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An Anisotropic Inf-Convolution BV type model for dynamic reconstruction.*

Maïtine Bergounioux[†] and E. Papoutsellis[†]

Abstract. We are interested in a spatial-temporal variational model for image sequences. The model involves a fitting data term to be adapted to different modalities such as denoising, deblurring or emission tomography. The regularizing term acts as an infimal-convolution type operator that takes into account the respective influence of space and time variables in a separate mode. We give existence and uniqueness results and provide optimality conditions via duality analysis.

Key words. Spatio-temporal Variational Regularization, Infimal Convolution Total Variation, Anisotropic Total variation, Optimality Conditions

AMS subject classifications. 65D18, 68U10,65K10

1. Introduction. In this paper, we examine variational inverse problems for dynamic image reconstruction. As in the context of image restoration, the goal regarding a video restoration is to recover a *clean* image sequence given a degraded dynamic datum. Certainly, one of the main differences between image and video restoration is the additional temporal domain where a collection of images-frames evolves over the time. Beside the spatial structures which are a significant factor on the output quality of the reconstruction, the time direction has an important role on the temporal consistency among the frames. Furthermore, in terms of video applications, one may consider applications inherited from the imaging context and extend them to the dynamical framework. To name a few, we have dynamic denoising, deblurring, inpainting, decompression and emission tomography such as Positron Emission Tomography and Magnetic Resonance Imaging.

The aim of this paper is to study variational regularization models in an infinite dimensional setting defined on a spatial-temporal domain. In particular, given a corrupted image sequence g , we look for a solution u , in a Banach space \mathcal{X} , to the following generic minimization problem

$$(1.1) \quad \inf_{u \in \mathcal{X}} \mathcal{N}(u) + \mathcal{H}(\mathcal{A}u, g).$$

The first and second terms represent the well known *regularizer* and the data fitting term (*fidelity*). The former imposes a certain prior structure (regularity) on the solution u and the latter is determined by the nature of degradation, e.g., a transformation through a continuous and linear operator \mathcal{A} with the presence of random noise, as well as the modality of the problem. Regarding image restoration, the minimization problem (1.1) has been extensively used and examined from both theoretical and numerical point of view for different applications. For instance, we refer the reader to the famous ROF variational model [36], where the use of functions of bounded variation (BV) and the total variation regularization (TV) was established

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35 in image processing. Moreover, it was analyzed in [1], [44] and several extensions have been
 36 proposed in [15, 8, 12, 16, 24]. Now, concerning variational problems on a spatial-temporal
 37 domain, one can witness significantly less work from a theoretical perspective compared to
 38 a numerical one. Indeed, there is a plethora of numerical algorithms in the literature for
 39 variational video processing. We refer the reader to some of them as [17, 38, 32].

40 A quite natural approach towards image sequence reconstruction is to apply the minimiza-
 41 tion problem (1.1), acting on every image-frame of the sequence individually. For example, we
 42 use the above problem in order to denoise each frame from a sequence corrupted by Gaussian
 43 noise. We choose a non-smooth regularizer as the *total variation measure* over the spatial
 44 domain $\Omega \subset \mathbb{R}^2$. It is known for the piecewise constant structures it imposes to the solution u
 45 and can eliminate efficiently the noise while preserving the edges of the images. It is defined
 46 as

$$47 \quad (1.2) \quad \mathcal{N}(u) = \alpha |Du|(\Omega) = \sup \left\{ \int_{\Omega} u \operatorname{div} \phi \, dx \, dt : \phi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^2), \|\phi\|_{\infty} \leq \alpha \right\},$$

48 weighted by a positive parameter α and

$$49 \quad (1.3) \quad \|\phi\|_{\infty} = \operatorname{ess\,sup}_{x \in \Omega} |\phi(x)|_r, \quad |\phi(x)|_r = \begin{cases} \sqrt{\phi_1^2(x) + \phi_2^2(x)}, & r = 2, \text{ (isotropic)} \\ \max\{|\phi_1(x)|, |\phi_2(x)|\}, & r = \infty, \text{ (anisotropic)}. \end{cases}$$

50 This parameter is responsible for a proper balancing between the regularizer and the fidelity
 51 term which is fixed as $\mathcal{H}(u, g) = \frac{1}{2} \|u - g\|_{L^2(\Omega)}^2$ in this case. Although, this solution produces
 52 a satisfying result on the spatial domain, it does not take into account the interaction between
 53 time and space and some time artifacts, e.g. *flickering*, will be introduced. Note that one
 54 can use the anisotropic norm instead of an isotropic one in (1.3). Although these norms are
 55 equivalent in a finite dimensional setting, they have different effects on the corresponding
 56 computed minimizers. In the isotropic case, sharp corners will not be allowed in the edge set
 57 and smooth corners prevail. On the other hand, corners in the direction of the unit vectors
 58 are favored in the anisotropic variant. For more details, we refer the reader to [29, 21, 34] on
 59 the properties and differences between these two corresponding minimizers.

60 A more sophisticated path, referred as 3D denoising, is to extend the domain taking into
 61 account the time activity and treat an image sequence as a 3D volume where the time plays
 62 the role of the third variable. In this case, we write

$$63 \quad (1.4) \quad \mathcal{N}(u) = |D_{\alpha} u|(Q) = \sup \left\{ \int_Q u \operatorname{div}_{\alpha} \phi \, dx \, dt : \phi \in \mathcal{C}_c^1(Q, \mathbb{R}^3), \|\phi\|_{\infty} \leq 1 \right\}$$

where $Q = \mathcal{T} \times \Omega \subset \mathbb{R}^3$ is the three-dimensional spatial-temporal domain with $\mathcal{T} = (0, T)$.
 Here, we have a positive vector $\alpha = (\alpha_1, \alpha_2)$ acting on the space and time respectively with

$$\operatorname{div}_{\alpha} = \alpha_1 \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right) + \alpha_2 \frac{\partial}{\partial t} = \alpha_1 \operatorname{div}_x + \alpha_2 \operatorname{div}_t$$

64 and the TV smoothness is applied along both the spatial and the temporal directions. An
 65 obvious question that rises on this particularly setting is the correlation between the space

66 and time. Video regularization approaches as in [17, 25, 30] combine spatial and temporal
 67 domains under the corresponding dynamic isotropic norm $\|\phi\|_\infty = \text{ess sup}_{x \in Q} |\phi(t, x)|_2$. Hence,
 68 space and time are interacting with each other and contribute under some weight parameters
 69 to the TV regularizer.

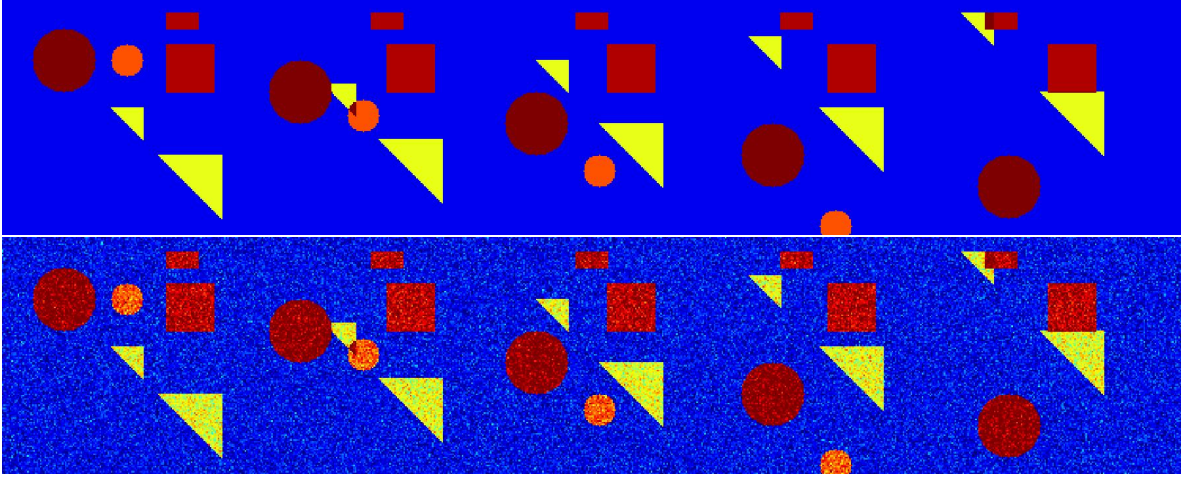


Figure 1.1: Image sequence of 5 frames and its noisy version corrupted with Gaussian noise. Geometrical shapes are moving in different directions with different moving speed.

70 However, this choice of norm is not very accurate concerning preservation of spatial and
 71 temporal discontinuities. Using the anisotropic norm, $\|\phi\|_\infty = \text{ess sup}_{x \in Q} |\phi(t, x)|_\infty$, where space
 72 and time are not correlated, has the advantage to focus on the discontinuities of Ω and \mathcal{T} in
 73 a separate mode respectively and preserve spatial and temporal details more accurately. In
 74 particular, we can decompose (1.4) into a spatial and a temporal total variation, see [2], and
 75 write

$$\begin{aligned}
 |Du|(Q) &= |D_x u|(Q) + |D_t u|(Q), \text{ with} \\
 |D_x u|(Q) &= \sup \left\{ \int_Q u \alpha_1 \left(\frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} \right) dx dt : \phi \in \mathcal{C}_c^1(Q, \mathbb{R}^3), \right. \\
 76 \quad (1.5) \quad &\quad \left. \max\{\sqrt{\phi_1^2(t, x) + \phi_2^2(t, x)}\} \leq 1 \right\}, \\
 |D_t u|(Q) &= \sup \left\{ \int_Q u \alpha_2 \frac{\partial \phi_3}{\partial x_3} dx dt : \phi \in \mathcal{C}_c^1(Q, \mathbb{R}^3), \max\{|\phi_3(t, x)|\} \leq 1 \right\}, \text{ and} \\
 |Du|(Q) &\leq |D_x u|(Q) + |D_t u|(Q) \leq \sqrt{2} |Du|(Q).
 \end{aligned}$$

77 This type of decomposition has already been proposed for several applications such as dynamic
 78 denoising, segmentation, video decompression and the reader is referred to [43, 38, 26, 17].
 79 Although, this paper is rather theoretical we would like to intrigue the reader with a simple
 80 numerical example. In Figure 1.1, we have an image sequence of 5 frames of several
 81 geometrical objects moving in different directions and speed under a constant background.

82 This is corrupted by Gaussian noise. Then, in order to compare between isotropic (1.4) and
 83 anisotropic (1.5) total variation spatial-temporal regularization we choose the same ratio $\frac{\alpha_1}{\alpha_2}$
 84 of the parameters for both cases that act in the space and time directions respectively.

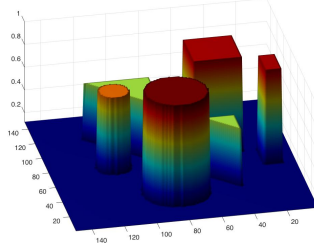
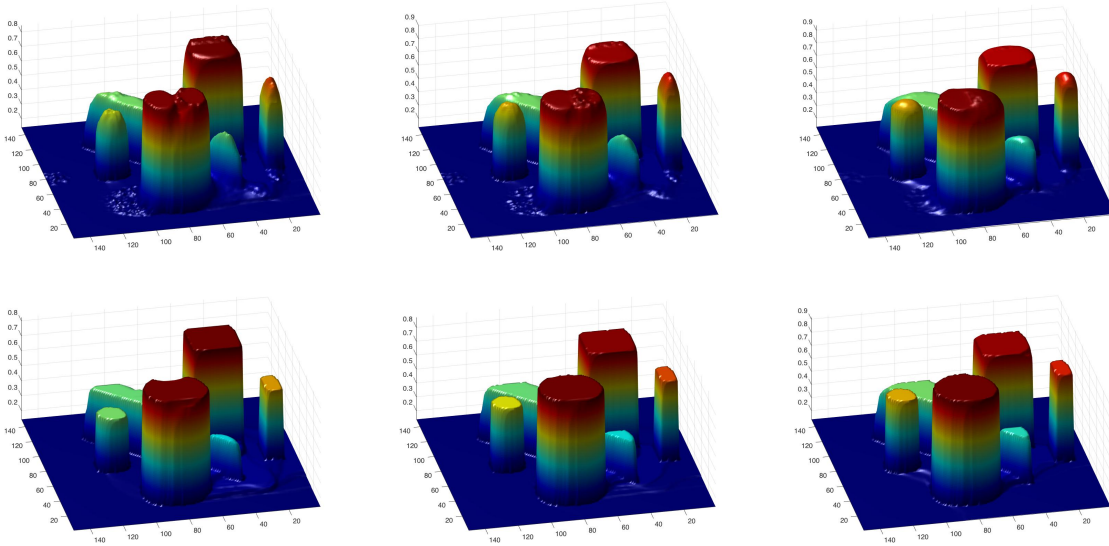


Figure 1.2: Surface plot of the 3rd frame of the *true* sequence.



(a) Isotropic vs Anisotropic TV $\frac{\alpha_1}{\alpha_2} = 5, \alpha_2 = 0.01$ (b) Isotropic vs Anisotropic TV $\frac{\alpha_1}{\alpha_2} = 10, \alpha_2 = 0.05$ (c) Isotropic vs Anisotropic TV $\frac{\alpha_1}{\alpha_2} = 50, \alpha_2 = 0.1$

Figure 1.3: Surface plots of the regularized solutions of (1.1) for the 3rd frame of Figure 1.1. We compare the ground truth, see Figure 1.2 with the isotropic (1.4) (1st row) and anisotropic (1.5) (2nd row) spatial-temporal total variation regularization. The parameters are the same for both cases.

85 In Figure 1.3, we present the surface plots of the 3rd frame of the corresponding regularized
 86 solutions of (1.1) with the squared L^2 norm fidelity term. We observe that for different ratios,
 87 anisotropic regularization is able to preserve the geometry of these objects.

88 Motivated by (1.5), we proceed with a further decomposition in terms of the test function

89 ϕ and define the following *decoupled* spatial-temporal total variation regularization,

$$90 \quad (1.6) \quad \mathcal{N}(u) = \alpha_1 \int_0^T \text{TV}_x(u)(t) dt + \alpha_2 \int_{\Omega} \text{TV}_t(u)(x) dx,$$

91 where $\text{TV}_x(u)(t) = (|Du|(\Omega))(t)$ and $\text{TV}_t(u)(x) = (|Du|(\mathcal{T}))(x)$ denote the spatial total
92 variation for every $t \in \mathcal{T}$ and the temporal total variation for every $x \in \Omega$ respectively. Note
93 that in the above formulations the test functions are defined in Ω and \mathcal{T} respectively.

94 Non-smooth regularization methods introduce different kind of modelling artifacts. As we
95 discussed above, a total variation regularizer tends to approximate non-constant noisy regions
96 with piecewise constant structures leading to the *staircasing* effect. This aspect is certainly
97 inherited in the dynamic framework and produces the *flickering* effect due to the staircasing
98 along the temporal dimension. In addition, one may observe some *ghost* artifacts on moving
99 objects, i.e., where certain features are overlapping between two consecutive frames. This is
100 due to the strong temporal regularization, namely when the ratio $\frac{\alpha_1}{\alpha_2}$ is relatively small, see
101 Figure 1.3. In order to overcome this kind of modelling artifacts, a combination of non-smooth
102 regularizers is used via the concept of the infimal convolution,

$$103 \quad (1.7) \quad \mathcal{N}(u) = F_1 \# F_2(u) = \inf_{v \in \mathcal{X}} F_1(u - v) + F_2(v).$$

104 This regularization functional is able to favor reconstructions with a relatively small F_1 or
105 F_2 contribution. In the imaging context, this is introduced in [15], where a first and second
106 order TV-based regularizers are combined in order to reduce the staircasing phenomenon.
107 Under this regularizer, the corresponding solution u of (1.1) promotes both piecewise constant
108 and smooth structures due to the presence of higher order derivatives and in fact provides a
109 certain decomposition between piecewise constant and smooth regions. On the other hand,
110 Holler and Kunisch in [25], extend the notion of infimal convolution in the context of dynamic
111 processing. In such a setting, they propose the use of total variation functionals as in (1.4) with
112 an isotropic relation on the spatial and temporal regularities. As in the imaging framework,
113 one can decompose an image sequence into a sequence that captures spatial information and
114 a sequence that encodes temporal activity. This type of spatial-temporal regularizer will be
115 discussed in Section 3 under our anisotropic formulation (1.6) of separate action in space and
116 time. Specifically, we propose the following infimal convolution total variation regularization
117 for an image sequence u given two positive vectors $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2)$,

$$118 \quad (1.8) \quad \mathcal{N}(u) = F_{\boldsymbol{\lambda}} \# F_{\boldsymbol{\mu}}(u) = \inf_{v \in \mathcal{X}} \int_0^T \lambda_1 \text{TV}_x(u - v)(t) dt + \int_{\Omega} \lambda_2 \text{TV}_t(u - v)(x) dx \\ + \int_0^T \mu_1 \text{TV}_x(v)(t) dt + \int_{\Omega} \mu_2 \text{TV}_t(v)(x) dx.$$

119 Depending on the choice of $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ one can enforce a certain regularity and either focus on space
120 or on time for the image sequences $u - v$ and v . For example, if one selects that $\lambda_1 = \mu_2 = \kappa$ and
121 $\lambda_2 = \mu_1 = 1$ with $\kappa > 1$ then the first two terms impose a TV smoothness more on the space
122 direction that in time for the $u - v$ term. For the other two terms, the TV smoothness acts
123 conversely for the v component. Therefore, it is a matter of proper balancing which is tuned

124 automatically via the infimal convolution and highlights the cost either on space or time.
 125 We would like to mention that the functionals in (1.7) are not necessarily total variational
 126 functionals and other combinations or high order functionals may be used, see for instance
 127 [39, 7].

128 Finally, we would like to emphasize on the nature of the positive parameters defined above.
 129 In the definitions (1.4), (1.5) and (1.8), we use parameters that are constant over the time do-
 130 main. Equivalently, every *frame* is penalized with the same constant. This is a fair assumption
 131 when the level of noise is assumed to be constant over time. However, in real world appli-
 132 cations this is not always the case. There are situations when the noise is signal-dependent
 133 e.g., Poisson noise and the noise-level variates over time. In the dynamic PET imaging and in
 134 particular in *list-mode* PET, see [42], data can be binned into sinograms allowing frame dura-
 135 tions to be determined after the acquisition. Under this approach, one has to choose between
 136 longer scans with good counting statistics and shorter scans that are noisy but preserving
 137 temporal resolution. A usual and fair choice is to select shorter scans in the beginning where
 138 there is a high activity of the radioactive tracer and longer scans at the end. For example,
 139 a 50 minutes acquisition in list mode rat-brain scans is rebinned into 27 frames under the
 140 following scheme: 4x10s, 4x20s, 4x60s, 14x180s, 1x120s, see [40]. Hence, our goal is to allow
 141 time dependent parameters on the above regularizers that can handle not only different levels
 142 of noise *per frame* (1st term) but also balance the temporal activity in terms of a non-uniform
 143 time domain discretization (2nd term), i.e.,

$$144 \quad (1.9) \quad \mathcal{N}(u) = \int_0^T \alpha_1(t) \text{TV}_x(u)(t) dt + \int_{\Omega} \text{TV}_t(\alpha_2(t)u)(x) dx.$$

145 Outline of the paper: The paper is organized as follows: we first recall some general properties
 146 of functions of bounded variation and fix the notations in terms of the dynamic framework. We
 147 continue with the definition of the regularizers used in this paper such as a *weighted* version
 148 of the spatial-temporal total variation as well as its extension to the infimal convolution. In
 149 addition, we define also the data fitting terms that are suitable for different applications. In
 150 Section 4, we examine the well-posedness (existence, uniqueness and stability) of the associated
 151 variation problem specifically for the infimal convolution regularizer and conclude in Section
 152 5, with the corresponding optimality conditions. Finally, we would like to mention that the
 153 nature of this paper is rather theoretical and we do not address any numerical issues. This
 154 will be done in a forthcoming paper.

155 **2. Preliminaries.** Let us denote $u : \mathcal{T} \times \Omega \rightarrow \mathbb{R}$, an image sequence defined on an open
 156 bounded set $\Omega \subset \mathbb{R}^d$ with smooth boundary representing the space domain with $d \geq 1$ and
 157 $\mathcal{T} = (0, T)$, $T > 0$ which represents the temporal domain. In this section, we recall some basic
 158 notations related to functions of bounded variation (BV) extended to the spatial-temporal
 159 context. In order to distinguish between spatial and temporal domains, we define the following

160 spaces

$$\begin{aligned}
& L^1(\mathcal{T}; \text{BV}(\Omega)) = \{u : \mathcal{T} \times \Omega \rightarrow \mathbb{R} \mid u(t, \cdot) \in \text{BV}(\Omega) \text{ a.e. } t \in \mathcal{T} \\
& \quad \text{and } t \mapsto \text{TV}_x(u)(t) \in L^1(\mathcal{T})\}, \\
161 \quad (2.1) \quad & L^1(\Omega; \text{BV}(\mathcal{T})) = \{u : \mathcal{T} \times \Omega \rightarrow \mathbb{R} \mid u(\cdot, x) \in \text{BV}(\mathcal{T}) \text{ a.e. } x \in \Omega \\
& \quad \text{and } x \mapsto \text{TV}_t(u)(x) \in L^1(\Omega) \}.
\end{aligned}$$

162 Here, TV_x and TV_t stand for the spatial and temporal total variation for every $t \in \mathcal{T}$ and
163 $x \in \Omega$ respectively. In particular, we have

$$\begin{aligned}
164 \quad (2.2) \quad & \text{TV}_x(u)(t) = \sup \left\{ \int_{\Omega} \xi(x) u(t, x) \, dx \mid \xi \in K_x \right\}, \\
& \text{TV}_t(u)(x) = \sup \left\{ \int_0^T \xi(t) u(t, x) \, dt \mid \xi \in K_t \right\},
\end{aligned}$$

165 with the corresponding sets

$$\begin{aligned}
166 \quad (2.3) \quad & K_x := \left\{ \xi = \text{div}_x(\varphi) \mid \varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^d), \|\varphi\|_{\infty, x} \leq 1 \right\}, \|\phi\|_{\infty, x} = \text{ess sup}_{x \in \Omega} |\phi(x)|_2 \\
& K_t := \left\{ \xi = \frac{d\varphi}{dt} \mid \varphi \in \mathcal{C}_c^1(\mathcal{T}, \mathbb{R}), \|\varphi\|_{\infty, t} \leq 1 \right\}, \|\phi\|_{\infty, t} = \text{ess sup}_{t \in \mathcal{T}} |\phi(t)|
\end{aligned}$$

167 where div_x is the divergence operator on the spatial domain and $|\cdot|_2$ is the isotropic-euclidean
168 norm in space. Finally, we define the space of functions of bounded variation on the spatial-
169 temporal domain Q , acting isotropically in these two directions i.e.,

$$\begin{aligned}
& \text{BV}(Q) = \{u \in L^1(Q) \mid \text{TV}(u) < \infty\}, \text{ where} \\
& \text{TV}(u) = \sup \left\{ \int_Q \xi(t, x) u(t, x) \, dx \, dt \mid \xi \in K \right\}, \text{ and} \\
170 \quad (2.4) \quad & K := \left\{ \xi = \text{div}_{(t,x)}(\varphi) \mid \varphi \in \mathcal{C}_c^1(Q, \mathbb{R} \times \mathbb{R}^d), \|\varphi\|_{\infty} \leq 1 \right\}, \\
& \|\phi\|_{\infty} = \text{ess sup}_{(t,x) \in Q} |\phi(t, x)|_2.
\end{aligned}$$

171 Note that $\text{div}_{(t,x)} = \frac{\partial}{\partial t} + \text{div}_x$. As we pointed out in the introduction, one may consider an
172 equivalent anisotropic norm using $|\phi(t, x)|_{\infty} = \max \left\{ \sqrt{\phi_1^2(t, x) + \phi_2^2(t, x)}, |\phi_3(t, x)| \right\} \leq 1$ and
173 all the following results are still true. In the following theorem, see [3, 5], we recall some useful
174 properties on the $\text{BV}(O)$ space, where O is a bounded, open set of \mathbb{R}^N (practically $O = \Omega$ with
175 $N = d$ or $O = Q$ with $N = d + 1$.)

Theorem 2.1. *Let $O \subset \mathbb{R}^N$, $N \geq 1$. The space $\text{BV}(O)$ endowed with the norm*

$$\|v\|_{\text{BV}(O)} := \|v\|_{L^1(O)} + \text{TV}(v)$$

is a Banach space.

(a) *For any $u \in \text{BV}(O)$ there exists a sequence $u_n \in \mathcal{C}^{\infty}(\bar{O})$ such that*

$$u_n \rightarrow u \text{ in } L^1(O) \text{ and } \text{TV}(u_n) \rightarrow \text{TV}(u).$$

(b) The mapping $u \mapsto \text{TV}(u)$ is lower semicontinuous from $\text{BV}(O)$ endowed with the $L^1(O)$ topology to \mathbb{R}^+ .

(c) $\text{BV}(O) \subset L^p(O)$ with continuous embedding, for $1 \leq p \leq \frac{N}{N-1}$ and we have the Poincaré-Wirtinger inequality (Remark 3.50 of [3]): there exists a constant C_O only depending on O such that for $1 \leq p \leq \frac{N}{N-1}$

$$\forall u \in \text{BV}(O) \quad \|u - \bar{u}\|_{L^p(O)} \leq C_O \text{TV}(u) ,$$

176 where \bar{u} is the mean value of u on O .

177 (d) $\text{BV}(O) \subset L^p(O)$ with compact embedding, for $1 \leq p < \frac{N}{N-1}$.

178 The lemma below is essential for the forthcoming analysis and relates the spaces (2.1) and
179 (2.4). It is based on the definitions above as well as of some tools in the proof of [22, Theorem
180 2, Section 5.10.2]. A similar result (but in a different context) can be found in [9, Lemma 3].

181 **Lemma 2.1.** We have $L^1(\mathcal{T}; \text{BV}(\Omega)) \cap L^1(\Omega; \text{BV}(\mathcal{T})) = \text{BV}(Q)$. Moreover, for every $u \in$
182 $\text{BV}(Q)$

$$183 \quad (2.5) \quad \text{TV}(u) \leq \int_0^T \text{TV}_x(u)(t) dt + \int_{\Omega} \text{TV}_t(u)(x) dx \leq \sqrt{2} \text{TV}(u).$$

Proof. We start with the first inclusion,

$$L^1(\mathcal{T}; \text{BV}(\Omega)) \cap L^1(\Omega; \text{BV}(\mathcal{T})) \subset \text{BV}(Q).$$

Let be $u \in L^1(\mathcal{T}; \text{BV}(\Omega)) \cap L^1(\Omega; \text{BV}(\mathcal{T}))$. For any $\xi \in K$ there exists $\varphi = (\varphi_1, \varphi_2) \in \mathcal{C}_c^1(Q, \mathbb{R} \times \mathbb{R}^d)$ such that $\|\varphi\|_{\infty} \leq 1$ and

$$\xi = \frac{\partial \varphi_1}{\partial t} + \text{div}_x \varphi_2 := \xi_1 + \xi_2$$

For every $t \in \mathcal{T}$, $\xi_2(t, \cdot) : x \mapsto \xi_2(t, x)$ belongs to K_x so that

$$\int_{\Omega} \xi_2(t, x) u(t, x) dx \leq \text{TV}_x(u)(t), \quad \text{a.e. } t \in \mathcal{T} ,$$

and

$$\int_0^T \int_{\Omega} \xi_2(t, x) u(t, x) dx dt \leq \int_0^T \text{TV}_x(u)(t) dt .$$

Similarly,

$$\int_{\Omega} \int_0^T \xi_1(t, x) u(t, x) dt dx \leq \int_{\Omega} \text{TV}_t(u)(x) dx .$$

184 Then, for every $\xi \in K$,

$$185 \quad \begin{aligned} \int_Q \xi(t, x) u(t, x) dt dx &= \int_0^T \int_{\Omega} \xi_2(t, x) u(t, x) dx dt + \int_{\Omega} \int_0^T \xi_1(t, x) u(t, x) dt dx \\ &\leq \int_0^T \text{TV}_x(u)(t) dt + \int_{\Omega} \text{TV}_t(u)(x) dx. \end{aligned}$$

The right hand side is finite independently of ξ since $u \in L^1(\mathcal{T}; \text{BV}(\Omega)) \cap L^1(\Omega; \text{BV}(\mathcal{T}))$. Therefore, $u \in \text{BV}(Q)$ and

$$\text{TV}(u) \leq \int_0^T \text{TV}_x(u)(t) dt + \int_{\Omega} \text{TV}_t(u)(x) dx .$$

Let us prove the converse inclusion. We first assume that $u \in W^{1,1}(Q)$. Then, using Fubini's theorem we get $t \mapsto \int_{\Omega} |\nabla_{t,x} u|(t, x) dx \in L^1(\mathcal{T})$ and $x \mapsto \int_0^T |\nabla_{t,x} u|(t, x) dt \in L^1(\Omega)$. Here, we write $|\nabla_{t,x} u|_2 = \sqrt{\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^d \left(\frac{\partial u}{\partial x_i}\right)^2}$ and

$$|\nabla_{t,x} u(t, x)|_2 \leq |\nabla_x u(t, x)|_2 + |\nabla_t u(t, x)| \leq \sqrt{2} |\nabla_{t,x} u(t, x)|_2.$$

186 Therefore, $t \mapsto \int_{\Omega} |\nabla_x u(t, x)|_2 dx \in L^1(\mathcal{T})$, $x \mapsto \int_0^T |\nabla_t u(t, x)| dt \in L^1(\Omega)$ and
187 $u \in L^1(\mathcal{T}; \text{BV}(\Omega)) \cap L^1(\Omega; \text{BV}(\mathcal{T}))$ with

$$188 \quad (2.6) \quad \text{TV}(u) \leq \int_0^T \text{TV}_x(u)(t) dt + \int_{\Omega} \text{TV}_t(u)(x) dx \leq \sqrt{2} \text{TV}(u). \\ 189$$

190 We now consider $u \in \text{BV}(Q)$ and show that $u \in L^1(\mathcal{T}; \text{BV}(\Omega))$. As $W^{1,1}(Q)$ is dense in
191 $\text{BV}(Q)$ in the sense of the intermediate convergence [5], there exists a sequence of functions
192 $u_k \in W^{1,1}(Q)$ such that u_k converges to u in $L^1(Q)$ and $\text{TV}(u_k) \rightarrow \text{TV}(u)$. From Fubini's
193 theorem, we infer that $u_k(t, \cdot)$ converges to $u(t, \cdot)$ in $L^1(\Omega)$, for almost every $t \in \mathcal{T}$ and $u_k(\cdot, x)$
194 converges to $u(\cdot, x)$ in $L^1(\mathcal{T})$, for almost every $x \in \Omega$. Moreover, $\text{TV}(u_k) \rightarrow \text{TV}(u)$ is bounded.
195 Using (2.6) and Fatou's Lemma we have that

$$196 \quad (2.7) \quad \int_0^T \liminf_{k \rightarrow \infty} \text{TV}_x(u_k)(t) dt + \int_{\Omega} \liminf_{k \rightarrow \infty} \text{TV}_t(u_k)(x) dx \\ \leq \liminf_{k \rightarrow \infty} \left(\int_0^T \text{TV}_x(u_k)(t) dt + \int_{\Omega} \text{TV}_t(u_k)(x) dx \right) = \sqrt{2} \text{TV}(u).$$

Then, $\liminf_{k \rightarrow \infty} \text{TV}_x(u_k)(t) < \infty$, a.e $t \in \mathcal{T}$ and $\liminf_{k \rightarrow \infty} \text{TV}_t(u_k)(x) < \infty$, a.e $x \in \Omega$. Now, for a.e. $t \in \mathcal{T}$, we have that

$$\forall \xi \in K_x, \quad \int_{\Omega} u_k(t, x) \xi(x) dx \leq \text{TV}_x(u_k)(t).$$

Hence,

$$\int_{\Omega} u(t, x) \xi(x) dx = \lim_{k \rightarrow +\infty} \int_{\Omega} u_k(t, x) \xi(x) dx \leq \liminf_{k \rightarrow \infty} \text{TV}_x(u_k)(t) < \infty,$$

and

$$\text{TV}_x(u)(t) = \sup_{\xi \in K_x} \int_{\Omega} u(t, x) \xi(x) dx \leq \liminf_{k \rightarrow \infty} \text{TV}_x(u_k)(t) < \infty.$$

This means $u(t, \cdot) \in \text{BV}(\Omega)$ a.e. $t \in \mathcal{T}$. In a similar way, we have that $u(\cdot, x) \in \text{BV}(\mathcal{T})$ a.e. $x \in \Omega$, since

$$\text{TV}_t(u)(x) = \sup_{\xi \in \mathcal{K}_t} \int_0^T u(t, x) \xi(t) dt \leq \liminf_{k \rightarrow \infty} \text{TV}_t(u_k)(x) < \infty.$$

197 Finally, using (2.7), we get

$$\begin{aligned} & \int_0^T \text{TV}_x(u)(t) dt + \int_{\Omega} \text{TV}_t(u)(x) dx \\ 198 & \leq \int_0^T \liminf_{k \rightarrow \infty} \text{TV}_x(u_k)(t) dt + \int_{\Omega} \liminf_{k \rightarrow \infty} \text{TV}_t(u_k)(x) dx \leq \sqrt{2} \text{TV}(u) . \end{aligned}$$

199 This ends the proof, and the inequality (2.6) is also valid for every $u \in \text{BV}(Q)$. \square

200 **Remark 2.1.** *The second inclusion of the previous lemma can be seen as a generalization*
201 *of a function of bounded variation “in the sense of Tonelli” denoted by TBV, see [18, 4]. For*
202 *instance, a function of two variables $h(x, y)$ is TBV on a rectangle $[a, b] \times [c, d]$ if and only if*
203 *$\text{TV}_x h(\cdot, y) < \infty$ for a.e. $y \in [c, d]$, $\text{TV}_y h(x, \cdot) < \infty$ for a.e. $x \in [a, b]$ and $\text{TV}_x h(\cdot, y) \in L^1([a, b])$,*
204 *$\text{TV}_y h(x, \cdot) \in L^1([c, d])$.*

205 **3. The variational model.** As already mentioned in the introduction we are interested in
206 the following variational problem

$$207 \quad (3.1) \quad \inf_{u \in \mathcal{X}} \mathcal{N}(u) + \mathcal{H}(g, \mathcal{A}u) ,$$

208 where $\mathcal{X} = \text{BV}(Q)$. In this section, we describe the choices of the regularizer term $\mathcal{N}(u)$ as
209 well as the data fitting term $\mathcal{H}(g, \mathcal{A}u)$. Recall that $\Omega \subset \mathbb{R}^d$ with $d \geq 1$, $\mathcal{T} = (0, T)$ with $T > 0$
210 and $Q = \mathcal{T} \times \Omega \subset \mathbb{R}^{d+1}$.

211 **3.1. Spatial-temporal regularizer.** In this section, we define the spatial-temporal total
212 variation and infimal convolution total variation regularizers weighted by time dependent pa-
213 rameters. Let α a positive time-dependent weight function $\alpha \in W^{1,\infty}(\mathcal{T})$ with
214 $0 < \alpha_{\min} \leq \alpha(t)$ a.e. $t \in \mathcal{T}$. For the spatial and temporal variations, we write $\Phi_{\alpha_1}(u)$ (in
215 space) as the $L^1(\mathcal{T})$ norm of $t \mapsto \alpha_1(t) \text{TV}_x(u)(t)$, i.e.,

$$216 \quad (3.2) \quad \forall u \in L^1(\mathcal{T}; \text{BV}(\Omega)), \quad \Phi_{\alpha_1}(u) = \int_0^T \text{TV}_x[\alpha_1 u](t) dt = \int_0^T \alpha_1(t) \text{TV}_x[u](t) dt,$$

217 and for temporal penalization, Ψ_{α_2} as

$$218 \quad (3.3) \quad \forall u \in L^1(\Omega; \text{BV}(\mathcal{T})), \quad \Psi_{\alpha_2}(v) = \int_{\Omega} \text{TV}_t[\alpha_2 u](x) dx.$$

219 Note that Ψ_{α_1} , Ψ_{α_2} are convex functionals and that the time dependent parameters α_1, α_2
220 will satisfy

$$221 \quad (3.4) \quad \begin{cases} \alpha_1, \alpha_2 \in W^{1,\infty}(\mathcal{T}) \text{ and there exists} \\ \alpha_{\min} > 0 \text{ s.t. } 0 < \alpha_{\min} \leq \alpha_i(t) \text{ a.e. } t \in \mathcal{T}, i = 1, 2. \end{cases}$$

222 Therefore, using Lemma 2.1 and equations (3.2),(3.3) we have the following:

223 **Definition 3.1.** Let $\mathcal{X} = \text{BV}(Q)$ and $\alpha = (\alpha_1, \alpha_2)$ that satisfy (3.4). We define the spatial-
224 temporal total variation regularizer F_α on \mathcal{X} as

$$225 \quad (3.5) \quad F_\alpha(u) = \Phi_{\alpha_1}(u) + \Psi_{\alpha_2}(u),$$

that is

$$F_\alpha(u) = \int_0^T \text{TV}_x[\alpha_1 u](t) dt + \int_\Omega \text{TV}_t[\alpha_2 u](x) dx.$$

Moreover, for the spatial-temporal infimal convolution total variation regularization we fix $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ that satisfy (3.4) and write

$$\forall u \in \mathcal{X}, \quad F_{\lambda\#\mu}(u) = \inf_{v \in \mathcal{X}} F_\lambda(u - v) + F_\mu(v).$$

226 **Proposition 3.1 (Lower semicontinuity of F_α).** For every $\alpha = (\alpha_1, \alpha_2)$ that satisfy (3.4),
227 the functionals Φ_{α_1} and Ψ_{α_2} are lower semicontinuous on $L^1(\mathcal{T}; \text{BV}(\Omega))$ and $L^1(\Omega; \text{BV}(\mathcal{T}))$
228 respectively, with respect to the $L^1(Q)$ topology. In particular, the functional F_α is lower
229 semicontinuous on $\text{BV}(Q)$ with respect to the L^1 topology. As a consequence, these functionals
230 are lower semicontinuous on $\text{BV}(Q)$ for any $L^p(Q)$ topology with $p \geq 1$.

231 *Proof.* We start with the lower semicontinuity of Φ_{α_1} . The proof is similar for the lower
232 semicontinuity of Ψ_{α_2} . Let $u_n \in L^1(\mathcal{T}; \text{BV}(\Omega))$ such that $u_n \rightarrow u$ in $L^1(Q)$.

233 If $\liminf_{n \rightarrow +\infty} \Phi_{\alpha_1}(u_n) = +\infty$ then the lower semicontinuity inequality is obviously satisfied.

234 Otherwise, one can extract a subsequence (still denoted u_n) such that
235 $\sup_n \Phi_{\alpha_1}(u_n) = \sup_n \int_0^T \text{TV}_x[\alpha_1 u_n](t) dt < +\infty$. Fatou's Lemma applied to the sequence
236 $\text{TV}_x(\alpha_1 u_n)$ gives

$$237 \quad \int_0^T \liminf_{n \rightarrow +\infty} \text{TV}_x[\alpha_1 u_n](t) dt \leq \liminf_{n \rightarrow +\infty} \int_0^T \text{TV}_x[\alpha_1 u_n](t) dt = \liminf_{n \rightarrow +\infty} \Phi_{\alpha_1}(u_n) < +\infty.$$

Moreover, for a.e. $t \in \mathcal{T}$ we have

$$\forall \xi \in K_x, \quad \text{TV}_x[\alpha_1 u_n](t) \geq \int_\Omega \alpha_1(t) \xi(x) u_n(t, x) dx.$$

As u_n strongly converges to u in $L^1(Q)$ then $u_n(t, x) \rightarrow u(t, x)$ in $L^1(\Omega)$ a.e. $t \in \mathcal{T}$ up to a subsequence. Therefore,

$$\forall \xi \in K_x, \text{ a.e. } t \in (0, T), \quad \liminf_{n \rightarrow +\infty} \text{TV}_x[\alpha_1 u_n](t) \geq \int_\Omega \alpha_1(t) \xi(x) u(t, x) dx,$$

and for almost every $t \in \mathcal{T}$

$$\liminf_{n \rightarrow +\infty} \text{TV}_x[\alpha_1 u_n](t) \geq \sup_{\xi \in K_x} \int_\Omega \alpha_1(t) \xi(x) u(t, x) dx = \text{TV}_x[\alpha_1 u](t).$$

Finally,

$$\Phi_{\alpha_1}(u) = \int_0^T \text{TV}_x[\alpha_1 u](t) dt \leq \int_0^T \liminf_{n \rightarrow +\infty} \text{TV}_x[\alpha_1 u_n](t) dt \leq \liminf_{n \rightarrow +\infty} \Phi_{\alpha_1}(u_n).$$

239 Eventually, the functional F_α is lower semicontinuous on $BV(Q)$ as the sum of two lower
 240 semicontinuous functionals. \square

241 Next result provides a relation between the total variation regularization which correlates
 242 space and time and the functional F_α where these directions are treated separately. It is a key
 243 result to prove well-posedness results in the forthcoming analysis.

244 **Theorem 3.1.** *Assume that $\alpha = (\alpha_1, \alpha_2)$ satisfy (3.4). Then, there exists positive constants*
 245 *C_α^-, C_α^+ depending on α , such that for every $u \in BV(Q)$*

$$246 \quad (3.6) \quad C_\alpha^- \text{TV}(\alpha_2 u) \leq F_\alpha(u) \leq C_\alpha^+ \text{TV}(\alpha_2 u) .$$

Proof. Let $\alpha_{max} = \max\{\|\alpha_1\|_{L^\infty(\mathcal{T})}, \|\alpha_2\|_{L^\infty(\mathcal{T})}\}$ and note that $\Phi_{\alpha_1}(u) = \Phi_1(\alpha_1 u)$, for every $u \in BV(Q)$. Then, we have that

$$\frac{\alpha_{min}}{\alpha_{max}} \Phi_1(\alpha_2 u) \leq \Phi_{\alpha_1}(u) \leq \frac{\alpha_{max}}{\alpha_{min}} \Phi_1(\alpha_2 u), \quad \forall u \in BV(Q).$$

247 Since $F_\alpha(u) = \Phi_{\alpha_1}(u) + \Psi_{\alpha_2}(u) = \Phi_1(\frac{\alpha_1}{\alpha_2} \alpha_2 u) + \Psi_1(\alpha_2 u)$ we conclude to

$$248 \quad \frac{\alpha_{min}}{\alpha_{max}} \Phi_1(\alpha_2 u) + \Psi_1(\alpha_2 u) \leq F_\alpha(u) \leq \frac{\alpha_{max}}{\alpha_{min}} \Phi_1(\alpha_2 u) + \Psi_1(\alpha_2 u) \Leftrightarrow$$

$$249 \quad \frac{\alpha_{min}}{\alpha_{max}} (\Phi_1(\alpha_2 u) + \Psi_1(\alpha_2 u)) \leq F_\alpha(u) \leq \frac{\alpha_{max}}{\alpha_{min}} (\Phi_1(\alpha_2 u) + \Psi_1(\alpha_2 u)),$$

250 since $\frac{\alpha_{min}}{\alpha_{max}} \leq 1$ and $\frac{\alpha_{max}}{\alpha_{min}} \geq 1$. Using (2.5) in Lemma 2.1, we obtain

$$\frac{\alpha_{min}}{\alpha_{max}} \text{TV}(\alpha_2 u) \leq F_\alpha(u) \leq \sqrt{2} \frac{\alpha_{max}}{\alpha_{min}} \text{TV}(\alpha_2 u).$$

251 \square

252 In (3.6), we observe that the time dependent parameter α_1 that acts on the spatial domain
 253 of F_α does not contribute to the correlated spatial-temporal total variation. In terms of the
 254 infimal convolution regularizer, a similar result is true when a certain assumption on the time
 255 dependent parameters is imposed.

256 **Proposition 3.2.** *Let $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ time dependent positive parameters*
 257 *that satisfy (3.4). Additionally, let $\kappa > 0$ such that $\mu_2 = \kappa \lambda_2$. Then, there exists constants*
 258 *$C_1, C_2 > 0$ depending on λ, μ and κ such that*

$$259 \quad (3.7) \quad \forall u \in BV(Q) \quad C_1 \text{TV}(\lambda_2 u) \leq F_\lambda \# F_\mu(u) \leq C_2 \text{TV}(\lambda_2 u).$$

260 *Proof.* Let be $u \in BV(Q)$, then for any $v \in BV(Q)$ using Theorem 3.1, we have that

$$261 \quad F_\lambda(u - v) + F_\mu(v) \geq C_\lambda^- \text{TV}(\lambda_2(u - v)) + C_\mu^- \text{TV}(\mu_2 v) = C_\lambda^- \text{TV}(\lambda_2(u - v)) + \kappa C_\mu^- \text{TV}(\lambda_2 v)$$

$$262 \quad \geq \min \{C_\lambda^-, \kappa C_\mu^-\} \left(\text{TV}(\lambda_2(u - v)) + \text{TV}(\lambda_2 v) \right) \geq C_1 \text{TV}(\lambda_2 u)$$

263

Passing to the infimum over $v \in BV(Q)$ and obtain the left-hand side of (3.7). On the other hand, we have that

$$\inf_{v \in BV(Q)} F_\lambda(u - v) + F_\mu(v) \leq F_\lambda(u) \leq C_\lambda^+ \text{TV}(\lambda_2 u) = C_2 \text{TV}(\lambda_2 u).$$

264 \square

265 **Remark 3.1.** *The assumption that there exists $\kappa > 0$ such that $\mu_2 = \kappa\lambda_2$ is a technical*
 266 *assumption. Namely, we are not able to give estimates for $F_\lambda \# F_\mu$ without it. Therefore*
 267 *it is crucial to establish results that follow in the sequel. However, it is not too restrictive.*
 268 *The parameters λ_1 and μ_1 that are responsible on the spatial TV regularization are still not*
 269 *correlated. This allows more freedom so as to handle the time dependent parameters between*
 270 *frames via Φ . On the other hand, the temporal TV regularization imposes that λ_2 and μ_2*
 271 *are equal up to a real κ that can be tuned as well. In fact, this provides a connection on the*
 272 *temporal structure of the image sequences $u - v$ and v .*
 273 *Moreover, we still deal with four parameters which gives freedom to the model but increases the*
 274 *complexity. However, it is possible to reduce this number with consistent additional relation*
 275 *between two of them, see for instance [7].*

276 The following is an immediate result when we consider constant parameters with respect to
 277 time.

278 **Corollary 3.1.** *Assume α, λ and μ are positive constant parameters. Then, we have the*
 279 *following relations for every $u \in \text{BV}(Q)$,*

$$280 \quad \alpha_{\min} C_\alpha^- \text{TV}(u) \leq F_\alpha(u) \leq \alpha_{\max} C_\alpha^+ \text{TV}(u)$$

$$281 \quad \lambda_{\min} C_1 \text{TV}(u) \leq F_\lambda \# F_\mu(u) \leq \lambda_{\max} C_2 \text{TV}(u),$$

283 where $\alpha_{\min} = \min\{\alpha_1, \alpha_2\}$ and $\alpha_{\max} = \max\{\alpha_1, \alpha_2\}$ and respectively for λ_{\min} and λ_{\max} .

284 **3.2. Fitting data term.** In this section, we describe the possible choices of the data fitting
 285 term depending on the degradation of the input dynamic datum g as well as the linear operator
 286 \mathcal{A} . Our setting is quite general and can be applied to any video denoising and deblurring
 287 application for instance, or even dynamic emission tomography (ET) such as Positron Emission
 288 Tomography (PET). We begin with two separate cases in terms of the linear operator \mathcal{A} .

289

290 **Case (1) :** $\mathcal{A} = A$

291

292 We consider a linear and continuous operator with the following assumptions:

$$293 \quad (3.8) \quad A \in \mathcal{L}(L^p(Q), L^q(Q)) \text{ with } 1 < p \leq \frac{d+1}{d}, \quad 1 \leq q < \infty,$$

$$294 \quad (3.9) \quad A\chi_Q \neq 0,$$

296 Under these conditions, one can use an identity operator (e.g., denoising) or a blurring (con-
 297 volution) operator (e.g., deblurring) to represent A . In the case of deblurring, we define
 298 $Au := h * u$, where h is a spatially invariant blurring kernel that remains constant over the
 299 time domain. Consequently, we assume that

$$300 \quad (3.10) \quad A(\alpha(t)u) = \alpha(t)A(u),$$

for a positive time dependent parameter α , see [17], i.e.,

$$A(\alpha(t)u(t, x)) = \alpha(t)A(u(t, x)) = \alpha(t)(h(x) * u(t, x)).$$

301 For (3.9), we assume that A does not annihilate constant functions which is an important tool
 302 to derive existence results. Now we define

$$303 \quad (3.11) \quad \mathcal{H}(g, \mathcal{A}u) = \frac{1}{q} \|Au - g\|_{L^q(Q)}^q \text{ with } g \in L^q(Q).$$

304 as our data fitting term. This is suitable for dynamic data corrupted by noise that follows
 305 Gaussian distribution ($q = 2$) or impulse noise ($q = 1$) for example, see also [10].

306

307

Case (2) : $\mathcal{A} = \mathcal{R}$

308

309 Here, we consider a linear operator related to emission imaging. The dynamic data that
 310 we obtained during a PET scan for instance, are connected through an integral (projection)
 311 operator known as the Radon transform \mathcal{R} . For every $t \in \mathcal{T}$, we write

$$312 \quad (3.12) \quad (\mathcal{R}u(\theta, s))(t) = \int_{x \cdot \theta = s} u(t, x) dx,$$

313 where $\{x \in \mathbb{R}^d : x \cdot \theta = s\}$ is the hyperplane perpendicular to $\theta \in \mathcal{S}^{d-1}$ with distance $s \in \mathbb{R}$
 314 from the origin. For $t \in \mathcal{T}$, $(\mathcal{R}u(\theta, s))(t)$ lies on $\{(\theta, s) : \theta \in \mathcal{S}^{d-1}, s \in \mathbb{R}\}$, a cylinder of
 315 dimension d and is often referred as *projection space* or *sinogram space*. In the dynamic
 316 framework, we set $\Sigma = \mathcal{T} \times \{(\theta, s) : \theta \in \mathcal{S}^{d-1}, s \in \mathbb{R}\}$ and the Radon transform is a continuous
 317 linear operator with

$$318 \quad (3.13) \quad \mathcal{R} : L^1(Q) \rightarrow L^1(\Sigma), \quad \|\mathcal{R}u\|_{L^1(\Sigma)} \leq C \|u\|_{L^1(Q)}.$$

319 We refer the reader to [31] for general continuity results of the Radon transform in L^p spaces.
 320 Furthermore, if $p \geq \frac{d+1}{d}$, the Radon transform is L^p discontinuous, since the function
 321 $u(x) = |x|^{-\frac{d+1}{p}} \frac{1}{\log(|x|)}$ belongs to $L^p(Q)$, for $x \in Q$ but is not integrable over any hyperplane,
 322 see [28, Th. 3.32].

323 During the PET acquisition process, a certain amount of events e.g., photon-emissions are
 324 collected by the scanner (detectors) and organized into the so-called temporal bins $g(\theta, s, t)$
 325 for every $t \in \mathcal{T}$. The associated noise in this data is called *photon noise* due to the ran-
 326 domness in the photo counting process and in fact, obeys the well-known Poisson probabilit-
 327 ity distribution. For this kind of noise we use the *Kullback-Leibler* divergence, see [10],[27],
 328 $D_{KL} : L^1(Q) \times L^1(Q) \rightarrow \mathbb{R}_+ \cup \{\infty\}$, defined as

$$329 \quad (3.14) \quad D_{KL}(u, v) = \int_Q \left(u \log \left(\frac{u}{v} \right) - u + v \right) dx dt, \quad \forall u, v \geq 0 \text{ a.e.}$$

330 This is in fact the *Bregman* distance of the *Boltzmann-Shannon* entropy, see [33]. We briefly
 331 recall some of the basic properties of the KL-functional which can be found in [11],[33] and
 332 will be used later.

333 **Lemma 3.1.** *The following properties hold true:*

- 334 (a) $D_{KL}(u, v)$ is nonnegative and equal to 0 if and only if $u = v$.
 335 (b) The function $(u, v) \mapsto D_{KL}(u, v)$ is convex.

336 (c) For fixed $u \in L^1_+(Q)$ (resp. $v \in L^1_+(Q)$), the function $D_{KL}(u, \cdot)$ (resp. $D_{KL}(\cdot, v)$) is
 337 weakly lower semicontinuous with respect to L^1 topology.

338 (d) For every $u, v \in L^1_+(Q)$

$$339 \quad (3.15) \quad \|u - v\|_{L^1(Q)}^2 \leq \left(\frac{2}{3} \|u\|_{L^1(Q)} + \frac{4}{3} \|v\|_{L^1(Q)} \right) D_{KL}(u, v).$$

340 Another crucial assumption is that the dynamic data g are bounded and bounded away from
 341 0. Equivalently, we assume

$$342 \quad (3.16) \quad g \in L^\infty(Q), \quad \text{with} \quad \inf_{\Sigma} g > 0.$$

343 The boundedness assumption is true since we deal with a finite acquisition time. Moreover,
 344 for a reasonably long counting process, where some million of photons are detected, all the
 345 PET detectors will record a certain amount of photons, even if it is relatively small in practice.
 346 Additionally, one has to consider a certain level of background noise. Hence, the assumptions
 347 of the boundedness and the boundedness away from 0 can describe a realistic emitted data
 348 and do not deviate from the ground-truth. For more details, we refer to [45, Chapters 10-12].
 349 Due to (3.12), we have a similar condition as in (3.10) and in addition we require that \mathcal{R} does
 350 not annihilate constant functions. Hence, we have that

$$351 \quad (3.17) \quad \mathcal{R}(\alpha(t)u) = \alpha(t)\mathcal{R}(u), \quad \mathcal{R}\chi_Q \neq 0.$$

352 To conclude, we define

$$353 \quad (3.18) \quad \mathcal{H}(g, \mathcal{A}u) = D_{KL}(g, \mathcal{R}u),$$

354 whose domain is the cone of positive functions, as our data fitting term. In practice, when we
 355 deal with the minimization problem (3.1), the fidelity is a reduced version of the KL-divergence,
 356 since we can neglect the terms that are independent of u . Indeed, we write

$$357 \quad (3.19) \quad \mathcal{H}(g, \mathcal{A}u) = \int_{\Sigma} \mathcal{R}u - g \log \mathcal{R}u \, d\theta \, ds \, dt,$$

358 since the $g \log g$, $-g$ do not count on the minimization problem (3.1). Let us mention that the
 359 domain of above expression is still the cone of positive functions since $u \geq 0$ a.e. implies that
 360 $\mathcal{R}u \geq 0$ a.e. Though we are mainly interested in the Radon transform case, one can replace \mathcal{R}
 361 with the identity or a blurring operator as in the previous case, suitable for Poisson denoising
 362 and deblurring and with the analogous assumptions (3.16), (3.17).

363 **4. Well posedness results.** In this section, we are interested in the well-posedness of
 364 the minimization problem (3.1) for the regularizers described in Section 3.1 and the different
 365 choices of the data fitting term in (3.11) and (3.19). We focus on the infimal convolution total
 366 variation regularizer case i.e., $\mathcal{N}(u) := F_\lambda \# F_\mu(u)$. In the case of the total variation regularizer,
 367 the forthcoming analysis is similar and most of the proofs are the same with minor adaptations.
 368 We prove well-posedness (existence, uniqueness and stability) via the direct method of calculus
 369 of variations for

$$370 \quad (\mathcal{P}) \quad \inf_{u \in \text{BV}(Q)} F_\lambda \# F_\mu(u) + \mathcal{H}(g, \mathcal{A}u).$$

371 In particular, we need the lower semicontinuity condition to be true for both the regularizing
372 and the fidelity term, together with some compactness properties.

373 **4.1. Lower semicontinuity of the inf-convolution operator.** Note that the lower semi-
374 continuity of the inf-convolution operator is not true in general, even if F_λ is, see [6, Example
375 12.13]. Additional assumptions have to be imposed such as coercivity on the underlying space
376 as well as exactness of the infimal convolution in order to be lower semicontinuous. The first
377 step is to show that the inf-convolution operator is *exact* in our case. This is the object of
378 next Lemma.

Lemma 4.1 (Exactness of $F_\lambda \# F_\mu$). *Assume that λ and μ verify (3.4) and there exists $\kappa > 0$ such that $\mu_2 = \kappa\lambda_2$. Then, for every $u \in \text{BV}(Q)$, there exists $v_u \in \text{BV}(Q)$ such that*

$$v_u \in \underset{v \in \text{BV}(Q)}{\text{argmin}} F_\lambda(u - v) + F_\mu(v) \text{ and } \int_Q \mu_2(t) v_u(t, x) dt dx = 0.$$

Proof. Fix $u \in \text{BV}(Q)$. Let v_n be a minimizing sequence of

$$\inf_{v \in \text{BV}(Q)} F_\lambda(u - v) + F_\mu(v).$$

Then $v_n \in \text{BV}(Q)$ and without loss of generality we may assume that the mean value of $\mu_2 v_n$ is

$$\overline{\mu_2 v_n} := \frac{1}{|Q|} \int_Q \mu_2(t) v_n(t, x) dx dt = 0.$$

379 Indeed, since $\mu_2 = \kappa\lambda_2$, it is easy to see that

$$380 F_\lambda \left(u - \left(v_n - \frac{1}{\mu_2} \overline{\mu_2 v_n} \right) \right) + F_\mu \left(v_n - \frac{1}{\mu_2} \overline{\mu_2 v_n} \right) = F_\lambda(u - v_n) + F_\mu(v_n),$$

381

382 so that $w_n := v_n - \frac{1}{\mu_2} \overline{\mu_2 v_n}$ is also a minimizing sequence that satisfies $\int_Q \mu_2 w_n dx dt = 0$.

383 As $F_\lambda(u - v_n) + F_\mu(v_n)$ is bounded and Theorem 3.1 yields that $\text{TV}(\mu_2 v_n)$ is bounded
384 as well. Moreover, we have $\|\mu_2 v_n\|_{L^1(Q)} \leq \text{TV}(\mu_2 v_n)$ from the Poincaré-Wirtinger inequality,
385 see Theorem 2.1. Hence, $(\mu_2 v_n)$ is BV-bounded. This implies that v_n is BV-bounded as well
386 (see Lemma 4.2 and (4.1)). Therefore, there exists $v_u \in \text{BV}(Q)$ such that, up to subsequence,
387 $v_n \xrightarrow{w^*} v_u$, i.e., $v_n \rightarrow v_u$ for the $L^1(Q)$ topology. We end the proof with the lower semicontinuity
388 of the functional with respect to the the $L^1(Q)$ topology, see Proposition 3.1. In addition, since
389 $\int_Q \mu_2(t) v_n(t, x) dx dt = 0$, we have from the L^1 convergence that $\int_Q \mu_2(t) v_u(t, x) dx dt = 0$ as
390 well. \square

391

392 Next Lemma provides an estimate on $u \in \text{BV}(Q)$ when α satisfies (3.4) and $\alpha u \in \text{BV}(Q)$.

393

394 **Lemma 4.2.** *Assume that $\alpha \in W^{1,\infty}(\mathcal{T})$ and that there exists $\alpha_{\min} > 0$ such that $0 <$
395 $\alpha_{\min} \leq \alpha(t)$ a.e. $t \in \mathcal{T}$, then $1/\alpha \in W^{1,\infty}(\mathcal{T})$. Moreover, if $\alpha u \in \text{BV}(Q)$ then $u \in \text{BV}(Q)$ as
396 well.*

397 *Proof.* Let α be in $W^{1,\infty}(\mathcal{T})$ such that $0 < \alpha_{\min} \leq \alpha(t)$ a.e. $t \in \mathcal{T}$. We use Proposition
 398 8.4 of [13] to infer that $1/\alpha \in W^{1,\infty}(\mathcal{T})$. Moreover, if $\alpha u \in \text{BV}(Q)$ then $u \in \text{BV}(Q)$. Indeed,
 399 since

$$400 \quad (4.1) \quad \|u\|_{\text{BV}(Q)} \leq \frac{2}{\alpha_{\min}} \|\alpha u\|_{L^1(Q)} + \frac{\|\alpha'\|_{L^\infty(\mathcal{T})}}{\alpha_{\min}^2} \text{TV}(\alpha u) \leq C_\alpha \|\alpha u\|_{\text{BV}(Q)}.$$

401 □
 402 Now we prove a lower semicontinuity result of $F_\lambda \# F_\mu$. Here, we use the exactness of $F_\lambda \# F_\mu$
 403 and the BV coercivity of one of its terms. For more details on the lower semicontinuity of the
 404 infimal convolution we refer to [41].

405 **Theorem 4.1.** *Assume that λ and μ verify (3.4) and there exists $\kappa > 0$ such that $\mu_2 = \kappa \lambda_2$.
 406 Then, the infimal-convolution $F_\lambda \# F_\mu$ operator is lower semicontinuous on $\text{BV}(Q)$ with respect
 407 to the L^1 topology. Precisely, if u_n is a sequence in $\text{BV}(Q)$ that converges to some u with
 408 respect to the strong L^1 topology then*

$$409 \quad (4.2) \quad F_\lambda \# F_\mu(u) \leq \liminf_{n \rightarrow +\infty} F_\lambda \# F_\mu(u_n).$$

Proof. Let $u_n \in \text{BV}(Q)$ such that $u_n \rightarrow u$ in $L^1(Q)$. If $\liminf_{n \rightarrow +\infty} F_\lambda \# F_\mu(u_n) = +\infty$ then
 relation (4.2) is satisfied. Otherwise, there exists a subsequence (denoted similarly) and a
 constant C such that $F_\lambda \# F_\mu(u_n) \leq C$. Since $F_\lambda \# F_\mu$ is exact, there exists $v_n \in \text{BV}(Q)$ such
 that

$$F_\lambda(u_n - v_n) + F_\mu(v_n) = F_\lambda \# F_\mu(u_n) \quad \text{and} \quad \int_Q \mu_2 v_n = 0.$$

We claim that $(\mu_2 v_n)$ is BV-bounded (that is $\|\mu_2 v_n\|_{\text{BV}(Q)}$ is uniformly bounded with respect
 to n). Indeed, Theorem (3.1) yields

$$\text{TV}(\mu_2 v_n) \leq \frac{1}{C_\mu^-} F_\mu(v_n) \leq \frac{C}{C_\mu^-}.$$

Using Poincaré-Wirtinger inequality, we have that

$$\|\mu_2 v_n\|_{L^1(Q)} \leq C_Q \text{TV}(\mu_2 v_n) \leq \frac{C}{C_\mu^-} C_Q.$$

Following similar steps as before, there exists a subsequence $v_n \xrightarrow{w^*} \tilde{v}$ in $\text{BV}(Q)$. Due to the
 lower semicontinuity F_λ and F_μ with respect to L^1 and its exactness, we have

$$F_\lambda(u - \tilde{v}) + F_\mu(\tilde{v}) \leq \liminf_{n \rightarrow +\infty} F_\lambda(u_n - v_n) + F_\mu(v_n) = \liminf_{n \rightarrow +\infty} F_\lambda \# F_\mu(u_n)$$

and since $F_\lambda \# F_\mu(u) \leq F_\lambda(u - \tilde{v}) + F_\mu(\tilde{v})$, we conclude that

$$F_\lambda \# F_\mu(u) \leq \liminf_{n \rightarrow +\infty} F_\lambda \# F_\mu(u_n).$$

410 □

411 **4.2. Well posedness.** Now we focus on the existence of a solution for (\mathcal{P}) . The proof is
 412 based on the corresponding results in [1, 44, 33] adapted to a spatial-temporal framework.

413 **Theorem 4.2 (Existence).** *Let λ, μ that satisfy (3.4) and that there exists a real number*
 414 *$\kappa > 0$ such that $\mu_2 = \kappa\lambda_2$. Then, there exists a solution to problem (\mathcal{P}) .*

Proof. Let $u_n \in \text{BV}(Q)$ be a minimizing sequence of problem (\mathcal{P}) . Then there exists $M > 0$ such that

$$F_\lambda \# F_\mu(u_n) + \mathcal{H}(g, Au_n) \leq M.$$

Using Proposition 3.2, we deduce that $\text{TV}(\lambda_2 u_n)$ is bounded. Let $w_n = \lambda_2 u_n$ then we have $\|w_n - \bar{w}_n\|_{L^p(Q)} \leq M_3$ with $1 \leq p \leq \frac{d+1}{d}$. Moreover, we have

$$\|w_n\|_{L^p(Q)} \leq \|w_n - \bar{w}_n\|_{L^p(Q)} + \|\bar{w}_n\|_{L^p(Q)} \leq M_3 + \left| \int_Q w_n dx dt \right|.$$

415 The goal is to prove that u_n is BV bounded. This is equivalent to find an estimate on the last
 416 term of the above inequality. To achieve this, we consider the two cases with respect to the
 417 choice of the fidelity term presented in Section 3.2.

418

419

$$\text{Case (1)} : \mathcal{H}(g, Au) = \frac{1}{q} \|Au - g\|_{L^q(Q)}^q$$

420

421

422

Recall that $g \in L^q(Q)$, $A \in \mathcal{L}(L^p(Q), L^q(Q))$ with $1 \leq p \leq \frac{d+1}{d}$, $1 \leq q < \infty$, and satisfy (3.9), (3.10). Then, one has that

$$\begin{aligned} 423 \quad \left| \int_Q w_n dx dt \right| \frac{\|A\chi_Q\|_{L^q(Q)}}{|Q|} &= \|A\bar{w}_n\|_{L^q(Q)} \leq \|A\bar{w}_n - Aw_n + Aw_n - \lambda_2 g + \lambda_2 g\|_{L^q(Q)} \\ 424 &\leq \|A\| \|w_n - \bar{w}_n\|_{L^p(Q)} + \|A(\lambda_2 u_n) - \lambda_2 g\|_{L^q(Q)} + \|\lambda_2 g\|_{L^q(Q)} \\ 425 &\leq \|A\| \|w_n - \bar{w}_n\|_{L^p(Q)} + \|\lambda_2\|_{L^\infty(\mathcal{T})} \left(\|Au_n - g\|_{L^q(Q)} + \|g\|_{L^q(Q)} \right) \\ 426 &\leq \|A\| M_3 + \|\lambda_2\|_{L^\infty(\mathcal{T})} \left((qM)^{1/q} + \|g\|_{L^q(Q)} \right) \leq M_4. \\ 427 \end{aligned}$$

428

429

430

431

432

Recall that, $g \in L^\infty(Q)$, $\inf_\Sigma g > 0$, $\mathcal{R} \in \mathcal{L}(L^1(Q), L^1(\Sigma))$ that satisfies (3.17) with an additional positivity constraint $u_n \geq 0$. Therefore, it suffices to bound $\int_Q w_n dx dt$. Since $\inf_\Sigma g > 0$, we employ (3.15) and using (3.13) we have

$$\begin{aligned} 433 \quad \|\mathcal{R}w_n - \lambda_2 g\|_{L^1(\Sigma)}^2 &\leq \left(\frac{2}{3} \|\lambda_2 g\|_{L^1(\Sigma)} + \frac{4}{3} \|\mathcal{R}w_n\|_{L^1(\Sigma)} \right) D_{KL}(\lambda_2 g, \lambda_2 \mathcal{R}u_n) \\ 434 &\leq \left(\frac{2}{3} \|\lambda_2\|_{L^\infty(\mathcal{T})} \|g\|_{L^1(\Sigma)} + \frac{4}{3} \|\mathcal{R}(w_n - \bar{w}_n) + \mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} \right) \|\lambda_2\|_{L^\infty(\mathcal{T})} D_{KL}(g, \mathcal{R}u_n) \\ 435 &\leq \left(\frac{2}{3} \|\lambda_2\|_{L^\infty(\mathcal{T})} \|g\|_{L^1(\Sigma)} + \frac{4}{3} \|\mathcal{R}\| \|w_n - \bar{w}_n\|_{L^1(Q)} + \frac{4}{3} \|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} \right) \|\lambda_2\|_{L^\infty(\mathcal{T})} M \end{aligned}$$

$$\leq \left(\frac{2}{3} \|\lambda_2\|_{L^\infty(\mathcal{T})} \|g\|_{L^1(\Sigma)} + \frac{4}{3} \|\mathcal{R}\| |Q|^{1/p'} M_3 + \frac{4}{3} \|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} \right) \|\lambda_2\|_{L^\infty(\mathcal{T})} M.$$

Hence,

$$(4.3) \quad \|\mathcal{R}w_n - \lambda_2 g\|_{L^1(\Sigma)}^2 \leq \left(M_5 + \frac{4}{3} \|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} \right) M_6.$$

On the other hand,

$$\begin{aligned} \|\mathcal{R}w_n - \lambda_2 g\|_{L^1(\Sigma)}^2 &\geq \left(\|\mathcal{R}(w_n - \bar{w}_n) - \lambda_2 g\|_{L^1(\Sigma)} - \|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} \right)^2 \\ &\geq \|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} \left(\|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} - 2 \|\mathcal{R}(w_n - \bar{w}_n) - \lambda_2 g\|_{L^1(\Sigma)} \right) \\ &\geq \|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} \left(\|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} - 2 \left(\|\mathcal{R}\| |Q|^{1/p'} M_3 + \|\lambda_2\|_{L^\infty(\mathcal{T})} \|g\|_{L^1(\Sigma)} \right) \right) \\ (4.4) \quad &\geq \|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} \left(\|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} - M_7 \right). \end{aligned}$$

Also, we have that

$$(4.5) \quad \|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} = \frac{\int_Q w_n dx dt}{|Q|} \|\mathcal{R}\chi_Q\|_{L^1(\Sigma)} \Leftrightarrow \|\mathcal{R}\bar{w}_n\|_{L^1(\Sigma)} = M_8 \|\bar{w}_n\|_{L^1(Q)}.$$

Combining (4.3), (4.4) and (4.5), we derive that

$$(4.6) \quad M_8 \|\bar{w}_n\|_{L^1(Q)} \left(M_8 \|\bar{w}_n\|_{L^1(Q)} - M_7 - \frac{4}{3} M_6 \right) \leq M_5 M_6.$$

Let $B = M_8 \|\bar{w}_n\|_{L^1(Q)} - M_7 - \frac{4}{3} M_6$. If $B \geq 1$, it is immediate from (4.6), that $\|\bar{w}_n\|_{L^1(Q)}$ is bounded. Otherwise, we have that

$$\|\bar{w}_n\|_{L^1(Q)} \leq \frac{1 + M_7 + \frac{4}{3} M_6}{M_8},$$

which is again bounded.

To conclude, we have proved that in both cases $w_n = \lambda_2 u_n$ is L^p bounded and hence is BV bounded. Using Lemma 4.2, u_n is BV and L^p bounded. Then, there exists subsequence still denoted by u_n such that $u_n \xrightarrow{w^*} u$ in BV, i.e., $u_n \rightarrow u$ in L^1 and $u_n \xrightarrow{w} u$ in L^p , $1 < p \leq \frac{d+1}{d}$. Theorem 4.1 yields that

$$F_\lambda \# F_\mu(u) \leq \liminf_{n \rightarrow \infty} F_\lambda \# F_\mu(u_n).$$

Moreover, due to the lower semicontinuity of the fidelity terms as well as the continuity of A and \mathcal{R} , we conclude that

$$\mathcal{H}(g, Au) \leq \liminf_{n \rightarrow \infty} \mathcal{H}(g, Au_n).$$

This means that u is a solution to (\mathcal{P}) . □

453 **Remark 4.1.** *To be consistent with the cases where either A or \mathcal{R} is the identity operator,*
 454 *let us mention that the BV-boundedness is immediate since*

$$455 \quad \|u_n\|_{L^q(Q)} \leq \|u_n - g\|_{L^q(Q)} + \|g\|_{L^q(Q)}$$

$$456 \quad \|u_n\|_{L^1(Q)} - \|g\|_{L^\infty(Q)} \log \|u_n\|_{L^1(Q)} - |Q| \leq D_{KL}(g, u_n).$$

458 *In the latter case, we use the Jensen's inequality and that $\frac{x-1}{x} \leq \log x$, for $x > 0$. We refer*
 459 *also to [27].*

460 **Theorem 4.3 (Uniqueness).** *Assume that λ and μ satisfy (3.4) and that there exists a real*
 461 *number $\kappa > 0$ such that $\mu_2 = \kappa\lambda_2$. In addition, assume that*
 462 *• either A is injective in Case (1) for (3.11) and $1 < q < \infty$, or*
 463 *• g satisfies (3.16) for Case (2).*
 464 *Then the solution to (\mathcal{P}) is unique.*

465 *Proof.* Note that $F_\lambda \# F_\mu$ is convex since F_λ and F_μ are convex. We first consider Case
 466 (1) : since $1 < q < \infty$ and A is injective then $u \mapsto \frac{1}{q} \|Au - g\|_{L^q(Q)}^q$ is strictly convex.
 467 In case (2), since $\inf_\Sigma g > 0$ and \mathcal{R} is injective, see for instance [28], then $u \mapsto D_{KL}(g, \mathcal{R}u)$ is
 468 strictly convex. In both cases, we have that the energy $u \mapsto \mathcal{E}(u) := (F_\lambda \# F_\mu)(u) + \mathcal{H}(Au, g)$ is
 469 strictly convex as a sum of a convex and a strictly convex terms. This gives uniqueness. \square

470 To conclude this section, we discuss the stability of minimizers of (\mathcal{P}) , see [1, 33, 37] for
 471 instance, with respect to a *small* perturbation on the data g . Let (g_n) be a perturbed dynamic
 472 data sequence such that

$$473 \quad (4.7) \quad \mathcal{H}(g_n, g \rightarrow 0) \Leftrightarrow \begin{cases} \|g_n - g\|_{L^q(Q)} \rightarrow 0, & g_n \in L^q(Q) \quad \text{Case (1)} \\ D_{KL}(g_n, g) \rightarrow 0, & g_n \in L^\infty(\Sigma) \quad \text{Case (2)} \end{cases}$$

474 and the corresponding perturbed minimization problem

$$475 \quad (4.8) \quad \inf_{u \in \text{BV}(Q)} (F_\lambda \# F_\mu)(u) + \mathcal{H}(g_n, Au).$$

476

477 **Theorem 4.4 (Stability).** *Assume the assumptions of Theorem 4.3 are fulfilled and that*
 478 *$\inf_\Sigma g_n > 0$ and $\log \mathcal{R}u \in L^\infty(\Sigma)$ in Case (2). Then problem (\mathcal{P}) is stable with respect to*
 479 *perturbations on g . Precisely, let (g_n) as in (4.7) and u, u_n be the solutions to (\mathcal{P}) and (4.8)*
 480 *respectively. Then, there exists a subsequence of (u_n) that converges to u in $\text{BV}(Q)$ - w^* .*

481 *Proof.* Since u_n minimizes (4.8), then for every $v \in \text{BV}(Q)$

$$482 \quad (4.9) \quad (F_\lambda \# F_\mu)(u_n) + \mathcal{H}(g_n, Au_n) \leq (F_\lambda \# F_\mu)(v) + \mathcal{H}(g_n, Av).$$

483 As in the previous proofs, we consider each case separately.

484

$$485 \quad \textbf{Case (1)} : \mathcal{H}(g, Au) = \frac{1}{q} \|Au - g\|_{L^q(Q)}^q$$

486

487 Since $g_n \rightarrow g$ in $L^q(Q)$, then there exists $n_0 \in \mathbb{N}$ such that $\|g - g_n\|_q^q \leq \frac{q}{2^{q-1}}$ for every
488 $n \geq n_0$. So, for every $n \geq n_0$

$$\begin{aligned}
489 \quad (F_\lambda \# F_\mu)(u_n) + \frac{1}{q} \|Au_n - g\|_{L^q(Q)}^q &\leq 2^{q-1} \left((F_\lambda \# F_\mu)(u_n) + \frac{1}{q} \|Au_n - g_n\|_{L^q(Q)}^q + \frac{1}{q} \|g_n - g\|_{L^q(Q)}^q \right) \\
490 &\leq 2^{q-1} \left((F_\lambda \# F_\mu)(u) + \frac{1}{q} \|Au - g_n\|_{L^q(Q)}^q + \frac{1}{q} \|g_n - g\|_{L^q(Q)}^q \right) \\
491 &\leq (M + 1).
\end{aligned}$$

Here, we used the convexity of the L^q norm ($q > 1$) and relation (4.9) with $v = u$. Following the proof of Theorem 4.2, we can prove that (u_n) is BV-bounded, L^p -bounded with $1 < p \leq \frac{d+1}{d}$ and (v_n) are BV-bounded. Therefore, we have that $u_n \rightarrow \tilde{u}$ in L^1 , $u_n \xrightarrow{w} \tilde{u}$ in L^p , with $1 < p \leq \frac{d+1}{d}$ and $v_n \rightarrow v$ in L^1 . It remains to show that \tilde{u} is a minimizer of (\mathcal{P}) . Theorem 4.1 yields that

$$(F_\lambda \# F_\mu)(\tilde{u}) \leq \liminf_{n \rightarrow \infty} (F_\lambda \# F_\mu)(u_n).$$

Moreover $Au_n - g_n \rightarrow A\tilde{u} - g$ in $L^q(Q)$. Since,

$$\forall v \in \text{BV}(Q) \quad (F_\lambda \# F_\mu)(u_n) + \frac{1}{q} \|Au_n - g_n\|_{L^q(Q)}^q \leq (F_\lambda \# F_\mu)(v) + \frac{1}{q} \|Av - g_n\|_{L^q(Q)}^q$$

493 we get for every $v \in \text{BV}(Q)$ that

$$\begin{aligned}
494 \quad (F_\lambda \# F_\mu)(\tilde{u}) + \frac{1}{q} \|A\tilde{u} - g\|_{L^q(Q)}^q &\leq \liminf_{n \rightarrow \infty} \left[(F_\lambda \# F_\mu)(u_n) + \frac{1}{q} \|Au_n - g_n\|_{L^q(Q)}^q \right] \\
495 &\leq \lim_{n \rightarrow \infty} (F_\lambda \# F_\mu)(v) + \frac{1}{q} \|Av - g_n\|_{L^q(Q)}^q \\
496 &\leq (F_\lambda \# F_\mu)(v) + \frac{1}{q} \|Av - g\|_{L^q(Q)}^q.
\end{aligned}$$

498 So \tilde{u} is a minimizer and we conclude with uniqueness that $u = \tilde{u}$.

499

$$\text{Case (2)} : \mathcal{H}(g, \mathcal{A}u) = D_{KL}(g, \mathcal{R}u)$$

500

502 We assumed that $g, g_n \in L^\infty(\Sigma)$, $\inf_\Sigma g, \inf_\Sigma g_n > 0$ and $\log \mathcal{R}u \in L^\infty(\Sigma)$. Using (4.9), for
503 every $v \in \text{BV}(Q)$, $v \geq 0$ a.e.,

$$504 \quad (4.10) \quad (F_\lambda \# F_\mu)(u_n) + D_{KL}(g_n, \mathcal{R}u_n) \leq (F_\lambda \# F_\mu)(v) + D_{KL}(g_n, \mathcal{R}v).$$

A short computation gives that

$$|D_{KL}(g_n, \mathcal{R}v) - D_{KL}(g, \mathcal{R}v) - D_{KL}(g_n, g)| \leq \|\log \mathcal{R}v - \log g\|_{L^\infty(\Sigma)} \|g_n - g\|_{L^1(\Sigma)}.$$

As $D_{KL}(g_n, g) \rightarrow 0$, then since Lemma 3.1 implies that $g_n \rightarrow g$ in $L^1(\Sigma)$. So

$$\forall v \in \text{BV}(Q), v \geq 0 \quad \lim_{n \rightarrow \infty} D_{KL}(g_n, \mathcal{R}v) = D_{KL}(g, \mathcal{R}v).$$

Moreover, we can prove using Theorem 4.2 Case (2), that u_n is bounded in $L^p(Q)$ with $1 < p \leq \frac{d+1}{d}$ by a constant depending on $\|g_n\|_{L^1(\Sigma)}$. This bound is uniform with respect to n since $\|g_n\|_{L^1(\Sigma)}$ is bounded. As before, u_n is BV bounded and there exists $\tilde{u} \in \text{BV}(Q)$ such that $u_n \rightarrow \tilde{u}$ in $L^1(Q)$. Hence, $\mathcal{R}u_n \rightarrow \mathcal{R}\tilde{u}$ in $L^1(\Sigma)$ as well as pointwise convergent almost everywhere in Σ . By Fatou's Lemma applied to the sequence $(g_n \log g_n - g_n \log \mathcal{R}u_n - g_n + \mathcal{R}u_n)_n$, we obtain

$$D_{KL}(g, \mathcal{R}\tilde{u}) \leq \liminf_{n \rightarrow \infty} D_{KL}(g_n, \mathcal{R}u_n).$$

505 Similarly to the previous case, we get for every $v \in \text{BV}(Q)$, $v \geq 0$ that

$$\begin{aligned} 506 \quad (F_\lambda \# F_\mu)(\tilde{u}) + D_{KL}(g, \mathcal{R}\tilde{u}) &\leq \liminf_{n \rightarrow \infty} (F_\lambda \# F_\mu)(u_n) + D_{KL}(g_n, \mathcal{R}u_n) \\ 507 &\leq \lim_{n \rightarrow \infty} (F_\lambda \# F_\mu)(v) + D_{KL}(g_n, \mathcal{R}v) \\ 508 &\leq (F_\lambda \# F_\mu)(v) + D_{KL}(g, \mathcal{R}v). \end{aligned}$$

510 By uniqueness, we conclude that $\tilde{u} = u$ is the minimizer of (\mathcal{P}) . □

511 **4.3. An equivalent formulation.** We end this section by providing an equivalent formu-
512 lation for (\mathcal{P}) that may be useful for numerical computations. The key tool is the exactness
513 of the inf-convolution operator. The original problem (\mathcal{P}) also reads

$$514 \quad (\mathcal{P}') \quad \inf_{(u,v) \in \text{BV}(Q) \times \text{BV}(Q)} F_\lambda(u-v) + F_\mu(v) + \mathcal{H}(g, \mathcal{A}u).$$

515

516 **Theorem 4.5 (Equivalence).** *Assume that λ and μ verify (3.4) and there exists $\kappa > 0$ such*
517 *that $\mu_2 = \kappa \lambda_2$.*

518 1. *If (\mathbf{u}, \mathbf{v}) is a solution of (\mathcal{P}') , then \mathbf{u} is a solution of (\mathcal{P}) and*

$$519 \quad (4.11) \quad F_\lambda(\mathbf{u} - \mathbf{v}) + F_\mu(\mathbf{v}) = F_\lambda \# F_\mu(\mathbf{u}) = \inf_{v \in \text{BV}(Q)} \{F_\lambda(\mathbf{u} - v) + F_\mu(v)\}.$$

520 2. *If \mathbf{u} is a solution of (\mathcal{P}) and equation (4.11) is verified for some $\mathbf{v} \in \text{BV}(Q)$, then (\mathbf{u}, \mathbf{v})*
521 *is a solution of (\mathcal{P}')*

522 *Proof.* Assume that (\mathbf{u}, \mathbf{v}) is a solution to (\mathcal{P}') . Then, for every $(u, v) \in \text{BV}(Q) \times \text{BV}(Q)$
523 we have

$$524 \quad (4.12) \quad F_\lambda(\mathbf{u} - \mathbf{v}) + F_\mu(\mathbf{v}) + \mathcal{H}(g, \mathcal{A}\mathbf{u}) \leq F_\lambda(u - v) + F_\mu(v) + \mathcal{H}(g, \mathcal{A}u).$$

Taking $u = \mathbf{u}$ gives

$$\forall v \in \text{BV}(Q), \quad F_\lambda(\mathbf{u} - \mathbf{v}) + F_\mu(\mathbf{v}) \leq F_\lambda(\mathbf{u} - v) + F_\mu(v),$$

that is $F_\lambda(\mathbf{u} - \mathbf{v}) + F_\mu(\mathbf{v}) = F_\lambda \# F_\mu(\mathbf{u})$. Let us fix $u \in \text{BV}(Q)$. Using (4.12), we obtain

$$\forall v \in \text{BV}(Q), \quad F_\lambda \# F_\mu(\mathbf{u}) + \mathcal{H}(g, \mathcal{A}\mathbf{u}) \leq F_\lambda(u - v) + F_\mu(v) + \mathcal{H}(g, \mathcal{A}u),$$

which results to

$$F_\lambda \# F_\mu(\mathbf{u}) + \mathcal{H}(g, \mathcal{A}\mathbf{u}) \leq \left(\inf_{v \in \text{BV}(Q)} F_\lambda(u - v) + F_\mu(v) \right) + \mathcal{H}(g, \mathcal{A}u) = F_\lambda \# F_\mu(u) + \mathcal{H}(g, \mathcal{A}u).$$

525 Therefore, \mathbf{u} is a solution to (\mathcal{P}) .

526 Conversely, assume \mathbf{u} is a solution to (\mathcal{P}) . As $F_\lambda \# F_\mu$ is exact at \mathbf{u} , there exists $\mathbf{v} \in \text{BV}(Q)$
 527 such that $F_\lambda(\mathbf{u} - \mathbf{v}) + F_\mu(\mathbf{v}) = F_\lambda \# F_\mu(\mathbf{u})$. Then, for every $(u, v) \in \text{BV}(Q) \times \text{BV}(Q)$

$$\begin{aligned} 528 \quad F_\lambda(\mathbf{u} - \mathbf{v}) + F_\mu(\mathbf{v}) + \mathcal{H}(g, \mathcal{A}\mathbf{u}) &= F_\lambda \# F_\mu(\mathbf{u}) + \mathcal{H}(g, \mathcal{A}\mathbf{u}) \leq F_\lambda \# F_\mu(u) + \mathcal{H}(g, \mathcal{A}u) \\ 529 &\leq F_\lambda(u - v) + F_\mu(v) + \mathcal{H}(g, \mathcal{A}u) . \end{aligned}$$

531 This proves that (\mathbf{u}, \mathbf{v}) is a solution to (\mathcal{P}') . \square

532 **5. Optimality conditions.** In the final section of this paper, we deal with the optimality
 533 conditions of (\mathcal{P}) . Optimality conditions are useful since they provide qualitative information
 534 on the solution of the minimization problem. In many cases, they are a useful tool to prove
 535 convergence of the algorithms and get error estimates independent on the discretization grid.
 536 Here, we use standard duality techniques based on the convex conjugate and the subdifferential
 537 of a functional in order to characterize the solutions. However, as we often deal with the dual
 538 of the underlying space, we prefer to use a reflexive framework since the dual of $\text{BV}(Q)$ is
 539 not easy to handle. Therefore we choose p with $1 \leq p < \frac{d+1}{d}$, so that $\text{BV}(Q)$ is compactly
 540 embedded in $L^p(Q)$.

We denote $\langle \cdot, \cdot \rangle_{p,p'}$ the duality product between $L^p(Q)$ and its dual $L^{p'}(Q)$ with $\frac{1}{p} + \frac{1}{p'} = 1$
 and

$$\forall u \in L^p(Q), \forall v \in L^{p'}(Q) \quad \langle u, v \rangle_{p,p'} = \int_Q u(t, x) v(t, x) dt dx .$$

541 We start by extending Φ_{α_1} , Ψ_{α_2} and F_α from their respective domains to $L^p(Q)$ as follows:

$$\begin{aligned} 542 \quad \tilde{\Phi}_{\alpha_1}(u) &= \begin{cases} \Phi_{\alpha_1}(u) & \text{if } u \in L^1(\mathcal{T}; \text{BV}(\Omega)), \\ +\infty & \text{else,} \end{cases}, \quad \tilde{\Psi}_{\alpha_2}(u) = \begin{cases} \Psi_{\alpha_2}(u) & \text{if } u \in L^1(\Omega; \text{BV}(\mathcal{T})), \\ +\infty & \text{else,} \end{cases}, \\ \tilde{F}_\alpha(u) &= \begin{cases} F_\alpha(u) & \text{if } u \in \text{BV}(Q), \\ +\infty & \text{if } u \in L^p(Q) \setminus \text{BV}(Q). \end{cases} \end{aligned}$$

543 We define the *extended* problem as

$$544 \quad (\mathcal{P}^*) \quad \inf_{u \in L^p(Q)} (\tilde{F}_\lambda \# \tilde{F}_\mu)(u) + \mathcal{H}(g, \mathcal{A}u) .$$

545 With the definition of \tilde{F}_λ , it is clear that problems (\mathcal{P}) and (\mathcal{P}^*) have the same solution set.
 546 So, we look for optimality conditions for (\mathcal{P}^*) . It is obvious that the lower semicontinuity for
 547 the extended regularizing terms as in Proposition 3.1 is still valid.

548 **Corollary 5.1.** *Let $\alpha = (\alpha_1, \alpha_2)$ that satisfies (3.4). The functionals $\tilde{\Phi}_{\alpha_1}$, $\tilde{\Psi}_{\alpha_2}$ and \tilde{F}_α are*
 549 *convex and lower semicontinuous on $L^p(Q)$.*

550 We begin with the Fenchel conjugates of the corresponding regularizing terms and then
 551 focus on the characterization of the subdifferential of $\tilde{F}_\lambda \# \tilde{F}_\mu(u) + \mathcal{H}(g, \mathcal{A}u)$.

552 **5.1. Fenchel conjugate of $\tilde{F}_\lambda \# \tilde{F}_\mu$.** One way to derive the optimality conditions of (\mathcal{P}^*) ,
 553 is by computing the subdifferentials of each term. A useful tool to achieve this goal is to
 554 compute the conjugate functionals. We start with the following theorem found in [5, Theorem
 555 9.5.1.].

Theorem 5.1. *If V is a normed space with dual space V' , and $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex and proper function, then*

$$\forall (u, u^*) \in V \times V' \quad u^* \in \partial f(u) \iff u \in \partial f^*(u^*) ,$$

where f^* is the Fenchel conjugate of f and the subdifferential of f at u is

$$\partial f(u) = \{u^* \in V^* \mid \forall v \in V, f(v) - f(u) \geq \langle u^*, v - u \rangle_{V', V}\}.$$

The first step is to compute the Fenchel conjugate of the regularizing term $\tilde{F}_\lambda \# \tilde{F}_\mu$ starting by \tilde{F}_λ . Let us focus on the computation of the Fenchel-conjugate of $\tilde{\Phi}_\lambda$. We consider the set

$$\mathcal{K}_x := \left\{ \xi = \operatorname{div}_x \varphi \mid \varphi \in L^\infty(\mathcal{T}; \mathcal{C}_c^1(\Omega, \mathbb{R}^d)), \|\varphi\|_\infty \leq 1 \right\} \subset L^\infty(Q).$$

556 We have the following lemma that provides a relation with the sets defined in (2.3).

557 **Lemma 5.1.** *We have $K_x \subset \mathcal{K}_x$, where K_x is given by (2.3). Conversely, any $\xi \in \mathcal{K}_x$
 558 verifies $\xi(t, \cdot) \in K_x$, for almost every $t \in \mathcal{T}$.*

559 *Proof.* Let be $\xi \in \mathcal{K}_x$. There exists $\varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^d)$ such that $\xi = \operatorname{div}_x \varphi$ and $\|\varphi\|_{\infty, x} \leq 1$.
 560 Let $\psi \in L^\infty(\mathcal{T}; \mathcal{C}_c^1(\Omega, \mathbb{R}^d))$ defined as $\psi(t, x) = \varphi(x)$, $(t, x) \in Q$. Then $\|\psi\|_\infty \leq 1$ and we may
 561 extend ξ on $\mathcal{T} \times \Omega$ with $\xi(t, x) = \xi(x) = \operatorname{div}_x \varphi(x) = \operatorname{div}_x \psi(t, x)$. \square
 562

Theorem 5.2 ($\tilde{\Phi}_\alpha$ Conjugate). *For every function α that satisfies (3.4), we have*

$$\tilde{\Phi}_\alpha^* = \mathbb{1}_{\overline{\alpha \mathcal{K}_x}}$$

563 where, $\mathbb{1}_C$ is the indicator function of the set C and $\overline{\mathcal{K}_x}$ is the $L^{p'}(Q)$ -closure of \mathcal{K}_x .

564 *Proof.* Note that for every $u^* \in L^{p'}(Q)$,

$$565 \quad (5.1) \quad \tilde{\Phi}_\alpha^*(u^*) = \sup_{v \in L^p(Q)} \langle u^*, v \rangle_{p, p'} - \tilde{\Phi}_\alpha(v) = \sup_{v \in \operatorname{BV}(Q)} \langle u^*, v \rangle_{p, p'} \Phi_\alpha(v).$$

Let $\xi \in \mathcal{K}_x$, then $\xi(t, \cdot) \in K_x$ for almost every $t \in \mathcal{T}$ and (2.2) gives

$$\int_\Omega \xi(t, x) u(t, x) \, dx \leq \sup_{\zeta \in K_x} \int_\Omega \zeta(x) u(t, x) \, dx = \operatorname{TV}_x(u)(t);$$

567 using (3.2), we obtain that

$$568 \quad (5.2) \quad \sup_{\xi \in \alpha \mathcal{K}_x} \langle \xi, u \rangle = \sup_{\xi \in \alpha \mathcal{K}_x} \int_0^T \int_\Omega \xi(t, x) u(t, x) \, dx \leq \Phi_\alpha(u).$$

569 As $\tilde{\Phi}_\alpha$ is positively homogeneous, then $\tilde{\Phi}_\alpha^*$ is the indicator of some closed subset $\tilde{\mathcal{K}}$ of $L^{p'}(Q)$
 570 (Corollary 13.2.1 of [35]).

571 • We first prove that $\alpha\overline{\mathcal{K}_x} \subset \tilde{\mathcal{K}}$. Let u^* be in $\alpha\mathcal{K}_x$. Using (5.1), (5.2) we have that for any
 572 $v \in \text{BV}(Q)$, $\Phi_\alpha(v) \geq \langle u^*, v \rangle$ and so $\tilde{\Phi}_\alpha^*(u^*) \leq 0$. As $\tilde{\Phi}_\alpha^*$ is an indicator function this means that
 573 $\tilde{\Phi}_\alpha^*(u^*) = 0$. So $u^* \in \tilde{\mathcal{K}}$ and $\alpha\mathcal{K}_x \subset \tilde{\mathcal{K}}$. As $\tilde{\mathcal{K}}$ is $L^{p'}(Q)$ -closed this gives $\alpha\overline{\mathcal{K}_x} \subset \tilde{\mathcal{K}}$.

574 • Let us prove the converse inclusion. Assume there exists $u^* \in \tilde{\mathcal{K}}$ such that $u^* \notin \alpha\overline{\mathcal{K}_x}$. One
 575 can separate u^* and $\alpha\overline{\mathcal{K}_x}$, see [13]: there exists $\omega \in \mathbb{R}$ and $u_0 \in L^p(Q)$ such that

$$\begin{aligned} 576 \quad & \langle u_0, u^* \rangle_{p,p'} > \omega \geq \sup_{v^* \in \alpha\overline{\mathcal{K}_x}} \langle u_0, v^* \rangle_{p,p'} \\ 577 \quad (5.3) \quad & \Rightarrow \sup_{v^* \in \alpha\overline{\mathcal{K}_x}} \langle u_0, v^* - u^* \rangle_{p,p'} < 0. \\ 578 \end{aligned}$$

579 On the other hand, since $\tilde{\Phi}_\alpha$ is convex and lower semicontinuous with respect to the L^p -
 580 topology, then by Fenchel-Moreau theorem we have that $\tilde{\Phi}_\alpha^{**} = \tilde{\Phi}_\alpha$. In particular we write
 581 that for all $u \in \text{BV}(Q)$,

$$582 \quad (5.4) \quad \Phi_\alpha(u) = \sup_{v^* \in L^{p'}(Q)} \langle u, v^* \rangle_{p,p'} - \tilde{\Phi}_\alpha^*(v^*) = \sup_{v^* \in L^{p'}(Q)} \langle u, v^* \rangle_{p,p'},$$

since $\tilde{\Phi}_\alpha^*(v^*) = 0$. Let us fix $t \in \mathcal{T}$, then

$$\forall \xi \in K_x, \quad \alpha(t)\xi(x)u(t, x) \leq \sup_{\zeta \in K_x} \alpha(t)\zeta(x)u(t, x) \quad \text{a.e. } x \in \Omega,$$

583 and taking the supremum we have that

$$\begin{aligned} 584 \quad & \sup_{\xi \in K_x} \int_{\Omega} \alpha(t)\xi(x)u(t, x) \, dx \leq \int_{\Omega} \sup_{\zeta \in K_x} \alpha(t)\zeta(x)u(t, x) \, dx, \\ 585 \quad & \text{TV}_x(\alpha u)(t) \leq \int_{\Omega} \sup_{\zeta \in K_x} \alpha(t)\zeta(x)u(t, x) \, dx. \\ 586 \end{aligned}$$

587 We subtract both sides by $\langle u, u^* \rangle_{p,p'}$ and integrate over the time domain \mathcal{T} to recover

$$\begin{aligned} 588 \quad & \int_0^T \text{TV}_x(\alpha u)(t) \, dt - \int_0^T \int_{\Omega} u^*(t, x)u(t, x) \, dx \, dt \leq \\ 589 \quad & \int_0^T \int_{\Omega} \left[\sup_{\zeta \in K_x} \alpha(t)\zeta(x) - u^*(t, x) \right] u(t, x) \, dx \, dt. \\ 590 \end{aligned}$$

591 Then, using (5.4) and Lemma 5.1, we have that for all $u \in \text{BV}(Q)$

$$\begin{aligned} 592 \quad & 0 \leq \Phi_\alpha(u) - \langle u, u^* \rangle_{p,p'} \leq \int_0^T \int_{\Omega} \left[\sup_{\zeta \in \alpha K_x} \zeta(x) - u^*(t, x) \right] u(t, x) \, dx \, dt \\ 593 \quad & \leq \int_0^T \int_{\Omega} \left[\sup_{\xi \in \alpha\mathcal{K}_x} \xi(t, x) - u^*(t, x) \right] u(t, x) \, dx \, dt. \\ 594 \end{aligned}$$

Hence, this implies

$$\forall u \in \text{BV}(Q), \quad \int_0^T \int_{\Omega} \left(\sup_{\xi \in \alpha \mathcal{K}_x} \xi(t, x) - u^*(t, x) \right) u(t, x) \, dx \, dt \geq 0.$$

As $\text{BV}(Q)$ is dense in $L^p(Q)$ with respect to the L^p -norm (since it includes $\mathcal{C}_c^1(Q)$) we get

$$\forall u \in L^p(Q), \quad \int_Q \left(\sup_{\xi \in \alpha \mathcal{K}_x} \xi(t, x) - u^*(t, x) \right) u(t, x) \, dx \, dt \geq 0.$$

In a similar way, choosing $-u$ instead of u we conclude that

$$\sup_{\xi \in \alpha \mathcal{K}_x} \xi - u^* = 0 \Rightarrow \sup_{\xi \in \alpha \mathcal{K}_x} \langle u_0, \xi - u^* \rangle_{p, p'} = 0$$

As $\alpha \mathcal{K}_x \subset \overline{\alpha \mathcal{K}_x}$, then

$$\sup_{\xi \in \overline{\alpha \mathcal{K}_x}} \langle u_0, \xi - u^* \rangle_{p, p'} \geq 0.$$

595 which is a contradiction by (5.3). \square

596 The following is the analogous result of the previous theorem for the $\tilde{\Psi}_\alpha$ functional and can
597 be proved similarly.

Theorem 5.3 ($\tilde{\Psi}_\alpha$ Conjugate). *For every function α that satisfies (3.4), we have*

$$\tilde{\Psi}_\alpha^* = \mathbb{1}_{\overline{\alpha \mathcal{K}_t}}, \quad \text{where } \mathcal{K}_t := \left\{ \xi = \frac{d\psi}{dt} \mid \psi \in L^\infty(\Omega, \mathcal{C}_c^1(0, T, \mathbb{R})), \|\psi\|_\infty \leq 1 \right\}.$$

Using the above theorems, we are able to compute the convex conjugate of the extended spatial-temporal total variation defined in 3.5. We use the following results for the convex conjugate of the infimal convolution and the convex conjugate of the sum, see [5, Chapter 9.4], i.e., for two proper, closed, convex functionals ϕ, ψ we have

$$(\phi \# \psi)^* = \phi^* + \psi^* \quad \text{and} \quad (\phi + \psi)^* = (\phi^* \# \psi^*)^*.$$

Corollary 5.2. *For every α that satisfies (3.4), we have that*

$$\tilde{F}_\alpha^* = \mathbb{1}_{\overline{\mathcal{K}_\alpha}} \quad \text{with} \quad \mathcal{K}_\alpha = \alpha_1 \overline{\mathcal{K}_x} + \alpha_2 \overline{\mathcal{K}_t}.$$

Proof. As $\tilde{F}_\alpha = \tilde{\Phi}_{\alpha_1} + \tilde{\Psi}_{\alpha_2}$ and $\tilde{\Phi}_{\alpha_1}, \tilde{\Psi}_{\alpha_2}$ are convex, lower semicontinuous, we have

$$\tilde{F}_\alpha^* = (\tilde{\Phi}_{\alpha_1} + \tilde{\Psi}_{\alpha_2})^* = (\tilde{\Phi}_{\alpha_1}^* \# \tilde{\Psi}_{\alpha_2}^*)^{**} = (\mathbb{1}_{\overline{\alpha_1 \mathcal{K}_x}} \# \mathbb{1}_{\overline{\alpha_2 \mathcal{K}_t}})^{**} = (\mathbb{1}_{\overline{\alpha_1 \mathcal{K}_x + \alpha_2 \mathcal{K}_t}})^{**} = (\mathbb{1}_{\mathcal{K}_\alpha})^{**},$$

598 where $\mathcal{K}_\alpha = \alpha_1 \overline{\mathcal{K}_x} + \alpha_2 \overline{\mathcal{K}_t}$. Moreover, one has that $(\mathbb{1}_{\mathcal{K}_\alpha})^{**} = \mathbb{1}_{\overline{\mathcal{K}_\alpha}}$, since the $(L^{p'})$ closure
599 $\overline{\mathcal{K}_\alpha}$ of \mathcal{K}_α is convex, see [35, Chapter 13]. \square

600

Corollary 5.3 ($\tilde{F}_\lambda \# \tilde{F}_\mu$ Conjugate). *For every λ, μ that satisfy (3.4), we have*

$$(\tilde{F}_\lambda \# \tilde{F}_\mu)^* = \mathbb{1}_{\overline{\mathcal{K}_\lambda \cap \mathcal{K}_\mu}},$$

601 where $\mathcal{K}_\lambda, \mathcal{K}_\mu$ are the corresponding sets defined in Corollary 5.2.

602 We have computed the convex conjugate of our proposed regularizer and we proceed now with
603 the optimality conditions of (\mathcal{P}) .

5.2. Optimality conditions for (\mathcal{P}) . Since the problem (\mathcal{P}^*) is convex we have that \mathbf{u} is the solution if and only if $0 \in \partial\mathcal{E}(\mathbf{u})$ where

$$\mathcal{E}(u) := (\tilde{F}_\lambda \# \tilde{F}_\mu)(u) + \mathcal{H}(g, Au).$$

604 We use the following result that allows to estimate the subdifferential of the sum of two
605 functionals, see [5, Theorem 9.5.4].

606 **Theorem 5.4.** *Let $(V, \|\cdot\|)$ be a normed space and let $f, h : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be two lower
607 semicontinuous, convex and proper functions.*

608 (a) *The following inclusion is always true: $\partial f + \partial h \subset \partial(f + h)$.*

609 (b) *If f is finite and continuous at a point of $\text{dom } h$, then we have: $\partial f + \partial h = \partial(f + h)$.*

5.2.1. Case (1). In this subsection we focus on the first case where the L^q fidelity term is $\mathcal{H}(g, Au) = \frac{1}{q} \|Au - g\|_{L^q(Q)}^q$ with $1 \leq q < +\infty$ and A satisfies assumptions (3.8), (3.9) and (3.10). Clearly, $\text{dom } \tilde{F}_\lambda \# \tilde{F}_\mu = \text{BV}(Q)$, $\text{dom } \mathcal{H}(g, Au) = L^p(Q)$ and $u \rightarrow \mathcal{H}(g, Au)$ is L^p continuous at $0 \in \text{BV}(Q)$. Therefore,

$$\partial\mathcal{E}(u) = \partial\tilde{F}_\lambda \# \tilde{F}_\mu(u) + \partial\mathcal{H}(g, Au).$$

610 Any u^* of $\partial\mathcal{E}(u)$ writes $u^* = u_1^* + u_2^*$ where $u_1^* \in \partial\tilde{F}_\lambda \# \tilde{F}_\mu(u)$ and $u_2^* \in \partial\mathcal{H}(g, Au)$. In the
611 sequel, we characterize the elements u_1^*, u_2^* . Starting with the subdifferential of $\tilde{F}_\lambda \# \tilde{F}_\mu$, it is
612 easy to check that for every $u \in \text{BV}(Q) \hookrightarrow L^p(Q)$, we get

$$613 \quad (5.5) \quad u_1^* \in \partial\tilde{F}_\lambda \# \tilde{F}_\mu(u) \iff u_1^* \in \overline{\mathcal{K}_\lambda} \cap \overline{\mathcal{K}_\mu} \text{ and } \forall v^* \in \overline{\mathcal{K}_\lambda} \cap \overline{\mathcal{K}_\mu}, \langle u, v^* - u_1^* \rangle_{p,p'} \leq 0,$$

where $\overline{\mathcal{K}_\lambda} \cap \overline{\mathcal{K}_\mu}$ is a closed convex subset of $L^{p'}(Q)$. Indeed, we use Theorem 5.1, Corollary 5.3 and that $\tilde{F}_\lambda \# \tilde{F}_\mu$ is convex and lower semicontinuous, to get

$$u \in \partial(\tilde{F}_\lambda \# \tilde{F}_\mu)^*(u_1^*) = \partial\mathbb{1}_{\overline{\mathcal{K}_\lambda} \cap \overline{\mathcal{K}_\mu}}(u_1^*).$$

614 The subdifferential of the data fitting term using [20, Proposition 5.7] is

$$615 \quad (5.6) \quad \partial\mathcal{H}(g, Au) = \begin{cases} A^*(Au - g)^{q-1}, & \text{if } 1 < q < \infty \\ \left\{ A^*z, \|z\|_{L^\infty(Q)} \leq 1, z \in \text{sign}(Au - g) \right\}, & \text{if } q = 1. \end{cases}$$

Note that in the latter case one has

$$\partial(\|\cdot - g\|_{L^1(Q)})(v) = \partial(\|\cdot\|_{L^1(Q)})(v - g) = \{z \in L^\infty(Q) \mid \|z\|_{L^\infty(Q)} \leq 1, z \in \text{sign}(v - g)\}.$$

Overall, we have that

$$0 \in \partial\mathcal{E}(u) \iff \exists u^* \in \partial\mathcal{H}(g, Au) \text{ such that } -u^* \in \partial\tilde{F}_\lambda \# \tilde{F}_\mu(u)$$

616 and one concludes to the following result:

617 **Theorem 5.5.** *A function $\mathbf{u} \in \text{BV}(Q)$ is a solution to (\mathcal{P}) if and only if*

1. $\forall v \in \overline{\mathcal{K}_\lambda} \cap \overline{\mathcal{K}_\mu} \quad \langle \mathbf{u}, A^*(A\mathbf{u} - g)^{q-1} - v \rangle_{p,p'} \leq 0, \text{ if } 1 < q < +\infty,$
- 618 2. $\forall v \in \overline{\mathcal{K}_\lambda} \cap \overline{\mathcal{K}_\mu}, \quad \langle \mathbf{u}, A^*z - v \rangle_{p,p'} \leq 0, \text{ if } q = 1 \text{ with}$
 $z \in L^\infty(Q), \|z\|_{L^\infty(Q)} \leq 1, z \in \text{sign}(A\mathbf{u} - g).$

619 **5.2.2. Optimality conditions for (\mathcal{P}) : case (2).** In this subsection we focus on the Kull-
 620 back Leibler divergence $\mathcal{H}(g, \mathcal{A}u) = D_{KL}(g, \mathcal{R}u)$ where $u \in L^1_+(Q)$, the positive cone of L^1 .
 621 We cannot follow the same strategy as before due to the limitations of this fidelity in terms
 622 of continuity. It is known that a proper, convex, lower semicontinuous is continuous if and
 623 only if the interior of its domain is not empty, i.e., $\text{int}(\text{dom}f) \neq \emptyset$, see [20]. In our case the
 624 effective domain is in fact nowhere dense and $D_{KL}(g, \mathcal{R}\cdot)$ is nowhere continuous in $L^1(\Omega)$, let
 625 alone in $L^p(\Omega)$, see [19, Remark 2.12]. Moreover, $F_\lambda \# F_\mu$ is not continuous with respect to the
 626 L^p norm.

627 Therefore we use $\text{BV}(Q)$ as the underlying functional space. In the sequel $\langle \cdot, \cdot \rangle$ de-
 628 notes the duality product between $\text{BV}(Q)'$ and $\text{BV}(Q)$. We use Theorem 5.4 again with
 629 $V = \text{BV}(Q)$, $f = F_\lambda \# F_\mu$ and $h = D_{KL}(g, \mathcal{R}\cdot)$. Indeed, f is lower semicontinuous due to
 630 Theorem 4.1 and h due to the continuity properties of both the Radon transform and the
 631 Kullback-Leibler divergence.

632 **Proposition 5.1.** *Assume that λ and μ satisfy (3.4) and that there exists a real number*
 633 *$\kappa > 0$ such that $\mu_2 = \kappa\lambda_2$. Then $F_\lambda \# F_\mu$ is continuous on $\text{BV}(Q)$ (and of course at any*
 634 *element of $\text{dom}f \cap \text{dom}h = \text{BV}_+(Q)$ the set of positive BV functions).*

635 *Proof.* Let u_1, u_2 be in $\text{BV}(Q)$. As $F_\lambda \# F_\mu$ is exact, there exists $v_1 \in \text{BV}(Q)$ such that
 636 $F_\lambda \# F_\mu(u_1) = F_\lambda(u_1 - v_1) + F_\mu(v_1)$. We get

$$\begin{aligned} 637 \quad F_\lambda \# F_\mu(u_2) &= \inf_{v \in \text{BV}(Q)} F_\lambda(u_2 - v) + F_\mu(v) \leq F_\lambda(u_2 - v_1) + F_\mu(v_1) \\ 638 &\leq F_\lambda(u_2 - u_1) + F_\lambda(u_1 - v_1) + F_\mu(v_1) \\ 639 &\leq F_\lambda(u_2 - u_1) + F_\lambda \# F_\mu(u_1). \end{aligned}$$

Similarly

$$F_\lambda \# F_\mu(u_1) \leq F_\lambda(u_1 - u_2) + F_\lambda \# F_\mu(u_2),$$

and using Theorem 3.1

$$|F_\lambda \# F_\mu(u_1) - F_\lambda \# F_\mu(u_2)| \leq F_\lambda(u_1 - u_2) \leq C_2 \text{TV}(\lambda_2(u_1 - u_2)).$$

641 Moreover

$$\begin{aligned} 642 \quad \text{TV}(\lambda_2(u_1 - u_2)) &\leq \|\lambda_2\|_{L^\infty(\mathcal{T})} \text{TV}(u_1 - u_2) + \left\| \lambda_2' \right\|_{L^\infty(\mathcal{T})} \|u_1 - u_2\|_{L^1(Q)} \\ 643 &\leq \|\lambda_2\|_{W^{1,\infty}(\mathcal{T})} \|u_1 - u_2\|_{\text{BV}}. \end{aligned}$$

This prove the continuity of $F_\lambda \# F_\mu$ on $\text{BV}(Q)$. \square

So $\mathbf{u} \in \text{BV}_+(Q)$ is a solution to (\mathcal{P}) if and only if

$$0 \in \partial(F_\lambda \# F_\mu)(\mathbf{u}) + \partial D_{KL}(g, \mathcal{R}\mathbf{u}).$$

645 Equivalently, there exists $\mathbf{u}^* \in \partial(F_\lambda \# F_\mu)(\mathbf{u})$ such that $-\mathbf{u}^* \in \partial D_{KL}(g, \mathcal{R}\cdot)(\mathbf{u})$. As usual,
 646 we have $\mathbf{u}^* \in \partial(F_\lambda \# F_\mu)(\mathbf{u}) \iff \mathbf{u} \in \partial(F_\lambda \# F_\mu)^*(\mathbf{u}^*)$. However, in this setting we are
 647 in different topology. Though we have computed F_λ^* for previous case, the computation of
 648 F_λ^* is still challenging. Indeed, we cannot use the arguments used in Theorem 5.2 since the

649 underlying topology is now the BV one and not the $L^p(Q)$ one any longer. In particular, we
 650 loose reflexivity as well as an integral representation on the duality product, see [23].

Since F_λ is positively homogeneous functional, we know there exists a closed convex subset of BV' that we call \mathbf{K}_λ such that $F_\lambda^* = \mathbb{1}_{\mathbf{K}_\lambda}(\mathbf{u}^*)$ is the indicator function of \mathbf{K}_λ . Unfortunately, we are not able to give an explicit description of \mathbf{K}_λ : we only know that $\mathcal{K}_\lambda \subset \mathbf{K}_\lambda$. We obtain

$$(F_\lambda \# F_\mu)^* = F_\lambda^* + F_\mu^* = \mathbb{1}_{\mathbf{K}_\lambda} + \mathbb{1}_{\mathbf{K}_\mu} = \mathbb{1}_{\mathbf{K}_\lambda \cap \mathbf{K}_\mu}.$$

Therefore,

$$\mathbf{u}^* \in \partial(F_\lambda \# F_\mu)(\mathbf{u}) \iff \mathbf{u}^* \in \mathbf{K}_\lambda \cap \mathbf{K}_\mu \text{ and } \forall w^* \in \mathbf{K}_\lambda \cap \mathbf{K}_\mu \quad \langle \mathbf{u}, w^* - \mathbf{u}^* \rangle \leq 0.$$

Next, we compute $\partial\mathcal{H}(g, \mathcal{R}\cdot)(\mathbf{u})$. Let be $w \in BV_+(Q)$

$$-\mathbf{u}^* \in \partial\mathcal{H}(g, \mathcal{R}\cdot)(\mathbf{u}) \implies \forall s > 0 \quad \frac{\mathcal{H}(g, \mathcal{R}(\mathbf{u} + s\mathbf{w})) - \mathcal{H}(g, \mathcal{R}\mathbf{u})}{s} \geq -\langle \mathbf{u}^*, \mathbf{w} \rangle.$$

Passing to the limit as $s \rightarrow 0$ gives $\langle \nabla\mathcal{H}(g, \mathcal{R}\cdot)(\mathbf{u}) + \mathbf{u}^*, \mathbf{w} \rangle \geq 0$.

Conversely, assume $\langle \nabla\mathcal{H}(g, \mathcal{R}\cdot)(\mathbf{u}) + \mathbf{u}^*, \mathbf{w} \rangle \geq 0$ for every $w \in BV_+(Q)$. Let be $w \in BV(Q)$: if $\mathbf{u} + w \leq 0$ then it is clear that

$$+\infty = \mathcal{H}(g, \mathcal{R}(\mathbf{u} + w)) - \mathcal{H}(g, \mathcal{R}\mathbf{u}) \geq \langle (-\mathbf{u}^*), \mathbf{w} \rangle.$$

Otherwise, by convexity

$$\mathcal{H}(g, \mathcal{R}(\mathbf{u} + w)) - \mathcal{H}(g, \mathcal{R}\mathbf{u}) \geq \langle \nabla\mathcal{H}(g, \mathcal{R}\cdot)(\mathbf{u}), \mathbf{w} \rangle \geq \langle (-\mathbf{u}^*), \mathbf{w} \rangle.$$

Therefore

$$-\mathbf{u}^* \in \partial\mathcal{H}(g, \mathcal{R}\cdot)(\mathbf{u}) \iff \forall w \in BV_+(Q), \quad \langle \nabla\mathcal{H}(g, \mathcal{R}\cdot)(\mathbf{u}) + \mathbf{u}^*, \mathbf{w} \rangle \geq 0.$$

A short computation gives

$$\nabla\mathcal{H}(g, \mathcal{R}\cdot)(\mathbf{u}) = \mathcal{R}^* \left(\mathbf{1}_\Sigma - \frac{g}{\mathcal{R}\mathbf{u}} \right).$$

Finally ,

$$-\mathbf{u}^* \in \partial\mathcal{H}(g, \mathcal{R}\cdot)(\mathbf{u}) \iff \forall w \in BV_+(Q), \quad \left\langle \mathcal{R}^* \left(\mathbf{1}_\Sigma - \frac{g}{\mathcal{R}\mathbf{u}} \right) + \mathbf{u}^*, \mathbf{w} \right\rangle \geq 0.$$

651 For this case, we conclude with the following optimality conditions

652 **Theorem 5.6.** *Let $\mathbf{u} \in BV_+(Q)$. Then \mathbf{u} is a solution to (P) if and only if there exists*
 653 *$\mathbf{u}^* \in \mathbf{K}_\lambda \cap \mathbf{K}_\mu \subset BV(Q)'$ such that*

$$654 \quad (5.7) \quad \forall w^* \in \mathbf{K}_\lambda \cap \mathbf{K}_\mu \quad \langle \mathbf{u}, w^* - \mathbf{u}^* \rangle \leq 0,$$

$$655 \quad (5.8) \quad \forall w \in BV_+(Q) \quad \left\langle \mathcal{R}^* \left(\mathbf{1}_\Sigma - \frac{g}{\mathcal{R}\mathbf{u}} \right) + \mathbf{u}^*, \mathbf{w} \right\rangle \geq 0.$$

656

657 **Remark 5.1.** *The difficulties met in order to establish the optimality conditions are closely*
 658 *related to the so-called two-norm discrepancy in control theory (see [14] for example). We have*
 659 *to deal with both the BV- norm and the L^p -norm. The qualification condition that we need to*
 660 *describe the subdifferentials is easy to satisfy with the BV-norm. However, the computation of*
 661 *the conjugate functions cannot be explicit within a non reflexive framework. On the contrary,*
 662 *the use of L^p -norm leads to a nice description of conjugate functions while the splitting of the*
 663 *differential cannot be done. In a discrete setting, these difficulties disappear of course.*

664 **6. Conclusion.** We perform a thorough analysis on the proposed spatial-temporal infimal-
 665 convolution regularizer under time dependent weight parameters. It acts in a separate mode
 666 on the spatial and temporal domains and it can be applied to a wide range of problems such as
 667 denoising, deblurring and emission tomography with different kind of noise (impulse, gaussian
 668 or Poisson). We focus on the well-posedness of the proposed minimization problem and provide
 669 existence, uniqueness and stability results into a very general framework. We further derive
 670 the optimality conditions using standard tools from duality theory. However, we have still
 671 to focus in depth on the characterization of the sets K_λ to have a clear insight of the dual
 672 variables. This implies that we have to deal with the dual of the BV space and use some
 673 integral representations as in [23]. Another issue is to describe carefully the discretization
 674 process and the dual problem in an appropriate way, especially with respect to isotropic or
 675 anisotropic spatial-temporal discrete norms. Finally, in a forthcoming paper, we shall perform
 676 numerics, especially for PET reconstruction, and compare this model to those that can be
 677 found in the literature such as [25].

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