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Jean-Pierre Cances, B Sissokho, Vahid Meghdadi, Ahmed D. Kora

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I. Note Technique: CSI estimation in massive MIMO systems and pilot contamination MRC receiver architecture

JP Cances, B Sissokho, V Meghdadi, A Kora
Xlim Limoges UMR 7252

1- Context and System:

We consider L cells, where each cell contains one base station equipped with M antennas and K single-antenna users. Assume that the L base station share the same frequency band. We consider uplink transmission, where the lth base station receives signals from all users in all cells (Fig.1). Then, the $M \times 1$ received vector at the lth base station is given by:

$$\boldsymbol{y}_l = \sqrt{p_u} \sum_{i=1}^{L} \boldsymbol{\Gamma}_{i,l} \boldsymbol{x}_i + \boldsymbol{n}_l \qquad (1)$$

Where $\boldsymbol{\Gamma}_{i,l}$ represents the $M \times K$ channel matrix between the lth base station and the K users in the ith cell, i.e., $\left[\boldsymbol{\Gamma}_{i,l}\right]_{m,k}$ is the channel coefficient between the mth antenna of the lth base station and the kth user in the ith cell. $\sqrt{p_u}\,\boldsymbol{x}_i$ is the $K \times 1$ transmitted vector of K users in the ith cell (the average power used by each user is p_u) and $\boldsymbol{n}_l$ contains $M \times 1$ additive white Gaussian noise (AWGN) samples. We assume that the elements of $\boldsymbol{n}_l$ are Gaussian distributed with zero mean and variance: $\mathcal{E}\left[\boldsymbol{n}_l \boldsymbol{n}_l^H\right] = \sigma_{\boldsymbol{n}_l}^2 \boldsymbol{I}_M$ and $\mathcal{E}\left[\boldsymbol{n}_l^H \boldsymbol{n}_l\right] = M \sigma_{\boldsymbol{n}_l}^2$.

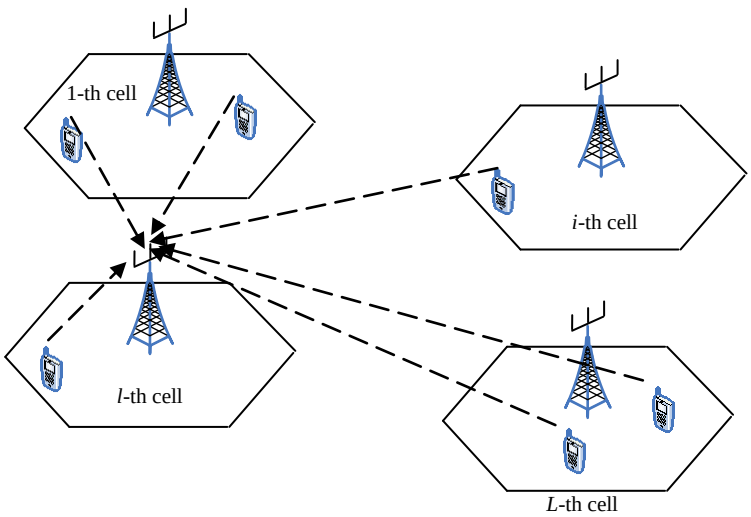

Fig. 1: Uplink transmission in multicell MU-MIMO system

2- Physical Channel Model:

Here, we introduce the finite-dimensional channel model that will be used throughout this technical note. The angular domain is divided into a large but finite number of directions P.

$P$is fixed regardless of the number of base station antennas ($P<M$). Each direction, corresponding to the angle: $\phi_k, \phi_k \in [-\pi/2, \pi/2], k = 1,..., P$,is associated with an $M \times 1$ array steering vector $\boldsymbol{a}(\phi_k)$ which is given by:

$$\boldsymbol{a}(\phi_k) = \frac{1}{\sqrt{P}}\left[e^{-jf_1(\phi_k)}, e^{-jf_2(\phi_k)}, \ldots, e^{-jf_M(\phi_k)}\right]^T \qquad (2)$$

Where $f_i(\phi)$ is some function of ϕ. Typically, for a regular phased array, we have:

$$f_i(\phi) = 2\pi \frac{d}{\lambda}(i-1)\sin(\phi)$$

The channel vector from kth user in the ith cell to the lth base station is then a linear combination of the steering vectors as follows: $\sum_{m=1}^{P} g_{ilkm}\, \boldsymbol{a}(\phi_m)$, where g_{ilkm} is the propagation coefficient from the kth user of the ith cell to the lth base station, associated with the physical direction m (direction of arrival ϕ_m). Let $\boldsymbol{G}_{i,l} \equiv [\boldsymbol{g}_{il1}, ..., \boldsymbol{g}_{ilK}]$ be a $P \times K$ matrix with $\boldsymbol{g}_{ilk} \equiv [g_{ilk1}, ..., g_{ilkP}]^T$ that contains the path gains from the kth user in the ith cell to the lth base station. The elements of $\boldsymbol{G}_{i,l}$ are assumed to be independent. Then, the$M \times K$ channel matrix between the lth base station and the K users in the ith cell is:

$$\boldsymbol{\Gamma}_{i,l} = \boldsymbol{A}\boldsymbol{G}_{i,l} \qquad (3)$$

Where $\boldsymbol{A} \equiv [\boldsymbol{a}(\phi_1), ..., \boldsymbol{a}(\phi_P)]$ is a full rank $M \times P$matrix. The propagation channel $\boldsymbol{G}_{i,l}$ models independent fast fading, geometric attenuation, and log-normal shadow fading. Its elements g_{ilkm} are given by:

$$g_{ilkm} = h_{ilkm}\sqrt{\beta_{ilk}} \qquad m = 1,2,\ldots,P \qquad (4)$$

Where h_{ilkm} is a fast fading coefficient assumed to be zero mean and have unit variance. $\sqrt{\beta_{ilk}}$ models the path loss and shadowing which are assumed to be independent of the direction m and to be constant and known a priori. This assumption is reasonable since the value β_{ilk} changes very slowly with time. Then, we have:

$$\boldsymbol{G}_{i,l} = \boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} \qquad (5)$$

Where $\boldsymbol{H}_{i,l}$ is the $P \times K$ matrix of fast fading coefficients between the K users in the ith cell and the lth base station, the columns $\boldsymbol{H}_{i,l}(:,k)$ are independent and we have:

$$\mathcal{E}\left[\boldsymbol{H}_{i,l}^{H}(:,k)\boldsymbol{H}_{i,l}(:,k)\right]=\mathcal{E}\left[\boldsymbol{h}_{ilk}^{H}\boldsymbol{h}_{ilk}\right]=\mathcal{E}\left[\sum_{m=1}^{P}\left|h_{il}(m,k)\right|^{2}\right]=\mathcal{E}\left[\sum_{m=1}^{P}\left|h_{ilkm}\right|^{2}\right]=1$$

Furthermore within one given column we consider the elements: $h_{il}(m,k)$ independent, i.e.: $\mathcal{E}\left[h_{ilkm}\, h_{ilkn}^{*}\right]=0$ if $m \neq n$. $\boldsymbol{D}_{i,l}$ is a $K \times K$ diagonal matrix whose diagonal elements are given by $\left[\boldsymbol{D}_{i,l}\right]_{k,k}=\beta_{ilk}$. Therefore, (1) can be written as:

$$\boldsymbol{y}_l=\sqrt{p_u}\,\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{G}_{i,l}\boldsymbol{x}_i+\boldsymbol{n}_l=\sqrt{p_u}\,\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\boldsymbol{x}_i+\boldsymbol{n}_l \qquad (6)$$

In the following sections we will use the following intermediate parameters: $\beta_{jl}=\sum_{k=1}^{K}\beta_{jlk}$, $\overline{\beta}_{ln}=\sum_{i=1}^{L}\beta_{iln}$, and $\hat{\beta}_l=\sum_{i=1}^{L}\beta_{il}$.

3- Channel Estimation:

Channel estimation is performed by using training sequences received on the uplink. A part of the coherence interval is used for the uplink training. All users in all cells simultaneously transmit pilot sequences of length τ symbols. The assumption on synchronized transmission represents the worst case from the pilot contamination point of view, but it makes no fundamental difference to assume unsynchronized transmission. We assume that the same set of pilot sequences is used in all L cells. Therefore, the pilot sequences used in the lth cell can be represented by a $K \times \tau$ matrix $\sqrt{p_p}\,\boldsymbol{\phi}_l=\sqrt{p_p}\,\boldsymbol{\phi}$ ($\tau \geq K$), which satisfies $\boldsymbol{\phi}\boldsymbol{\phi}^{H}=\boldsymbol{I}_K$ where: $p_p=\tau\, p_u$. From (6), the received pilot matrix at the lth base can be written as:

$$\boldsymbol{Y}_{p,l}=\sqrt{p_p}\,\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\boldsymbol{\phi}+\boldsymbol{N}_l \qquad (7)$$

$\boldsymbol{N}_l$ is the resulting noise matrix of size $M \times \tau$ and of course we have: $\mathcal{E}\left[\boldsymbol{N}_l\right]=\boldsymbol{0}_{M\times\tau}$ $\mathcal{E}\left[\boldsymbol{N}_l\boldsymbol{N}_l^{H}\right]=\tau\,\sigma_{\boldsymbol{n}_l}^{2}\boldsymbol{I}_M$ and $\mathcal{E}\left[\boldsymbol{N}_l^{H}\boldsymbol{N}_l\right]=M\,\sigma_{\boldsymbol{n}_l}^{2}\boldsymbol{I}_\tau$.

3.1. Minimum Mean-Square Error (MMSE) Estimation:

We assume that the base station uses here MMSE estimation. The received pilot matrix $\boldsymbol{Y}_{p,l}$ is multiplied by $\boldsymbol{\phi}^{H}$ at the right side to obtain: $\tilde{\boldsymbol{Y}}_{p,l}=\boldsymbol{Y}_{p,l}\,\boldsymbol{\phi}^{H}$:

$$\begin{aligned}\tilde{\boldsymbol{Y}}_{p,l} &= \boldsymbol{Y}_{p,l}\,\boldsymbol{\phi}^H \\ &= \sqrt{p_p}\,\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{N}_l\,\boldsymbol{\phi}^H \qquad (8) \\ &= \sqrt{p_p}\,\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{W}_l\end{aligned}$$

With: $\boldsymbol{W}_l = \boldsymbol{N}_l\,\boldsymbol{\phi}^H$. Since we consider here *M*-PSK modulation format for the transmit pilot matrixwe can denote $\boldsymbol{\phi}$ as:

$$\boldsymbol{\phi} = \frac{1}{\sqrt{\tau}}\begin{bmatrix} e^{j\theta_{11}} & e^{j\theta_{12}} & \cdots & e^{j\theta_{1p}} & \cdots & e^{j\theta_{1\tau}} \\ e^{j\theta_{21}} & e^{j\theta_{22}} & \cdots & e^{j\theta_{2p}} & \cdots & e^{j\theta_{2\tau}} \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ e^{j\theta_{m1}} & e^{j\theta_{m2}} & \cdots & e^{j\theta_{mp}} & \cdots & e^{j\theta_{m\tau}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{j\theta_{K1}} & e^{j\theta_{K2}} & \cdots & e^{j\theta_{Kp}} & \cdots & e^{j\theta_{K\tau}} \end{bmatrix}$$

Hence, we have: $\boldsymbol{\phi}(k,m) = a_{k,m} = \frac{1}{\sqrt{\tau}}e^{j\theta_{k,m}}$ and $\boldsymbol{\phi}^H(n,k) = b_{n,k} = \frac{1}{\sqrt{\tau}}e^{-j\theta_{k,n}}$. The matrix product $\boldsymbol{N}_l\,\boldsymbol{\phi}^H$ yields:

$$\boldsymbol{N}_l\,\boldsymbol{\phi}^H(n,m) = \sum_{k=1}^{\tau} n_l(n,k)\,\boldsymbol{\phi}^H(k,m) = \frac{1}{\sqrt{\tau}}\sum_{k=1}^{\tau} n_l(n,k)\,e^{-j\theta_{m,k}}$$

In the same way we obtain:

$$\boldsymbol{\phi}\,\boldsymbol{N}_l^H(n,m) = \sum_{k=1}^{\tau}\phi(n,k)\,n_l^*(m,k) = \frac{1}{\sqrt{\tau}}\sum_{k=1}^{\tau} n_l^*(m,k)\,e^{j\theta_{n,k}}$$

The matrix product: $\boldsymbol{W}_l\,\boldsymbol{W}_l^H = \boldsymbol{N}_l\,\boldsymbol{\phi}^H\boldsymbol{N}_l\,\boldsymbol{\phi}^H\boldsymbol{\phi}\,\boldsymbol{N}_l^H$ is thus equal to:

$$\begin{aligned}\boldsymbol{N}_l\,\boldsymbol{\phi}^H\boldsymbol{\phi}\,\boldsymbol{N}_l^H(n,m) &= \sum_{k=1}^{K}\boldsymbol{N}_l\,\boldsymbol{\phi}^H(n,k)\,\boldsymbol{\phi}\,\boldsymbol{N}_l^H(k,m) \\ &= \sum_{k=1}^{K}\frac{1}{\sqrt{\tau}}\sum_{s=1}^{\tau} n_l(n,s)\,e^{-j\theta_{k,s}}\,\frac{1}{\sqrt{\tau}}\sum_{r=1}^{\tau} n_l^*(m,r)\,e^{j\theta_{k,r}} \\ &= \frac{1}{\tau}\sum_{k=1}^{K}\sum_{s=1}^{\tau}\sum_{r=1}^{\tau} n_l(n,s)\,n_l^*(m,r)\,e^{j(\theta_{k,r}-\theta_{k,s})}\end{aligned}$$

Taking the expectation, we obtain:

$$\mathcal{E}\left[\boldsymbol{N}_l \boldsymbol{\phi}^H \boldsymbol{\phi} \boldsymbol{N}_l^H(n,m)\right] = \frac{1}{\tau}\sum_{k=1}^{K}\sum_{s=1}^{\tau}\sum_{r=1}^{\tau}\mathcal{E}\left[n_l(n,s)n_l^*(m,r)\right]e^{j(\theta_{k,r}-\theta_{k,s})}$$

Clearly, $\mathcal{E}\left[n_l(n,s)n_l^*(m,r)\right] = \sigma_{\boldsymbol{n}_l}^2$ if and only if : $m = n$ and $r = s$. In these conditions we obtain:

$$\mathcal{E}\left[\boldsymbol{N}_l \boldsymbol{\phi}^H \boldsymbol{\phi} \boldsymbol{N}_l^H(n,n)\right] = \frac{1}{\tau}\sum_{k=1}^{K}\tau\,\sigma_{\boldsymbol{n}_l}^2 = K\sigma_{\boldsymbol{n}_l}^2$$

We eventually obtain : $\mathcal{E}\left[\boldsymbol{W}_l \boldsymbol{W}_l^H\right] = \mathcal{E}\left[\boldsymbol{N}_l \boldsymbol{\phi}^H \boldsymbol{\phi} \boldsymbol{N}_l^H\right] = K\sigma_{\boldsymbol{n}_l}^2\,\boldsymbol{I}_M$

Concerning the other quantity $\mathcal{E}\left[\boldsymbol{W}_l^H \boldsymbol{W}_l\right]$, we can obtain directly:

$$\mathcal{E}\left[\boldsymbol{W}_l^H \boldsymbol{W}_l\right] = \mathcal{E}\left[\boldsymbol{\phi}\boldsymbol{N}_l^H\,\boldsymbol{N}_l\boldsymbol{\phi}^H\right] = \boldsymbol{\phi}\mathcal{E}\left[\boldsymbol{N}_l^H \boldsymbol{N}_l\right]\boldsymbol{\phi}^H = M\sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_K$$

Remark : If we only consider one column (typically the *n*th column) of W_l denoted as $\boldsymbol{w}_{ln}$, we have :

$$\mathcal{E}\left\{\boldsymbol{w}_{ln}\boldsymbol{w}_{ln}^H\right\} = \sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_M \text{ and } \mathcal{E}\left\{\boldsymbol{w}_{ln}^H\boldsymbol{w}_{ln}\right\} = M\sigma_{\boldsymbol{n}_l}^2$$

We are now coming back to the channel estimation problem. Since $\boldsymbol{H}_{l,l}$ has independent columns, we can estimate each column of $\boldsymbol{H}_{l,l}$ independently. Let $\tilde{\boldsymbol{y}}_{pln}$ be the *n*th column of $\tilde{\boldsymbol{Y}}_{p,l}$. Then:

$$\tilde{\boldsymbol{y}}_{pln} = \sqrt{p_p}\boldsymbol{A}\boldsymbol{h}_{lln}\beta_{lln}^{1/2} + \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq\boldsymbol{l}}^{\boldsymbol{L}}\boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln} \quad (9)$$

Where $\boldsymbol{h}_{iln}$ and $\boldsymbol{w}_{ln}$ are the *n*th column of $\boldsymbol{H}_{i,l}$ and W_l respectively. Denote by: $\boldsymbol{z}_{ln} \equiv \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq\boldsymbol{l}}^{\boldsymbol{L}}\boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln}$, then the MMSE estimate of $\boldsymbol{h}_{iln}$ is given by :

$$\hat{\boldsymbol{h}}_{lln} = \beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H(p_p\beta_{lln}\boldsymbol{A}\boldsymbol{A}^H + \boldsymbol{R}_{\boldsymbol{z}_{ln}})^{-1}\tilde{\boldsymbol{y}}_{pln} \quad (10)$$

With: $\boldsymbol{R}_{\boldsymbol{z}_{ln}} = \mathcal{E}\left\{\boldsymbol{z}_{ln}\boldsymbol{z}_{ln}^H\right\} = p_p\boldsymbol{A}\boldsymbol{A}^H\sum_{\boldsymbol{i}=1,\boldsymbol{i}\neq\boldsymbol{l}}^{\boldsymbol{L}}\beta_{iln} + \sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_M \quad (11)$

Reporting (11) into (10), we obtain:

$$\hat{\boldsymbol{h}}_{lln} = \beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H\left(p_p\beta_{lln}\boldsymbol{A}\boldsymbol{A}^H + p_p\boldsymbol{A}\boldsymbol{A}^H\sum_{\boldsymbol{i}=1,\boldsymbol{i}\neq\boldsymbol{l}}^{\boldsymbol{L}}\beta_{iln} + \sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$

$$= \beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H\left(p_p\boldsymbol{A}\boldsymbol{A}^H\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln} + \sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$

We can rewrite $\hat{\boldsymbol{h}}_{lln}$ in the following form :

$$\hat{\boldsymbol{h}}_{lln} = \frac{\beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H}{\sigma_{\boldsymbol{n}_l}^2}\left(\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}\boldsymbol{A}\boldsymbol{A}^H + \boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$

$$= \frac{\beta_{lln}^{1/2}\sqrt{p_p}}{\sigma_{\boldsymbol{n}_l}^2}\boldsymbol{A}^H\left(\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}^H + \boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$

$$= \left(\frac{\beta_{lln}^{1/2}\sqrt{p_p}}{\sigma_{\boldsymbol{n}_l}^2}\sqrt{\frac{\sigma_{\boldsymbol{n}_l}^2}{p_p\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}}\right)\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}^H\left(\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}^H + \boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$

$$= \left(\frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{\boldsymbol{n}_l}^2\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}}\right)\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}^H\left(\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}\sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}\sum_{\boldsymbol{i}=1}^{\boldsymbol{L}}\beta_{iln}}\boldsymbol{A}^H + \boldsymbol{I}_M\right)^{-1}\tilde{\boldsymbol{y}}_{pln}$$

The former equation can be written: $\hat{\boldsymbol{h}}_{lln} = \lambda\boldsymbol{B}^H(\boldsymbol{B}\boldsymbol{B}^H + \boldsymbol{I}_M)^{-1}\tilde{\boldsymbol{y}}_{pln}$, with: $\boldsymbol{B} = \sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}}\sqrt{\sum_{i=1}^{L}\beta_{iln}}\boldsymbol{A}$, and $\lambda = \frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{\boldsymbol{n}_l}^2\sum_{i=1}^{L}\beta_{iln}}}$. Using the matrix inversion lemma, we have:

$$\boldsymbol{P}\left(\boldsymbol{I}_m + \boldsymbol{Q}\boldsymbol{P}\right)^{-1} = \left(\boldsymbol{I}_n + \boldsymbol{P}\boldsymbol{Q}\right)^{-1}\boldsymbol{P}$$

With the following sizes for $\boldsymbol{P}$ and $\boldsymbol{Q}$: $[\boldsymbol{Q}]_{m\times n}, [\boldsymbol{P}]_{n\times m}$. We use this lemma with: $\boldsymbol{P} = \boldsymbol{B}^H$, $\boldsymbol{Q} = \boldsymbol{B}$, we obtain: $\boldsymbol{B}^H(\boldsymbol{B}\boldsymbol{B}^H + \boldsymbol{I}_M)^{-1} = (\boldsymbol{I}_P + \boldsymbol{B}^H\boldsymbol{B})^{-1}\boldsymbol{B}^H$, so we eventually obtain:

$$\hat{\boldsymbol{h}}_{lln} = \frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{\boldsymbol{n}_l}^2 \sum_{i=1}^{L} \beta_{iln}}} \left(\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2} \boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} + \boldsymbol{I}_P \right)^{-1} \sqrt{\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2}} \sqrt{\sum_{i=1}^{L} \beta_{iln}} \boldsymbol{A}^H \tilde{\boldsymbol{y}}_{pln}$$

$$\hat{\boldsymbol{h}}_{lln} = \frac{\sqrt{p_p}\beta_{lln}^{1/2}}{\sigma_{\boldsymbol{n}_l}^2} \left(\frac{p_p}{\sigma_{\boldsymbol{n}_l}^2} \boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} + \boldsymbol{I}_P \right)^{-1} \boldsymbol{A}^H \tilde{\boldsymbol{y}}_{pln} \qquad (12)$$

$$\hat{\boldsymbol{h}}_{lln} = \sqrt{p_p}\beta_{lln}^{1/2} \left(p_p \boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} + \sigma_{\boldsymbol{n}_l}^2 \boldsymbol{I}_P \right)^{-1} \boldsymbol{A}^H \tilde{\boldsymbol{y}}_{pln}$$

The *k*th diagonal element of $p_p \boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln}$ in (12) equals: $\frac{Mp_p}{P} \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln}$. Since the uplink is typically interference-limited we have: $\frac{Mp_p}{P} \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} >> \sigma_{\boldsymbol{n}_l}^2$, therefore, $\hat{\boldsymbol{h}}_{lln}$ can be approximated as:

$$\hat{\boldsymbol{h}}_{lln} = \beta_{lln}^{1/2} \sqrt{p_p} \left(p_p \boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} \right)^{-1} \boldsymbol{A}^H \tilde{\boldsymbol{y}}_{pln} \qquad (13)$$

Thus, the MMSE estimate of $\boldsymbol{H}_{l,l}$ is:

$$\hat{\boldsymbol{H}}_{l,l} = p_p^{-1/2} (\boldsymbol{A}^H \boldsymbol{A})^{-1} \boldsymbol{A}^H \tilde{\boldsymbol{Y}}_{p,l} \boldsymbol{D}_l^{-1} \boldsymbol{D}_{ll}^{1/2} \qquad (14)$$

Where: $\boldsymbol{D}_l \equiv \sum_{i=1}^{L} \boldsymbol{D}_{il}$. Then, the estimate of the physical channel matrix between the *l*th base station and the *K* users in the *l*th cell is given by:

$$\hat{\Gamma}_{l,l} = \boldsymbol{A}\hat{\boldsymbol{H}}_{l,l} \boldsymbol{D}_{ll}^{1/2} = p_p^{-1/2} \Pi_{\boldsymbol{A}} \tilde{\boldsymbol{Y}}_{p,l} \boldsymbol{D}_l^{-1} \boldsymbol{D}_{ll} \qquad (15)$$

Where $\Pi_{\boldsymbol{A}} \equiv \boldsymbol{A}\left(\boldsymbol{A}^H \boldsymbol{A}\right)^{-1} \boldsymbol{A}^H$ is the orthogonal projection onto $\boldsymbol{A}$. We can see that since post-multiplication of $\boldsymbol{Y}_{p,l}$ with $\boldsymbol{\phi}^H$ means just multiplication with the pseudo-inverse $\left(\boldsymbol{\phi}\boldsymbol{\phi}^H = \boldsymbol{I}_K\right)$, $\tilde{\boldsymbol{Y}}_{p,l}$ is the conventional least-squares channel estimate. The MMSE channel estimator that we derived thus performs conventional channel estimation and then projects the estimate onto the physical (beamspace) model for the array.

3.2. Bayesian estimator with MAP rule:

We start from equation (7):

$$\tilde{\boldsymbol{Y}}_{p,l} = \boldsymbol{Y}_{p,l} \boldsymbol{\phi}^H = \sqrt{p_p} \boldsymbol{A} \sum_{i=1}^{L} \boldsymbol{H}_{i,l} \boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{W}_l$$

Let $\tilde{\boldsymbol{y}}_{pln}$ be the nth column of $\tilde{\boldsymbol{Y}}_{p,l}$. Then:

$$\tilde{\boldsymbol{y}}_{pln} = \sqrt{p_p}\boldsymbol{A}\boldsymbol{h}_{\boldsymbol{lln}}\beta_{lln}^{1/2} + \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}=1,i\neq l}^{\boldsymbol{L}}\boldsymbol{h}_{\boldsymbol{iln}}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln} \qquad (16)$$

Where $\boldsymbol{h}_{\boldsymbol{iln}}$ and $\boldsymbol{w}_{ln}$ are the nth column of $\boldsymbol{H}_{i,l}$ and W_l respectively. Denote by: $\boldsymbol{z}_{ln} \equiv \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq \boldsymbol{l}}^{\boldsymbol{L}}\boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln}$, we can write:

$$\tilde{\boldsymbol{y}}_{pln} = \sqrt{p_p}\boldsymbol{A}\boldsymbol{h}_{lln}\beta_{lln}^{1/2} + \boldsymbol{z}_{ln} \qquad (17)$$

Applying Bayes' rule, the conditional distribution of the channel $\boldsymbol{h}_{lln}$, given the observed received training signal $\tilde{\boldsymbol{y}}_{pln}$, is:

$$p(\boldsymbol{h}_{lln}|\tilde{\boldsymbol{y}}_{pln}) = \frac{p(\boldsymbol{h}_{lln})p(\tilde{\boldsymbol{y}}_{pln}|\boldsymbol{h}_{lln})}{p(\tilde{\boldsymbol{y}}_{pln})} = p(\boldsymbol{h}_{lln})p(\tilde{\boldsymbol{y}}_{pln}|\boldsymbol{h}_{lln}) \qquad (18)$$

We use the Gaussian probability density function (PDF) of the random vector $\boldsymbol{h}_{lln}$ and assume its elements: $h_{lln}(1),...,h_{lln}(i),...,h_{lln}(P)$, are mutually independent, giving the joint pdf :

$$p(\boldsymbol{h}_{lln}) = \frac{\exp[-\boldsymbol{h}_{lln}^H \boldsymbol{R}_{\boldsymbol{h}_{lln}}^{-1} \boldsymbol{h}_{lln}]}{\left(\sqrt{2\pi}\right)^P \det \boldsymbol{R}_{\boldsymbol{h}_{lln}}} \qquad (19)$$

Since the elements: $h_{lln}(1),...,h_{lln}(i),...,h_{lln}(P)$ are mutually independent this reduces to:

$$p(\boldsymbol{h}_{lln}) = \frac{\exp[-\boldsymbol{h}_{lln}^H \boldsymbol{h}_{lln}]}{\left(\sqrt{2\pi}\right)^P} \qquad (20)$$

We also have:

$$p(\tilde{\boldsymbol{y}}_{pln}|\boldsymbol{h}_{lln}) = \frac{\exp[-(\tilde{\boldsymbol{y}}_{pln} - \sqrt{p_p}\boldsymbol{A}\boldsymbol{h}_{\boldsymbol{lln}}\beta_{lln}^{1/2})^H \boldsymbol{R}_{\boldsymbol{z}_{ln}}^{-1}(\tilde{\boldsymbol{y}}_{pln} - \sqrt{p_p}\boldsymbol{A}\boldsymbol{h}_{\boldsymbol{lln}}\beta_{lln}^{1/2})]}{\pi^P \det \boldsymbol{R}_{\boldsymbol{z}_{ln}}} \qquad (21)$$

Combining (20) and (21) and reporting into (18) we obtain:

$$p(\boldsymbol{h}_{lln}|\tilde{\boldsymbol{y}}_{pln}) = \frac{\exp(-f(\boldsymbol{h}_{lln}))}{(\sqrt{2\pi})^P \pi^P \det \boldsymbol{R}_{\boldsymbol{z}_{ln}}} \qquad (22)$$

With: $\boldsymbol{R}_{\boldsymbol{z}_{ln}} = \mathcal{E}\left[(\sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq \boldsymbol{l}}^{\boldsymbol{L}} \boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln})(\sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq \boldsymbol{l}}^{\boldsymbol{L}} \boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln})^H \right]$ and:

$$f(\boldsymbol{h}_{lln}) = \boldsymbol{h}_{lln}^H \boldsymbol{h}_{lln} + (\tilde{\boldsymbol{y}}_{pln} - \sqrt{p_p}\boldsymbol{A}\boldsymbol{h}_{\boldsymbol{lln}}\beta_{lln}^{1/2})^H \boldsymbol{R}_{\boldsymbol{z}_{ln}}^{-1}(\tilde{\boldsymbol{y}}_{pln} - \sqrt{p_p}\boldsymbol{A}\boldsymbol{h}_{\boldsymbol{lln}}\beta_{lln}^{1/2})$$

The ML estimation $\hat{\boldsymbol{h}}_{lln}$ of $\boldsymbol{h}_{lln}$ is given by:

$$\begin{aligned}
\hat{\boldsymbol{h}}_{lln} &= \arg\max_{\boldsymbol{h}_{lln}\in C^{P\times 1}} \exp(-f(\boldsymbol{h}_{lln})) \\
\hat{\boldsymbol{h}}_{lln} &= \arg\min_{\boldsymbol{h}_{lln}\in C^{P\times 1}} f(\boldsymbol{h}_{lln}) \\
\hat{\boldsymbol{h}}_{lln} &= \beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H \left(p_p\beta_{lln}\boldsymbol{A}\boldsymbol{A}^H + \boldsymbol{R}_{\boldsymbol{z}_{ln}} \right)^{-1} \tilde{\boldsymbol{y}}_{pln}
\end{aligned} \tag{23}$$

We have clearly:

$$\begin{aligned}
\boldsymbol{R}_{\boldsymbol{z}_{ln}} &= \mathcal{E}\left[(\sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq \boldsymbol{l}}^{\boldsymbol{L}} \boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln})(\sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq \boldsymbol{l}}^{\boldsymbol{L}} \boldsymbol{h}_{iln}\beta_{iln}^{1/2} + \boldsymbol{w}_{ln})^H \right] \\
\boldsymbol{R}_{\boldsymbol{z}_{ln}} &= \sigma_{\boldsymbol{n}_l}^2 \boldsymbol{I}_M + p_p \sum_{i\neq l}^{L} \beta_{iln}\boldsymbol{A}\boldsymbol{A}^H
\end{aligned} \tag{24}$$

And we find again:

$$\hat{\boldsymbol{h}}_{lln} = \beta_{lln}^{1/2}\sqrt{p_p}\boldsymbol{A}^H \left(p_p\boldsymbol{A}\boldsymbol{A}^H \sum_{\boldsymbol{i}=1}^{\boldsymbol{L}} \beta_{iln} + \sigma_{\boldsymbol{n}_l}^2 \boldsymbol{I}_M \right)^{-1} \tilde{\boldsymbol{y}}_{pln} \tag{25}$$

To obtain (23) and (25) we have applied (39) and (40) (see Appendix) with: $\boldsymbol{R}_{\boldsymbol{h}} = \boldsymbol{I}$, $\boldsymbol{R}_{\boldsymbol{n}} = \boldsymbol{R}_{\boldsymbol{z}_{ln}}$, $\boldsymbol{S} = \sqrt{p_p}\beta_{lln}^{1/2}\boldsymbol{A}$.

3.3. Bayesian estimation with vectorized model:

We start from equation (8):

$$\begin{aligned}
\tilde{\boldsymbol{Y}}_{p,l} &= \boldsymbol{Y}_{p,l}\boldsymbol{\phi}^* = \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}=1}^{L} \boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{N}_l\boldsymbol{\phi}^* = \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}=1}^{L} \boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{W}_l \\
\tilde{\boldsymbol{Y}}_{p,l} &= \sqrt{p_p}\boldsymbol{A}\boldsymbol{H}_{l,l}\boldsymbol{D}_{l,l}^{1/2} + \sqrt{p_p}\boldsymbol{A}\sum_{\boldsymbol{i}\neq l}^{L} \boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{W}_l
\end{aligned}$$

Starting from this equation we begin by vectorizing the received matrix $\tilde{\boldsymbol{Y}}_{p,l}$ of size $M\times K$, we obtain the vector:

$$\tilde{\boldsymbol{y}}_{p,l} = \begin{pmatrix} \tilde{\boldsymbol{Y}}_{p,l}(:,1) \\ \vdots \\ \tilde{\boldsymbol{Y}}_{p,l}(:,k) \\ \vdots \\ \tilde{\boldsymbol{Y}}_{p,l}(:,K) \end{pmatrix}_{MK\times 1} \qquad (26)$$

With:

$$\begin{aligned} \tilde{\boldsymbol{Y}}_{p,l}(:,k) &= \sqrt{p_p}\left[\boldsymbol{A}\boldsymbol{H}_{l,l}\boldsymbol{D}_{l,l}^{1/2}\right](:,k) + \sqrt{p_p}\left[\boldsymbol{A}\sum_{\boldsymbol{i}\neq l}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\right](:,k) + \boldsymbol{W}_l(:,k) \\ \tilde{\boldsymbol{Y}}_{p,l}(:,k) &= \sqrt{p_p}\left[\boldsymbol{A}\boldsymbol{H}_{l,l}\boldsymbol{D}_{l,l}^{1/2}\right](:,k) + \boldsymbol{Z}_l(:,k) \end{aligned} \qquad (27)$$

With: $\boldsymbol{Z}_l(:,k) \equiv \sqrt{p_p}\left[\boldsymbol{A}\sum_{\boldsymbol{i}\neq l}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\right](:,k) + \boldsymbol{W}_l(:,k)$

Clearly, we have the property:

$$\left[\boldsymbol{A}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\right](:,k) = \boldsymbol{A}\boldsymbol{H}_{i,l}(:,k)\beta_{ilk}^{1/2} \qquad (28)$$

Applying Bayes' rule, the conditional distribution of the channel $\boldsymbol{H}_{l,l}(:,k)$ given the observed received training signal $\tilde{\boldsymbol{y}}_{p,l}$ is:

$$\begin{aligned} & p(\boldsymbol{H}_{l,l}(:,k)\big|\tilde{\boldsymbol{Y}}_{p,l}(:,k)) \\ &= \frac{p(\boldsymbol{H}_{l,l}(:,k))\, p(\tilde{\boldsymbol{Y}}_{p,l}(:,k)\big|\boldsymbol{H}_{l,l}(:,k))}{p(\tilde{\boldsymbol{Y}}_{p,l}(:,k))} = p(\boldsymbol{H}_{l,l}(:,k))\, p(\tilde{\boldsymbol{Y}}_{p,l}(:,k)\big|\boldsymbol{H}_{l,l}(:,k)) \end{aligned} \qquad (29)$$

We use the Gaussian probability density function (pdf) of the random vector $\boldsymbol{H}_{l,l}(:,k)$ and assume its elements: $H_{l,l}(1,k),...,H_{l,l}(1,k),...,H_{l,l}(P,k)$ are mutually independent, giving the joint pdf:

$$p(\boldsymbol{H}_{l,l}(:,k)) = \frac{\exp[-\boldsymbol{H}_{l,l}^{H}(:,k)\,\boldsymbol{R}_{\boldsymbol{H}_{l,l}(:,k)}^{-1}\,\boldsymbol{H}_{l,l}^{H}(:,k)]}{(\sqrt{2\pi})^{P}\det \boldsymbol{R}_{\boldsymbol{H}_{l,l}(:,k)}} \qquad (30)$$

Since the elements: $H_{l,l}(1,k),...,H_{l,l}(m,k),...,H_{l,l}(P,k)$ are mutually independent, this reduces to:

$$p(\boldsymbol{H}_{l,l}(:,k)) = \frac{\exp[-\boldsymbol{H}_{l,l}^{H}(:,k)\,\boldsymbol{H}_{l,l}^{H}(:,k)]}{(\sqrt{2\pi})^{P}} \qquad (31)$$

We also have:

$$\begin{aligned} & p(\tilde{\boldsymbol{Y}}_{p,l}(:,k)\big|\boldsymbol{H}_{l,l}(:,k)) \\ & = \frac{\exp[-(\tilde{\boldsymbol{Y}}_{p,l}(:,k) - \boldsymbol{A}\boldsymbol{H}_{l,l}(:,k)\beta_{llk}^{1/2})^{H}\,\boldsymbol{R}_{\boldsymbol{Z}_l(:,k)}^{-1}(\tilde{\boldsymbol{Y}}_{p,l}(:,k) - \boldsymbol{A}\boldsymbol{H}_{l,l}(:,k)\beta_{llk}^{1/2})]}{(\pi)^{P}\det \boldsymbol{R}_{\boldsymbol{Z}_l(:,k)}} \end{aligned} \qquad (32)$$

Combining (31) and (32) and reporting into (29) we obtain:

$$p(\boldsymbol{H}_{l,l}(:,k)\big|\tilde{\boldsymbol{Y}}_{p,l}(:,k)) = \frac{\exp(-f(\boldsymbol{H}_{l,l}(:,k)))}{AB} \qquad (33)$$

Where: $A \equiv \left(\sqrt{2\pi}\right)^{P}$, $B \equiv (\pi)^{P}\det \boldsymbol{R}_{\boldsymbol{Z}_l(:,k)}$, this yields :

$$\begin{aligned} & f(\boldsymbol{H}_{l,l}(:,k)) \\ & \equiv \boldsymbol{H}_{l,l}^{H}(:,k)\,\boldsymbol{H}_{l,l}^{H}(:,k) + (\tilde{\boldsymbol{Y}}_{p,l}(:,k) - \boldsymbol{A}\boldsymbol{H}_{l,l}(:,k)\beta_{llk}^{1/2})^{H}\,\boldsymbol{R}_{\boldsymbol{Z}_l(:,k)}^{-1}(\tilde{\boldsymbol{Y}}_{p,l}(:,k) - \boldsymbol{A}\boldsymbol{H}_{l,l}(:,k)\beta_{llk}^{1/2}) \end{aligned} \qquad (34)$$

Using the maximum a posteriori (MAP) decision rule, the Bayesian estimator yields the most probable value of channel estimation given the observation: $\tilde{\boldsymbol{Y}}_{p,l}(:,k)$.

$$\begin{aligned} & \hat{\boldsymbol{H}}_{l,l}^{H}(:,k) = \arg\max_{\boldsymbol{H}_{l,l}^{H}(:,k)}\, p(\boldsymbol{H}_{l,l}(:,k)\big|\tilde{\boldsymbol{Y}}_{p,l}(:,k)) \\ & \hat{\boldsymbol{H}}_{l,l}^{H}(:,k) = \arg\min_{\boldsymbol{H}_{l,l}^{H}(:,k)}\, f(\boldsymbol{H}_{l,l}(:,k)) \\ & \hat{\boldsymbol{H}}_{l,l}^{H}(:,k) = \sqrt{p_p}\,\beta_{llk}^{1/2}\boldsymbol{A}^{H}(p_p\beta_{llk}\boldsymbol{A}\boldsymbol{A}^{H} + \boldsymbol{R}_{\boldsymbol{Z}_l(:,k)})^{-1}\tilde{\boldsymbol{Y}}_{p,l}(:,k) \end{aligned} \qquad (35)$$

It is straightforward to show that (see equation (24)):

$$\boldsymbol{R}_{\boldsymbol{Z}_l(:,k)} = \sigma_{\boldsymbol{n}_l}^{2}\boldsymbol{I}_M + p_p\sum_{i\neq l}^{L}\beta_{ilk}\boldsymbol{A}\boldsymbol{A}^{H} \qquad (36)$$

And we find again:

$$\hat{\boldsymbol{H}}_{l,l}(:,k) = \sqrt{p_p}\,\beta_{lln}^{1/2}\boldsymbol{A}^{H}(\sigma_{\boldsymbol{n}_l}^{2}\boldsymbol{I}_M + p_p\sum_{i=1}^{L}\beta_{ilk}\boldsymbol{A}\boldsymbol{A}^{H})^{-1}\tilde{\boldsymbol{Y}}_{p,l}(:,k) \qquad (37)$$

<u>Reminder Appendix 1</u>:

If we have the following equivalent model:

$$\boldsymbol{y} = \boldsymbol{Sh} + \boldsymbol{n} \quad (38)$$

With the following pdf :

$$p(\boldsymbol{h}|\boldsymbol{y}) = \frac{\exp(-l(\boldsymbol{h}))}{K}$$

$$l(\boldsymbol{h}) = \boldsymbol{h}^H \boldsymbol{R}_h^{-1} \boldsymbol{h} + (\boldsymbol{y} - \boldsymbol{Sh})^H \boldsymbol{R}_n^{-1} (\boldsymbol{y} - \boldsymbol{Sh}) \quad (39)$$

The Bayesian estimator yields the most probable value given the observation $\boldsymbol{y}$ and this yields:

$$\hat{\boldsymbol{h}} = \boldsymbol{R}_h \boldsymbol{S}^H (\boldsymbol{R}_n + \boldsymbol{R}_h \boldsymbol{S}\, \boldsymbol{S}^H)^{-1} \boldsymbol{y} \quad (40)$$

4- Uplink Data Transmission:

We consider the context of an uplink transmission where the base station uses the channel estimates to perform signal processing techniques such as MRC (Maximum Ratio Combining), ZF (Zero Forcing) or MMSE (Minimum Mean Square Error).

4.1. Maximum Ratio Combining:

In this case the *l*th base station processes the received signal by multiplying it by the conjugate transpose (Hermitian) of the channel estimate. From (6) and (14), we obtain:

$$\begin{aligned} \boldsymbol{r}_l &= \hat{\Gamma}_{ll}^H \boldsymbol{y}_l \\ &= p_p^{-1/2} \boldsymbol{D}_{ll} \boldsymbol{D}_l^{-1} \left(\sqrt{p_p}\, \boldsymbol{A} \sum_{i=1}^{L} \boldsymbol{H}_{i,l} \boldsymbol{D}_{i,l}^{1/2} + \boldsymbol{W}_l \right)^H \Pi_{\boldsymbol{A}} \left(\sqrt{p_u}\, \boldsymbol{A} \sum_{\boldsymbol{j}=1}^{L} \boldsymbol{H}_{j,l} \boldsymbol{D}_{j,l}^{1/2} \boldsymbol{x}_j + \boldsymbol{n}_l \right) \end{aligned} \quad (41)$$

With: $\Pi_{\boldsymbol{A}} \equiv \boldsymbol{A}(\boldsymbol{A}^H \boldsymbol{A})^{-1} \boldsymbol{A}^H$

Remark : $\Pi_{\boldsymbol{A}}^H \equiv \left[\boldsymbol{A}(\boldsymbol{A}^H \boldsymbol{A})^{-1} \boldsymbol{A}^H \right]^H = \boldsymbol{A} \left[(\boldsymbol{A}^H \boldsymbol{A})^{-1} \right]^H \boldsymbol{A}^H = \boldsymbol{A} \left[(\boldsymbol{A}^H \boldsymbol{A})^{-1} \right] \boldsymbol{A}^H = \Pi_{\boldsymbol{A}}$

To further proceed with (41) we directly suppose that we operate with a huge number of antennas, i.e. we suppose that *M* grows towards infinity. We simplify first the quantity: $\Pi_{\boldsymbol{A}}$; to do this we consider the case of a uniform array. The response vector is thus given by:

$$\boldsymbol{a}(\phi_k) = \frac{1}{\sqrt{P}} \left[1, e^{-j2\pi d \sin(\phi_k)/\lambda}, \ldots, e^{-j2\pi (M-1) d \sin(\phi_k)/\lambda} \right]^T \quad (42)$$

d is the antenna spacing between radiating elements and λ is the wavelength. Then, we have, for $k \neq l$:

$$\begin{aligned}\frac{1}{M}\boldsymbol{a}(\phi_k)^H\boldsymbol{a}(\phi_l) &= \frac{1}{MP}\sum_{m=0}^{M-1} e^{j2\pi d\, m\,(\sin(\phi_k)-\sin(\phi_l))/\lambda} \\ &= \frac{1}{MP}\frac{1-e^{j2\pi d\, M\,(\sin(\phi_k)-\sin(\phi_l))/\lambda}}{1-e^{j2\pi d\,(\sin(\phi_k)-\sin(\phi_l))/\lambda}} \\ &= \frac{1}{P}e^{j2\pi d\,(M-1)(\sin(\phi_k)-\sin(\phi_l))/\lambda}\left[\frac{\sin\left(\pi d\, M(\sin(\phi_k)-\sin(\phi_l))/\lambda\right)}{M\sin\left(\pi d\,(\sin(\phi_k)-\sin(\phi_l))/\lambda\right)}\right]\end{aligned} \tag{43}$$

From (43) it is clear that, for $k \neq l$: $\lim_{M\to\infty}\frac{1}{M}\boldsymbol{a}(\phi_k)^H\boldsymbol{a}(\phi_l) = 0$

For $k = l$, we have immediately, $\lim_{M\to\infty}\frac{1}{M}\boldsymbol{a}(\phi_l)^H\boldsymbol{a}(\phi_l) = 1/P$. Therefore, we deduce: $\lim_{M\to\infty}\frac{1}{M}\boldsymbol{A}^H\boldsymbol{A} = \frac{1}{P}\boldsymbol{I}_P$. Furthermore, we have: $\lim_{M\to\infty}\text{Trace}\left[\boldsymbol{A}^H\boldsymbol{A}\right] = \frac{M}{P}P = M$.
Using this property, we can write:

$$\boldsymbol{\Pi}_{\boldsymbol{A}} \equiv \boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H \approx_{M\to\infty} \boldsymbol{A}(\frac{M}{P}\boldsymbol{I}_P)^{-1}\boldsymbol{A}^H = \frac{P}{M}\boldsymbol{A}\boldsymbol{A}^H \tag{44}$$

Remark: In the general case, we have:

$$\boldsymbol{A}^H\boldsymbol{A} = \frac{1}{\sqrt{P}}\begin{bmatrix} e^{jf_1(\phi_1)} & e^{jf_2(\phi_1)} & \dots & e^{jf_M(\phi_1)} \\ e^{jf_1(\phi_2)} & e^{jf_2(\phi_2)} & \dots & e^{jf_M(\phi_2)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{jf_1(\phi_P)} & e^{jf_2(\phi_P)} & \dots & e^{jf_M(\phi_P)} \end{bmatrix} = \frac{1}{P}\begin{bmatrix} M & \sum_{i=1}^{M} e^{j(f_i(\phi_1)-f_i(\phi_2))} & \dots & \sum_{i=1}^{M} e^{j(f_i(\phi_1)-f_i(\phi_P))} \\ \sum_{i=1}^{M} e^{j(f_i(\phi_2)-f_i(\phi_1))} & M & \dots & \sum_{i=1}^{M} e^{j(f_i(\phi_2)-f_i(\phi_P))} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{M} e^{j(f_i(\phi_P)-f_i(\phi_1))} & \sum_{i=1}^{M} e^{j(f_i(\phi_P)-f_i(\phi_2))} & \dots & M \end{bmatrix}$$

Concerning the product: $\boldsymbol{A}\boldsymbol{A}^H$, we obtain:

$$\frac{1}{\sqrt{P}}\begin{bmatrix} e^{jf_1(\phi_1)} & e^{jf_2(\phi_1)} & \cdots & e^{jf_M(\phi_1)} \\ e^{jf_1(\phi_2)} & e^{jf_2(\phi_2)} & \cdots & e^{jf_M(\phi_2)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{jf_1(\phi_P)} & e^{jf_2(\phi_P)} & \cdots & e^{jf_M(\phi_P)} \end{bmatrix}$$

$$\boldsymbol{A}\,\boldsymbol{A}^{\boldsymbol{H}} = \frac{1}{\sqrt{P}}\begin{bmatrix} e^{-jf_1(\phi_1)} & e^{-jf_1(\phi_2)} & \cdots & e^{-jf_1(\phi_P)} \\ e^{-jf_2(\phi_1)} & e^{-jf_2(\phi_2)} & \cdots & e^{-jf_2(\phi_P)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-jf_M(\phi_1)} & e^{-jf_M(\phi_2)} & \cdots & e^{-jf_M(\phi_P)} \end{bmatrix} = \frac{1}{P}\begin{bmatrix} P & \sum_{i=1}^{P} e^{j(f_2(\phi_i)-f_1(\phi_i))} & \cdots & \sum_{i=1}^{P} e^{j(f_M(\phi_i)-f_1(\phi_i))} \\ \sum_{i=1}^{P} e^{j(f_1(\phi_i)-f_2(\phi_i))} & P & \cdots & \sum_{i=1}^{P} e^{j(f_M(\phi_i)-f_2(\phi_i))} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^{P} e^{j(f_1(\phi_i)-f_M(\phi_i))} & \sum_{i=1}^{P} e^{j(f_1(\phi_i)-f_2(\phi_i))} & \cdots & P \end{bmatrix}$$

So, the term $\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}}(m,n)$ is equal to (in the case of a linear phased array):

$$\boldsymbol{A}\,\boldsymbol{A}^{\boldsymbol{H}}(m,n) = \frac{1}{P}\sum_{i=1}^{P} e^{j(f_n(\phi_i)-f_m(\phi_i))} = \frac{1}{P}\sum_{i=1}^{P} e^{j2\pi((n-1)\sin(\phi_i)-(m-1)\sin(\phi_i))/\lambda}$$

$$\boldsymbol{A}\,\boldsymbol{A}^{\boldsymbol{H}}(m,n) = \frac{1}{P}\sum_{i=1}^{P} e^{j2\pi((n-m)\sin(\phi_i))/\lambda}$$

A coarse approximation of matrix $\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}}$ in the case where $P \to \infty$ could be: $\boldsymbol{A}\boldsymbol{A}^{\boldsymbol{H}} \approx \boldsymbol{I}_M$.
Using (43) and (44) into (41) and the fact that as $M \to \infty$ the products of uncorrelated quantities can be ignored [5], we have:

$$\frac{\boldsymbol{r}_l}{\sqrt{p_u}\,M} \xrightarrow{M\to\infty} \boldsymbol{D}_{ll}\boldsymbol{D}_l^{-1}\left(\sum_{i=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\right)^H \frac{\boldsymbol{A}^H\boldsymbol{A}}{M}\left(\sum_{j=1}^{L}\boldsymbol{H}_{j,l}\boldsymbol{D}_{j,l}^{1/2}\boldsymbol{x}_j\right) \qquad (45)$$

Then, using: $\lim_{M\to\infty}\frac{1}{M}\boldsymbol{A}^H\boldsymbol{A} = \frac{1}{P}\boldsymbol{I}_P$, it is straightforward to obtain :

$$\frac{\boldsymbol{r}_l}{\sqrt{p_u}\,M} \xrightarrow{M\to\infty} \frac{1}{P}\boldsymbol{D}_{ll}\boldsymbol{D}_l^{-1}\left(\sum_{i=1}^{L}\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}\right)^H\left(\sum_{j=1}^{L}\boldsymbol{H}_{j,l}\boldsymbol{D}_{j,l}^{1/2}\boldsymbol{x}_j\right) \qquad (46)$$

The interpretation of (46) is quite simple: since the elements of $\boldsymbol{H}_{i,l}\boldsymbol{D}_{i,l}^{1/2}$ are independent, the above result reveals that the performance of the system under the finite-dimensional channel model with P angular bins and with an unlimited number of base station antennas is the same as the performance under an uncorrelated channel model with P antennas.

4.2. Analysis of the Pilot Contamination effect:

To gain further insight into the pilot contamination effect, we derive lower bounds on the achievable rate for a finite and an infinite number of base station antennas (the analysis requires that $M>P$). To obtain these lower bounds we use the techniques of [3,4,6]. We assume an uniform array at the base station, and the elements of $\boldsymbol{H}_{i,l}$ are i.i.d Gaussian random variables. Let r_{ln} be the nth element of the received $K \times 1$ vector $\boldsymbol{r}_l$ and denote $\tilde{r}_{ln} \equiv r_{ln}\sqrt{p_p}\sum_{i=1}^{L}\beta_{iln}/\beta_{lln}$, we deduce from (41) that:

$$\tilde{r}_{ln} = \left(\sqrt{p_p}\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{g}_{iln} + \boldsymbol{w}_{ln}\right)^H \boldsymbol{\Pi}_{\boldsymbol{A}}\left(\sqrt{p_u}\boldsymbol{A}\sum_{j=1}^{L}\boldsymbol{G}_{j,l}\boldsymbol{x}_j + \boldsymbol{n}_l\right) \quad (47)$$

Remember that we have: $\boldsymbol{G}_{j,l} \equiv \left[\boldsymbol{g}_{jl1},\ldots,\boldsymbol{g}_{jlK}\right]$ which is a $P \times K$ matrix with: $\boldsymbol{g}_{jlk} \equiv \left[g_{jlk1},\ldots,g_{jlkP}\right]^T$ which is a vector that contains the path gains from the kth user in the ith cell to the lth base station. Equation (47) can be rewritten as:

$$\begin{aligned}
\tilde{r}_{ln} &= \left(\sqrt{p_p}\boldsymbol{A}\sum_{i=1}^{L}\boldsymbol{g}_{iln} + \boldsymbol{w}_{ln}\right)^H \boldsymbol{\Pi}_{\boldsymbol{A}}\left(\sqrt{p_u}\boldsymbol{A}\sum_{j=1}^{L}\boldsymbol{G}_{j,l}\boldsymbol{x}_j + \boldsymbol{n}_l\right) \\
&= \left(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{g}_{iln}^H\boldsymbol{A}^H + \boldsymbol{w}_{ln}^H\right)\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H\left(\sqrt{p_u}\boldsymbol{A}\sum_{j=1}^{L}\boldsymbol{G}_{j,l}\boldsymbol{x}_j + \boldsymbol{n}_l\right) \\
&= \left(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{g}_{iln}^H\boldsymbol{A}^H + \boldsymbol{w}_{ln}^H\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H\right)\left(\sqrt{p_u}\boldsymbol{A}\sum_{j=1}^{L}\boldsymbol{G}_{j,l}\boldsymbol{x}_j + \boldsymbol{n}_l\right) \qquad (48) \\
&= \sqrt{p_p p_u}\sum_{i=1}^{L}\boldsymbol{g}_{iln}^H\boldsymbol{A}^H\boldsymbol{A}\sum_{j=1}^{L}\boldsymbol{G}_{j,l}\boldsymbol{x}_j + \sqrt{p_u}\boldsymbol{w}_{ln}^H\boldsymbol{A}\sum_{j=1}^{L}\boldsymbol{G}_{j,l}\boldsymbol{x}_j + \sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{g}_{iln}^H\boldsymbol{A}^H\boldsymbol{n}_l + \boldsymbol{w}_{ln}^H\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H\boldsymbol{n}_l
\end{aligned}$$

Equation (48) is eventually equivalent to:

$$\tilde{r}_{ln} = \boldsymbol{\alpha}_{ln}^T\boldsymbol{x}_l + \sum_{j\neq l}^{L}\boldsymbol{\alpha}_{jn}^T\boldsymbol{x}_j + z_{ln} \qquad (49)$$

With: $\boldsymbol{\alpha}_{jn}^T \equiv \sqrt{p_p p_u}\sum_{i=1}^{L}\boldsymbol{g}_{iln}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{G}_{j,l} + \sqrt{p_u}\boldsymbol{w}_{ln}^H\boldsymbol{A}\boldsymbol{G}_{j,l}$ and $z_{ln} = (\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{g}_{iln}^H\boldsymbol{A}^H + \boldsymbol{w}_{ln}^H)\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{n}_l$

We have:

$$\mathcal{E}\left\{\boldsymbol{\alpha}_{ln}^T\right\} \equiv \sqrt{p_p p_u}\,\mathcal{E}\left\{\boldsymbol{g}_{lln}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{g}_{lln}\right\}\boldsymbol{e}_n^T \qquad (50)$$

Where $\boldsymbol{e}_n$ is the nth column of the $K \times K$ identity matrix. By adding and subtracting $\mathcal{E}\left\{\boldsymbol{\alpha}_{ln}^T\right\}$ into (49) and using (50) we obtain:

$$\begin{aligned}\tilde{r}_{ln} &= \mathcal{E}\left\{\boldsymbol{\alpha}_{ln}^T\right\}\boldsymbol{x}_l + (\boldsymbol{\alpha}_{ln}^T - \mathcal{E}\left\{\boldsymbol{\alpha}_{ln}^T\right\})\boldsymbol{x}_l + \sum_{j\neq l}^{L}\boldsymbol{\alpha}_{jn}^T\boldsymbol{x}_j + z_{ln} \\ &= \sqrt{p_p p_u}\,\mathcal{E}\left\{\boldsymbol{g}_{l\,ln}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{g}_{l\,ln}\right\}x_{ln} + (\boldsymbol{\alpha}_{ln}^T - \mathcal{E}\left\{\boldsymbol{\alpha}_{ln}^T\right\})\boldsymbol{x}_l + \sum_{j\neq l}^{L}\boldsymbol{\alpha}_{jn}^T\boldsymbol{x}_j + z_{ln}\end{aligned} \tag{51}$$

Let define the function $C(x)$ as: $C(x) = \log_2(1+x)$, the nth user in the lth cell can achieve an uplink rate of at least:

$$R_{ln} = C\left(\frac{p_p p_u\left|\mathcal{E}\left\{\boldsymbol{g}_{l\,ln}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{g}_{l\,ln}\right\}\right|^2}{\sum_{j=1}^{L}\mathcal{E}\left\{\left\|\boldsymbol{\alpha}_{jn}^T\right\|^2\right\} - \left\|\mathcal{E}\left\{\boldsymbol{\alpha}_{\mathrm{ln}}^T\right\}\right\|^2 + \mathcal{E}\left\{\left|z_{ln}\right|^2\right\}}\right) \tag{52}$$

In this case, the achievable uplink rate of the nth user in the lth cell in (52) can be lower bounded by (60) since we have:

$$\begin{aligned}\left\|\mathcal{E}\left\{\boldsymbol{\alpha}_{ln}^T\right\}\right\|^2 &= \left\|\sqrt{p_p p_u}\,\mathcal{E}\left\{\boldsymbol{g}_{l\,ln}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{g}_{l\,ln}\right\}\boldsymbol{e}_n^T\right\|^2 \\ &= p_p p_u \mathrm{Trace}\left[\mathcal{E}\left\{\boldsymbol{g}_{l\,ln}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{g}_{l\,ln}\right\}\boldsymbol{e}_n^T\boldsymbol{e}_n\mathcal{E}\left\{\boldsymbol{g}_{l\,ln}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{g}_{l\,ln}\right\}\right] \\ &= p_p p_u\left[\mathcal{E}\left\{\boldsymbol{g}_{l\,ln}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{g}_{l\,ln}\right\}\right]^2\mathrm{Trace}\left[\boldsymbol{e}_n^T\boldsymbol{e}_n\right] \\ &= p_p p_u\left[\mathcal{E}\left\{\boldsymbol{g}_{l\,ln}^H\boldsymbol{A}^H\boldsymbol{A}\boldsymbol{g}_{l\,ln}\right\}\right]^2\end{aligned}$$

We study the product $\boldsymbol{g}_{l\,ln}^H\boldsymbol{A}^H$ by considering only one column of matrix: $\boldsymbol{A}^H$, we have :

$$\boldsymbol{A} \equiv [\boldsymbol{a}(\phi_1),\ldots,\boldsymbol{a}(\phi_P)] = \frac{1}{\sqrt{P}}\begin{bmatrix} e^{-jf_1(\phi_1)} & e^{-jf_1(\phi_2)} & \cdots & e^{-jf_1(\phi_P)} \\ e^{-jf_2(\phi_1)} & e^{-jf_2(\phi_2)} & \cdots & e^{-jf_2(\phi_P)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-jf_M(\phi_1)} & e^{-jf_M(\phi_2)} & \cdots & e^{-jf_M(\phi_P)} \end{bmatrix}$$

Hence, we obtain:

$$\boldsymbol{g}_{l\,ln}^H\boldsymbol{A}^H(:,k) = \frac{1}{\sqrt{P}}\left[g_{l\,ln1}^*,\ldots,g_{l\,lnP}^*\right]\begin{bmatrix} e^{jf_k(\phi_1)} \\ e^{jf_k(\phi_2)} \\ \vdots \\ e^{jf_k(\phi_P)} \end{bmatrix} = \frac{1}{\sqrt{P}}\sum_{j=1}^{P} g_{l\,ln\,j}^* e^{jf_k(\phi_j)}$$

In this case, we obtain:

$$\boldsymbol{g}_{l\,ln}^{H}\boldsymbol{A}^{H}=\frac{1}{\sqrt{P}}\left[\sum_{j=1}^{P}g_{l\,ln\,j}^{*}e^{jf_1(\phi_j)},\ldots,\sum_{j=1}^{P}g_{l\,ln\,j}^{*}e^{jf_k(\phi_j)},\ldots,\sum_{j=1}^{P}g_{l\,ln\,j}^{*}e^{jf_M(\phi_j)}\right]_{1\times M}$$

And it is straightforward to calculate:

$$\begin{aligned}
&\mathcal{E}\left\{\boldsymbol{g}_{l\,ln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{l\,ln}\right\}\\
&=\frac{1}{P}\mathcal{E}\left\{\left(\sum_{j=1}^{P}g_{l\,ln\,j}^{*}e^{jf_1(\phi_j)}\right)\left(\sum_{j=1}^{P}g_{l\,ln\,j}e^{-jf_1(\phi_j)}\right)+\ldots+\left(\sum_{j=1}^{P}g_{l\,ln\,j}^{*}e^{jf_M(\phi_j)}\right)\left(\sum_{j=1}^{P}g_{l\,ln\,j}e^{-jf_M(\phi_j)}\right)\right\} \qquad (53)\\
&=\frac{M}{P}\sum_{j=1}^{P}\mathcal{E}\left[\left|g_{l\,ln\,j}\right|^2\right]=\frac{M}{P}\sum_{j=1}^{P}\mathcal{E}\left[\beta_{lln}\left|h_{l\,ln\,j}\right|^2\right]=\frac{M}{P}\beta_{lln}\sum_{j=1}^{P}\mathcal{E}\left[\left|h_{l\,ln\,j}\right|^2\right]=\frac{M}{P}\beta_{lln}
\end{aligned}$$

Hence, we obtain:

$$\left[\mathcal{E}\left\{\boldsymbol{g}_{l\,ln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{l\,ln}\right\}\right]^2=\frac{M^2}{P^2}\beta_{lln}^2 \qquad (54)$$

- Concerning: $\mathcal{E}\left\{\left|z_{ln}\right|^2\right\}$, we have :

$$\begin{aligned}
\mathcal{E}\left\{\left|z_{ln}\right|^2\right\}&=\mathcal{E}\left\{\left|(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}+\boldsymbol{w}_{ln}^{H})\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{n}_l\right|^2\right\}\\
&=\mathcal{E}\left\{(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}+\boldsymbol{w}_{ln}^{H})\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{n}_l\boldsymbol{n}_l^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{A}\boldsymbol{g}_{i\,ln}+\boldsymbol{w}_{ln}^{H})\right\}\\
&=\mathcal{E}\left\{(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}+\boldsymbol{w}_{ln}^{H})\boldsymbol{\Pi}_{\boldsymbol{A}}\mathcal{E}\left[\boldsymbol{n}_l\boldsymbol{n}_l^{H}\right]\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{A}\boldsymbol{g}_{i\,ln}+\boldsymbol{w}_{ln}^{H})\right\}\\
&=\mathcal{E}\left\{(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}+\boldsymbol{w}_{ln}^{H})\boldsymbol{\Pi}_{\boldsymbol{A}}\sigma_{\boldsymbol{n}_l}^2\boldsymbol{I}_M\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{A}\boldsymbol{g}_{i\,ln}+\boldsymbol{w}_{ln}^{H})\right\}\\
&=\sigma_{\boldsymbol{n}_l}^2\mathcal{E}\left\{(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}+\boldsymbol{w}_{ln}^{H})\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{A}\boldsymbol{g}_{i\,ln}+\boldsymbol{w}_{ln}^{H})\right\}\\
&=\sigma_{\boldsymbol{n}_l}^2\mathcal{E}\left\{(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}+\boldsymbol{w}_{ln}^{H})\boldsymbol{\Pi}_{\boldsymbol{A}}(\sqrt{p_p}\sum_{i=1}^{L}\boldsymbol{A}\boldsymbol{g}_{i\,ln}+\boldsymbol{w}_{ln}^{H})\right\}
\end{aligned}$$

Remark :

$$\begin{aligned}
\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}&=\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H\left[\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H\right]^H\\
&=\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H\\
&=\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H=\boldsymbol{\Pi}_{\boldsymbol{A}}
\end{aligned}$$

We use the independency of columns $\boldsymbol{g}_{i\,ln}$ and $\boldsymbol{g}_{j\,ln}$, i.e.: $\mathcal{E}\left[\boldsymbol{g}_{iln}^{H}\,\boldsymbol{g}_{jln}\right]=0$, for $i \neq j$.

$$\begin{aligned}\mathcal{E}\left\{|z_{ln}|^2\right\} &= \sigma_{\boldsymbol{n}_l}^2\left\{\mathcal{E}\left[p_p\sum_{i=1}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{A}\,\boldsymbol{g}_{i\,ln}\right]+\mathcal{E}\left[\boldsymbol{w}_{ln}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{\Pi}_{\boldsymbol{A}}^{H}\boldsymbol{w}_{ln}\right]\right\}\\ &= \sigma_{\boldsymbol{n}_l}^2\left\{\mathcal{E}\left[p_p\sum_{i=1}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{i\,ln}\right]+\mathcal{E}\left[\boldsymbol{w}_{ln}^{H}\boldsymbol{\Pi}_{\boldsymbol{A}}\boldsymbol{w}_{ln}\right]\right\}\\ &= \sigma_{\boldsymbol{n}_l}^2\left\{\mathcal{E}\left[p_p\sum_{i=1}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\;\boldsymbol{g}_{i\,ln}\right]+\mathcal{E}\left[\boldsymbol{w}_{ln}^{H}\boldsymbol{A}(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}\boldsymbol{A}^{H}\boldsymbol{w}_{ln}\right]\right\}\end{aligned}$$

Due to (53), we have :

$$\mathcal{E}\left\{\boldsymbol{g}_{l\,ln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{l\,ln}\right\}=\frac{M}{P}\beta_{lln}$$

The singular value decomposition of matrix $(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}$ yields :

$$(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}=\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{H}$$

$\boldsymbol{D}$ is a diagonal matrix which can be written as: $\boldsymbol{D}=\mathrm{diag}\left[\lambda_1,\lambda_2,...,\lambda_k,...\lambda_P\right]$ with $\lambda_i=1/\beta_i$, i = 1,…,P designing the eigenvalues of matrix $\boldsymbol{A}^{H}\boldsymbol{A}$. Thus, we have:

$$\mathcal{E}\left[\,\boldsymbol{w}_{ln}^{H}\boldsymbol{A}(\boldsymbol{A}^{H}\boldsymbol{A})^{-1}\boldsymbol{A}^{H}\boldsymbol{w}_{ln}\right]=\mathcal{E}\left[\,\boldsymbol{w}_{ln}^{H}\boldsymbol{A}\boldsymbol{U}\boldsymbol{D}\boldsymbol{U}^{H}\boldsymbol{A}^{H}\boldsymbol{w}_{ln}\right]=\mathcal{E}\left[\,\boldsymbol{w}_{ln}^{H}\boldsymbol{A}\boldsymbol{U}\boldsymbol{D}\left[\,\boldsymbol{w}_{ln}^{\boldsymbol{H}}\boldsymbol{A}\boldsymbol{U}\right]^{H}\right]$$

We have:

$$\begin{aligned}\boldsymbol{w}_{ln}^{H}\boldsymbol{A} &= \frac{1}{\sqrt{P}}\left[w_{ln}^{*}(1),w_{ln}^{*}(2),...,w_{ln}^{*}(M)\right]\begin{bmatrix} e^{-jf_1(\phi_1)} & e^{-jf_1(\phi_2)} & \cdots & e^{-jf_1(\phi_P)}\\ e^{-jf_2(\phi_1)} & e^{-jf_2(\phi_2)} & \cdots & e^{-jf_2(\phi_P)}\\ \vdots & \vdots & \ddots & \vdots\\ e^{-jf_M(\phi_1)} & e^{-jf_M(\phi_2)} & \cdots & e^{-jf_M(\phi_P)}\end{bmatrix}\\ &= \frac{1}{\sqrt{P}}\left[w_{ln}^{*}(1)e^{-jf_1(\phi_1)}+...+w_{ln}^{*}(M)e^{-jf_M(\phi_1)},...,w_{ln}^{*}(1)e^{-jf_1(\phi_P)}+...+w_{ln}^{*}(M)e^{-jf_M(\phi_P)}\right]\end{aligned}$$

Hence, the kth column of matrix $\boldsymbol{w}_{ln}^{H}\boldsymbol{A}\boldsymbol{U}$ is thus written as:

$$\begin{aligned}&\left(\boldsymbol{w}_{ln}^{H}\boldsymbol{A}\boldsymbol{U}\right)(:,k)\\ &=\frac{1}{\sqrt{P}}\left[\sum_{m=1}^{P}U(m,k)\sum_{i=1}^{M}w_{ln}^{*}(i)e^{-jf_i(\phi_m)}\right]\end{aligned}$$

In these conditions we have:

$$\mathcal{E}\left[\left(\boldsymbol{w}_{ln}^{H}\boldsymbol{AU}\right)(:,k)\,\boldsymbol{D}\left(\boldsymbol{w}_{ln}^{H}\boldsymbol{AU}\right)^{H}(:,k)\right]$$

$$=\frac{1}{P}\mathcal{E}\left[\sum_{m=1}^{P}U(m,k)\sum_{i=1}^{M}w_{ln}^{*}(i)e^{-jf_i(\phi_m)}\lambda_k\sum_{p=1}^{P}U^{*}(p,k)\sum_{j=1}^{M}w_{ln}(j)e^{jf_j(\phi_p)}\right]$$

$$=\frac{1}{P}\left[\lambda_k\sum_{m=1}^{P}U(m,k)\sum_{p=1}^{P}U^{*}(p,k)\mathcal{E}\left[\sum_{i=1}^{M}w_{ln}^{*}(i)e^{-jf_i(\phi_m)}\sum_{j=1}^{M}w_{ln}(j)e^{jf_j(\phi_p)}\right]\right]$$

$$=\frac{1}{P}\left[\lambda_k\sum_{m=1}^{P}U(m,k)\sum_{p=1}^{P}U^{*}(p,k)\sum_{j=1}^{M}\sum_{i=1}^{M}\mathcal{E}\left[w_{ln}^{*}(i)\,w_{ln}(j)\right]e^{jf_j(\phi_p)}e^{-jf_i(\phi_m)}\right]$$

$$=\frac{1}{P}\left[\lambda_k\sum_{m=1}^{P}U(m,k)\sum_{p=1}^{P}U^{*}(p,k)\sum_{i=1}^{M}\sigma_{\boldsymbol{n}_l}^{2}\;e^{j[f_i(\phi_p)-f_i(\phi_m)]}\right]$$

$$=\frac{\lambda_k\sigma_{\boldsymbol{n}_l}^{2}}{P}\sum_{m=1}^{P}U(m,k)\sum_{p=1}^{P}U^{*}(p,k)\sum_{i=1}^{M}e^{j[f_i(\phi_p)-f_i(\phi_m)]}$$

And we eventually obtain:

$$\mathcal{E}\left[\boldsymbol{w}_{ln}^{H}\boldsymbol{AUD}\left[\boldsymbol{w}_{ln}^{\boldsymbol{H}}\boldsymbol{AU}\right]^{H}\right]=\frac{\sigma_{\boldsymbol{n}_l}^{2}}{P}\sum_{k=1}^{P}\sum_{p=1}^{P}\sum_{m=1}^{P}\sum_{i=1}^{M}\lambda_k U(m,k)U^{*}(p,k)\,e^{j[f_i(\phi_p)-f_i(\phi_m)]} \qquad (55)$$

Concerning: $\mathcal{E}\left\{|z_{ln}|^2\right\}$, this yields :

$$\mathcal{E}\left\{|z_{ln}|^2\right\}=\frac{\sigma_{\boldsymbol{n}_l}^{2}M}{P}\,p_p\sum_{i=1}^{L}\beta_{iln}+\frac{\sigma_{\boldsymbol{n}_l}^{2}}{P}\sum_{k=1}^{P}\sum_{p=1}^{P}\sum_{m=1}^{P}\sum_{i=1}^{M}\lambda_k U(m,k)U^{*}(p,k)\,e^{j[f_i(\phi_p)-f_i(\phi_m)]}$$

$$\mathcal{E}\left\{|z_{ln}|^2\right\}=\frac{M\sigma_{\boldsymbol{n}_l}^{2}}{P}\left(p_p\sum_{i=1}^{L}\beta_{iln}+\frac{\sigma_{\boldsymbol{n}_l}^{2}}{M}\sum_{k=1}^{P}\sum_{p=1}^{P}\sum_{m=1}^{P}\sum_{i=1}^{M}\lambda_k U(m,k)U^{*}(p,k)\,e^{j[f_i(\phi_p)-f_i(\phi_m)]}\right) \qquad (56)$$

$$\mathcal{E}\left\{|z_{ln}|^2\right\}=\frac{M\sigma_{\boldsymbol{n}_l}^{2}}{P}\left(p_p\overline{\beta}_{ln}+\frac{\sigma_{\boldsymbol{n}_l}^{2}}{M}\sum_{k=1}^{P}\sum_{p=1}^{P}\sum_{m=1}^{P}\sum_{i=1}^{M}\lambda_k U(m,k)U^{*}(p,k)\,e^{j[f_i(\phi_p)-f_i(\phi_m)]}\right)$$

Using the notation: $S_{U^2(:,k)}^{f_i}\equiv\sum_{p=1}^{P}\sum_{m=1}^{P}\sum_{i=1}^{M}U(m,k)U^{*}(p,k)\,e^{j[f_i(\phi_p)-f_i(\phi_m)]}$, we obtain :

$$\mathcal{E}\left\{|z_{ln}|^2\right\}=\frac{M\sigma_{\boldsymbol{n}_l}^{2}}{P}\left(p_p\overline{\beta}_{ln}+\frac{\sigma_{\boldsymbol{n}_l}^{2}}{M}\sum_{k=1}^{P}S_{U^2(:,k)}^{f_i}\right) \qquad (57)$$

Remark: in the case of a uniform array, we have:

$$f_i(\phi_p)=\frac{2\pi(i-1)d}{\lambda}\sin(\phi_p)$$

And we obtain:

$$\mathcal{E}\left\{ |z_{ln}|^2 \right\} = \frac{M\sigma_{\boldsymbol{n}_l}^2}{P}\left(p_p \overline{\beta}_{ln} + \frac{\sigma_{\boldsymbol{n}_l}^2}{M}\sum_{k=1}^{P}\sum_{p=1}^{P}\sum_{m=1}^{P}\sum_{i=1}^{M}\lambda_k U(m,k)U^*(p,k)\, e^{\frac{j2\pi(i-1)d}{\lambda}[\sin(\phi_p)-\sin(\phi_m)]} \right) \quad (58)$$

Remark: UsingLemma 1(Eq. (59)), it is possible to obtain a simpler expression. We have:

$$\begin{aligned}
&\mathcal{E}\left[\boldsymbol{w}_{ln}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{\Pi}_{\boldsymbol{A}}^H \boldsymbol{w}_{ln}\right] \\
&= \frac{1}{M}\mathrm{Trace}\left[\boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{\Pi}_{\boldsymbol{A}}^H\right]\mathcal{E}\left[\boldsymbol{w}_{ln}^H \boldsymbol{w}_{ln}\right] \\
&= \frac{1}{M}\mathrm{Trace}\left[\boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{\Pi}_{\boldsymbol{A}}^H\right] M\sigma_{\boldsymbol{n}_l}^2 \qquad \text{(see (59))} \\
&= \sigma_{\boldsymbol{n}_l}^2 \mathrm{Trace}\left[\boldsymbol{\Pi}_{\boldsymbol{A}}\right]
\end{aligned}$$

To proceed further on with this expression, we need to calculate:

$$\begin{aligned}
&\mathrm{Trace}\left[\boldsymbol{\Pi}_{\boldsymbol{A}}\right] = \mathrm{Trace}\left[\boldsymbol{A}(\boldsymbol{A}^H\boldsymbol{A})^{-1}\boldsymbol{A}^H\right] \\
&\approx_{M\to\infty} \frac{P}{M}\mathrm{Trace}\left[\boldsymbol{A}\boldsymbol{A}^H\right] = \frac{P}{M}M = P
\end{aligned}$$

And, hence we obtain:

$$\mathcal{E}\left\{ |z_{ln}|^2 \right\}_{M\to\infty} = \sigma_{\boldsymbol{n}_l}^2\left(p_p \frac{M}{P}\overline{\beta}_{ln} + \sigma_{\boldsymbol{n}_l}^2 P \right) \qquad (58)$$

- Concerning: $\mathcal{E}\left(\left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right)$, we have:

$$\begin{aligned}
&\mathcal{E}\left(\left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right) = \mathcal{E}\left(\boldsymbol{\alpha}_{jn}^T \boldsymbol{\alpha}_{jn}^*\right) \\
&= \mathcal{E}\left(\left(\sqrt{p_p p_u}\sum_{i=1}^{L}\boldsymbol{g}_{iln}^H \boldsymbol{A}^H \boldsymbol{A}\boldsymbol{G}_{j,l} + \sqrt{p_u}\boldsymbol{w}_{ln}^H \boldsymbol{A}\boldsymbol{G}_{j,l}\right)\left(\sqrt{p_p p_u}\sum_{m=1}^{L}\boldsymbol{G}_{j,l}^H \boldsymbol{A}^H \boldsymbol{A}\boldsymbol{g}_{mln}^H + \sqrt{p_u}\boldsymbol{G}_{j,l}^H \boldsymbol{A}^H \boldsymbol{w}_{ln}\right)\right) \\
&= p_p p_u \mathcal{E}\left(\sum_{i=1}^{L}\boldsymbol{g}_{iln}^H \boldsymbol{A}^H \boldsymbol{A}\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^H \sum_{m=1}^{L}\boldsymbol{A}^H \boldsymbol{A}\boldsymbol{g}_{mln}\right) + p_u \mathcal{E}\left(\boldsymbol{w}_{ln}^H \boldsymbol{A}\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^H \boldsymbol{A}^H \boldsymbol{w}_{ln}\right) \\
&= p_p p_u \mathcal{E}\left(\sum_{i=1,i\neq j}^{L}\boldsymbol{g}_{iln}^H \boldsymbol{A}^H \boldsymbol{A}\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^H\right]\sum_{m=1}^{L}\boldsymbol{A}^H \boldsymbol{A}\boldsymbol{g}_{mln}\right) + p_p p_u \mathcal{E}\left(\boldsymbol{g}_{jln}^H \boldsymbol{A}^H \boldsymbol{A}\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^H \boldsymbol{A}^H \boldsymbol{A}\boldsymbol{g}_{jln}\right) \\
&\quad + p_u \mathcal{E}\left(\boldsymbol{w}_{ln}^H \boldsymbol{A}\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^H\right]\boldsymbol{A}^H \boldsymbol{w}_{ln}\right)
\end{aligned}$$

Concerning the product: $\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}$, we have:

$$\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}=\begin{bmatrix} g_{jl11} & g_{jl21} & \cdots & g_{jlk1} & \cdots & g_{jlK1} \\ g_{jl12} & g_{jl22} & \cdots & g_{jlk2} & \cdots & g_{jlK2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{jl1m} & g_{jl2m} & \cdots & g_{jlkm} & \cdots & g_{jlKm} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{jl1P} & g_{jl1P} & \cdots & g_{jlkP} & \cdots & g_{jlkP} \end{bmatrix}\begin{bmatrix} g_{jl11}^{*} & g_{jl12}^{*} & \cdots & g_{jl1k}^{*} & \cdots & g_{jl1P}^{*} \\ g_{jl21}^{*} & g_{jl22}^{*} & \cdots & g_{jl2k}^{*} & \cdots & g_{jl2P}^{*} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{jlm1}^{*} & g_{jlm2}^{*} & \cdots & g_{jlmk}^{*} & \cdots & g_{jlmP}^{*} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{jlK1}^{*} & g_{jlK2}^{*} & \cdots & g_{jlKk}^{*} & \cdots & g_{jlKP}^{*} \end{bmatrix}$$

$$=\begin{bmatrix} \sum_{k=1}^{K}\left|g_{jlk1}\right|^{2} & \sum_{k=1}^{K}g_{jlk1}g_{jlk2}^{*} & \cdots & \sum_{k=1}^{K}g_{jlk1}g_{jlkm}^{*} & \cdots & \sum_{k=1}^{K}g_{jlk1}g_{jlkP}^{*} \\ \sum_{k=1}^{K}g_{jlk2}g_{jlk1}^{*} & \sum_{k=1}^{K}\left|g_{jlk2}\right|^{2} & \cdots & \sum_{k=1}^{K}g_{jlk2}g_{jlkm}^{*} & \cdots & \sum_{k=1}^{K}g_{jlk2}g_{jlkP}^{*} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^{K}g_{jlkm}g_{jlk1}^{*} & \sum_{k=1}^{K}g_{jlkm}g_{jlk2}^{*} & \cdots & \sum_{k=1}^{K}\left|g_{jlkm}\right|^{2} & \cdots & \sum_{k=1}^{K}g_{jlkm}g_{jlkP}^{*} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^{K}g_{jlkP}g_{jlk1}^{*} & \sum_{k=1}^{K}g_{jlkP}g_{jlk2}^{*} & \cdots & \sum_{k=1}^{K}g_{jlkP}g_{jlkm}^{*} & \cdots & \sum_{k=1}^{K}\left|g_{jlkP}\right|^{2} \end{bmatrix}$$

So, we obtain:

$$\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\right]=\operatorname{diag}\left[\mathcal{E}\left[\sum_{k=1}^{K}\left|g_{jlk1}\right|^{2}\right],\mathcal{E}\left[\sum_{k=1}^{K}\left|g_{jlk2}\right|^{2}\right],\ldots,\mathcal{E}\left[\sum_{k=1}^{K}\left|g_{jlkm}\right|^{2}\right],\ldots,\mathcal{E}\left[\sum_{k=1}^{K}\left|g_{jlkP}\right|^{2}\right]\right]$$

With:

$$\mathcal{E}\left[\sum_{k=1}^{K}\left|g_{jlkm}\right|^{2}\right]=\sum_{k=1}^{K}\beta_{jlk}\mathcal{E}\left[\left|h_{jlkm}\right|^{2}\right]$$

Thus :

$$\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\right]=\operatorname{diag}\left[\sum_{k=1}^{K}\beta_{jlk}\mathcal{E}\left[\left|h_{jlk1}\right|^{2}\right],\sum_{k=1}^{K}\beta_{jlk}\mathcal{E}\left[\left|h_{jlk2}\right|^{2}\right],\ldots,\sum_{k=1}^{K}\beta_{jlk}\mathcal{E}\left[\left|h_{jlkm}\right|^{2}\right],\ldots,\sum_{k=1}^{K}\beta_{jlk}\mathcal{E}\left[\left|h_{jlkP}\right|^{2}\right]\right]$$

Then, we use the following lemma:

Lemma 1: If we consider a given complex random vector $\boldsymbol{V}$ of size $K \times 1$ with i.i.d random variables of variance σ^2 and a complex deterministic matrix $\boldsymbol{W}$ of size $K \times M$, then we have:

$$\begin{aligned}&\mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{W}\boldsymbol{W}^{H}\boldsymbol{V}\right]\\&=\frac{1}{K}\mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{V}\right]\operatorname{Trace}\left(\boldsymbol{W}\boldsymbol{W}^{H}\right)\end{aligned}\qquad(59)$$

Proof:

We take $\boldsymbol{V}$ as: $\boldsymbol{V} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_K \end{pmatrix}$ and $\boldsymbol{W}$ as: $\boldsymbol{W} = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} & \cdots & w_{1M} \\ w_{21} & w_{22} & \vdots & w_{2n} & \cdots & w_{2M} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{p1} & w_{p2} & \cdots & w_{pn} & \cdots & w_{pM} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{K1} & w_{K2} & \cdots & w_{Kn} & \cdots & w_{KM} \end{pmatrix}$, in these conditions, we have :

$\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{W}$

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} & \cdots & w_{1M} \\ w_{21} & w_{22} & \vdots & w_{2n} & \cdots & w_{2M} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{p1} & w_{p2} & \cdots & w_{pn} & \cdots & w_{pM} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{K1} & w_{K2} & \cdots & w_{Kn} & \cdots & w_{KM} \end{pmatrix}$$

$$= \begin{pmatrix} d_1^* & d_2^* & \cdots & d_m^* & \cdots & d_K^* \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{K} w_{k1} d_k^* & \sum_{k=1}^{K} w_{k2} d_k^* & \cdots & \sum_{k=1}^{K} w_{km} d_k^* & \cdots & \sum_{k=1}^{K} w_{kM} d_k^* \end{pmatrix}$$

And, we have:

$$\mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{W}\boldsymbol{W}^{H}\boldsymbol{V}\right]$$

$$= \mathcal{E}\left(\left|\sum_{k=1}^{K} w_{k1} d_k^*\right|^2 + \left|\sum_{k=1}^{K} w_{k2} d_k^*\right|^2 + \cdots + \left|\sum_{k=1}^{K} w_{km} d_k^*\right|^2 + \cdots + \left|\sum_{k=1}^{K} w_{kM} d_k^*\right|^2 \right)$$

$$= \sum_{k=1}^{K} |w_{k1}|^2 \mathcal{E}\left[\left|d_k^*\right|^2\right] + \sum_{k=1}^{K} |w_{k2}|^2 \mathcal{E}\left[\left|d_k^*\right|^2\right] + \cdots + \sum_{k=1}^{K} |w_{km}|^2 \mathcal{E}\left[\left|d_k^*\right|^2\right] + \cdots + \sum_{k=1}^{K} |w_{kM}|^2 \mathcal{E}\left[\left|d_k^*\right|^2\right]$$

$$= \sigma^2 \left[\sum_{k=1}^{K} |w_{k1}|^2 + |w_{k2}|^2 + \cdots + |w_{km}|^2 + \cdots + |w_{kM}|^2\right]$$

$$= \frac{1}{K} \mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{V}\right] \mathrm{Trace}\left(\boldsymbol{W}\boldsymbol{W}^{H}\right)$$

Remark: It is possible to generalize this lemma to the case of a random matrix $\boldsymbol{V}$ of size $K \times M$, and with a matrix $\boldsymbol{W}$ of size $K \times P$. We still have:

$$\mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{W}\boldsymbol{W}^{H}\boldsymbol{V}\right]$$
$$=\frac{1}{K}\mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{V}\right]\mathrm{Trace}\left(\boldsymbol{W}\boldsymbol{W}^{H}\right)$$

We take $\boldsymbol{V}$ as: $\boldsymbol{V}=\begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1m} & \cdots & d_{1M} \\ d_{21} & d_{22} & \cdots & d_{2m} & \cdots & d_{2M} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{k1} & d_{k2} & \cdots & d_{km} & \cdots & d_{kM} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{K1} & d_{K2} & \cdots & d_{Km} & \cdots & d_{KM} \end{pmatrix}$ and $\boldsymbol{W}$ as:

$\boldsymbol{W}=\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} & \cdots & w_{1P} \\ w_{21} & w_{22} & \vdots & w_{2n} & \cdots & w_{2P} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} & \cdots & w_{nP} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{K1} & w_{K2} & \cdots & w_{Kn} & \cdots & w_{KP} \end{pmatrix}$, in these conditions we have:

$\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{W}$

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} & \cdots & w_{1P} \\ w_{21} & w_{22} & \vdots & w_{2n} & \cdots & w_{2P} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} & \cdots & w_{nP} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{K1} & w_{K2} & \cdots & w_{Kn} & \cdots & w_{KP} \end{pmatrix}$$

$$=\begin{pmatrix} d_{11}^{*} & d_{21}^{*} & \cdots & d_{k1}^{*} & \cdots & d_{K1}^{*} \\ d_{12}^{*} & d_{22}^{*} & \cdots & d_{k2}^{*} & \cdots & d_{K2}^{*} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{1m}^{*} & d_{2m}^{*} & \cdots & d_{km}^{*} & \cdots & d_{Km}^{*} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{1M}^{*} & d_{2M}^{*} & \cdots & d_{kM}^{*} & \cdots & d_{KM}^{*} \end{pmatrix} \Rightarrow \boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{W}(m,n)=\sum_{k=1}^{K} d_{km}^{*} w_{kn} = a_{m,n}$$

The general term of matrix $\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{W}\boldsymbol{W}^{H}\boldsymbol{V}$ is thus written as:

$$\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{W}\boldsymbol{W}^{H}\boldsymbol{V}(m,n)=\sum a_{m,k} b_{k,n} = \sum_{k=1}^{M} a_{m,k} a_{n,k}^{*} = \sum_{k=1}^{M}\left(\sum_{j=1}^{K} d_{jm}^{*} w_{jk}\right)\left(\sum_{i=1}^{K} d_{in} w_{ik}^{*}\right)$$

Hence, we obtain :

$$\mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{W}\,\boldsymbol{W}^{H}\boldsymbol{V}(m,n)\right] = \sum_{k=1}^{M}\mathcal{E}\left[\left(\sum_{j=1}^{K} d_{jm}^{*} w_{jk}\right)\left(\sum_{i=1}^{K} d_{in} w_{ik}^{*}\right)\right] = \sum_{k=1}^{M}\sum_{i}\sum_{j}\mathcal{E}\left[d_{jm}^{*} d_{in}\right] w_{jk} w_{ik}^{*}$$

Clearly, we have: $\mathcal{E}\left[d_{jm}^{*} d_{in}\right] = \sigma^{2}$ if $i = j$ and $\boldsymbol{m} = \boldsymbol{n}$, otherwise $\mathcal{E}\left[d_{jm}^{*} d_{in}\right] = 0$. Hence we obtain:

$$\mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{W}\,\boldsymbol{W}^{H}\boldsymbol{V}\right] = \mathrm{Diag}\left[\sigma^{2}\mathrm{Trace}\left(\boldsymbol{W}\,\boldsymbol{W}^{H}\right),\ldots,\sigma^{2}\mathrm{Trace}\left(\boldsymbol{W}\,\boldsymbol{W}^{H}\right)\right] = \frac{1}{K}\mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{V}\right]\mathrm{Trace}\left(\boldsymbol{W}\,\boldsymbol{W}^{H}\right)$$

$$\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{V}$$

$$\begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} & \cdots & d_{1M} \\ d_{21} & d_{22} & \vdots & d_{2n} & \cdots & d_{2M} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} & \cdots & d_{nM} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{K1} & d_{K2} & \cdots & d_{Kn} & \cdots & d_{KM} \end{pmatrix}$$

$$= \begin{pmatrix} d_{11}^{*} & d_{21}^{*} & \cdots & d_{k1}^{*} & \cdots & d_{K1}^{*} \\ d_{12}^{*} & d_{22}^{*} & \cdots & d_{k2}^{*} & \cdots & d_{K2}^{*} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{1m}^{*} & d_{2m}^{*} & \cdots & d_{km}^{*} & \cdots & d_{Km}^{*} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{1M}^{*} & d_{2M}^{*} & \cdots & d_{kM}^{*} & \cdots & d_{KM}^{*} \end{pmatrix} \Rightarrow \boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{V}(n,n) = \sum_{k=1}^{K} d_{kn}^{*} d_{kn}$$

Hence:

$$\mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{V}(n,n)\right] = K\sigma^{2}$$

And : $\mathcal{E}\left[\boldsymbol{V}^{\boldsymbol{H}}\boldsymbol{V}\right] = K\sigma^{2}\boldsymbol{I}_{K}$ End of proof.

Using this lemma here, we obtain:

$$\mathcal{E}\left(\sum_{i=1,i\neq j}^{L}\boldsymbol{g}_{iln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\right]\sum_{m=1}^{L}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{m\,ln}\right)$$
$$=\frac{1}{P}\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\right]\right]\sum_{i=1,i\neq j}^{L}\mathcal{E}\left[\boldsymbol{g}_{i\,ln}^{H}\,\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{iln}\right]$$
$$=\frac{1}{P^{2}}\left(\sum_{i=1,i\neq j}^{L}\mathcal{E}\left[\boldsymbol{g}_{i\,ln}^{H}\;\boldsymbol{g}_{i\,ln}\right]\right)\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)\right]\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\right]\right]$$
$$=\frac{1}{P^{2}}\left(\sum_{i=1,i\neq j}^{L}\mathcal{E}\left[\boldsymbol{g}_{i\,ln}^{H}\;\boldsymbol{g}_{iln}\right]\right)\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\right]\right]$$

$$\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\right]\right]$$
$$=\mathrm{Trace}\left[\mathrm{diag}\left[\sum_{k=1}^{K}\beta_{jlk}\mathcal{E}\left[\left|h_{jlk1}\right|^{2}\right],\sum_{k=1}^{K}\beta_{jlk}\mathcal{E}\left[\left|h_{jlk2}\right|^{2}\right],\cdots\sum_{k=1}^{K}\beta_{jlk}\mathcal{E}\left[\left|h_{jlkm}\right|^{2}\right],\cdots,\sum_{k=1}^{K}\beta_{jlk}\mathcal{E}\left[\left|h_{jlkP}\right|^{2}\right]\right]\right]$$
$$=\sum_{k=1}^{K}\beta_{jlk}\mathcal{E}\left[\sum_{m=1}^{P}\left|h_{jlkm}\right|^{2}\right]$$
$$=\sum_{k=1}^{K}\beta_{jlk}=\beta_{jl}$$

Hence:

$$\mathcal{E}\left(\sum_{i=1,i\neq j}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\right]\sum_{m=1}^{L}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{m\,ln}\right)$$
$$=\frac{1}{P^{2}}\left(\sum_{i=1,i\neq j}^{L}\mathcal{E}\left[\boldsymbol{g}_{i\,ln}^{H}\;\boldsymbol{g}_{i\,ln}\right]\right)\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]\beta_{jl}$$

And, using:

$$\sum_{i=1,i\neq j}^{L}\mathcal{E}\left[\boldsymbol{g}_{i\,ln}^{H}\;\boldsymbol{g}_{i\,ln}\right]=\sum_{i=1,i\neq j}^{L}\sum_{m=1}^{P}\mathcal{E}\left[\left|g_{ilnm}\right|^{2}\right]=\sum_{i=1,i\neq j}^{L}\beta_{iln}\sum_{m=1}^{P}\mathcal{E}\left[\left|h_{ilnm}\right|^{2}\right]=\sum_{i=1,i\neq j}^{L}\beta_{iln}$$

We eventually obtain:

$$\begin{aligned}&\mathcal{E}\left(\sum_{i=1,i\neq j}^{L}\boldsymbol{g}_{iln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\right]\sum_{m=1}^{L}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{m\,ln}\right)\\&=\frac{1}{P^{2}}\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]\beta_{jl}\sum_{i=1,i\neq j}^{L}\beta_{iln}\end{aligned}\qquad(60)$$

Using the approximation: $\lim_{M\to\infty}\frac{1}{M}\boldsymbol{A}^{H}\boldsymbol{A}=\frac{1}{P}\boldsymbol{I}_{P}$, we obtain:

$$\lim_{M\to\infty}\mathcal{E}\left(\sum_{i=1,i\neq j}^{L}\boldsymbol{g}_{i\,ln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\mathcal{E}\left[\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\right]\sum_{m=1}^{L}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{m\,ln}\right)$$
$$=\frac{M^{2}}{P^{3}}\beta_{jl}\sum_{i=1,i\neq j}^{L}\beta_{iln}=\frac{M^{2}\beta_{jl}\sum_{i=1,i\neq j}^{L}\beta_{iln}}{P^{3}} \tag{61}$$

Concerning the term: $\mathcal{E}\left(\boldsymbol{g}_{j\,ln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{j\,ln}\right)$, we have:

$$\begin{aligned}
&\mathcal{E}\left(\boldsymbol{g}_{jln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\boldsymbol{G}_{j,l}\boldsymbol{G}_{j,l}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jln}\right)\\
&=\mathcal{E}\left(\boldsymbol{g}_{jln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\sum_{p=1}^{K}\boldsymbol{G}_{j,l}(:,p)\boldsymbol{G}_{j,l}^{H}(:,p)\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jln}\right)\\
&=\mathcal{E}\left(\boldsymbol{g}_{jln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\sum_{p=1}^{K}\boldsymbol{g}_{jlp}\boldsymbol{g}_{jlp}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jln}\right)\\
&=\mathcal{E}\left(\boldsymbol{g}_{jln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\sum_{p=1,\,p\neq n}^{K}\boldsymbol{g}_{jlp}\boldsymbol{g}_{jlp}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jln}\right)+\mathcal{E}\left(\boldsymbol{g}_{jln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jln}\boldsymbol{g}_{jln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jln}\right)
\end{aligned} \tag{62}$$

For $p\neq n$ we have:

$$\begin{aligned}
&\mathcal{E}\left(\boldsymbol{g}_{jln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jlp}\boldsymbol{g}_{jlp}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jln}\right)\\
&=\mathcal{E}\left(\boldsymbol{g}_{jln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\mathcal{E}\left[\boldsymbol{g}_{jlp}\boldsymbol{g}_{jlp}^{H}\right]\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jln}\right)\\
&=\frac{1}{P}\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{g}_{jlp}\boldsymbol{g}_{jlp}^{H}\right]\right]\mathcal{E}\left[\boldsymbol{g}_{jln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\;\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jln}\right]\\
&=\frac{1}{P^{2}}\mathcal{E}\left[\boldsymbol{g}_{jln}^{H}\,\boldsymbol{g}_{jln}\right]\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)\right]\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{g}_{jlp}\boldsymbol{g}_{jlp}^{H}\right]\right]\\
&=\frac{1}{P^{2}}\mathcal{E}\left[\boldsymbol{g}_{jln}^{H}\,\boldsymbol{g}_{jln}\right]\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{g}_{jlp}\boldsymbol{g}_{jlp}^{H}\right]\right]
\end{aligned}$$

With:

$$\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{g}_{jlp}\boldsymbol{g}_{jlp}^{H}\right]\right]=\mathcal{E}\left[\left|\boldsymbol{g}_{jlp1}\right|^{2}+\left|\boldsymbol{g}_{jlp2}\right|^{2}+\cdots+\left|\boldsymbol{g}_{jlpP}\right|^{2}\right]=\beta_{jlp}$$

So:

$$\begin{aligned}
&\mathcal{E}\left(\boldsymbol{g}_{jln}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jlp}\boldsymbol{g}_{jlp}^{H}\boldsymbol{A}^{H}\boldsymbol{A}\,\boldsymbol{g}_{jln}\right)\\
&=\frac{1}{P^{2}}\mathcal{E}\left[\boldsymbol{g}_{jln}^{H}\,\boldsymbol{g}_{jln}\right]\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]\beta_{jlp} \quad (63)\\
&=\frac{1}{P^{2}}\,\mathrm{Trace}\left[\left(\boldsymbol{A}^{H}\boldsymbol{A}\right)^{2}\right]\beta_{jlp}\beta_{jln}
\end{aligned}$$

And:

$$
\begin{aligned}
&\mathcal{E}\left(\boldsymbol{g}_{jln}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \sum_{p=1, p\neq n}^{K} \boldsymbol{g}_{jlp} \boldsymbol{g}_{jlp}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{g}_{jln} \right) \\
&= \frac{1}{P^2} \operatorname{Trace}\left[\left(\boldsymbol{A}^{H} \boldsymbol{A}\right)^2 \right] \beta_{jln} \sum_{p=1, p\neq n}^{K} \beta_{jlp} \\
&= \frac{1}{P^2} \operatorname{Trace}\left[\left(\boldsymbol{A}^{H} \boldsymbol{A}\right)^2 \right] \beta_{jln} \sum_{p=1}^{K} \beta_{jlp} - \frac{1}{P^2} \operatorname{Trace}\left[\left(\boldsymbol{A}^{H} \boldsymbol{A}\right)^2 \right] \beta_{jln}^2 \qquad (64) \\
&= \frac{1}{P^2} \operatorname{Trace}\left[\left(\boldsymbol{A}^{H} \boldsymbol{A}\right)^2 \right] \beta_{jln} \beta_{jl} - \frac{1}{P^2} \operatorname{Trace}\left[\left(\boldsymbol{A}^{H} \boldsymbol{A}\right)^2 \right] \beta_{jln}^2 \\
&= \frac{1}{P^2} \operatorname{Trace}\left[\left(\boldsymbol{A}^{H} \boldsymbol{A}\right)^2 \right] \left(\beta_{jln} \beta_{jl} - \beta_{jln}^2\right)
\end{aligned}
$$

The only term we have to evaluate now is:

$$
\begin{aligned}
&\mathcal{E}\left(\boldsymbol{g}_{jln}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{g}_{jln} \boldsymbol{g}_{jln}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{g}_{jln} \right) \\
&= \mathcal{E}\left(\left[\boldsymbol{g}_{jln}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{g}_{jln} \right]^2 \right)
\end{aligned}
$$

We have:

$$
\begin{aligned}
&\boldsymbol{g}_{jln}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{g}_{jln} \\
&= \frac{1}{P} \left\{ \left(\sum_{m=1}^{P} g_{jlnm}^{*} e^{jf_1(\phi_m)} \right) \left(\sum_{m=1}^{P} g_{jlnk} e^{-jf_1(\phi_k)} \right) + \cdots + \left(\sum_{m=1}^{P} g_{jlnm}^{*} e^{jf_M(\phi_m)} \right) \left(\sum_{m=1}^{P} g_{jlnk} e^{-jf_M(\phi_k)} \right) \right\} \\
&= \frac{1}{P} \left\{ \sum_{k} \sum_{m} g_{jlnm}^{*} g_{jlnk} e^{j[f_1(\phi_m) - f_1(\phi_k)]} + \cdots + \sum_{k} \sum_{m} g_{jlnm}^{*} g_{jlnk} e^{j[f_M(\phi_m) - f_M(\phi_k)]} \right\} \\
&= \frac{1}{P} \left\{ A_{m,k}^{1} + A_{m,k}^{2} + \cdots + A_{m,k}^{M} \right\}
\end{aligned}
$$

With : $A_{m,k}^{r} = \sum_{k} \sum_{m} g_{jlnm}^{*} g_{jlnk} e^{j[f_r(\phi_m) - f_r(\phi_k)]}$

And:

$$
\begin{aligned}
&\left[\boldsymbol{g}_{jln}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{g}_{jln} \right]^2 \\
&= \frac{1}{P^2} \left[\left(A_{m,k}^{1}\right)^2 + \left(A_{m,k}^{2}\right)^2 + \cdots + \left(A_{m,k}^{M}\right)^2 + \sum_{r} \sum_{t} A_{m,k}^{r} A_{m,k}^{t} \right]
\end{aligned}
$$

We have:

$$
\begin{aligned}
\mathcal{E}\left[\left(A_{m,k}^{n}\right)^{2}\right] &= \mathcal{E}\left[\left(\sum_{k}\sum_{m} g_{jlnm}^{*} g_{jlnk} e^{j[f_n(\phi_m)-f_n(\phi_k)]}\right)\left(\sum_{s}\sum_{t} g_{jlns}^{*} g_{jlnt} e^{j[f_n(\phi_s)-f_n(\phi_t)]}\right)\right] \\
&= \mathcal{E}\left[\left(\sum_{k}\sum_{m}\sum_{s}\sum_{t} g_{jlnm}^{*} g_{jlnk} g_{jlns}^{*} g_{jlnt} e^{j[f_n(\phi_m)-f_n(\phi_k)+f_n(\phi_s)-f_n(\phi_t)]}\right)\right] \\
&= \sum_{k}\sum_{m}\sum_{s}\sum_{t} \mathcal{E}\left[g_{jlnm}^{*} g_{jlnk} g_{jlns}^{*} g_{jlnt}\right] e^{j[f_n(\phi_m)-f_n(\phi_k)+f_n(\phi_s)-f_n(\phi_t)]} \\
&= \sum_{k}\sum_{s} \mathcal{E}\left[\left|g_{jlnk}\right|^{2}\right] \mathcal{E}\left[\left|g_{jlns}\right|^{2}\right] \\
&= \beta_{jln}^{2}
\end{aligned}
$$

Then, we have to calculate:

$$
\begin{aligned}
\mathcal{E}\left[A_{m,k}^{r} A_{m,k}^{t}\right] &= \mathcal{E}\left[\left(\sum_{k}\sum_{m} g_{jlnm}^{*} g_{jlnk} e^{j[f_r(\phi_m)-f_r(\phi_k)]}\right)\left(\sum_{s}\sum_{p} g_{jlns}^{*} g_{jlnp} e^{j[f_t(\phi_s)-f_t(\phi_p)]}\right)\right] \\
&= \sum\sum\sum\sum \mathcal{E}\left[g_{jlnm}^{*} g_{jlnk} g_{jlns}^{*} g_{jlnp}\right] e^{j[f_r(\phi_m)-f_r(\phi_k)+f_t(\phi_s)-f_t(\phi_p)]} \\
&= \sum\sum \mathcal{E}\left[g_{jlnm}^{*} g_{jlnk}\right] \mathcal{E}\left[g_{jlns}^{*} g_{jlns}\right] \\
&= \beta_{jln}^{2}
\end{aligned}
$$

With these approximations, we obtain:

$$
\begin{aligned}
&\mathcal{E}\left[\boldsymbol{g}_{jln}^{H} \boldsymbol{A}^{H} \boldsymbol{A} \boldsymbol{g}_{jln}\right]^{2} \\
&\approx \frac{1}{P^{2}}\left[M\beta_{jln}^{2} + M^{2}\beta_{jln}^{2}\right] \qquad (65) \\
&= \frac{M(M+1)\beta_{jln}^{2}}{P^{2}}
\end{aligned}
$$

In the same manner we obtain:

$$
\begin{aligned}
&\mathcal{E}\left(\boldsymbol{w}_{ln}^{H} \boldsymbol{A} \mathcal{E}\left[\boldsymbol{G}_{j,l} \boldsymbol{G}_{\boldsymbol{j,l}}^{H}\right] \boldsymbol{A}^{H} \boldsymbol{w}_{ln}\right) \\
&= \frac{1}{P}\mathrm{Trace}\left[\mathcal{E}\left[\boldsymbol{G}_{j,l} \boldsymbol{G}_{\boldsymbol{j,l}}^{H}\right] \mathcal{E}\left[\boldsymbol{w}_{ln}^{H} \boldsymbol{A} \boldsymbol{A}^{H} \boldsymbol{w}_{ln}\right]\right] \\
&= \frac{\beta_{jl}}{P}\frac{1}{P}\sum P\,\mathcal{E}\left[\left|w_{ln}(1)\right|^{2} + \left|w_{ln}(1)\right|^{2} + \left|w_{ln}(M)\right|^{2}\right] \qquad (66)\ \text{(see (68))} \\
&= \frac{M\sigma_{\boldsymbol{n}_l}^{2}\beta_{jl}}{P}
\end{aligned}
$$

We deduce:

$$\mathcal{E}\left(\left\|\boldsymbol{\alpha}_{jn}^{T}\right\|^{2}\right)=\frac{p_p p_u}{P^2}\operatorname{Trace}\left[\left(\boldsymbol{A}^H\boldsymbol{A}\right)^2\right](\bar{\beta}_{ln}\beta_{jl}-\beta_{jl}^2)+\frac{p_p p_u M(M+1)}{P^2}\beta_{jln}^2+\frac{p_u M\sigma_{\boldsymbol{n}_l}^2\beta_{jl}}{P} \quad (67)$$

The calculation of $\boldsymbol{w}_{ln}^H\boldsymbol{A}$ yields:

$$\boldsymbol{w}_{ln}^H\boldsymbol{A}=\frac{1}{\sqrt{P}}\left[w_{ln}^*(1),w_{ln}^*(2),\ldots,w_{ln}^*(M)\right]\begin{bmatrix} e^{-jf_1(\phi_1)} & e^{-jf_1(\phi_2)} & \cdots & e^{-jf_1(\phi_P)} \\ e^{-jf_2(\phi_1)} & e^{-jf_2(\phi_2)} & \cdots & e^{-jf_2(\phi_P)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-jf_M(\phi_1)} & e^{-jf_M(\phi_2)} & \cdots & e^{-jf_M(\phi_P)} \end{bmatrix}$$

$$=\frac{1}{\sqrt{P}}\left[w_{ln}^*(1)e^{-jf_1(\phi_1)}+\ldots+w_{ln}^*(M)e^{-jf_M(\phi_1)},\ldots,w_{ln}^*(1)e^{-jf_1(\phi_P)}+\ldots+w_{ln}^*(M)e^{-jf_M(\phi_P)}\right]$$

And, in these conditions, we have:

$$\begin{aligned}&\mathcal{E}\left[\boldsymbol{w}_{ln}^H\boldsymbol{A}\boldsymbol{A}^H\boldsymbol{w}_{ln}\right]\\&=\frac{1}{P}\sum P\mathcal{E}\left[\left|w_{ln}(1)\right|^2+\left|w_{ln}(2)\right|^2+\cdots+\left|w_{ln}(M)\right|^2\right] \qquad (68)\\&=M\sigma_{\boldsymbol{n}_l}^2\end{aligned}$$

<u>Remark</u> : Using the lemma of appendix 2, we can directly obtain:

$$\mathcal{E}\left[\boldsymbol{w}_{ln}^H\boldsymbol{A}\boldsymbol{A}^H\boldsymbol{w}_{ln}\right]=\frac{1}{M}\mathcal{E}\left[\boldsymbol{w}_{ln}^H\boldsymbol{w}_{ln}\right]\operatorname{Trace}\left[\boldsymbol{A}\boldsymbol{A}^H\right]=\frac{1}{M}M\sigma_{\boldsymbol{n}_l}^2M=M\sigma_{\boldsymbol{n}_l}^2$$

We obtain the uplink rate of the *n*th user in the *l*th cell as:

$$R_{ln}=C\left(\frac{p_p p_u M^2/P^2\beta_{lln}^2}{\sum_{j=1}^{L}\left(p_p p_u/P^2\operatorname{Trace}\left[\left(\boldsymbol{A}^H\boldsymbol{A}\right)^2\right]\left(\bar{\beta}_{ln}\beta_{jl}-\beta_{jln}^2\right)+p_p p_u M(M+1)\beta_{jln}^2/P^2+p_u M\sigma_{\boldsymbol{n}_l}^2\beta_{jl}/P\right)-p_p p_u M^2/P^2\beta_{lln}^2+M\sigma_{\boldsymbol{n}_l}^2/P\left(p_p\bar{\beta}_{ln}+\frac{\sigma_{\boldsymbol{n}_l}^2}{M}\sum_{k=1}^{P}S_{U^2(:,k)}^{f_i}\right)}\right)$$

This can be simplified as:

$$R_{ln}=C\left(\frac{\beta_{lln}^2}{\sum_{j=1}^{L}\left(\operatorname{Trace}\left[\left(\boldsymbol{A}^H\boldsymbol{A}\right)^2\right]\left(\bar{\beta}_{ln}\beta_{jl}-\beta_{jln}^2\right)/M^2+M(M+1)\beta_{jln}^2/M^2+\sigma_{\boldsymbol{n}_l}^2\beta_{jl}P/Mp_p\right)-\beta_{lln}^2+\sigma_{\boldsymbol{n}_l}^2P/M\left(\bar{\beta}_{ln}/p_u+\frac{\sigma_{\boldsymbol{n}_l}^2}{p_p p_u M}\sum_{k=1}^{P}S_{U^2(:,k)}^{f_i}\right)}\right)$$

With: $C(x)=\log_2(1+x)$ and $S_{U^2(:,k)}^{f_i}$ which has been defined in equation (57).

If we examine the behaviour of the former expression for an unlimited number of base station antennas, we have clearly:

$$\lim_{M\to\infty} \frac{1}{M^2} \text{Trace}\left[\boldsymbol{A}^H \boldsymbol{A}\right]^2 = \frac{1}{P^2} \text{Trace}\left[\boldsymbol{I}_P\right] = \frac{1}{P}$$

Concerning the term at the extreme side of the denominator we have to resort to the original definition:

$$\begin{aligned}
\mathcal{E}\left\{\left|z_{ln}\right|^2\right\} &= \sigma_{\boldsymbol{n}_l}^2 \left\{\mathcal{E}\left[p_p \sum_{i=1}^{L} \boldsymbol{g}_{iln}^H \boldsymbol{A}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{A}\, \boldsymbol{g}_{iln}\right] + \mathcal{E}\left[\boldsymbol{w}_{ln}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{\Pi}_{\boldsymbol{A}}^H \boldsymbol{w}_{ln}\right]\right\} \\
&= \sigma_{\boldsymbol{n}_l}^2 \left\{\mathcal{E}\left[p_p \sum_{i=1}^{L} \boldsymbol{g}_{iln}^H \boldsymbol{A}^H \boldsymbol{A}(\boldsymbol{A}^H \boldsymbol{A})^{-1} \boldsymbol{A}^H \boldsymbol{A}\, \boldsymbol{g}_{iln}\right] + \mathcal{E}\left[\boldsymbol{w}_{ln}^H \boldsymbol{\Pi}_{\boldsymbol{A}} \boldsymbol{w}_{ln}\right]\right\} \\
&= \sigma_{\boldsymbol{n}_l}^2 \left\{\mathcal{E}\left[p_p \sum_{i=1}^{L} \boldsymbol{g}_{iln}^H \boldsymbol{A}^H \boldsymbol{A}\ \boldsymbol{g}_{iln}\right] + \mathcal{E}\left[\boldsymbol{w}_{ln}^H \boldsymbol{A}(\boldsymbol{A}^H \boldsymbol{A})^{-1} \boldsymbol{A}^H \boldsymbol{w}_{ln}\right]\right\}
\end{aligned}$$

We have:

$$\mathcal{E}\left[\boldsymbol{w}_{ln}^H \boldsymbol{A}(\boldsymbol{A}^H \boldsymbol{A})^{-1} \boldsymbol{A}^H \boldsymbol{w}_{ln}\right] \approx_{M\to\infty} \mathcal{E}\left[\boldsymbol{w}_{ln}^H \boldsymbol{A} \frac{P}{M} \boldsymbol{I}_P\, \boldsymbol{A}^H \boldsymbol{w}_{ln}\right] = \frac{P}{M} \mathcal{E}\left[\boldsymbol{w}_{ln}^H \boldsymbol{A}\ \boldsymbol{A}^H \boldsymbol{w}_{ln}\right]$$

We have (see (68)): $\mathcal{E}\left[\boldsymbol{w}_{ln}^H \boldsymbol{A}\boldsymbol{A}^H \boldsymbol{w}_{ln}\right] = M\sigma_{\boldsymbol{n}_l}^2$, so we obtain :

$$\mathcal{E}\left[\boldsymbol{w}_{ln}^H \boldsymbol{A}(\boldsymbol{A}^H \boldsymbol{A})^{-1} \boldsymbol{A}^H \boldsymbol{w}_{ln}\right] \approx_{M\to\infty} P\sigma_{\boldsymbol{n}_l}^2$$

Reporting all these obtained equivalent values into the former expression, we obtain:

$$R_{ln} \approx_{M\to\infty} C\left(\frac{\beta_{lln}^2}{\sum_{j=1}^{L}\left(\left(\bar{\beta}_{ln}\beta_{jl} - \beta_{jln}^2\right)/P + \beta_{jln}^2\right) - \beta_{lln}^2}\right) \quad (69)$$

Which can be rewritten as:

$$R_{ln} \approx_{M\to\infty} C\left(\frac{\beta_{lln}^2}{\bar{\beta}_{ln}\sum_{j=1}^{L}\beta_{jl}/P + \sum_{j=1}^{L}\beta_{jln}^2(1-1/P) - \beta_{lln}^2}\right) \quad (70)$$

If P also goes to infinity, the lower bound on the achievable rate becomes:

$$R_{ln} \approx_{M \to \infty} C \left(\frac{\beta_{lln}^2}{\sum_{j=1, j \neq l}^{L} \beta_{jln}^2} \right) \qquad (71)$$