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I. **Note Technique: CSI estimation in massive MIMO systems and pilot contamination for Zero Forcing receiver architectures**

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**1- Context and System:**

We consider  $L$  cells, where each cell contains one base station equipped with  $M$  antennas and  $K$  single-antenna users. Assume that the  $L$  base stations share the same frequency band. We consider uplink transmission, where the  $l$ th base station receives signals from all users in all cells (Fig.1). Then, the  $M \times 1$  received vector at the  $l$ th base station is given by:

$$\mathbf{y}_l = \sqrt{p_u} \sum_{i=1}^L \mathbf{\Gamma}_{i,l} \mathbf{x}_i + \mathbf{n}_l \quad (1)$$

Where  $\mathbf{\Gamma}_{i,l}$  represents the  $M \times K$  channel matrix between the  $l$ th base station and the  $K$  users in the  $i$ th cell, i.e.,  $[\mathbf{\Gamma}_{i,l}]_{m,k}$  is the channel coefficient between the  $m$ th antenna of the  $l$ th base station and the  $k$ th user in the  $i$ th cell.  $\sqrt{p_u} \mathbf{x}_i$  is the  $K \times 1$  transmitted vector of  $K$  users in the  $i$ th cell (the average power used by each user is  $p_u$ ) and  $\mathbf{n}_l$  contains  $M \times 1$  additive white Gaussian noise (AWGN) samples. We assume that the elements of  $\mathbf{n}_l$  are Gaussian distributed with zero mean and variance:  $\mathcal{E}[\mathbf{n}_l \mathbf{n}_l^H] = \sigma_{n_l}^2 \mathbf{I}_M$  and  $\mathcal{E}[\mathbf{n}_l^H \mathbf{n}_l] = M \sigma_{n_l}^2$ .

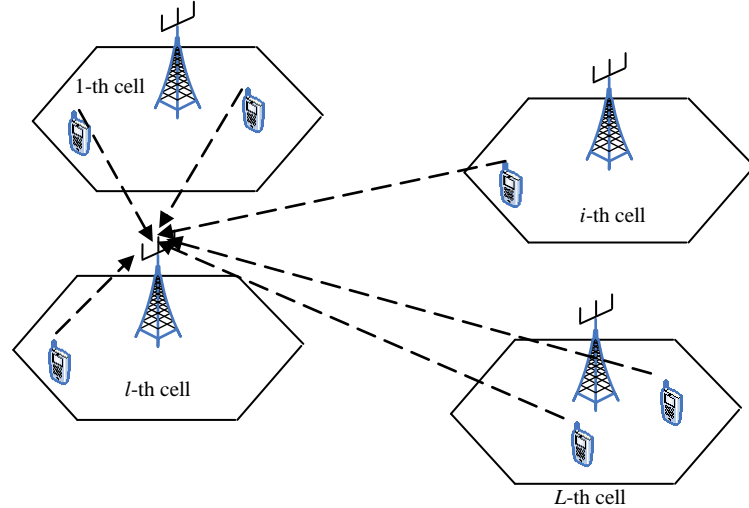


Fig. 1: Uplink transmission in multicell MU-MIMO system

**2- Physical Channel Model:**

Here, we introduce the finite-dimensional channel model that will be used throughout this technical note. The angular domain is divided into a large but finite number of directions  $P$ .

$P$  is fixed regardless of the number of base station antennas ( $P < M$ ). Each direction, corresponding to the angle:  $\phi_k, \phi_k \in [-\pi/2, \pi/2], k = 1, \dots, P$ , is associated with an  $M \times 1$  array steering vector  $\mathbf{a}(\phi_k)$  which is given by:

$$\mathbf{a}(\phi_k) = \frac{1}{\sqrt{P}} \left[ e^{-j f_1(\phi_k)}, e^{-j f_2(\phi_k)}, \dots, e^{-j f_M(\phi_k)} \right]^T \quad (2)$$

Where  $f_i(\phi)$  is some function of  $\phi$ . Typically, for a regular phased array, we have:

$$f_i(\phi) = 2\pi \frac{d}{\lambda} (i-1) \sin(\phi)$$

The channel vector from  $k$ th user in the  $i$ th cell to the  $l$ th base station is then a linear combination of the steering vectors as follows:  $\sum_{m=1}^P g_{ilm} \mathbf{a}(\phi_m)$ , where  $g_{ilm}$  is the propagation coefficient from the  $k$ th user of the  $i$ th cell to the  $l$ th base station, associated with the physical direction  $m$  (direction of arrival  $\phi_m$ ). Let  $\mathbf{G}_{i,l} \equiv [\mathbf{g}_{i,l1}, \dots, \mathbf{g}_{i,lK}]$  be a  $P \times K$  matrix with  $\mathbf{g}_{ilk} \equiv [g_{ilk1}, \dots, g_{ilkP}]^T$  that contains the path gains from the  $k$ th user in the  $i$ th cell to the  $l$ th base station. The elements of  $\mathbf{G}_{i,l}$  are assumed to be independent. Then, the  $M \times K$  channel matrix between the  $l$ th base station and the  $K$  users in the  $i$ th cell is:

$$\mathbf{\Gamma}_{i,l} = \mathbf{A} \mathbf{G}_{i,l} \quad (3)$$

Where  $\mathbf{A} \equiv [\mathbf{a}(\phi_1), \dots, \mathbf{a}(\phi_P)]$  is a full rank  $M \times P$  matrix. The propagation channel  $\mathbf{G}_{i,l}$  models independent fast fading, geometric attenuation, and log-normal shadow fading. Its elements  $g_{ilm}$  are given by:

$$g_{ilm} = h_{ilm} \sqrt{\beta_{ilk}} \quad m = 1, 2, \dots, P \quad (4)$$

Where  $h_{ilm}$  is a fast fading coefficient assumed to be zero mean and have unit variance.  $\sqrt{\beta_{ilk}}$  models the path loss and shadowing which are assumed to be independent of the direction  $m$  and to be constant and known a priori. This assumption is reasonable since the value  $\beta_{ilk}$  changes very slowly with time. Then, we have:

$$\mathbf{G}_{i,l} = \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \quad (5)$$

Where  $\mathbf{H}_{i,l}$  is the  $P \times K$  matrix of fast fading coefficients between the  $K$  users in the  $i$ th cell and the  $l$ th base station, the columns  $\mathbf{H}_{i,l}(:, k)$  are independent and we have:

$$\mathcal{E}[\mathbf{H}_{i,l}^H(:,k)\mathbf{H}_{i,l}(:,k)] = \mathcal{E}[\mathbf{h}_{ilk}^H\mathbf{h}_{ilk}] = \mathcal{E}\left[\sum_{m=1}^P|h_{il}(m,k)|^2\right] = \mathcal{E}\left[\sum_{m=1}^P|h_{ilk m}|^2\right] = 1$$

Furthermore within one given column we consider the elements:  $h_{il}(m,k)$  independent, i.e.:

$\mathcal{E}[h_{ilk m} h_{ilk n}^*] = 0$  if  $m \neq n$ .  $\mathbf{D}_{i,l}$  is a  $K \times K$  diagonal matrix whose diagonal elements are given by  $[\mathbf{D}_{i,l}]_{k,k} = \beta_{ilk}$ . Therefore, (1) can be written as:

$$\mathbf{y}_l = \sqrt{p_u} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{x}_i + \mathbf{n}_l = \sqrt{p_u} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \mathbf{x}_i + \mathbf{n}_l \quad (6)$$

In the following sections we will use the following intermediate parameters:  $\beta_{jl} = \sum_{k=1}^K \beta_{jlk}$ ,

$$\bar{\beta}_{ln} = \sum_{i=1}^L \beta_{iln}, \text{ and } \hat{\beta}_l = \sum_{i=1}^L \beta_{il}.$$

### **3- Channel Estimation:**

Channel estimation is performed by using training sequences received on the uplink. A part of the coherence interval is used for the uplink training. All users in all cells simultaneously transmit pilot sequences of length  $\tau$  symbols. The assumption on synchronized transmission represents the worst case from the pilot contamination point of view, but it makes no fundamental difference to assume unsynchronized transmission. We assume that the same set of pilot sequences is used in all  $L$  cells. Therefore, the pilot sequences used in the  $l$ th cell can be represented by a  $K \times \tau$  matrix  $\sqrt{p_p} \boldsymbol{\phi}_l = \sqrt{p_p} \boldsymbol{\phi}$  ( $\tau \geq K$ ), which satisfies  $\boldsymbol{\phi} \boldsymbol{\phi}^H = \mathbf{I}_K$  where:  $p_p = \tau p_u$ . From (6), the received pilot matrix at the  $l$ th base can be written as:

$$\mathbf{Y}_{p,l} = \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \boldsymbol{\phi} + \mathbf{N}_l \quad (7)$$

$\mathbf{N}_l$  is the resulting noise matrix of size  $M \times \tau$  and of course we have:  $\mathcal{E}[\mathbf{N}_l] = \mathbf{0}_{M \times \tau}$   
 $\mathcal{E}[\mathbf{N}_l \mathbf{N}_l^H] = \tau \sigma_n^2 \mathbf{I}_M$  and  $\mathcal{E}[\mathbf{N}_l^H \mathbf{N}_l] = M \sigma_n^2 \mathbf{I}_\tau$ .

#### **3.1. Minimum Mean-Square Error (MMSE) Estimation:**

We assume that the base station uses here MMSE estimation. The received pilot matrix  $\mathbf{Y}_{p,l}$  is multiplied by  $\boldsymbol{\phi}^H$  at the right side to obtain:  $\tilde{\mathbf{Y}}_{p,l} = \mathbf{Y}_{p,l} \boldsymbol{\phi}^H$ :

$$\begin{aligned}
\tilde{\mathbf{Y}}_{p,l} &= \mathbf{Y}_{p,l} \boldsymbol{\phi}^H \\
&= \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{N}_l \boldsymbol{\phi}^H \quad (8) \\
&= \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{W}_l
\end{aligned}$$

With:  $\mathbf{W}_l = \mathbf{N}_l \boldsymbol{\phi}^H$ . Since we consider here  $M$ -PSK modulation format for the transmit pilot matrix we can denote  $\boldsymbol{\phi}$  as:

$$\boldsymbol{\phi} = \frac{1}{\sqrt{\tau}} \begin{bmatrix} e^{j\theta_{11}} & e^{j\theta_{12}} & \dots & e^{j\theta_{1p}} & \dots & e^{j\theta_{1\tau}} \\ e^{j\theta_{21}} & e^{j\theta_{22}} & \dots & e^{j\theta_{2p}} & \dots & e^{j\theta_{2\tau}} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ e^{j\theta_{m1}} & e^{j\theta_{m2}} & \dots & e^{j\theta_{mp}} & \dots & e^{j\theta_{m\tau}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{j\theta_{K1}} & e^{j\theta_{K2}} & \dots & e^{j\theta_{Kp}} & \dots & e^{j\theta_{K\tau}} \end{bmatrix}$$

Hence, we have:  $\phi(k, m) = a_{k,m} = \frac{1}{\sqrt{\tau}} e^{j\theta_{k,m}}$  and  $\phi^H(n, k) = b_{n,k} = \frac{1}{\sqrt{\tau}} e^{-j\theta_{k,n}}$ . The matrix product  $\mathbf{N}_l \boldsymbol{\phi}^H$  yields:

$$\mathbf{N}_l \boldsymbol{\phi}^H(n, m) = \sum_{k=1}^{\tau} n_l(n, k) \phi^H(k, m) = \frac{1}{\sqrt{\tau}} \sum_{k=1}^{\tau} n_l(n, k) e^{-j\theta_{m,k}}$$

In the same way we obtain:

$$\boldsymbol{\phi} \mathbf{N}_l^H(n, m) = \sum_{k=1}^{\tau} \phi(n, k) n_l^*(m, k) = \frac{1}{\sqrt{\tau}} \sum_{k=1}^{\tau} n_l^*(m, k) e^{j\theta_{n,k}}$$

The matrix product:  $\mathbf{W}_l \mathbf{W}_l^H = \mathbf{N}_l \boldsymbol{\phi}^H \mathbf{N}_l \boldsymbol{\phi} \mathbf{N}_l^H$  is thus equal to:

$$\begin{aligned}
\mathbf{N}_l \boldsymbol{\phi}^H \boldsymbol{\phi} \mathbf{N}_l^H(n, m) &= \sum_{k=1}^K \mathbf{N}_l \boldsymbol{\phi}^H(n, k) \boldsymbol{\phi} \mathbf{N}_l^H(k, m) \\
&= \sum_{k=1}^K \frac{1}{\sqrt{\tau}} \sum_{s=1}^{\tau} n_l(n, s) e^{-j\theta_{k,s}} \frac{1}{\sqrt{\tau}} \sum_{r=1}^{\tau} n_l^*(m, r) e^{j\theta_{k,r}} \\
&= \frac{1}{\tau} \sum_{k=1}^K \sum_{s=1}^{\tau} \sum_{r=1}^{\tau} n_l(n, s) n_l^*(m, r) e^{j(\theta_{k,r} - \theta_{k,s})}
\end{aligned}$$

Taking the expectation, we obtain:

$$\mathcal{E}\left[N_l \boldsymbol{\phi}^H \boldsymbol{\phi} N_l^H(n, m)\right] = \frac{1}{\tau} \sum_{k=1}^K \sum_{s=1}^{\tau} \sum_{r=1}^{\tau} \mathcal{E}\left[n_l(n, s) n_l^*(m, r)\right] e^{j(\theta_{k,r} - \theta_{k,s})}$$

Clearly,  $\mathcal{E}\left[n_l(n, s) n_l^*(m, r)\right] = \sigma_{n_l}^2$  if and only if:  $m=n$  and  $r=s$ . In these conditions we obtain:

$$\mathcal{E}\left[N_l \boldsymbol{\phi}^H \boldsymbol{\phi} N_l^H(n, n)\right] = \frac{1}{\tau} \sum_{k=1}^K \tau \sigma_{n_l}^2 = K \sigma_{n_l}^2$$

We eventually obtain:  $\mathcal{E}\left[\mathbf{W}_l \mathbf{W}_l^H\right] = \mathcal{E}\left[N_l \boldsymbol{\phi}^H \boldsymbol{\phi} N_l^H\right] = K \sigma_{n_l}^2 \mathbf{I}_M$

Concerning the other quantity  $\mathcal{E}\left[\mathbf{W}_l^H \mathbf{W}_l\right]$ , we can obtain directly:

$$\mathcal{E}\left[\mathbf{W}_l^H \mathbf{W}_l\right] = \mathcal{E}\left[\boldsymbol{\phi} N_l^H N_l \boldsymbol{\phi}^H\right] = \boldsymbol{\phi} \mathcal{E}\left[N_l^H N_l\right] \boldsymbol{\phi}^H = M \sigma_{n_l}^2 \mathbf{I}_K$$

**Remark :** If we only consider one column (typically the  $n$ th column) of  $\mathbf{W}_l$  denoted as  $\mathbf{w}_{ln}$ , we have :

$$\mathcal{E}\left\{\mathbf{w}_{ln} \mathbf{w}_{ln}^H\right\} = \sigma_{n_l}^2 \mathbf{I}_M \text{ and } \mathcal{E}\left\{\mathbf{w}_{ln}^H \mathbf{w}_{ln}\right\} = M \sigma_{n_l}^2$$

We are now coming back to the channel estimation problem. Since  $\mathbf{H}_{l,l}$  has independent columns, we can estimate each column of  $\mathbf{H}_{l,l}$  independently. Let  $\tilde{\mathbf{y}}_{pln}$  be the  $n$ th column of  $\tilde{\mathbf{Y}}_{p,l}$ . Then:

$$\tilde{\mathbf{y}}_{pln} = \sqrt{p_p} \mathbf{A} \mathbf{h}_{lln} \beta_{lln}^{1/2} + \sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln} \quad (9)$$

Where  $\mathbf{h}_{iln}$  and  $\mathbf{w}_{ln}$  are the  $n$ th column of  $\mathbf{H}_{i,l}$  and  $\mathbf{W}_l$  respectively. Denote by:

$\mathbf{z}_{ln} \equiv \sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln}$ , then the MMSE estimate of  $\mathbf{h}_{iln}$  is given by :

$$\hat{\mathbf{h}}_{lln} = \beta_{lln}^{1/2} \sqrt{p_p} \mathbf{A}^H \left( p_p \beta_{lln} \mathbf{A} \mathbf{A}^H + \mathbf{R}_{z_{ln}} \right)^{-1} \tilde{\mathbf{y}}_{pln} \quad (10)$$

With:  $\mathbf{R}_{z_{ln}} = \mathcal{E}\left\{\mathbf{z}_{ln} \mathbf{z}_{ln}^H\right\} = p_p \mathbf{A} \mathbf{A}^H \sum_{i=1, i \neq l}^L \beta_{iln} + \sigma_{n_l}^2 \mathbf{I}_M \quad (11)$

Reporting (11) into (10), we obtain:

$$\begin{aligned}\hat{\mathbf{h}}_{lln} &= \beta_{lln}^{1/2} \sqrt{p_p} \mathbf{A}^H \left( p_p \beta_{lln} \mathbf{A} \mathbf{A}^H + p_p \mathbf{A} \mathbf{A}^H \sum_{i=1, i \neq l}^L \beta_{iln} + \sigma_{n_i}^2 \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln} \\ &= \beta_{lln}^{1/2} \sqrt{p_p} \mathbf{A}^H \left( p_p \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} + \sigma_{n_i}^2 \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln}\end{aligned}$$

We can rewrite  $\hat{\mathbf{h}}_{lln}$  in the following form :

$$\begin{aligned}\hat{\mathbf{h}}_{lln} &= \frac{\beta_{lln}^{1/2} \sqrt{p_p} \mathbf{A}^H}{\sigma_{n_i}^2} \left( \frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln} \mathbf{A} \mathbf{A}^H + \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln} \\ &= \frac{\beta_{lln}^{1/2} \sqrt{p_p}}{\sigma_{n_i}^2} \mathbf{A}^H \left( \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A} \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A}^H + \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln} \\ &= \left( \frac{\beta_{lln}^{1/2} \sqrt{p_p}}{\sigma_{n_i}^2} \sqrt{\frac{\sigma_{n_i}^2}{p_p \sum_{i=1}^L \beta_{iln}}} \right) \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A}^H \left( \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A} \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A}^H + \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln} \\ &= \left( \frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{n_i}^2 \sum_{i=1}^L \beta_{iln}}} \right) \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A}^H \left( \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A} \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A}^H + \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln}\end{aligned}$$

The former equation can be written:  $\hat{\mathbf{h}}_{lln} = \lambda \mathbf{B}^H (\mathbf{B} \mathbf{B}^H + \mathbf{I}_M)^{-1} \tilde{\mathbf{y}}_{pln}$ , with:  $\mathbf{B} = \sqrt{\frac{p_p}{\sigma_{n_i}^2}} \sqrt{\sum_{i=1}^L \beta_{iln}} \mathbf{A}$ ,

and  $\lambda = \frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{n_i}^2 \sum_{i=1}^L \beta_{iln}}}$ . Using the matrix inversion lemma, we have:

$$\mathbf{P} (\mathbf{I}_m + \mathbf{Q} \mathbf{P})^{-1} = (\mathbf{I}_n + \mathbf{P} \mathbf{Q})^{-1} \mathbf{P}$$

With the following sizes for  $\mathbf{P}$  and  $\mathbf{Q}$ :  $[\mathbf{Q}]_{m \times n}, [\mathbf{P}]_{n \times m}$ . We use this lemma with:  $\mathbf{P} = \mathbf{B}^H$ ,  $\mathbf{Q} = \mathbf{B}$ , we obtain:  $\mathbf{B}^H (\mathbf{B} \mathbf{B}^H + \mathbf{I}_M)^{-1} = (\mathbf{I}_P + \mathbf{B}^H \mathbf{B})^{-1} \mathbf{B}^H$ , so we eventually obtain:

$$\begin{aligned}
\hat{\mathbf{h}}_{lln} &= \frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{n_l}^2 \sum_{i=1}^L \beta_{iln}}} \left( \frac{p_p}{\sigma_{n_l}^2} \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} + \mathbf{I}_P \right)^{-1} \sqrt{\frac{p_p}{\sigma_{n_l}^2}} \sqrt{\sum_{i=1}^L \beta_{iln}} \mathbf{A}^H \tilde{\mathbf{y}}_{p,ln} \\
\hat{\mathbf{h}}_{lln} &= \frac{\sqrt{p_p} \beta_{lln}^{1/2}}{\sigma_{n_l}^2} \left( \frac{p_p}{\sigma_{n_l}^2} \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} + \mathbf{I}_P \right)^{-1} \mathbf{A}^H \tilde{\mathbf{y}}_{p,ln} \\
\hat{\mathbf{h}}_{lln} &= \sqrt{p_p} \beta_{lln}^{1/2} \left( p_p \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} + \sigma_{n_l}^2 \mathbf{I}_P \right)^{-1} \mathbf{A}^H \tilde{\mathbf{y}}_{p,ln}
\end{aligned} \tag{12}$$

The  $k$ th diagonal element of  $p_p \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln}$  in (12) equals:  $\frac{M p_p}{P} \sum_{i=1}^L \beta_{iln}$ . Since the uplink is typically interference-limited we have:  $\frac{M p_p}{P} \sum_{i=1}^L \beta_{iln} \gg \sigma_{n_l}^2$ , therefore,  $\hat{\mathbf{h}}_{lln}$  can be approximated as:

$$\hat{\mathbf{h}}_{lln} = \beta_{lln}^{1/2} \sqrt{p_p} \left( p_p \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} \right)^{-1} \mathbf{A}^H \tilde{\mathbf{y}}_{p,ln} \tag{13}$$

Thus, the MMSE estimate of  $\mathbf{H}_{l,l}$  is:

$$\hat{\mathbf{H}}_{l,l} = p_p^{-1/2} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll}^{1/2} \tag{14}$$

Where:  $\mathbf{D}_l \equiv \sum_{i=1}^L \mathbf{D}_{il}$ . Then, the estimate of the physical channel matrix between the  $l$ th base station and the  $K$  users in the  $l$ th cell is given by:

$$\hat{\Gamma}_{l,l} = \mathbf{A} \hat{\mathbf{H}}_{l,l} \mathbf{D}_{ll}^{1/2} = p_p^{-1/2} \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll} \tag{15}$$

Where  $\mathbf{\Pi}_A \equiv \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H$  is the orthogonal projection onto  $\mathbf{A}$ . We can see that since post-multiplication of  $\mathbf{Y}_{p,l}$  with  $\boldsymbol{\phi}^H$  means just multiplication with the pseudo-inverse ( $\boldsymbol{\phi} \boldsymbol{\phi}^H = \mathbf{I}_K$ ),  $\tilde{\mathbf{Y}}_{p,l}$  is the conventional least-squares channel estimate. The MMSE channel estimator that we derived thus performs conventional channel estimation and then projects the estimate onto the physical (beam-space) model for the array.

### **3.2. Bayesian estimator with MAP rule:**

We start from equation (7):

$$\tilde{\mathbf{Y}}_{p,l} = \mathbf{Y}_{p,l} \boldsymbol{\phi}^H = \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{W}_l$$



Let  $\tilde{\mathbf{y}}_{pln}$  be the  $n$ th column of  $\tilde{\mathbf{Y}}_{p,l}$ . Then:

$$\tilde{\mathbf{y}}_{pln} = \sqrt{P_p} \mathbf{A} \mathbf{h}_{lln} \beta_{lln}^{1/2} + \sqrt{P_p} \mathbf{A} \sum_{i=1, i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln} \quad (16)$$

Where  $\mathbf{h}_{iln}$  and  $\mathbf{w}_{ln}$  are the  $n$ th column of  $\mathbf{H}_{i,l}$  and  $\mathbf{W}_l$  respectively. Denote by:  $\mathbf{z}_{ln} \equiv \sqrt{P_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln}$ , we can write:

$$\tilde{\mathbf{y}}_{pln} = \sqrt{P_p} \mathbf{A} \mathbf{h}_{lln} \beta_{lln}^{1/2} + \mathbf{z}_{ln} \quad (17)$$

Applying Bayes' rule, the conditional distribution of the channel  $\mathbf{h}_{lln}$ , given the observed received training signal  $\tilde{\mathbf{y}}_{pln}$ , is:

$$p(\mathbf{h}_{lln} | \tilde{\mathbf{y}}_{pln}) = \frac{p(\mathbf{h}_{lln}) p(\tilde{\mathbf{y}}_{pln} | \mathbf{h}_{lln})}{p(\tilde{\mathbf{y}}_{pln})} = p(\mathbf{h}_{lln}) p(\tilde{\mathbf{y}}_{pln} | \mathbf{h}_{lln}) \quad (18)$$

We use the Gaussian probability density function (PDF) of the random vector  $\mathbf{h}_{lln}$  and assume its elements:  $h_{lln}(1), \dots, h_{lln}(i), \dots, h_{lln}(P)$ , are mutually independent, giving the joint pdf :

$$p(\mathbf{h}_{lln}) = \frac{\exp[-\mathbf{h}_{lln}^H \mathbf{R}_{h_{lln}}^{-1} \mathbf{h}_{lln}]}{(\sqrt{2\pi})^P \det \mathbf{R}_{h_{lln}}} \quad (19)$$

Since the elements:  $h_{lln}(1), \dots, h_{lln}(i), \dots, h_{lln}(P)$  are mutually independent this reduces to:

$$p(\mathbf{h}_{lln}) = \frac{\exp[-\mathbf{h}_{lln}^H \mathbf{h}_{lln}]}{(\sqrt{2\pi})^P} \quad (20)$$

We also have:

$$p(\tilde{\mathbf{y}}_{pln} | \mathbf{h}_{lln}) = \frac{\exp[-(\tilde{\mathbf{y}}_{pln} - \sqrt{P_p} \mathbf{A} \mathbf{h}_{lln} \beta_{lln}^{1/2})^H \mathbf{R}_{z_{ln}}^{-1} (\tilde{\mathbf{y}}_{pln} - \sqrt{P_p} \mathbf{A} \mathbf{h}_{lln} \beta_{lln}^{1/2})]}{\pi^P \det \mathbf{R}_{z_{ln}}} \quad (21)$$

Combining (20) and (21) and reporting into (18) we obtain:

$$p(\mathbf{h}_{lln} | \tilde{\mathbf{y}}_{pln}) = \frac{\exp(-f(\mathbf{h}_{lln}))}{(\sqrt{2\pi})^P \pi^P \det \mathbf{R}_{z_{ln}}} \quad (22)$$

With:  $\mathbf{R}_{z_{ln}} = \mathcal{E} \left[ (\sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln}) (\sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln})^H \right]$  and:  
 $f(\mathbf{h}_{ln}) = \mathbf{h}_{ln}^H \mathbf{h}_{ln} + (\tilde{\mathbf{y}}_{p_{ln}} - \sqrt{p_p} \mathbf{A} \mathbf{h}_{ln} \beta_{ln}^{1/2})^H \mathbf{R}_{z_{ln}}^{-1} (\tilde{\mathbf{y}}_{p_{ln}} - \sqrt{p_p} \mathbf{A} \mathbf{h}_{ln} \beta_{ln}^{1/2})$

The ML estimation  $\hat{\mathbf{h}}_{ln}$  of  $\mathbf{h}_{ln}$  is given by:

$$\begin{aligned} \hat{\mathbf{h}}_{ln} &= \arg \max_{\mathbf{h}_{ln} \in \mathbb{C}^{p \times 1}} \exp(-f(\mathbf{h}_{ln})) \\ \hat{\mathbf{h}}_{ln} &= \arg \min_{\mathbf{h}_{ln} \in \mathbb{C}^{p \times 1}} f(\mathbf{h}_{ln}) \\ \hat{\mathbf{h}}_{ln} &= \beta_{ln}^{1/2} \sqrt{p_p} \mathbf{A}^H \left( p_p \beta_{ln} \mathbf{A} \mathbf{A}^H + \mathbf{R}_{z_{ln}} \right)^{-1} \tilde{\mathbf{y}}_{p_{ln}} \end{aligned} \quad (23)$$

We have clearly:

$$\begin{aligned} \mathbf{R}_{z_{ln}} &= \mathcal{E} \left[ (\sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln}) (\sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln})^H \right] \\ \mathbf{R}_{z_{ln}} &= \sigma_{n_l}^2 \mathbf{I}_M + p_p \sum_{i \neq l}^L \beta_{iln} \mathbf{A} \mathbf{A}^H \end{aligned} \quad (24)$$

And we find again:

$$\hat{\mathbf{h}}_{ln} = \beta_{ln}^{1/2} \sqrt{p_p} \mathbf{A}^H \left( p_p \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} + \sigma_{n_l}^2 \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{p_{ln}} \quad (25)$$

To obtain (23) and (25) we have applied (39) and (40) (see Appendix) with:  $\mathbf{R}_h = \mathbf{I}$ ,  $\mathbf{R}_n = \mathbf{R}_{z_{ln}}$ ,  
 $\mathbf{S} = \sqrt{p_p} \beta_{ln}^{1/2} \mathbf{A}$ .

### **3.3. Bayesian estimation with vectorized model:**

We start from equation (8):

$$\begin{aligned} \tilde{\mathbf{Y}}_{p,l} &= \mathbf{Y}_{p,l} \boldsymbol{\phi}^* = \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{N}_l \boldsymbol{\phi}^* = \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{W}_l \\ \tilde{\mathbf{Y}}_{p,l} &= \sqrt{p_p} \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} + \sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{W}_l \end{aligned}$$

Starting from this equation we begin by vectorizing the received matrix  $\tilde{\mathbf{Y}}_{p,l}$  of size  $M \times K$ , we obtain the vector:

$$\tilde{\mathbf{y}}_{p,l} = \begin{pmatrix} \tilde{\mathbf{Y}}_{p,l}(:,1) \\ \vdots \\ \tilde{\mathbf{Y}}_{p,l}(:,k) \\ \vdots \\ \tilde{\mathbf{Y}}_{p,l}(:,K) \end{pmatrix}_{MK \times 1} \quad (26)$$

With:

$$\begin{aligned} \tilde{\mathbf{Y}}_{p,l}(:,k) &= \sqrt{p_p} \left[ \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \right](:,k) + \sqrt{p_p} \left[ \mathbf{A} \sum_{i \neq l}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \right](:,k) + \mathbf{W}_l(:,k) \\ \tilde{\mathbf{Y}}_{p,l}(:,k) &= \sqrt{p_p} \left[ \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \right](:,k) + \mathbf{Z}_l(:,k) \end{aligned} \quad (27)$$

$$\text{With: } \mathbf{Z}_l(:,k) \equiv \sqrt{p_p} \left[ \mathbf{A} \sum_{i \neq l}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \right](:,k) + \mathbf{W}_l(:,k)$$

Clearly, we have the property:

$$\left[ \mathbf{A} \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \right](:,k) = \mathbf{A} \mathbf{H}_{i,l}(:,k) \beta_{ilk}^{1/2} \quad (28)$$

Applying Bayes' rule, the conditional distribution of the channel  $\mathbf{H}_{l,l}(:,k)$  given the observed received training signal  $\tilde{\mathbf{y}}_{p,l}$  is:

$$\begin{aligned} p(\mathbf{H}_{l,l}(:,k) | \tilde{\mathbf{Y}}_{p,l}(:,k)) \\ = \frac{p(\mathbf{H}_{l,l}(:,k)) p(\tilde{\mathbf{Y}}_{p,l}(:,k) | \mathbf{H}_{l,l}(:,k))}{p(\tilde{\mathbf{Y}}_{p,l}(:,k))} = p(\mathbf{H}_{l,l}(:,k)) p(\tilde{\mathbf{Y}}_{p,l}(:,k) | \mathbf{H}_{l,l}(:,k)) \end{aligned} \quad (29)$$

We use the Gaussian probability density function (pdf) of the random vector  $\mathbf{H}_{l,l}(:,k)$  and assume its elements:  $H_{l,l}(1,k), \dots, H_{l,l}(1,k), \dots, H_{l,l}(P,k)$  are mutually independent, giving the joint pdf:

$$p(\mathbf{H}_{l,l}(:,k)) = \frac{\exp[-\mathbf{H}_{l,l}^H(:,k) \mathbf{R}_{\mathbf{H}_{l,l}(:,k)}^{-1} \mathbf{H}_{l,l}^H(:,k)]}{(\sqrt{2\pi})^P \det \mathbf{R}_{\mathbf{H}_{l,l}(:,k)}} \quad (30)$$

Since the elements:  $H_{l,l}(1,k), \dots, H_{l,l}(m,k), \dots, H_{l,l}(P,k)$  are mutually independent, this reduces to:

$$p(\mathbf{H}_{l,l}(:,k)) = \frac{\exp[-\mathbf{H}_{l,l}^H(:,k) \mathbf{H}_{l,l}^H(:,k)]}{(\sqrt{2\pi})^P} \quad (31)$$

We also have:

$$\begin{aligned} & p(\tilde{\mathbf{Y}}_{p,l}(:,k) | \mathbf{H}_{l,l}(:,k)) \\ &= \frac{\exp[-(\tilde{\mathbf{Y}}_{p,l}(:,k) - \mathbf{A}\mathbf{H}_{l,l}(:,k)\beta_{llk}^{1/2})^H \mathbf{R}_{\mathbf{Z}_l(:,k)}^{-1} (\tilde{\mathbf{Y}}_{p,l}(:,k) - \mathbf{A}\mathbf{H}_{l,l}(:,k)\beta_{llk}^{1/2})]}{(\pi)^P \det \mathbf{R}_{\mathbf{Z}_l(:,k)}} \end{aligned} \quad (32)$$

Combining (31) and (32) and reporting into (29) we obtain:

$$p(\mathbf{H}_{l,l}(:,k) | \tilde{\mathbf{Y}}_{p,l}(:,k)) = \frac{\exp(-f(\mathbf{H}_{l,l}(:,k)))}{AB} \quad (33)$$

Where:  $A \equiv (\sqrt{2\pi})^P$ ,  $B \equiv (\pi)^P \det \mathbf{R}_{\mathbf{Z}_l(:,k)}$ , this yields :

$$\begin{aligned} & f(\mathbf{H}_{l,l}(:,k)) \\ & \equiv \mathbf{H}_{l,l}^H(:,k) \mathbf{H}_{l,l}^H(:,k) + (\tilde{\mathbf{Y}}_{p,l}(:,k) - \mathbf{A}\mathbf{H}_{l,l}(:,k)\beta_{llk}^{1/2})^H \mathbf{R}_{\mathbf{Z}_l(:,k)}^{-1} (\tilde{\mathbf{Y}}_{p,l}(:,k) - \mathbf{A}\mathbf{H}_{l,l}(:,k)\beta_{llk}^{1/2}) \end{aligned} \quad (34)$$

Using the maximum a posteriori (MAP) decision rule, the Bayesian estimator yields the most probable value of channel estimation given the observation:  $\tilde{\mathbf{Y}}_{p,l}(:,k)$ .

$$\begin{aligned} \hat{\mathbf{H}}_{l,l}^H(:,k) &= \arg \max_{\mathbf{H}_{l,l}^H(:,k)} p(\mathbf{H}_{l,l}(:,k) | \tilde{\mathbf{Y}}_{p,l}(:,k)) \\ \hat{\mathbf{H}}_{l,l}^H(:,k) &= \arg \min_{\mathbf{H}_{l,l}^H(:,k)} f(\mathbf{H}_{l,l}(:,k)) \\ \hat{\mathbf{H}}_{l,l}^H(:,k) &= \sqrt{p_p} \beta_{llk}^{1/2} \mathbf{A}^H (p_p \beta_{llk} \mathbf{A} \mathbf{A}^H + \mathbf{R}_{\mathbf{Z}_l(:,k)})^{-1} \tilde{\mathbf{Y}}_{p,l}(:,k) \end{aligned} \quad (35)$$

It is straightforward to show that (see equation (24)):

$$\mathbf{R}_{\mathbf{Z}_l(:,k)} = \sigma_{n_l}^2 \mathbf{I}_M + p_p \sum_{i \neq l} \beta_{ilk} \mathbf{A} \mathbf{A}^H \quad (36)$$

And we find again:

$$\hat{\mathbf{H}}_{l,l}^H(:,k) = \sqrt{p_p} \beta_{llk}^{1/2} \mathbf{A}^H (\sigma_{n_l}^2 \mathbf{I}_M + p_p \sum_{i=1}^L \beta_{ilk} \mathbf{A} \mathbf{A}^H)^{-1} \tilde{\mathbf{Y}}_{p,l}(:,k) \quad (37)$$

### **Reminder Appendix 1:**

If we have the following equivalent model:

$$\mathbf{y} = \mathbf{S}\mathbf{h} + \mathbf{n} \quad (38)$$

With the following pdf :

$$p(\mathbf{h}|\mathbf{y}) = \frac{\exp(-l(\mathbf{h}))}{K}$$

$$l(\mathbf{h}) = \mathbf{h}^H \mathbf{R}_h^{-1} \mathbf{h} + (\mathbf{y} - \mathbf{S}\mathbf{h})^H \mathbf{R}_n^{-1} (\mathbf{y} - \mathbf{S}\mathbf{h}) \quad (39)$$

The Bayesian estimator yields the most probable value given the observation  $\mathbf{y}$  and this yields:

$$\hat{\mathbf{h}} = \mathbf{R}_h \mathbf{S}^H (\mathbf{R}_n + \mathbf{R}_h \mathbf{S} \mathbf{S}^H)^{-1} \mathbf{y} \quad (40)$$

#### **4- Uplink Data Transmission:**

We consider the context of an uplink transmission where the base station uses the channel estimates to perform signal processing techniques such as MRC (Maximum Ratio Combining), ZF (Zero Forcing) or MMSE (Minimum Mean Square Error).

##### **4.1. Zero forcing Receiver:**

In this case the  $l$ th base station processes the received signal by multiplying it by the pseudo-invert of the channel estimate. From (6) and (14), we obtain:

$$\mathbf{r}_l = (\hat{\Gamma}_l^H \hat{\Gamma}_l)^{-1} \hat{\Gamma}_l^H \mathbf{y}_l \quad (41)$$

With:  $\hat{\Gamma}_l = \mathbf{A}_{M \times P} [\hat{\mathbf{H}}_l \mathbf{D}_l^{1/2}]_{P \times K} = p_p^{-1/2} \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_l$ .  $\mathbf{\Pi}_A$  is of size  $M \times M$ ,  $\tilde{\mathbf{Y}}_{p,l}$  is of size  $M \times K$ . The product  $\hat{\Gamma}_l^H \hat{\Gamma}_l$  can be written as:

$$\begin{aligned} \hat{\Gamma}_l^H \hat{\Gamma}_l &= p_p^{-1/2} [\mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_l]^H p_p^{-1/2} \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_l \\ &= p_p^{-1} \mathbf{D}_l^H (\mathbf{D}_l^{-1})^H \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_l \end{aligned} \quad (42)$$

Of course, we have:  $\mathbf{\Pi}_A^H \mathbf{\Pi}_A = \mathbf{I}_M$ , so (42) reduces to :

$$\begin{aligned} \hat{\Gamma}_l^H \hat{\Gamma}_l &= p_p^{-1} \mathbf{D}_l^H (\mathbf{D}_l^{-1})^H \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_l \\ &= \frac{1}{p_p} \mathbf{D}_l \mathbf{D}_l^{-1} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_l \end{aligned} \quad (43)$$

In these conditions, we have:

$$\begin{aligned} \left[ \hat{\Gamma}_u^H \hat{\Gamma}_u \right]^{-1} &= p_p \left[ \mathbf{D}_u (\mathbf{D}_l^{-1})^H \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_u \right]^{-1} \\ &= p_p \mathbf{D}_u^{-1} \mathbf{D}_l \left[ \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \mathbf{D}_l \mathbf{D}_u^{-1} \end{aligned} \quad (44)$$

Then, we obtain:

$$\begin{aligned} &\left[ \hat{\Gamma}_u^H \hat{\Gamma}_u \right]^{-1} \hat{\Gamma}_u^H \\ &= p_p \mathbf{D}_u^{-1} \mathbf{D}_l \left[ \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \mathbf{D}_l \mathbf{D}_u^{-1} p_p^{-1/2} \left[ \Pi_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_u \right]^H \\ &= p_p^{1/2} \mathbf{D}_u^{-1} \mathbf{D}_l \left[ \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \mathbf{D}_l \mathbf{D}_u^{-1} \mathbf{D}_u \mathbf{D}_l^{-1} \tilde{\mathbf{Y}}_{p,l}^H \Pi_A^H \\ &= p_p^{1/2} \mathbf{D}_u^{-1} \mathbf{D}_l \left[ \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \tilde{\mathbf{Y}}_{p,l}^H \Pi_A^H \end{aligned} \quad (45)$$

And:

$$\begin{aligned} \mathbf{r}_l &= \left( \hat{\Gamma}_u^H \hat{\Gamma}_u \right)^{-1} \hat{\Gamma}_u^H \mathbf{y}_l \\ &= p_p^{1/2} \mathbf{D}_u^{-1} \mathbf{D}_l \left[ \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \tilde{\mathbf{Y}}_{p,l}^H \Pi_A^H \mathbf{y}_l \end{aligned} \quad (46)$$

Let  $\tilde{\mathbf{r}}_l$  the received  $K \times 1$  vector:  $\tilde{\mathbf{r}}_l \equiv \frac{1}{\sqrt{p_p}} \mathbf{D}_l^{-1} \mathbf{D}_u \mathbf{r}_l$ , we deduce from (46) that :

$$\tilde{\mathbf{r}}_l = \left[ \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \mathbf{y}_l \quad (47)$$

#### 4.2. Analysis of the Pilot Contamination effect:

We denote as  $\mathbf{M}_{p,l}^A$  the matrix:  $\left[ \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \right]_{K \times K}^{-1}$ . In this case, if we look at the  $n$ th component of vector  $\tilde{\mathbf{r}}_l$ , we have :

$$\begin{aligned} \tilde{\mathbf{r}}_l(n) &= \tilde{r}_{ln} \\ &= \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \mathbf{y}_l \\ &= \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \left[ \sqrt{p_u} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \mathbf{x}_i + \mathbf{n}_l \right] \\ &= \sqrt{p_u} \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \mathbf{x}_l + \sqrt{p_u} \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \mathbf{A} \sum_{i \neq l}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \mathbf{x}_i + \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \mathbf{n}_l \\ &= \boldsymbol{\alpha}_{ln}^T \mathbf{x}_l + \sum_{j \neq l}^L \boldsymbol{\alpha}_{jn}^T \mathbf{x}_j + z_{ln} \end{aligned} \quad (48)$$

With:  $\boldsymbol{\alpha}_{jn}^T \cong \sqrt{p_u} \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{H}_{j,l} \mathbf{D}_{j,l}^{1/2}$  and  $z_{ln} \cong \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{n}_l$ . We have :

$$\mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\} = \mathcal{E}\left[\sqrt{p_u} \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2}\right] \mathbf{e}_n \quad (49)$$

Where  $\mathbf{e}_n$  is the  $n$ th column of the  $K \times K$  identity matrix. By adding and subtracting  $\mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}$  from  $\boldsymbol{\alpha}_{ln}^T$  in (79), we obtain:

$$\tilde{r}_{ln} = \mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\} \mathbf{x}_l + (\boldsymbol{\alpha}_{ln}^T - \mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}) \mathbf{x}_l + \sum_{j \neq l} \boldsymbol{\alpha}_{jn}^T \mathbf{x}_j + z_{ln} \quad (50)$$

We obtain again the capacity of the  $n$ th user in the  $l$ th cell:

$$R_{ln} = C \left( \frac{\|\mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}\|^2}{\sum_{j=1}^L \mathcal{E}\{\|\boldsymbol{\alpha}_{jn}^T\|^2\} - \|\mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}\|^2 + \mathcal{E}\{|z_{ln}|^2\}} \right) \quad (51)$$

To develop (51) we detail first the calculation of the matrix:  $(\mathbf{M}_{p,l}^A)^{-1} = \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l}$ , we have :

$$\begin{aligned} (\mathbf{M}_{p,l}^A)^{-1} &= \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \\ &= \left( \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \left( \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right) \\ &= \left( \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H + \mathbf{W}_l^H \right) \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \left( \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right) \\ &= p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \dots \\ &\quad + \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \\ &= p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \end{aligned}$$

Clearly, this reduces to:

$$\begin{aligned}
& \mathcal{E}[\tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l}] \\
&= p_p \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{j=1}^L \mathbf{G}_{j,l} \right] + \mathcal{E}[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l] \\
&= p_p \sum_{i,j}^L \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}] + \mathcal{E}[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l]
\end{aligned}$$

To further proceed this quantity, we have to calculate:

$$\mathbf{G}_{j,l}^H \mathbf{A}^H = \frac{1}{\sqrt{P}} \begin{bmatrix} e^{jf_1(\phi_1)} & e^{jf_2(\phi_1)} & \dots & e^{jf_M(\phi_1)} \\ e^{jf_1(\phi_2)} & e^{jf_2(\phi_2)} & \dots & e^{jf_M(\phi_2)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{jf_1(\phi_p)} & e^{jf_2(\phi_p)} & \dots & e^{jf_M(\phi_p)} \end{bmatrix} \begin{bmatrix} \sum_{k=1}^P g_{j1k}^* e^{jf_1(\phi_k)} & \sum_{k=1}^P g_{j1k}^* e^{jf_2(\phi_k)} & \dots & \sum_{k=1}^P g_{j1k}^* e^{jf_M(\phi_k)} \\ \sum_{k=1}^P g_{j2k}^* e^{jf_1(\phi_k)} & \sum_{k=1}^P g_{j2k}^* e^{jf_2(\phi_k)} & \dots & \sum_{k=1}^P g_{j2k}^* e^{jf_M(\phi_k)} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^P g_{jKk}^* e^{jf_1(\phi_k)} & \sum_{k=1}^P g_{jKk}^* e^{jf_2(\phi_k)} & \dots & \sum_{k=1}^P g_{jKk}^* e^{jf_M(\phi_k)} \end{bmatrix}$$

The  $(m,n)$ th element of  $\mathbf{G}_{i,l}^H \mathbf{A}^H$  is thus written as :

$$[\mathbf{G}_{i,l}^H \mathbf{A}^H(m,n)]_{K \times M} = a_{m,n} = \frac{1}{\sqrt{P}} \sum_{k=1}^P g_{ilmk}^* e^{jf_n(\phi_k)} \quad (52)$$

Concerning the matrix:  $\mathbf{A} \mathbf{G}_{j,l}$ , we obtain :

$$\mathbf{A} \mathbf{G}_{j,l} = \frac{1}{\sqrt{P}} \begin{bmatrix} \sum_{n=1}^P e^{-jf_1(\phi_n)} g_{j1n} & \sum_{n=1}^P e^{-jf_1(\phi_n)} g_{j2n} & \dots & \sum_{n=1}^P e^{-jf_1(\phi_n)} g_{jkn} & \dots & \sum_{n=1}^P e^{-jf_1(\phi_n)} g_{jKn} \\ \sum_{n=1}^P e^{-jf_2(\phi_n)} g_{j1n} & \sum_{n=1}^P e^{-jf_2(\phi_n)} g_{j2n} & \dots & \sum_{n=1}^P e^{-jf_2(\phi_n)} g_{jkn} & \dots & \sum_{n=1}^P e^{-jf_2(\phi_n)} g_{jKn} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sum_{n=1}^P e^{-jf_m(\phi_n)} g_{j1n} & \sum_{n=1}^P e^{-jf_m(\phi_n)} g_{j2n} & \dots & \sum_{n=1}^P e^{-jf_m(\phi_n)} g_{jkn} & \dots & \sum_{n=1}^P e^{-jf_m(\phi_n)} g_{jKn} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ \sum_{n=1}^P e^{-jf_M(\phi_n)} g_{j1n} & \sum_{n=1}^P e^{-jf_M(\phi_n)} g_{j2n} & \dots & \sum_{n=1}^P e^{-jf_M(\phi_n)} g_{jkn} & \dots & \sum_{n=1}^P e^{-jf_M(\phi_n)} g_{jKn} \end{bmatrix}$$

Thus, the  $(m,k)$ th element of  $\mathbf{A} \mathbf{G}_{j,l}$  is thus written as :



$$\left[ \mathbf{A}\mathbf{G}_{j,l}(m,k) \right]_{M \times K} = \frac{1}{\sqrt{P}} \sum_{n=1}^P g_{jkn} e^{-jf_n(\phi_n)} \quad (53)$$

And the  $(m,n)$ th element of matrix  $\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}$  is equal to:

$$\begin{aligned} \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] (m,n) &= \sum_{k=1}^M a_{m,k} b_{k,n} \\ &= \frac{1}{P} \sum_{k=1}^M \left( \sum_{r=1}^P g_{ilmr}^* e^{jf_k(\phi_r)} \right) \left( \sum_{q=1}^P g_{jlnq} e^{-jf_k(\phi_q)} \right) \\ &= \frac{1}{P} \sum_{k=1}^M \left( \sum_{r=1}^P \sum_{q=1}^P g_{ilmr}^* g_{jlnq} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \\ &= \frac{1}{P} \sum_{k=1}^M \left( \sum_{r=1}^P \sum_{q=1}^P h_{ilmr}^* h_{jlnq} \sqrt{\beta_{ilm} \beta_{jln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \end{aligned} \quad (54)$$

Hence, we have:

$$\mathcal{E} \left[ \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] (m,n) \right] = \frac{1}{P} \sum_{k=1}^M \left( \sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[ h_{ilmr}^* h_{jlnq} \right] \sqrt{\beta_{ilm} \beta_{jln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \quad (55)$$

For  $i = j$ , we obtain:

$$\mathcal{E} \left[ \left[ \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] (m,n) \right] = \frac{1}{P} \sum_{k=1}^M \left( \sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[ h_{jlmr}^* h_{jlnq} \right] \sqrt{\beta_{jlm} \beta_{jln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right)$$

The quantity:  $\mathcal{E} \left[ h_{ilmr}^* h_{jlnq} \right]$  is equal to zero for  $i \neq j$  and for  $i = j$ , due to the independency between columns, we have  $\mathcal{E} \left[ h_{ilmr}^* h_{jlnq} \right] = 0$  if  $m \neq n$ . For  $i = j$  and  $m = n$  we have:

$$\begin{aligned} \mathcal{E} \left[ \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] (n,n) \right] &= \frac{1}{P} \sum_{k=1}^M \left( \sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[ h_{ilmr}^* h_{ilnq} \right] \beta_{iln} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \\ \mathcal{E} \left[ \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] (n,n) \right] &= \frac{1}{P} \sum_{k=1}^M \left( \sum_{r=1}^P \mathcal{E} \left[ |h_{ilmr}|^2 \beta_{iln} \right] \right) = \frac{M \beta_{iln}}{P} \end{aligned} \quad (56)$$

Concerning the quantity:  $\mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right]$ , we have already developed the calculations in the former section,

$$\begin{aligned}
& \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= \frac{1}{M} \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{W}_l \right] \text{Trace} \left( \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right) \quad (57) \\
&= \frac{1}{M} M \sigma_{n_l}^2 \mathbf{I}_K \text{Trace} \left[ \mathbf{\Pi}_A \right] \approx_{M \rightarrow \infty} \sigma_{n_l}^2 P \mathbf{I}_K
\end{aligned}$$

We eventually obtain:

$$\mathcal{E} \left[ \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] = \text{diag} \left( \rho_{1,l}, \rho_{2,l}, \dots, \rho_{K,l} \right) \quad (58)$$

With:

$$\begin{aligned}
\rho_{n,l} &= p_p \frac{M}{P} \sum_{i=1}^L \beta_{in} + \sigma_{n_l}^2 P \\
\rho_{n,l} &= p_p \frac{M}{P} \bar{\beta}_{in} + \sigma_{n_l}^2 P
\end{aligned}$$

We come back to the calculation of:  $\mathcal{E} \left\{ \boldsymbol{\alpha}_{ln}^T \right\}$ , remember we have :

$$\begin{aligned}
\mathcal{E} \left\{ \boldsymbol{\alpha}_{ln}^T \right\} &= \mathcal{E} \left[ \sqrt{p_u} \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \right] \mathbf{e}_n \\
\mathcal{E} \left\{ \boldsymbol{\alpha}_{ln}^T \right\} &\approx \sqrt{p_u} \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \right] \mathcal{E} \left[ \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{G}_{l,l} \right] \mathbf{e}_n \\
&= \sqrt{p_u} \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \right] \mathcal{E} \left[ \left( \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{G}_{l,l} \right] \mathbf{e}_n \\
&= \sqrt{p_u} \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \right] \mathcal{E} \left[ \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{G}_{l,l} \right] \mathbf{e}_n \\
&= \sqrt{p_p p_u} \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \right] \mathcal{E} \left[ \mathbf{G}_{l,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{l,l} \right] \mathbf{e}_n
\end{aligned}$$

We have already found that (see equation (56)):

$$\mathcal{E} \left[ \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] (n,n) \right] = \frac{M \beta_{in}}{P}$$

Using:  $\mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \right] \approx 1 / \rho_{n,l}$ , we eventually obtain:

$$\begin{aligned}
\mathcal{E} \{ \boldsymbol{\alpha}_{ln}^T \} &\approx \sqrt{p_p p_u} \frac{M \beta_{ln}}{P \rho_{n,l}} \mathbf{e}_n \\
&= \sqrt{p_p p_u} \frac{M \beta_{ln}}{P \left[ p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_l}^2 P \right]} \mathbf{e}_n \\
&= \sqrt{p_p p_u} \frac{M \beta_{ln}}{p_p M \bar{\beta}_{ln} + P^2 \sigma_{n_l}^2} \mathbf{e}_n \\
&= \sqrt{p_p p_u} \frac{\beta_{ln}}{p_p \bar{\beta}_{ln} + P^2 \sigma_{n_l}^2 / M} \mathbf{e}_n
\end{aligned} \tag{59}$$

Concerning the quantity:  $\mathcal{E} \left\{ \left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right\}$ , we have:

$$\begin{aligned}
\mathcal{E} \left\{ \left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right\} &= \mathcal{E} \left( \boldsymbol{\alpha}_{jn}^T \boldsymbol{\alpha}_{jn}^* \right) \\
&= p_u \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right] \\
&= p_u \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right]
\end{aligned}$$

We have:

$$\begin{aligned}
&\tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \\
&= \left( \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \left( \sqrt{p_p} \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} + \mathbf{W}_l \right) \\
&= \left( \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \left( \sqrt{p_p} \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} + \mathbf{W}_l \right)
\end{aligned}$$

And this yields:

$$\begin{aligned}
&\mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right] \\
&= \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \left( \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \left( \sqrt{p_p} \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} + \mathbf{W}_l \right) \mathbf{M}_{p,l}^A(n,:)^H \right] \\
&= \mathcal{E} \left[ p_p \mathbf{M}_{p,l}^A(n,:) \sum_{i=1, i \neq j}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1, m \neq j}^L \mathbf{G}_{m,l} \mathbf{M}_{p,l}^A(n,:)^H \right] + \dots \\
&\dots + \mathcal{E} \left[ p_p \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{M}_{p,l}^A(n,:)^H \right] + \dots \\
&\dots + \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{M}_{p,l}^A(n,:)^H \right]
\end{aligned}$$

And, we have:

$$\begin{aligned}
& \mathcal{E} \left[ p_p \mathbf{M}_{p,l}^A(n,:) \sum_{i=1, i \neq j}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1, m \neq j}^L \mathbf{G}_{m,l} \mathbf{M}_{p,l}^A(n,:) \right] \\
&= \frac{p_p}{P} \text{Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \sum_{i=1, i \neq j}^L \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{M}_{p,l}^A(n,:) \right] \\
&= \frac{p_p}{P^2} \text{Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \text{Trace} \left[ \left( \mathbf{A}^H \mathbf{A} \right)^2 \right] \sum_{i=1, i \neq j}^L \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \mathbf{M}_{p,l}^A(n,:) \right] \\
&= \frac{p_p}{P^2 K} \text{Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \text{Trace} \left[ \left( \mathbf{A}^H \mathbf{A} \right)^2 \right] \text{Trace} \left[ \mathcal{E} \left[ \sum_{i=1, i \neq j}^L \mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \right] \right] \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:) \right] \\
&= \frac{p_p}{P^2 K} \text{Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \text{Trace} \left[ \left( \mathbf{A}^H \mathbf{A} \right)^2 \right] \left[ \sum_{i=1, i \neq j}^L \text{Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \right] \right] \right] \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:) \right] \\
&= \frac{p_p \text{Trace} \left[ \left( \mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left( \sum_{i=1, i \neq j}^L \beta_{il} \right) \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:) \right]}{P^2 K} \\
&= \frac{p_p \text{Trace} \left[ \left( \mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left( \sum_{i=1, i \neq j}^L \beta_{il} \right) \gamma_{n,l}}{P^2 K}
\end{aligned}$$

With :  $\mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:) \right] \approx \gamma_{n,l}$

We have (already demonstrated for the MRC receiver):

$$\mathcal{E} \left[ \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] = \text{diag} \left[ \sum_{k=1}^K \beta_{jlk} \mathcal{E} \left[ |h_{jlk1}|^2 \right], \sum_{k=1}^K \beta_{jlk} \mathcal{E} \left[ |h_{jlk2}|^2 \right], \dots, \sum_{k=1}^K \beta_{jlk} \mathcal{E} \left[ |h_{jlkM}|^2 \right], \dots, \sum_{k=1}^K \beta_{jlk} \mathcal{E} \left[ |h_{jlkP}|^2 \right] \right]$$

And :  $\text{Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] = \sum_{k=1}^K \beta_{jlk} = \beta_{jl}$ . Concerning the quantity  $\mathcal{E} \left[ \mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \right]$ , we have :

$$\mathbf{G}_{j,l}^H \mathbf{G}_{j,l} = \begin{bmatrix} \mathcal{G}_{j11}^* & \mathcal{G}_{j12}^* & \cdots & \mathcal{G}_{j1k}^* & \cdots & \mathcal{G}_{j1P}^* \\ \mathcal{G}_{j21}^* & \mathcal{G}_{j22}^* & \cdots & \mathcal{G}_{j2k}^* & \cdots & \mathcal{G}_{j2P}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{G}_{jlm}^* & \mathcal{G}_{jlm}^* & \cdots & \mathcal{G}_{jlm}^* & \cdots & \mathcal{G}_{jlm}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{G}_{j1P}^* & \mathcal{G}_{j1P}^* & \cdots & \mathcal{G}_{j1P}^* & \cdots & \mathcal{G}_{j1P}^* \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^P |\mathcal{G}_{j1k}|^2 & \sum_{k=1}^P \mathcal{G}_{j1k}^* \mathcal{G}_{j2k} & \cdots & \sum_{k=1}^P \mathcal{G}_{j1k}^* \mathcal{G}_{jmk} & \cdots & \sum_{k=1}^P \mathcal{G}_{j1k}^* \mathcal{G}_{jKk} \\ \sum_{k=1}^P \mathcal{G}_{j2k} \mathcal{G}_{j1k}^* & \sum_{k=1}^P |\mathcal{G}_{j2k}|^2 & \cdots & \sum_{k=1}^P \mathcal{G}_{j2k} \mathcal{G}_{jmk} & \cdots & \sum_{k=1}^P \mathcal{G}_{j2k} \mathcal{G}_{jKk} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^P \mathcal{G}_{jmk} \mathcal{G}_{j1k}^* & \sum_{k=1}^P \mathcal{G}_{jmk} \mathcal{G}_{j2k} & \cdots & \sum_{k=1}^P |\mathcal{G}_{jmk}|^2 & \cdots & \sum_{k=1}^P \mathcal{G}_{jmk} \mathcal{G}_{jKk} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^P \mathcal{G}_{jKk} \mathcal{G}_{j1k}^* & \sum_{k=1}^P \mathcal{G}_{jKk} \mathcal{G}_{j2k} & \cdots & \sum_{k=1}^P \mathcal{G}_{jKk} \mathcal{G}_{jmk} & \cdots & \sum_{k=1}^P |\mathcal{G}_{jKk}|^2 \end{bmatrix}$$

$$\begin{aligned}
\mathcal{E} \left[ \mathbf{G}_{j,l}^H \mathbf{G}_{j,l} \right] &= \text{diag} \left[ \mathcal{E} \left[ \sum_{k=1}^P |\mathcal{G}_{j1k}|^2 \right], \mathcal{E} \left[ \sum_{k=1}^P |\mathcal{G}_{j2k}|^2 \right], \dots, \mathcal{E} \left[ \sum_{k=1}^P |\mathcal{G}_{jmk}|^2 \right], \dots, \mathcal{E} \left[ \sum_{k=1}^P |\mathcal{G}_{jKk}|^2 \right] \right] \\
&= \text{diag} \left[ \beta_{j1} \mathcal{E} \left[ \sum_{k=1}^P |h_{j1k}|^2 \right], \beta_{j2} \mathcal{E} \left[ \sum_{k=1}^P |h_{j2k}|^2 \right], \dots, \beta_{jm} \mathcal{E} \left[ \sum_{k=1}^P |h_{jmk}|^2 \right], \dots, \beta_{jK} \mathcal{E} \left[ \sum_{k=1}^P |h_{jKk}|^2 \right] \right] \\
&= \text{diag} \left[ \beta_{j1}, \beta_{j2}, \dots, \beta_{jm}, \dots, \beta_{jK} \right]
\end{aligned}$$

$$\text{And: Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{j,l}^H \mathbf{G}_{j,l} \right] \right] = \sum_{k=1}^K \beta_{jlk} = \beta_{jl}.$$

Concerning the term:  $\mathcal{E} \left[ p_p \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{M}_{p,l}^A(n,:) \right]^H$ , we have:

$$\begin{aligned}
&\mathcal{E} \left[ p_p \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{M}_{p,l}^A(n,:) \right]^H \\
&= \frac{p_p}{K} \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:) \right]^H \text{Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \right] \\
&= \frac{p_p \gamma_{n,l}}{K} \text{Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right]^2 \right]
\end{aligned}$$

We use the following approximation (the demonstration is given later):

$$\begin{aligned} & \mathcal{E} \left[ \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \\ &= \frac{M^2 \beta_{jl}}{P^2} \text{diag}(\beta_{jl1}, \beta_{jl2}, \dots, \beta_{jlK}) \end{aligned}$$

Hence this yields:

$$\begin{aligned} & \mathcal{E} \left[ p_p \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{M}_{p,l}^A(n,:)^H \right] \\ &= \frac{p_p}{K} \text{Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right]^2 \right] \gamma_{n,l} \\ &= \frac{p_p}{K} \frac{M^2 \beta_{jl}^2}{P^2} \gamma_{n,l} \end{aligned}$$

Concerning the last quantity, we have :

$$\begin{aligned} & \mathcal{E} \left[ \mathbf{M}_{p,l}^A(n,:) \mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{M}_{p,l}^A(n,:)^H \right] \\ &= \frac{\gamma_{n,l}}{K} \text{Trace} \left[ \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \right] \right] \\ &= \frac{\gamma_{n,l}}{K} \frac{M^2 \sigma_{n_l}^2}{P} \beta_{jl} \\ &= \frac{M^2 \sigma_{n_l}^2}{KP} \beta_{jl} \gamma_{n,l} \end{aligned}$$

We have:

$$\begin{aligned} & \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \right] \\ &= \frac{1}{K} \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{W}_l \right] \text{Trace} \left( \mathcal{E} \left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] \right) \\ &= \frac{M \sigma_{n_l}^2}{K} \frac{M}{P} \beta_{jl} \mathbf{I}_K = \frac{M^2 \sigma_{n_l}^2}{KP} \beta_{jl} \mathbf{I}_K \end{aligned}$$

Concerning the quantity:  $\mathcal{E} \left( \left[ \mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:) \right]^H \right)$ , we have:

$$\begin{aligned} & \mathcal{E} \left( \left[ \mathbf{M}_{p,l}^A \mathbf{M}_{p,l}^A \right]^H \right) = \mathcal{E} \left( \left[ \left( \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right)^{-1} \left( \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right)^{-1 H} \right] \right) \\ &= \mathcal{E} \left( \left[ \left( \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right)^H \left( \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right) \right]^{-1} \right) \\ &= \mathcal{E} \left( \left[ \left( \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right) \left( \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right) \right]^{-1} \right) \\ &= \mathcal{E} \left( \left[ \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \right) = \mathcal{E} \left( \left[ \left( \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right)^2 \right]^{-1} \right) \end{aligned}$$

We have:

$$\begin{aligned}
& \left( \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right)^2 \\
&= \left[ p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right]^2 \\
&= p_p^2 \left( \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left( \sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) + \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \dots \\
&\quad \dots + p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \sum_{m=1}^L \mathbf{G}_{m,l}^H \mathbf{A}^H \mathbf{W}_l + p_p \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} + p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \dots \\
&\quad \dots + p_p^{3/2} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l + p_p^{3/2} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \dots \\
&\quad \dots + p_p \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l + \dots \\
&\quad \dots + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + p_p^{3/2} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \dots \\
&\quad \dots + \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \dots \\
&\quad \dots + p_p^{3/2} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \dots \\
&\quad \dots + p_p \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l
\end{aligned}$$

When we take the expectation of the different terms in the former expression we make the following assumptions, due to the independency between channel parameters and additive noise we neglect the corresponding expectation term each time the stochastic variable appears an odd number of times in the calculation. Using this rule, it is not difficult to demonstrate that the following expectation terms are equal to zero:

$$\begin{aligned}
& \mathcal{E} \left( \left[ \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^2 \right) \\
& \approx p_p^2 \mathcal{E} \left[ \left( \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left( \sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) \right] \\
& \quad \dots + \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + p_p \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \sum_{m=1}^L \mathbf{G}_{m,l}^H \mathbf{A}^H \mathbf{W}_l \right] + \dots \\
& \quad \dots + p_p \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right] + p_p \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + \dots \\
& \quad \dots + p_p \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] + p_p \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] + \dots \\
& \quad \dots + p_p \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \right]
\end{aligned}$$

$\mathcal{E} \left( \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right)$   
 $\mathcal{E} \left( \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \right)$   
 $\mathcal{E} \left( \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l} \right)$   
 $\mathcal{E} \left( \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l} \right)$

This yields:

Due to the independency between channel parameters and additive noise, we can further simplify using:

$$\mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \sum_{m=1}^L \mathbf{G}_{m,l}^H \mathbf{A}^H \mathbf{W}_l \right] \approx \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \sum_{m=1}^L \mathcal{E} [\mathbf{W}_l \mathbf{G}_{m,l}^H] \mathbf{A}^H \mathbf{W}_l \right] \approx \mathbf{0}$$

And, for the same reason:

$$\mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right] \approx \mathbf{0}$$

This reduces the calculation of  $\mathcal{E} \left( \left[ \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^2 \right)$  and this gives:

$$\begin{aligned} & \mathcal{E} \left( \left[ \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^2 \right) \\ & \approx p_p^2 \mathcal{E} \left[ \left( \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left( \sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) \right] + \dots \\ & \dots + \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + \dots \\ & \dots + p_p \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + \dots \\ & \dots + p_p \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] + \dots \\ & \dots + p_p \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] + p_p \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \right] \end{aligned}$$



$$\begin{aligned}
& \mathcal{E} \left( \left[ \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^2 \right) \\
& \approx p_p^2 \mathcal{E} \left[ \left( \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left( \sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) \right] + \dots \\
& \dots + \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + 2p_p \sum_{i=1}^L \mathcal{E} \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + \dots \\
& \dots + p_p \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[ \mathbf{W}_l \mathbf{W}_l^H \right] \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] + p_p \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathcal{E} \left[ \mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \right] \mathbf{A}^H \mathbf{W}_l \right]
\end{aligned}$$

Concerning the first term in the expectation, we take a particular case with  $L = 3$  before a complete generalization. We use the notation:  $\mathbf{a}_i = \mathbf{A} \mathbf{G}_{i,l}$ , then, we obtain :

$$\begin{aligned}
& \mathcal{E} \left[ \left( \sum_{i=1}^3 \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^3 \mathbf{G}_{k,l} \right) \left( \sum_{j=1}^3 \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^3 \mathbf{G}_{m,l} \right) \right] \\
& = \mathcal{E} \left[ \left( \sum_{j=1}^3 \mathbf{a}_j^H \sum_{i=1}^3 \mathbf{a}_i \right) \left( \sum_{m=1}^3 \mathbf{a}_m^H \sum_{k=1}^3 \mathbf{a}_k \right) \right] \\
& = \mathcal{E} \left[ \left( \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_3^H \mathbf{a}_2 \right)^2 \right] \\
& = \mathcal{E} \left[ \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_1 + \dots \right. \\
& \dots + \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_3^H \mathbf{a}_2 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_1 + \dots \\
& \dots + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_3^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_2^H \mathbf{a}_1 + \dots \\
& \dots + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_1^H \mathbf{a}_3 + \dots \\
& \dots + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_1 + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_3^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_3 \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_1^H \mathbf{a}_3 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_3 \mathbf{a}_1^H \mathbf{a}_2 + \dots \\
& \dots + \mathbf{a}_1^H \mathbf{a}_3 \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_3 \mathbf{a}_2^H \mathbf{a}_1 + \mathbf{a}_1^H \mathbf{a}_3 \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_1^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_2 + \mathbf{a}_2^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_1 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_2^H \mathbf{a}_1 \mathbf{a}_3^H \mathbf{a}_3 + \dots \\
& \dots + \mathbf{a}_2^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_2^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_1 \mathbf{a}_2^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_1 \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_1 \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_1 \mathbf{a}_3^H \mathbf{a}_2 + \mathbf{a}_2^H \mathbf{a}_3 \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_3 \mathbf{a}_2^H \mathbf{a}_2 + \dots \\
& \dots + \mathbf{a}_2^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_3 \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_2^H \mathbf{a}_3 \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_3 \mathbf{a}_2^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_3 \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_1 + \dots \\
& \dots + \mathbf{a}_3^H \mathbf{a}_1 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_1 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_1 \mathbf{a}_2^H \mathbf{a}_1 + \mathbf{a}_3^H \mathbf{a}_1 \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_1 \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_3^H \mathbf{a}_1 \mathbf{a}_3^H \mathbf{a}_2 + \dots \\
& \dots + \mathbf{a}_3^H \mathbf{a}_2 \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_3^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_2 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_2 \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_2 \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_1 + \mathbf{a}_3^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_2 \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_3^H \mathbf{a}_2 \mathbf{a}_3^H \mathbf{a}_2 ] \\
& = \mathcal{E} \left[ \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_1 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_2 \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_1^H \mathbf{a}_1 + \dots \right. \\
& \dots + \left. \mathbf{a}_3^H \mathbf{a}_3 \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_1 + \mathbf{a}_1^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_2^H \mathbf{a}_3 \mathbf{a}_3^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_1 \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_2 \mathbf{a}_2^H \mathbf{a}_3 \right]
\end{aligned}$$

In fact, due to the independency between different channels, we have the following results:

$$\mathcal{E}[\mathbf{a}_i^H \mathbf{a}_i \mathbf{a}_i^H \mathbf{a}_j] = 0, \mathcal{E}[\mathbf{a}_i^H \mathbf{a}_j \mathbf{a}_j^H \mathbf{a}_j] = 0, \mathcal{E}[\mathbf{a}_i^H \mathbf{a}_j \mathbf{a}_j^H \mathbf{a}_k] = 0, \mathcal{E}[\mathbf{a}_i^H \mathbf{a}_j \mathbf{a}_i^H \mathbf{a}_j] = 0$$

We have to calculate three kinds of terms:

-1) The first one is:

$$\mathcal{E}[\mathbf{a}_i^H \mathbf{a}_i \mathbf{a}_i^H \mathbf{a}_i] = \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}]$$

Concerning first matrix:  $\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}$ , the general term of matrix is:

$$\begin{aligned} \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] (m, n) &= a_{m,n} \\ &= \frac{1}{P} \sum_{k=1}^M \left( \sum_{r=1}^P \sum_{q=1}^P h_{ilmr}^* h_{ilnq} \sqrt{\beta_{ilm} \beta_{iln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \\ &= \frac{1}{P} \sum_{k=1}^M A_{r,q}^{i,k}(m, n) \end{aligned}$$

With:

$$A_{r,q}^{i,k}(m, n) = \sum_{r=1}^P \sum_{q=1}^P h_{ilmr}^* h_{ilnq} \sqrt{\beta_{ilm} \beta_{iln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]}$$

We have, in this case:

$$\mathcal{E} \left( \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] (m, n) \right) = \frac{1}{P} \sum_{k=1}^M \left( \sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[ h_{ilmr}^* h_{ilnq} \right] \sqrt{\beta_{ilm} \beta_{iln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right)$$

Of course, this quantity is not equal to zero only and only if  $n = m$  and  $r = q$ , this yields :

$$\begin{aligned} \mathcal{E} \left( \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] (n, n) \right) &= \frac{1}{P} \sum_{k=1}^M \left( \sum_{r=1}^P \mathcal{E} \left[ |h_{ilmr}|^2 \right] \beta_{iln} \right) \\ &= \frac{1}{P} \sum_{k=1}^M \beta_{iln} \\ &= \frac{M}{P} \beta_{iln} \end{aligned}$$

Then, we obtain:

$$\mathcal{E} \left( \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \right) = \frac{M}{P} \text{diag}(\beta_{il1}, \beta_{il2}, \dots, \beta_{ilK})$$

In these conditions, the general term of matrix  $\left( \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right)^2$  is:

$$\begin{aligned} \left( \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right)^2 (m, n) &= \sum_{p=1}^K a_{m,p} a_{p,n} \\ &= \sum_{p=1}^K \left( \frac{1}{P} \sum_{k=1}^M A_{r,q}^{j,k}(m, p) \right) \left( \frac{1}{P} \sum_{u=1}^M A_{s,t}^{j,u}(p, n) \right) \\ &= \frac{1}{P^2} \sum_{p=1}^K \sum_{k=1}^M \sum_{u=1}^M A_{r,q}^{j,k}(m, p) A_{s,t}^{j,u}(p, n) \end{aligned}$$

We have:

$$\begin{aligned}
& \mathcal{E} \left[ A_{r,q}^{j,k}(m,p) A_{s,t}^{j,u}(p,n) \right] \\
&= \mathcal{E} \left[ \left( \sum_{r=1}^P \sum_{q=1}^P h_{jlmr}^* h_{jlpq} \sqrt{\beta_{jlm} \beta_{jlp}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \left( \sum_{s=1}^P \sum_{t=1}^P h_{jlp s}^* h_{jln t} \sqrt{\beta_{jlp} \beta_{jln}} e^{j[f_u(\phi_s) - f_u(\phi_t)]} \right) \right] \\
&= \sum_{r=1}^P \sum_{q=1}^P \sum_{s=1}^P \sum_{t=1}^P \mathcal{E} \left[ h_{jlmr}^* h_{jlpq} h_{jlp s}^* h_{jln t} \right] \beta_{jlp} \sqrt{\beta_{jlm} \beta_{jln}} e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_s) - f_u(\phi_t)]}
\end{aligned}$$

The term  $\mathcal{E} \left[ h_{jlmr}^* h_{jlpq} h_{jlp s}^* h_{jln t} \right]$  is not equal to zero when:  $n = m$  and  $s = q$  and  $r = t$ . In these conditions we have:

$$\begin{aligned}
& \mathcal{E} \left[ A_{r,q}^{j,k}(n,p) A_{s,t}^{j,u}(p,n) \right] \\
&= \sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[ h_{jlmr}^* h_{jlpq} h_{jlpq}^* h_{jlnr} \right] \beta_{jlp} \beta_{jln} e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \\
&= \sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[ |h_{jlmr}|^2 \right] \mathcal{E} \left[ |h_{jlpq}|^2 \right] \beta_{jlp} \beta_{jln} e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \\
&= \beta_{jlp} \beta_{jln} \sum_{q=1}^P \mathcal{E} \left[ |h_{jlpq}|^2 \right] e^{j[f_u(\phi_q) - f_k(\phi_q)]} \sum_{r=1}^P \mathcal{E} \left[ |h_{jlmr}|^2 \right] e^{j[f_k(\phi_r) - f_u(\phi_r)]}
\end{aligned}$$

We have:

$$\begin{aligned}
e^{j[f_u(\phi_q) - f_k(\phi_q)]} &= e^{j \left[ 2\pi \frac{d}{\lambda} (u-1) \sin(\phi_q) - 2\pi \frac{d}{\lambda} (k-1) \sin(\phi_q) \right]} = e^{j \left[ 2\pi \frac{d}{\lambda} (u-k) \sin(\phi_q) \right]} \\
e^{j[f_k(\phi_r) - f_u(\phi_r)]} &= e^{j \left[ 2\pi \frac{d}{\lambda} (k-1) \sin(\phi_r) - 2\pi \frac{d}{\lambda} (u-1) \sin(\phi_r) \right]} = e^{j \left[ 2\pi \frac{d}{\lambda} (k-u) \sin(\phi_r) \right]}
\end{aligned}$$

This yields:

$$\begin{aligned}
S &= \sum_{k=1}^M \sum_{u=1, u \neq k}^M e^{j \left[ 2\pi \frac{d}{\lambda} (u-k) (\sin(\phi_q) - \sin(\phi_r)) \right]} = \left[ \sum_{k=1}^M e^{-j 2\pi \frac{d}{\lambda} k (\sin(\phi_q) - \sin(\phi_r))} \right] \left[ \sum_{u=1, u \neq k}^M e^{j 2\pi \frac{d}{\lambda} u (\sin(\phi_q) - \sin(\phi_r))} \right] \\
&= \left[ \sum_{k=1}^M e^{-j 2\pi \frac{d}{\lambda} k (\sin(\phi_q) - \sin(\phi_r))} \right] \left[ \sum_{u=1}^M e^{j 2\pi \frac{d}{\lambda} u (\sin(\phi_q) - \sin(\phi_r))} - e^{j 2\pi \frac{d}{\lambda} k (\sin(\phi_q) - \sin(\phi_r))} \right] \\
&= \left( \sum_{k=1}^M e^{-j 2\pi \frac{d}{\lambda} k (\sin(\phi_q) - \sin(\phi_r))} \right) \left( \sum_{u=1}^M e^{j 2\pi \frac{d}{\lambda} u (\sin(\phi_q) - \sin(\phi_r))} \right) - M \\
&= \left( e^{-j 2\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))} \frac{1 - e^{-j 2\pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))}}{1 - e^{-j 2\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))}} \right) \left( e^{j 2\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))} \frac{1 - e^{j 2\pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))}}{1 - e^{j 2\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))}} \right) - M \\
&= \left( \frac{\sin \left[ \pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]}{\sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right)^2 - M \\
&= M^2 \left( \frac{\sin \left[ \pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right)^2 - M
\end{aligned}$$

Using this result, and supposing  $\mathcal{E} \left[ |h_{jlpq}|^2 \right] = \mathcal{E} \left[ |h_{jlnr}|^2 \right] = 1/P \quad \forall r, \forall q$ , (this means that the contributions of each multipath component is the same) we obtain:

$$\begin{aligned}
&\mathcal{E} \left[ \left( \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right)^2 (n, n) \right] \\
&= \frac{1}{P^2} \sum_{p=1}^K \sum_{k=1}^M \sum_{u=1}^M \mathcal{E} \left[ A_{r,q}^{j,k}(n, p) A_{s,t}^{j,u}(p, n) \right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{P^2} \sum_{p=1}^K \sum_{k=1}^M \sum_{u=1}^M \beta_{jlp} \beta_{jln} \sum_{r=1}^P \mathcal{E} \left[ |h_{jlnr}|^2 \right] \sum_{q=1}^P \mathcal{E} \left[ |h_{jlpq}|^2 \right] e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \\
&= \frac{1}{P^2} \beta_{jln} \sum_{p=1}^K \beta_{jlp} \sum_{k=1}^M \left[ \sum_{r=1}^P \mathcal{E} \left[ |h_{jlnr}|^2 \right] \sum_{q=1}^P \mathcal{E} \left[ |h_{jlpq}|^2 \right] \right] + \dots \\
&\dots + \frac{1}{P^2} \beta_{jln} \sum_{p=1}^K \beta_{jlp} \sum_{k=1}^M \sum_{u=1, u \neq k}^M \left[ \sum_{r=1}^P \mathcal{E} \left[ |h_{jlnr}|^2 \right] \sum_{q=1}^P \mathcal{E} \left[ |h_{jlpq}|^2 \right] e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \right] \\
&= \frac{M}{P^2} \beta_{jln} \beta_{jl} + \frac{1}{P^4} \beta_{jln} \beta_{jl} \sum_{k=1}^M \sum_{u=1, u \neq k}^M \left[ \sum_{r=1}^P \sum_{q=1}^P e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \right] \\
&= \frac{M}{P^2} \beta_{jln} \beta_{jl} + \frac{1}{P^4} \beta_{jln} \beta_{jl} \sum_{r=1}^P \sum_{q=1}^P \left[ \sum_{k=1}^M \sum_{u=1, u \neq k}^M e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \right] \\
&= \frac{M}{P^2} \beta_{jln} \beta_{jl} + \frac{1}{P^4} \beta_{jln} \beta_{jl} \sum_{r=1}^P \sum_{q=1}^P \left[ M^2 \frac{\left( \frac{\sin \left[ \pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right)^2}{M} - M \right] \\
&= \frac{M}{P^2} \beta_{jln} \beta_{jl} + \frac{M^2}{P^4} \beta_{jln} \beta_{jl} \sum_{r=1}^P \sum_{q=1}^P \left[ \frac{\left( \frac{\sin \left[ \pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right)^2}{M} - \frac{1}{M} \right] \\
&= \frac{M^2}{P^4} \beta_{jln} \beta_{jl} \sum_{r=1}^P \sum_{q=1}^P \left( \frac{\sin \left[ \pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right)^2
\end{aligned}$$

Using the notation:  $S_{r,q} = \sum_{r=1}^P \sum_{q=1}^P \left( \frac{\sin \left[ \pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right)^2$ , we obtain eventually :

$$\begin{aligned}
&\mathcal{E} \left[ \left( \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right)^2 (n, n) \right] \\
&\approx \frac{M^2}{P^4} \beta_{jln} \beta_{jl} S_{r,q}
\end{aligned}$$

This yields:

$$\begin{aligned}
&\mathcal{E} \left[ \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \\
&= \frac{M^2 \beta_{jl} S_{r,q}}{P^4} \text{diag}(\beta_{j11}, \beta_{j12}, \dots, \beta_{j1K})
\end{aligned}$$

And:

$$\begin{aligned} & \mathcal{E} \left[ \sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \\ &= \frac{M^2 S_{r,q}}{P^4} \sum_{j=1}^L \beta_{jl} \text{diag}(\beta_{j11}, \beta_{j12}, \dots, \beta_{j1K}) \end{aligned}$$

-2) The second one is:

$$\mathcal{E}[\mathbf{a}_i^H \mathbf{a}_i \mathbf{a}_j^H \mathbf{a}_j] = \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}]$$

If we suppose that the channels are independent, we can write:

$$\begin{aligned} \mathcal{E}[\mathbf{a}_i^H \mathbf{a}_i \mathbf{a}_j^H \mathbf{a}_j] &= \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}] = \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}] \mathcal{E}[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}] \\ &= \frac{M^2}{P^2} \text{diag}(\beta_{i11}, \beta_{i12}, \dots, \beta_{i1K}) \text{diag}(\beta_{j11}, \beta_{j12}, \dots, \beta_{j1K}) \\ &= \frac{M^2}{P^2} \text{diag}(\beta_{i11} \beta_{j11}, \beta_{i12} \beta_{j12}, \dots, \beta_{i1K} \beta_{j1K}) \end{aligned}$$

-3) The third one is:

$$\mathcal{E}[\mathbf{a}_i^H \mathbf{a}_j \mathbf{a}_j^H \mathbf{a}_i] = \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}] = \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E}[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H] \mathbf{A} \mathbf{G}_{i,l}]$$

The general term of matrix  $\mathbf{A} \mathbf{G}_{j,l}$  is written as :

$$[\mathbf{A} \mathbf{G}_{j,l}(m, k)]_{M \times K} = \frac{1}{\sqrt{P}} \sum_{s=1}^P g_{j l k s} e^{-j m (\phi_s)} = a_{m,k}$$

And we have :

$$[\mathbf{G}_{j,l}^H \mathbf{A}^H(n, p)]_{K \times M} = \frac{1}{\sqrt{P}} \sum_{r=1}^P g_{j l n r}^* e^{j p (\phi_r)} = b_{n,p}$$

The general term of matrix  $\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H$  is thus :

$$\begin{aligned}
\left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] (m, p) &= \sum a_{m,k} b_{k,p} \\
&= \sum_{k=1}^K \left( \frac{1}{\sqrt{P}} \sum_{s=1}^P g_{j l k s} e^{-j f_m(\phi_s)} \right) \left( \frac{1}{\sqrt{P}} \sum_{r=1}^P g_{j l k r}^* e^{j f_p(\phi_r)} \right) \\
&= \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \sum_{s=1}^P g_{j l k r}^* g_{j l k s} e^{j f_p(\phi_r) - j f_m(\phi_s)} \\
&= \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \sum_{s=1}^P h_{j l k r}^* h_{j l k s} \sqrt{\beta_{j l k} \beta_{j l k}} e^{j f_p(\phi_r) - j f_m(\phi_s)}
\end{aligned}$$

Hence, we have :

$$\mathcal{E} \left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] (m, p) = \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \sum_{s=1}^P \mathcal{E} \left[ h_{j l k r}^* h_{j l k s} \right] \beta_{j l k} e^{j f_p(\phi_r) - j f_m(\phi_s)}$$

For  $m=p$ , we obtain:

$$\mathcal{E} \left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] (m, m) = \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \mathcal{E} \left[ |h_{j l k r}|^2 \right] \beta_{j l k} = \frac{1}{P} \sum_{k=1}^K \beta_{j l k} = \frac{1}{P} \beta_{j l}$$

However, if  $m \neq p$ , we have:

$$\begin{aligned}
\mathcal{E} \left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] (m, p) &= \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \mathcal{E} \left[ |h_{j l k r}|^2 \right] \beta_{j l k} e^{j f_p(\phi_r) - j f_m(\phi_r)} \\
&= \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \mathcal{E} \left[ |h_{j l k r}|^2 \right] \beta_{j l k} e^{j 2\pi \frac{d}{\lambda} (p-m) \sin(\phi_r)} \\
&= \frac{1}{P^2} \sum_{k=1}^K \beta_{j l k} \sum_{r=1}^P e^{j 2\pi \frac{d}{\lambda} (p-m) \sin(\phi_r)} \\
&= \frac{\beta_{j l}}{P^2} \sum_{r=1}^P e^{j 2\pi \frac{d}{\lambda} (p-m) \sin(\phi_r)}
\end{aligned}$$

And this quantity is not necessarily equal to zero. We can approximate  $\mathcal{E} \left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right]$  as :

$$\mathcal{E} \left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] = \frac{1}{P} \beta_{j l} \mathbf{I}_M$$

The  $(m,n)$ th element of  $\mathbf{G}_{i,l}^H \mathbf{A}^H$  is written as :

$$\left[ \mathbf{G}_{i,l}^H \mathbf{A}^H (m, n) \right]_{K \times M} = a_{m,n} = \frac{1}{\sqrt{P}} \sum_{k=1}^P g_{i l m k}^* e^{j f_n(\phi_k)}$$

And we obtain:

$$\begin{aligned}
\mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] (m, p) &= \sum_{r=1}^M a_{m,r} b_{r,p} \\
&= \sum_{r=1}^M \frac{1}{\sqrt{P}} \left[ \sum_{k=1}^P g_{ilmk}^* e^{j f_r(\phi_k)} \right] \frac{\beta_{jl}}{P^2} \sum_{q=1}^P e^{j 2\pi \frac{d}{\lambda} (p-r) \sin(\phi_q)} \\
&= \sum_{r=1}^M \frac{1}{\sqrt{P}} \left[ \sum_{k=1}^P g_{ilmk}^* e^{j 2\pi \frac{d}{\lambda} (r-1) \sin(\phi_k)} \right] \frac{\beta_{jl}}{P^2} \sum_{q=1}^P e^{j 2\pi \frac{d}{\lambda} (p-r) \sin(\phi_q)} \\
&= \frac{\beta_{jl}}{P^{5/2}} \sum_{r=1}^M \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j 2\pi \frac{d}{\lambda} (r-1) \sin(\phi_k)} e^{j 2\pi \frac{d}{\lambda} (p-r) \sin(\phi_q)} \\
&= \frac{\beta_{jl}}{P^{5/2}} \sum_{r=1}^M \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j 2\pi \frac{d}{\lambda} r (\sin(\phi_k) - \sin(\phi_q))} e^{j 2\pi \frac{d}{\lambda} (p \sin(\phi_q) - \sin(\phi_k))} \\
&= \frac{\beta_{jl}}{P^{5/2}} \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j 2\pi \frac{d}{\lambda} (p \sin(\phi_q) - \sin(\phi_k))} \sum_{r=1}^M e^{j 2\pi \frac{d}{\lambda} r (\sin(\phi_k) - \sin(\phi_q))} \\
&= \frac{\beta_{jl}}{P^{5/2}} \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j 2\pi \frac{d}{\lambda} (p \sin(\phi_q) - \sin(\phi_k))} e^{j 2\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q))} \frac{1 - e^{j 2\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q))}}{1 - e^{j 2\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q))}} \\
&= \frac{M \beta_{jl}}{P^{5/2}} \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j 2\pi \frac{d}{\lambda} (p-1) \sin(\phi_q)} e^{j 2\pi \frac{d}{\lambda} (M-1) (\sin(\phi_k) - \sin(\phi_q))} \frac{\sin \left[ \pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \\
&= \frac{M \beta_{jl}}{P^{5/2}} \sum_{k=1}^P g_{ilmk}^* e^{j \pi \frac{d}{\lambda} (M-1) \sin(\phi_k)} \sum_{q=1}^P e^{j \pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[ \pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \\
&= \frac{M \beta_{jl}}{P^{5/2}} \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j \pi \frac{d}{\lambda} [(M-1) \sin(\phi_k) + (2p-M-1) \sin(\phi_q)]} \frac{\sin \left[ \pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]}
\end{aligned}$$

The general term of matrix  $\mathbf{A} \mathbf{G}_{i,l}$  is written as :

$$\left[ \mathbf{A} \mathbf{G}_{i,l} (p, n) \right]_{M \times K} = \frac{1}{\sqrt{P}} \sum_{s=1}^P g_{ilns} e^{-j f_p(\phi_s)} = a_{p,n}$$

We have :



$$\begin{aligned}
& \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] \mathbf{A} \mathbf{G}_{i,l} \right] (m, n) \\
&= \sum_{p=1}^M \frac{M \beta_{jl}}{P^{5/2}} \sum_{k=1}^P \sum_{s=1}^P \mathcal{E} \left[ \mathbf{g}_{ilmk} \mathbf{g}_{ilns} \right] e^{j\pi \frac{d}{\lambda} (M-1) \sin(\phi_k)} e^{-j2\pi \frac{d}{\lambda} (p-1) \sin(\phi_s)} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[ \pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \frac{1}{\sqrt{P}} \sum_{s=1}^P \mathbf{g}_{ilns} e^{-j\pi \frac{d}{\lambda} \sin(\phi_s)} \\
&= \frac{M \beta_{jl}}{P^3} \sum_{p=1}^M \sum_{k=1}^P \sum_{s=1}^P \mathcal{E} \left[ \mathbf{g}_{ilmk} \mathbf{g}_{ilns} \right] e^{j\pi \frac{d}{\lambda} (M-1) \sin(\phi_k)} e^{-j2\pi \frac{d}{\lambda} (p-1) \sin(\phi_s)} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[ \pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]}
\end{aligned}$$

We obtain eventually :

$$\begin{aligned}
& \mathcal{E} \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] \mathbf{A} \mathbf{G}_{i,l} \right] (m, n) \\
&= \frac{M \beta_{jl}}{P^3} \sum_{p=1}^M \sum_{k=1}^P \sum_{s=1}^P \mathcal{E} \left[ \mathbf{g}_{ilmk} \mathbf{g}_{ilns} \right] e^{j\pi \frac{d}{\lambda} (M-1) \sin(\phi_k)} e^{-j2\pi \frac{d}{\lambda} (p-1) \sin(\phi_s)} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[ \pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]}
\end{aligned}$$

The quantity is not equal to zero when  $m = n$  and  $k = s$ , this entails:

$$\begin{aligned}
& \mathcal{E} \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[ \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] \mathbf{A} \mathbf{G}_{i,l} \right] (n, n) \\
&= \frac{M \beta_{jl}}{P^3} \sum_{p=1}^M \sum_{k=1}^P \frac{\beta_{iln}}{P} e^{j\pi \frac{d}{\lambda} (M-1) \sin(\phi_k)} e^{-j2\pi \frac{d}{\lambda} (p-1) \sin(\phi_k)} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[ \pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \\
&= \frac{M \beta_{jl} \beta_{iln}}{P^4} \sum_{p=1}^M \sum_{k=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_k) [M-2p+1]} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[ \pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \\
&= \frac{M \beta_{jl} \beta_{iln}}{P^4} \sum_{p=1}^M \sum_{k=1}^P \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} [M-2p+1] (\sin(\phi_k) - \sin(\phi_q))} \frac{\sin \left[ \pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]}
\end{aligned}$$

Using the notation:  $T_{p,k,q} = \sum_{p=1}^M \sum_{k=1}^P \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} [M-2p+1] (\sin(\phi_k) - \sin(\phi_q))} \frac{\sin \left[ \pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[ \pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]}$ ,

we obtain :

$$\begin{aligned}
& \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E}[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H] \mathbf{A} \mathbf{G}_{i,l}] \\
&= \frac{M}{P^4} \beta_{jl} T_{p,k,q} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK})
\end{aligned}$$

In fact, for the sake of simplicity we will use a lower bound for the different expectation forms due to the difficulty of obtaining asymptotic expressions. We will have in this case:

$$\begin{aligned}
& \mathcal{E} \left[ \left( \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right)^2 (n, n) \right] \\
&> \frac{M}{P^2} \beta_{jl} \beta_{jl}
\end{aligned}$$

And:

$$\begin{aligned}
& \mathcal{E} \left[ \sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \\
&> \frac{M}{P^2} \sum_{j=1}^L \beta_{jl} \text{diag}(\beta_{j1}, \beta_{j2}, \dots, \beta_{jK})
\end{aligned}$$

And, in the same way :

$$\begin{aligned}
& \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E}[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H] \mathbf{A} \mathbf{G}_{i,l}] \\
&> \frac{M}{P^2} \beta_{jl} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK})
\end{aligned}$$

Using these lower bounds we obtain in the general case:

$$\begin{aligned}
& \mathcal{E} \left[ \left( \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left( \sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) \right] \\
&= \sum_{i=1}^L \mathcal{E}(\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}) + \sum_{i=1}^L \sum_{j=1, j \neq i}^L \mathcal{E}(\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}) + \dots \\
&\dots + \sum_{i=1}^L \sum_{j=1, j \neq i}^L \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}] \\
&> \sum_{i=1}^L \frac{M \beta_{il}}{P^2} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) + \sum_{i=1}^L \sum_{j=1, j \neq i}^L \frac{M^2}{P^2} \text{diag}(\beta_{i1} \beta_{j1}, \beta_{i2} \beta_{j2}, \dots, \beta_{iK} \beta_{jK}) + \\
&\dots + \sum_{i=1}^L \sum_{j=1, j \neq i}^L \frac{M}{P^2} \beta_{jl} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \\
&> \frac{M}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[ \beta_{il} \mathbf{I}_K + M \sum_{j=1, j \neq i}^L \text{diag}(\beta_{j1}, \beta_{j2}, \dots, \beta_{jK}) + \sum_{j=1, j \neq i}^L \beta_{jl} \mathbf{I}_K \right]
\end{aligned}$$

$$\begin{aligned}
&> \frac{M}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[ \sum_{j=1}^L \beta_{ji} \mathbf{I}_K + M \sum_{j=1, j \neq i}^L \text{diag}(\beta_{j1}, \beta_{j2}, \dots, \beta_{jK}) \right] \\
&> \frac{M}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[ \hat{\beta}_i \mathbf{I}_K + M \sum_{j=1, j \neq i}^L \text{diag}(\beta_{j1}, \beta_{j2}, \dots, \beta_{jK}) \right] \\
&> \frac{M^2}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[ \frac{\hat{\beta}_i}{M} \mathbf{I}_K + \sum_{j=1, j \neq i}^L \text{diag}(\beta_{j1}, \beta_{j2}, \dots, \beta_{jK}) \right] \\
&> \frac{M^2}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[ \text{diag} \left( \sum_{j=1, j \neq i}^L \beta_{ji} + \frac{\hat{\beta}_i}{M}, \sum_{j=1, j \neq i}^L \beta_{j2} + \frac{\hat{\beta}_i}{M}, \dots, \sum_{j=1, j \neq i}^L \beta_{jK} + \frac{\hat{\beta}_i}{M} \right) \right]
\end{aligned}$$

We obtain a lower bound which is quite accurate.

The following term is:  $\mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right]$ , we have:

$$\begin{aligned}
&\mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&\approx \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathcal{E} \left[ \mathbf{W}_l \mathbf{W}_l^H \right] \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= K \sigma_{n_l}^2 \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right]
\end{aligned}$$

$$\Pi_A \equiv \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \approx_{M \rightarrow \infty} \mathbf{A} \left( \frac{M}{P} \mathbf{I}_P \right)^{-1} \mathbf{A}^H = \frac{P}{M} \mathbf{A} \mathbf{A}^H$$

$$\begin{aligned}
&\mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= \frac{1}{M} \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{W}_l \right] \text{Trace} \left( \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right) \\
&= \frac{1}{M} M \sigma_{n_l}^2 \mathbf{I}_K \text{Trace} \left[ \Pi_A \right] \approx_{M \rightarrow \infty} \sigma_{n_l}^2 \frac{P}{M} \text{Trace}(\mathbf{A} \mathbf{A}^H) \mathbf{I}_K
\end{aligned}$$

We eventually obtain:

$$\begin{aligned}
&\mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= \frac{KP}{M} \sigma_{n_l}^4 \text{Trace}(\mathbf{A} \mathbf{A}^H) \mathbf{I}_K
\end{aligned}$$

Then, we find :

$$\begin{aligned}
& \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= \sum_{i=1}^L \mathcal{E} \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= \sum_{i=1}^L \frac{M}{P} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \sigma_{n_i}^2 \frac{P}{M} \text{Trace}(\mathbf{A} \mathbf{A}^H) \mathbf{I}_K \\
&= \sigma_{n_i}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) \text{diag} \left( \sum_{i=1}^L \beta_{i1}, \sum_{i=1}^L \beta_{i2}, \dots, \sum_{i=1}^L \beta_{iK} \right) \\
&= \sigma_{n_i}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K})
\end{aligned}$$

$$\begin{aligned}
& \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] \\
&= \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \sum_{i=1}^L \mathcal{E} \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \\
&= \sigma_{n_i}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K})
\end{aligned}$$

The remaining terms are :  $\mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right]$  and  $\mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \right]$ , we have :

$$\mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] = \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[ \mathbf{W}_l \mathbf{W}_l^H \right] \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right]$$

And using :  $\mathcal{E} \left[ \mathbf{W}_l \mathbf{W}_l^H \right] = K \sigma_{n_i}^2 \mathbf{I}_M$ , this yields :

$$\begin{aligned}
& \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] = K \sigma_{n_i}^2 \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] = K \sigma_{n_i}^2 \sum_{i=1}^L \mathcal{E} \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \\
&= K \sigma_{n_i}^2 \sum_{i=1}^L \frac{M}{P} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) = \frac{K \sigma_{n_i}^2 M}{P} \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K})
\end{aligned}$$

And, for the last term :

$$\begin{aligned}
& \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathcal{E} \left[ \mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \right] \mathbf{A}^H \mathbf{W}_l \right] = \frac{1}{P} \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \mathbf{A}^H \mathbf{W}_l \right] \text{Trace} \left[ \sum_{i=1}^L \mathcal{E} \left[ \mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \right] \right] \\
& = \frac{1}{P} \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \mathbf{A}^H \mathbf{W}_l \right] \sum_{i=1}^L \text{Trace} \left[ \mathcal{E} \left[ \mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \right] \right] \\
& = \frac{\text{Trace}(\mathbf{A} \mathbf{A}^H)}{MP} \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{W}_l \right] \sum_{i=1}^L \beta_{il} \\
& = M \sigma_{n_l}^2 \mathbf{I}_K \frac{\text{Trace}(\mathbf{A} \mathbf{A}^H)}{MP} \sum_{i=1}^L \beta_{il} \\
& = \sigma_{n_l}^2 \frac{\text{Trace}(\mathbf{A} \mathbf{A}^H)}{P} \sum_{i=1}^L \beta_{il} \mathbf{I}_K \\
& = \frac{\sigma_{n_l}^2 \hat{\beta}_l \text{Trace}(\mathbf{A} \mathbf{A}^H)}{P} \mathbf{I}_K
\end{aligned}$$

We eventually obtain :

$$\begin{aligned}
& p_p^2 \mathcal{E} \left[ \left( \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left( \sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) \right] + \dots \\
& \dots + \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + 2p_p \sum_{i=1}^L \mathcal{E} \left[ \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + \dots \\
& \dots + p_p \mathcal{E} \left[ \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[ \mathbf{W}_l \mathbf{W}_l^H \right] \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] + p_p \mathcal{E} \left[ \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathcal{E} \left[ \mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \right] \mathbf{A}^H \mathbf{W}_l \right] \\
& \mathcal{E} \left( \left[ \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^2 \right) \\
& = p_p^2 \frac{M^2}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[ \text{diag} \left( \sum_{j=1, j \neq i}^L \beta_{j1} + \frac{\hat{\beta}_1}{M}, \sum_{j=1, j \neq i}^L \beta_{j2} + \frac{\hat{\beta}_2}{M}, \dots, \sum_{j=1, j \neq i}^L \beta_{jK} + \frac{\hat{\beta}_K}{M} \right) \right] + \dots \\
& \dots + \frac{KP}{M} \sigma_{n_l}^4 \text{Trace}(\mathbf{A} \mathbf{A}^H) \mathbf{I}_K + 2p_p \sigma_{n_l}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K}) + p_p \frac{K \sigma_{n_l}^2 M}{P} \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K}) + \dots \\
& \dots + p_p \frac{\sigma_{n_l}^2 \hat{\beta}_l \text{Trace}(\mathbf{A} \mathbf{A}^H)}{P} \mathbf{I}_K \\
& = p_p^2 \frac{M^2}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[ \text{diag} \left( \bar{\beta}_{11} - \beta_{i1} + \frac{\hat{\beta}_1}{M}, \bar{\beta}_{12} - \beta_{i2} + \frac{\hat{\beta}_2}{M}, \dots, \bar{\beta}_{1K} - \beta_{iK} + \frac{\hat{\beta}_K}{M} \right) \right] + \dots \\
& \dots + \left( 2p_p \sigma_{n_l}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) + p_p \frac{K \sigma_{n_l}^2 M}{P} \right) \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K}) + \text{Trace}(\mathbf{A} \mathbf{A}^H) \left( \frac{KP}{M} \sigma_{n_l}^4 + p_p \frac{\sigma_{n_l}^2 \hat{\beta}_l}{P} \right) \mathbf{I}_K
\end{aligned}$$

We obtain in these conditions:

$$\mathcal{E} \left[ \mathbf{M}_{p,l}^A(n, :) \mathbf{M}_{p,l}^A(n, :)^H \right] \approx \gamma_{n,l}$$

Considering the case where  $M$  is sufficiently high, we have  $\mathbf{A}\mathbf{A}^H \approx \mathbf{I}_M$  and this yields:

$$\gamma_{n,l} = \frac{1}{p_p^2 \frac{M^2}{P^2} \left( \sum_{i=1}^L \beta_{iln} \left[ \bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] \right) + p_p M \sigma_{n_l}^2 \left( 2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \sigma_{n_l}^2 M \left( p_p \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \sigma_{n_l}^2 \right)}$$

$$\gamma_{n,l} = \frac{P^2}{p_p^2 M^2 \left[ \left( \sum_{i=1}^L \beta_{iln} \left[ \bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_l}^2}{Mp_p} \left( 2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \sigma_{n_l}^2 \frac{P^2}{p_p M} \left( p_p \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \sigma_{n_l}^2 \right) \right]}$$

This can be simplified as:

$$\gamma_{n,l} = \frac{P^2}{p_p^2 M^2 \left[ \left( \sum_{i=1}^L \beta_{iln} \left[ \bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_l}^2}{Mp_p} \left[ \left( 2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \frac{\sigma_{n_l}^2}{p_p} \right] \right]}$$

$$\gamma_{n,l} \approx_{M \rightarrow \infty} \frac{P^2}{p_p^2 M^2 \left[ \left( \sum_{i=1}^L \beta_{iln} \left[ \bar{\beta}_{ln} - \beta_{iln} \right] \right) \right]}$$

And, we eventually obtain:

$$\mathcal{E} \left\{ \left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right\} = \mathcal{E} \left( \boldsymbol{\alpha}_{jn}^T \boldsymbol{\alpha}_{jn}^* \right)$$

$$= \frac{\gamma_{n,l} p_u p_p \text{Trace} \left[ \left( \mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left( \sum_{i=1, i \neq j}^L \beta_{il} \right)}{P^2 K} + \frac{\gamma_{n,l} p_u p_p M^2 \beta_{jl}^2}{KP^2} + \frac{\sigma_{n_l}^2 p_u M^2 \beta_{jl} \gamma_{n,l}}{KP}$$

$$= \frac{\gamma_{n,l} \left[ p_u p_p \text{Trace} \left[ \left( \mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left( \sum_{i=1, i \neq j}^L \beta_{il} \right) + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_l}^2 p_u M^2 P \beta_{jl} \right]}{P^2 K} \quad (60)$$

$$= \frac{\left[ p_u p_p \text{Trace} \left[ \left( \mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left( \sum_{i=1, i \neq j}^L \beta_{il} \right) + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_l}^2 p_u M^2 P \beta_{jl} \right]}{p_p^2 M^2 K \left[ \left( \sum_{i=1}^L \beta_{iln} \left[ \bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_l}^2}{Mp_p} \left[ \left( 2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \frac{\sigma_{n_l}^2}{p_p} \right] \right]}$$

Considering the case where  $M$  goes towards infinity, we have:

$$\lim_{M \rightarrow \infty} \frac{1}{M} \mathbf{A}^H \mathbf{A} = \frac{1}{P} \mathbf{I}_P$$

This yields:  $\lim_{M \rightarrow \infty} \mathbf{A}^H \mathbf{A} = \frac{M}{P} \mathbf{I}_P$  and  $\lim_{M \rightarrow \infty} \text{Trace} \left[ \left( \mathbf{A}^H \mathbf{A} \right)^2 \right] = \frac{M^2}{P^2} P = \frac{M^2}{P}$  and we obtain:

$$\begin{aligned}
& \mathcal{E} \left\{ \left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right\} \\
&= \frac{\left[ p_u p_p \beta_{jl} \left( \sum_{i=1, i \neq j}^L \beta_{il} \right) M^2 / P + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_i}^2 p_u M^2 P \beta_{jl} \right]}{p_p^2 M^2 K \left[ \left( \sum_{i=1}^L \beta_{in} \left[ \bar{\beta}_{ln} - \beta_{in} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_i}^2}{M p_p} \left[ \left( 2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \frac{\hat{\beta}_l}{P} + \frac{K P}{M} \frac{\sigma_{n_i}^2}{p_p} \right] \right]} \\
&= \frac{\left[ p_u p_p \beta_{jl} \left( \sum_{i=1, i \neq j}^L \beta_{il} \right) / P + p_u p_p \beta_{jl}^2 + \sigma_{n_i}^2 p_u P \beta_{jl} \right]}{p_p^2 K \left[ \left( \sum_{i=1}^L \beta_{in} \left[ \bar{\beta}_{ln} - \beta_{in} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_i}^2}{M p_p} \left[ \left( 2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \frac{\hat{\beta}_l}{P} + \frac{K P}{M} \frac{\sigma_{n_i}^2}{p_p} \right] \right]} \\
&\approx_{M \rightarrow \infty} \frac{p_u p_p \beta_{jl} \left( \sum_{i=1, i \neq j}^L \beta_{il} \right) / P + p_u p_p \beta_{jl}^2 + \sigma_{n_i}^2 p_u P \beta_{jl}}{p_p^2 K \sum_{i=1}^L \beta_{in} \left[ \bar{\beta}_{ln} - \beta_{in} \right]} \\
&= \frac{p_u}{p_p} \beta_{jl} \frac{\left( \hat{\beta}_l - \beta_{jl} \right) / P + \beta_{jl} + \frac{\sigma_{n_i}^2}{P} P}{K \sum_{i=1}^L \beta_{in} \left[ \bar{\beta}_{ln} - \beta_{in} \right]} \tag{61}
\end{aligned}$$

Concerning the last term:  $\mathcal{E} \left[ |z_{ln}|^2 \right] = \mathcal{E} \left[ \left| \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{n}_l \right|^2 \right]$ , we have :

$$\begin{aligned}
& \mathcal{E} \left\{ |z_{ln}|^2 \right\} = \mathcal{E} \left\{ \left| \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{n}_l \right|^2 \right\} \\
&= \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{n}_l \mathbf{n}_l^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathcal{E} \left[ \mathbf{n}_l \mathbf{n}_l^H \right] \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \sigma_{n_i}^2 \mathbf{I}_M \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \sigma_{n_i}^2 \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \sigma_{n_i}^2 \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \mathcal{E} \left[ \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \sigma_{n_i}^2 \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \text{diag} \left( \rho_{1,l}, \rho_{2,l}, \dots, \rho_{K,l} \right) \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \frac{\sigma_{n_i}^2}{K} \sum_{i=1}^K \rho_{i,l} \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:)^H \right\} \tag{62} \\
&\approx \frac{\sigma_{n_i}^2 \gamma_{n,l} \sum_{i=1}^K \rho_{i,l}}{K}
\end{aligned}$$

We have: 
$$\frac{\sum_{i=1}^K \rho_{i,l}}{K} = \frac{1}{K} \sum_{n=1}^K \left[ p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_l}^2 P \right] = \frac{p_p M}{KP} \sum_{n=1}^K \bar{\beta}_{ln} + \sigma_{n_l}^2 P$$

$$\mathcal{E} \left\{ |z_{ln}|^2 \right\} = \frac{\left( p_p \frac{M}{KP} \sum_{n=1}^K \bar{\beta}_{ln} + \sigma_{n_l}^2 P \right) \sigma_{n_l}^2 P^2}{p_p^2 M^2 \left[ \left( \sum_{i=1}^L \beta_{iln} \left[ \bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_l}^2}{M p_p} \left[ \left( 2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \frac{\sigma_{n_l}^2}{p_p} \right] \right]}$$

The final expression for the uplink capacity in the case of a ZF receiver is then equal to:

$$R_{ln} = C \left( \frac{\|\mathcal{E}(\boldsymbol{\alpha}_{ln}^T)\|^2}{\sum_{j=1}^L \mathcal{E}\{\|\boldsymbol{\alpha}_{jn}^T\|^2\} - \|\mathcal{E}(\boldsymbol{\alpha}_{ln}^T)\|^2 + \mathcal{E}(|z_{ln}|^2)} \right) \left( \frac{p_p p_u \beta_{ln}^2}{\left( p_p \bar{\beta}_{ln} + P^2 \sigma_{n_l}^2 / M \right)^2} \right) \left( \frac{p_u \sum_{j=1}^L \beta_{jl} \frac{(\hat{\beta}_l - \beta_{jl}) / P + \beta_{jl} + \frac{\sigma_{n_l}^2}{P} P}{K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} - \frac{p_p p_u \beta_{ln}^2}{\left( p_p \bar{\beta}_{ln} + P^2 \sigma_{n_l}^2 / M \right)^2} + \frac{\sigma_{n_l}^2 \gamma_{n,l} \sum_{i=1}^K \rho_{i,l}}{K}}{\right)} \right) \quad (63)$$

When  $M$  goes towards infinity, we obtain:

$$R_{ln} = C \left( \frac{\|\mathcal{E}(\boldsymbol{\alpha}_{ln}^T)\|^2}{\sum_{j=1}^L \mathcal{E}\{\|\boldsymbol{\alpha}_{jn}^T\|^2\} - \|\mathcal{E}(\boldsymbol{\alpha}_{ln}^T)\|^2 + \mathcal{E}(|z_{ln}|^2)} \right) \left( \frac{p_u \beta_{ln}^2}{p_p (\bar{\beta}_{ln})^2} \right) \left( \frac{p_u \sum_{j=1}^L \beta_{jl} \frac{(\hat{\beta}_l - \beta_{jl}) / P + \beta_{jl} + \frac{\sigma_{n_l}^2}{P} P}{K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} - \frac{p_u \beta_{ln}^2}{p_p (\bar{\beta}_{ln})^2}}{\right)} \right) \quad (64)$$



At high SNR we can neglect the term  $\frac{\sigma_{n_l}^2}{P_p} P \beta_{jl}$  and, doing this, when we consider also the case where  $P$  goes towards infinity, we obtain:

$$\begin{aligned}
 R_{ln} \approx_{M \rightarrow \infty} C & \left( \frac{\frac{P_u}{P_p} \frac{\beta_{lln}^2}{(\bar{\beta}_{ln})^2}}{\frac{P_u}{P_p} \frac{\sum_{j=1}^L \beta_{jl}^2}{K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} - \frac{P_u}{P_p} \frac{\beta_{lln}^2}{(\bar{\beta}_{ln})^2}} \right) \\
 R_{ln} \approx_{M \rightarrow \infty} C & \left( \frac{\frac{\beta_{lln}^2}{(\bar{\beta}_{ln})^2}}{\frac{\sum_{j=1}^L \beta_{jl}^2}{K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} - \frac{\beta_{lln}^2}{(\bar{\beta}_{ln})^2}} \right) = \left( \frac{\beta_{lln}^2 \left[ K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}] \right]}{(\bar{\beta}_{ln})^2 \sum_{j=1}^L \beta_{jl}^2 - \beta_{lln}^2 K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} \right)
 \end{aligned} \tag{65}$$