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An Inf-Convolution BV type model for dynamic reconstruction. *

Maitine Bergounioux[†] and E. Papoutsellis[†]

Abstract. We are interested in a spatial temporal variational model for image sequences. The model involves a fitting data term to be adapted to different modalities such as denoising, deblurring or emission tomography. The regularizing term acts as an infimal-convolution type operator that takes into account the respective influence of time and space variables. We give existence and uniqueness results and provide optimality conditions via duality analysis. In a forthcoming paper, see [5], we deal with the numerical realisation of the proposed model and focus on dynamic Positron Emission Tomography (PET) reconstruction.

Key words. variational model, Inf-convolution, Total variation, PET, 4D imaging

AMS subject classifications. 65D18, 68U10,65K10

1. Introduction. In this paper, we are interested in describing a variational model for denoising/deblurring and/or emission tomography (ET) reconstruction of vector-valued images. What we call vector-valued images are usually color images, multi-spectral images, images acquired at different time intervals as videos. We focus on dynamic medical imaging as PET or functional MR images. There are many methods that handle videos or dynamical processes and the majority of them focus on the numerics. Here we aim to describe a variational model in an infinite dimensional setting, which, to our knowledge, has not been done yet. In a the dynamical setting, spatial and temporal components contribute differently, therefore we proceed with a non global description for both of them. Though, we have in mind a specific application to dynamic PET, we present a model flexible enough to address many applications.

We cannot quote the numerous papers on videos, since there is a huge literature, even if we restrict ourselves to variational methods. Let us mention however the paper by Holler and Kunisch [15], where the authors consider the model we investigate in a semi-discrete setting. In addition, for dynamical PET applications, an active contour method with gradient vector flow has been developed in [17, 18] but the underlying variational model has not been explored. In this work, we do not address numerical issues since our concern is purely theoretical: we aim to describe a powerful variational model and perform mathematical analysis (well-posedness and optimality conditions). We will present numerical tests together with comparison with classical semi-discrete models/methods in the PET context in a forthcoming paper [5].

Let us denote $u : (0, T) \times \Omega \rightarrow \mathbb{R}$, the dynamic image with respect to the space domain $\Omega \subset \mathbb{R}^d$ with $d \geq 1$ and with $T > 0$. In the following, we write $Q \subset \mathbb{R}^{d+1}$ instead of $(0, T) \times \Omega$. We focus on variational methods applied on a spatial-temporal domain and more precisely we

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consider the following generic minimization problem:

$$(1.1) \quad \inf_{u \in \mathcal{X}} \mathcal{N}(u) + \mathcal{H}(g, Au) ,$$

where,

- \mathcal{X} is a suitable space where the minimization is well defined.
- $\mathcal{N}(u)$ is the regularizing term, that imposes a prior structure on the solution u
- $\mathcal{H}(g, Au)$ is the fitting data term and is determined by the modality and the induced noise. We assume that the input data $g \in L^\infty(Q)$ is degraded through a continuous and linear operation A and with an additional random noise. In the forthcoming analysis, we consider two cases of the linear operator A . We set $A = \mathcal{A}$, a general bounded and linear operator

$$(1.2) \quad \mathcal{A} \in \mathcal{L}(L^p(Q), L^q(Q)) \text{ with } 1 \leq p \leq \frac{d+1}{d} \text{ and } 1 \leq q < \infty.$$

Moreover, we set $A = \mathcal{R}$, i.e., the Radon transform. Depending on the choice of the noise and the operator A , we have the following fidelity terms:

- **Gaussian/Impulse noise.** If the noise follows a Gaussian distribution, it is known that a suitable distance is the squared L^2 norm, namely

$$\mathcal{H}(g, Au) = \int_Q |g - Au|^2 dx dt.$$

In the case of impulse noise or “salt and pepper noise”, a non-smooth fidelity term is appropriate, see [24],[25], that is

$$\mathcal{H}(g, Au) = \int_Q |g - Au| dx dt$$

(i.e. the L^1 norm). Here, we consider a more general case:

$$\mathcal{H}(g, Au) = \frac{1}{q} \|Au - g\|_{L^q(Q)}^q.$$

- **Poisson noise.** In the case where the input data g follows a Poisson distribution, we use the so-called *Kullback-Leibler* (KL) divergence $D_{KL} : L^1(Q) \times L^1(Q) \rightarrow \mathbb{R}_+$, see for instance [8],[21], defined by

$$(1.3) \quad D_{KL}(u, v) = \int_Q \left(u \log \left(\frac{u}{v} \right) - u + v \right) dx dt, \quad \forall u, v \geq 0 \text{ a.e.}$$

We briefly recall some of the basic properties of KL-functional which can be found in [9],[27] and will be used later:

Lemma 1. (a) *Since $\ln x \leq x - 1$, for $x > 0$ the function $D_{KL}(u, v)$ is non-negative and equal to 0 if and only if $u = v$.*

(b) *The function $(u, v) \mapsto D_{KL}(u, v)$ is convex.*

- (c) For fixed $u \in L^1(Q)$ (resp. $v \in L^1(Q)$), the function $D_{KL}(u, \cdot)$ (resp. $D_{KL}(\cdot, v)$) is weakly lower semicontinuous with respect to L^1 topology.
- (d) The following estimate is true:

$$(1.4) \quad \|u - v\|_{L^1(Q)}^2 \leq \left(\frac{2}{3} \|u\|_{L^1(Q)} + \frac{4}{3} \|v\|_{L^1(Q)} \right) D_{KL}(u, v).$$

Poisson data occur in a plethora of applications where images are obtained by means of counting particles e.g. photons, that arrive to a measuring equipment device. In medical imaging for instance we have the Positron Emission Tomography (PET), Single-photon emission computed tomography (SPECT), see [31] or in a general context [32]. Moreover, some astronomical images are characterized by this kind of behaviour, see [7, 20].

Here, we focus on reconstructing dynamic raw data that appear in Emission Tomography (PET, SPECT). Raw data are usually corrupted with Poisson noise (or commonly referred as the photon counting noise) and they are connected through an integral (projection) operator known as the Radon transform \mathcal{R} , see [23]. For every $t \in (0, T)$, we have that

$$(1.5) \quad (\mathcal{R}u(\theta, s))(t) = \int_{x \cdot \theta = s} u(t, x) dx,$$

where $\{x \in \mathbb{R}^d : x \cdot \theta = s\}$ is the hyperplane perpendicular to $\theta \in \mathcal{S}^{d-1}$ with distance $s \in \mathbb{R}$ from the origin. For every $t \in (0, T)$, the Radon transform $(\mathcal{R}u(\theta, s))(t)$ lies on the $\{(\theta, s) : \theta \in \mathcal{S}^{d-1}, s \in \mathbb{R}\}$ a cylinder of dimension d and is often referred as *projection space* or *sinogram space*.

In a dynamic framework, we set $\Sigma = (0, T) \times \{(\theta, s) : \theta \in \mathcal{S}^{d-1}, s \in \mathbb{R}\}$ and the Radon transform is a continuous linear operator with

$$(1.6) \quad \mathcal{R} : L^1(Q) \rightarrow L^1(\Sigma), \quad \|\mathcal{R}u\|_{L^1(\Sigma)} \leq C \|u\|_{L^1(Q)}.$$

We refer the reader to [26] for general continuity results of the Radon transform in L^p spaces. Recall that, if $p \geq \frac{d+1}{d}$, the Radon transform is L^p discontinuous, since the function $u(x) = |x|^{-\frac{d+1}{p}} \frac{1}{\log(|x|)}$ belongs to $L^p(Q)$, for $x \in Q$ but is not integrable over any hyperplane, see [22, Th. 3.32]. Now, the fidelity term in the case of Poisson noise is a reduced version of the KL-divergence, since we can neglect the terms that are independent of $u \in \mathcal{X}$. Indeed, we write

$$(1.7) \quad \mathcal{H}(g, \mathcal{R}u) = \int_{\Sigma} \mathcal{R}u - g \log \mathcal{R}u d\theta ds dt$$

since the $g \log g$, $-g$ do not count on the minimization problem (1.1). Finally, let us mention that the above expression is well-defined since $u \geq 0$ implies that $\mathcal{R}u \geq 0$ and this constraint is essential on (1.1).

Overall, we may consider different problems depending on the choice of the operators \mathcal{A} and \mathcal{R} . For instance, if $\mathcal{A} = Id$ (the identity operator) we focus on the denoising model of data

corrupted by Gaussian ($q=2$), impulse ($q=1$) or Poisson noise i.e., $\mathcal{R} = Id$. In addition, if $\mathcal{A}u = h \star u$ with $h \in L^\infty(Q)$ is a convolution operator, we are interested on the deblurring process.

We have already discussed about the different fidelity terms in (1.1) and now we turn our attention to the choice of the regulariser. For instance, in the work by Schaeffer and al. [29], a space-time total variation term is considered as a regularizer. However, the authors focus on the approximation schemes and deal with semi-discretized minimizers. Most of the papers on dynamic and/or video image analysis focus on numerical realization and algorithms as [19] in a MRI context, or [16] with the use of high order differentiation for compressed sensing. Besides these *classical methods* we propose to use the following regularization term:

$$(1.8) \quad \mathcal{N}_{\lambda,\mu}(u) = F_\lambda \# F_\mu(u) := \inf_{v \in \mathcal{X}} F_\lambda(u - v) + F_\mu(v),$$

where \mathcal{X} is a suitable functional space and F_λ, F_μ are total variation type functionals on space and time weighted by some time dependent parameters λ, μ . Regularization via inf-convolution is a classical tool, since it is closely related to dual formulation (see [4, 13] for example). The concept of an infimal convolution penalisation on a spatial-temporal domain has been introduced (in a slightly different form) in [15] within a semi-discrete (with respect to time) framework. We also refer the reader to [30], a recent work on a spatial-temporal infimal convolution using high-order regularisers. However, the authors in [15] investigate inf-convolution of n -th order and its connection with Total Generalized Variation (TGV), see [10], while we have restricted ourselves to $n = 2$.

The paper is organized as follows: we first recall general results and give useful tools to deal with the dynamic framework. Then we focus on the infimal-convolution regularizing term and provide lower semi-continuity results. We prove also that the functional we consider is equivalent to the total variation on $(0, T) \times \Omega$. We next give an existence theorem in the restrictive case where the parameters are constant as well as some stability results. Last section is devoted to the dual formulation in an extended framework: this allows to generalize the previous existence theorem to time dependent parameters and derive the optimality conditions.

2. Preliminaries. In this section, we recall some basic notations related to functions of bounded variation (BV) extended to a spatial-temporal space. Recall that $\Omega \subset \mathbb{R}^d$, $d \geq 1$ is a bounded open set with smooth boundary and $Q = (0, T) \times \Omega$; then we define,

$$\begin{aligned} L^1(0, T; BV(\Omega)) &= \{u : (0, T) \times \Omega \rightarrow \mathbb{R} \mid u(t, \cdot) \in BV(\Omega) \text{ a.e. } t \in (0, T) \\ &\quad \text{and } t \mapsto TV_x(u)(t) \in L^1(0, T)\} \\ L^1(\Omega; BV(0, T)) &= \{v : (0, T) \times \Omega \rightarrow \mathbb{R} \mid v(\cdot, x) \in BV(0, T) \text{ a.e. } x \in \Omega \\ &\quad \text{and } x \mapsto TV_t(v)(x) \in L^1(\Omega)\}, \end{aligned}$$

where TV_x (respectively TV_t) stands for the standard total variation with respect to the space variable (respectively the time variable). Precisely, we set

$$(2.1) \quad K_x := \left\{ \xi = \operatorname{div}_x(\varphi) \mid \varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^d), \|\varphi\|_{\infty, x} \leq 1 \right\},$$

$$(2.2) \quad K_t := \left\{ \xi = \frac{d\varphi}{dt} \mid \varphi \in \mathcal{C}_c^1(0, T, \mathbb{R}), \|\varphi\|_{\infty, t} \leq 1 \right\},$$

and write respectively for a.e. $t \in (0, T)$ and $x \in \Omega$

$$(2.3) \quad \begin{aligned} TV_x[v](t) &= \sup \left\{ \int_{\Omega} \xi(t, x)v(t, x) dx \mid \xi \in K_x \right\}, \\ TV_t[v](x) &= \sup \left\{ \int_0^T \xi(t, x)v(t, x) dt \mid \xi \in K_t \right\}. \end{aligned}$$

Note that $\|\cdot\|_{\infty, t}$ (resp. $\|\cdot\|_{\infty, x}$) norms stand for the norm of $L^\infty(0, T)$ (resp. $L^\infty(\Omega)$). For a function of bounded variation on the spatial-temporal domain Q , we write

$$(2.4) \quad BV(Q) = \{u \in L^1(Q) \mid TV(u) < \infty\}$$

where

$$(2.5) \quad TV[v] = \sup \left\{ \int_Q \xi(t, x)v(t, x) dx dt \mid \xi \in K \right\}.$$

with

$$(2.6) \quad K := \left\{ \xi = \operatorname{div}_{(t, x)}(\varphi) \mid \varphi \in \mathcal{C}_c^1(Q, \mathbb{R} \times \mathbb{R}^d), \|\varphi\|_{\infty} \leq 1 \right\}.$$

In the following theorem, we recall some useful properties on the $BV(O)$ space, where O is an bounded, open set of \mathbb{R}^N (practically $O = \Omega$ with $N = d$ or $O = Q$ with $N = d + 1$.)

Theorem 2.1. *Let $O \subset \mathbb{R}^N$, $N \geq 1$. The space $BV(O)$ endowed with the norm*

$$\|v\|_{BV(O)} := \|v\|_{L^1(O)} + TV(v)$$

is a Banach space.

- *The mapping $u \mapsto TV[u]$ is lower semi-continuous from $BV(O)$ endowed with the $L^1(O)$ topology to \mathbb{R}^+ .*
- *$BV(O) \subset L^p(O)$ with continuous embedding, for $1 \leq p \leq \frac{N}{N-1}$ and*
- *$BV(O) \subset L^p(O)$ with compact embedding, for $1 \leq p < \frac{N}{N-1}$.*

The lemma below is essential for the forthcoming analysis. It is based on the definitions above as well as of some tools in the proof of [14, Theorem 2, Section 5.10.2]. A similar result but in a different context can be found in [6, Lemma 3].

Lemma 2. *We have $L^1(0, T; BV(\Omega)) \cap L^1(\Omega; BV(0, T)) = BV(Q)$.*

Proof. We start with the first inclusion,

$$L^1(0, T; BV(\Omega)) \cap L^1(\Omega; BV(0, T)) \subset BV(Q).$$

Let be $u \in L^1(0, T; BV(\Omega)) \cap L^1(\Omega; BV(0, T))$. For any $\xi \in K$ there exists $\varphi = (\varphi_1, \varphi_2) \in \mathcal{C}_c^1(Q, \mathbb{R} \times \mathbb{R}^d)$ such that $\|\varphi\|_\infty \leq 1$ and

$$\xi = \frac{\partial \varphi_1}{\partial t} + \operatorname{div}_x \varphi_2 := \xi_1 + \xi_2$$

Note that for every $t \in (0, T)$, $\xi_2(t, \cdot) : x \mapsto \xi_2(t, x)$ belongs to K_x so that

$$\int_{\Omega} \xi_2(t, x) u(t, x) dx \leq TV_x(u)(t) \quad \text{a.e. } t \in (0, T),$$

and

$$\int_0^T \int_{\Omega} \xi_2(t, x) u(t, x) dx dt \leq \int_0^T TV_x(u)(t) dt.$$

Similarly,

$$\int_{\Omega} \int_0^T \xi_1(t, x) u(t, x) dt dx \leq \int_{\Omega} TV_t(u)(x) dx.$$

Then, for every $\xi \in K$

$$\begin{aligned} \int_Q \xi(t, x) u(t, x) dt dx &= \int_0^T \int_{\Omega} \xi_2(t, x) u(t, x) dx dt + \int_{\Omega} \int_0^T \xi_1(t, x) u(t, x) dt dx \\ &\leq \int_0^T TV_x(u)(t) dt + \int_{\Omega} TV_t(u)(x) dx. \end{aligned}$$

The right hand side is finite independently of ξ since $u \in L^1(0, T; BV(\Omega)) \cap L^1(\Omega; BV(0, T))$. Therefore, $u \in BV(Q)$ and

$$(2.7) \quad TV(u) \leq \int_0^T TV_x(u)(t) dt + \int_{\Omega} TV_t(u)(x) dx.$$

Let us prove the converse inclusion. We first assume that $u \in W^{1,1}(Q)$. Then with Fubini's theorem we get

$$t \mapsto \int_{\Omega} |\nabla_{t,x} u|(t, x) dx \in L^1(0, T), \quad \text{and } x \mapsto \int_0^T |\nabla_{t,x} u|(t, x) dt \in L^1(\Omega).$$

Recall that

$$|\nabla_{t,x} u| = \sqrt{\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^d \left(\frac{\partial u}{\partial x_i}\right)^2},$$

so that

$$\max(|\nabla_t u(t, x)|, |\nabla_x u(t, x)|) \leq |\nabla_{t,x} u(t, x)| \leq |\nabla_t u(t, x)| + |\nabla_x u(t, x)|.$$

Therefore

$$t \mapsto \int_{\Omega} |\nabla_x u|(t, x) dx \in L^1(0, T), \quad \text{and } x \mapsto \int_0^T |\nabla_x u|(t, x) dt \in L^1(\Omega),$$

and we get $u \in L^1(0, T; BV(\Omega)) \cap L^1(\Omega; BV(0, T))$ with

$$(2.8) \quad \max \left(\int_0^T TV_x(u)(t) dt, \int_{\Omega} TV_t(u)(x) dx \right) \leq TV(u) \\ \leq \int_0^T TV_x(u)(t) dt + \int_{\Omega} TV_t(u)(x) dx.$$

We now consider $u \in BV(Q)$ and show that $u \in L^1(0, T, BV(\Omega))$. The other inclusion can be proved similarly. As $W^{1,1}(Q)$ is dense in $BV(Q)$ in the sense of the intermediate convergence [4], there exists a sequence of functions $u_k \in W^{1,1}(Q)$ such that u_k converges to u in $L^1(Q)$ and $TV(u_k) \rightarrow TV(u)$. With Fubini's theorem we infer that $u_k(t, \cdot)$ converges to $u(t, \cdot)$ in $L^1(\Omega)$, for almost every $t \in (0, T)$ (up to a subsequence). Moreover, $TV(u_k) \rightarrow TV(u)$ is bounded. Using (2.8) we claim that $\int_0^T TV_x(u_k)(t) dt$ is bounded as well and with Fatou's Lemma we get

$$(2.9) \quad \int_0^T \liminf_{k \rightarrow \infty} TV_x(u_k)(t) dt \leq \liminf_{k \rightarrow \infty} \int_0^T TV_x(u_k)(t) dt \\ \leq \liminf_{k \rightarrow \infty} TV(u_k) = TV(u) < +\infty$$

and $\liminf_{k \rightarrow \infty} TV_x(u_k)(t) < \infty$ a.e $t \in (0, T)$. Now, for a.e $t \in (0, T)$ we get

$$\forall \xi \in K_x \quad \int_{\Omega} u_k(t, x) \xi(x) dx \leq TV_x(u_k)(t).$$

So, for every $\xi \in K_x$

$$\int_{\Omega} u(t, x) \xi(x) dx = \lim_{k \rightarrow +\infty} \int_{\Omega} u_k(t, x) \xi(x) dx \leq \liminf_{k \rightarrow +\infty} TV_x(u_k)(t) < +\infty.$$

Therefore

$$TV_x(u)(t) = \sup_{\xi \in K_x} \int_{\Omega} u(t, x) \xi(x) dx \leq \liminf_{k \rightarrow +\infty} TV_x(u_k)(t) < +\infty.$$

This means $u(t, \cdot) \in BV(\Omega)$ a.e $t \in (0, T)$. Using (2.9), we get

$$\int_0^T TV_x(u)(t) dt \leq \int_0^T \liminf_{k \rightarrow +\infty} TV_x(u_k)(t) dt \leq TV(u).$$

This ends the proof, and the inequality (2.8) is also valid for every $u \in BV(Q)$. \square

Remark 2.1. The second inclusion of the previous lemma can be seen as a generalization of a function of bounded variation "in the sense of Tonelli" denoted by *TBV*, see [12, 3]. For instance, a function of two variables $h(x, y)$ is *TBV* on a rectangle $[a, b] \times [c, d]$ if and only if $TV_x h(\cdot, y) < \infty$ for a.e $y \in [c, d]$, $TV_y h(x, \cdot) < \infty$ for a.e $x \in [a, b]$ and $TV_x h(\cdot, y) \in L^1([a, b])$, $TV_y h(x, \cdot) \in L^1([c, d])$.

Remark 2.2. Note that in Lemma (2), we prove also

$$TV(u) \leq \int_0^T TV_x(u)(t) dt + \int_{\Omega} TV_t(u)(x) dx \leq 2TV(u).$$

2.1. The $F_\lambda \# F_\mu$ regularizer . In this section, we proceed with the analysis of the proposed regularizer, that is

$$(2.10) \quad \mathcal{N}_{\lambda,\mu}(u) = F_\lambda \# F_\mu(u) := \inf_{v \in \mathcal{X}} F_\lambda(u - v) + F_\mu(v).$$

In the sequel we always deal with functions α such that

$$(2.11) \quad \alpha \in W^{1,\infty}(0, T) \text{ and } 0 < \alpha_{min} \leq \alpha(t) \leq \alpha_{max} < 1 \text{ a.e. } t \in (0, T),$$

where α_{min} and α_{max} are real numbers. Indeed, it is important to allow time dependent parameters since acquisition snapshots may not be uniformly distributed. This is the case, for example, for a dynamical TEP process. Once the problem is discretized, these parameters involve the time step which is not constant.

We define $\Phi_\alpha(u)$ (in space) as the $L^1(0, T)$ -norm of $t \mapsto \alpha(t)TV_x[u](t)$, i.e.,

$$(2.12) \quad \forall u \in L^1(0, T; BV(\Omega)), \quad \Phi_\alpha(u) = \int_0^T TV_x[\alpha u](t) dt = \int_0^T \alpha(t)TV_x[u](t) dt,$$

and for a penalization on the temporal domain, we define Ψ_α as

$$(2.13) \quad \Psi_\alpha(v) = \int_\Omega TV_t[\alpha v](x) dx, \quad \forall v \in L^1(\Omega; BV(0, T))$$

Using Lemma 2 and equations (2.12),(2.13) we have the following:

Definition 2.1. Let $\mathcal{X} = BV(Q)$ and $\alpha = (\alpha_1, \alpha_2) \in W^{1,\infty}(0, T) \times W^{1,\infty}(0, T)$. We define F_α on \mathcal{X} as

$$(2.14) \quad F_\alpha(u) = \Phi_{\alpha_1}(u) + \Psi_{\alpha_2}(u), \quad \forall u \in \mathcal{X}$$

that is

$$F_\alpha(u) = \int_0^T TV_x[\alpha_1 u](t) dt + \int_\Omega TV_t[\alpha_2 u](x) dx.$$

Moreover, for $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ where $\lambda_i, \mu_i, i = 1, 2$ satisfy (2.11) we define

$$F_\lambda \# F_\mu(u) := \inf_{v \in \mathcal{X}} F_\lambda(u - v) + F_\mu(v)$$

Remark 2.3. Note that if α is a constant function then

$$\forall m \in \mathbb{R} \quad F_\alpha(u + m) = F_\alpha(u) .$$

Proposition 2.1. The functionals Φ_{α_1} and Ψ_{α_2} are convex and lower semi-continuous on $L^1(0, T; BV(\Omega))$ and $L^1(\Omega; BV(0, T))$ respectively, with respect to the $L^1(Q)$ topology. In particular, for any $\alpha = (\alpha_1, \alpha_2)$ such that α_i satisfies (2.11), the functional F_α is lower semi-continuous on $BV(Q)$ with respect to the L^1 topology. In particular, they are both lower semi-continuous on $BV(Q)$ for any $L^p(Q)$ topology with $p \geq 1$.

Proof. We start with the lower semicontinuity of Φ_{α_1} . The proof is similar for the lower semicontinuity of Ψ_{α_2} . Let $u_n \in L^1(0, T; BV(\Omega))$ such that $u_n \rightarrow u$ in $L^1(Q)$ and $u_n(t, x) \in BV(\Omega)$ a.e. $t \in (0, T)$ and $TV_x(u_n)(t) \in L^1(0, T)$. If $\liminf_{n \rightarrow +\infty} \Phi_{\alpha_1}(u_n) = +\infty$ then the lower semi-continuity inequality is obviously satisfied. Otherwise, one can extract a subsequence (still denoted u_n) such that

$$\sup_n \Phi_{\alpha_1}(u_n) = \sup_n \int_0^T TV_x[\alpha_1 u_n](t) dt < +\infty ,$$

Fatou's Lemma applied to the sequence $TV_x(\alpha_1 u_n)$ gives

$$\int_0^T \liminf_{n \rightarrow +\infty} TV_x[\alpha_1 u_n](t) dt \leq \liminf_{n \rightarrow +\infty} \int_0^T TV_x[\alpha_1 u_n](t) dt = \liminf_{n \rightarrow +\infty} \Phi_{\alpha_1}(u_n) < +\infty .$$

Moreover, for a.e. $t \in (0, T)$ we get

$$\forall \xi \in K_x, \quad TV_x[\alpha_1 u_n](t) \geq \int_{\Omega} \alpha_1(t) \xi(x) u_n(t, x) dx .$$

As u_n strongly converges to u in $L^1(Q)$ then $u_n(t, x) \rightarrow u(t, x)$ in $L^1(\Omega)$ a.e. $t \in (0, T)$ up to a subsequence. Therefore

$$\forall \xi \in K_x, \text{ a.e. } t \in (0, T), \quad \liminf_{n \rightarrow +\infty} TV_x[\alpha_1 u_n](t) \geq \int_{\Omega} \alpha_1(t) \xi(x) u(t, x) dx ,$$

and , for almost every $t \in (0, T)$

$$\liminf_{n \rightarrow +\infty} TV_x[\alpha_1 u_n](t) \geq \sup_{\xi \in K_x} \int_{\Omega} \alpha_1(t) \xi(x) u(t, x) dx = TV_x[\alpha_1 u](t) .$$

Finally,

$$\Phi_{\alpha_1}(u) = \int_0^T TV_x[\alpha_1 u](t) dt \leq \int_0^T \liminf_{n \rightarrow +\infty} TV_x[\alpha_1 u_n](t) dt \leq \liminf_{n \rightarrow +\infty} \Phi_{\alpha_1}(u_n) .$$

Eventually, the functional F_{α} is lower semicontinuous on $BV(Q)$ as the sum of two lower semicontinuous functionals. \square

Next result provides an equivalence relation between the spatial-temporal total variation regularization and the functional F_{α} . We have already an equivalence relation in the case where $\alpha_1 = \alpha_2 = 1$, see Remark 2.2.

Proposition 2.2. *Assume that $\alpha = (\alpha_1, \alpha_2)$ satisfies (2.11). The functional F_{α} is equivalent to the total variation in $BV(Q)$ and there exists constants C_1, C_2 such that, for every $u \in BV(Q)$,*

$$C_1 TV(u) \leq F_{\alpha}(u) \leq C_2 TV(u) .$$

Proof. From (2.11) and Remark 2.2, we obtain that

$$\begin{aligned} F_{\alpha}(u) &= \Phi_{\alpha_1}(u) + \Psi_{\alpha_2}(u) \geq \alpha_{\min} \left(\int_0^T TV_x(u)(t) dt + \int_{\Omega} TV_t(u)(x) dx \right) \\ &\geq \alpha_{\min} TV(u) . \end{aligned}$$

Similarly, for the right hand side we have that $F_{\alpha}(u) \leq 2\alpha_{\max} TV(u)$. \square

Remark 2.4. 1. We just prove that F_α is equivalent to the total variation, so that the model we investigate is a classical ICTV (Infimal Convolution Total Variation) one. However, the use of F_α allows to take the different roles on the time and space variable into account, with parameters acting as weight functions. This give some freedom to the model, allowing to adapt these parameters to the discretization steps, as in [15] for example where the authors have a similar approach, using an inf-convolution process to take time and space information into account. Using our notations they consider the time-discretized formulation of

$$\int_0^T TV_x(\kappa(u-v))(t) dt + \int_\Omega TV_t((u-v))(x) dx$$

$$\text{and } \int_0^T TV_x(v)(t) dt + \int_\Omega TV_t(\kappa v)(x) dx$$

where $\kappa > 1$. This coincides with (2.10) when $\lambda = (\kappa, 1)$ and $\mu = (1, \kappa)$.

Furthermore, the use of F_α allows to consider many discretization processes. As an example, a classical approximation of the total variation is

$$TV(u) \simeq \int_Q |\nabla_{t,x} u| dt dx = \int_Q \sqrt{\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^d \left(\frac{\partial u}{\partial x_i}\right)^2} dt dx,$$

while F_α may be approximated by

$$F_\alpha(u) \simeq \int_Q \left(|\nabla_t u| + \sqrt{\sum_{i=1}^d \left(\frac{\partial u}{\partial x_i}\right)^2} \right) dt dx$$

whose behavior is different from the numerical point of view.

2. Practically speaking, we set $\alpha_1 = a$ and $\alpha_2 = 1 - a$ with $a \in (0, 1)$. In the limit case where $a = 1$ we have

$$F_{(1,0)}(u) = \int_0^T TV_x[u](t) dt \text{ and } F_{(0,1)}(u) = \int_\Omega TV_t[u](x) dx.$$

Finally, a similar result is also true for the infimal-convolution regularizer (2.10).

Proposition 2.3. *There exists constants $C_1, C_2 > 0$ such that*

$$\forall u \in \mathcal{X} \quad C_1 TV(u) \leq \mathcal{N}_{\lambda,\mu}(u) \leq C_2 TV(u).$$

Proof. Let be $u \in \mathcal{X}$. Then for any $v \in \mathcal{X}$

$$C_1 TV(u) \leq C_1 TV(u-v) + C_1 TV(v) \leq F_\lambda(u-v) + F_\mu(v)$$

Passing to the infimum gives $C_1 TV(u) \leq \mathcal{N}_{\lambda,\mu}(u)$. On the other hand,

$$\mathcal{N}_{\lambda,\mu}(u) = \inf_{v \in \mathcal{X}} F_\lambda(u-v) + F_\mu(v) \leq F_\mu(u) \leq C_2 TV(u).$$

□

3. Well posedness results. In this section, we are interested in the well-posedness of the minimization problem (1.1).

Recall that $Q = (0, T) \times \Omega \subset \mathbb{R}^{d+1}$, $\mathcal{X} = \text{BV}(Q)$ with $1 \leq p \leq \frac{d+1}{d}$. Also, we fix two linear and continuous operators with $\mathcal{A} \in \mathcal{L}(L^p(Q), L^q(Q))$, $1 \leq q < \infty$ and $\mathcal{R} \in \mathcal{L}(L^1(Q), L^1(\Sigma))$ for the Radon transform. A general fidelity term covering both cases is defined below

$$(3.1) \quad \mathcal{H}(g, Au) = \begin{cases} \frac{1}{q} \|Au - g\|_{L^q(Q)}^q, & \text{if } A = \mathcal{A}, \\ D_{KL}(g, \mathcal{R}u) + \mathbb{1}_{\{u \geq 0\}}(u), & \text{if } A = \mathcal{R}. \end{cases}$$

Note that for the second case we assume that $\inf_{\Sigma} g > 0$. Using (3.1), (2.10), then (1.1) becomes

$$(\mathcal{P}) \quad \inf_{u \in \mathcal{X}} F_{\lambda} \# F_{\mu}(u) + \mathcal{H}(g, Au)$$

which is equivalent to

$$(3.2) \quad \inf_{(u, v) \in \mathcal{X} \times \mathcal{X}} F_{\lambda}(u - v) + F_{\mu}(v) + \mathcal{H}(g, Au).$$

We begin by verifying the existence and uniqueness of (3.2) using some classical arguments of [1] adapted to the spatial-temporal framework. In terms of the Kullback-Leibler data fidelity we follow [27], [28].

Theorem 3.1. *Assume that λ, μ are constants and $g \in L^{\infty}(A(Q))$ with $A\chi_Q \neq 0$. Then, the minimization problem (3.2) admits (at least) a solution pair $(u_*, v_*) \in \mathcal{X} \times \mathcal{X}$.*

Proof. Let $(u_n, v_n) \in \mathcal{X} \times \mathcal{X}$ a minimizing sequence and set

$$(3.3) \quad J(u_n, v_n) = F_{\lambda}(u_n - v_n) + F_{\mu}(v_n) + \mathcal{H}(g, Au_n).$$

Since $g \in L^{\infty}(A(Q))$ the infimum is finite and there exists a constant $C > 0$ such that $J(u_n, v_n) \leq C$ and we get $F_{\lambda}(u - v_n), F_{\mu}(v) \leq C$ as well. In the following, we use the same constant $C > 0$. Using Proposition 2.2 there exists $C > 0$ such that $TV(v_n) \leq C$. Since, λ, μ are constants, we assume without loss of generality that $\int_Q v = 0$. Indeed, we have that

$$F_{\lambda}(u - v + c) + F_{\mu}(v + c) = F_{\lambda}(u - v) + F_{\mu}(v).$$

From the Poincaré inequality, see [2], we obtain that $\|v_n\|_{L^1(Q)} \leq TV(v_n) \leq C$ and that (v_n) is BV-bounded. Then, there exists $v_* \in \text{BV}(Q)$ such that, up to subsequence, $v_{n_k} \xrightarrow{w^*} v_*$. It suffices to prove that u_n is BV-bounded. From the Poincaré inequality, we have that

$$\begin{aligned} \|u_n - \bar{u}_n\|_{L^p(Q)} &\leq CTV(u_n) \leq CTV(u_n - v_n) + CTV(v_n) \\ &\leq CF_{\lambda}(u_n - v_n) + F_{\mu}(v_n) \leq C \end{aligned}$$

where \bar{u}_n is the mean value of u_n . Moreover, we have that

$$(3.4) \quad \|u_n\|_{L^p(Q)} \leq \|u_n - \bar{u}_n\|_{L^p(Q)} + \left| \int_Q u_n dx dt \right| \leq C + \left| \int_Q u_n dx dt \right|.$$

We consider two cases with respect to the choice of the fidelity term.

- **If $A = \mathcal{A}$:** Since $\mathcal{A}\chi_Q \neq 0$, we have the following estimate

$$\begin{aligned} \left| \int_Q u_n dx dt \right| \frac{\|\mathcal{A}\chi_Q\|_{L^q(Q)}}{|Q|} &= \|\mathcal{A}\bar{u}_n\|_{L^q(Q)} \leq \|\mathcal{A}\bar{u}_n - \mathcal{A}u_n + \mathcal{A}u_n - g + g\|_{L^q(Q)} \\ &\leq \|\mathcal{A}\| \|u_n - \bar{u}_n\|_{L^p(Q)} + \|\mathcal{A}u - g\|_{L^q(Q)} + \|g\|_{L^q(Q)} \\ &\leq C \end{aligned}$$

- **If $A = \mathcal{R}$:** In this case, we have an additional constraint on u_n , that is $u_n \geq 0$ and therefore it suffices to bound $\int_Q u_n dx dt = |Q| \|\bar{u}_n\|_{L^1(Q)}$. We use the estimate in (1.4) and set $u_n - \bar{u}_n = w_n$. Then

$$(3.5) \quad \begin{aligned} \|\mathcal{R}w_n + \mathcal{R}\bar{u}_n - g\|_{L^1(\Sigma)}^2 &\leq \left(\frac{2}{3} \|g\|_{L^1(\Sigma)} + \frac{4}{3} \|\mathcal{R}w_n + \mathcal{R}\bar{u}_n\|_{L^1(\Sigma)} \right) D_{KL}(g, \mathcal{R}u_n) \\ &\leq \left(\frac{2}{3} \|g\|_{L^1(\Sigma)} + \frac{4}{3} \|A\| C + \frac{4}{3} \|\mathcal{A}\bar{u}_n\|_{L^1(\Sigma)} \right) C. \end{aligned}$$

On the other hand,

$$(3.6) \quad \begin{aligned} \|\mathcal{R}w_n + \mathcal{R}\bar{u}_n - g\|_{L^1(\Sigma)}^2 &\geq \left(\|\mathcal{R}w_n - g\|_{L^1(\Sigma)} - \|\mathcal{R}\bar{u}_n\|_{L^1(\Sigma)} \right)^2 \\ &\geq \|\mathcal{R}\bar{u}_n\|_{L^1(\Sigma)} \left(\|\mathcal{R}\bar{u}_n\|_{L^1(\Sigma)} - 2\|\mathcal{R}w_n - g\|_{L^1(\Sigma)} \right) \\ &\geq \|\mathcal{R}\bar{u}_n\|_{L^1(\Sigma)} \left(\|\mathcal{R}\bar{u}_n\|_{L^1(\Sigma)} - 2(\|A\| C + \|g\|_{L^1(\Sigma)}) \right). \end{aligned}$$

Similar to the previous case, one has

$$(3.7) \quad \|\mathcal{R}\bar{u}_n\|_{L^1(Q)} = \frac{\int_Q u_n dx dt}{|Q|} \|\mathcal{R}\chi_Q\|_{L^1(\Sigma)} \Leftrightarrow \|\mathcal{R}\bar{u}_n\|_{L^1(\Sigma)} = C \|\bar{u}_n\|_{L^1(Q)}$$

Combining (3.5), (3.6) using (3.7), we derive that

$$(3.8) \quad \begin{aligned} C \|\bar{u}_n\|_{L^1(Q)} \left(C \|\bar{u}_n\|_{L^1(Q)} - 2(\|A\| C + \|g\|_{L^1(\Sigma)}) - \frac{4}{3} C \right) \\ \leq \left(\frac{2}{3} \|g\|_{L^1(\Sigma)} + \frac{4}{3} \|A\| C \right) C \end{aligned}$$

Let $B = C \|\bar{u}_n\|_{L^1(Q)} - 2(\|A\| C + \|g\|_{L^1(\Sigma)}) - \frac{4}{3} C$. If $B \geq 1$, it is immediate from (3.8), that $\|\bar{u}_n\|_{L^1(Q)}$ is bounded. Otherwise, we have that

$$\|\bar{u}_n\|_{L^1(Q)} \leq \frac{2(\|A\| C + \|g\|_{L^1(\Sigma)}) + \frac{4}{3} C + 1}{C}$$

which is again bounded.

For both cases (u_n) is L^p -bounded with $1 \leq p \leq \frac{d+1}{d}$. Then, it is also BV bounded and there exists $u_{n_k} \xrightarrow{w^*} v_*$ in BV, $u_{n_k} \xrightarrow{w} u_*$ in L^p , $1 \leq p \leq \frac{d+1}{d}$. Now, using the continuities of \mathcal{A} and \mathcal{R} , we obtain that $\mathcal{A}u_{n_k} \xrightarrow{w} \mathcal{A}u_*$ in L^q and $\mathcal{R}u_{n_k} \xrightarrow{w} \mathcal{R}u_*$ in L^1 . Equivalently, we have that

$$\frac{1}{q} \|\mathcal{A}u_* - g\|_{L^q(Q)}^q \leq \liminf_{k \rightarrow \infty} \frac{1}{q} \|\mathcal{A}u_{n_k} - g\|_{L^q(Q)}^q$$

and also using Lemma 1

$$D_{KL}(g, \mathcal{R}u_*) \leq \liminf_{k \rightarrow \infty} D_{KL}(g, \mathcal{R}u_{n_k}).$$

Finally, due to the lower semicontinuity of F_λ, F_μ we conclude that

$$J(u_*, v_*) \leq \liminf_{k \rightarrow \infty} J(u_{n_k}, v_{n_k}).$$

□

We continue with the uniqueness of the minimizers of (3.2). It suffices to prove that objective functional $J(u, v)$ in (3.3) is strictly convex in $\mathcal{X} \times \mathcal{X}$. By the definition of the fidelity term (3.1) this is true for every $1 < q < \infty$. Furthermore, the Kullback-Leibler divergence is strictly convex since $\inf_{\Sigma} g > 0$ and the Radon transform is injective on L^1 , see for instance [22]. Hence, we need only an injectivity assumption on \mathcal{A} . Moreover, if $(u, v) \in \mathcal{X} \times \mathcal{X}$ is a minimizer then also $(u, v + c) \in \mathcal{X} \times \mathcal{X}$ with $c \in \mathbb{R}$. Therefore, we have proved the following:

Theorem 3.2. *If $\mathcal{A} \in \mathcal{L}(L^p(Q), L^q(Q))$, with $1 < q < +\infty$, is an injective operator then Problem (3.2) with constants λ, μ admits a unique minimizer $(u, v) \in \mathcal{X} \times \mathcal{X}$ except in the direction $(0, c)$, $c \in \mathbb{R}$.*

To conclude, we focus on the stability of minimizers of (3.2) with respect to a small perturbation on the data g . Let (g_n) be a perturbed data sequence with $\mathcal{H}(g_n, g) \rightarrow 0$ and (u_n, v_n) be a solution to

$$(3.9) \quad \inf_{(u, v) \in \mathcal{X} \times \mathcal{X}} J_n(u, v) \text{ with } J_n(u, v) := F_\lambda(u - v) + F_\mu(v) + \mathcal{H}(g_n, \mathcal{A}u).$$

Theorem 3.3. *The problem (3.2) is stable with respect to g perturbations. Precisely, if (u, v) and (u_n, v_n) are solutions to (3.2) and (3.9) respectively, there exist subsequences (denoted similarly) converging to (u, v) in $BV-w^*$ and $u_n \xrightarrow{w} u$ in L^p with $1 \leq p \leq \frac{d+1}{d}$. Note that for the Kullback-Leibler case, we have to assume also that $\inf_{\Sigma} g_n > 0$ and $\log \mathcal{R}u \in L^\infty(\Sigma)$.*

Proof. Since $(u_n, v_n) \in \mathcal{X} \times \mathcal{X}$ is a solution of (3.9), then

$$(3.10) \quad J_n(u_n, v_n) \leq J_n(u, v), \quad \forall (u, v) \in \mathcal{X} \times \mathcal{X}.$$

• **If $A = \mathcal{A}$** , there exists $M > 0$ such that

$$\begin{aligned} F_\lambda(u_n - v_n) + F_\mu(v_n) + \frac{1}{q} \|\mathcal{A}u_n - g_n\|_{L^q(Q)}^q &\leq F_\lambda(u - v) + F_\mu(v) + \frac{1}{q} \|\mathcal{A}u - g_n\|_{L^q(Q)}^q \\ &\leq CTV(u) + 2CTV(v) + \frac{1}{q} \|\mathcal{A}u - g_n + g - g\|_{L^q(Q)}^q \\ &\leq CTV(u) + 2CTV(v) + \frac{2^q}{q} (\|\mathcal{A}u - g\|_{L^q(Q)}^q + \|g_n - g\|_{L^q(Q)}^q) \leq M. \end{aligned}$$

As in the previous proofs, we obtain subsequences still denoted $(u_n), (v_n)$ and $(u_*, v_*) \in \mathcal{X} \times \mathcal{X}$ such that $(u_n, v_n) \xrightarrow{w^*} (u_*, v_*)$ in BV and $u_n \xrightarrow{w} u_*$ in L^p with $1 \leq p \leq \frac{d+1}{d}$. Then, $\mathcal{A}u_n - g_n \xrightarrow{w} \mathcal{A}u_* - g$ in L^q ; using (3.10) and the lower semicontinuity of the corresponding functionals we conclude that

$$\begin{aligned} & F_\lambda(u_* - v_*) + F_\mu(v_*) + \frac{1}{q} \|\mathcal{A}u_* - g\|_{L^q(Q)}^q \\ & \leq \liminf_{n \rightarrow \infty} F_\lambda(u_n - v_n) + F_\mu(v_n) + \frac{1}{q} \|\mathcal{A}u_n - g_n\|_{L^q(Q)}^q \\ & \leq \limsup_{n \rightarrow \infty} F_\lambda(u_n - v_n) + F_\mu(v_n) + \frac{1}{q} \|\mathcal{A}u_n - g_n\|_{L^q(Q)}^q \\ & \leq \limsup_{n \rightarrow \infty} F_\lambda(u - v) + F_\mu(v) + \frac{1}{q} \|\mathcal{A}u - g_n\|_{L^q(Q)}^q \\ & = F_\lambda(u - v) + F_\mu(v) + \frac{1}{q} \|\mathcal{A}u - g\|_{L^q(Q)}^q \quad \text{for all } (u, v) \in \mathcal{X} \times \mathcal{X}. \end{aligned}$$

Equivalently (u_*, v_*) is a minimizer of (3.2).

• If $A = \mathcal{R}$, we get $D_{KL}(g_n, g) \rightarrow 0$ and $\|g_n - g\|_{L^1(Q)} \rightarrow 0$ by (1.4); so

$$(3.11) \quad F_\lambda(u_n - v_n) + F_\mu(v_n) + D_{KL}(g_n, \mathcal{R}u_n) \leq F_\lambda(u - v) + F_\mu(v) + D_{KL}(g_n, \mathcal{R}u),$$

for all $(u, v) \in \mathcal{X} \times \mathcal{X}$ with a.e. $u \geq 0$. Furthermore,

$$|D_{KL}(g_n, \mathcal{R}u) - D_{KL}(g, \mathcal{R}u) - D_{KL}(g_n, g)| \leq \|\log \mathcal{R}u - \log g\|_{L^\infty(\Sigma)} \|g_n - g\|_{L^1(\Sigma)}.$$

Since $g, \log \mathcal{R}u \in L^\infty(\Sigma)$ we pass to the limit and obtain

$$\lim_{n \rightarrow \infty} D_{KL}(g_n, \mathcal{R}u) = D_{KL}(g, \mathcal{R}u);$$

so the right hand side of (3.11) is bounded.

As before, there exists $(u_n, v_n) \xrightarrow{w^*} (u_*, v_*)$ in BV and $u_n \xrightarrow{w} u_*$ in L^p with $1 \leq p \leq \frac{d+1}{d}$. Here, we use both the strong convergence in L^1 (that is $\mathcal{R}u_n \rightarrow \mathcal{R}u_*$ and $g_n \rightarrow g$) and the pointwise convergence almost everywhere.

Applying Fatou's Lemma to $(g_n \log g_n - g_n \log \mathcal{R}u_n - g_n + \mathcal{R}u_n)_n$, one gets

$$D_{KL}(g, \mathcal{R}u_*) \leq \liminf_{n \rightarrow \infty} D_{KL}(g_n, \mathcal{R}u_n).$$

The end of the proof is similar to the previous case. \square

Finally, let us address the importance of the additional conditions on the data g and the perturbed data g_n for the Kullback-Leibler divergence case. In theory, conditions such as $\inf_\Sigma g, \inf_\Sigma g_n > 0$ are necessary in order to the KL fidelity terms to be well defined. In practice, these are not significantly restrictive. In emission tomography for instance, this can be achieved if we confine ourselves to the lines of the Radon transform intersecting the support of u . In addition, under a finite time acquisition process the requirement that $g \in L^\infty(\Sigma)$ is also a valid assumption.

4. Optimality conditions via duality .

4.1. Extension to $L^p(Q)$. Here, we choose p such that $1 < p < \frac{d+1}{d}$ so that $\text{BV}(Q) \subset L^p(Q)$ with compact embedding. We denote $\langle \cdot, \cdot \rangle_{p,p'}$ the duality product between $L^p(Q)$ and its dual $L^{p'}(Q)$ (with $\frac{1}{p} + \frac{1}{p'} = 1$). Note that

$$\forall u \in L^p(Q), \forall v \in L^{p'}(Q) \quad \langle u, v \rangle_{p,p'} = \int_Q u(t, x) v(t, x) dt dx .$$

We start by extending Φ_{α_1} , Ψ_{α_2} and F_α from their respective domains to $L^p(Q)$ as follows:

$$\tilde{\Phi}_{\alpha_1}(u) = \begin{cases} \Phi_{\alpha_1}(u) & \text{if } u \in L^1(0, T; \text{BV}(\Omega)) \\ +\infty & \text{else.} \end{cases},$$

$$\tilde{\Psi}_{\alpha_2}(u) = \begin{cases} \Psi_{\alpha_2}(u) & \text{if } u \in L^1(\Omega; \text{BV}(0, T)) \\ +\infty & \text{else.} \end{cases}$$

and

$$\tilde{F}_\alpha(u) = \begin{cases} F_\alpha(u) & \text{if } u \in \text{BV}(Q) \\ +\infty & \text{if } u \in L^p(Q) \setminus \text{BV}(Q) \end{cases} .$$

Proposition 4.1. *The functionals $\tilde{\Phi}_\alpha$ and $\tilde{\Psi}_\alpha$ are convex and lower semi-continuous on $L^p(Q)$ with respect to the $L^1(Q)$ topology.*

Proof. Let $(u_n)_{n \geq 0}$ in $L^p(Q)$ be a $L^1(Q)$ -strongly convergent sequence to $u \in L^1(Q)$. If $\liminf_{n \rightarrow +\infty} \tilde{\Phi}_\alpha(u_n) = +\infty$ then the lower semi-continuity inequality is obviously satisfied. Otherwise, one can extract a subsequence (still denoted u_n) such that $\tilde{\Phi}_\alpha(u_n) = \Phi_\alpha(u_n)$ is bounded. This means that $u_n \in L^1(0, T; \text{BV}(\Omega))$ for every n . The end of the proof is the same as the proof of Proposition 2.1 \square

Corollary 4.1. *For any $\alpha = (\alpha_1, \alpha_2)$ such that α_i satisfies (2.11), the functional \tilde{F}_α is lower semi-continuous on $L^p(Q)$ with respect to the L^1 topology.*

Now, we prove that the extended infimal-convolution $\tilde{\mathcal{N}}_{\lambda, \mu}(u) = \tilde{F}_\lambda \# \tilde{F}_\mu(u)$ is lower semi-continuous with respect to the $L^p(Q)$ topology using the following

Theorem 4.1 (Proposition 15.1.7 [4]). *Let V be a reflexive space and $f, g : V \rightarrow]-\infty, +\infty]$ be two proper convex lower semicontinuous functions. Assume*

(i) *compactness: if $t_n \rightarrow +\infty$, u_n and v_n converge weakly, $u_n + v_n \rightarrow 0$ strongly, and $f(t_n u_n) + g(t_n v_n)$ is bounded from above, then u_n and v_n converge strongly;*

(ii) *compatibility: if $f^\infty(v) + g^\infty(-v) \leq 0$ then $f^\infty(-v) + g^\infty(v) \leq 0$.*

Then the inf-convolution $f \# g$ is lower semicontinuous.

Here f^∞ denotes the recession function of f ([4] p. 555), where f is lsc. It is defined as following

$$f^\infty(v) = \lim_{t \rightarrow +\infty} \frac{f(v_0 + tv)}{t},$$

where v_0 is any element such that $f(v_0) < +\infty$. Here we can choose $v_0 = 0$.

Theorem 4.2 ($\tilde{\mathcal{N}}_{\lambda,\mu}$ semi-continuity). *For every λ, μ whose components satisfy (2.11), $\tilde{F}_\lambda \# \tilde{F}_\mu(u)$ is lower semi-continuous with respect to the $L^p(Q)$ - topology.*

Proof. We choose $V = L^p(Q)$ in Theorem 4.1, $f = \tilde{F}_\lambda$ and $g = \tilde{F}_\mu$. We have seen in Proposition 4.1 that for any α satisfying assumption (2.11) then \tilde{F}_α is lsc on $L^p(Q)$ with respect to the $L^p(Q)$. As F_α is positively homogeneous we get $\tilde{F}_\alpha(tv) = |t|\tilde{F}_\alpha(v)$ and $\tilde{F}_\alpha^\infty = \tilde{F}_\alpha$. Moreover F_α is even so point (ii) of Theorem 4.1 is ensured.

Now consider $t_n \rightarrow +\infty$, u_n and v_n that converge weakly in $L^p(Q)$ such that $u_n + v_n \rightarrow 0$ strongly, and $\tilde{F}_\lambda(t_n u_n) + \tilde{F}_\mu(t_n v_n)$ is bounded from above. So $t_n u_n$ and $t_n v_n$ are in $BV(Q)$ and with Proposition 2.2, this means that $TV(t_n u_n)$ and $TV(t_n v_n)$ are bounded from above and (since $t_n \rightarrow +\infty$) $TV(u_n) \rightarrow 0$ and $TV(v_n) \rightarrow 0$. Moreover u_n and v_n converge weakly in $L^p(Q)$ so they are bounded in $L^p(Q)$ and in $L^1(Q)$. So u_n and v_n are bounded in $BV(Q)$. Therefore they strongly converge in $L^p(Q)$ since $BV(Q)$ is compactly embedded in $L^p(Q)$. \square

In the previous section, we showed an existence result under the assumption that the parameters λ and μ are constant. Here, working with the extended functionals we may neglect this assumption. Moreover, we obtain a uniqueness as well.

Theorem 4.3. *Assume $\lambda = (\lambda_1, \lambda_2)$ and $\mu = (\mu_1, \mu_2)$ are such that λ_i, μ_i satisfies (2.11) and $\mathcal{X} = L^p(Q)$ with $1 < p < \frac{d+1}{d}$. Then (3.2) has a unique solution.*

Proof. Let $u_n \in L^p(Q)$ be a minimizing sequence : $\tilde{\mathcal{N}}_{\lambda,\mu}(u_n) + \mathcal{H}(g, Au_n)$ converges to $\inf(\mathcal{P})$. Therefore $\tilde{\mathcal{N}}_{\lambda,\mu}(u_n)$ is bounded, which means that $u_n \in BV(Q)$ and that $TV(u_n)$ is bounded as well, see Proposition 2.2. It remains to prove that (u_n) is bounded in $L^p(Q)$, hence BV bounded. The rest of the proof is identical to Theorem 3.1 if we consider again two different cases with respect to the data fitting term i.e., $A = \mathcal{A}, \mathcal{R}$. The uniqueness is guaranteed if $A \in \mathcal{L}(L^p(Q), L^q(Q))$ is an injective operator with $1 < q < \infty$ and $\inf_\Sigma g > 0$ for the Kullback-Leibler case.

4.2. Fenchel conjugate of $\tilde{\mathcal{N}}_{\lambda,\mu} = \tilde{F}_\lambda \# \tilde{F}_\mu$. In order to derive the optimality conditions of (3.2), we have to compute the subdifferentials of each term. A useful tool to achieve this goal is to compute the conjugate functionals. Indeed, we know

Theorem 4.4. [4, Theorem 9.5.1.] *If V is a normed space with dual space V' , and $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semi-continuous convex and proper function, then*

$$\forall (u, u^*) \in V \times V' \quad u^* \in \partial f(u) \iff u \in \partial f^*(u^*) ,$$

where f^* is the Fenchel conjugate of f and $\partial f(u)$ is the sub-differential of f at u :

$$\partial f(u) = \{u^* \in V^* \mid \forall v \in V \quad f(v) - f(u) \geq \langle u^*, v - u \rangle_{V',V} \} .$$

The first step is to compute the Fenchel conjugate of $\tilde{\mathcal{N}} = \tilde{F}_\lambda \# \tilde{F}_\mu$ starting by \tilde{F}_λ . Let us focus on the computation of the Fenchel-conjugate of $\tilde{\Phi}_\lambda$. We set

$$\mathcal{K}_x := \left\{ \xi = \operatorname{div}_x \phi \mid \phi \in L^\infty(0, T; \mathcal{C}_c^1(\Omega, \mathbb{R}^d)), \|\phi\|_\infty \leq 1 \right\} .$$

Note that $\mathcal{K}_x \subset L^\infty(Q)$. In what follows, we identify functions defined on Ω with functions defined on $(0, T) \times \Omega$ which are constant with respect to time :

$$\forall t \in (0, T) \quad \xi(t, x) = \xi(x) \text{ a. e. } x \in \Omega.$$

With this identification, we get the following result that will be used in the next proof.

Lemma 3. *We get*

$$K_x \subset \mathcal{K}_x ,$$

where K_x is given by (2.1).

Conversely, any $\xi \in \mathcal{K}_x$ verifies $\xi(t, \cdot) \in K_x$, for almost every $t \in (0, T)$.

Proof. Let be $\xi \in K_x$. There exists $\varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^d)$ such that $\xi = \operatorname{div}_x \varphi$ and $\|\varphi\|_{\infty, x} \leq 1$. Let $\tilde{\varphi} \in \mathcal{C}_c^1(Q, \mathbb{R}^d)$ be defined as $\tilde{\varphi}(t, x) = \varphi(x)$, $(t, x) \in Q$. Then $\|\tilde{\varphi}\|_\infty \leq 1$ and $\xi = \operatorname{div}_x \tilde{\varphi} = \operatorname{div}_x \varphi$. This ends the proof. \square

Theorem 4.5 ($\tilde{\Phi}_\alpha$ Conjugate). *For every function α such that (2.11) is satisfied, we get*

$$\tilde{\Phi}_\alpha^* = \mathbb{1}_{\alpha \overline{\mathcal{K}_x}}$$

where, $\mathbb{1}_C$ is the indicatrix function of the set C and $\overline{\mathcal{K}_x}$ is the $L^{p'}$ -closure of \mathcal{K}_x .

Proof. We first remark that, for every $u^* \in L^{p'}(Q)$

$$(4.1) \quad \tilde{\Phi}_\alpha^*(u^*) = \sup_{v \in L^p(Q)} \langle v, u^* \rangle_{p, p'} - \tilde{\Phi}_\alpha(v) = \sup_{v \in \operatorname{BV}(Q)} \langle v, u^* \rangle_{p, p'} - \Phi_\alpha(v).$$

Recall that, for any $u \in \operatorname{BV}(Q)$,

$$TV_x(u)(t) = \sup_{\xi \in K_x} \int_\Omega \xi(x) u(t, x) dx \text{ a.e. } t .$$

Let be $\xi \in \mathcal{K}_x$, then $\xi(t, \cdot) \in K_x$ for almost every $t \in (0, T)$.

$$\int_\Omega \xi(t, x) u(t, x) dx \leq \sup_{\zeta \in K_x} \int_\Omega \zeta(x) u(t, x) dx = TV_x(u)(t).$$

So

$$(4.2) \quad \Phi_\alpha(u) \geq \sup_{\xi \in \alpha \mathcal{K}_x} \int_0^T \int_\Omega \xi(t, x) u(t, x) dx = \sup_{\xi \in \alpha \mathcal{K}_x} \langle \xi, u \rangle_{p, p'} .$$

As $\tilde{\Phi}_\alpha$ is positively homogeneous, then $\tilde{\Phi}_\alpha^*$ is the indicatrix of some closed subset $\tilde{\mathcal{K}}$ of $L^{p'}(Q)$.

• We first prove that $\alpha \overline{\mathcal{K}_x} \subset \tilde{\mathcal{K}}$. Let u^* be in $\alpha \mathcal{K}_x$. Using (4.1),

$$\tilde{\Phi}_\alpha^*(u^*) = \sup_{u \in \operatorname{BV}(Q)} \langle u^*, u \rangle_{p, p'} - \Phi_\alpha(u) .$$

Equation (4.2) gives that $\Phi_\alpha(u) \geq \langle u^*, u \rangle_{p,p'}$ for any $u \in \text{BV}(Q)$ and so $\tilde{\Phi}_\alpha^*(u^*) \leq 0$. As $\tilde{\Phi}_\alpha^*$ is an indicatrix function this means that $\tilde{\Phi}_\alpha^*(u^*) = 0$. So $u^* \in \tilde{\mathcal{K}}$ and $\alpha\mathcal{K}_x \subset \tilde{\mathcal{K}}$. As $\tilde{\mathcal{K}}$ is $L^{p'}$ -closed this gives $\alpha\overline{\mathcal{K}_x} \subset \tilde{\mathcal{K}}$.

• Let us prove the converse inclusion. Assume there exists $u^* \in \tilde{\mathcal{K}}$ such that $u^* \notin \alpha\overline{\mathcal{K}_x}$. One can separate u^* and $\alpha\overline{\mathcal{K}_x}$, see [11], there exists $\omega \in \mathbb{R}$ and $u_0 \in L^p(Q)$ such that

$$\langle u^*, u_0 \rangle_{p,p'} > \omega \geq \sup_{v^* \in \alpha\overline{\mathcal{K}_x}} \langle v^*, u_0 \rangle_{p,p'}.$$

Hence, we have that

$$(4.3) \quad \sup_{v^* \in \alpha\overline{\mathcal{K}_x}} \langle v^* - u^*, u_0 \rangle_{p,p'} < 0.$$

On the other hand, one can proceed with the following: as $\tilde{\Phi}_\alpha$ is convex and lower semi-continuous with respect to the L^p -topology, we get $\tilde{\Phi}_\alpha^{**} = \tilde{\Phi}_\alpha$. This gives in particular

$$\forall u \in \text{BV}(Q), \quad \Phi_\alpha(u) = \sup_{v^* \in L^{p'}(Q)} \langle v^*, u \rangle_{p,p'} - \tilde{\Phi}_\alpha^*(v^*).$$

As $u^* \in \tilde{\mathcal{K}}$ then $\tilde{\Phi}_\alpha^*(u^*) = 0$ and we obtain

$$(4.4) \quad \forall u \in \text{BV}(Q), \quad \Phi_\alpha(u) \geq \langle u^*, u \rangle_{p,p'}.$$

Let us fix $t \in (0, T)$. We get

$$\forall \xi \in K_x \quad \alpha(t)\xi(x)u(t, x) \leq \sup_{\zeta \in K_x} \alpha(t)\zeta(x)u(t, x) \quad \text{a. e. } x \in \Omega.$$

Then,

$$\begin{aligned} \forall \xi \in K_x \quad \int_{\Omega} \alpha(t)\xi(x)u(t, x) \, dx &\leq \int_{\Omega} \sup_{\zeta \in K_x} \alpha(t)\zeta(x)u(t, x) \, dx, \\ \sup_{\xi \in K_x} \int_{\Omega} \alpha(t)\xi(x)u(t, x) \, dx &\leq \int_{\Omega} \sup_{\zeta \in K_x} \alpha(t)\zeta(x)u(t, x) \, dx, \\ TV_x(\alpha u)(t) &\leq \int_{\Omega} \sup_{\zeta \in K_x} \alpha(t)\zeta(x)u(t, x) \, dx \\ TV_x(\alpha u)(t) - \int_{\Omega} u^*(t, x)u(t, x) \, dx &\leq \int_{\Omega} \left[\sup_{\zeta \in K_x} \alpha(t)\zeta(x) - u^*(t, x) \right] u(t, x) \, dx. \end{aligned}$$

Integrate over $(0, T)$, we obtain

$$\begin{aligned} \int_0^T TV_x(\alpha u)(t) \, dt - \int_0^T \int_{\Omega} u^*(t, x)u(t, x) \, dx \, dt &\leq \\ \int_0^T \int_{\Omega} \left[\sup_{\zeta \in K_x} \alpha(t)\zeta(x) - u^*(t, x) \right] u(t, x) \, dx \, dt. \end{aligned}$$

Then,

$$\begin{aligned} \Phi_\alpha(u) - \langle u^*, u \rangle_{p,p'} &\leq \int_0^T \int_\Omega \left[\sup_{\zeta \in \alpha K_x} \zeta(x) - u^*(t, x) \right] u(t, x) \, dx \, dt \\ &\leq \int_0^T \int_\Omega \left[\sup_{\xi \in \alpha \mathcal{K}_x} \zeta(x) - u^*(t, x) \right] u(t, x) \, dx \, dt, \end{aligned}$$

since $\alpha K_x \subset \alpha \mathcal{K}_x$ (with Lemma 3). With relation (4.4) this implies

$$\forall u \in \text{BV}(Q) \quad \int_0^T \int_\Omega \left(\sup_{\xi \in \alpha \mathcal{K}_x} \xi(t, x) - u^*(t, x) \right) u(t, x) \, dx \, dt \geq 0.$$

As $\text{BV}(Q)$ is dense in $L^p(Q)$ with respect to the L^p -norm (since it includes $\mathcal{C}^1(Q)$) we get

$$\forall u \in L^p(Q) \quad \int_Q \left(\sup_{\xi \in \alpha \mathcal{K}_x} \xi(t, x) - u^*(t, x) \right) u(t, x) \, dx \, dt \geq 0.$$

In a similar way, choosing $-u$ instead of u we conclude that

$$\sup_{\xi \in \alpha \mathcal{K}_x} \xi - u^* = 0.$$

Therefore,

$$\sup_{\xi \in \alpha \mathcal{K}_x} \langle \xi - u^*, u_0 \rangle_{p,p'} = 0.$$

As $\alpha \mathcal{K}_x \subset \overline{\alpha \mathcal{K}_x}$ then

$$\sup_{\xi \in \overline{\alpha \mathcal{K}_x}} \langle \xi - u^*, u_0 \rangle_{p,p'} \geq 0.$$

which gives a contradiction. with equation (4.3). □

We can prove similarly the following

Theorem 4.6 ($\tilde{\Psi}_\alpha$ Conjugate). *For every α such that (2.11) is satisfied, we get*

$$\tilde{\Psi}_\alpha^* = \mathbb{1}_{\overline{\alpha \mathcal{K}_t}}$$

where,

$$\mathcal{K}_t := \left\{ \xi = \frac{d\psi}{dt} \mid \psi \in L^\infty(\Omega, \mathcal{C}_c^1(0, T, \mathbb{R})), \|\psi\|_\infty \leq 1 \right\}.$$

Corollary 4.2. *For every $\lambda = (\lambda_1, \lambda_2)$ such that (2.11) is satisfied for each component, we get*

$$\tilde{F}_\lambda^*(u^*) = \mathbb{1}_{\overline{\mathcal{K}_\lambda}}(u^*)$$

for every $u^* \in L^{p'}(Q)$, where

$$(4.5) \quad \mathcal{K}_\lambda = \left\{ \xi = \lambda_1 \operatorname{div}_x \varphi + \lambda_2 \frac{d\psi}{dt} \mid \varphi \in L^\infty(0, T; \mathcal{C}_c^1(\Omega, \mathbb{R}^d)), \psi \in L^\infty(\Omega, \mathcal{C}_c^1(0, T, \mathbb{R})), \|\varphi\|_\infty \leq 1, \|\psi\|_\infty \leq 1 \right\}$$

Proof. As $\tilde{F}_\lambda = \tilde{\Phi}_{\lambda_1} + \tilde{\Psi}_{\lambda_2}$ which are convex, lower semicontinuous, we get [4, Th. 9.4.1]

$$\tilde{F}_\lambda^* = \tilde{\Phi}_{\lambda_1}^* \# \tilde{\Psi}_{\lambda_2}^* = \mathbb{1}_{\lambda_1 \overline{\mathcal{K}_x}} \# \mathbb{1}_{\lambda_2 \overline{\mathcal{K}_t}} = \mathbb{1}_{\lambda_1 \overline{\mathcal{K}_x} + \lambda_2 \overline{\mathcal{K}_t}} = \mathbb{1}_{\overline{\mathcal{K}_\lambda}},$$

where $\mathcal{K}_\lambda = \lambda_1 \mathcal{K}_x + \lambda_2 \mathcal{K}_t$. We have already observed that $\tilde{F}_\lambda^* = F_\lambda^*$ on $L^{p'}(Q)$. □

Corollary 4.3 ($\tilde{\mathcal{N}}_{\lambda,\mu}$ Conjugate). *For every λ, μ such that (2.11) is satisfied, we get*

$$\tilde{\mathcal{N}}_{\lambda,\mu}^* = \mathbb{1}_{\overline{\mathcal{K}_\lambda \cap \mathcal{K}_\mu}}.$$

4.3. Optimality conditions for (\mathcal{P}) . Now we are ready to use Theorem 4.4 with $V = L^p(Q)$. Since the problem (3.2) is (strictly) convex we get that u is the solution to this problem if and only if $0 \in \partial \mathcal{J}(u)$ where

$$\mathcal{J}(u) := \tilde{\mathcal{N}}_{\lambda,\mu}(u) + \mathcal{H}(g, Au).$$

Clearly, $\text{dom } \tilde{\mathcal{N}}_{\lambda,\mu} = \text{BV}(Q)$, $\text{dom } \mathcal{H}(g, Au) = L^p(Q)$ and $\text{dom } \mathcal{H}(g, \mathcal{R}u) = L_+^1(Q)$. Since, $Q \subset \mathbb{R}^{d+1}$ is bounded, we have that $\|\mathcal{R}u\|_{L^1(\Sigma)} \leq C \|u\|_{L^1(Q)} \leq \tilde{C} \|u\|_{L^p(Q)}$ with $1 < p < \frac{d+1}{d}$. Recall that the Radon transform is not necessarily defined for $p \geq \frac{d+1}{d}$, see the introduction. We may also write that $\text{dom } \mathcal{H}(g, Au) = L^p(Q)$ or $L_+^p(Q)$ with $1 < p < \frac{d+1}{d}$. Therefore, Theorem 9.5.4 of [4] may be applied and

$$\partial \mathcal{J}(u) = \partial \tilde{\mathcal{N}}_{\lambda,\mu}(u) + \partial \mathcal{H}(g, Au).$$

Any u^* of $\partial \mathcal{J}(u)$ writes $u^* = u_1^* + u_2^*$ where $u_1^* \in \partial \tilde{\mathcal{N}}_{\lambda,\mu}(u)$ and $u_2^* \in \partial \mathcal{H}(g, Au)$. In the sequel, we characterise the elements u_1^*, u_2^* .

Starting with the $\tilde{\mathcal{N}}_{\lambda,\mu}$ -subdifferential, it is easy to check that for every $u \in \text{BV}(Q) \hookrightarrow L^p(Q)$, we get

$$(4.6) \quad u_1^* \in \partial \tilde{\mathcal{N}}_{\lambda,\mu}(u) \subset L^{p'}(Q) \iff u_1^* \in \mathcal{K}_{\lambda,\mu} \text{ and } \forall v^* \in \mathcal{K}_{\lambda,\mu} \quad \langle v^* - u_1^*, u \rangle_{p,p'} \leq 0.$$

where $\mathcal{K}_{\lambda,\mu} := \overline{\mathcal{K}_\lambda} \cap \overline{\mathcal{K}_\mu}$ is a closed convex subset of $L^{p'}(Q)$. Indeed, we use Theorem 4.4, Corollary 4.3 and that $\tilde{\mathcal{N}}_{\lambda,\mu}$ is convex and lower semi-continuous, to get

$$u \in \partial \tilde{\mathcal{N}}_{\lambda,\mu}^*(u_1^*) = \partial \mathbb{1}_{\mathcal{K}_{\lambda,\mu}}(u_1^*).$$

Moreover, it is well known that the subdifferential of $\mathbb{1}_{L_+^p}$ is the normal cone, namely

$$\begin{aligned} \mathbf{N}_+(u) &= \{v^* \in L^{p'}(Q) \mid \forall v \in L_+^p, \langle v^*, v - u \rangle_{p,p'} \leq 0\} \\ &= \{v^* \in L^{p'}(Q) \mid \forall v \in L_+^p, \langle v^*, v \rangle_{p,p'} \leq 0\}; \end{aligned}$$

indeed, as $u \geq 0$, for any $v^* \in \mathbf{N}_+(u)$ choosing $v = 0$ and $v = 2u$ gives

$$\langle v^*, u \rangle_{p,p'} = 0.$$

Since $\langle v^*, v \rangle \leq 0$ and $v \in L_+^p$ then it is immediate that $v^* \leq 0$ and so $\mathbf{N}_+(u) \subset L_-^p$. For the distance term, recall that $1 \leq q < \infty$ and

$$\mathcal{H}(g, Au) = \begin{cases} \frac{1}{q} \|Au - g\|_{L^q(Q)}^q, & \text{if } A = \mathcal{A}, \\ \int_{\Sigma} (\mathcal{R}u - g \log \mathcal{R}u)(t, x) \, dx \, dt + \mathbb{1}_{L_+^p}(u), & \text{if } A = \mathcal{R}. \end{cases}$$

• In the case where $q > 1$, $\mathcal{H}(g, Au)$ it is differentiable on its domain and

$$\partial \mathcal{H}(g, Au) = \begin{cases} \mathcal{A}^*(Au - g)^{q-1}, & \text{if } A = \mathcal{A}, \\ \mathcal{R}^* \left(\mathbf{1}_{\Sigma} - \frac{g}{\mathcal{R}u} \right) + \partial \mathbb{1}_{L_+^p}(u), & \text{if } A = \mathcal{R}, \end{cases}$$

where $\mathbf{1}_{\Sigma}$ is the characteristic function of Σ . In the case where $A = \mathcal{R}$, we have applied once again Theorem 9.5.4 of [4] since the functional

$$u \mapsto \int_{\Sigma} (\mathcal{R}u - g \log \mathcal{R}u)(t, x) \, dx \, dt$$

is continuous on $L_+^p(Q)$.

• If $q = 1$, then $\mathcal{H}(g, Au) = N_g(\mathcal{A}u - g)$ where $N_g(v) = \|v - g\|_{L^1(Q)}$; we get

$$\mathcal{H}(g, Au) = \mathcal{A}^* \partial N_g(\mathcal{A}u).$$

Let us compute $\partial N_g(v)$ for any $v \in L^1(Q)$:

$$\begin{aligned} \partial N_g(v) &= \left\{ z \in L^\infty(Q) \mid \|w - g\|_{L^1(Q)} \geq \|v - g\|_{L^1(Q)} + \langle z, w - v \rangle, \forall w \in L^1(Q) \right\} \\ &= \left\{ z \in L^\infty(Q) \mid \|v\|_{L^1(Q)} \geq \|v - g\|_{L^1(Q)} + \langle z, w - (v - g) \rangle, \forall w \in L^1(Q) \right\} \\ &= \partial(\|\cdot\|_{L^1(Q)})(v - g). \end{aligned}$$

It is well known that

$$\begin{aligned} \partial(\|\cdot\|_{L^1(Q)})(v) &= \{z \in L^\infty(Q) \mid \|z\|_{L^\infty(Q)} \leq 1, \langle z, v \rangle = \|v\|_{L^1(Q)}\} \\ &= \{z \in L^\infty(Q) \mid \|z\|_{L^\infty(Q)} \leq 1, z \in \text{sign}(v)\}. \end{aligned}$$

So

$$\partial N_g(\mathcal{A}u) = \left\{ z \in L^\infty(Q) : \|z\|_{L^\infty(Q)} \leq 1, z \in \text{sign}(\mathcal{A}u - g) \right\}.$$

Hence,

$$u^* \in \mathcal{H}(g, Au) \Leftrightarrow u^* = \mathcal{A}^* z, \|z\|_{L^\infty(Q)} \leq 1, z \in \text{sign}(\mathcal{A}u - g).$$

Finally

$$\partial \mathcal{H}(g, Au) = \begin{cases} \mathcal{A}^*(Au - g)^{q-1}, & \text{if } A = \mathcal{A}, 1 < q < \infty \\ \left\{ \mathcal{A}^* z, \|z\|_{L^\infty(Q)} \leq 1, z \in \text{sign}(\mathcal{A}u - g) \right\}, & \text{if } A = \mathcal{A}, q = 1 \\ \mathcal{R}^* \left(\mathbf{1}_{\Sigma} - \frac{g}{\mathcal{R}u} \right) + \partial \mathbb{1}_{L_+^p}(u), & \text{if } A = \mathcal{R}. \end{cases}$$

Hence, for the subdifferential of the fidelity term, we prove the following

Proposition 4.2. For any $u \in L^p(Q)$,

$$u_2^* \in \partial\mathcal{H}(g, Au) \iff u_2^* = u_4^* + u_3^*,$$

where $u_3^* = 0$ if $A = \mathcal{A}$ (no positivity constraint) and $u_3^* \leq 0$ with

$$(4.7) \quad \langle u, u_3^* \rangle_{p,p'} = 0 .$$

if $A = \mathcal{R}$. Here

$$(4.8) \quad u_4^* \in \begin{cases} \{\mathcal{A}^*(Au - g)^{q-1}\}, & \text{if } A = \mathcal{A}, 1 < q < \infty \\ \{\mathcal{A}^*z, \|z\|_{L^\infty(Q)} \leq 1, z \in \text{sign}(Au - g)\}, & \text{if } A = \mathcal{A}, q = 1 \\ \{\mathcal{R}^*\left(\mathbf{1}_\Sigma - \frac{g}{\mathcal{R}u}\right)\}, & \text{if } A = \mathcal{R}. \end{cases}$$

We finally obtain,

$$\begin{aligned} 0 \in \partial\mathcal{J}(u) &\iff u_1^* = -u_2^* = -u_4^* - u_3^*, \\ &\iff -u_4^* - u_3^* \in \partial\tilde{\mathcal{N}}_{\lambda,\mu}(u), \text{ with } u_3^* \begin{cases} = 0 & \text{if } A = \mathcal{A} \\ \leq 0 \text{ and (4.7)} & \text{if } A = \mathcal{R} \end{cases} , \\ &\iff \forall v \in \mathcal{K}_{\lambda,\mu} \quad \langle u_4^* + v, u \rangle_{p,p'} \leq -\langle u_3^*, u \rangle_{p,p'} . \end{aligned}$$

With relation (4.7) and the fact that $v \in \mathcal{K}_{\lambda,\mu} \iff -v \in \mathcal{K}_{\lambda,\mu}$ we get

$$0 \in \partial\mathcal{J}(u) \iff \forall v \in \mathcal{K}_{\lambda,\mu} \quad \langle u_4^* - v, u \rangle_{p,p'} \leq 0 .$$

Theorem 4.7. A function $\mathbf{u} \in \text{BV}(Q)$ is a unique solution to (3.2) if and only if

- Case $A = \mathcal{A}$ and $1 < q < +\infty$

$$\forall v \in \mathcal{K}_{\lambda,\mu} \quad \langle \mathcal{A}^*(\mathcal{A}\mathbf{u} - g)^{q-1} - v, \mathbf{u} \rangle_{p,p'} \leq 0,$$

- Case $A = \mathcal{A}$ and $q = 1$

$$\forall v \in \mathcal{K}_{\lambda,\mu}, \quad \langle \mathcal{A}^*z - v, \mathbf{u} \rangle_{p,p'} \leq 0, \|z\|_{L^\infty(Q)} \leq 1, z \in \text{sign}(\mathcal{A}\mathbf{u} - g),$$

- Case $A = \mathcal{R}$

$$\forall v \in \mathcal{K}_{\lambda,\mu}, \quad \left\langle \mathcal{R}^*\left(\mathbf{1}_\Sigma - \frac{g}{\mathcal{R}\mathbf{u}}\right) - v, \mathbf{u} \right\rangle_{p,p'} \leq 0 .$$

Here $\mathcal{K}_{\lambda,\mu} = \overline{\mathcal{K}_\lambda} \cap \overline{\mathcal{K}_\mu}$, where the closure is taken in the sense of the $L^{p'}$ -norm and $\mathcal{K}_\lambda, \mathcal{K}_\mu$ are defined by (4.5).

4.4. Optimality conditions in the decoupled case (3.2) . We may also use the equivalent formulation (3.2) to get the optimality conditions. In a similar manner, we deal with the extended functions on $\mathcal{X} = L^p$ topology framework. If (\mathbf{u}, \mathbf{v}) is a solution to problem (3.2), then we clearly have

$$\begin{cases} \mathbf{u} = \operatorname{argmin}_{u \in \mathcal{X}} \tilde{F}_\lambda(u - \mathbf{v}) + \mathcal{H}(g, Au), \\ \mathbf{v} = \operatorname{argmin}_{v \in \mathcal{X}} \tilde{F}_\lambda(\mathbf{u} - v) + \tilde{F}_\mu(v). \end{cases}$$

As the functionals are convex and lower semicontinuous with respect to the L^p topology with $1 \leq p \leq \frac{d+1}{d}$, the necessary and sufficient condition is

$$(4.9) \quad \begin{cases} 0 \in \partial \left(\tilde{F}_\lambda \circ \tau_{\mathbf{v}} \right) (\mathbf{u}) + \mathcal{H}(g, A\mathbf{u}), \\ 0 \in \partial \left(\tilde{F}_\lambda \circ \tau_{\mathbf{u}} \right) (\mathbf{v}) + \tilde{F}_\mu(\mathbf{v}), \end{cases}$$

where $\tau_u(v) = v - u$. A usual we use

$$u^* \in \partial(\tilde{F}_\lambda \circ \tau_{\mathbf{v}})(\mathbf{u}) \iff \mathbf{u} \in \partial \left(\tilde{F}_\lambda \circ \tau_{\mathbf{v}} \right)^* (u^*).$$

A simple computation gives $\partial(\tilde{F}_\lambda \circ \tau_{\mathbf{v}})^* = \partial\tilde{F}_\lambda^* + \mathbf{v}$ and we get

$$u^* \in \partial(\tilde{F}_\lambda \circ \tau_{\mathbf{v}})(\mathbf{u}) \iff \mathbf{u} - \mathbf{v} \in \partial 1_{\overline{\mathcal{K}_\lambda}}(u^*).$$

Similarly

$$v^* \in \partial(\tilde{F}_\lambda \circ \tau_{\mathbf{u}})(\mathbf{v}) \iff \mathbf{v} - \mathbf{u} \in \partial 1_{\overline{\mathcal{K}_\lambda}}(v^*).$$

Let us focus on the first equation of (4.9):

$$\begin{aligned} 0 \in \partial \left[\tilde{F}_\lambda \circ \tau_{\mathbf{v}} \right] (\mathbf{u}) + \mathcal{H}(g, A\mathbf{u}) &\iff -u_4^* - u_3^* \in \partial(\tilde{F}_\lambda \circ \tau_{\mathbf{v}})(\mathbf{u}) \\ &\iff \mathbf{u} - \mathbf{v} \in \partial 1_{\overline{\mathcal{K}_\lambda}}(-u_4^* - u_3^*) \\ &\iff \forall w \in \overline{\mathcal{K}_\lambda} \quad \langle \mathbf{u} - \mathbf{v}, w + u_4^* + u_3^* \rangle_{p,p'} \leq 0, \\ &\iff \forall w \in \overline{\mathcal{K}_\lambda} \quad \langle \mathbf{v} - \mathbf{u}, w - u_4^* - u_3^* \rangle_{p,p'} \leq 0, \end{aligned}$$

where u_4^* is given by (4.8). Once again we used $w \in \overline{\mathcal{K}_\lambda} \iff -w \in \overline{\mathcal{K}_\lambda}$. Here, $u_3^* = 0$ if $A = \mathcal{A}$ and $u_3^* \leq 0, \langle \mathbf{u}, u_3^* \rangle_{p,p'} = 0$ if $A = \mathcal{R}$. Moreover

$$\begin{aligned} -u_4^* - u_3^* \in \partial(\tilde{F}_\lambda \circ \tau_{\mathbf{v}})(\mathbf{u}) \\ &\iff (\tilde{F}_\lambda \circ \tau_{\mathbf{v}})(\mathbf{u}) + (\tilde{F}_\lambda \circ \tau_{\mathbf{v}})^*(-u_4^* - u_3^*) = -\langle \mathbf{u}, u_4^* + u_3^* \rangle_{p,p'} \\ &\iff \tilde{F}_\lambda(\mathbf{u} - \mathbf{v}) = -\langle \mathbf{u}, u_4^* \rangle_{p,p'} . \end{aligned}$$

We consider now the second equation of (4.9):

$$\begin{aligned}
0 \in \partial \left[\tilde{F}_\lambda \circ \tau_{\mathbf{u}}(\mathbf{v}) + \tilde{F}_\mu(\mathbf{v}) \right] &\iff \\
\exists w^* \in \partial \tilde{F}_\mu(\mathbf{v}) \text{ such that } -w^* \in \partial(\tilde{F}_\lambda \circ \tau_{\mathbf{u}})(\mathbf{v}) &\iff \\
\mathbf{v} \in \partial 1_{\overline{\mathcal{K}_\mu}}(w^*) \cap \left(\partial 1_{\overline{\mathcal{K}_\lambda}}(-w^*) + \mathbf{u} \right) &\iff \\
\mathbf{v} \in \partial 1_{\overline{\mathcal{K}_\mu}}(w^*) \text{ and } \mathbf{v} - \mathbf{u} \in \partial 1_{\overline{\mathcal{K}_\lambda}}(-w^*) &\iff \\
\forall w \in \overline{\mathcal{K}_\lambda} \langle \mathbf{v} - \mathbf{u}, w + w^* \rangle_{p,p'} \leq 0 \text{ and } \forall w \in \overline{\mathcal{K}_\mu} \langle v_*, w - w^* \rangle_{p,p'} \leq 0.
\end{aligned}$$

Moreover,

$$\mathbf{v} \in \partial 1_{\overline{\mathcal{K}_\mu}}(w^*) \iff w^* \in \partial \tilde{F}_\mu(\mathbf{v}) \iff 1_{\overline{\mathcal{K}_\mu}}(w^*) + F_\mu(\mathbf{v}) = \langle v_*, w^* \rangle_{p,p'} .$$

This gives $\langle v_*, w^* \rangle_{p,p'} = \tilde{F}_\mu(\mathbf{v})$. Similarly $\langle \mathbf{v} - \mathbf{u}, w^* \rangle_{p,p'} = -\tilde{F}_\lambda(\mathbf{v} - \mathbf{u})$. We finally obtain

Theorem 4.8. *For any solution (\mathbf{u}, \mathbf{v}) to (3.2), there exists $u_3^* \in L^p(Q)$ and $w^* \in \mathcal{K}_{\lambda,\mu}$ such that*

$$(4.10) \quad \forall w \in \overline{\mathcal{K}_\lambda} \quad \langle \mathbf{u} - \mathbf{v}, u_4^* + u_3^* - w \rangle_{p,p'} \leq 0,$$

$$(4.11) \quad \forall w \in \overline{\mathcal{K}_\lambda} \quad \langle \mathbf{v} - \mathbf{u}, w + w^* \rangle_{p,p'} \leq 0,$$

$$(4.12) \quad \forall w \in \overline{\mathcal{K}_\mu} \quad \langle \mathbf{v}, w - w^* \rangle_{p,p'} \leq 0,$$

$$(4.13) \quad \begin{cases} u_3^* = 0 \text{ if } A = \mathcal{A}, \\ u_3^* \leq 0, \mathbf{u} \geq 0, \langle \mathbf{u}, u_3^* \rangle_{p,p'} = 0 \text{ if } A = \mathcal{R}, \end{cases} \quad \text{and } F_\lambda(\mathbf{u} - \mathbf{v}) = -\langle \mathbf{u}, w^* \rangle_{p,p'}$$

$$(4.14) \quad \langle \mathbf{v}, w^* \rangle_{p,p'} = F_\mu(\mathbf{v}) \text{ and } \langle \mathbf{v} - \mathbf{u}, w^* \rangle_{p,p'} = -F_\lambda(\mathbf{v} - \mathbf{u})$$

Here u_4^* is given by (4.8) and we used that $\tilde{F}_\mu(\mathbf{v}) = F_\mu(\mathbf{v})$ and $\tilde{F}_\lambda(\mathbf{u} - \mathbf{v}) = F_\lambda(\mathbf{u} - \mathbf{v})$.

Remark 4.1. • With equation (4.14), equation (4.12) gives

$$\forall w \in \overline{\mathcal{K}_\mu} \quad \langle \mathbf{v}, w \rangle_{p,p'} \leq F_\mu(\mathbf{v});$$

similarly equation (4.11) gives

$$\forall w \in \overline{\mathcal{K}_\lambda} \quad \langle \mathbf{v} - \mathbf{u}, w \rangle_{p,p'} \leq F_\lambda(\mathbf{v} - \mathbf{u}).$$

Eventually, equation (4.14) implies

$$\langle \mathbf{u}, w^* \rangle_{p,p'} = F_\lambda(\mathbf{v} - \mathbf{u}) + F_\mu(\mathbf{v}) = (F_\lambda \# F_\mu)(\mathbf{u}) .$$

• In the case where $p = 2$ or if we consider a discretized problem then equation (4.10) is equivalent to

$$u_4^* + u_3^* = \text{Proj}_{\overline{\mathcal{K}_\lambda}}(u_4^* + u_3^* + \mathbf{v} - \mathbf{u}) ;$$

equation (4.11) is equivalent to

$$w^* = -\text{Proj}_{\mathcal{K}_\lambda}(\mathbf{v} - \mathbf{u} - w^*)$$

and equation (4.12) is equivalent to

$$w^* = \text{Proj}_{\mathcal{K}_\mu}(\mathbf{v} + w^*) .$$

5. Conclusion. The model we have investigated is now better understood from a mathematical point of view. We get well-posedness results and optimality conditions that allow to use primal-dual algorithm. We provide two methods dealing with two specific cases :

- the parameters are supposed to be constant and we get results for any L^p topology with $1 \leq p \leq \frac{d+1}{d}$; this allows to recover the L^2 framework if $d = 1$;
- the parameters may be time dependent which allows to include non uniform time discretization step in them. However, we can only consider $1 < p < \frac{d+1}{d}$. The value $p = 1$ is excluded for reflexivity reason and $p = \frac{d+1}{d}$ for lack of compactness.

Next issue is to describe carefully the discretization process and the dual problem in a appropriate way. Next, we shall perform numerics using classical performant methods to compare this model to those that can be found in the literature and in [15] in particular. This will be addressed in a forthcoming paper.

REFERENCES

- [1] R. ACAR AND C. R. VOGEL, *Analysis of Bounded Variation Penalty Methods for Ill-Posed Problems*, Inverse Problems, 10 (1994), pp. 1217–1229. <http://dx.doi.org/10.1088/0266-5611/10/6/003>.
- [2] L. AMBROSIO, N. FUSCO, AND D. PALLARA, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000.
- [3] J. APPELL, J. BANAS, AND N. MERENTES DÍAZ, *Bounded Variation and Around*, De Gruyter series in nonlinear analysis and applications, The Clarendon Press Oxford University Press, Berlin, 2013.
- [4] H. ATTOUCH, G. BUTTAZZO, AND G. MICHAILLE, *Variational analysis in Sobolev and BV spaces*, vol. 6 of MPS/SIAM Series on Optimization, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2006. Applications to PDEs and optimization.
- [5] M. BERGOUNIOUX, E. PAPOUTSELLIS, AND C. TAUBER, *Spatial-temporal regularisation in dynamic image processing and applications to positron emission tomography*, tech. report, 2016.
- [6] M. BERGOUNIOUX AND E. TRÉLAT, *A variational method using fractional order Hilbert spaces for tomographic reconstruction of blurred and noised binary images*, Journal of Functional Analysis, 259 (2010), pp. 2296 – 2332. <http://dx.doi.org/10.1016/j.jfa.2010.05.016>.
- [7] M. BERTERO, P. BOCCACCI, G. DESIDERÁ, AND G. VICIDOMINI, *Image deblurring with poisson data: from cells to galaxies*, Inverse Problems, 25 (2009), p. 123006. <http://stacks.iop.org/0266-5611/25/i=12/a=123006>.
- [8] M. BERTERO, H. LANTÉRI, AND L. ZANNI, *Iterative image reconstruction: a point of view*, 2007.
- [9] J. M. BORWEIN AND A. S. LEWIS, *Convergence of best entropy estimates*, SIAM Journal on Optimization, 1 (1991), pp. 191–205, doi:10.1137/0801014. <http://dx.doi.org/10.1137/0801014>.
- [10] K. BREDIES, K. KUNISCH, AND T. POCK, *Total Generalized Variation*, SIAM Journal on Imaging Sciences, 3 (2010), pp. 492–526. <http://dx.doi.org/10.1137/090769521>.
- [11] H. BREZIS, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer-Verlag New York, 2011. <http://dx.doi.org/10.1007/978-0-387-70914-7>.

- [12] J. A. CLARKSON AND R. C. ADAMS, *On definitions of bounded variation for functions of two variables*, Transactions of the American Mathematical Society, 35 (1933), pp. 824–854. <http://www.jstor.org/stable/1989593>.
- [13] I. EKELAND AND R. TEMAM, *Convex analysis and variational problems*, SIAM, 1976.
- [14] L. C. EVANS AND R. F. GARIEPY, *Measure theory and fine properties of functions*, CRC press, 2015.
- [15] M. HOLLER AND K. KUNISCH, *On infimal convolution of tv-type functionals and applications to video and image reconstruction*, SIAM Journal on Imaging Sciences, 7 (2014), pp. 2258–2300. <http://dx.doi.org/10.1137/130948793>.
- [16] M. S. HOSSEINI AND K. N. PLATANIOTIS, *High-accuracy total variation with application to compressed video sensing*, IEEE Transactions on Image Processing, 23 (2014), pp. 3869–3884.
- [17] V. JAOUEN, P. GONZALEZ, S. CHALON, D. GUILLOTEAU, I. BUVAT, AND C. TAUBER, *4dgvf segmentation of vector-valued images*, in 2014 IEEE International Conference on Image Processing (ICIP), IEEE, 2014, pp. 11–15.
- [18] V. JAOUEN, P. GONZALEZ, S. STUTE, D. GUILLOTEAU, I. BUVAT, AND C. TAUBER, *Vector-based active surfaces for segmentation of dynamic pet images*, in 2013 IEEE 10th International Symposium on Biomedical Imaging, IEEE, 2013, pp. 61–64.
- [19] G. LANDI AND E.-L. PICCOLOMINI, *A total variation regularization strategy in dynamic mri*, Optimization Methods and Software, 20 (2005), pp. 545–558.
- [20] H. LANTÉRI AND C. THEYS, *Restoration of astrophysical images—the case of poisson data with additive gaussian noise*, EURASIP Journal on Advances in Signal Processing, 2005 (2005), pp. 1–14. <http://dx.doi.org/10.1155/ASP.2005.2500>.
- [21] T. LE, R. CHARTRAND, AND T. J. ASAKI, *A variational approach to reconstructing images corrupted by poisson noise*, Journal of Mathematical Imaging and Vision, 27 (2007), pp. 257–263. <http://dx.doi.org/10.1007/s10851-007-0652-y>.
- [22] A. MARKOE, *Analytic Tomography*, Cambridge University Press, 2014. <http://dx.doi.org/10.1017/CBO9780511530012>.
- [23] F. NATTERER, *The mathematics of computerized tomography*, vol. 32, Siam, 1986.
- [24] M. NIKOLOVA, *Minimizers of cost-functions involving nonsmooth data-fidelity terms. application to the processing of outliers*, SIAM Journal on Numerical Analysis, 40 (2002), pp. 965–994. <http://dx.doi.org/10.1137/S0036142901389165>.
- [25] M. NIKOLOVA, *A variational approach to remove outliers and impulse noise*, Journal of Mathematical Imaging and Vision, 20 (2004), pp. 99–120. <http://dx.doi.org/10.1023/B:JMIV.0000011326.88682.e5>.
- [26] D. OBERLIN AND E. STEIN, *Mapping properties of the radon transform*, Indiana Univ. Math. J., 31 (1982), pp. 641–650.
- [27] E. RESMERITA AND R. S. ANDERSSEN, *Joint additive Kullback-Leibler residual minimization and regularization for linear inverse problems*, Mathematical Methods in the Applied Sciences, 30 (2007), pp. 1527–1544.
- [28] A. SAWATZKY, *(Nonlocal) total variation in Medical imaging*, PhD thesis, University of Münster, 2011.
- [29] H. SCHAEFFER, Y. YANG, AND S. OSHER, *Space-time regularization for video decompression*, SIAM Journal on Imaging Sciences, 8 (2015), pp. 373–402.
- [30] M. SCHLOEGL, M. HOLLER, A. SCHWARZL, K. BREDIES, AND R. STOLLBERGER, *Infimal convolution of total generalized variation functionals for dynamic mri*, Magnetic Resonance in Medicine, (2016). <http://dx.doi.org/10.1002/mrm.26352>.
- [31] Y. VARDI, L. A. SHEPP, AND L. KAUFMAN, *A statistical model for positron emission tomography*, Journal of the American Statistical Association, 80 (1985), pp. 8–20. <http://www.jstor.org/stable/2288030>.
- [32] M. N. WERNICK AND J. N. AARSVOLD, *Emission tomography : the fundamentals of PET and SPECT*, Elsevier Academic Press, 2004.