x \right| + \frac{1}{\sqrt{2\varepsilon}} \|u(t,\cdot,a)\|_{L^2_x(\mathbb{R})} \\ &\leq \left| \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \partial_y u(t,y,a) |\varepsilon - y| \, \mathrm{d}y \right| + \frac{1}{\sqrt{2\varepsilon}} \|u(t,\cdot,a)\|_{L^2_x(\mathbb{R})} \\ &\leq \frac{\varepsilon^{1/2}}{2\sqrt{3}} \, \|\partial_x u(t,\cdot,a)\|_{L^2_x(\mathbb{R})} + \frac{1}{\sqrt{2\varepsilon}} \|u(t,\cdot,a)\|_{L^2_x(\mathbb{R})}\end{aligned}$$

and conclude that

$$\int a^{\gamma-1}u^2(t,0,a)\,\mathrm{d}a \le \frac{1}{6}\left\|\varepsilon^{1/2}a^{(\gamma-1)/2}\partial_x u(t,\cdot,\cdot)\right\|^2_{L^2_{x,a}} + \left\|\varepsilon^{-1/2}a^{(\gamma-1)/2}u(t,\cdot,\cdot)\right\|^2_{L^2_{x,a}}.$$

We choose ε depending on a as $\varepsilon = \eta a^{2(\gamma-1)}$ with η small to be fixed, and deduce

$$\int a^{\gamma-1}u^2(t,0,a)\,\mathrm{d}a \le \frac{\eta}{6}\left\|a^{(\gamma-1)}\partial_x u(t,\cdot,\cdot)\right\|^2_{L^2_{x,a}} + \frac{1}{\eta}\left\|u(t,\cdot,\cdot)\right\|^2_{L^2_{x,a}}.$$

Finally we use that for $\gamma \ge 3/2$ we have $2(\gamma-1)\ge 1$ and therefore on $[0,a_0]$ (remember that the support condition on a is propagated according to Lemma 32) we have $a^{2(\gamma-1)} \le Ca$. Plugging above we get

$$\int a^{\gamma-1}u^2(t,0,a)\,\mathrm{d}a \le \frac{C\eta}{6}\left\|a^{1/2}\partial_x u(t,\cdot,\cdot)\right\|^2_{L^2_{x,a}} + \frac{1}{\eta}\left\|u(t,\cdot,\cdot)\right\|^2_{L^2_{x,a}}.$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint u^2\,\mathrm{d}x\,\mathrm{d}a \le \left(\frac{C\eta\gamma}{6}-1\right)\iint a|\partial_x u|^2\,\mathrm{d}x\,\mathrm{d}a + \frac{\gamma}{\eta}\left\|u\right\|^2_{L^2_{x,a}}.$$

By choosing $\eta < 6/(C\gamma)$, this proves that the L^2 norm exists for all times if it is finite initially.

Case (c): blow-up

First easy step is to compute the evolution for $v(t,a) := u(t,0,a)$. We Fourier transform equation (10) in x:

$$\begin{aligned}\partial_t\hat{u}(t,\xi,a) &= -a|\xi|^2\hat{u}(t,\xi,a) + \partial_a\left(a^\gamma u(t,0,a)\right)\\ &= -a|\xi|^2\hat{u}(t,\xi,a) + \partial_a\left(\frac{a^\gamma}{2\pi}\int_{\mathbb{R}}\hat{u}(t,\eta,a)\,\mathrm{d}\eta\right),\end{aligned}$$

and use Duhamel principle where the last term in the right-hand side is treated as a source term:

$$\hat{u}(t,\xi,a) = e^{-ta|\xi|^2}\hat{u}(0,\xi,a) + \int_0^t \frac{e^{-(t-s)a|\xi|^2}}{2\pi}\partial_a\left(a^\gamma\int_{\mathbb{R}}\hat{u}(s,\eta,a)\,\mathrm{d}\eta\right)\mathrm{d}s\,. \tag{13}$$

Let

$$v(t,a) := \frac{1}{2\pi}\int\hat{u}(t,\eta,a)\,\mathrm{d}\eta = u(t,0,a)\,,$$

from (13) we deduce the identity:

$$v(t,a) = w(t,a) + C_1\int_0^t \frac{1}{\sqrt{a(t-s)}}\partial_a\left(a^\gamma v(s,a)\right)\mathrm{d}s\,, \tag{14}$$

where $C_1 > 0$ is an explicit constant and w is defined by

$$w(t,a) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-ta|\xi|^2}\hat{u}(0,\xi,a)\,\mathrm{d}\xi\,.$$

Then, since $u(s,0,a) \geq 0$, the key remark is that $v(s,a) \geq 0$. Moreover, since the second term in the right-hand side of (14) has zero integral against $a^{1/2}$, one has

$$\int_{\mathbb{R}_+} a^{1/2}v(t,a)\,\mathrm{d}a = \int_{\mathbb{R}_+} a^{1/2}w(t,a)\,\mathrm{d}a \leq C_2\,.$$

We have therefore a moment bound to start with: $\int a^{1/2}v(t,a)\,\mathrm{d}a$ remains bounded for all times. This already shows that singularity can only form at $(0,0)$. Hence, we are here looking only at the value at $x=0$, since for $x \neq 0$ the evolution is a diffusion in x which does not create singularities. Therefore, choosing $M \in (0,1/2)$ so that $0 < M+\gamma-1/2 < M$ (which is possible since $\gamma < 3/2$) we have

$$\begin{aligned}\int_{\mathbb{R}_+} a^M v(t,a)\,\mathrm{d}a &= \int_{\mathbb{R}_+} a^M w(t,a)\,\mathrm{d}a + C_1\int_0^t\int_{\mathbb{R}_+}\frac{a^{M-1/2}}{\sqrt{(t-s)}}\partial_a\left(a^\gamma v(s,a)\right)\mathrm{d}a\,\mathrm{d}s\\ &= \int_{\mathbb{R}_+} a^M w(t,a)\,\mathrm{d}a\\ &\quad - C_1\left(M-\frac{1}{2}\right)\int_0^t\int_{\mathbb{R}_+}\frac{a^{M+\gamma-3/2}}{\sqrt{(t-s)}}v(s,a)\,\mathrm{d}a\,\mathrm{d}s\,.\end{aligned}$$

Note that since M satisfies $M-1/2+\gamma > 0$, in the previous computation the boundary term due to the integration by parts vanishes:

$$\int_0^t\left[\frac{a^{M+\gamma-1/2}}{\sqrt{(t-s)}}v(s,a)\right]_{a=0}^{a=+\infty}\mathrm{d}s = 0\,.$$

Furthermore, there exists some positive constants C_3 and η, with $M-\eta \in (0,1/2)$, such that $-(M-1/2)a^{M+\gamma-3/2} \geq C_3 a^{M-\eta}$ on the compact support $[0,a_0]$. And we deduce that

$$\begin{aligned}\int_{\mathbb{R}_+} a^M v(t,a)\,\mathrm{d}a \geq &\int a^M w(t,a)\,\mathrm{d}a\\ &+\ C_4(1+T)^{-1/2}\int_0^t\int_{\mathbb{R}_+} a^{M-\eta}v(s,a)\,\mathrm{d}a\,\mathrm{d}s\end{aligned}$$

for a constant C_4. We have by interpolation (using $M-\eta < M < 1/2$)

$$\int_{\mathbb{R}_+} a^M v(s,a)\,\mathrm{d}a \leq \left(\int_{\mathbb{R}_+} a^{1/2}v(s,a)\,\mathrm{d}a\right)^\theta\left(\int_{\mathbb{R}_+} a^{M-\eta}v(s,a)\,\mathrm{d}a\right)^{1-\theta}$$

for some $\theta \in (0,1)$. Finally, using the bound on the 1/2-moment, we obtain that

$$\begin{aligned}\int_{\mathbb{R}_+} a^M v(t,a)\,\mathrm{d}a \geq &\int_{\mathbb{R}_+} a^M w(t,a)\,\mathrm{d}a\\ &+\ C_5(1+T)^{-1/2}\int_0^t\left(\int_{\mathbb{R}_+} a^M v(s,a)\,\mathrm{d}a\right)^{1/(1-\theta)}\mathrm{d}s\end{aligned}$$

for some constant $C_5 > 0$. This means that $Y(t) := \int a^M v(t,a)\,\mathrm{d}a$ satisfies on $[0,T]$:

$$Y(0) > 0 \quad \text{ and } \quad Y'(t) \geq C_5(1+T)^{-1/2} Y(t)^{1+\alpha}\,,$$

where $1 + \alpha := 1/(1-\theta) > 1$. This implies on $[0,T]$ that

$$Y(t) \geq \left[\frac{1}{Y(0)^\alpha - \alpha C_5 (1+T)^{-1/2} t}\right]^{1/\alpha}$$

At $t = T/2$ we have $\alpha C_5(1+T)^{-1/2} t = \alpha C_5 (1+T)^{-1/2} T/2$ goes to infinity as T goes to infinity, hence by taking T large enough, we find that $Y(t)$ becomes infinite in finite time.

Using next that

$$\int_{\mathbb{R}_+} a^M v(t,a)\,\mathrm{d}a \leq \int_{\mathbb{R}_+} v(t,a)\,\mathrm{d}a + \int_{a\geq 1} a^{1/2} v(t,a)\,\mathrm{d}a\,,$$

together with $\int_{\mathbb{R}_+} a^{1/2} v(t,a)\,\mathrm{d}a < \infty$, we deduce that the L^1 norm can not stay finite for all time. The same reasoning holds true for all L^p with Holder inequality. □

4.4 Boundary value problem associated to (10)

In [1], an explicit solution was obtained by using Laplace transform on the boundary value problem associated to (10) for a particular initial condition $n_0(x,a) = \delta_{(0,a_0)}$. In this section we recall such a result and we compare it with the result of Theorem 33.

First, we transform equation (10), which includes a singular coefficient, namely $\delta_{x=0}$, into a boundary value problem with regular coefficients. In this part, we will use both notations $n(t,x,a), p(t)$ and $n_t(x,a), p_t$.

Proposition 34. *Assume that $f(0) = 0$, $a_0 > 0$ and that $f(a_0) \neq 0$. Assume in addition that $n_0(x,a) = \delta_{(0,a_0)}$, $p_0 = 0$ and $q_0 = 0$, then (n_t, p_t, q_t) given in Lemma 27 satisfies the following problem in the classical sense:*

$$\begin{cases} \partial_t n(t,x,a) = a\ \partial_x^2 n(t,x,a) \ \ \textit{for } (t,x,a) \in \mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R}_+^*\,, \\ a\left(\partial_x n(t,0^+,a) - \partial_x n(t,0^-,a)\right) = \partial_a \left(f(a) n(t,0,a)\right)\,, a \in \mathbb{R}_+^*\,. \end{cases} \tag{15}$$

Furthermore, one has

$\lim_{a\to 0}\left(f(a)n(t,0,a)\right) = -p'(t)$, *with* $p'(t) = 0$ *if* $f(a) > 0$,

$\lim_{a\to +\infty}\left(f(a)n(t,0,a)\right) = q'(t)$, *with* $q'(t) = 0$ *if* $f(a) < 0$.

Proof. We use Lemma 27 with $n_0(x,a) = \delta_{(0,a_0)}$ and we assume that $f(0) = 0$. From the space symmetry of the process with respect to the origin, we first notice that $n_t(x,a) = n_t(-x,a)$ for all $(t,x,a) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}_+$. Next, as it is classical in such situations, in order to obtain (15), we multiply equation (10) by particular test functions φ and we integrate by parts: respectively $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R}_\pm^* \times \mathbb{R}_+^*\right)$ and $\varphi \in \mathcal{C}_c^\infty\left(\mathbb{R} \times \mathbb{R}_+^*\right)$. Using the smoothing effect of the heat equation, we can see that Equation (15) is satisfied in the classical sense. □

Let us now introduce some notations that will be useful for the computation of the explicit solution to problem (15): H will denote the Heaviside function, $H(a-a_0)=0$ if $a<a_0$, $H(a-a_0)=1$ if $a>a_0$ and Z will be the function defined by

$$Z(x,a)=\frac{|x|}{\sqrt{a}}+2\int_{a_0}^{a}\frac{\sqrt{a'}}{f(a')}\,\mathrm{d}a'. \tag{16}$$

Proposition 35. *The boundary value problem* (15) *with* $n_0(x,a)=\delta_{(0,a_0)}$ *and* $p_0=0$ *admits the following explicit solution* (n_t,p_t)*:*

1. *Local deceleration* ($f(a)<0$)*:*

$$n_t(x,a)=H(a_0-a)\,\frac{Z(x,a)}{|f(a)|\sqrt{4\pi t^3}}\,e^{-\frac{Z(x,a)^2}{4t}} \tag{17}$$

$$\textit{and}\quad p_t=\int_0^t \mathrm{erfc}\Big(\frac{1}{\sqrt{s}}\int_0^{a_0}\frac{\sqrt{a'}}{|f(a')|}\,\mathrm{d}a'\Big)\,\mathrm{d}s\,,$$

2. *Local acceleration* ($f(a)>0$)*:*

$$n_t(x,a)=H(a-a_0)\frac{Z}{f(a)\sqrt{4\pi t^3}}\,e^{-\frac{Z(x,a)^2}{4t}}\qquad \textit{and}\qquad p_t=0\,. \tag{18}$$

Proof. Taking the Laplace transform in t, the Fourier transform in x of (15) with $n_0(x,a)=\delta_{(0,a_0)}$ and using the equality

$$\mathcal{L}^t(n)(0,a,\lambda)=\frac{1}{2\pi}\int_{\mathbb{R}}\tilde{n}_\lambda(\xi,a,\lambda)e^{i\xi\times 0}\,\mathrm{d}\xi\,,$$

we can compute

$$\tilde{n}_\lambda(\xi,a)=\frac{1}{\lambda+a|\xi|^2}\partial_a\left(f(a)\left(\frac{1}{2\pi}\int_{\mathbb{R}}\tilde{n}_\lambda(\xi',a)\,\mathrm{d}\xi'\right)\right)+\frac{\delta_{a=a_0}}{\lambda+a|\xi|^2}\,, \tag{19}$$

where we have denoted by $\tilde{n}_\lambda(\xi,a)=\int_{\mathbb{R}}\mathcal{L}^t(n)(x,a,\lambda)e^{-i\xi x}\,\mathrm{d}x$ the Fourier transform in x of the Laplace transform in t of n_t, which is denoted by $\mathcal{L}^t(n)(x,a,\lambda)=\int_{\mathbb{R}_+}n_t(x,a)e^{-\lambda t}\,\mathrm{d}t$.

Consequently after integration we obtain

$$\begin{aligned}\int_{\mathbb{R}}\tilde{n}_\lambda(\xi,a)\,\mathrm{d}\xi&=\left[\partial_a\left(f(a)\left(\frac{1}{2\pi}\int_{\mathbb{R}}\tilde{n}_\lambda(\xi',a)\,\mathrm{d}\xi'\right)\right)+\delta_{a=a_0}\right]\int_{\mathbb{R}}\frac{1}{\lambda+a|\xi|^2}\,\mathrm{d}\xi\\&=\frac{\pi}{\sqrt{\lambda a}}\left[\partial_a\left(f(a)\left(\frac{1}{2\pi}\int_{\mathbb{R}}\tilde{n}_\lambda(\xi,a)\,\mathrm{d}\xi\right)\right)+\delta_{a=a_0}\right]\,.\end{aligned} \tag{20}$$

With the previous expression, $\int_{\mathbb{R}}\tilde{n}_\lambda(\xi,a)\,\mathrm{d}\xi$ can be computed, hence the expression of $\tilde{n}_\lambda$ can be deduced by using (19). Finally, by inverting first the Fourier transform and then the Laplace tranform, we can compute n_t. From (19) and (20) it follows that

$$\begin{aligned}\tilde{n}_\lambda(\xi,a)&=\frac{1}{\pi}\frac{\sqrt{\lambda a}}{\lambda+a\xi^2}\int_{\mathbb{R}}\tilde{n}_\lambda(\xi,a)\,\mathrm{d}\xi\\&=C(a,a_0,\lambda)\frac{f(a_0)}{2\pi f(a)}\frac{\sqrt{\lambda a}}{\lambda+a\xi^2}e^{2\sqrt{\lambda}\int_{a_0}^{a}\frac{\sqrt{a'}}{f(a')}\,\mathrm{d}a'}\,,\end{aligned}$$

where

$$C(a,a_0,\lambda)=\begin{cases}C(\lambda)H(a-a_0) & \text{if } f>0\,,\\ C(\lambda)H(a_0-a) & \text{if } f<0\,.\end{cases}$$

$C(\lambda)$ is determined by the jump of $\int_{\mathbb{R}} \tilde{n}_\lambda(\xi,a)\,\mathrm{d}\xi$ at $a=a_0$:

$$\int_{\mathbb{R}} \tilde{n}_\lambda(\xi,a_0^+)\,\mathrm{d}\xi-\int_{\mathbb{R}} \tilde{n}_\lambda(\xi,a_0^-)\,\mathrm{d}\xi=\frac{1}{f(a_0)}\,, \tag{21}$$

from which we deduce that $C(\lambda)=1/|f(a_0)|$, hence

$$\tilde{n}_\lambda(\xi,a)=C(a,a_0)\frac{1}{2\pi|f(a)|}\frac{\sqrt{\lambda a}}{\lambda+a\xi^2}e^{2\sqrt{\lambda}\int_{a_0}^a \frac{\sqrt{a'}}{f(a')}\,\mathrm{d}a'}\,.$$

Next, using the Fourier inverse transform, it yields that

$$\begin{aligned}\mathcal{L}^t(n)(x,a,\lambda) &= C(a,a_0)\frac{1}{|f(a)|}e^{-\left|\sqrt{\frac{\lambda}{a}}x\right|+2\sqrt{\lambda}\int_{a_0}^a \frac{\sqrt{a'}}{f(a')}\,\mathrm{d}a'}\\ &= C(a,a_0)\frac{1}{|f(a)|}e^{-\sqrt{\lambda}Z(x,a)}\,,\end{aligned}$$

where Z is given by (16).

Laplace inverting this latter expression, we obtain the joint distribution given by (17) if $f<0$ and by (18) if $f>0$. The expression of p_t is straigtforward. □

Remark 36. *Unsurprisingly the blow-up character of the solution to* (10) *given in Theorem 33 can also be observed on the explicit solution given in Proposition 35. Indeed the symptom of blow-up in a L^p framework corresponds to $p_t\neq 0$. In the case where $f(a)=-a^\gamma$, we see that $\int_0^a a'^{1/2-\gamma}\,\mathrm{d}a'<\infty$ if $\gamma\in(0,3/2)$. Since Laplace transform inversion requires a specific initial condition the result given in Theorem 33 is more general.*

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