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# Sums of the digits in bases 2 and 3

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*To Robert Tichy, for his 60th birthday*

## Abstract

Let  $b \geq 2$  be an integer and let  $s_b(n)$  denote the sum of the digits of the representation of an integer  $n$  in base  $b$ . For sufficiently large  $N$ , one has

$$\text{Card}\{n \leq N : |s_3(n) - s_2(n)| \leq 0.1457205 \log n\} > N^{0.970359}.$$

The proof only uses the separate (or marginal) distributions of the values of  $s_2(n)$  and  $s_3(n)$ .

## 1 Introduction

For integers  $b \geq 2$  and  $n \geq 0$ , we denote by “the sum of the digits of  $n$  in base  $b$ ” the quantity

$$s_b(n) = \sum_{j \geq 0} \varepsilon_j, \text{ where } n = \sum_{j \geq 0} \varepsilon_j b^j \text{ with } \forall j : \varepsilon_j \in \{0, 1, \dots, b-1\}.$$

Our attention on the question of the proximity of  $s_2(n)$  and  $s_3(n)$  comes from the apparently non related question of the distribution of the last non zero digit of  $n!$  in base 12 (cf. [2] and [3]).<sup>1</sup>

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<sup>1</sup>Indeed, if the last non zero digit of  $n!$  in base 12 belongs to  $\{1, 2, 5, 7, 10, 11\}$  then  $|s_3(n) - s_2(n)| \leq 1$ ; this seems to occur infinitely many times.

Computation shows that there are 48 266 671 607 positive integers up to  $10^{12}$  for which  $s_2(n) = s_3(n)$ , but it seems to be unknown whether there are infinitely many integers  $n$  for which  $s_2(n) = s_3(n)$  or even for which  $|s_2(n) - s_3(n)|$  is significantly small.

We do not know the first appearance of the result we quote as Theorem 1; in any case, it is a straightforward application of the fairly general main result of N. L. Bassily and I. Kátai [1]. We recall that a sequence  $\mathcal{A} \subset \mathbb{N}$  of integers is said to have asymptotic natural density 1 if

$$\text{Card}\{n \leq N : n \in \mathcal{A}\} = N + o(N).$$

**Theorem 1.** *Let  $\psi$  be a function tending to infinity with its argument. The sequence of natural numbers  $n$  for which*

$$\begin{aligned} \left(\frac{1}{\log 3} - \frac{1}{\log 4}\right) \log n - \psi(n) \sqrt{\log n} &\leq s_3(n) - s_2(n) \\ &\leq \left(\frac{1}{\log 3} - \frac{1}{\log 4}\right) \log n + \psi(n) \sqrt{\log n} \end{aligned}$$

*has asymptotic natural density 1.*

Our main result is that there exist infinitely many  $n$  for which  $|s_3(n) - s_2(n)|$  is significantly smaller than  $\left(\frac{1}{\log 3} - \frac{1}{\log 4}\right) \log n = 0.18889\dots \log n$ . More precisely we have the following:

**Theorem 2.** *For sufficiently large  $N$ , one has*

$$\text{Card}\{n \leq N : |s_3(n) - s_2(n)| \leq 0.1457205 \log n\} > N^{0.970359}. \quad (1)$$

The mere information we use in proving Theorem 2 is the knowledge of the separate (or marginal) distributions of  $(s_2(n))_n$  and  $(s_3(n))_n$ , without using any further information concerning their joint distribution.

In Section 2, we provide a heuristic approach to Theorems 1 and 2; the actual distribution of  $(s_2(n))_n$  and  $(s_3(n))_n$  is studied in Section 3. The proof of Theorem 2 is given in Sections 4.

Let us formulate three remarks as a conclusion to this introductory section.

It seems that our present knowledge of the joint distribution of  $s_2$  and  $s_3$  (cf. for example C. Stewart [5] for a Diophantine approach or M. Drmota [4] for a probabilistic one) does not permit us to improve on Theorem 2.

Theorem 2 can be extended to any pair of distinct bases, say  $q_1$  and  $q_2$ : more than computation, the Authors have deliberately chosen to present an idea to the Dedicatee.

Although we could not prove it, we believe that Theorem 2 represents the limit of our method.

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## 2 A heuristic approach

As a warm-up for the actual proofs, we sketch a heuristic approach. A positive integer  $n$  may be expressed as

$$n = \sum_{j=0}^{J(n)} \varepsilon_j(n) b^j, \text{ with } J(n) = \left\lfloor \frac{\log n}{\log b} \right\rfloor.$$

If we consider an interval of integers around  $N$ , the smaller is  $j$  the more equidistributed are the  $\varepsilon_j(n)$ ’s, and the smaller are the elements of a family  $\mathcal{J} = \{j_1 < j_2 < \dots < j_s\}$  the more independent are the  $\varepsilon_j(n)$ ’s for  $j \in \mathcal{J}$ . Thus a first model for  $s_b(n)$  for  $n$  around  $N$  is to consider a sum of  $\left\lfloor \frac{\log N}{\log b} \right\rfloor$  independent random variables uniformly distributed in  $\{0, 1, \dots, b-1\}$ . Thinking of the central limit theorem, we even consider a continuous model, representing  $s_b(n)$ , for  $n$  around  $N$  by a Gaussian random variable  $S_{b,N}$  with expectation and variance given by

$$\mathbb{E}(S_{b,N}) = \frac{(b-1) \log N}{2 \log b} \text{ and } \mathbb{V}(S_{b,N}) = \frac{(b^2-1) \log N}{12 \log b}.$$

In particular

$$\mathbb{E}(S_{2,N}) = \frac{\log N}{\log 4} \text{ and } \mathbb{E}(S_{3,N}) = \frac{\log N}{\log 3},$$

and their standard deviations have the order of magnitude  $\sqrt{\log N}$ .

*Towards Theorem 1.* In [1], it is proved that a central limit theorem actually holds for  $s_b$ ; more precisely, the following proposition is the special case of the first relation in the main Theorem of [1], with  $f(n) = s_b(n)$  and  $P(X) = X$ .

**Proposition 1.** *For any positive  $y$ , as  $x$  tend to infinity, one has*

$$\frac{1}{x} \text{Card} \left\{ n < x : |s_b(n) - \mathbb{E}(S_{b,n})| < y (\mathbb{V}(S_{b,n}))^{1/2} \right\} \rightarrow \frac{1}{\sqrt{2\pi}} \int_{-y}^y e^{-t^2/2} dt.$$

Theorem 1 easily follows from Proposition 1: the set under our consideration is the intersection of 2 sets of density 1.

*Towards Theorem 2.* If we wish to deal with a difference  $|s_3(n) - s_2(n)| < u \log n$  for some  $u < \left( \frac{1}{\log 3} - \frac{1}{\log 4} \right)$  we must, by what we have seen above, consider events of asymptotic probability zero, which means that a heuristic approach must be substantiated by a rigorous proof. Our key remark is that the variance of  $S_{3,N}$  is larger than that of  $S_{2,N}$ ; this implies the following: the probability that  $S_{3,N}$  is at a distance  $d$  from its mean is larger than the probability that  $S_{2,N}$  is at a distance  $d$  from its mean. So, we have the hope to find some  $u < \left( \frac{1}{\log 3} - \frac{1}{\log 4} \right)$  such that the probability that  $|S_{2,N} - \mathbb{E}(S_{2,N})| > u \log N$  is smaller than the probability that  $S_{3,N}$  is very close to  $\mathbb{E}(S_{2,N})$ . This will imply that for some  $\omega$  we have  $|S_{3,N}(\omega) - S_{2,N}(\omega)| \leq u \log N$ .

### 3 On the distribution of the values of $s_2(n)$ and $s_3(n)$

In order to prove Theorem 2 we need

- an upper bound for the tail of the distribution of  $s_2$ ,
- a lower bound for the tail of the distribution of  $s_3$ .

### 3.1 Upper bound for the tail of the distribution of $s_2$

**Proposition 2.** *Let  $\lambda \in (0, 1)$ . For any*

$$\nu > 1 - ((1 - \lambda) \log(1 - \lambda) + (1 + \lambda) \log(1 + \lambda)) / \log 4$$

*and any sufficiently large integer  $H$ , we have*

$$\text{Card}\{n < 2^{2H} : |s_2(n) - H| \geq \lambda H\} \leq 2^{2H\nu}. \quad (2)$$

*Proof.* When  $b = 2$ , the distribution of the values of  $s_2(n)$  is simply binomial; we thus get

$$\text{Card}\{0 \leq n < 2^{2H} : s_2(n) = m\} = \binom{2H}{m}.$$

Using the fact that the sequence  $\binom{2H}{m}$  is symmetric and unimodal plus Stirling's formula, we obtain that when  $m \leq (1 - \lambda)H$  or  $m \geq (1 + \lambda)H$ , one has

$$\begin{aligned} \binom{2H}{m} &\leq H^{O(1)} \frac{(2H)^{2H}}{((1 - \lambda)H)^{(1 - \lambda)H} ((1 + \lambda)H)^{(1 + \lambda)H}} \\ &\leq H^{O(1)} \left( \frac{2^2}{(1 - \lambda)^{(1 - \lambda)} (1 + \lambda)^{(1 + \lambda)}} \right)^H \\ &\leq H^{O(1)} \left( 2^{(1 - ((1 - \lambda) \log(1 - \lambda) + (1 + \lambda) \log(1 + \lambda))) / 2 \log 2} \right)^{2H}. \end{aligned}$$

Relation (2) comes from the above inequality and the fact that the left hand side of (2) is the sum of at most  $2H$  such terms.  $\square$

### 3.2 Lower bound for the tail of the distribution of $s_3$

**Proposition 3.** *Let  $L$  be sufficiently large an integer. We have*

$$\text{Card}\{n < 3^L : s_3(n) = \lfloor L \log 3 / \log 4 \rfloor\} \geq 3^{0.970359238L}. \quad (3)$$

*Proof.* The positive integer  $L$  being given, we write any integer  $n \in [0, 3^L)$  in its non necessarily proper representation, as a chain of exactly  $L$  characters,  $\ell_i(n)$  of them being equal to  $i$ , for  $i \in \{0, 1, 2\}$ , the sum  $\ell_0(n) + \ell_1(n) + \ell_2(n)$  being equal to  $L$ , the total number of digits in this representation<sup>2</sup>. One has

$$\text{Card}\{0 \leq n < 3^L : s_3(n) = m\} = \sum_{\substack{\ell_0 + \ell_1 + \ell_2 = L \\ \ell_1 + 2\ell_2 = m}} \frac{L!}{\ell_0! \ell_1! \ell_2!}. \quad (4)$$

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<sup>2</sup>For example, when  $L = 5$ , the number "sixty" will be represented as 02020. Happy palindromic birthday, Robert!

In order to get a lower bound for the left hand side of (4), it is enough to select one term in its right hand side. We choose

$$l_2 = \lfloor 0.235001144L \rfloor; l_1 = \lfloor L \log 3 / \log 4 \rfloor - 2 l_2; l_0 = L - l_1 - l_2.$$

A straightforward application of Stirling's formula, similar to the one used in the previous subsection, leads to (3).  $\square$

## 4 Proof of Theorem 2

Let  $N$  be sufficiently large an integer. We let  $K = \lfloor \log N / \log 3 \rfloor - 2$  and  $H = \lfloor (K - 1) \log 3 / \log 4 \rfloor + 2$ . We notice that we have

$$N/81 \leq 3^{K-1} < 3^K < 2^{2H} \leq N. \quad (5)$$

We use Proposition 2 with  $\lambda = 0.14572049 \log 4$ , which leads to

$$\text{Card}\{n \leq 2^{2H} : |s_2(n) - H| \geq \lambda H\} \leq 2^{0.970359230 \times 2H} \leq N^{0.970359230}. \quad (6)$$

For any  $n \in [2 \cdot 3^{K-1}, 3^K)$  we have  $s_3(n) = 2 + s_3(n - 2 \cdot 3^{K-1})$  and so it follows from Proposition 3 that we have

$$\begin{aligned} & \text{Card}\{n \in [2 \cdot 3^{K-1}, 3^K) : s_3(n) = H\} \\ &= \text{Card}\{n < 3^{K-1} : s_3(n) = H - 2\} \\ &= \text{Card}\{n < 3^{K-1} : s_3(n) = \lfloor (K - 1) \log 3 / \log 4 \rfloor\} \\ &\geq 3^{0.970359238(K-1)} \geq N^{0.970359237}. \end{aligned}$$

This implies that we have

$$\text{Card}\{n \leq 2^{2H} : s_3(n) = H\} \geq N^{0.970359237}. \quad (7)$$

From (6) and (7), we deduce that for  $N$  sufficiently large, we have

$$\text{Card}\{n \leq N : |s_2(n) - s_3(n)| \leq 0.1457205 \log n\} \geq N^{0.970359}.$$

$\square$

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