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FLATNESS OF MULTI-INPUT CONTROL-AFFINE SYSTEMS LINEARIZABLE VIA ONE-FOLD PROLONGATION*

FLORENTINA NICOLAU[†] AND WITOLD RESPONDEK[‡]

Abstract. We study flatness of multi-input control-affine systems. We give a geometric characterization of systems that become static feedback linearizable after an invertible one-fold prolongation of a suitably chosen control. They form a particular class of flat systems. Namely, they are of differential weight $n + m + 1$, where n is the dimension of the state-space and m is the number of controls. We propose conditions (verifiable by differentiation and algebraic operations) describing that class and provide a system of PDE's giving all minimal flat outputs. We illustrate our results by an example of the quadrotor helicopter.

Key words. flatness, flat outputs, differential weight, linearization.

1. Introduction. In this paper, we study flatness of nonlinear control systems of the form

$$\Xi : \dot{x} = F(x, u),$$

where x is the state defined on a open subset X of \mathbb{R}^n and u is the control taking values in an open subset U of \mathbb{R}^m (more generally, an n -dimensional manifold X and an m -dimensional manifold U , respectively). The dynamics F are smooth and the word smooth will always mean C^∞ -smooth.

The notion of flatness was introduced in control theory in the 1990's, by Fliess, Lévine, Martin and Rouchon [8, 9], see also [13, 14, 17, 24], and has attracted a lot of attention because of its multiple applications in the problem of constructive controllability and motion planning (see, e.g. [10, 16, 25, 29, 23, 18, 26, 27]). Flat systems form a class of control systems whose set of trajectories can be parametrized by m functions and their time-derivatives, m being the number of controls. More precisely, the system $\Xi : \dot{x} = F(x, u)$ is *flat* if we can find m functions, $\varphi_i(x, u, \dots, u^{(l)})$ such that

$$(1.1) \quad x = \gamma(\varphi, \dots, \varphi^{(s-1)}) \text{ and } u = \delta(\varphi, \dots, \varphi^{(s)}),$$

for a certain integer s , where $\varphi = (\varphi_1, \dots, \varphi_m)$ is called a *flat output*. Therefore the time-evolution of all state and control variables can be determined from that of flat outputs without integration and all trajectories of the system can be completely parameterized. A similar notion, of systems of undetermined differential equations integrable without integration, has been studied by Hilbert [11] and Cartan [6], see also [29], where connections between Cartan prolongations and flatness were studied.

Flatness is closely related to the notion of feedback linearization. It is well known that systems linearizable via invertible static feedback are flat. Their description (1.1) uses the minimal possible, which is $n + m$, number of time-derivatives of the

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components of the flat output φ . In general, a flat system is not linearizable by static feedback, with the exception of the single-input case where flatness reduces to static feedback linearization, see [7] and [24]. For any flat system that is not static feedback linearizable, the minimal number of time-derivatives of φ_i needed to express x and u (which is called the differential weight [26]) is thus greater than $n + m$ and measures actually the smallest possible dimension of a precompensator linearizing dynamically the system. Therefore the simplest systems for which the differential weight is greater than $n + m$ are systems linearizable via one-dimensional precompensator, thus of differential weight $n + m + 1$. They form the class that we are studying in the paper: our goal is to give a geometric verifiable characterization of control-affine systems that become static feedback linearizable after an invertible one-fold prolongation of a suitably chosen control.

The paper is organized as follows. In Section 2, we recall the definition of flatness and define the notion of differential weight of a flat system. In Section 3, we give our main results: we characterize control-affine systems that become static feedback linearizable after an invertible one-fold prolongation. They form a particular class of flat systems, that is, flat systems of differential weight $n + m + 1$. We provide necessary and sufficient conditions for flatness of differential weight $n + m + 1$ and explain in Section 3.2 how to verify them. We describe all minimal flat outputs in Section 4. For all results presented in Sections 3 and 4, we assumed that all ranks involved are constant. In Section 5 we address the issue of the importance of the constant rank assumption and clarify for which results it is necessary. We illustrate our results by an example of the quadrotor helicopter in Section 6 and provide proofs in Section 7.

2. Flatness. The fundamental property of flat systems is that all their solutions may be parametrized by a finite number of functions and their time-derivatives. Fix an integer $l \geq -1$ and denote $U^l = U \times \mathbb{R}^{ml}$ and $\bar{u}^l = (u, \dot{u}, \dots, u^{(l)})$. For $l = -1$, the set U^{-1} is empty and \bar{u}^{-1} is an empty sequence.

DEFINITION 2.1. *The system $\Xi : \dot{x} = F(x, u)$ is flat at $(x_0, \bar{u}_0^l) \in X \times U^l$, for $l \geq -1$, if there exists a neighborhood \mathcal{O}^l of (x_0, \bar{u}_0^l) and m smooth functions $\varphi_i = \varphi_i(x, u, \dot{u}, \dots, u^{(l)})$, $1 \leq i \leq m$, defined in \mathcal{O}^l , having the following property: there exist an integer s and smooth functions γ_i , $1 \leq i \leq n$, and δ_j , $1 \leq j \leq m$, such that*

$$x_i = \gamma_i(\varphi, \dot{\varphi}, \dots, \varphi^{(s-1)}) \text{ and } u_j = \delta_j(\varphi, \dot{\varphi}, \dots, \varphi^{(s)})$$

for any C^{l+s} -control $u(t)$ and corresponding trajectory $x(t)$ that satisfy $(x(t), u(t), \dots, u^{(l)}(t)) \in \mathcal{O}^l$, where $\varphi = (\varphi_1, \dots, \varphi_m)$ and is called a flat output.

Whenever necessary to specify the number of derivatives of u on which the components of the flat outputs φ depend, we say that the system Ξ is $(x, u, \dots, u^{(r)})$ -flat if the r -th-derivative is the highest involved. In the particular case $\varphi_i = \varphi_i(x)$, for $1 \leq i \leq m$, we say that the system is x -flat.

In general, r is not greater than the integer l needed to define the neighborhood \mathcal{O}^l . In our study, r will be proved to be -1 , i.e., the flat outputs depend on x only, and l is -1 or 0 .

The minimal number of derivatives of components of a flat output, needed to express x and u , will be called the differential weight of that flat output and is formalized as follows. By definition, for any flat output φ of Ξ there exist integers s_1, \dots, s_m

such that

$$\begin{aligned} x &= \gamma(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(s_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(s_m)}) \\ u &= \delta(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(s_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(s_m)}). \end{aligned}$$

Moreover, we can choose (s_1, \dots, s_m) , γ and δ such that (see [26]) if for any other m -tuple $(\tilde{s}_1, \dots, \tilde{s}_m)$ and functions $\tilde{\gamma}$ and $\tilde{\delta}$, we have

$$\begin{aligned} x &= \tilde{\gamma}(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(\tilde{s}_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(\tilde{s}_m)}) \\ u &= \tilde{\delta}(\varphi_1, \dot{\varphi}_1, \dots, \varphi_1^{(\tilde{s}_1)}, \dots, \varphi_m, \dot{\varphi}_m, \dots, \varphi_m^{(\tilde{s}_m)}), \end{aligned}$$

then $s_i \leq \tilde{s}_i$, for $1 \leq i \leq m$. We will call $\sum_{i=1}^m (s_i + 1) = m + \sum_{i=1}^m s_i$ the differential weight of φ . A flat output of Ξ is called *minimal* if its differential weight is the lowest among all flat outputs of Ξ . We define the *differential weight* of a flat system to be equal to the differential weight of a minimal flat output.

The differential weight is $n+m+p$, where $p \geq 0$, can be interpreted as the minimal dimension of a precompensator that dynamically linearizes the system. Indeed, $p = 0$ corresponds to static feedback linearizable systems (see Theorem 2.2 below) and the case $p = 1$ is the subject of this paper.

Consider a control-affine system

$$(2.1) \quad \Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x),$$

where f and g_1, \dots, g_m are smooth vector fields on X . The system Σ is linearizable by static feedback if it is equivalent via a diffeomorphism $z = \phi(x)$ and an invertible static feedback transformation, $u = \alpha(x) + \beta(x)v$, to a linear controllable system $\Lambda : \dot{z} = Az + Bv$.

The problem of static feedback linearization was solved by Brockett [3] (for a smaller class of transformations) and then by Jakubczyk and Respondek [15] and, independently, by Hunt and Su [12], who gave geometric necessary and sufficient conditions. The following theorem recalls their result and, furthermore, gives an equivalent way of describing static feedback linearizable systems from the point of view of differential weight.

Define inductively the sequence of distributions $\mathcal{D}^{i+1} = \mathcal{D}^i + [f, \mathcal{D}^i]$, where \mathcal{D}^0 is given by $\mathcal{D}^0 = \text{span}\{g_1, \dots, g_m\}$ and denote $[f, \mathcal{D}^i] = \{[f, \xi] : \xi \in \mathcal{D}^i\}$.

THEOREM 2.2. *The following conditions are equivalent:*

- (FL1) Σ is locally static feedback linearizable, around $x_0 \in X$;
- (FL2) Σ is locally static feedback equivalent, around $x_0 \in X$, to the Brunovský canonical form

$$(Br) : \begin{cases} \dot{z}_i^j &= z_i^{j+1} \\ \dot{z}_i^{\rho_i} &= v_i \end{cases}$$

where $1 \leq i \leq m$, $1 \leq j \leq \rho_i - 1$, and $\sum_{i=1}^m \rho_i = n$;

- (FL3) For any $q \geq 0$, the distributions \mathcal{D}^q are of constant rank, around $x_0 \in X$, involutive, and $\mathcal{D}^{n-1} = TX$;
- (FL4) Σ is flat at $x_0 \in X$, of differential weight $n + m$.

The geometry of static feedback linearizable systems is given by the following sequence of nested involutive distributions:

$$\mathcal{D}^0 \subset \mathcal{D}^1 \subset \dots \subset \mathcal{D}^{n-1} = TX.$$

It is well known that a feedback linearizable system is static feedback equivalent to the Brunovsky canonical form, see [4], and is clearly flat with $\varphi = (\varphi_1, \dots, \varphi_m) = (z_1^1, \dots, z_m^1)$ being a minimal flat output (of differential weight $n + m$). Therefore, for static feedback linearizable systems, the representation of all states and controls uses the minimal possible, which is $n + m$, number of time-derivatives of φ_i and an equivalent way of describing them is that they are flat systems of differential weight $n + m$.

In general, a flat system is not linearizable by static feedback, with the exception of the single-input case. Any single input-system is flat if and only if it is static feedback linearizable, see [7, 24], and thus of differential weight $n + 1$. Flat systems can be seen as a generalization of static feedback linearizable systems. Namely they are linearizable via dynamic, invertible and endogenous feedback, see [9, 8, 17, 24]. Our goal is thus to describe the simplest flat systems that are not static feedback linearizable: control-affine systems that become static feedback linearizable after an invertible one-fold prolongation, which is the simplest dynamic feedback. They are flat systems of differential weight $n + m + 1$, see Proposition 3.1 below. In this paper, we will completely characterize them (actually, almost completely, since for two particular sub-cases we do not provide verifiable conditions, see Section 3.2) and show how their geometry differs and how it reminds that given by the involutive distributions \mathcal{D}^i for static feedback linearizable systems.

3. Main results. Throughout, we make the following assumption:

(Assumption 1) From now on, unless stated otherwise, we assume that all ranks involved are constant in a neighborhood of a given $x_0 \in X$ (or $(x_0, \bar{u}_0^l) \in X \times U^l$). All results presented here are valid on an open and dense subset of either X or $X \times U$ (or $X \times U^l$) and hold locally, around any given point of that set.

REMARK 1. *The studied systems may display ranks that are not constant around some points, but the results presented here are valid only around points where the ranks are constant. We discuss in Section 5 why (Assumption 1) is important, for which results it is necessary and for which it can be weakened or neglected.*

PROPOSITION 3.1.

Consider a control system $\Xi : \dot{x} = F(x, u)$. The following conditions are equivalent:

- (i) Ξ is flat at (x_0, \bar{u}_0^l) , of differential weight $n + m + 1$, for a certain $l \geq -1$;
- (ii) Ξ is x -flat at x_0 or (x_0, u_0) , of differential weight $n + m + 1$;
- (iii) There exists, around x_0 , an invertible static feedback transformation $u = \psi(x, \tilde{u})$ bringing the system Ξ into $\tilde{\Xi} : \dot{x} = \tilde{F}(x, \tilde{u}) = F(x, \psi(x, \tilde{u}))$, such that the prolongation

$$\tilde{\Xi}^{(1,0,\dots,0)} : \begin{cases} \dot{x} &= \tilde{F}(x, y_1, v_2, \dots, v_m) \\ \dot{y}_1 &= v_1 \end{cases}$$

is locally static feedback linearizable, with $y_1 = \tilde{u}_1$, $v_i = \tilde{u}_i$, for $2 \leq i \leq m$.

Moreover, if Ξ is a control-affine system of the form $\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$, then the equivalences (i) \iff (ii) \iff (iii) hold with the general feedback $u = \psi(x, \tilde{u})$ being replaced by $u = \psi(x, \tilde{u}) = \alpha(x) + \beta(x)\tilde{u}$, the system Ξ by $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \sum_{i=1}^m \tilde{u}_i \tilde{g}_i(x)$ and the prolongation $\tilde{\Xi}^{(1,0,\dots,0)}$ by

$$\tilde{\Sigma}^{(1,0,\dots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^m v_i \tilde{g}_i(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = \tilde{u}_1$, $v_i = \tilde{u}_i$, for $2 \leq i \leq m$, $\tilde{f} = f + \alpha g$ and $\tilde{g} = g\beta$, where $g = (g_1, \dots, g_m)$ and $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_m)$.

The proofs of Proposition 3.1 and of all theorems of this section are given in Section 7.

A system Ξ satisfying (iii) is called dynamically linearizable via invertible one-fold prolongation. Notice that $\tilde{\Xi}^{(1,0,\dots,0)}$ is, as indicated by the notation, obtained by prolonging the control \tilde{u}_1 as $v_1 = \dot{\tilde{u}}_1$ and keeping $v_i = \tilde{u}_i$, for $2 \leq i \leq m$. The above result asserts that for systems of differential weight $n + m + 1$, flatness and x -flatness coincide and that, moreover, these properties are equivalent to linearizability via the simplest dynamic feedback, namely invertible one-fold preintegration.

To simplify the exposition of the paper, from now on, we will consider the control-affine case only. The generalization for the control-nonlinear systems is straightforward.

Before giving our main theorems, let us introduce the notion of corank that will be frequently used in the rest of the paper.

REMARK 2. If $\mathcal{A} \subset \mathcal{B}$, the corank of the inclusion $\mathcal{A} \subset \mathcal{B}$, denoted by $\text{cork}(\mathcal{A} \subset \mathcal{B})$, equals the rank of the quotient \mathcal{B}/\mathcal{A} , i.e., $\text{cork}(\mathcal{A} \subset \mathcal{B}) = \text{rk}(\mathcal{B}/\mathcal{A})$. Let \mathcal{A} and \mathcal{B} be two distributions of constant rank and f a vector field. Denote $[\mathcal{A}, \mathcal{B}] = \{[a, b] : a \in \mathcal{A}, b \in \mathcal{B}\}$ and $[f, \mathcal{B}] = \{[f, b] : b \in \mathcal{B}\}$. Clearly, \mathcal{A} and \mathcal{B} are subdistributions of $[\mathcal{A}, \mathcal{B}]$ because we take all $a \in \mathcal{A}$ and all $b \in \mathcal{B}$ and not just generators.

From now on, we deal only with systems that are not static feedback linearizable. Therefore one of the distributions \mathcal{D}^i fails to satisfy condition (FL3) of Theorem 2.2. Flat systems are always accessible so $\mathcal{D}^{n-1} = TX$ holds and all distributions \mathcal{D}^i are supposed to be of constant rank, see (Assumption 1) above. So there exists an integer k such that \mathcal{D}^k is not involutive. Suppose that k is the smallest integer satisfying that property. The integer k plays a fundamental role in our study.

Our main result describing flat systems of differential weight $n + m + 1$ is given by the two following theorems corresponding to the first noninvolutive distribution \mathcal{D}^k being either \mathcal{D}^0 , i.e., $k = 0$ (Theorem 3.3) or \mathcal{D}^k , for $k \geq 1$ (Theorem 3.2). These two cases have slightly different geometries, but we are able to merge them into one general result, Theorem 3.4, whose conditions although compact are less readable and do not allow the reader to see the differences between flat systems of differential weight $n + m + 1$ with $k \geq 1$ and those with $k = 0$. Namely, if $k \geq 1$ we never face singularities in the control space (i.e., even if the minimal flat outputs are defined locally around a given x_0 , they are always global with respect to the control). This is no longer the case if $k = 0$ and the prolongation always creates singularities in the control space. In order to highlight these differences, we start by presenting the two cases separately.

THEOREM 3.2. Assume $k \geq 1$. The control system Σ given by (2.1), is flat at x_0 , of differential weight $n + m + 1$, if and only if it satisfies around x_0 :

- (A1) There exists an involutive distribution \mathcal{H}^k verifying $\mathcal{D}^{k-1} \subset \mathcal{H}^k \subset \mathcal{D}^k$, with the second inclusion of corank one;
(A2) The distributions \mathcal{H}^i , for $i \geq k+1$, are involutive, where $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$;
(A3) There exists ρ such that $\mathcal{H}^\rho = TX$.

The distributions $\mathcal{D}^0, \dots, \mathcal{D}^k$ are feedback invariant and, if \mathcal{H}^k exists, so are $\mathcal{H}^k, \dots, \mathcal{H}^\rho$. Therefore the geometry of systems described by the previous theorem can be summarized by the following sequence of inclusions:

$$\begin{array}{ccccccc} \mathcal{D}^0 \subset \dots \subset \mathcal{D}^{k-1} & \subset & \mathcal{D}^k & \subset & \overline{\mathcal{D}}^k & & \\ & \subset & \cup & & \cap & & \\ & & \mathcal{H}^k & \subset & \mathcal{H}^{k+1} & \subset \dots \subset \mathcal{H}^\rho = TX & \end{array}$$

where all distributions, except \mathcal{D}^k , are involutive, $\overline{\mathcal{D}}^k$ is the involutive closure of \mathcal{D}^k and the inclusion $\mathcal{H}^k \subset \mathcal{D}^k$ is of corank one. The main structural condition is the existence of a corank one involutive subdistribution \mathcal{H}^k in \mathcal{D}^k containing \mathcal{D}^{k-1} . We will discuss in Section 3.2, the uniqueness of \mathcal{H}^k and provide its construction. The inclusion $\mathcal{D}^{k-1} \subset \mathcal{H}^k$ yields $\mathcal{D}^k \subset \mathcal{H}^{k+1}$ which gives $\overline{\mathcal{D}}^k \subset \mathcal{H}^{k+1}$ (since \mathcal{H}^{k+1} is involutive by (A2)). Notice also that the inclusion $\mathcal{D}^{k-1} \subset \mathcal{H}^k$ is of corank at least one. Otherwise, $\mathcal{D}^{k-1} = \mathcal{H}^k$ which would imply $\mathcal{D}^k = \mathcal{H}^{k+1}$ and thus \mathcal{D}^k would be involutive. It is clear that in the particular case $\overline{\mathcal{D}}^k = TX$, we have $\rho = k + 1$.

If $k = 0$, i.e., the first noninvolutive distribution is \mathcal{D}^0 , then a similar result holds but in the chain of involutive subdistributions $\mathcal{H}^0 \subset \mathcal{H}^1 \subset \mathcal{H}^2 \subset \dots$ (playing the role of $\mathcal{H}^k \subset \mathcal{H}^{k+1} \subset \mathcal{H}^{k+2} \subset \dots$), the distribution \mathcal{H}^1 is not defined as $\mathcal{H}^1 = \mathcal{H}^0 + [f, \mathcal{H}^0]$, but as $\mathcal{H}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] + [f, \mathcal{H}^0]$, (compare (A2) and (A2)') and satisfies an additional condition (A4)' which as we will see plays a double role (of a nonsingularity condition and of a structural condition). In fact, flat systems with $k = 0$ may exhibit singularities in the control space (created by one-fold prolongation of the to-be-prolonged control) defined by

$$U_{sing}(x) = \{u(x) \in \mathbb{R}^m : \text{rk}(\mathcal{D}^0 + [f + \sum_{i=1}^m u_i g_i, \mathcal{H}^0])(x) < \text{rk} \mathcal{H}^1(x)\}$$

and excluded by (A4)'. To describe the singular controls, apply an invertible feedback $u = \beta \tilde{u}$ such that $\mathcal{H}^0 = \text{span}\{h_2, \dots, h_m\}$ and $\mathcal{D}^0 = \text{span}\{\tilde{g}_1, h_2, \dots, h_m\}$, where $(\tilde{g}_1, h_2, \dots, h_m) = (g_1, g_2, \dots, g_m)\beta$. Denote $\text{rk} \mathcal{H}^1 = r + m$ and for any $2 \leq i_1 < \dots < i_r \leq m$, put

$$c_{i_1, \dots, i_r} = (ad_f h_{i_1} + \tilde{u}_1 [\tilde{g}_1, h_{i_1}]) \wedge \dots \wedge (ad_f h_{i_r} + \tilde{u}_1 [\tilde{g}_1, h_{i_r}]) \wedge \tilde{g}_1 \wedge h_2 \wedge \dots \wedge h_m.$$

Then the set U_{sing} can be written as:

$$\begin{aligned} U_{sing}(x) &= \{\tilde{u}(x) \in \mathbb{R}^m : \text{rk span}\{\tilde{g}_1, h_j, [f + \tilde{u}_1 \tilde{g}_1, h_j], 2 \leq j \leq m\}(x) < \text{rk} \mathcal{H}^1(x)\} \\ &= \{\tilde{u}(x) \in \mathbb{R}^m : c_{i_1, \dots, i_r}(x, \tilde{u}_1) = 0, 2 \leq i_1 < \dots < i_r \leq m\}, \end{aligned}$$

where $\tilde{u}(x) = (\tilde{u}_1(x), \tilde{u}_2(x), \dots, \tilde{u}_m(x))$ and thus $\tilde{u}_2(x), \dots, \tilde{u}_m(x)$ are arbitrary. Now it is clear that for any $x \in X$, the singular set $U_{sing}(x)$ is an algebraic (empty or not) subset of \mathbb{R}^m . In the particular case of $\text{rk} \mathcal{H}^1 = 2m - 1$, i.e, $\text{cork}(\mathcal{D}^0 \subset \mathcal{H}^1) = r = m - 1$ is maximal possible, the singular set $U_{sing}(x)$ is the zero level set of the polynomial $c_{2, \dots, m}(x, \tilde{u}_1)$ of degree $m - 1$ with respect to \tilde{u}_1 . So for each fixed $x \in X$, the singular controls $U_{sing}(x)$ form the union of μ affine hyperplanes in \mathbb{R}^m , where $0 \leq \mu \leq m - 1$ is the number of distinct real roots of the polynomial $c_{2, \dots, m}(x, \tilde{u}_1)$. It follows that

if m is even, then the one-fold prolongation always creates singularities in the control space (in the particular case of two controls $m = 2$, the singular set is an affine line in \mathbb{R}^2 , see [21]) but if m is odd, then singularities of control may be absent, see normal forms in Subsection 6.1.

Before giving our result, we introduce $\mathcal{H}_u^1 = \mathcal{D}^0 + [f + \sum_{i=1}^m u_i g_i, \mathcal{H}^0]$, which, for each fixed $u \in \mathbb{R}^m$, is a distribution defined on X .

THEOREM 3.3. *Assume $k = 0$. The system Σ given by (2.1), is flat at (x_0, u_0) , of differential weight $n + m + 1$, if and only if it satisfies around (x_0, u_0) :*

- (A1)' *There exists an involutive distribution $\mathcal{H}^0 \subset \mathcal{D}^0$, of corank one;*
- (A2)' *The distributions \mathcal{H}^i , for $i \geq 1$, are involutive, where $\mathcal{H}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] + [f, \mathcal{H}^0]$ and $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$, for $i \geq 2$;*
- (A3)' *There exists ρ such that $\mathcal{H}^\rho = TX$;*
- (A4)' *$\mathcal{H}_u^1(x) = \mathcal{H}^1(x)$, for any (x, u) in a neighborhood of (x_0, u_0) .*

Like in Theorem 3.2, if $\overline{\mathcal{D}}^0 = TX$, then $\rho = 1$. Notice that the condition (A4)' plays a double role. First, it is a structural condition since it assures that for all values of $u \in \mathbb{R}^m$, we obtain the same distribution $\mathcal{H}_u^1 = \mathcal{H}^1$, in other words, \mathcal{H}_u^1 does not depend on u . Now recall that since $k = 0$, the distribution $\mathcal{D}^0 = \text{span}\{\tilde{g}_1, h_j, 2 \leq j \leq m\}$ (where we used the notations introduced just before Theorem 3.3) is noninvolutive, thus the rank of $\text{span}\{\tilde{g}_1, h_j, [f + u_1 \tilde{g}_1 + \sum_{i=2}^m u_i h_i, h_j], 2 \leq j \leq m\}(x)$ could a priori drop at $u = u_0$. From (A4)', it is immediate that $u_0 \notin U_{\text{sing}}(x_0)$, where $U_{\text{sing}}(x_0) = \{u \in \mathbb{R}^m : \text{rk}(\text{span}\{g_1, h_j, [f + u_1 g_1 + \sum_{i=2}^m u_i h_i, h_j], 2 \leq j \leq m\})(x_0) < \text{rk } \mathcal{H}^1(x_0)\}$. Hence (A4)' is also a regularity condition, since it excludes the singular controls, that is, the controls u for which $\text{rk } \mathcal{H}_u^1(x_0) < \text{rk } \mathcal{H}^1(x_0)$.

The cases $k = 0$ and $k \geq 1$ are similar, but they have slightly different geometries. Even if at first sight, it seems not possible to merge them (because of different definitions of the distributions \mathcal{H}^1 and \mathcal{H}^{k+1} and of a possible existence of singularities in the control space for $k = 0$), the following result enables us to unify them. Theorem 3.4 is based on the observation that in both cases, we actually have $\mathcal{H}^{k+1} = \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] + [f, \mathcal{H}^k]$ (by definition of \mathcal{H}^1 , for $k = 0$, and as a direct consequence of the definition of \mathcal{H}^{k+1} , for $k \geq 1$, see the comments following Theorem 3.2).

Similarly to $\mathcal{H}_u^1(x)$, defined before the Theorem 3.3, for u fixed in \mathbb{R}^m , we consider the distribution $\mathcal{H}_u^{k+1} = \mathcal{D}^k + [f + \sum_{i=1}^m u_i g_i, \mathcal{H}^k]$, defined on X . We have thus introduced a family of distributions defined on X and parameterized by u .

THEOREM 3.4. *The system Σ , given by (2.1), is flat either at x_0 , if $k \geq 1$, or at (x_0, u_0) , if $k = 0$, of differential weight $n + m + 1$, if and only if it satisfies around (x_0, u_0) :*

- (A1)'' *There exists an involutive distribution \mathcal{H}^k verifying $\mathcal{D}^{k-1} \subset \mathcal{H}^k \subset \mathcal{D}^k$, with the second inclusion of corank one (and where $\mathcal{D}^{-1} = \{0\}$, if $k = 0$);*
- (A2)'' *The distributions \mathcal{H}^i , for $i \geq k + 1$, are involutive, where $\mathcal{H}^{k+1} = \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] + [f, \mathcal{H}^k]$ and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, for $i \geq k + 1$;*
- (A3)'' *There exists ρ such that $\mathcal{H}^\rho = TX$;*
- (A4)'' *$(\mathcal{D}^k + [f + \sum_{i=1}^m u_i g_i, \mathcal{H}^k])(x, u) = \mathcal{H}^{k+1}(x)$.*

Notice that taking in the above theorem $k = 0$ gives Theorem 3.3. Observe also that the role of condition (A4)'' changes with k . For $k = 0$, it immediately gives that the distribution $\mathcal{H}_u^1 = \mathcal{D}^0 + [f + \sum_{i=1}^m u_i g_i, \mathcal{H}^0]$ does not depend on u and that $u_0 \notin U_{\text{sing}}(x_0)$, excluding singularities in the control space. So, in this case, (A4)'' plays the role of both a structural and a regularity condition. If $k \geq 1$, it can be easily

shown that $\mathcal{H}_u^{k+1} = \mathcal{D}^k + [f + \sum_{i=1}^m u_i g_i, \mathcal{H}^k]$ does not depend on the control u and that (A4)'' can actually be written as $\mathcal{D}^k + [f, \mathcal{H}^k] = \mathcal{H}^{k+1}$. So, for $k \geq 1$, item (A4)'' is only a structural condition assuring that, modulo \mathcal{D}^k , the directions in $[\mathcal{D}^k, \mathcal{D}^k]$ that are not in \mathcal{D}^k are in fact in $[f, \mathcal{H}^k]$, implying that $\bar{\mathcal{D}}^k \subset \mathcal{H}^{k+1} = \mathcal{H}^k + [f, \mathcal{H}^k]$.

The proofs of Theorems 3.2 and 3.3 are given in Section 7, whereas Theorem 3.4 is their direct consequence. The crucial problem of constructing the involutive subdistribution $\mathcal{H}^k \subset \mathcal{D}^k$ will be treated in Section 3.2.

The previous theorems enable us to define the control u_p (which is given up to a multiplicative function) to be prolonged in order to obtain the locally static feedback linearizable $\tilde{\Sigma}^{(1,0,\dots,0)}$. In the next section, we will explain the construction of u_p .

3.1. To-be-prolonged control. We will construct in this section the control u_p to be prolonged (preintegrated) in order to dynamically linearize the system.

According to the following lemma (that we will prove as a part of Proposition 7.2(ii) in Section 7), to the involutive subdistribution \mathcal{H}^k , where $k \geq 1$, we can associate a unique corank one subdistribution \mathcal{H} in \mathcal{D}^0 that plays a crucial role in defining the to-be-prolonged control. If $k = 0$, we simply put $\mathcal{H} = \mathcal{H}^0$.

LEMMA 3.5. *Assume $k \geq 1$ and suppose that \mathcal{D}^k contains an involutive subdistribution \mathcal{H}^k , of corank one satisfying $\mathcal{D}^{k-1} \subset \mathcal{H}^k$. Then there exists a distribution \mathcal{H} , uniquely associated to \mathcal{H}^k , such that $\mathcal{H} \subset \mathcal{D}^0$ is of corank one and $\mathcal{H}^k = \mathcal{D}^{k-1} + \text{ad}_f^k \mathcal{H}$. Moreover, all distributions $\mathcal{H}^i = \mathcal{D}^{i-1} + \text{ad}_f^i \mathcal{H}$, for $0 \leq i \leq k-1$, where $\mathcal{D}^{-1} = \{0\}$ and $\mathcal{H}^0 = \mathcal{H}$, are involutive and are feedback invariant, that is, do not change if we replace f by $f + \sum_{i=1}^m \alpha_i g_i$.*

Since $\text{rk } \mathcal{H} = m-1$, we can find m functions β_1, \dots, β_m (not vanishing simultaneously) such that $u_1(x)\beta_1(x) + \dots + u_m(x)\beta_m(x) = 0$ if and only if $\sum_{i=1}^m u_i(x)g_i(x) \in \mathcal{H}(x)$. The to-be-prolonged control u_p (becoming \tilde{u}_1 after feedback) that needs to be preintegrated in order to dynamically linearize the system is

$$u_p = \tilde{u}_1 = u_1(x)\beta_1(x) + \dots + u_m(x)\beta_m(x)$$

and we put $v_1 = \frac{d}{dt}u_p = \frac{d}{dt}\tilde{u}_1$. Therefore u_p is not unique and given up to multiplication by a non-vanishing function. Indeed, if u_p is a to-be-prolonged control, then so is $\tilde{u}_p = u_1(x)\tilde{\beta}_1(x) + \dots + u_m(x)\tilde{\beta}_m(x)$, where $\tilde{\beta}_i = \gamma\beta_i$ and $\gamma(x) \neq 0$. What is thus canonical is not a to-be-prolonged control $u_p = \tilde{u}_1 = u_1(x)\beta_1(x) + \dots + u_m(x)\beta_m(x)$, or the \mathbb{R}^m -valued vector function $(\beta_1(x), \dots, \beta_m(x))$ defining it, but, respectively, the collection of the to-be-prolonged controls $\gamma(x)u_p$ and the field of lines $[\beta_1(x) : \beta_2(x) : \dots : \beta_m(x)]$ in \mathbb{R}^m , where the latter denotes projective coordinates in \mathbb{R}^m .

Finding u_p requires knowing β_1, \dots, β_m , which in turn is reduced to calculating \mathcal{H} and, finally, to constructing the involutive subdistribution \mathcal{H}^k . The latter problem is solved in the next section.

3.2. Verification of the conditions. Theorems 3.2 and Theorems 3.3 (stated together as Theorem 3.4) describe all flat systems of differential weight $n + m + 1$. In order to verify their conditions, we have to check whether the distribution \mathcal{D}^k (respectively \mathcal{D}^0) contains an involutive subdistribution \mathcal{H}^k (respectively \mathcal{H}^0) of corank one. We will see that the corank r of the inclusion $\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]$ plays an important role in conditions verifications. Recall that, according to the Remark 2, cork $(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k])$ simply means $\text{rk}(\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k])/\mathcal{D}^k$.

In fact, if $r \geq 2$, then the existence of \mathcal{H}^k (and its construction, if it exists) is given by Proposition 3.6 below. Also in the case $r = 1$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, the

involutive subdistribution \mathcal{H}^k can be uniquely identified as $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$, where \mathcal{C}^k is the characteristic distribution of \mathcal{D}^k and is defined below. For both cases, we thus get verifiable necessary and sufficient conditions for flatness of differential weight $n + m + 1$, stated as Theorem 3.7. If $r = 1$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, we will introduce a new index l . If l exists, then it takes the role of k and leads to checkable conditions given by Theorem 3.8. If l does not exist, then still three sub-cases are possible. For the first one, we provide verifiable conditions and for the second, we are not able to distinguish between flatness of differential weight $n + m + 1$ and $n + m + 2$, respectively in Theorem 3.9 (vi) and (vii). For the third sub-case (defined at the end of this Section), we are not able to give verifiable conditions for flatness of differential weight $n + m + 1$.

Consider a distribution \mathcal{D} of rank d , defined on a manifold X of dimension n and define its annihilator $\mathcal{D}^\perp = \{\omega \in \Lambda^1(X) : \langle \omega, f \rangle = 0, \forall f \in \mathcal{D}\}$, where $\Lambda^1(X)$ is the space of smooth differentials 1-forms on X . Let $\text{cork}(\mathcal{D} \subset \mathcal{D} + [\mathcal{D}, \mathcal{D}]) = r$ and let $\omega_1, \dots, \omega_r, \omega_{r+1}, \dots, \omega_s$, where $s = n - d$, be differential 1-forms such that locally $\mathcal{D}^\perp = \text{span}\{\omega_1, \dots, \omega_s\}$ and $(\mathcal{D} + [\mathcal{D}, \mathcal{D}])^\perp = \text{span}\{\omega_{r+1}, \dots, \omega_s\}$.

The *Engel rank* of \mathcal{D} equals 1 at x if and only if \mathcal{D} is non involutive and $(d\omega_i \wedge d\omega_j)(x) = 0 \text{ mod } \mathcal{D}^\perp$, for any $1 \leq i, j \leq s$. For any $\omega \in \mathcal{D}^\perp$, we define $\mathcal{W}(\omega) = \{f \in \mathcal{D} : f \lrcorner \omega \in \mathcal{D}^\perp\}$, where \lrcorner is the interior product. The characteristic distribution $\mathcal{C} = \{f \in \mathcal{D} : [f, \mathcal{D}] \subset \mathcal{D}\}$ of \mathcal{D} is given by

$$\mathcal{C} = \bigcap_{i=1}^s \mathcal{W}(\omega_i).$$

It follows directly from the Jacobi identity that the characteristic distribution is always involutive. Define the distribution

$$\mathcal{B} = \sum_{i=1}^r \mathcal{W}(\omega_i).$$

Although the distributions $\mathcal{W}(\omega_i)$ depend on the choice of ω_i 's, the distribution \mathcal{B} does not and we have the following result [22] based on [5].

PROPOSITION 3.6. *Consider a distribution \mathcal{D} of rank d and let $\text{cork}(\mathcal{D} \subset \mathcal{D} + [\mathcal{D}, \mathcal{D}]) = r$.*

- (i) *Assume $r \geq 3$. The distribution \mathcal{D} contains an involutive subdistribution \mathcal{H} of corank one if and only if it satisfies*
 - (ISD1) *The Engel rank of \mathcal{D} equals one;*
 - (ISD2) *The characteristic distribution \mathcal{C} of \mathcal{D} has rank $d - r - 1$.**Moreover, that involutive subdistribution is unique and is given by $\mathcal{H} = \mathcal{B}$.*
- (ii) *Assume $r = 2$. The distribution \mathcal{D} contains a corank one involutive subdistribution \mathcal{H} if and only if \mathcal{D} verifies (ISD1)-(ISD2) and the distribution \mathcal{B} is involutive. Then \mathcal{H} is unique and given by $\mathcal{H} = \mathcal{B}$.*
- (iii) *Assume $r = 1$. The distribution \mathcal{D} contains an involutive subdistribution of corank one \mathcal{H} if and only if it satisfies the condition (ISD2). In the case $r = 1$, if an involutive subdistribution of corank one \mathcal{H} exists, it is never unique.*

The above conditions are easy to check and a unique involutive subdistribution of corank one can be constructed if $r \geq 2$, i.e., $\text{cork}(\mathcal{D} \subset \mathcal{D} + [\mathcal{D}, \mathcal{D}]) \geq 2$.

Therefore, we can check (verifying (ISD1)-(ISD2) for $\mathcal{D} = \mathcal{D}^k$ and, only if $r \geq 2$, the involutivity of \mathcal{B}) whether an involutive subdistribution \mathcal{H}^k of corank one in \mathcal{D}^k exists and if it exists, then it is unique and can be explicitly calculated. As a consequence, for any given control-affine system satisfying $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$,

the conditions of Theorems 3.2, 3.3 and 3.4 are verifiable and we can thus check whether the system is flat with the differential weight $n + m + 1$. Moreover, the verification involves differentiation and algebraic operations only, without solving PDE's or bringing the system into a normal form. Moreover, under the same assumption, it can be shown (via the Jacobi identity, see Proposition 7.2 in Section 7.1) that if \mathcal{D}^k contains an involutive subdistribution \mathcal{H}^k of corank one, then \mathcal{H}^k satisfies the following inclusion $\mathcal{D}^{k-1} \subset \mathcal{H}^k$ and we no longer have to suppose it in the statement of the theorems.

Let us now consider the case $r = 1$, that is, $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) = 1$. In that case, according to Proposition 3.6(iii), if an involutive subdistribution \mathcal{H}^k of corank one of \mathcal{D}^k exists, then it is never unique. It is easy to see that not all choices of an involutive subdistribution \mathcal{H}^k lead to dynamically feedback linearizable systems via invertible one-fold prolongation. A natural question arises: how to identify the "right" subdistribution \mathcal{H}^k (that is, the subdistribution \mathcal{H}^k that leads to a static feedback linearizable prolongation) in the case $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) = 1$?

The involutivity of \mathcal{D}^k can be lost in two different ways: either $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ (which makes sense only if $k \geq 1$) or $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and there exist $1 \leq i, j \leq m$ such that $[ad_f^k g_i, ad_f^k g_j] \notin \mathcal{D}^k$. As asserts Theorem 3.7 (ii) below, in the case $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, the corank one involutive subdistribution \mathcal{H}^k can be uniquely identified by another argument. Namely, $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$, where \mathcal{C}^k is the characteristic distribution (defined above) of \mathcal{D}^k . The subdistribution \mathcal{H}^k has to verify some additional conditions analogous to those of Theorem 3.2.

THEOREM 3.7. *Assume $k \geq 0$ and consider the control system Σ , given by (2.1).*

- (i) *Suppose that $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$. The system Σ is flat at x_0 (at (x_0, u_0) , if $k = 0$) of differential weight $n+m+1$, if and only if \mathcal{D}^k satisfies either item (i) or item (ii) of Proposition 3.6 and its unique involutive subdistribution \mathcal{H}^k , given by that proposition, satisfies the conditions (A1)'' – (A4)'' of Theorem 3.4 (or equivalently, satisfies the conditions (A1) – (A3) of Theorem 3.2, if $k \geq 1$, or the conditions (A1)' – (A4)' of Theorem 3.3, if $k = 0$).*
- (ii) *Suppose that $k \geq 1$, $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) = 1$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. The system Σ is flat at x_0 , of differential weight $n+m+1$, if and only if the following conditions are satisfied:*
 - (C1) $\text{rk } \mathcal{C}^k = \text{rk } \mathcal{D}^k - 2$, where \mathcal{C}^k is the characteristic distribution of \mathcal{D}^k ;
 - (C2) $\text{rk}(\mathcal{C}^k \cap \mathcal{D}^{k-1}) = \text{rk } \mathcal{D}^{k-1} - 1$;
 - (C3) *The distributions \mathcal{H}^i , for $i \geq k$, are involutive, where $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$ and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$;*
 - (C4) *There exists ρ such that $\mathcal{H}^\rho = TX$.*

For the case treated by item (ii) of the above theorem, the to-be-prolonged control is defined exactly as explained in Section 3.1. The proof of Theorem 3.7(i) is a direct consequence of Theorems 3.2 and 3.3, and of Proposition 3.6. We present the proof of Theorem 3.7(ii) in Section 7.

It can be shown that in the case $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ (no matter what is the value of $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k])$), the involutive subdistribution \mathcal{H}^k can always be defined as above, i.e., the computation of \mathcal{H}^k using the procedure given by Proposition 3.6 and that provided by conditions (C1) – (C3) of the above theorem are equivalent if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. This is not valid anymore if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$; indeed, in that case, we have $\mathcal{D}^{k-1} \subset \mathcal{C}^k$, the condition (C2) is not verified and (C3) would give $\mathcal{H}^k = \mathcal{C}^k$. Notice that in the case $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, the inclusion $\mathcal{C}^k \subset \mathcal{H}^k$ is always satisfied

(implying $\mathcal{D}^{k-1} \subset \mathcal{H}^k$) and is of corank one if additionally $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) = 1$, i.e., $\mathcal{H}^k = \mathcal{C}^k + \text{span}\{g\}$, where g is a vector field belonging to \mathcal{D}^k , but not to \mathcal{C}^k .

If $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) = 1$, $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and there exist $1 \leq i, j \leq m$ such that $[ad_f^k g_i, ad_f^k g_j] \notin \mathcal{D}^k$, any corank one involutive subdistribution \mathcal{H}^k may serve to define a control (different distributions yield different controls) whose prolongation gives a static feedback linearizable system. Thus, in order to verify flatness of differential weight $n + m + 1$, we have to construct a corank one involutive subdistribution \mathcal{H}^k (condition (A1)), check the involutivity of all distributions \mathcal{H}^{k+i} defined with the help of \mathcal{H}^k (condition (A2)) and the existence of ρ such that $\mathcal{H}^\rho = TX$ (condition (A3)). If (A2) – (A3) are satisfied for that choice of \mathcal{H}^k , then the system is flat of differential weight $n + m + 1$ and becomes static feedback linearizable after the prolongation of the control u_p associated to \mathcal{H}^k (see Section 3.1). If this is not the case, construct another corank one involutive subdistribution $\tilde{\mathcal{H}}^k$ and repeat the same procedure for $\tilde{\mathcal{H}}^k$. The problem that we are facing with this algorithm, is the definition of a simple criterion to decide when to stop, i.e., to conclude that the system is not flat of differential weight $n + m + 1$. We will explain next how we may overcome this difficulty and propose verifiable conditions for flatness of differential weight $n + m + 1$, for almost all subcases of the particular case $\text{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ (where $\mathcal{D}^{-1} = \{0\}$, if $k = 0$). To this end, let l denote the smallest integer such that either

$$(l\text{-cork}) \quad \text{cork}(\mathcal{D}^l \subset [\mathcal{D}^l, \mathcal{D}^l]) \geq 2$$

or

$$(l\text{-struct}) \quad \text{cork}(\mathcal{D}^l \subset [\mathcal{D}^l, \mathcal{D}^l]) = 1 \text{ and } [\mathcal{D}^{l-1}, \mathcal{D}^l] \not\subset \mathcal{D}^l.$$

Under our assumptions, if l exists, then $l \geq k + 1$ but it may not exist, in which case all distributions \mathcal{D}^i , for $i \geq k$, are either involutive or satisfy $\text{cork}(\mathcal{D}^i \subset [\mathcal{D}^i, \mathcal{D}^i]) = 1$ and $[\mathcal{D}^{i-1}, \mathcal{D}^i] \subset \mathcal{D}^i$. It can be shown that the distributions \mathcal{D}^i , for $k \leq i \leq l$, are in fact feedback invariant. If l exists, then we will use the distribution \mathcal{D}^l (instead of \mathcal{D}^k) to give the conditions for flatness of differential weight $n + m + 1$. The main idea is that, instead of constructing the subdistribution \mathcal{H}^k , we will uniquely identify \mathcal{H}^l and construct, with its help, the sequence of distributions \mathcal{H}^i , $i \geq 0$, see Theorem 3.8. The obtained conditions are similar to those of Theorems 3.4 and 3.7 but with the integer k being replaced by l .

THEOREM 3.8. *Consider the control system Σ , given by (2.1). Assume $k \geq 0$, $\text{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and the integer l exists.*

(iii) *Suppose that $k \geq 1$ and l satisfies (l-cork). The system Σ is flat at x_0 of differential weight $n + m + 1$, if and only if \mathcal{D}^l satisfies either item (i) or item (ii) of Proposition 3.6 and its unique involutive subdistribution \mathcal{H}^l , given by that proposition, fulfils the conditions (A1)'' – (A4)'' of Theorem 3.4 (or equivalently, satisfies the conditions (A1) – (A3) of Theorem 3.2) and additionally:*

(A5)'' *The distributions $\mathcal{H}^i = \mathcal{D}^{i-1} + ad_f^i \mathcal{H}$, for $k \leq i \leq l - 1$, are involutive, where \mathcal{H} is the corank one subdistribution of \mathcal{D}^0 , uniquely associated to \mathcal{H}^l , such that $\mathcal{H}^l = \mathcal{D}^{l-1} + ad_f^l \mathcal{H}$.*

(iv) *Suppose that $k \geq 1$ and l satisfies (l-struct). The system Σ is flat at x_0 of differential weight $n + m + 1$, if and only if \mathcal{D}^l satisfies the conditions (C1) – (C4) of Theorem 3.7 (ii) and, additionally, condition (A5)'', of the above item.*

(v) *Suppose that $k = 0$ and l satisfies (l-cork) (resp. (l-struct)). The system Σ is flat at (x_0, u_0) of differential weight $n + m + 1$, if and only if \mathcal{D}^l satisfies either*

item (i) or item (ii) of Proposition 3.6 (resp. the conditions (C1) – (C4) of Theorem 3.7 (ii)) and the corank one subdistribution \mathcal{H} of \mathcal{D}^0 , uniquely associated to \mathcal{H}^l by $\mathcal{H}^l = \mathcal{D}^{l-1} + \text{ad}_f^l \mathcal{H}$, satisfies the conditions (A1)'' – (A4)'' of Theorem 3.4, where \mathcal{H}^l is the involutive subdistribution of \mathcal{D}^l , given by Proposition 3.6 (respect. by condition (C3) of Theorem 3.7 (ii)).

If l does not exist, then all distributions \mathcal{D}^i , for $i \geq 0$, are feedback invariant and we will denote by \mathcal{C}^k the characteristic distribution of \mathcal{D}^k . Since $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, it follows immediately that \mathcal{D}^{k-1} is contained in \mathcal{C}^k . Moreover, it can be shown that

$$\text{rk } \mathcal{C}^k = \text{rk } \mathcal{D}^k - 2$$

is necessary for flatness of differential weight $n + m + 1$, we can thus assume that this relation holds for Theorem 3.9. Under that assumption, it can be proven that there exist vector fields $h_3, \dots, h_m \in \mathcal{D}^0$ such that

$$\mathcal{C}^k = \mathcal{D}^{k-1} + \text{span} \{ \text{ad}_f^k h_j, 3 \leq j \leq m \}.$$

We introduce the following sequence of distributions:

$$\mathcal{E}^0 = \text{span} \{ h_3, \dots, h_m \} \text{ and } \mathcal{E}^{i+1} = \mathcal{D}^i + \text{span} \{ \text{ad}_f^{i+1} h_j, 3 \leq j \leq m \}, \text{ for } i \geq 0.$$

Let s be the smallest integer such that \mathcal{E}^s is not involutive. Notice that the integer s may not exist, i.e., all distributions \mathcal{E}^i , for $i \geq 0$, are involutive, and in that case, we take $s = \infty$. It can be easily shown that all distributions \mathcal{E}^i , for $0 \leq i \leq s$, are in fact feedback invariant. If s exists and $[\mathcal{E}^s, \mathcal{E}^s] \neq TX$, the obtained conditions are given in terms of the distribution \mathcal{E}^s , see Theorem 3.9(vi), and remind very much those for two-input control systems that are flat of differential weight $n + 3$, see [21, 19].

If s does not exist (i.e., all distributions \mathcal{E}^i are involutive) and there exists ρ such that $\mathcal{E}^\rho = TX$, then the system actually becomes static feedback linearizable after two prolongations without any additional condition and thus, it is flat of differential weight at most $n + m + 2$. Indeed, apply an invertible static feedback to bring the system Σ into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \tilde{u}_2 \tilde{g}_2(x) + \sum_{i=3}^m \tilde{u}_i h_i$. Then the prolonged system

$$\tilde{\Sigma}^{(1,1,0,\dots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + y_2 \tilde{g}_2(x) + \sum_{i=3}^m y_i h_i \\ \dot{y}_1 &= v_1 \\ \dot{y}_2 &= v_2 \end{cases}$$

where $y_i = \tilde{u}_i$, for $1 \leq i \leq 2$, and $v_j = \tilde{u}_j$, for $3 \leq j \leq m$, is static feedback linearizable, since all its linearizability distributions are of the form $\mathcal{D}_p^i = \text{span} \left\{ \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right\} + \mathcal{E}^i$ and thus involutive.

THEOREM 3.9. *Consider the control system Σ , given by (2.1). Assume that $k \geq 0$, $\text{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, $\text{cork}(\mathcal{C}^k \subset \mathcal{D}^k) = 2$ and the integer l does not exist.*

- (vi) *Suppose that $k \geq 0$, the integer s exists and $[\mathcal{E}^s, \mathcal{E}^s] \neq TX$. The system Σ is flat at x_0 (at (x_0, u_0) , if $k = 0$) of differential weight $n + m + 1$, if and only if*
- (D1) $\text{rk } \bar{\mathcal{E}}^s = \text{rk } \mathcal{E}^s + 1$, where $\bar{\mathcal{E}}^s$ is the involutive closure of \mathcal{E}^s ;
 - (D2) $\text{rk}(\bar{\mathcal{E}}^s + \mathcal{D}^s) = \text{rk } \mathcal{D}^s = \text{rk } \bar{\mathcal{E}}^s + 1$, implying the existence of a vector field $h_2 \in \mathcal{D}^0$ such that $h_2 \notin \mathcal{E}^0$ and $\text{ad}_f^s h_2 \in \bar{\mathcal{E}}^s$;

- (D3) The distributions \mathcal{H}^i , for $i \geq s-1$, are involutive, where $\mathcal{H}^{s-1} = \mathcal{E}^{s-1} + \text{span}\{ad_f^{s-1}h_2\}$ and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, for $i \geq s-1$, if $s \geq 1$, (resp., the distributions \mathcal{H}^i , for $i \geq 0$, are involutive, where $\mathcal{H}^0 = \mathcal{E}^0 + \text{span}\{h_2\}$, $\mathcal{H}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] + [f, \mathcal{H}^0]$ and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, for $i \geq 1$, if $s = 0$);
- (D4) There exists ρ such that $\mathcal{H}^\rho = TX$;
- (D5) $\mathcal{H}_u^1(x) = \mathcal{H}^1(x)$, around (x_0, u_0) , where $\mathcal{H}_u^1(x) = \mathcal{D}^0 + [f + \sum_{i=1}^m u_i g_i, \mathcal{H}^0]$, if $k = 0$.

(vii) Suppose that $k \geq 0$ and the integer s does not exist. The system Σ is flat at (x_0, u_0) of differential weight at most $n + m + 2$.

The proofs of Theorems 3.8 and 3.9 follow a similar line as those of, respectively, Theorem 3.2 and Theorem 3.1 in [21] and are left to the reader.

Now, notice that, in the case of two-input control-affine systems, i.e., $m = 2$, any corank one involutive subdistribution \mathcal{H}^k of \mathcal{D}^k satisfies $\text{cork}(\mathcal{D}^k \subset \mathcal{H}^{k+1}) = 1$, therefore, $\overline{\mathcal{D}^k} = \mathcal{H}^{k+1}$ and we necessarily have $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) = 1$. Thus neither item (i) nor item (ii) of Proposition 3.6 occurs for two-input flat systems of differential weight $n + m + 1 = n + 3$. Thus we cannot check flatness of differential weight $n + 3$ using Theorem 3.7(i). On the other hand, Theorem 3.7 (ii) covers the case $m = 2$, but only if $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$. In [21] (see also [19]), we treat the case $m = 2$ in its full generality. Namely, we define (by another method) the involutive subdistribution \mathcal{H}^k in all cases satisfying $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] \neq TX$ (no matter whether $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$ or $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and $[ad_f^k g_1, ad_f^k g_2] \notin \mathcal{D}^k$). Moreover, in the particular case $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, the subdistribution \mathcal{H}^k is defined as in Theorem 3.7 (ii). Finally, if $\mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k] = TX$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$, we have shown, in [21], that the system is flat of differential weight $n+3$ without any additional structural condition.

To summarize, the conditions of Theorems 3.2-3.4, restated as in Theorems 3.7 - 3.9, are always checkable (with the help of Proposition 3.6) in terms of the vector fields of the original system, except for three particular cases: $m = 2$, for which the authors presented in [21], see also [19], verifiable necessary and sufficient conditions for flatness of differential weight $n+3$, and the case $m \geq 3$, if $\text{cork}(\mathcal{D}^k \subset [\mathcal{D}^k, \mathcal{D}^k]) = 1$ and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \subset \mathcal{D}^k$ and either l does not exist and $[\mathcal{E}^s, \mathcal{E}^s] = TX$ or both l and s do not exist. The last two sub-cases need a separate analysis (for the last one, we only do not know to distinguish between flatness of differential weight $n + m + 1$ and $n + m + 2$). Moreover, the verification of conditions involves differentiation and algebraic operations only.

4. Calculating flat outputs. The goal of this section is to answer the question whether a given m -tuple of smooth functions forms a minimal x -flat output.

Recall that, according to Lemma 3.5 in Section 3.1, we can always construct the following sequence of nested involutive distributions:

$$(4.1) \quad \mathcal{H}^0 \subset \mathcal{H}^1 \subset \dots \subset \mathcal{H}^{\rho-1} \subset \mathcal{H}^\rho = TX$$

where either $k = 0$ and then \mathcal{H}^0 is the involutive corank one subdistribution of \mathcal{D}^0 , the distribution \mathcal{H}^1 is defined by $\mathcal{H}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] + [f, \mathcal{H}^0]$ and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, for $1 \leq i \leq \rho - 1$, or $k \geq 1$ and then \mathcal{H}^0 is the involutive corank one subdistribution of \mathcal{D}^0 associated to \mathcal{H}^k and $\mathcal{H}^i = \mathcal{D}^{i-1} + ad_f^i \mathcal{H}$, $1 \leq i \leq k - 1$. If $k \geq 1$, then, for $i \geq 2$, we actually have $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$. For both cases, ρ stands the smallest integer such that $\mathcal{H}^\rho = TX$. We will denote by r_j the corank of the inclusion

$\mathcal{H}^{\rho-j} \subset \mathcal{H}^{\rho-j+1}$, for $1 \leq j \leq \rho$. We clearly have $1 \leq r_1 \leq r_2 \leq \dots \leq r_\rho \leq m$ and we put $r_0 = 0$.

We can now state our result describing all minimal x -flat outputs of differential weight $n + m + 1$. The following proposition answers the question whether a given m -tuple of smooth functions $(\varphi_1, \dots, \varphi_{r_\rho}, \psi_{r_\rho+1}, \dots, \psi_m)$ forms a minimal x -flat output and holds for both cases $k = 0$ and $k \geq 1$. If $r_\rho = m$, then in the above m -tuple the functions ψ_l are missing.

PROPOSITION 4.1. *Consider the control system Σ , given by (2.1), that is flat at x_0 (resp. at (x_0, u_0) , if $k = 0$), of differential weight $n + m + 1$. Then an m -tuple $(\varphi_1, \dots, \varphi_{r_\rho}, \psi_{r_\rho+1}, \dots, \psi_m)$ of smooth functions defined on a neighborhood of x_0 is a minimal x -flat output at x_0 (resp. at (x_0, u_0)) if and only if (after permuting them, if necessary):*

(FO1) for $1 \leq j \leq \rho$, the differentials $d\varphi_i$ annihilate $\mathcal{H}^{\rho-j}$, where $1 \leq i \leq r_j$;

(FO2) the differentials $d\varphi_i^{(q)}$ and $d\psi_l$ are independent at x_0 , where $r_\rho + 1 \leq l \leq m$, and $1 \leq j \leq \rho$, $0 \leq q \leq \rho - j$, $r_{j-1} + 1 \leq i \leq r_j$.

A proof of Proposition 4.1 is given in Section 7.

5. Constant rank assumptions and systems with singularities. Recall that for all results presented in this paper we have supposed in (Assumption 1) that all ranks involved are constant in a neighborhood of a given $x_0 \in X$ (or $(x_0, \bar{u}_0^l) \in X \times U^l$). A natural question is whether (some of) our results hold around points of the singular sets where certain ranks drop. In particular, which ranks are allowed to vary and which have to be constant.

The constant rank assumption is necessary for Proposition 3.1, claiming the equivalence between flatness of differential weight $n + m + 1$, x -flatness of differential weight $n + m + 1$, and dynamic feedback linearization via invertible one-fold prolongation. Without (Assumption 1), that equivalence no longer holds as the following example shows.

Example. Consider the following control system

$$(S) \begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = x_1 + x_2 u_2, \end{cases}$$

where $x \in \mathbb{R}^2$ and $u \in \mathbb{R}^2$. It is easy to see that (S) is (x, u) -flat at (x_0, u_0) of differential weight $n + m + 1 = 5$, where $x_0 = 0$ and $u_0 \in \mathbb{R}^2$, with (x_2, u_2) a minimal flat output, but it is not x -flat at (x_0, u_0) .

So the equivalence $(i) \iff (ii)$ does not hold. Neither holds $(ii) \iff (iii)$ because the system prolonged via $u_1 = v_1$, $\dot{u}_2 = v_2$, whose state is (x_1, x_2, u_2) and controls are (v_1, v_2) , is static feedback linearizable at $x_0 = 0 \in \mathbb{R}^2$. The reason for which the equivalences $(i) \iff (ii)$ and $(ii) \iff (iii)$ are not valid is that the distribution $\mathcal{D}^0 = \text{span} \left\{ \frac{\partial}{\partial x_1}, x_2 \frac{\partial}{\partial x_2} \right\}$ is not of constant rank around $x_0 = 0 \in \mathbb{R}^2$.

To understand the role of the constant rank assumption for Theorems 3.2, 3.3, 3.4, recall that if a system is static feedback linearizable (i.e., of differential weight $n + m$), then the distributions \mathcal{D}^i of Theorem 2.2 have to be of constant rank. It turns out that if a system Σ is flat of differential weight $n + m + 1$ around (x_0, u_0) , then the distributions \mathcal{D}^i need not be of constant rank as shows the system (S) of the above example that is flat of differential weight $n + m + 1 = 2 + 2 + 1 = 5$ around $(x_0, u_0) = (0, u_0)$, for any u_0 , but whose distribution \mathcal{D}^0 is not of constant rank

around x_0 . Even under the stronger requirement of x -flatness, the distributions \mathcal{D}^i of Theorems 3.2 and 3.4 need not be of constant rank. Indeed, for the system (S) prolonged via $\dot{u}_1 = v_1$, $\dot{u}_2 = v_2$, the distribution \mathcal{D}^1 is not of constant rank although the system is x -flat.

On the other hand, if we study linearizability under invertible one-fold prolongation (instead of flatness), then we can reformulate Theorems 3.2-3.4 around any (x_0, u_0) (around which \mathcal{D}^i are of constant rank or not). It turns out that the ranks of the involutive distributions \mathcal{H}^i have to be constant, while that of \mathcal{D}^k may vary (but not those of \mathcal{D}^i , for $i \leq k-1$). Namely, the following theorem, an analogue of Theorem 3.3, holds for the problem of linearization via an invertible one-fold prolongation.

Let k be the smallest integer such that the distribution \mathcal{D}^k does not satisfy the static feedback linearizability conditions of Theorem 2.2, i.e., the smallest k such that either \mathcal{D}^k is not involutive or not of constant rank, and denote $d = \max \dim(\mathcal{D}^k(x)/\mathcal{D}^{k-1}(x))$ around x_0 . By the latter we mean that there exists a neighborhood of x_0 such that the dimension of the quotient is not greater than d and in any neighborhood of x_0 there are points at which it is d .

THEOREM 5.1. *Suppose $k \geq 1$. The system Σ is locally, around (x_0, u_0) , feedback linearizable via an invertible one-fold prolongation if and only if there exists a $(m \times d)$ -matrix $\beta = (\beta_{ij})$, where β_{ij} are C^∞ -smooth functions in a neighborhood of x_0 , $\text{rk } \beta(x) = d$, and such that*

- (A0)_s $\text{span}\{ad_f^k g_1, \dots, ad_f^k g_m\}(x) = \text{span}\{ad_f^k \tilde{g}_1, \dots, ad_f^k \tilde{g}_d\}(x) \text{ mod } \mathcal{D}^{k-1}(x)$, for any x in a neighborhood of x_0 , where $\tilde{g} = g\beta$, with $g = (g_1, \dots, g_m)$ and $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_d) = (\tilde{g}_1, h_2, \dots, h_d)$;
- (A1)_s The distribution $\mathcal{H}^k = \mathcal{D}^{k-1} + \text{span}\{ad_f^k h_2, \dots, ad_f^k h_d\}$ is involutive and of constant rank $\text{rk } \mathcal{H}^k = \text{rk } \mathcal{D}^{k-1} + d - 1$;
- (A2)_s The distributions \mathcal{H}^i , for $i \geq k+1$, where $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$, are involutive and of constant rank;
- (A3)_s There exists ρ such that $\mathcal{H}^\rho = TX$.

Notice that, indeed, the involutive distributions \mathcal{H}^i have to be of constant rank. On the other hand, the rank of \mathcal{D}^k may vary. Conditions (A0)_s - (A1)_s actually imply that \mathcal{D}^k is generated, as a module, by the involutive sub-distribution of constant rank \mathcal{H}^k and just one more vector field of the form $ad_f^k \tilde{g}_1$ that may or may not vanish at x_0 (for $\tilde{g}_1 = \sum_{i=1}^m g_i \beta_{1,i}$, not all $\beta_{1,i}$ vanishing at x_0). If it does vanish, then the rank of \mathcal{D}^k , indeed, drops at x_0 . The proof of the above theorem follows the same line as that of Theorem 3.2. The fact that \mathcal{H}^i have to be of constant rank is a direct consequence of the constant rank of the linearizability distributions \mathcal{D}_p^i associated to the static feedback linearizable prolongation $\tilde{\Sigma}^{(1,0,\dots,0)}$ (see the proof of Theorem 3.2). In the same way as in the proof of Theorem 3.2, we deduce that $\mathcal{D}^{k-1} \subset \mathcal{H}^k \subset \mathcal{D}^k$, but now the codimension of the inclusion $\mathcal{H}^k(x) \subset \mathcal{D}^k(x)$ is either 1 or 0 and, moreover, the distribution \mathcal{D}^k has to be generated by \mathcal{H}^k and just one more vector field $ad_f^k \tilde{g}_1$ that satisfies condition (A0)_s.

Example. We will show that the conditions of Theorem 5.1 cannot be replaced by their pointwise analogue, that is, by the requirement: there exists an involutive distribution \mathcal{H}^k of constant rank, satisfying (A2)_s and (A3)_s, such that

$\text{codim}(\mathcal{H}^k(x) \subset \mathcal{D}^k(x)) \leq 1$. To this aim, consider the system

$$\begin{aligned}\dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_1 + \frac{1}{2}(x_4^2 + x_5^2) \\ \dot{x}_3 &= u_1 \\ \dot{x}_4 &= u_2 \\ \dot{x}_5 &= u_3.\end{aligned}$$

We have $\mathcal{D}^0 = \text{span}\{\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}\}$ and $\mathcal{D}^1 = \mathcal{D}^0 + \text{span}\{\frac{\partial}{\partial x_1}, x_4 \frac{\partial}{\partial x_2}, x_5 \frac{\partial}{\partial x_2}\}$, so $4 \leq \dim \mathcal{D}^1(x) \leq 5$ and the involutive distribution $\mathcal{H}^1 = \text{span}\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}\}$ satisfies $\mathcal{D}^0 \subset \mathcal{H}^1 \subset \mathcal{D}^1$ and $\text{rk } \mathcal{H}^1(x) = 4$. Moreover, $\mathcal{H}^2 = \mathcal{H}^1 + [f, \mathcal{H}^1] = TX$. Nevertheless the system is not static feedback linearizable via an invertible one-fold prolongation. An attempt would be to put $\tilde{x}_1 = x_1 + \frac{1}{2}(x_4^2 + x_5^2)$ giving $\dot{\tilde{x}}_1 = x_3 + x_4 u_2 + x_5 u_3$. But then $\dot{\tilde{x}}_1$ depends on both \dot{u}_2 and \dot{u}_3 implying that two one-fold prolongations are needed.

To see the reason, notice that $\text{rk } \mathcal{D}^0 = 3$, the maximal dimension of $\mathcal{D}^1(x)/\mathcal{D}^0(x)$, in a neighborhood of $0 \in \mathbb{R}^5$, is 2, and there does not exist any smooth vector field $\tilde{g}_1 = \beta_1 g_1 + \beta_2 g_2 + \beta_3 g_3$ (with β_i smooth and not all vanishing at $0 \in \mathbb{R}^5$) such that $\text{span}\{ad_f \tilde{g}_1\} = \text{span}\{x_4 \frac{\partial}{\partial x_2}, x_5 \frac{\partial}{\partial x_2}\} \text{ mod } \mathcal{D}^0$, so \mathcal{D}^1 cannot be generated by the involutive sub-distribution \mathcal{H}^1 of constant rank 4 and just one more vector field $ad_f \tilde{g}_1$. Thus (because of Theorem 5.1) we are, indeed, not able to linearize the system via an invertible one-fold prolongation.

6. Examples.

6.1. An example: normal form for $k = 0$ and singularities in the control space. Consider a three-input control-affine systems Σ flat of differential weight $n + 3 + 1$ at (x_0, u_0) , with $k = 0$ and $\text{cork}(\mathcal{D}^0 \subset \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]) = 2$. Σ is locally static feedback equivalent, around $x_0 \in X$, to the the following normal form in a neighborhood of $z_0 \in \mathbb{R}^n$

$$(NF_{m=3}^{k=0}) \left\{ \begin{array}{llll} \dot{z}_1^1 & = & z_1^2 & \dot{z}_j^1 & = & z_j^2 \\ & & \vdots & & & \vdots \\ \dot{z}_1^{\rho_1-1} & = & z_1^{\rho_1} & \dot{z}_j^{\rho_j-1} & = & z_j^{\rho_j} \\ \dot{z}_1^{\rho_1} & = & \tilde{u}_1 & \dot{z}_j^{\rho_j} & = & a_j(z) + z_j^{\rho_j+1} \tilde{u}_1 \\ & & & \dot{z}_j^{\rho_j+1} & = & \tilde{u}_j \end{array} \right.$$

where $j = 2$ or 3 , $\rho_1 + \rho_2 + \rho_3 = n$, and a_j are arbitrary smooth functions such that $\text{rk} \left(\frac{\partial(a_j(z) + z_j^{\rho_j+1} \tilde{u}_1)}{\partial z_i^{\rho_i+1}} \right) (z_0, \tilde{u}_0) = 2$, where $2 \leq i, j \leq 3$.

The above normal form is generalizing the Brunovský canonical form. Namely, for three-input control systems, at most two components (i.e., at most only one component for each chain) are replaced by arbitrary (nonlinear) functions. It is easy to see that $(NF_{m=3}^{k=0})$ is flat with the top variables $\varphi = (z_1^1, z_2^1, z_3^1)$ being minimal flat outputs of differential weight $n + 3 + 1$ and that $(NF_{m=3}^{k=0})$ becomes locally static feedback linearizable after a one-fold prolongation of \tilde{u}_1 , which is the to-be-prolonged control. Moreover, if we replace \tilde{u}_1 by $\hat{u}_1 = \beta(z) \tilde{u}_1$, with $\beta(z) \neq 0$, and we prolong \hat{u}_1 instead of \tilde{u}_1 , the prolonged system is also locally static feedback linearizable. The normal form $(NF_{m=3}^{k=0})$ allows us to see that in the case $k = 0$ (and according to our results only

in that case!), the precompensator may create singularities in the control space (depending on state). Indeed, the controls \tilde{u}_0 satisfying $\text{rk}\left(\frac{\partial a_j(z)+z_j^{\rho_j+1}\tilde{u}_1}{\partial z_i^{\rho_i+1}}\right)(z_0, \tilde{u}_0) < 2$, where $2 \leq i, j \leq 3$, are singular for $(NF_{m=3}^{k=0})$, an invariant description of that set of singular controls being given by U_{sing} . It follows that $(NF_{m=3}^{k=0})$ ceases to be flat of differential weigh $n+3+1$ at (z_0, \tilde{u}_0) , with \tilde{u}_0 for which the above rank is at most 1 or, equivalently, $\det\left(\begin{array}{cc} \frac{\partial a_1}{\partial z_1^{\rho_1+1}} + \tilde{u}_1 & \frac{\partial a_1}{\partial z_2^{\rho_2+1}} \\ \frac{\partial a_2}{\partial z_1^{\rho_1+1}} & \frac{\partial a_2}{\partial z_2^{\rho_2+1}} + \tilde{u}_1 \end{array}\right) = 0$. For each fixed $z \in \mathbb{R}^n$, the determinant is a polynomial of \tilde{u}_1 of degree 2, so it may have 2, 1, or 0 distinct real roots. Therefore for a given $z \in \mathbb{R}^n$, the values of the singular controls, respectively, form two affine planes in the control space \mathbb{R}^3 , one affine plane in \mathbb{R}^3 , or are absent.

6.2. Quadrotor helicopter. A quadrotor is a four rotor helicopter. Assume that a body frame is fixed at the center of gravity of the quadrotor, with the z -axis pointing up-wards. The body frame is related to the inertial frame by a position vector (x_1, y_1, z_1) and 3 angles (θ, ψ, φ) representing pitch, roll and yaw, respectively. The equations of motion are given by the following control system [1, 2] (see also [28], where a quadrotor with a cable-suspended load is considered):

$$\Sigma_{QH} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = u_1(\cos \varphi \sin \theta \cos \psi + \sin \varphi \sin \psi) \\ \dot{y}_1 = y_2 \\ \dot{y}_2 = u_1(\sin \varphi \sin \theta \cos \psi - \cos \varphi \sin \psi) \\ \dot{z}_1 = z_2 \\ \dot{z}_2 = -g + u_1(\cos \theta \cos \psi) \\ \dot{\theta} = u_2 \\ \dot{\psi} = u_3 \\ \dot{\varphi} = u_4 \end{cases}$$

The control u_1 represents the total thrust on the body in the z -axis, u_2 and u_3 are the pitch and roll inputs and u_4 is the yawing moment. The quadrotor helicopter has been shown to be flat, with (x_1, y_1, z_1, φ) a flat output, see [2]. The system is not static feedback linearizable, but it becomes static feedback linearizable after an invertible one-fold prolongation. To illustrate our results, fix $\xi_0 \in X = \mathbb{R}^6 \times \text{SO}(3)$ such that $(\cos \theta \cos \psi \cos \varphi (\cos \varphi \sin \theta \cos \psi + \sin \varphi \sin \psi))(\xi_0) \neq 0$. In order to simplify the bracket computations, we apply the following static feedback transformation (which is supposed invertible around the nominal point ξ_0)

$$\begin{aligned} \tilde{u}_1 &= u_1(\cos \varphi \sin \theta \cos \psi + \sin \varphi \sin \psi) \\ \tilde{u}_i &= u_i, \quad 2 \leq i \leq 4, \end{aligned}$$

and get

$$\tilde{\Sigma}_{QH} : \begin{cases} \dot{x}_1 = x_2 & \dot{y}_1 = y_2 \\ \dot{x}_2 = \tilde{u}_1 & \dot{y}_2 = \tilde{u}_1 a(\theta, \psi, \varphi) \\ & \dot{\theta} = \tilde{u}_2 \\ \dot{z}_1 = z_2 & \dot{\varphi} = \tilde{u}_4, \\ \dot{z}_2 = -g + \tilde{u}_1 b(\theta, \psi, \varphi) \\ \dot{\psi} = \tilde{u}_3, \end{cases}$$

where

$$a = \frac{\sin \varphi \sin \theta \cos \psi - \cos \varphi \sin \psi}{\cos \varphi \sin \theta \cos \psi + \sin \varphi \sin \psi} \quad \text{and} \quad b = \frac{\cos \theta \cos \psi}{\cos \varphi \sin \theta \cos \psi + \sin \varphi \sin \psi}.$$

The distribution

$$\mathcal{D}^0 = \text{span} \left\{ \frac{\partial}{\partial x_2} + a \frac{\partial}{\partial y_2} + b \frac{\partial}{\partial z_2}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \varphi} \right\}$$

is not involutive. Indeed, the vector fields g_i , $1 \leq i \leq 4$, $[g_1, g_2]$ and $[g_1, g_3]$ are independent at ξ_0 (provided that $\cos \theta_0 \cos \psi_0 \cos \varphi_0 \neq 0$, which is verified according to our assumption). We obtain

$$\mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] = \text{span} \left\{ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \varphi}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial z_2} \right\}.$$

Here $k = 0$ and $\text{cork}(\mathcal{D}^0 \subset [\mathcal{D}^0, \mathcal{D}^0]) = 2$, consequently we are in the case of Theorem 3.3. It is immediate to identify the unique corank one involutive subdistribution of \mathcal{D}^0 , that is $\mathcal{H}^0 = \text{span} \left\{ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \psi}, \frac{\partial}{\partial \varphi} \right\}$.

We have $\mathcal{H}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] + [f, \mathcal{H}^0] = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0]$ (since $[f, g_i] = 0$, for $2 \leq i \leq 4$), which is clearly involutive, and $\mathcal{H}^2 = TX$. The system $\tilde{\Sigma}_{QH}$ satisfies all conditions of Theorem 3.3, hence the corresponding prolongation given by

$$\tilde{\Sigma}_{QH}^{(1,0,0,0)} : \begin{cases} \dot{x}_1 = x_2 & \dot{y}_1 = y_2 \\ \dot{x}_2 = \tilde{u}_1 & \dot{y}_2 = \tilde{u}_1 a(\theta, \psi, \varphi) \\ \dot{\tilde{u}}_1 = v_1 & \dot{z}_1 = z_2 \\ & \dot{z}_2 = -g + \tilde{u}_1 b(\theta, \psi, \varphi) \\ & \dot{\theta} = v_2 \\ & \dot{\psi} = v_3 \\ & \dot{\varphi} = v_4, \end{cases}$$

where $v_i = \tilde{u}_i$, for $2 \leq i \leq 4$, is locally static feedback linearizable. Indeed, applying the following change of coordinates $\tilde{\theta} = \tilde{u}_1 a(\theta, \psi, \varphi)$ and $\tilde{\psi} = -g + \tilde{u}_1 b(\theta, \psi, \varphi)$ (which is valid in a neighborhood of ξ_0 and for $\tilde{u}_{10} \neq 0$) and a suitable feedback transformation, we get

$$\tilde{\Sigma}_{QH}^{(1,0,0,0)} : \begin{cases} \dot{x}_1 = x_2 & \dot{y}_1 = y_2 & \dot{z}_1 = z_2 & \dot{\varphi} = \tilde{v}_4 \\ \dot{x}_2 = w & \dot{y}_2 = \tilde{\theta} & \dot{z}_2 = \tilde{\psi} \\ \dot{w} = \tilde{v}_1 & \dot{\tilde{\theta}} = \tilde{v}_2 & \dot{\tilde{\psi}} = \tilde{v}_3, \end{cases}$$

which is the Brunovsky canonical form with (x_1, y_1, z_1, φ) playing the role of the top variables. From this, it is obvious that (x_1, y_1, z_1, φ) is a minimal flat output, i.e., of differential weight $n + m + 1 = 9 + 4 + 1 = 14$.

7. Proofs.

7.1. Notations and useful results . Consider a control system of the form

$$\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x) = f(x) + u_1 g_1(x) + \sum_{i=2}^m u_i h_i(x),$$

where the change of notation is to distinguish the first control (respectively, the first vector field g_1) from the remaining controls u_i (respectively, remaining vector fields g_i), for $2 \leq i \leq m$. By $\Sigma^{(1,0,\dots,0)}$ we will denote the system Σ with one-fold prolongation of the first control, that is,

$$\Sigma^{(1,0,\dots,0)} : \begin{cases} \dot{x} = f(x) + y_1 g_1(x) + \sum_{i=2}^m v_i h_i(x) \\ \dot{y}_1 = v_1 \end{cases}$$

with $y_1 = u_1$ and $v_i = u_i$, for $2 \leq i \leq m$.

To $\Sigma^{(1,0,\dots,0)}$, we associate the distributions $\mathcal{D}_p^{i+1} = \mathcal{D}_p^i + [F, \mathcal{D}_p^i]$, for $i \geq 0$, where $F = \sum_{i=1}^n (f_i + y_1 g_{1i}) \frac{\partial}{\partial x_i}$ stands for the drift of $\Sigma^{(1,0,\dots,0)}$ and $\mathcal{D}_p^0 = \text{span} \left\{ \frac{\partial}{\partial y_1}, \sum_{i=1}^n h_{ji} \frac{\partial}{\partial x_i}, 2 \leq j \leq m \right\}$ is the distribution spanned by the control vector fields of the prolonged system, the subindex p referring to the prolonged system $\Sigma^{(1,0,\dots,0)}$.

In our proofs we will need the two following technical results. Consider the control system Σ , given by (2.1), and let \mathcal{D}^k be the first noninvolutive distribution.

PROPOSITION 7.1. *Assume that Σ is dynamically linearizable via invertible one-fold prolongation. If $k \geq 1$, then $\text{rk } \mathcal{D}^k - \text{rk } \mathcal{D}^{k-1} \geq 2$.*

Proof. Assume $\text{rk } \mathcal{D}^k - \text{rk } \mathcal{D}^{k-1} = 1$ and let l be the smallest integer such that $\text{rk } \mathcal{D}^l - \text{rk } \mathcal{D}^{l-1} = 1$. It is clear that $1 \leq l \leq k$. Since Σ is dynamically linearizable via invertible one-fold prolongation, there exists an invertible static feedback transformation, $u(x) = \alpha(x) + \beta(x)\tilde{u}$, bringing Σ into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \sum_{i=2}^m \tilde{u}_i \tilde{h}_i(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0,\dots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^m v_i \tilde{h}_i(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_i = \tilde{u}_i$, for $2 \leq i \leq m$, is locally static feedback linearizable. For simplicity of notation, we will drop the tildes, but we will keep distinguishing g_1 from h_i (which could also be denoted g_i) whose controls are not preintegrated.

Since $\Sigma^{(1,0,\dots,0)}$ is locally static feedback linearizable, for any $i \geq 0$ the distributions \mathcal{D}_p^i are involutive, of constant rank, and there exists an integer ρ such that $\text{rk } \mathcal{D}_p^\rho = n + 1$. We have

$$\begin{aligned} \mathcal{D}_p^0 &= \text{span} \left\{ \frac{\partial}{\partial y_1}, h_j, 2 \leq j \leq m \right\}, \\ \mathcal{D}_p^1 &= \text{span} \left\{ \frac{\partial}{\partial y_1}, g_1, h_j, \text{ad}_f h_j + y_1 [g_1, h_j], 2 \leq j \leq m \right\}. \end{aligned}$$

For $k \geq 1$, the distribution $\mathcal{D}^0 = \text{span} \{g_1, h_j, 2 \leq j \leq m\}$ is involutive, thus $[g_1, h_j] \in \mathcal{D}^0$, for $2 \leq j \leq m$, and $\mathcal{D}_p^1 = \text{span} \left\{ \frac{\partial}{\partial y_1}, g_1, h_j, \text{ad}_f h_j, 2 \leq j \leq m \right\}$. It is easy to prove (by an induction argument) that, for $1 \leq i \leq l$,

$$\mathcal{D}_p^i = \text{span} \left\{ \frac{\partial}{\partial y_1}, g_1, \dots, \text{ad}_f^{i-1} g_1, h_j, \dots, \text{ad}_f^i h_j, 2 \leq j \leq m \right\}.$$

We have $\mathcal{D}^{l-1} = \text{span} \{g_1, \dots, \text{ad}_f^{l-1} g_1, h_j, \dots, \text{ad}_f^{l-1} h_j, 2 \leq j \leq m\}$ and by the definition of l either $\text{ad}_f^l h_j \in \mathcal{D}^{l-1}$, for all $2 \leq j \leq m$, i.e., $\text{ad}_f^l g_1 \notin \mathcal{D}^{l-1}$, or there exists an integer $2 \leq s \leq m$ such that $\text{ad}_f^l h_s \notin \mathcal{D}^{l-1}$.

In the first case:

$$\mathcal{D}_p^j = \text{span} \left\{ \frac{\partial}{\partial y_1} \right\} + \mathcal{D}^{j-1}, \text{ for } j \geq l,$$

The involutivity of the distribution \mathcal{D}_p^j , associated to the prolonged system, implies that of \mathcal{D}^{j-1} . For $j = k + 1$, it contradicts the fact that \mathcal{D}^k is noninvolutive.

In the second case, there exists an integer $2 \leq s \leq m$ such that $\text{ad}_f^l h_s \notin \mathcal{D}^{l-1}$. Since $\text{rk } \mathcal{D}^l = \text{rk } \mathcal{D}^{l-1} + 1$, we deduce that $\mathcal{D}^l = \text{span} \{g_1, \dots, \text{ad}_f^{l-1} g_1, h_j, \dots, \text{ad}_f^{l-1} h_j, \text{ad}_f^l h_s, 2 \leq j \leq m\}$. Moreover, for $\Sigma^{(1,0,\dots,0)}$, we have

$$\mathcal{D}_p^j = \text{span} \left\{ \frac{\partial}{\partial y_1} \right\} + \mathcal{D}^j, \text{ for } j \geq l,$$

and the involutivity of \mathcal{D}_p^j implies that of \mathcal{D}^j . For $j = k$, it follows that \mathcal{D}^k is involutive, which contradicts the assumption of noninvolutivity of \mathcal{D}^k . Thus l , if it exists, satisfies $l \geq k + 1$ and $\text{rk } \mathcal{D}^k - \text{rk } \mathcal{D}^{k-1} \geq 2$.

PROPOSITION 7.2. *Assume $k \geq 1$ and suppose that \mathcal{D}^k contains an involutive subdistribution \mathcal{H}^k , of corank one.*

- (i) *If $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$, then \mathcal{H}^k satisfies $\mathcal{D}^{k-1} \subset \mathcal{H}^k$.*
- (ii) *If \mathcal{H}^k satisfies $\mathcal{D}^{k-1} \subset \mathcal{H}^k$, then there exists a distribution \mathcal{H} , uniquely associated to \mathcal{H}^k , such that $\mathcal{H} \subset \mathcal{D}^0$ is of corank one and $\mathcal{H}^k = \mathcal{D}^{k-1} + \text{ad}_f^k \mathcal{H}$. Moreover, all distributions $\mathcal{H}^i = \mathcal{D}^{i-1} + \text{ad}_f^i \mathcal{H}$, for $0 \leq i \leq k-1$, where $\mathcal{D}^{-1} = \{0\}$ and $\mathcal{H}^0 = \mathcal{H}$, are involutive and are feedback invariant, that is, do not change if we replace f by $f + \sum_{i=1}^m \alpha_i g_i$.*

REMARK 3. *Notice that for $1 \leq i \leq k-1$, we actually have $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$. Assume that there exists an integer ρ such that $\mathcal{H}^\rho = TX$ and suppose that ρ is the smallest integer satisfying that property. If we denote by r_j the corank of the inclusion $\mathcal{H}^{\rho-j} \subset \mathcal{H}^{\rho-j+1}$, for $1 \leq j \leq \rho$, we clearly have $1 \leq r_1 \leq r_2 \leq \dots \leq r_\rho \leq m$.*

Proof of (i). By $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$ and according to Proposition 3.6, if the distribution \mathcal{D}^k contains an involutive subdistribution \mathcal{H}^k , of corank one, then \mathcal{H}^k is unique. Using the Jacobi identity, it is easy to show that $\mathcal{D}^{k-2} \subset \mathcal{H}^k$. To prove (i), suppose $\mathcal{D}^{k-1} \not\subset \mathcal{H}^k$, i.e., there exists a vector field $v \in \mathcal{D}^{k-1}$, of the form $v = \sum_{i=1}^m \alpha_i \text{ad}_f^{k-1} g_i \text{ mod } \mathcal{D}^{k-2}$, satisfying $\mathcal{D}^k = \mathcal{H}^k + \text{span}\{v\}$, where α_i are smooth functions, not vanishing simultaneously and such that $v \notin \mathcal{D}^{k-2}$. The vector field v can also be written as $v = \text{ad}_f^{k-1}(\sum_{i=1}^m \alpha_i g_i) \text{ mod } \mathcal{D}^{k-2}$ and we can always assume, without loss of generality, that α_1 is nonzero and $\text{ad}_f^{k-1} g_1 \notin \mathcal{D}^{k-2}$. So replacing g_1 by $\sum_{i=1}^m \alpha_i g_i$, we have $\mathcal{D}^k = \mathcal{H}^k + \text{span}\{\text{ad}_f^{k-1} g_1\}$. From this, we deduce that the involutive subdistribution \mathcal{H}^k is given by

$$\mathcal{H}^k = \text{span}\{g_1, \dots, \text{ad}_f^{k-2} g_1, \text{ad}_f^k g_1, g_j, \dots, \text{ad}_f^k g_j, 2 \leq j \leq m\}.$$

Thus, the new directions, completing \mathcal{D}^k to $\overline{\mathcal{D}^k}$, where $\overline{\mathcal{D}^k}$ is the involutive closure of \mathcal{D}^k , are of the form $[\text{ad}_f^k g_i, \text{ad}_f^{k-1} g_1]$ for some indices i such that $1 \leq i \leq m$, and since $\text{cork}(\mathcal{D}^k \subset \mathcal{D}^k + [\mathcal{D}^k, \mathcal{D}^k]) \geq 2$, there are at least two integers i satisfying that property. Therefore at least one of them, say s , satisfies $s \neq 0$, and we have $[\text{ad}_f^k g_s, \text{ad}_f^{k-1} g_1] \notin \mathcal{D}^k$. Applying the Jacobi identity, we obtain

$$\begin{aligned} [\text{ad}_f^k g_s, \text{ad}_f^{k-1} g_1] &= [[f, \text{ad}_f^{k-1} g_s], \text{ad}_f^{k-1} g_1] = [[f, \text{ad}_f^{k-1} g_1], \text{ad}_f^{k-1} g_s] + [f, [\text{ad}_f^{k-1} g_s, \text{ad}_f^{k-1} g_1]] \\ &= [\text{ad}_f^k g_1, \text{ad}_f^{k-1} g_s] \text{ mod } \mathcal{D}^k \end{aligned}$$

and since the vector fields $\text{ad}_f^k g_1$ and $\text{ad}_f^{k-1} g_s$ belong to \mathcal{H}^k , which is involutive, $[\text{ad}_f^k g_1, \text{ad}_f^{k-1} g_s] \in \mathcal{H}^k$. It follows immediately that $[\text{ad}_f^k g_s, \text{ad}_f^{k-1} g_1] \in \mathcal{D}^k$, which contradicts our assumption. Therefore, the inclusion $\mathcal{D}^{k-1} \subset \mathcal{H}^k$ holds.

Proof of (ii). Let us first show the existence of the distribution \mathcal{H} . Denote $\text{cork}(\mathcal{D}^{k-1} \subset \mathcal{D}^k) = q$ and suppose that the vector fields $g_i \in \mathcal{D}^0$, for $1 \leq i \leq q$, satisfy

$$\mathcal{D}^k = \mathcal{D}^{k-1} + \text{span}\{\text{ad}_f^k g_i, 1 \leq i \leq q\}.$$

Thus there exist smooth functions β_j^i such that $\text{ad}_f^k g_j = \sum_{i=1}^q \beta_j^i \text{ad}_f^k g_i \text{ mod } \mathcal{D}^{k-1}$, for $q+1 \leq j \leq m$. It follows $\text{ad}_f^k(g_j - \sum_{i=1}^q \beta_j^i g_i) = 0 \text{ mod } \mathcal{D}^{k-1}$. Denote $h_j = g_j -$

$\sum_{i=1}^r \beta_j^i g_i$, for $q+1 \leq j \leq m$. We clearly have $\mathcal{D}^0 = \text{span}\{g_1, \dots, g_q, h_{q+1}, \dots, h_m\}$, with h_j such that $ad_f^k h_j \in \mathcal{D}^{k-1}$, for $q+1 \leq j \leq m$.

Since $\mathcal{D}^{k-1} \subset \mathcal{H}^k$ and $\mathcal{H}^k \subset \mathcal{D}^k$ is of corank one, there exist smooth functions λ_j^i , for $1 \leq i, j \leq q$, such that the $q \times q$ -matrix $\Lambda = (\lambda_j^i)$ is invertible and the distributions \mathcal{H}^k and \mathcal{D}^k verify

$$\mathcal{H}^k = \mathcal{D}^{k-1} + \text{span}\left\{\sum_{i=1}^q \lambda_j^i ad_f^k g_i, 2 \leq j \leq q\right\},$$

$$\mathcal{D}^k = \mathcal{H}^k + \text{span}\left\{\sum_{i=1}^q \lambda_1^i ad_f^k g_i\right\}.$$

Denote $\tilde{g}_1 = \sum_{i=1}^q \lambda_1^i g_i$ and $h_j = \sum_{i=1}^q \lambda_j^i g_i$, for $2 \leq j \leq q$. We put $\mathcal{H} = \text{span}\{h_j, 2 \leq j \leq m\}$, which is clearly of corank one in $\mathcal{D}^0 = \text{span}\{\tilde{g}_1, h_j, 2 \leq j \leq m\}$ and satisfies $\mathcal{H}^k = \mathcal{D}^{k-1} + ad_f^k \mathcal{H}$.

We will prove next the involutivity of all distributions \mathcal{H}^i , for $0 \leq i \leq k-1$. Assume that the distribution \mathcal{H}^{k-1} given by

$$\mathcal{H}^{k-1} = \mathcal{D}^{k-2} + ad_f^{k-1} \mathcal{H} = \mathcal{D}^{k-2} + \text{span}\{ad_f^{k-1} h_j, 2 \leq j \leq m\}$$

is not involutive. Since the inclusion $\mathcal{H}^{k-1} \subset \mathcal{D}^{k-1}$ is of corank one and \mathcal{D}^{k-1} is involutive, it follows that $\overline{\mathcal{H}^{k-1}} = \mathcal{D}^{k-1}$. Moreover, $\mathcal{D}^{k-2} \subset \mathcal{H}^{k-1}$ and \mathcal{D}^{k-2} involutive imply that the new direction completing \mathcal{H}^{k-1} to its involutive closure is given by a vector field of the form $[ad_f^l h_i, ad_f^{k-1} h_j]$ or of the form $[ad_f^s \tilde{g}_1, ad_f^{k-1} h_j]$, where $2 \leq i, j \leq m$, $0 \leq l \leq k-1$ and $0 \leq s \leq k-2$, and is necessarily collinear with $ad_f^{k-1} \tilde{g}_1 \text{ mod } \mathcal{H}^{k-1}$.

Let us suppose that there exist two integers $2 \leq i, j \leq m$ such that $[ad_f^l h_i, ad_f^{k-1} h_j] \notin \mathcal{H}^{k-1}$. Hence, there exists a non zero smooth function α such that $[ad_f^l h_i, ad_f^{k-1} h_j] = \alpha ad_f^{k-1} \tilde{g}_1 \text{ mod } \mathcal{H}^{k-1}$. From this, applying the Jacobi identity and the involutivity of \mathcal{H}^k , it follows

$$\begin{aligned} [ad_f^l h_i, ad_f^k h_j] &= [ad_f^l h_i, [f, ad_f^{k-1} h_j]] = [f, [ad_f^l h_i, ad_f^{k-1} h_j]] - [ad_f^{l+1} h_i, ad_f^{k-1} h_j] \\ &= [f, \alpha ad_f^{k-1} \tilde{g}_1] \text{ mod } \mathcal{H}^k = \alpha ad_f^k \tilde{g}_1 \text{ mod } \mathcal{H}^k. \end{aligned}$$

On the other hand, $[ad_f^l h_i, ad_f^k h_j] \in \mathcal{H}^k$, and consequently $ad_f^k \tilde{g}_1 \in \mathcal{H}^k$, which contradicts our assumption, otherwise $\mathcal{D}^k = \mathcal{H}^k$ and \mathcal{D}^k would be involutive. We conclude that \mathcal{H}^{k-1} is involutive. Following the same line, the involutivity of \mathcal{H}^i implies that of \mathcal{H}^{i-1} , for $1 \leq i \leq k-1$.

An analogous reasoning applies if $[ad_f^l \tilde{g}_1, ad_f^{k-1} h_j] \notin \mathcal{H}^{k-1}$.

7.2. Proof of Proposition 3.1. We will show the implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i).

(i) \Rightarrow (ii). Consider the control system $\Xi : \dot{x} = F(x, u)$ and assume that Ξ is flat at (x_0, \bar{u}_0^l) , of differential weight $n + m + 1$. Let $\varphi = (\varphi_1, \dots, \varphi_m)$ be a minimal flat output. We will denote by s_i the order of the highest derivative of φ_i , for $1 \leq i \leq m$, involved in the expression of x and u , i.e.,

$$(7.1) \quad x = \gamma(\bar{\varphi}_1^{s_1}, \dots, \bar{\varphi}_m^{s_m}) \text{ and } u = \delta(\bar{\varphi}_1^{s_1}, \dots, \bar{\varphi}_m^{s_m}),$$

where $\tilde{\varphi}_i^j = (\varphi_i, \dot{\varphi}_i, \dots, \varphi_i^{(j)})$ and $\sum_{i=1}^m s_i + m = n + m + 1$. We will denote by $d.w.(\varphi)$ the differential weight of φ so $d.w.(\varphi) = n + m + 1$. Denote $\mathcal{X} = \text{span}\{dx_1, \dots, dx_n\}$ and $\mathcal{U} = \text{span}\{du_1, \dots, du_m\}$.

Assume that there exists $\varphi_q = \varphi_q(x, u, \dot{u}, \dots, u^{(j)})$, where $j \geq 1$. The differential weight of φ being $n + m + 1$ implies that, clearly, $s_q = 0$. Indeed, if $s_q \geq 1$, then $d\varphi_q \wedge \dots \wedge d\varphi_q^{(s_q)} \neq 0 \pmod{\mathcal{X} + \mathcal{U}}$ and $d.w.(\varphi)$ would be $n + m + s_q + 1 > n + m + 1$. Denote $\theta = \varphi_q(x, u, \dot{u}, \dots, u^{(j)})$. If there exists a flat output φ_i such that $d\varphi_i \wedge d\theta \neq 0 \pmod{\mathcal{X} + \mathcal{U}}$, then $d.w.(\varphi)$ would be at least $n + m + 2$. We thus have $\varphi_i = \varphi_i(x, u, \theta)$, for $1 \leq i \leq m$, and we bring together the components φ_i that depend explicitly on θ by permuting φ_i 's such that $\varphi_i = \varphi_i(x, u)$, for $1 \leq i \leq p$, and $\varphi_j = \varphi_j(x, u, \theta)$, for $p + 1 \leq j \leq m$, where $\frac{\partial \varphi_j}{\partial \theta} \neq 0$. We assume, without loss of generality, that $q = m$, i.e., $\varphi_m = \theta$. Clearly, $s_i = 0$, for $p + 1 \leq i \leq m$ (if not, $d\varphi_i \wedge d\dot{\varphi}_i \neq 0 \pmod{\mathcal{X} + \mathcal{U}}$ contradicting $d.w.(\varphi) = n + m + 1$).

Let ρ_i , for $1 \leq i \leq p$, be the relative degree of φ_i , that is, the smallest integer such that the derivative $\varphi_i^{(\rho_i)}$ depends explicitly on the control u . In particular, $\rho_i = 0$, if φ_i depends explicitly on u . For $1 \leq i \leq p$ and $1 \leq j \leq m$, denote $\text{rk}\left(\frac{\partial \varphi_i}{\partial u_j}\right) = r_0$ and $\text{rk}\left(\frac{\partial \varphi_i^{(\rho_i)}}{\partial u_j}\right) = r_1$ and assume that there exist $r_2 - r_0$ flat outputs that do not depend on θ and whose relative degree $\rho_i \geq 1$. Clearly, $0 \leq r_0 \leq r_1 \leq r_2 \leq p$. Permute φ_i , for $1 \leq i \leq p$, and apply an invertible static feedback $u = \psi(x, v)$ such that

$$\begin{aligned} \varphi_i &= v_i, & 1 \leq i \leq r_0, \\ \varphi_i^{(\rho_i)} &= v_i, & r_0 + 1 \leq i \leq r_1. \end{aligned}$$

By a supplementary permutation we get $\rho_i \geq 1$, for $r_0 + 1 \leq i \leq r_2$, and for those indices we introduce $z_i^j = L_F^{j-1} \varphi_i$, for $1 \leq j \leq \rho_i$. Let w be complementary coordinates, that is, $\dim z + \dim w = n$. The system in the (z, w) -coordinates reads

$$(7.2) \quad \begin{aligned} \dot{z}_i^j &= z_i^{j+1}, & \dot{z}_i^j &= z_i^{j+1}, \\ \dot{z}_i^{\rho_i} &= v_i, \quad r_0 + 1 \leq i \leq r_1, & \dot{z}_i^{\rho_i} &= a_i(z, w, v_1, \dots, v_{r_1}), \quad r_1 + 1 \leq i \leq r_2, \\ \dot{w} &= G(z, w, v), \end{aligned}$$

for some smooth functions a_i and a smooth map G , and is flat with flat outputs

$$(7.3) \quad \begin{aligned} \varphi_i &= v_i, & 1 \leq i \leq r_0, \\ \varphi_i &= z_i^1, & r_0 + 1 \leq i \leq r_2, \\ \varphi_i &= b_i(z, w, v_1, \dots, v_{r_0}), & r_2 + 1 \leq i \leq p, \\ \varphi_i &= c_i(z, w, v, \theta), & p + 1 \leq i \leq m, \end{aligned}$$

for some smooth functions b_i and c_i . The z -part is affected by at most r_1 controls (by $v_{r_0+1}, \dots, v_{r_1}$ and perhaps by some among v_1, \dots, v_{r_0}). So the remaining $m - r_1$ controls v_i , for $r_1 + 1 \leq i \leq m$, have to be present in the w -part implying that $\dim w \geq m - r_1$. For $r_1 + 1 \leq i \leq p$, the functions $\varphi_i^{(\rho_i)}$ (equal either a_i or b_i) depend explicitly on (some of) v_1, \dots, v_{r_1} so $d\varphi_i^{(\rho_i+1)} \notin \mathcal{X} + \mathcal{U}$ and thus $s_i \leq \rho_i + 1$ and, moreover, if $s_i = \rho_i + 1$, then $d\varphi_i^{(\rho_i+1)} \wedge d\theta = 0 \pmod{\mathcal{X} + \mathcal{U}}$. From flatness it follows that using at most $2(p - r_1)$ functions $\varphi_i^{(\rho_i)}$, $\varphi_i^{(\rho_i+1)}$, for $r_1 + 1 \leq i \leq p$ (equal, respectively, either a_i and \dot{a}_i or b_i and \dot{b}_i), and $m - p$ functions $\varphi_j = c_j$, $p + 1 \leq j \leq m$, we should be able to express (via functions that depend also on z_i^j and v_1, \dots, v_{r_1}) at

least $m - r_1$ state variables w_j and $m - r_1$ control variables v_i , for $r_1 + 1 \leq i \leq m$. So we need

$$2(p - r_1) + m - p \geq 2(m - r_1),$$

which is equivalent to $p \geq m$, yielding a contradiction since $p \leq m - 1$. Therefore the components φ_i of any minimal flat output do not depend on w^j , for $j \geq 1$.

It remains to consider the case of Ξ being (x, u) -flat. As we have just proved, we can bring the system Ξ into the form (7.2), with $p = m$, whose minimal flat output is given by (7.3), with the components $\varphi_i = c_i$ absent. The system is flat so we need derivatives (at least of first order) of all functions a_i , for $r_1 + 1 \leq i \leq r_2$, and of all functions b_i , for $r_2 + 1 \leq i \leq m$, to express $m - r_1$ controls v_{r_1+1}, \dots, v_m . By the definition of the relative degree, all functions a_i and b_i depend explicitly on some of v_q . It follows that, for $s \geq 1$, $a_i^{(s)}$ and $b_i^{(s)}$ depend on $\dot{v}_q, \dots, v_q^{(q)}$. Notice that, obviously, $dv_q^{(j)}$, for $1 \leq j \leq s$, are independent modulo $\mathcal{X} + \mathcal{U}$. The differential weight of φ is $n + m + 1$ so in order to express the remaining controls v_{r_1+1}, \dots, v_m , firstly, only first order derivatives of a_i and b_i may be involved and, secondly, only one control among v_1, \dots, v_{r_1} , say v_l , can be explicitly present in a_i and b_i (and thus only $d\dot{v}_l \notin \mathcal{X} + \mathcal{U}$, assuring that $d.w.(\varphi) = n + m + 1$).

We will consider two cases depending on whether the control v_l (whose derivative \dot{v}_l is involved) satisfies either $1 \leq l \leq r_0$ or $r_0 + 1 \leq l \leq r_1$.

Consider the case $r_0 + 1 \leq l \leq r_1$. The functions a_i are of the form $a_i = a_i(z, w, v_l)$ but the functions b_i are absent since, on one hand, they have to (nontrivially) depend on v_l but, on the other hand, they depend on v_1, \dots, v_{r_0} only (by the definition of r_0). It follows that $r_2 = m$. The system takes the form

$$\begin{aligned} \dot{z}_i^j &= z_i^{j+1}, & \dot{z}_i^j &= z_i^{j+1}, \\ \dot{z}_i^{\rho_i} &= v_i, \quad r_0 + 1 \leq i \leq r_1, & \dot{z}_i^{\rho_i} &= a_i(z, w, v_l), \quad r_1 + 1 \leq i \leq m, \\ \dot{w} & & \dot{w} &= G(z, w, v), \end{aligned}$$

for some smooth functions a_i , and is flat with flat outputs

$$\begin{aligned} \varphi_i &= v_i, \quad 1 \leq i \leq r_0, \\ \varphi_i &= z_i^1, \quad r_0 + 1 \leq i \leq m. \end{aligned}$$

Now notice that the first r_0 controls v_1, \dots, v_{r_0} and the last $m - r_1$ controls v_{r_1+1}, \dots, v_m do not affect the z -subsystem, so they must be present in the w -subsystem. Therefore, we have $\dim w \geq r_0 + m - r_1$. So for flatness we should be able to express (at least) $r_0 + m - r_1$ components of w and $m - r_1$ controls v_{r_1+1}, \dots, v_m with the help of $2(m - r_1)$ functions a_i and \dot{a}_i , for $r_1 + 1 \leq i \leq m$ (using the functions z_i^j and the controls v_1, \dots, v_{r_0} as well). To do so, we need

$$2(m - r_1) \geq 2(m - r_1) + r_0,$$

implying $r_0 = 0$. Moreover, it follows that $\dim w = m - r_1$ so by a suitable invertible static feedback and permuting the variables to rename v_l as v_1 we can bring the system into the form

$$\begin{aligned} \dot{z}_i^j &= z_i^{j+1}, & \dot{z}_i^j &= z_i^{j+1}, \\ \dot{z}_i^{\rho_i} &= v_i, \quad 1 \leq i \leq r_1, & \dot{z}_i^{\rho_i} &= a_i(z, w, v_1), \\ \dot{w}_i & & \dot{w}_i &= v_i, \quad r_1 + 1 \leq i \leq m, \end{aligned}$$

which is x -flat with flat outputs $\varphi_i = z_i^1$, for $1 \leq i \leq m$.

Now we will consider the case $1 \leq l \leq r_0$. The functions a_i and b_i (the latter may now exist contrary to the previous case) are of the form $a_i = a_i(z, w, v_l)$ and $b_i = b_i(z, w, v_l)$, respectively, and depend nontrivially on v_l . The system takes the form

$$\begin{aligned} \dot{z}_i^j &= z_i^{j+1}, & \dot{z}_i^j &= z_i^{j+1}, \\ \dot{z}_i^{\rho_i} &= v_i, \quad r_0 + 1 \leq i \leq r_1, & \dot{z}_i^{\rho_i} &= a_i(z, w, v_l), \quad r_1 + 1 \leq i \leq r_2, \\ \dot{w} &= G(z, w, v), & & \end{aligned}$$

for some smooth functions a_i and a smooth map G , and is (x, u) -flat with flat outputs

$$\begin{aligned} \varphi_i &= v_i, & 1 \leq i \leq r_0, \\ \varphi_i &= z_i^1, & r_0 + 1 \leq i \leq r_2 \\ \varphi_i &= b_i(z, w, v_l), & r_2 + 1 \leq i \leq m. \end{aligned}$$

Now notice that the first $r_0 - 1$ controls among v_1, \dots, v_{r_0} (all but v_l that is present in $\dot{z}_i^{\rho_i}$, for $r_1 + 1 \leq i \leq r_2$) and the last $m - r_1$ controls v_{r_1+1}, \dots, v_m do not affect the z -subsystem, so they must be present in the w -subsystem. Therefore we have $\dim w \geq r_0 - 1 + m - r_1$. So for flatness we should be able to express (at least) $r_0 - 1 + m - r_1$ components of w and $m - r_1$ controls v_{r_1+1}, \dots, v_m with the help of $2(m - r_1)$ functions a_i, \dot{a}_i, b_j , and \dot{b}_j , where $r_1 + 1 \leq i \leq r_2$ and $r_2 + 1 \leq j \leq m$ (using the functions z_i^j and the controls v_1, \dots, v_{r_0} as well). To do so, we need

$$2(m - r_1) \geq 2(m - r_1) + r_0 - 1,$$

implying $r_0 = 0$ or $r_0 = 1$. The first case is impossible since $1 \leq l \leq r_0$. Therefore $r_0 = 1$ implying $l = 1$. Moreover, it follows that $\dim w = m - r_1$ so by a suitable invertible static feedback we can bring the system into the form

$$\begin{aligned} \dot{z}_i^j &= z_i^{j+1}, & \dot{z}_i^j &= z_i^{j+1}, \\ \dot{z}_i^{\rho_i} &= v_i, & \dot{z}_i^{\rho_i} &= a_i(z, w, v_1), & \dot{w}_i &= v_i, \\ & 2 \leq i \leq r_1, & & r_1 + 1 \leq i \leq r_2, & & r_1 + 1 \leq i \leq m, \end{aligned}$$

and is (x, u) -flat with minimal flat outputs

$$\begin{aligned} \varphi_1 &= v_1, \\ \varphi_i &= z_i^1, & 2 \leq i \leq r_2 \\ \varphi_i &= b_i(z, w, v_1), & r_2 + 1 \leq i \leq m. \end{aligned}$$

We will show that this system is also x -flat. To this end, observe that $\text{rk}\left(\frac{\partial a}{\partial(v_1, w)}\right) = r_2 - r_1$, where $a = (a_{r_1+1}, \dots, a_{r_2})$, and that all a_i depend explicitly on v_1 . Therefore there exist $m - r_2 + 1$ components w_i of w such that $\text{rk}\left(\frac{\partial(a, w^2)}{\partial(v_1, w)}\right) = m - r_1 + 1$, where $w^2 = (w_i)$, with $i \in I_2 \subset \{r_1 + 1, \dots, m\}$ and the set of indices I_2 is of cardinality $m - r_2 + 1$. Let $I_1 \subset \{r_1 + 1, \dots, m\}$ be the set of remaining indices. We claim that the system is x -flat with flat outputs $\psi_i = z_i^1$, for $2 \leq i \leq r_2$, and $\psi_i = w_i$, for $i \in I_2$. Indeed, differentiating $(\rho_i - 1)$ -times $\psi_i = z_i^1$, for $2 \leq i \leq r_2$, we get $z_i^j, v_2, \dots, v_{r_1}$, and $a_{r_1+1}, \dots, a_{r_2}$. Since $\text{rk}\left(\frac{\partial(a, w^2)}{\partial(v_1, w)}\right) = m - r_1 + 1$, it follows by the implicit function theorem that knowing a_j and $\psi_i = w_i$, for $i \in I_2$, we can express v_1 and the components w_i , for $i \in I_1$. Knowing all components w_i , for $r_1 + 1 \leq i \leq m$, we get all remaining controls via $\dot{w}_i = v_i$.

Notice that in the just considered case (for which we have proven to have $l = r_0 = 1$), we can bring the system together with the components ψ_i 's of its flat output to the following form. Rename $z_{r_1+1}^j$ as z_1^j and a_{r_1+1} as a_1 , then w_i , for $i \in I_2$, as $z_{r_1+1}^1, \dots, z_{m-r_2+r_1+1}^1$, respectively, and w_i , for $i \in I_1$, as $w_{m-r_2+r_1+2}, \dots, w_m$, respectively, and, finally, the old z_i^j , for $r_1 + 2 \leq i \leq r_2$, as $z_{m-r_2+r_1+2}^j, \dots, z_m^j$. Applying the invertible static feedback $\tilde{v}_1 = a_1(z, w, v_1)$, which transforms a_i into $\tilde{a}_i(z, w, \tilde{v}_1)$, and keeping the notation v_1 and a_1 for the tilde-variables, we bring the system into

$$\begin{aligned} \dot{z}_i^j &= z_i^{j+1}, & \dot{z}_i^j &= z_i^{j+1}, \\ \dot{z}_i^{\rho_i} &= v_i, \quad 1 \leq i \leq q_1, & \dot{z}_i^{\rho_i} &= a_i(z, w, v_1), \\ & & \dot{w}_i &= v_i, \quad q_1 + 1 \leq i \leq m, \end{aligned}$$

where $q_1 = m - r_2 + r_1 + 1$ (and hence $\dim w = r_2 - r_1 - 1$) and $\rho_i = 1$, for $r_1 + 1 \leq i \leq q_1$, which is x -flat with minimal flat output $\psi_i = z_i^1$, for $1 \leq i \leq m$.

(ii) \Rightarrow (iii). We will use the notations from the proof of the implication (i) \Rightarrow (ii) above. Assume that the system is x -flat, let $\varphi_1(x), \dots, \varphi_m(x)$ be components of its flat output, ρ_i their relative degrees and denote $\text{rk} \left(\frac{\partial \varphi_i^{(\rho_i)}}{\partial u_j} \right) = r$ (we use r instead of r_1 because the two other integers are trivial: $r_0 = 0$ since the system is x -flat and thus $r_2 = m$). As above, we introduce the functions $z_i^j = L_F^{j-1} \varphi_i$, apply an invertible static feedback and permute the flat outputs to get $\dot{z}_i^{\rho_i} = v_i$, for $1 \leq i \leq r$ and $\dot{z}_i^{\rho_i} = a_i(z, w, v_1, \dots, v_r)$, for $r + 1 \leq i \leq m$, where w completes the z_i^j 's to a coordinate system. The system is x -flat of differential weight $n = m + 1$, so using the same argument as above (in the first case $r_0 + 1 \leq l \leq r_1$), we prove that all functions a_i depend on one control only, say v_1 , and that their first derivatives only can be used to express the missing controls v_{r+1}, \dots, v_m . So the system can be brought via an invertible static feedback to the form

$$(7.4) \quad \begin{aligned} \dot{z}_i^j &= z_i^{j+1}, & \dot{z}_i^j &= z_i^{j+1}, \\ \dot{z}_i^{\rho_i} &= v_i, \quad 1 \leq i \leq r, & \dot{z}_i^{\rho_i} &= a_i(z, w, v_1), \\ & & \dot{w}_i &= v_i, \quad r + 1 \leq i \leq m, \end{aligned}$$

with x -flat outputs $\varphi_i = z_i^1$, for $1 \leq i \leq m$.

Notice that the above form perfectly coincides with both forms that we have obtained when proving the implication (i) \Rightarrow (ii). Indeed, in the case $r_0 + 1 \leq l \leq r_1$ we have proved that the system is x -flat and admits the above form with $r = r_1$. In the case $1 \leq l \leq r_0$, we have proved that although the system is (x, u) -flat with differential weight $n + m + 1$, it is also x -flat and admits the above form with $r = q_1$.

Obviously the system becomes static feedback linearizable via the preintegration $v_1 = y_1$, $\dot{y}_1 = \tilde{v}_1$, $\tilde{v}_i = v_i$, $2 \leq i \leq m$. Notice that, if Ξ is the control-affine system Σ , then the feedback transformation bringing Σ into the above form is actually a control-affine transformation $u = \alpha(z) + \beta(z)v$ yielding $\dot{z}_i^{\rho_i} = v_i$, $1 \leq i \leq r$, and $\dot{z}_i^{\rho_i} = a_i(z, w, v_1) = \alpha_i(z, w) + \beta_i(z, w)v_1$, for $r + 1 \leq i \leq m$.

(iii) \Rightarrow (i). Suppose that the one-fold prolongation of the first control of $\Xi : \dot{x} = F(x, u)$, given by

$$\Xi^{(1,0,\dots,0)} : \begin{cases} \dot{x} &= F(x, y_1, v_2, \dots, v_m) \\ \dot{y}_1 &= v_1 \end{cases}$$

where $u_1 = y_1$ and $u_i = v_i$, $2 \leq i \leq m$, is locally static feedback linearizable. We will prove that Ξ is flat of differential weight $n + m + 1$.

$\Xi^{(1,0,\dots,0)}$ is equivalent via a diffeomorphism $z = \phi(x, y_1)$ and an invertible transformation $v = \psi(x, y_1, \tilde{v})$, to the Brunovský canonical form

$$\begin{aligned} \dot{z}_i^j &= z_i^{j+1}, \quad 1 \leq j \leq \rho_i - 1, \\ \dot{z}_i^{\rho_i} &= \tilde{v}_i, \quad 1 \leq i \leq m, \end{aligned}$$

where $\dim z = \sum_{i=1}^m \rho_i = n + 1$, for which $\varphi = (z_1^1, \dots, z_m^1)$ is a minimal flat output of differential weight $n + m + 1$. It follows that $z = (\bar{\varphi}_1^{\rho_1-1}, \dots, \bar{\varphi}_m^{\rho_m-1}) = \phi(x, y_1)$, where $\bar{\varphi}_i^j = (\varphi_i, \dot{\varphi}_i, \dots, \varphi_i^{(j)})$, thus for the original variables x and the first component $u_1 = y_1$ of u , we have $(x, u_1)^t = \phi^{-1}(z) = \phi^{-1}(\bar{\varphi}_1^{\rho_1-1}, \dots, \bar{\varphi}_m^{\rho_m-1})$. Moreover, $\tilde{v} = (\varphi_1^{(\rho_1)}, \dots, \varphi_m^{(\rho_m)})$, the map being invertible with respect to $v = (v_1, \dots, v_m)$, so we deduce that $u_i = v_i = \delta_i(\bar{\varphi}_1^{\rho_1}, \dots, \bar{\varphi}_m^{\rho_m})$, for $2 \leq i \leq m$ (for suitable components δ_i of the inverse), showing that φ is a flat output of Ξ of differential weight $n + m + 1$.

7.3. Proof of Theorem 3.2. Necessity. Let us consider a flat control system Σ : $\dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$, of differential weight $n + m + 1$. According to Proposition 3.1, there exists an invertible feedback transformation $u = \alpha(x) + \beta(x)\tilde{u}$, bringing Σ into the form $\tilde{\Sigma}$: $\dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \sum_{i=2}^m \tilde{u}_i \tilde{h}_i(x)$, such that the prolongation

$$\tilde{\Sigma}^{(1,0,\dots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^m v_i \tilde{h}_i(x) \\ \dot{y}_1 &= v_1, \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_j = \tilde{u}_j$, for $2 \leq j \leq m$, is locally static feedback linearizable. For simplicity of notation, we will drop the tildes redbut we will keep distinguishing g_1 from h_j (which could also be denoted g_j , $2 \leq j \leq m$) whose controls are not preintegrated. Recall that, see Section 7.1, that \mathcal{D}_p^i denote the linearizability distributions of the prolonged system $\Sigma^{(1,0,\dots,0)}$. Since $\Sigma^{(1,0,\dots,0)}$ is locally static feedback linearizable, \mathcal{D}_p^i are involutive, of constant rank, for any $i \geq 0$, and there exists an integer ρ such that $\text{rk } \mathcal{D}_p^\rho = n + 1$. We have

$$\begin{aligned} \mathcal{D}_p^0 &= \text{span} \left\{ \frac{\partial}{\partial y_1}, h_j, 2 \leq j \leq m \right\}, \\ \mathcal{D}_p^1 &= \text{span} \left\{ \frac{\partial}{\partial y_1}, g_1, h_j, \text{ad}_f h_j + y_1 [g_1, h_j], 2 \leq j \leq m \right\}. \end{aligned}$$

Since $k \geq 1$, the distribution $\mathcal{D}^0 = \text{span} \{g_1, h_j, 2 \leq j \leq m\}$ is involutive, thus $[g_1, h_j] \in \mathcal{D}^0$ and hence $\mathcal{D}_p^1 = \text{span} \left\{ \frac{\partial}{\partial y_1}, g_1, h_j, \text{ad}_f h_j, 2 \leq j \leq m \right\}$. It is easy to prove (by an induction argument) that, for $1 \leq i \leq k$,

$$\mathcal{D}_p^i = \text{span} \left\{ \frac{\partial}{\partial y_1}, g_1, \dots, \text{ad}_f^{i-1} g_1, h_j, \dots, \text{ad}_f^i h_j, 2 \leq j \leq m \right\}.$$

Define

$$\mathcal{H}^k = \text{span} \{g_1, \dots, \text{ad}_f^{k-1} g_1, h_j, \dots, \text{ad}_f^k h_j, 2 \leq j \leq m\}.$$

Since the intersection of involutive distributions is an involutive distribution, $\mathcal{H}^i = \mathcal{D}_p^i \cap TX = \text{span} \{g_1, \dots, \text{ad}_f^{i-1} g_1, h_j, \dots, \text{ad}_f^i h_j, 2 \leq j \leq m\}$ is involutive, for $1 \leq i \leq k$. In particular, we have \mathcal{H}^k involutive. Moreover \mathcal{H}^k is a well defined distribution on X (it does not depend on y). It is immediate that $\mathcal{D}^{k-1} \subset \mathcal{H}^k \subset \mathcal{D}^k$, where the second inclusion is of corank one, otherwise $\mathcal{H}^k = \mathcal{D}^k$ and \mathcal{D}^k would be involutive or $\mathcal{H}^k = \mathcal{D}^{k-1}$ and $\text{rk } \mathcal{D}^k - \text{rk } \mathcal{D}^{k-1} = 1$, which contradicts Proposition 7.1 asserting that

$\text{rk } \mathcal{D}^k - \text{rk } \mathcal{D}^{k-1} \geq 2$ is necessary for flatness of differential weight $n + m + 1$. Recall that $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$, for $i \geq k + 1$. We have

$$\mathcal{D}_p^{k+1} = \text{span} \left\{ \frac{\partial}{\partial y_1} \right\} + \mathcal{H}^k + [f, \mathcal{H}^k] = \text{span} \left\{ \frac{\partial}{\partial y_1} \right\} + \mathcal{H}^{k+1}$$

and by an induction argument

$$\mathcal{D}_p^{k+i} = \text{span} \left\{ \frac{\partial}{\partial y_1} \right\} + \mathcal{H}^{k+i}, \quad i \geq 2.$$

Consequently, the involutivity of \mathcal{D}_p^{k+i} implies that of \mathcal{H}^{k+i} , for $i \geq 1$. Moreover, $\text{rk } \mathcal{D}_p^\rho = n + 1$, proving that $\text{rk } \mathcal{H}^\rho = n$, i.e., $\mathcal{H}^\rho = TX$.

Sufficiency. Consider a control system satisfying (A1) – (A3) and let $\mathcal{H}^0 = \text{span} \{h_j, 2 \leq j \leq m\}$ be the distribution defined by Proposition 7.2(ii). This system is static feedback equivalent to $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + \sum_{i=2}^m u_i h_i(x)$. By the same proposition, the involutivity of $\mathcal{H}^i = \mathcal{D}^{i-1} + \text{ad}_f^i \mathcal{H}$ follows for $0 \leq i \leq k - 1$. It is immediate to see that the prolongation

$$\Sigma^{(1,0,\dots,0)} : \begin{cases} \dot{x} &= f(x) + y_1 g_1(x) + \sum_{i=2}^m v_i h_i(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = u_1$ and $v_j = u_j$, for $2 \leq j \leq m$, is locally static feedback linearizable. Indeed, the linearizability distributions \mathcal{D}_p^i , associated to $\Sigma^{(1,0,\dots,0)}$, are of the form

$$\mathcal{D}_p^i = \text{span} \left\{ \frac{\partial}{\partial y_1} \right\} + \mathcal{H}^i, \quad i \geq 0,$$

and the involutivity of \mathcal{H}^i implies that of \mathcal{D}_p^i , because \mathcal{H}^i does not depend on y_1 . Moreover, $\text{rk } \mathcal{H}^\rho = n$, thus $\text{rk } \mathcal{D}_p^\rho = n + 1$ and $\Sigma^{(1,0,\dots,0)}$ is locally static feedback linearizable. By Proposition 3.1, the system Σ is flat of differential weight $n + m + 1$.

7.4. Proof of Theorem 3.3. Necessity.

Repeating the beginning of the necessity part of the Proof of Theorem 3.2, we conclude that the linearizability distributions \mathcal{D}_p^i of the prolonged system $\Sigma^{(1,0,\dots,0)}$ (we drop the tildes) are involutive, of constant rank, for any $i \geq 0$, and there exists an integer ρ such that $\text{rk } \mathcal{D}_p^\rho = n + 1$. We have

$$\mathcal{D}_p^0 = \text{span} \left\{ \frac{\partial}{\partial y_1}, h_j, 2 \leq j \leq m \right\}$$

involutive. It follows immediately that

$$\mathcal{H}^0 = \text{span} \{h_j, 2 \leq j \leq m\}$$

is a well defined distribution on X , is involutive (as intersection of involutive distributions $\mathcal{H}^0 = \mathcal{D}_p^0 \cap TX$) and of corank one in \mathcal{D}^0 . This shows (A1)'. The distribution

$$\mathcal{D}_p^1 = \text{span} \left\{ \frac{\partial}{\partial y_1}, g_1, h_j, \text{ad}_f h_j + y_1 [g_1, h_j], 2 \leq j \leq m \right\}$$

is involutive and we deduce that $[g_1, h_j] \in \mathcal{D}_p^1$ and $\text{ad}_f h_j \in \mathcal{D}_p^1$. Thus

$$\mathcal{D}_p^1 = \text{span} \left\{ \frac{\partial}{\partial y_1}, g_1, h_j, \text{ad}_f h_j, [g_1, h_j], 2 \leq j \leq m \right\} = \text{span} \left\{ \frac{\partial}{\partial y_1} \right\} + \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] + [f, \mathcal{H}^0]$$

and, in particular, we deduce that $\mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] + [f, \mathcal{H}^0] = \text{span}\{g_1, h_j, \text{ad}_f h_j + y_1[g_1, h_j], 2 \leq j \leq m\}$. From this, it is immediate that

$$\mathcal{H}^1(x) = \text{span}\{g_1, h_j, \text{ad}_f h_j + u_1[g_1, h_j], 2 \leq j \leq m\} = \mathcal{H}_u^1(x),$$

around (x_0, u_0) , implying (A4)'.

The involutivity of \mathcal{D}_p^1 implies that of $\mathcal{H}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] + [f, \mathcal{H}^0]$, because $\mathcal{H}^1 = \mathcal{D}_p^1 \cap TX$ is the intersection of two involutive distributions.

The rest of the proof follows the same line as that of Theorem 3.2.

Sufficiency. Consider a control system $\Sigma : \dot{x} = f(x) + u_1 g_1(x) + \sum_{i=2}^m u_i h_i(x)$ satisfying (A1)' – (A4)', where the corank one involutive subdistribution is given by $\mathcal{H}^0 = \text{span}\{h_j, 2 \leq j \leq m\}$. We will prove that the prolongation

$$\Sigma^{(1,0,\dots,0)} : \begin{cases} \dot{x} &= f(x) + y_1 g_1(x) + \sum_{i=2}^m v_i h_i(x) \\ \dot{y}_1 &= v_1 \end{cases}$$

with $y_1 = u_1$ and $v_i = u_i$, for $2 \leq i \leq m$, is locally static feedback linearizable, around (x_0, y_0) . We have $\mathcal{D}_p^0 = \text{span}\{\frac{\partial}{\partial y_1}, h_j, 2 \leq j \leq m\} = \text{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^0$, which is clearly involutive (since so is \mathcal{H}^0 by (A1)'), and

$$\mathcal{D}_p^1 = \text{span}\{\frac{\partial}{\partial y_1}, g_1, h_j, \text{ad}_f h_j + y_1[g_1, h_j], 2 \leq j \leq m\}.$$

According to (A4)' we have, around (x_0, u_0) ,

$$\mathcal{H}_u^1(x) = \text{span}\{g_1, h_j, [f + u_1 g_1 + \sum_{i=2}^m u_i h_i, h_j], 2 \leq j \leq m\} = \mathcal{H}^1(x),$$

and thus

$$\mathcal{D}_p^1 = \text{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^1.$$

It follows, by an induction argument, that all linearizability distributions \mathcal{D}_p^i , associated to $\Sigma^{(1,0,\dots,0)}$, are of the form

$$\mathcal{D}_p^i = \text{span}\{\frac{\partial}{\partial y_1}\} + \mathcal{H}^i, \quad i \geq 1,$$

and the involutivity of \mathcal{H}^i implies that of \mathcal{D}_p^i . Moreover, $\text{rk } H^\rho = n$, thus $\text{rk } \mathcal{D}_p^\rho = n+1$ and $\Sigma^{(1,0,\dots,0)}$ is locally static feedback linearizable. By Proposition 3.1, the system Σ is flat of differential weight $n + m + 1$.

7.5. Proof of Theorem 3.7 (ii). Before giving the proof of Theorem 3.7 (ii), notice that under the assumption \mathcal{D}^i involutive, for all $0 \leq i \leq k-1$, we have $\mathcal{D}^{k-2} \subset \mathcal{C}^k$, where \mathcal{C}^k is the characteristic distribution of \mathcal{D}^k . We will use that property in our proof.

Necessity. Repeating the beginning of the necessity part of the Proof of Theorem 3.2, we conclude that the linearizability distributions \mathcal{D}_p^i of the prolonged system $\Sigma^{(1,0,\dots,0)}$ (we drop the tildes) are involutive, of constant rank, for any $i \geq 0$, and there exists an integer ρ such that $\text{rk } \mathcal{D}_p^\rho = n+1$. Since $k \geq 1$, the distribution $\mathcal{D}^0 = \text{span}\{g_1, h_j, 2 \leq j \leq m\}$ is involutive, thus $[g_1, h_j] \in \mathcal{D}^0$ and hence $\mathcal{D}_p^1 =$

$\text{span}\{\frac{\partial}{\partial y_1}, g_1, h_j, ad_f h_j, 2 \leq j \leq m\}$. It is easy to prove (by an induction argument) that, for $1 \leq i \leq k$,

$$\mathcal{D}_p^i = \text{span}\{\frac{\partial}{\partial y_1}, g_1, \dots, ad_f^{i-1} g_1, h_j, \dots, ad_f^i h_j, 2 \leq j \leq m\}.$$

Since the intersection of involutive distributions is an involutive distribution, $\mathcal{D}_p^i \cap TX = \text{span}\{g_1, \dots, ad_f^{i-1} g_1, h_j, \dots, ad_f^i h_j, 2 \leq j \leq m\}$ is involutive, for $1 \leq i \leq k$. We deduce that the distribution

$$\mathcal{E}^k = \text{span}\{g_1, \dots, ad_f^{k-1} g_1, h_j, \dots, ad_f^k h_j, 2 \leq j \leq m\}$$

is involutive. Next we will prove that $\mathcal{E}^k = \mathcal{H}^k$ (recall that $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1}$, where \mathcal{C}^k is the characteristic distribution of \mathcal{D}^k).

It is immediate that $\mathcal{D}^{k-1} \subset \mathcal{E}^k \subset \mathcal{D}^k$, where the second inclusion is of corank one, otherwise $\mathcal{E}^k = \mathcal{D}^k$ and \mathcal{D}^k would be involutive, which contradicts our hypothesis.

Applying the Jacobi identity, it can be proved that $[ad_f^{k-1} h_j, ad_f^k g_1] \in \mathcal{D}^k$, for all $2 \leq j \leq m$, and since \mathcal{E}^k is involutive, we immediately have $[ad_f^{k-1} h_j, \mathcal{D}^k] \subset \mathcal{D}^k$, for $2 \leq j \leq m$. Thus $ad_f^{k-1} h_j \in \mathcal{C}^k$, for all $2 \leq j \leq m$, where \mathcal{C}^k is the characteristic distribution of \mathcal{D}^k . Moreover, since $\mathcal{D}^k = \mathcal{E}^k + \text{span}\{ad_f^k g_1\}$ is non-involutive and $[\mathcal{D}^{k-1}, \mathcal{D}^k] \not\subset \mathcal{D}^k$, we deduce that the new direction completing \mathcal{D}^k to $\overline{\mathcal{D}^k}$ is given by $[ad_f^{k-1} g_1, ad_f^k g_1] \notin \mathcal{D}^k$. Hence there exists smooth functions α_j such that $[ad_f^k h_j, ad_f^k g_1] = \alpha_j [ad_f^{k-1} g_1, ad_f^k g_1] \text{ mod } \mathcal{D}^k$, for $2 \leq j \leq m$. It follows $[ad_f^k h_j - \alpha_j ad_f^{k-1} g_1, ad_f^k g_1] = 0 \text{ mod } \mathcal{D}^k$. It is easy to show that

$$\mathcal{C}^k = \mathcal{D}^{k-2} + \text{span}\{ad_f^{k-1} h_j, ad_f^k h_j - \alpha_j ad_f^{k-1} g_1, 2 \leq j \leq m\}$$

thus $\mathcal{H}^k = \mathcal{C}^k + \mathcal{D}^{k-1} = \text{span}\{g_1, \dots, ad_f^{k-1} g_1, h_j, \dots, ad_f^k h_j, 2 \leq j \leq m\}$, $\text{rk } \mathcal{C}^k = \text{rk } \mathcal{D}^k - 2 = \mathcal{E}^k$ proving involutivity of \mathcal{H}^k and implying $\text{rk}(\mathcal{C}^k \cap \mathcal{D}^{k-1}) = \text{rk } \mathcal{D}^{k-1} - 1$.

The proof of the involutivity of \mathcal{H}^i , for $i \geq k+1$ and of the existence of ρ such that $\mathcal{H}^\rho = TX$ follows the same line as that of Theorem 3.2.

Sufficiency. Consider a control system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$ satisfying (C1)-(C4). We start our proof by showing that conditions (C1)-(C2) enable us to define a distribution \mathcal{H} such that $\mathcal{H} \subset \mathcal{D}^0$, of corank one, and $\mathcal{H}^k = \mathcal{D}^{k-1} + ad_f^k \mathcal{H}$.

To this aim, let us denote by r the corank of $\mathcal{D}^{k-2} \subset \mathcal{D}^{k-1}$. Assume that the vector fields $g_i \in \mathcal{D}^0$, for $1 \leq i \leq r$, satisfy

$$\mathcal{D}^{k-1} = \mathcal{D}^{k-2} + \text{span}\{ad_f^{k-1} g_i, 1 \leq i \leq r\}.$$

Applying similar arguments to those used in the proof of Proposition 7.2(ii), we can define $m-r$ vector fields h_j , for $r+1 \leq j \leq m$, such that $\mathcal{D}^0 = \text{span}\{g_1, \dots, g_r, h_{r+1}, \dots, h_m\}$ and $ad_f^{k-1} h_j \in \mathcal{D}^{k-2}$, for $r+1 \leq j \leq m$.

It is clear that $\mathcal{D}^{k-2} \subset \mathcal{C}^k$ and, since $\text{rk}(\mathcal{C}^k \cap \mathcal{D}^{k-1}) = \text{rk } \mathcal{D}^{k-1} - 1$, we have

$$\mathcal{C}^k \cap \mathcal{D}^{k-1} = \mathcal{D}^{k-2} + \text{span}\{c_j, 1 \leq j \leq r-1\},$$

where the vector fields c_j are of the form

$$c_j = \sum_{i=1}^r \lambda_j^i ad_f^{k-1} g_i = ad_f^{k-1} \left(\sum_{i=1}^r \lambda_j^i g_i \right) \text{ mod } \mathcal{D}^{k-2},$$

with λ_j^i smooth functions such that the matrix $\Lambda = (\lambda_j^i)$, where $1 \leq i \leq r$ and $1 \leq j \leq r-1$, is of full rank $r-1$. Denote $h_{j+1} = \sum_{i=1}^r \lambda_j^i g_i$, for $1 \leq j \leq r-1$, and suppose, without loss of generality, that g_1 is independent with them.

Since $ad_f^{k-1} h_j \in \mathcal{C}^k$, for $2 \leq j \leq m$, we have $[ad_f^{k-1} h_j, \mathcal{D}^k] \subset \mathcal{D}^k$. From this it can be shown, applying the Jacobi identity, that $[ad_f^{k-1} g_1, ad_f^k h_j] \in \mathcal{D}^k$, for $2 \leq j \leq m$. Therefore, the new direction completing \mathcal{D}^k to $\overline{\mathcal{D}}^k = \mathcal{D}^k + [\mathcal{D}^{k-1}, \mathcal{D}^k]$ is given by $[ad_f^{k-1} g_1, ad_f^k g_1]$ and there exist smooth functions α_j such that $[ad_f^k h_j, ad_f^k g_1] = \alpha_j [ad_f^{k-1} g_1, ad_f^k g_1] \bmod \mathcal{D}^k$, for $2 \leq j \leq m$. This gives $[ad_f^k h_j - \alpha_j ad_f^{k-1} g_1, ad_f^k g_1] = 0 \bmod \mathcal{D}^k$ and it can be easily verified that the characteristic distribution \mathcal{C}^k is given by

$$\mathcal{C}^k = \mathcal{D}^{k-2} + \text{span} \{ad_f^{k-1} h_j, ad_f^k h_j - \alpha_j ad_f^{k-1} g_1, 2 \leq j \leq m\}.$$

It follows immediately

$$\mathcal{H}^k = \mathcal{D}^{k-1} + \text{span} \{ad_f^k h_j, 2 \leq j \leq m\} = \mathcal{D}^{k-1} + ad_f^k \mathcal{H},$$

where the corank one subdistribution \mathcal{H} of \mathcal{D}^0 is given by

$$\mathcal{H} = \text{span} \{h_j, 2 \leq j \leq m\}.$$

The involutivity of \mathcal{H}^k implies that of all distributions $\mathcal{H}^i = \mathcal{D}^{i-1} + ad_f^i \mathcal{H}$, for $0 \leq i \leq k-1$, where $\mathcal{D}^{-1} = \{0\}$ and $\mathcal{H}^0 = \mathcal{H}$. The proof of that statement follows by the same method as that used in the proof of Proposition 7.2(ii).

We are now in position to show that the control system $\Sigma : \dot{x} = f(x) + \sum_{i=1}^m u_i g_i(x)$ is dynamically linearizable via an invertible one-fold prolongation. Transform Σ via an invertible static feedback into the form $\tilde{\Sigma} : \dot{x} = \tilde{f}(x) + \tilde{u}_1 \tilde{g}_1(x) + \sum_{i=2}^m \tilde{u}_i h_i(x)$, where the vector fields h_i are defined as above. Applying the same arguments as in the proof of Theorem 3.2, it is immediate to see that the prolongation

$$\tilde{\Sigma}^{(1,0,\dots,0)} : \begin{cases} \dot{x} &= \tilde{f}(x) + y_1 \tilde{g}_1(x) + \sum_{i=2}^m v_i h_i(x) \\ \dot{y}_1 &= v_1, \end{cases}$$

with $y_1 = \tilde{u}_1$ and $v_j = \tilde{u}_j$, for $2 \leq j \leq m$, is locally static feedback linearizable.

7.6. Proof of Proposition 4.1. Recall that to any flat system of differential weight $n+m+1$ we can attach, according to Lemma 3.5 in Section 3.1, the following sequence of nested involutive distributions:

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \dots \subset \mathcal{H}^{\rho-1} \subset \mathcal{H}^\rho = TX$$

where either $k=0$ and then \mathcal{H}^0 is the involutive corank one subdistribution of \mathcal{D}^0 , the distribution \mathcal{H}^1 is defined by $\mathcal{H}^1 = \mathcal{D}^0 + [\mathcal{D}^0, \mathcal{D}^0] + [f, \mathcal{H}^0]$ and $\mathcal{H}^{i+1} = \mathcal{H}^i + [f, \mathcal{H}^i]$, for $1 \leq i \leq \rho-1$, or $k \geq 1$ and then \mathcal{H}^0 is the involutive corank one subdistribution of \mathcal{D}^0 associated to \mathcal{H}^k and $\mathcal{H}^i = \mathcal{D}^{i-1} + ad_f^i \mathcal{H}$, $1 \leq i \leq k-1$. If $k \geq 1$, then, for $i \geq 2$, we actually have $\mathcal{H}^i = \mathcal{H}^{i-1} + [f, \mathcal{H}^{i-1}]$. For both cases, ρ stands for the smallest integer such that $\mathcal{H}^\rho = TX$. Recall that we denote by r_j the corank of the inclusion $\mathcal{H}^{\rho-j} \subset \mathcal{H}^{\rho-j+1}$, for $1 \leq j \leq \rho$. We clearly have $1 \leq r_1 \leq r_2 \leq \dots \leq r_\rho \leq m$. We will prove Proposition 4.1 in the case $k \geq 1$. If $k=0$, then similar arguments apply.

Necessity. Let $(\varphi_1, \dots, \varphi_{r_1}, \psi_{r_1+1}, \dots, \psi_m)$ be a minimal flat output of Σ , defined on a neighborhood \mathcal{X} of x_0 . According to the proof of Proposition 7.2, around any

point of \mathcal{X}' , an open and dense subset of \mathcal{X} , there exists a valid local change of coordinates in which the system, after applying a suitable invertible static feedback, takes the form (7.4), with φ_j and ψ_l being equal to the top variables z_i^1 , for $1 \leq i \leq m$, where ψ_l correspond to z_i^1 such that $\rho_i = 1$. The system under consideration is control-affine so, compare the comment following (7.4), we have $\dot{z}_i^{\rho_i} = a_i(z) + b_i(z)v_1$ and, by changing coordinates and applying a static feedback, we can assume $a_i = z_i^{\rho_i+1}$ (since we consider the case $k \geq 1$). For $0 \leq j \leq \rho$, we have

$$\begin{aligned} \mathcal{H}^j &= \text{span} \left\{ \frac{\partial}{\partial z_1^{\rho_1-j+1}}, \dots, \frac{\partial}{\partial z_1^{\rho_1}} \right\} \\ &+ \text{span} \left\{ \frac{\partial}{\partial z_i^{\rho_i-j}}, \dots, \frac{\partial}{\partial z_i^{\rho_i}}, \quad 2 \leq i \leq r \right\} \\ &+ \text{span} \left\{ \frac{\partial}{\partial z_i^{\rho_i-j+1}}, \dots, \frac{\partial}{\partial z_i^{\rho_i+1}}, \quad r+1 \leq i \leq m \right\}. \end{aligned}$$

It follows that $\mathcal{H}^{\rho-1}$ (recall that ρ is the smallest integer such that $\mathcal{H}^\rho = TX$) is annihilated by dz_i^1 such that $\rho_i = \rho + 1$, if $2 \leq i \leq r$, and $\rho_i = \rho$, if either $i = 1$ or $r+1 \leq i \leq m$. There are r_1 differentials of the form dz_i^1 satisfying one of the above conditions. Similarly, we show that \mathcal{H}^j is annihilated by dz_i^l such that $1 \leq l \leq \rho_i - j - 1$, if $2 \leq i \leq r$, and $1 \leq l \leq \rho_i - j$, if either $i = 1$ or $r+1 \leq i \leq m$. Among them there are $r_{\rho-j}$ differentials of the form dz_i^1 . It follows that r_j differentials of the form dz_i^1 annihilate $\mathcal{H}^{\rho-j}$, for $1 \leq j \leq \rho$, which (after a permutation, if necessary) are, respectively, $dz_i^1 = d\varphi_i$ proving that the functions $\varphi_1, \dots, \varphi_{r_\rho}$ satisfy condition (FO1) on \mathcal{X}' . Since all functions φ_i , for $1 \leq i \leq r_1$, as well as all involved distributions \mathcal{H}^j are defined on \mathcal{X} , by continuity (FO1) is valid on \mathcal{X} .

The differentials of flat outputs and those of their successive time-derivatives are always independent, so (FO2) follows.

Sufficiency. Let $(\varphi_1, \dots, \varphi_{r_\rho}, \psi_{r_{\rho+1}}, \dots, \psi_m)$ be an m -tuple of functions satisfying conditions (FO1)-(FO2). For any $1 \leq j \leq \rho$, define the functions $z_i^j = L_f^{l-j} \varphi_i$, where $j \leq l \leq \rho$ and $r_{j-1} + 1 \leq i \leq r_j$. Notice that now the variables z_i^j are indexed not the same way as those of the form (7.4), in particular, in the actual definition, the z_i^1 exist only for $1 \leq i \leq r_1$, the z_i^2 only for $1 \leq i \leq r_2$ and, in general, z_i^j only for $1 \leq i \leq r_j$. The differentials $dz_i^l = dL_f^{l-j} \varphi_i$ are independent because of (FO2) and we have $\dim z^j = r_j = \text{cork}(\mathcal{H}^{\rho-j} \subset \mathcal{H}^{\rho-j+1})$, for $1 \leq j \leq \rho$. Since $\text{rk } \mathcal{H}^0 = m - 1$, it follows that $\text{rank } \mathcal{H}^\rho = n = (m - 1) + r_1 + \dots + r_\rho$. Complete (z^1, \dots, z^ρ) by $z_i^{\rho+1}$, for $2 \leq i \leq m$, such that $z = (z^1, \dots, z^\rho, z^{\rho+1})$ forms a local coordinate system around z_0 .

Using condition (FO1) and $\mathcal{H}^j = \mathcal{H}^{j-1} + [f, \mathcal{H}^{j-1}]$ (which is valid for $j \geq 2$) we conclude that the differentials dz_i^1, \dots, dz_i^j annihilate the distribution $\mathcal{H}^{\rho-j}$. Recall that $\mathcal{D}^0 = \text{span} \{g_1, \dots, g_m\} \subset \mathcal{H}^1$, so the relative degree of z_i^j , for $1 \leq j \leq \rho$ is at least $\rho - j + 1$. It follows that in the z -coordinates, the system takes the form

$$\begin{aligned} \dot{z}_i^1 &= z_i^2 && \text{for } 1 \leq i \leq r_1 \\ \dot{z}_i^2 &= z_i^3 && \text{for } 1 \leq i \leq r_2 \\ &\vdots && \\ \dot{z}_i^{\rho-1} &= z_i^\rho && \text{for } 1 \leq i \leq r_{\rho-1} \\ \dot{z}_i^\rho &= f_i^\rho(z) + \sum_{j=1}^m u_j g_{ji}^\rho(z) && \text{for } 1 \leq i \leq r_\rho \\ \dot{z}_i^{\rho+1} &= f_i^{\rho+1}(z) + \sum_{j=1}^m u_j g_{ji}^{\rho+1}(z) && \text{for } 2 \leq i \leq m. \end{aligned}$$

The involutive distribution $\mathcal{H}^0 = \text{span} \left\{ \frac{\partial}{\partial z_i^{\rho+1}}, 2 \leq i \leq m \right\}$ is of corank one in $\mathcal{D}^0 = \text{span} \{g_1, \dots, g_m\}$ so by a suitable static feedback transformation $u = \alpha(z) + \beta(z)v$ and a permutation of the z -variables we can transform the z^ρ - and $z^{\rho+1}$ -subsystems into

$$\begin{aligned} \dot{z}_{i^*}^\rho &= v_{i^*} \\ \dot{z}_i^\rho &= a_i(z) + b_i(z)v_{i^*} \quad \text{for } 1 \leq i \leq r_\rho, \quad i \neq i^* \\ \dot{z}_i^{\rho+1} &= v_i \quad \text{for } 1 \leq i \leq m, \quad i \neq i^* \end{aligned}$$

(the relative degree of φ_{i^*} leading to v_{i^*} can be any between 1 and ρ). Recall that $\mathcal{H}^1 = \mathcal{D}^0 + [f, \mathcal{H}^0]$ is of rank $m + r_\rho - 1$. It follows that $\text{rk} \left(\frac{\partial a_i}{\partial z_j^{\rho+1}} \right) = r_\rho - 1$, where $1 \leq i \leq r_\rho$ and $1 \leq j \leq m$, with $i, j \neq i^*$. We introduce the new coordinates (but keep the same notation for them) $z_i^{\rho+1} = a_i(z)$, for $1 \leq i \leq r_\rho$, $i \neq i^*$, and apply a suitable static feedback (keeping the notation v_i for the modified controls), to get

$$\begin{aligned} \dot{z}_{i^*}^\rho &= v_{i^*} \\ \dot{z}_i^\rho &= z_i^{\rho+1} + b_i(z)v_{i^*} \quad \text{for } 1 \leq i \leq r_\rho, \quad i \neq i^* \\ \dot{z}_i^{\rho+1} &= v_i \quad \text{for } 1 \leq i \leq m, \quad i \neq i^*. \end{aligned}$$

Notice that the assumption $k \geq 1$ implies that the components $b_i(z)$ do not depend on $z_i^{\rho+1}$, for $i \neq i^*$. Now if $r_\rho = m$, then, clearly, the functions $\varphi_i = z_i^j$, where $1 \leq j \leq \rho$ and $r_{j-1} + 1 \leq i \leq r_j$, are flat outputs of the system around z_0 .

If $r_\rho < m$, then let $\psi_{r_\rho+1}, \dots, \psi_m$ be any functions satisfying (FO2). We have $z_i^l = L_f^{l-j} \varphi_i$, where $1 \leq j \leq l \leq \rho$ and $r_{j-1} + 1 \leq i \leq r_j$. Taking the next derivatives $L_f^{l-j+1} \varphi_i$ we can express the functions $z_i^{\rho+1}$, for $1 \leq i \leq r_\rho$, $i \neq i^*$. It thus follows by (FO2) that $\text{rk} \left(\frac{\partial \psi_j}{\partial z_i^{\rho+1}} \right) = m - r_\rho$, where $r_\rho + 1 \leq i, j \leq m$. Therefore the ψ_i 's, together with $L_f^{j-1} \varphi_i$'s with $j \leq \rho + 1$, allow to express the components $z_i^{\rho+1}$, for $r_\rho \leq i \leq m$ and differentiating them one time will yield the corresponding v_i . Notice that any functions ψ_i satisfying the above rank condition will work, which explains why they do not have to fulfil any structural condition but just (FO2).

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