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Laure Cardoulis, Michel Cristofol

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Abstract

In this paper we prove a stability result for the reconstruction of the potential q associated with the operator $\partial_t - \Delta + q$ in an infinite guide using a finite number of localized observations.

keywords : Inverse problem ; parabolic equation ; unbounded guide
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1 Introduction

Let ω be a bounded domain in $\mathbb{R}^{n-1}$, $n \geq 2$. Denote by $\Omega := \mathbb{R} \times \omega$ and $Q = \Omega \times (0,T)$, $\Sigma = \partial\Omega \times (0,T)$. We consider the following problem

$$\begin{cases} \partial_t u - \Delta u + qu = 0 \ \text{ in } \ Q, \\ u = g \ \text{ on } \ \Sigma \\ u(x,0) = u_0(x) \ \text{ in } \ \Omega, \end{cases} \tag{1}$$

where u_0 and g are sufficiently smooth positive functions and q is a bounded coefficient defined in Ω. Our problem can be stated as follows:
Let $l > 0$ and denote by $\Omega_* = (-*, *) \times \omega$. We determine the coefficient q on Ω_l from a finite number of measurements of the solution u of the system (1) on a lateral subset of $\partial\Omega_L$ for $L > l$ and from the knowledge of the solution at the time $\frac{T}{2}$. In the area of inverse problems, the classical understanding of finite number of measurements is formulated with respect to the infinite number of measurements involved by the Dirichlet to Neumann method.
The major novelty of this article is to obtain a Hölder stability result for the potential $q(x)$ in terms of a finite number of observations of the solution u

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of (1) on a bounded part of the boundary for a problem stated in an infinite guide.
The problem of the reconstruction of zeroth order term for parabolic operators has already been studied but most of the papers have investigated the case of bounded domains. For approaches based on Carleman estimates we can cite [4], [8] (see also [7] as a survey on this topic). Another approach based on pointwise observations in the one dimensional case can be found in [5]. The situation of unbounded domains is very few addressed: we can cite the reference [2] in which the authors use the notion of asymptotic spread of propagation as observations in the one dimensional case for periodic potentials.
In this paper we use the technique of Carleman estimates by defining special weight functions adapted to the case of an unbounded guide. For this, we adapt ideas from [3]. This article is organized as follows. In section 2, we precise our notations and the conditions required for the weight functions. In section 3 we state our main result. In section 4, we derive an adapted global Carleman estimate for our problem and finally in section 5 we prove our stability inequality.

2 Settings and hypotheses

We denote by $Q_* = \Omega_* \times (0,T) = (-*,*) \times \omega \times (0,T)$ and define the operator

$$Au = \partial_t u - \Delta u + qu.$$

Denote by $x = (x_1, ..., x_n) \in \Omega$ and $x' = (x_2, ..., x_n) \in \omega$.
Let $l > 0$, we are going to carry out special weight functions allowing us to avoid observations on the cross section of the wave guide in our inverse problem. For this we consider some positive real $L > l$, and we choose $a \in \mathbb{R}^n \setminus \Omega$ such that if $d(x) = |x' - a'|^2 - x_1^2$ for $x \in \Omega_L$, then

$$d > 0 \text{ in } \Omega_L, \quad |\nabla d| > 0 \text{ in } \overline{\Omega_L}. \tag{2}$$

Moreover we define $\Gamma_L = \{x \in \partial\Omega_L, < x - a, \nu(x) > \geq 0\}$ and $\gamma_L = \Gamma_L \cap \partial\Omega$. Here $< ., . >$ denotes the usual scalar product in $\mathbb{R}^n$ and $\nu(x)$ is the outwards unit normal vector to $\partial\Omega_L$ at x. From [7]-[8] we consider weight functions as follows, for $\lambda > 0$, $t \in (0,T)$,

$$\psi(x,t) = d(x) - \left(t - \frac{T}{2}\right)^2 + M_1 \text{ where } M_1 > \sup_{0<t<T} (t - T/2)^2 = (T/2)^2,$$

$$\text{and } \phi(x,t) = e^{\lambda\psi(x,t)}.$$

First we define $\beta_0 := \inf_{x\in\Omega_l} \psi(x, \frac{T}{2}) = \inf_{x\in\Omega_l} (|x'-a'|^2 - x_1^2) + M_1$ and $\beta_1 > 0$ such that

$$\beta_1^2 := \sup_{x\in\Omega_L} (|x'-a'|^2 - x_1^2) - \inf_{x\in\Omega_l} (|x'-a'|^2 - x_1^2).$$

Note that $\beta_1^2 = \sup_{x'\in\omega} |x'-a'|^2 - \inf_{x'\in\omega} |x'-a'|^2 + l^2$. Then, more precisely, we consider L and $T = 2L$ sufficiently large such that $\beta_2 := T/2 - \beta_1 > 0$ (even if it means changing a in order to keep the condition (2)). We get

$$\left(\frac{T}{2}\right)^2 \geq \beta_1^2 + \beta_2^2 = \sup_{x\in\Omega_L} (|x'-a'|^2 - x_1^2) - \inf_{x\in\Omega_l} (|x'-a'|^2 - x_1^2) + \beta_2^2,$$

and so

$$\left(\frac{T}{2}\right)^2 \geq \sup_{x\in\Omega_L} (|x'-a'|^2 - x_1^2) + M_1 - \beta_0 + \beta_2^2.$$

Then for all $x \in \Omega_L$, $\psi(x,T) \leq |x'-a'|^2 - x_1^2 - \sup_{x\in\Omega_L}(|x'-a'|^2 - x_1^2) + \beta_0 - \beta_2^2 \leq \beta_0 - \beta_2^2$. Thus there exists $\epsilon > 0$ such that, for all $x \in \Omega_L$ and $t \in ((0,2\epsilon) \cup (T-2\epsilon, T))$, $\psi(x,t) < \beta_0$. We choose ϵ small enough such that $l \leq L - 2\epsilon$. Due to the symmetric role played by $t - \frac{T}{2}$ and x_1 in the formulation of ψ, by the same way we have

$$\text{for all } x \in ((-L,-L+2\epsilon) \cup (L-2\epsilon, L)) \times \omega \text{ and } t \in (0,T),\ \psi(x,t) < \beta_0.$$

We set: $O_{L,\epsilon} = (\Omega_L \times ((0,2\epsilon) \cup (T-2\epsilon, T))) \cup (((-L,-L+2\epsilon) \cup (L-2\epsilon, L)) \times \omega \times (0,T))$. Therefore, if we denote by $d_0 = \min_{\Omega_l} \phi(., \frac{T}{2})$, $d_1 = \max_{O_{L,\epsilon}} \phi$, $d_2 = \max_{\overline{\Omega_L}} \phi(., \frac{T}{2})$ we get

$$d_1 < d_0 < d_2. \tag{3}$$

3 Main result

The method of Carleman estimate used in this paper requires solutions of the problem (1) with a minimum of regularity. Indeed the Buckgheim-Klibanov method [1] implies several time differentiations of the equation of system (1). We assume in the following that $q \in C^0(\Omega) \cap L^\infty(\Omega)$, and that u is an element of $\mathcal{H} = C^0(0,T,H^2(\Omega)) \cap H^3(0,T,H^2(\Omega))$ such that $\|u\|_{\mathcal{H}} < M$ for given $M > 0$. We will use the following notations: Let $\alpha = (\alpha_1, \cdots, \alpha_n)$ be

a multi-index with $\alpha_i \in \mathbb{N} \cup \{0\}$. We set $\partial_x^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and we define

$$H^{2,1}(Q_L) = \{u \in L^2(Q_L), \partial_x^\alpha \partial_t^{\alpha_{n+1}} u \in L^2(Q_L), |\alpha| + 2\alpha_{n+1} \leq 2\}.$$

We set $\frac{\partial u}{\partial \nu} = \nu \cdot \nabla u$. We can state our main result.

Theorem 1. *Assume that u_j for $j = 1, 2$ are solutions of (1) where q_j and $u_{0,j}$ are substituted respectively to q and u_0. Assume also that q_1, q_2 are bounded and continuous potentials defined on Ω. Then, for any $l > 0$, there exist $L > 0$ and $T > 0$ such that*

$$\|q_1 - q_2\|^2_{L^2(\Omega_l)} \leq K \left(\|(u_1 - u_2)(., T/2)\|^2_{H^2(\Omega_L)} + \int_{\gamma_L \times (0,T)} \sum_{k=1}^{2} \left| \frac{\partial(\partial_t^k (u_1 - u_2))}{\partial \nu} \right|^2 \right)^\kappa. \tag{4}$$

Here, $K > 0$ and $\kappa \in (0, 1)$ are two constants depending only on ω, l, M, M_1, T and a.

We stress out that, as in [3], the observation data are required on the lateral boundary γ_L and not on the whole boundary $\partial\Omega_L$. We underline that this stability result for the potential is not obtained on $\Omega = \mathbb{R} \times \omega$ but on $\Omega_l = (-l, l) \times \omega$, for an arbitrary $l > 0$, and that the observation domains Ω_L and γ_L, depend on l.

4 Global Carleman Inequality for a parabolic equation in a cylindrical domain

We recall here a global Carleman-type estimate proved in Yuan-Yamamoto [8], Yamamoto ([7] Theorem 7.3 p.48). Let $s > 0$ and denote by $LHS(u) := \int_{Q_L} \left(\frac{1}{s\phi}(|\partial_t u|^2 + |\Delta u|^2) \ + s\lambda^2 \phi \ |\nabla u|^2 + s^3 \lambda^4 \phi^3 |u|^2 \right) e^{2s\phi}$, $Au := f$ and $Obs_{\Gamma_L \times (0,T)}(u) := \int_{\Gamma_L \times (0,T)} \ |\frac{\partial u}{\partial \nu}|^2 e^{2s\phi}$. In the following parts, C will be a generic positive constant.

Proposition 4.1. *There exist positive constants λ_0, s_0 and $C = C(\lambda_0, s_0)$ such that*

$$LHS(u) \leq C \|e^{s\phi} f\|^2_{L^2(Q_L)} + Cs\lambda \ Obs_{\Gamma_L \times (0,T)}(u), \tag{5}$$

for all $s > s_0, \lambda > \lambda_0$ and all $u \in H^{2,1}(Q_L)$ satisfying $u(., 0) = u(., T) = 0$ in Ω, $u = 0$ on $\partial\Omega_L \times (0, T)$.

Then we deduce the following Carleman inequality

Proposition 4.2. *There exist positive constants λ_0, s_0 and $C = C(\lambda_0, s_0)$ such that*

$$LHS(u) \leq C\|e^{s\phi} f\|^2_{L^2(Q_L)} + Cs^3\lambda^4 e^{2sd_1}\|u\|^2_{H^{2,1}(Q_L)} + Cs\lambda\ Obs_{\gamma_L\times(0,T)}(u), \tag{6}$$

for all $s > s_0, \lambda > \lambda_0$ and all $u \in H^{2,1}(Q_L)$ satisfying $u(.,0) = u(.,T) = 0$ in Ω, $u = 0$ on $\partial\Omega_L \times (0,T)$.

Proof. Let χ, η cut-off functions be defined by $|\chi| \leq 1, |\eta| \leq 1$, $\eta(t) = 0$ if $t \in (0,\epsilon) \cup (T-\epsilon, T), \eta(t) = 1$ if $t \in \times(2\epsilon, T-2\epsilon)$, $\chi(x) = 0$ if $x \in ((-\infty, -L+\epsilon) \cup (L-\epsilon, +\infty)) \times \omega$, $\chi(x) = 1$ if $x \in (-L+2\epsilon, L-2\epsilon) \times \omega$. Recall that $\partial_t u - \Delta u + qu = f$. We consider $y = \eta\chi u$ and we get

$$\partial_t y - \Delta y + qy = h \text{ with } h = \eta\chi f + \eta R(u) + (\partial_t\eta)\chi u,$$

where R is the first order differential operator defined by $R(u) = -(\Delta\chi)u - 2\nabla\chi \cdot \nabla u$. Then we can apply the previous Carleman estimate (5) and we deduce that there exists a positive constant C such that

$$LHS(y) \leq C\|e^{s\phi} h\|^2_{L^2(Q_L)} + Cs\lambda\ Obs_{\Gamma_L\times(0,T)}(y).$$

Thanks to the cut-off functions the term $Obs_{\Gamma_L\times(0,T)}(y)$ can be rewritten in the form $Obs_{\gamma_L\times(0,T)}(u)$. Moreover

$$\|e^{s\phi}\eta R(u)\|^2_{L^2(Q_L)} \leq Ce^{2sd_1}\|u\|^2_{L^2(0,T,H^1(\Omega_L))}$$

$$\text{and } \|e^{s\phi}(\partial_t\eta)\chi u\|^2_{L^2(Q_L)} \leq Ce^{2sd_1}\|u\|^2_{L^2(0,T,L^2(\Omega_L))}.$$

Then we obtain

$$LHS(y) \leq C\|e^{s\phi} f\|^2_{L^2(Q_L)} + Ce^{2sd_1}\|u\|^2_{L^2(0,T,H^1(\Omega_L))} + Cs\lambda\ Obs_{\gamma_L\times(0,T)}(u). \tag{7}$$

Now we deal with $LHS(y)$. For $j = 0, 1, 2$, (with $\nabla^0 u = u$, $\nabla^1 u = \nabla u$, $\nabla^2 u = \Delta u$) since $\chi u = (1-\eta)\chi u + y$,

$$\|(s\phi)^{3/2-j}\lambda^{2-j}e^{s\phi}\nabla^j(\chi u)\|_{L^2(Q_L)} \leq \|(s\phi)^{3/2-j}\lambda^{2-j}e^{s\phi}(1-\eta)\nabla^j(\chi u)\|_{L^2(Q_L)}$$

$$+\|(s\phi)^{3/2-j}\lambda^{2-j}e^{s\phi}\nabla^j y\|_{L^2(Q_L)},$$

and so

$$\|(s\phi)^{3/2-j}\lambda^{2-j}e^{s\phi}\nabla^j(\chi u)\|_{L^2(Q_L)} \leq e^{sd_1}\|(s\phi)^{3/2-j}\lambda^{2-j}u\|_{H^{2,1}(Q_L)}$$

$$+\|(s\phi)^{3/2-j}\lambda^{2-j}e^{s\phi}\nabla^j y\|_{L^2(Q_L)}.$$

Doing the same for the term $\partial_t(\chi u)$ we deduce that there exists a positive constant C such that

$$LHS(\chi u) \leq C(e^{2sd_1}\|(s\phi)^{-1/2}u\|^2_{H^{2,1}(Q_L)}+e^{2sd_1}\sum_{j=0}^{1}\|(s\phi)^{3/2-j}\lambda^{2-j}\nabla^j u\|^2_{L^2(Q_L)}$$

$$+LHS(y))$$

and $$LHS(\chi u) \leq C(s^3\lambda^4 e^{2sd_1}\|u\|^2_{H^{2,1}(Q_L)} + LHS(y)).$$

Then by the identities $\partial_t u = \partial_t(\chi u) + (1-\chi)\partial_t u$,
$\nabla u = \nabla(\chi u) + (1-\chi)\nabla u - u\nabla\chi$,
$\Delta u = \Delta(\chi u) + (1-\chi)\Delta u - 2\nabla\chi\cdot\nabla u - u\Delta\chi$, we get

$$LHS(u) \leq C(LHS(\chi u) + s^3\lambda^4 e^{2sd_1}\|u\|_{H^{2,1}(Q_L)})$$

$$\leq C(s^3\lambda^4 e^{2sd_1}\|u\|^2_{H^{2,1}(Q_L)} + LHS(y)).$$

Then, from (7), we end up the proof. □

5 Inverse Problem

Now we deal with the Carleman estimate proved in Proposition 4.2 in order to get a stability inequality for the potential, which implies a uniqueness result. First we recall the following classical lemma (see [3]) and from now on, we will use the notation:
$w(\frac{T}{2}) = w(.,\frac{T}{2})$ for any function w.

Lemma 2. *There exist some positive constants C, s_2 such that*

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})}|z(T/2)|^2 \leq Cs\lambda^2\int_{Q_L} e^{2s\phi}|z|^2 + \frac{C}{s}\int_{Q_L} e^{2s\phi}|\partial_t z|^2,$$

for all $s \geq s_2, \lambda$ and $z \in H^1(0,T;L^2(\Omega_L))$.

Consider now the following systems

$$\begin{cases} \partial_t u_1 - \Delta u_1 + q_1 u_1 = 0 \text{ in } Q, \\ u_1 = g \text{ on } \Sigma, \\ u_1(x,0) = u_{0,1}(x) \text{ in } \Omega, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t u_2 - \Delta u_2 + q_2 u_2 = 0 \text{ in } Q, \\ u_2 = g \text{ on } \Sigma, \\ u_2(x,0) = u_{0,2}(x) \text{ in } \Omega. \end{cases} \tag{8}$$

We recall that $g, u_{0,1}$ and $u_{0,2}$ are positive functions. Denote by

$$y = u_1 - u_2, q = q_2 - q_1, z = \chi\eta y, z_1 = \partial_t z, z_2 = \partial_t^2 z.$$

Note that $\partial_t y - \Delta y + q_1 y = q u_2$, $\partial_t(\eta y) - \Delta(\eta y) + q_1\eta y = q\eta u_2 + y\partial_t\eta$ and

$$\partial_t z - \Delta z + q_1 z = q\chi\eta u_2 - 2\nabla\chi\cdot\nabla(\eta y) - \eta y\Delta\chi + \chi y\partial_t\eta, \tag{9}$$

$$\partial_t z_1 - \Delta z_1 + q_1 z_1 = f_1 := q\chi\partial_t(\eta u_2) - 2\nabla\chi\cdot\nabla(\partial_t(\eta y)) - \partial_t(\eta y)\Delta\chi + \chi\partial_t(y\partial_t\eta), \tag{10}$$

$$\partial_t z_2 - \Delta z_2 + q_1 z_2 = f_2 := q\chi\partial_t^2(\eta u_2) - 2\nabla\chi\cdot\nabla(\partial_t^2(\eta y)) - \partial_t^2(\eta y)\Delta\chi + \chi\partial_t^2(y\partial_t\eta). \tag{11}$$

We have from (9)

$$\partial_t z(T/2) - \Delta z(T/2) + q_1 z(T/2) = q\chi u_2(T/2) - 2\nabla\chi\cdot\nabla(y(T/2)) - y(T/2)\Delta\chi.$$

Then there exists a positive constant C such that, for all $s > 0$,

$$\int_{\Omega_L} e^{2s\phi(T/2)} q^2\chi^2 |u_2(T/2)|^2 \le Ce^{2sd_2}(\|z(T/2)\|^2_{H^2(\Omega_L)} + \|y(T/2)\|^2_{H^1(\Omega_L)})$$

$$+C\int_{\Omega_L} e^{2s\phi(T/2)}|\partial_t z(T/2)|^2.$$

But $\int_{\Omega_L} e^{2s\phi(T/2)}|\partial_t z(T/2)|^2 = \int_{\Omega_L} e^{2s\phi(T/2)}|z_1(T/2)|^2$. Using Lemma 2 we get

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})} q^2\chi^2 |u_2(T/2)|^2 \le Ce^{2sd_2}F(T/2) + Cs\lambda^2\int_{Q_L} e^{2s\phi}|z_1|^2 + \frac{C}{s}\int_{Q_L} e^{2s\phi}|z_2|^2, \tag{12}$$

with $F(\frac{T}{2}) = \|z(\frac{T}{2})\|^2_{H^2(\Omega_L)} + \|y(\frac{T}{2})\|^2_{H^1(\Omega_L)}$. Moreover by the Carleman inequality (6) for $z_i, i = 1, 2$ given by (10)-(11), for s sufficiently large, we have

$$\int_{Q_L} e^{2s\phi}|z_i|^2 \le \frac{C}{s^3\lambda^4}\int_{Q_L} e^{2s\phi}|f_i|^2 + Ce^{2sd_1}\|z_i\|^2_{H^{2,1}(Q_L)} + \frac{C}{s^2\lambda^3}Obs_{\gamma_L\times(0,T)}(z_i). \tag{13}$$

Combining (12)-(13) we get

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})} q^2\chi^2 |u_2(T/2)|^2 \le Ce^{2sd_2}F(T/2) + \frac{C}{s^2\lambda^2}\int_{Q_L} e^{2s\phi}(|f_1|^2 + \frac{1}{s^2\lambda^2}|f_2|^2)$$

$$+Ce^{2sd_1}(s\lambda^2\|z_1\|^2_{H^{2,1}(Q_L)} + \frac{1}{s}\|z_2\|^2_{H^{2,1}(Q_L)}) + \frac{C}{s\lambda}\int_{\gamma_L\times(0,T)} e^{2s\phi}(|\frac{\partial z_1}{\partial\nu}|^2 + \frac{1}{s^2\lambda^2}|\frac{\partial z_2}{\partial\nu}|^2).$$

Note that the conditions $u_{0,2} > 0$ and $g \geq 0$ imply that a sufficiently regular solution u_2 to the second system in (8), is strictly positive (by the maximum principle for the parabolic equation, see [6], Theorem 13.5 p.128). Then

$$\int_{\Omega_L} e^{2s\phi(\frac{T}{2})} q^2 \chi^2 \leq C e^{2sd_2} F(T/2) + C e^{2sd_1} s\lambda^2 (\|z_1\|^2_{H^{2,1}(Q_L)} + \|z_2\|^2_{H^{2,1}(Q_L)})$$

$$+ \frac{C}{s^2\lambda^2} \int_{\Omega_L} e^{2s\phi(\frac{T}{2})} q^2 \chi^2 + \frac{C}{s\lambda} \int_{\gamma_L \times (0,T)} (|\frac{\partial z_1}{\partial \nu}|^2 + |\frac{\partial z_2}{\partial \nu}|^2) e^{2s\phi} + \frac{C}{s^2\lambda^2} \int_{O_{L,\epsilon}} e^{2s\phi}.$$

Since $e^{2s\phi} \leq e^{2sd_2}$ on $\overline{\Omega_L}$ and $e^{2s\phi} \leq e^{2sd_1}$ on $O_{L,\epsilon}$, we get for s sufficiently large

$$e^{2sd_0} \|q\|^2_{L^2(\Omega_l)} \leq C(e^{2sd_2} B(u_1, u_2, T/2) + s\lambda^2 e^{2sd_1}),$$

with $B(u_1, u_2, \frac{T}{2}) = F(\frac{T}{2}) + \int_{\gamma_L \times (0,T)} |\frac{\partial}{\partial \nu}(\partial_t(\chi\eta(u_1 - u_2)))|^2 + |\frac{\partial}{\partial \nu}(\partial_t^2(\chi\eta(u_1 - u_2)))|^2$. This inequality can be rewritten in the following form

$$\|q\|^2_{L^2(\Omega_l)} \leq C(e^{2s(d_2 - d_0)} B(u_1, u_2, T/2) + s\lambda^2 e^{2s(d_1 - d_0)}).$$

Then, recalling that $d_1 - d_0 < 0$ and $d_2 - d_0 > 0$ we get our stability result (4).

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