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I. **Note Technique: CSI estimation in massive MIMO systems and pilot contamination for different kinds of receiver architectures**

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1- Context and System:

We consider L cells, where each cell contains one base station equipped with M antennas and K single-antenna users. Assume that the L base stations share the same frequency band. We consider uplink transmission, where the l th base station receives signals from all users in all cells (Fig.1). Then, the $M \times 1$ received vector at the l th base station is given by:

$$\mathbf{y}_l = \sqrt{p_u} \sum_{i=1}^L \mathbf{\Gamma}_{i,l} \mathbf{x}_i + \mathbf{n}_l \quad (1)$$

Where $\mathbf{\Gamma}_{i,l}$ represents the $M \times K$ channel matrix between the l th base station and the K users in the i th cell, i.e., $[\mathbf{\Gamma}_{i,l}]_{m,k}$ is the channel coefficient between the m th antenna of the l th base station and the k th user in the i th cell. $\sqrt{p_u} \mathbf{x}_i$ is the $K \times 1$ transmitted vector of K users in the i th cell (the average power used by each user is p_u) and \mathbf{n}_l contains $M \times 1$ additive white Gaussian noise (AWGN) samples. We assume that the elements of \mathbf{n}_l are Gaussian distributed with zero mean and variance: $\mathcal{E}[\mathbf{n}_l \mathbf{n}_l^H] = \sigma_{n_l}^2 \mathbf{I}_M$ and $\mathcal{E}[\mathbf{n}_l^H \mathbf{n}_l] = M \sigma_{n_l}^2$.

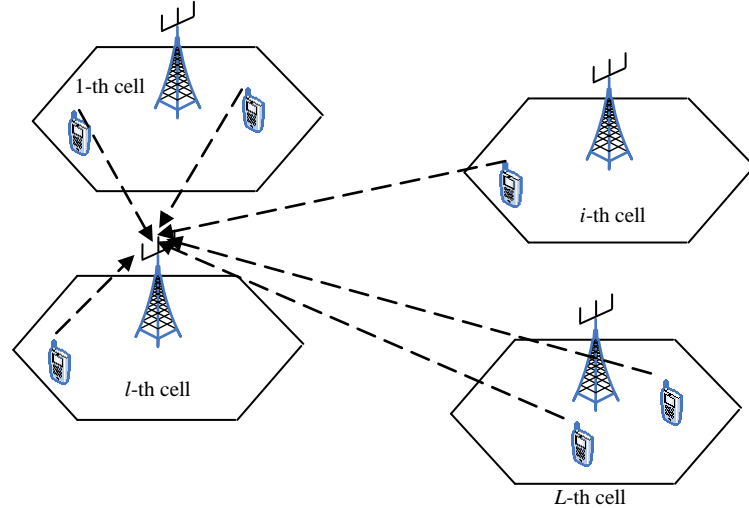


Fig. 1: Uplink transmission in multicell MU-MIMO system

2- Physical Channel Model:

Here, we introduce the finite-dimensional channel model that will be used throughout this technical note. The angular domain is divided into a large but finite number of directions P .

P is fixed regardless of the number of base station antennas ($P < M$). Each direction, corresponding to the angle: $\phi_k, \phi_k \in [-\pi/2, \pi/2], k = 1, \dots, P$, is associated with an $M \times 1$ array steering vector $\mathbf{a}(\phi_k)$ which is given by:

$$\mathbf{a}(\phi_k) = \frac{1}{\sqrt{P}} \left[e^{-j f_1(\phi_k)}, e^{-j f_2(\phi_k)}, \dots, e^{-j f_M(\phi_k)} \right]^T \quad (2)$$

Where $f_i(\phi)$ is some function of ϕ . Typically, for a regular phased array, we have:

$$f_i(\phi) = 2\pi \frac{d}{\lambda} (i-1) \sin(\phi)$$

The channel vector from k th user in the i th cell to the l th base station is then a linear combination of the steering vectors as follows: $\sum_{m=1}^P g_{ilk m} \mathbf{a}(\phi_m)$, where $g_{ilk m}$ is the propagation coefficient from the k th user of the i th cell to the l th base station, associated with the physical direction m (direction of arrival ϕ_m). Let $\mathbf{G}_{i,l} \equiv [\mathbf{g}_{i,l1}, \dots, \mathbf{g}_{i,lK}]$ be a $P \times K$ matrix with $\mathbf{g}_{ilk} \equiv [g_{ilk1}, \dots, g_{ilkP}]^T$ that contains the path gains from the k th user in the i th cell to the l th base station. The elements of $\mathbf{G}_{i,l}$ are assumed to be independent. Then, the $M \times K$ channel matrix between the l th base station and the K users in the i th cell is:

$$\mathbf{\Gamma}_{i,l} = \mathbf{A} \mathbf{G}_{i,l} \quad (3)$$

Where $\mathbf{A} \equiv [\mathbf{a}(\phi_1), \dots, \mathbf{a}(\phi_P)]$ is a full rank $M \times P$ matrix. The propagation channel $\mathbf{G}_{i,l}$ models independent fast fading, geometric attenuation, and log-normal shadow fading. Its elements $g_{ilk m}$ are given by:

$$g_{ilk m} = h_{ilk m} \sqrt{\beta_{ilk}} \quad m = 1, 2, \dots, P \quad (4)$$

Where $h_{ilk m}$ is a fast fading coefficient assumed to be zero mean and have unit variance. $\sqrt{\beta_{ilk}}$ models the path loss and shadowing which are assumed to be independent of the direction m and to be constant and known a priori. This assumption is reasonable since the value β_{ilk} changes very slowly with time. Then, we have:

$$\mathbf{G}_{i,l} = \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \quad (5)$$

Where $\mathbf{H}_{i,l}$ is the $P \times K$ matrix of fast fading coefficients between the K users in the i th cell and the l th base station, the columns $\mathbf{H}_{i,l}(:, k)$ are independent and we have:

$$\mathcal{E}\left[\mathbf{H}_{i,l}^H(:,k)\mathbf{H}_{i,l}(:,k)\right] = \mathcal{E}\left[\mathbf{h}_{ilk}^H\mathbf{h}_{ilk}\right] = \mathcal{E}\left[\sum_{m=1}^P|h_{il}(m,k)|^2\right] = \mathcal{E}\left[\sum_{m=1}^P|h_{ilkml}|^2\right] = 1$$

Furthermore within one given column we consider the elements: $h_{il}(m,k)$ independent, i.e.:

$\mathcal{E}[h_{ilkml}h_{ilknl}^*] = 0$ if $m \neq n$. $\mathbf{D}_{i,l}$ is a $K \times K$ diagonal matrix whose diagonal elements are given by $[\mathbf{D}_{i,l}]_{k,k} = \beta_{ilk}$. Therefore, (1) can be written as:

$$\mathbf{y}_l = \sqrt{p_u}\mathbf{A}\sum_{i=1}^L\mathbf{G}_{i,l}\mathbf{x}_i + \mathbf{n}_l = \sqrt{p_u}\mathbf{A}\sum_{i=1}^L\mathbf{H}_{i,l}\mathbf{D}_{i,l}^{1/2}\mathbf{x}_i + \mathbf{n}_l \quad (6)$$

In the following sections we will use the following intermediate parameters: $\beta_{jl} = \sum_{k=1}^K\beta_{jlk}$,

$$\bar{\beta}_{ln} = \sum_{i=1}^L\beta_{iln}, \text{ and } \hat{\beta}_l = \sum_{i=1}^L\beta_{il}.$$

3- Channel Estimation:

Channel estimation is performed by using training sequences received on the uplink. A part of the coherence interval is used for the uplink training. All users in all cells simultaneously transmit pilot sequences of length τ symbols. The assumption on synchronized transmission represents the worst case from the pilot contamination point of view, but it makes no fundamental difference to assume unsynchronized transmission. We assume that the same set of pilot sequences is used in all L cells. Therefore, the pilot sequences used in the l th cell can be represented by a $K \times \tau$ matrix $\sqrt{p_p}\boldsymbol{\phi}_l = \sqrt{p_p}\boldsymbol{\phi}$ ($\tau \geq K$), which satisfies $\boldsymbol{\phi}\boldsymbol{\phi}^H = \mathbf{I}_K$ where: $p_p = \tau p_u$. From (6), the received pilot matrix at the l th base can be written as:

$$\mathbf{Y}_{p,l} = \sqrt{p_p}\mathbf{A}\sum_{i=1}^L\mathbf{H}_{i,l}\mathbf{D}_{i,l}^{1/2}\boldsymbol{\phi} + \mathbf{N}_l \quad (7)$$

\mathbf{N}_l is the resulting noise matrix of size $M \times \tau$ and of course we have: $\mathcal{E}[\mathbf{N}_l] = \mathbf{0}_{M \times \tau}$
 $\mathcal{E}[\mathbf{N}_l\mathbf{N}_l^H] = \tau\sigma_n^2\mathbf{I}_M$ and $\mathcal{E}[\mathbf{N}_l^H\mathbf{N}_l] = M\sigma_n^2\mathbf{I}_\tau$.

3.1. Minimum Mean-Square Error (MMSE) Estimation:

We assume that the base station uses here MMSE estimation. The received pilot matrix $\mathbf{Y}_{p,l}$ is multiplied by $\boldsymbol{\phi}^H$ at the right side to obtain: $\tilde{\mathbf{Y}}_{p,l} = \mathbf{Y}_{p,l}\boldsymbol{\phi}^H$:

$$\begin{aligned}
\tilde{\mathbf{Y}}_{p,l} &= \mathbf{Y}_{p,l} \boldsymbol{\phi}^H \\
&= \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{N}_l \boldsymbol{\phi}^H \quad (8) \\
&= \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{W}_l
\end{aligned}$$

With: $\mathbf{W}_l = \mathbf{N}_l \boldsymbol{\phi}^H$. Since we consider here M -PSK modulation format for the transmit pilot matrix we can denote $\boldsymbol{\phi}$ as:

$$\boldsymbol{\phi} = \frac{1}{\sqrt{\tau}} \begin{bmatrix} e^{j\theta_{11}} & e^{j\theta_{12}} & \dots & e^{j\theta_{1p}} & \dots & e^{j\theta_{1\tau}} \\ e^{j\theta_{21}} & e^{j\theta_{22}} & \dots & e^{j\theta_{2p}} & \dots & e^{j\theta_{2\tau}} \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ e^{j\theta_{m1}} & e^{j\theta_{m2}} & \dots & e^{j\theta_{mp}} & \dots & e^{j\theta_{m\tau}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{j\theta_{K1}} & e^{j\theta_{K2}} & \dots & e^{j\theta_{Kp}} & \dots & e^{j\theta_{K\tau}} \end{bmatrix}$$

Hence, we have: $\phi(k, m) = a_{k,m} = \frac{1}{\sqrt{\tau}} e^{j\theta_{k,m}}$ and $\phi^H(n, k) = b_{n,k} = \frac{1}{\sqrt{\tau}} e^{-j\theta_{k,n}}$. The matrix product $\mathbf{N}_l \boldsymbol{\phi}^H$ yields:

$$\mathbf{N}_l \boldsymbol{\phi}^H(n, m) = \sum_{k=1}^{\tau} n_l(n, k) \phi^H(k, m) = \frac{1}{\sqrt{\tau}} \sum_{k=1}^{\tau} n_l(n, k) e^{-j\theta_{m,k}}$$

In the same way we obtain:

$$\boldsymbol{\phi} \mathbf{N}_l^H(n, m) = \sum_{k=1}^{\tau} \phi(n, k) n_l^*(m, k) = \frac{1}{\sqrt{\tau}} \sum_{k=1}^{\tau} n_l^*(m, k) e^{j\theta_{n,k}}$$

The matrix product: $\mathbf{W}_l \mathbf{W}_l^H = \mathbf{N}_l \boldsymbol{\phi}^H \mathbf{N}_l \boldsymbol{\phi} \mathbf{N}_l^H$ is thus equal to:

$$\begin{aligned}
\mathbf{N}_l \boldsymbol{\phi}^H \boldsymbol{\phi} \mathbf{N}_l^H(n, m) &= \sum_{k=1}^K \mathbf{N}_l \boldsymbol{\phi}^H(n, k) \boldsymbol{\phi} \mathbf{N}_l^H(k, m) \\
&= \sum_{k=1}^K \frac{1}{\sqrt{\tau}} \sum_{s=1}^{\tau} n_l(n, s) e^{-j\theta_{k,s}} \frac{1}{\sqrt{\tau}} \sum_{r=1}^{\tau} n_l^*(m, r) e^{j\theta_{k,r}} \\
&= \frac{1}{\tau} \sum_{k=1}^K \sum_{s=1}^{\tau} \sum_{r=1}^{\tau} n_l(n, s) n_l^*(m, r) e^{j(\theta_{k,r} - \theta_{k,s})}
\end{aligned}$$

Taking the expectation, we obtain:

$$\mathcal{E}\left[N_l \boldsymbol{\phi}^H \boldsymbol{\phi} N_l^H(n, m)\right] = \frac{1}{\tau} \sum_{k=1}^K \sum_{s=1}^{\tau} \sum_{r=1}^{\tau} \mathcal{E}\left[n_l(n, s) n_l^*(m, r)\right] e^{j(\theta_{k,r} - \theta_{k,s})}$$

Clearly, $\mathcal{E}\left[n_l(n, s) n_l^*(m, r)\right] = \sigma_{n_l}^2$ if and only if: $m=n$ and $r=s$. In these conditions we obtain:

$$\mathcal{E}\left[N_l \boldsymbol{\phi}^H \boldsymbol{\phi} N_l^H(n, n)\right] = \frac{1}{\tau} \sum_{k=1}^K \tau \sigma_{n_l}^2 = K \sigma_{n_l}^2$$

We eventually obtain: $\mathcal{E}\left[\mathbf{W}_l \mathbf{W}_l^H\right] = \mathcal{E}\left[N_l \boldsymbol{\phi}^H \boldsymbol{\phi} N_l^H\right] = K \sigma_{n_l}^2 \mathbf{I}_M$

Concerning the other quantity $\mathcal{E}\left[\mathbf{W}_l^H \mathbf{W}_l\right]$, we can obtain directly:

$$\mathcal{E}\left[\mathbf{W}_l^H \mathbf{W}_l\right] = \mathcal{E}\left[\boldsymbol{\phi} N_l^H N_l \boldsymbol{\phi}^H\right] = \boldsymbol{\phi} \mathcal{E}\left[N_l^H N_l\right] \boldsymbol{\phi}^H = M \sigma_{n_l}^2 \mathbf{I}_K$$

Remark : If we only consider one column (typically the n th column) of \mathbf{W}_l denoted as \mathbf{w}_{ln} , we have :

$$\mathcal{E}\left\{\mathbf{w}_{ln} \mathbf{w}_{ln}^H\right\} = \sigma_{n_l}^2 \mathbf{I}_M \text{ and } \mathcal{E}\left\{\mathbf{w}_{ln}^H \mathbf{w}_{ln}\right\} = M \sigma_{n_l}^2$$

We are now coming back to the channel estimation problem. Since $\mathbf{H}_{l,l}$ has independent columns, we can estimate each column of $\mathbf{H}_{l,l}$ independently. Let $\tilde{\mathbf{y}}_{pln}$ be the n th column of $\tilde{\mathbf{Y}}_{p,l}$. Then:

$$\tilde{\mathbf{y}}_{pln} = \sqrt{p_p} \mathbf{A} \mathbf{h}_{lln} \beta_{lln}^{1/2} + \sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln} \quad (9)$$

Where \mathbf{h}_{iln} and \mathbf{w}_{ln} are the n th column of $\mathbf{H}_{i,l}$ and \mathbf{W}_l respectively. Denote by:

$\mathbf{z}_{ln} \equiv \sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln}$, then the MMSE estimate of \mathbf{h}_{iln} is given by :

$$\hat{\mathbf{h}}_{lln} = \beta_{lln}^{1/2} \sqrt{p_p} \mathbf{A}^H \left(p_p \beta_{lln} \mathbf{A} \mathbf{A}^H + \mathbf{R}_{z_{ln}} \right)^{-1} \tilde{\mathbf{y}}_{pln} \quad (10)$$

With: $\mathbf{R}_{z_{ln}} = \mathcal{E}\left\{\mathbf{z}_{ln} \mathbf{z}_{ln}^H\right\} = p_p \mathbf{A} \mathbf{A}^H \sum_{i=1, i \neq l}^L \beta_{iln} + \sigma_{n_l}^2 \mathbf{I}_M \quad (11)$

Reporting (11) into (10), we obtain:

$$\begin{aligned}\hat{\mathbf{h}}_{lln} &= \beta_{lln}^{1/2} \sqrt{p_p} \mathbf{A}^H \left(p_p \beta_{lln} \mathbf{A} \mathbf{A}^H + p_p \mathbf{A} \mathbf{A}^H \sum_{i=1, i \neq l}^L \beta_{iln} + \sigma_{n_i}^2 \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln} \\ &= \beta_{lln}^{1/2} \sqrt{p_p} \mathbf{A}^H \left(p_p \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} + \sigma_{n_i}^2 \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln}\end{aligned}$$

We can rewrite $\hat{\mathbf{h}}_{lln}$ in the following form :

$$\begin{aligned}\hat{\mathbf{h}}_{lln} &= \frac{\beta_{lln}^{1/2} \sqrt{p_p} \mathbf{A}^H}{\sigma_{n_i}^2} \left(\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln} \mathbf{A} \mathbf{A}^H + \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln} \\ &= \frac{\beta_{lln}^{1/2} \sqrt{p_p}}{\sigma_{n_i}^2} \mathbf{A}^H \left(\sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A} \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A}^H + \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln} \\ &= \left(\frac{\beta_{lln}^{1/2} \sqrt{p_p}}{\sigma_{n_i}^2} \sqrt{\frac{\sigma_{n_i}^2}{p_p \sum_{i=1}^L \beta_{iln}}} \right) \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A}^H \left(\sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A} \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A}^H + \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln} \\ &= \left(\frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{n_i}^2 \sum_{i=1}^L \beta_{iln}}} \right) \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A}^H \left(\sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A} \sqrt{\frac{p_p}{\sigma_{n_i}^2} \sum_{i=1}^L \beta_{iln}} \mathbf{A}^H + \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{pln}\end{aligned}$$

The former equation can be written: $\hat{\mathbf{h}}_{lln} = \lambda \mathbf{B}^H (\mathbf{B} \mathbf{B}^H + \mathbf{I}_M)^{-1} \tilde{\mathbf{y}}_{pln}$, with: $\mathbf{B} = \sqrt{\frac{p_p}{\sigma_{n_i}^2}} \sqrt{\sum_{i=1}^L \beta_{iln}} \mathbf{A}$,

and $\lambda = \frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{n_i}^2 \sum_{i=1}^L \beta_{iln}}}$. Using the matrix inversion lemma, we have:

$$\mathbf{P} (\mathbf{I}_m + \mathbf{Q} \mathbf{P})^{-1} = (\mathbf{I}_n + \mathbf{P} \mathbf{Q})^{-1} \mathbf{P}$$

With the following sizes for \mathbf{P} and \mathbf{Q} : $[\mathbf{Q}]_{m \times n}, [\mathbf{P}]_{n \times m}$. We use this lemma with: $\mathbf{P} = \mathbf{B}^H$, $\mathbf{Q} = \mathbf{B}$, we obtain: $\mathbf{B}^H (\mathbf{B} \mathbf{B}^H + \mathbf{I}_M)^{-1} = (\mathbf{I}_P + \mathbf{B}^H \mathbf{B})^{-1} \mathbf{B}^H$, so we eventually obtain:

$$\begin{aligned}
\hat{\mathbf{h}}_{lln} &= \frac{\beta_{lln}^{1/2}}{\sqrt{\sigma_{n_l}^2 \sum_{i=1}^L \beta_{iln}}} \left(\frac{p_p}{\sigma_{n_l}^2} \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} + \mathbf{I}_P \right)^{-1} \sqrt{\frac{p_p}{\sigma_{n_l}^2}} \sqrt{\sum_{i=1}^L \beta_{iln}} \mathbf{A}^H \tilde{\mathbf{y}}_{pln} \\
\hat{\mathbf{h}}_{lln} &= \frac{\sqrt{p_p} \beta_{lln}^{1/2}}{\sigma_{n_l}^2} \left(\frac{p_p}{\sigma_{n_l}^2} \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} + \mathbf{I}_P \right)^{-1} \mathbf{A}^H \tilde{\mathbf{y}}_{pln} \\
\hat{\mathbf{h}}_{lln} &= \sqrt{p_p} \beta_{lln}^{1/2} \left(p_p \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} + \sigma_{n_l}^2 \mathbf{I}_P \right)^{-1} \mathbf{A}^H \tilde{\mathbf{y}}_{pln}
\end{aligned} \tag{12}$$

The k th diagonal element of $p_p \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln}$ in (12) equals: $\frac{M p_p}{P} \sum_{i=1}^L \beta_{iln}$. Since the uplink is typically interference-limited we have: $\frac{M p_p}{P} \sum_{i=1}^L \beta_{iln} \gg \sigma_{n_l}^2$, therefore, $\hat{\mathbf{h}}_{lln}$ can be approximated as:

$$\hat{\mathbf{h}}_{lln} = \beta_{lln}^{1/2} \sqrt{p_p} \left(p_p \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} \right)^{-1} \mathbf{A}^H \tilde{\mathbf{y}}_{pln} \tag{13}$$

Thus, the MMSE estimate of $\mathbf{H}_{l,l}$ is:

$$\hat{\mathbf{H}}_{l,l} = p_p^{-1/2} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll}^{1/2} \tag{14}$$

Where: $\mathbf{D}_l \equiv \sum_{i=1}^L \mathbf{D}_{il}$. Then, the estimate of the physical channel matrix between the l th base station and the K users in the l th cell is given by:

$$\hat{\Gamma}_{l,l} = \mathbf{A} \hat{\mathbf{H}}_{l,l} \mathbf{D}_{ll}^{1/2} = p_p^{-1/2} \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll} \tag{15}$$

Where $\mathbf{\Pi}_A \equiv \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H$ is the orthogonal projection onto \mathbf{A} . We can see that since post-multiplication of $\mathbf{Y}_{p,l}$ with $\boldsymbol{\phi}^H$ means just multiplication with the pseudo-inverse ($\boldsymbol{\phi} \boldsymbol{\phi}^H = \mathbf{I}_K$), $\tilde{\mathbf{Y}}_{p,l}$ is the conventional least-squares channel estimate. The MMSE channel estimator that we derived thus performs conventional channel estimation and then projects the estimate onto the physical (beam-space) model for the array.

3.2. Bayesian estimator with MAP rule:

We start from equation (7):

$$\tilde{\mathbf{Y}}_{p,l} = \mathbf{Y}_{p,l} \boldsymbol{\phi}^H = \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{W}_l$$

Let $\tilde{\mathbf{y}}_{pln}$ be the n th column of $\tilde{\mathbf{Y}}_{p,l}$. Then:

$$\tilde{\mathbf{y}}_{pln} = \sqrt{P_p} \mathbf{A} \mathbf{h}_{lln} \beta_{lln}^{1/2} + \sqrt{P_p} \mathbf{A} \sum_{i=1, i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln} \quad (16)$$

Where \mathbf{h}_{iln} and \mathbf{w}_{ln} are the n th column of $\mathbf{H}_{i,l}$ and \mathbf{W}_l respectively. Denote by: $\mathbf{z}_{ln} \equiv \sqrt{P_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln}$, we can write:

$$\tilde{\mathbf{y}}_{pln} = \sqrt{P_p} \mathbf{A} \mathbf{h}_{lln} \beta_{lln}^{1/2} + \mathbf{z}_{ln} \quad (17)$$

Applying Bayes' rule, the conditional distribution of the channel \mathbf{h}_{lln} , given the observed received training signal $\tilde{\mathbf{y}}_{pln}$, is:

$$p(\mathbf{h}_{lln} | \tilde{\mathbf{y}}_{pln}) = \frac{p(\mathbf{h}_{lln}) p(\tilde{\mathbf{y}}_{pln} | \mathbf{h}_{lln})}{p(\tilde{\mathbf{y}}_{pln})} = p(\mathbf{h}_{lln}) p(\tilde{\mathbf{y}}_{pln} | \mathbf{h}_{lln}) \quad (18)$$

We use the Gaussian probability density function (PDF) of the random vector \mathbf{h}_{lln} and assume its elements: $h_{lln}(1), \dots, h_{lln}(i), \dots, h_{lln}(P)$, are mutually independent, giving the joint pdf :

$$p(\mathbf{h}_{lln}) = \frac{\exp[-\mathbf{h}_{lln}^H \mathbf{R}_{h_{lln}}^{-1} \mathbf{h}_{lln}]}{(\sqrt{2\pi})^P \det \mathbf{R}_{h_{lln}}} \quad (19)$$

Since the elements: $h_{lln}(1), \dots, h_{lln}(i), \dots, h_{lln}(P)$ are mutually independent this reduces to:

$$p(\mathbf{h}_{lln}) = \frac{\exp[-\mathbf{h}_{lln}^H \mathbf{h}_{lln}]}{(\sqrt{2\pi})^P} \quad (20)$$

We also have:

$$p(\tilde{\mathbf{y}}_{pln} | \mathbf{h}_{lln}) = \frac{\exp[-(\tilde{\mathbf{y}}_{pln} - \sqrt{P_p} \mathbf{A} \mathbf{h}_{lln} \beta_{lln}^{1/2})^H \mathbf{R}_{z_{ln}}^{-1} (\tilde{\mathbf{y}}_{pln} - \sqrt{P_p} \mathbf{A} \mathbf{h}_{lln} \beta_{lln}^{1/2})]}{\pi^P \det \mathbf{R}_{z_{ln}}} \quad (21)$$

Combining (20) and (21) and reporting into (18) we obtain:

$$p(\mathbf{h}_{lln} | \tilde{\mathbf{y}}_{pln}) = \frac{\exp(-f(\mathbf{h}_{lln}))}{(\sqrt{2\pi})^P \pi^P \det \mathbf{R}_{z_{ln}}} \quad (22)$$

With: $\mathbf{R}_{z_{ln}} = \mathcal{E} \left[(\sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln}) (\sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln})^H \right]$ and:
 $f(\mathbf{h}_{ln}) = \mathbf{h}_{ln}^H \mathbf{h}_{ln} + (\tilde{\mathbf{y}}_{p_{ln}} - \sqrt{p_p} \mathbf{A} \mathbf{h}_{ln} \beta_{ln}^{1/2})^H \mathbf{R}_{z_{ln}}^{-1} (\tilde{\mathbf{y}}_{p_{ln}} - \sqrt{p_p} \mathbf{A} \mathbf{h}_{ln} \beta_{ln}^{1/2})$

The ML estimation $\hat{\mathbf{h}}_{ln}$ of \mathbf{h}_{ln} is given by:

$$\begin{aligned} \hat{\mathbf{h}}_{ln} &= \arg \max_{\mathbf{h}_{ln} \in \mathbb{C}^{p \times 1}} \exp(-f(\mathbf{h}_{ln})) \\ \hat{\mathbf{h}}_{ln} &= \arg \min_{\mathbf{h}_{ln} \in \mathbb{C}^{p \times 1}} f(\mathbf{h}_{ln}) \\ \hat{\mathbf{h}}_{ln} &= \beta_{ln}^{1/2} \sqrt{p_p} \mathbf{A}^H \left(p_p \beta_{ln} \mathbf{A} \mathbf{A}^H + \mathbf{R}_{z_{ln}} \right)^{-1} \tilde{\mathbf{y}}_{p_{ln}} \end{aligned} \quad (23)$$

We have clearly:

$$\begin{aligned} \mathbf{R}_{z_{ln}} &= \mathcal{E} \left[(\sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln}) (\sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{h}_{iln} \beta_{iln}^{1/2} + \mathbf{w}_{ln})^H \right] \\ \mathbf{R}_{z_{ln}} &= \sigma_{n_l}^2 \mathbf{I}_M + p_p \sum_{i \neq l}^L \beta_{iln} \mathbf{A} \mathbf{A}^H \end{aligned} \quad (24)$$

And we find again:

$$\hat{\mathbf{h}}_{ln} = \beta_{ln}^{1/2} \sqrt{p_p} \mathbf{A}^H \left(p_p \mathbf{A} \mathbf{A}^H \sum_{i=1}^L \beta_{iln} + \sigma_{n_l}^2 \mathbf{I}_M \right)^{-1} \tilde{\mathbf{y}}_{p_{ln}} \quad (25)$$

To obtain (23) and (25) we have applied (39) and (40) (see Appendix) with: $\mathbf{R}_h = \mathbf{I}$, $\mathbf{R}_n = \mathbf{R}_{z_{ln}}$,
 $\mathbf{S} = \sqrt{p_p} \beta_{ln}^{1/2} \mathbf{A}$.

3.3. Bayesian estimation with vectorized model:

We start from equation (8):

$$\begin{aligned} \tilde{\mathbf{Y}}_{p,l} &= \mathbf{Y}_{p,l} \boldsymbol{\phi}^* = \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{N}_l \boldsymbol{\phi}^* = \sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{W}_l \\ \tilde{\mathbf{Y}}_{p,l} &= \sqrt{p_p} \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} + \sqrt{p_p} \mathbf{A} \sum_{i \neq l}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{W}_l \end{aligned}$$

Starting from this equation we begin by vectorizing the received matrix $\tilde{\mathbf{Y}}_{p,l}$ of size $M \times K$, we obtain the vector:

$$\tilde{\mathbf{y}}_{p,l} = \begin{pmatrix} \tilde{\mathbf{Y}}_{p,l}(:,1) \\ \vdots \\ \tilde{\mathbf{Y}}_{p,l}(:,k) \\ \vdots \\ \tilde{\mathbf{Y}}_{p,l}(:,K) \end{pmatrix}_{MK \times 1} \quad (26)$$

With:

$$\begin{aligned} \tilde{\mathbf{Y}}_{p,l}(:,k) &= \sqrt{p_p} \left[\mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \right](:,k) + \sqrt{p_p} \left[\mathbf{A} \sum_{i \neq l}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \right](:,k) + \mathbf{W}_l(:,k) \\ \tilde{\mathbf{Y}}_{p,l}(:,k) &= \sqrt{p_p} \left[\mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \right](:,k) + \mathbf{Z}_l(:,k) \end{aligned} \quad (27)$$

$$\text{With: } \mathbf{Z}_l(:,k) \equiv \sqrt{p_p} \left[\mathbf{A} \sum_{i \neq l}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \right](:,k) + \mathbf{W}_l(:,k)$$

Clearly, we have the property:

$$\left[\mathbf{A} \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \right](:,k) = \mathbf{A} \mathbf{H}_{i,l}(:,k) \beta_{ilk}^{1/2} \quad (28)$$

Applying Bayes' rule, the conditional distribution of the channel $\mathbf{H}_{l,l}(:,k)$ given the observed received training signal $\tilde{\mathbf{y}}_{p,l}$ is:

$$\begin{aligned} p(\mathbf{H}_{l,l}(:,k) | \tilde{\mathbf{Y}}_{p,l}(:,k)) \\ = \frac{p(\mathbf{H}_{l,l}(:,k)) p(\tilde{\mathbf{Y}}_{p,l}(:,k) | \mathbf{H}_{l,l}(:,k))}{p(\tilde{\mathbf{Y}}_{p,l}(:,k))} = p(\mathbf{H}_{l,l}(:,k)) p(\tilde{\mathbf{Y}}_{p,l}(:,k) | \mathbf{H}_{l,l}(:,k)) \end{aligned} \quad (29)$$

We use the Gaussian probability density function (pdf) of the random vector $\mathbf{H}_{l,l}(:,k)$ and assume its elements: $H_{l,l}(1,k), \dots, H_{l,l}(1,k), \dots, H_{l,l}(P,k)$ are mutually independent, giving the joint pdf:

$$p(\mathbf{H}_{l,l}(:,k)) = \frac{\exp[-\mathbf{H}_{l,l}^H(:,k) \mathbf{R}_{\mathbf{H}_{l,l}(:,k)}^{-1} \mathbf{H}_{l,l}^H(:,k)]}{(\sqrt{2\pi})^P \det \mathbf{R}_{\mathbf{H}_{l,l}(:,k)}} \quad (30)$$

Since the elements: $H_{l,l}(1,k), \dots, H_{l,l}(m,k), \dots, H_{l,l}(P,k)$ are mutually independent, this reduces to:

$$p(\mathbf{H}_{l,l}(:,k)) = \frac{\exp[-\mathbf{H}_{l,l}^H(:,k) \mathbf{H}_{l,l}^H(:,k)]}{(\sqrt{2\pi})^P} \quad (31)$$

We also have:

$$\begin{aligned} & p(\tilde{\mathbf{Y}}_{p,l}(:,k) | \mathbf{H}_{l,l}(:,k)) \\ &= \frac{\exp[-(\tilde{\mathbf{Y}}_{p,l}(:,k) - \mathbf{A}\mathbf{H}_{l,l}(:,k)\beta_{llk}^{1/2})^H \mathbf{R}_{\mathbf{Z}_l(:,k)}^{-1} (\tilde{\mathbf{Y}}_{p,l}(:,k) - \mathbf{A}\mathbf{H}_{l,l}(:,k)\beta_{llk}^{1/2})]}{(\pi)^P \det \mathbf{R}_{\mathbf{Z}_l(:,k)}} \end{aligned} \quad (32)$$

Combining (31) and (32) and reporting into (29) we obtain:

$$p(\mathbf{H}_{l,l}(:,k) | \tilde{\mathbf{Y}}_{p,l}(:,k)) = \frac{\exp(-f(\mathbf{H}_{l,l}(:,k)))}{AB} \quad (33)$$

Where: $A \equiv (\sqrt{2\pi})^P$, $B \equiv (\pi)^P \det \mathbf{R}_{\mathbf{Z}_l(:,k)}$, this yields :

$$\begin{aligned} & f(\mathbf{H}_{l,l}(:,k)) \\ & \equiv \mathbf{H}_{l,l}^H(:,k) \mathbf{H}_{l,l}^H(:,k) + (\tilde{\mathbf{Y}}_{p,l}(:,k) - \mathbf{A}\mathbf{H}_{l,l}(:,k)\beta_{llk}^{1/2})^H \mathbf{R}_{\mathbf{Z}_l(:,k)}^{-1} (\tilde{\mathbf{Y}}_{p,l}(:,k) - \mathbf{A}\mathbf{H}_{l,l}(:,k)\beta_{llk}^{1/2}) \end{aligned} \quad (34)$$

Using the maximum a posteriori (MAP) decision rule, the Bayesian estimator yields the most probable value of channel estimation given the observation: $\tilde{\mathbf{Y}}_{p,l}(:,k)$.

$$\begin{aligned} \hat{\mathbf{H}}_{l,l}^H(:,k) &= \arg \max_{\mathbf{H}_{l,l}^H(:,k)} p(\mathbf{H}_{l,l}(:,k) | \tilde{\mathbf{Y}}_{p,l}(:,k)) \\ \hat{\mathbf{H}}_{l,l}^H(:,k) &= \arg \min_{\mathbf{H}_{l,l}^H(:,k)} f(\mathbf{H}_{l,l}(:,k)) \\ \hat{\mathbf{H}}_{l,l}^H(:,k) &= \sqrt{p_p} \beta_{llk}^{1/2} \mathbf{A}^H (p_p \beta_{llk} \mathbf{A} \mathbf{A}^H + \mathbf{R}_{\mathbf{Z}_l(:,k)})^{-1} \tilde{\mathbf{Y}}_{p,l}(:,k) \end{aligned} \quad (35)$$

It is straightforward to show that (see equation (24)):

$$\mathbf{R}_{\mathbf{Z}_l(:,k)} = \sigma_{n_l}^2 \mathbf{I}_M + p_p \sum_{i \neq l} \beta_{ilk} \mathbf{A} \mathbf{A}^H \quad (36)$$

And we find again:

$$\hat{\mathbf{H}}_{l,l}^H(:,k) = \sqrt{p_p} \beta_{llk}^{1/2} \mathbf{A}^H (\sigma_{n_l}^2 \mathbf{I}_M + p_p \sum_{i=1}^L \beta_{ilk} \mathbf{A} \mathbf{A}^H)^{-1} \tilde{\mathbf{Y}}_{p,l}(:,k) \quad (37)$$

Reminder Appendix 1:

If we have the following equivalent model:

$$\mathbf{y} = \mathbf{S}\mathbf{h} + \mathbf{n} \quad (38)$$

With the following pdf :

$$p(\mathbf{h}|\mathbf{y}) = \frac{\exp(-l(\mathbf{h}))}{K}$$

$$l(\mathbf{h}) = \mathbf{h}^H \mathbf{R}_h^{-1} \mathbf{h} + (\mathbf{y} - \mathbf{S}\mathbf{h})^H \mathbf{R}_n^{-1} (\mathbf{y} - \mathbf{S}\mathbf{h}) \quad (39)$$

The Bayesian estimator yields the most probable value given the observation \mathbf{y} and this yields:

$$\hat{\mathbf{h}} = \mathbf{R}_h \mathbf{S}^H (\mathbf{R}_n + \mathbf{R}_h \mathbf{S} \mathbf{S}^H)^{-1} \mathbf{y} \quad (40)$$

4- Uplink Data Transmission:

We consider the context of an uplink transmission where the base station uses the channel estimates to perform signal processing techniques such as MRC (Maximum Ratio Combining), ZF (Zero Forcing) or MMSE (Minimum Mean Square Error).

4.1. Maximum Ratio Combining:

In this case the l th base station processes the received signal by multiplying it by the conjugate transpose (Hermitian) of the channel estimate. From (6) and (14), we obtain:

$$\begin{aligned} \mathbf{r}_l &= \hat{\mathbf{\Gamma}}_{ll}^H \mathbf{y}_l \\ &= p_p^{-1/2} \mathbf{D}_{ll} \mathbf{D}_l^{-1} \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} + \mathbf{W}_l \right)^H \mathbf{\Pi}_A \left(\sqrt{p_u} \mathbf{A} \sum_{j=1}^L \mathbf{H}_{j,l} \mathbf{D}_{j,l}^{1/2} \mathbf{x}_j + \mathbf{n}_l \right) \end{aligned} \quad (41)$$

With: $\mathbf{\Pi}_A \equiv \mathbf{A}(\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H$

Remark : $\mathbf{\Pi}_A^H \equiv \left[\mathbf{A}(\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right]^H = \mathbf{A} \left[(\mathbf{A}^H \mathbf{A})^{-1} \right]^H \mathbf{A}^H = \mathbf{A} \left[(\mathbf{A}^H \mathbf{A})^{-1} \right] \mathbf{A}^H = \mathbf{\Pi}_A$

To further proceed with (41) we directly suppose that we operate with a huge number of antennas, i.e. we suppose that M grows towards infinity. We simplify first the quantity: $\mathbf{\Pi}_A$; to do this we consider the case of a uniform array. The response vector is thus given by:

$$\mathbf{a}(\phi_k) = \frac{1}{\sqrt{P}} \left[1, e^{-j2\pi d \sin(\phi_k)/\lambda}, \dots, e^{-j2\pi(M-1)d \sin(\phi_k)/\lambda} \right]^T \quad (42)$$

d is the antenna spacing between radiating elements and λ is the wavelength. Then, we have, for $k \neq l$:

$$\begin{aligned} \frac{1}{M} \mathbf{a}(\phi_k)^H \mathbf{a}(\phi_l) &= \frac{1}{MP} \sum_{m=0}^{M-1} e^{j2\pi d m (\sin(\phi_k) - \sin(\phi_l)) / \lambda} \\ &= \frac{1}{MP} \frac{1 - e^{j2\pi d M (\sin(\phi_k) - \sin(\phi_l)) / \lambda}}{1 - e^{j2\pi d (\sin(\phi_k) - \sin(\phi_l)) / \lambda}} \\ &= \frac{1}{P} e^{j2\pi d (M-1) (\sin(\phi_k) - \sin(\phi_l)) / \lambda} \left[\frac{\sin(\pi d M (\sin(\phi_k) - \sin(\phi_l)) / \lambda)}{M \sin(\pi d (\sin(\phi_k) - \sin(\phi_l)) / \lambda)} \right] \end{aligned} \quad (43)$$

From (43) it is clear that, for $k \neq l$: $\lim_{M \rightarrow \infty} \frac{1}{M} \mathbf{a}(\phi_k)^H \mathbf{a}(\phi_l) = 0$

For $k = l$, we have immediately, $\lim_{M \rightarrow \infty} \frac{1}{M} \mathbf{a}(\phi_l)^H \mathbf{a}(\phi_l) = 1/P$. Therefore, we deduce:

$$\lim_{M \rightarrow \infty} \frac{1}{M} \mathbf{A}^H \mathbf{A} = \frac{1}{P} \mathbf{I}_P. \text{ Furthermore, we have: } \lim_{M \rightarrow \infty} \text{Trace} [\mathbf{A}^H \mathbf{A}] = \frac{M}{P} P = M.$$

Using this property, we can write:

$$\mathbf{\Pi}_A \equiv \mathbf{A}(\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \approx_{M \rightarrow \infty} \mathbf{A} \left(\frac{M}{P} \mathbf{I}_P \right)^{-1} \mathbf{A}^H = \frac{P}{M} \mathbf{A} \mathbf{A}^H \quad (44)$$

Remark: In the general case, we have:

$$\begin{aligned} \frac{1}{\sqrt{P}} \begin{bmatrix} e^{-jf_1(\phi_1)} & e^{-jf_1(\phi_2)} & \dots & e^{-jf_1(\phi_P)} \\ e^{-jf_2(\phi_1)} & e^{-jf_2(\phi_2)} & \dots & e^{-jf_2(\phi_P)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-jf_M(\phi_1)} & e^{-jf_M(\phi_2)} & \dots & e^{-jf_M(\phi_P)} \end{bmatrix} \\ \mathbf{A}^H \mathbf{A} = \frac{1}{\sqrt{P}} \begin{bmatrix} e^{jf_1(\phi_1)} & e^{jf_2(\phi_1)} & \dots & e^{jf_M(\phi_1)} \\ e^{jf_1(\phi_2)} & e^{jf_2(\phi_2)} & \dots & e^{jf_M(\phi_2)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{jf_1(\phi_P)} & e^{jf_2(\phi_P)} & \dots & e^{jf_M(\phi_P)} \end{bmatrix} = \frac{1}{P} \begin{bmatrix} M & \sum_{i=1}^M e^{j(f_i(\phi_1) - f_i(\phi_2))} & \dots & \sum_{i=1}^M e^{j(f_i(\phi_1) - f_i(\phi_P))} \\ \sum_{i=1}^M e^{j(f_i(\phi_2) - f_i(\phi_1))} & M & \dots & \sum_{i=1}^M e^{j(f_i(\phi_2) - f_i(\phi_P))} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^M e^{j(f_i(\phi_P) - f_i(\phi_1))} & \sum_{i=1}^M e^{j(f_i(\phi_P) - f_i(\phi_2))} & \dots & M \end{bmatrix} \end{aligned}$$

Concerning the product: $\mathbf{A} \mathbf{A}^H$, we obtain:

$$\mathbf{A}\mathbf{A}^H = \frac{1}{\sqrt{P}} \begin{bmatrix} e^{-jf_1(\phi_1)} & e^{-jf_1(\phi_2)} & \dots & e^{-jf_1(\phi_P)} \\ e^{-jf_2(\phi_1)} & e^{-jf_2(\phi_2)} & \dots & e^{-jf_2(\phi_P)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-jf_M(\phi_1)} & e^{-jf_M(\phi_2)} & \dots & e^{-jf_M(\phi_P)} \end{bmatrix} = \frac{1}{P} \begin{bmatrix} P & \sum_{i=1}^P e^{j(f_2(\phi_i)-f_1(\phi_i))} & \dots & \sum_{i=1}^P e^{j(f_M(\phi_i)-f_1(\phi_i))} \\ \sum_{i=1}^P e^{j(f_1(\phi_i)-f_2(\phi_i))} & P & \dots & \sum_{i=1}^P e^{j(f_M(\phi_i)-f_2(\phi_i))} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=1}^P e^{j(f_1(\phi_i)-f_M(\phi_i))} & \sum_{i=1}^P e^{j(f_1(\phi_i)-f_2(\phi_i))} & \dots & P \end{bmatrix}$$

So, the term $\mathbf{A}\mathbf{A}^H(m,n)$ is equal to (in the case of a linear phased array):

$$\mathbf{A}\mathbf{A}^H(m,n) = \frac{1}{P} \sum_{i=1}^P e^{j(f_n(\phi_i)-f_m(\phi_i))} = \frac{1}{P} \sum_{i=1}^P e^{j2\pi((n-1)\sin(\phi_i)-(m-1)\sin(\phi_i))/\lambda}$$

$$\mathbf{A}\mathbf{A}^H(m,n) = \frac{1}{P} \sum_{i=1}^P e^{j2\pi((n-m)\sin(\phi_i))/\lambda}$$

A coarse approximation of matrix $\mathbf{A}\mathbf{A}^H$ in the case where $P \rightarrow \infty$ could be: $\mathbf{A}\mathbf{A}^H \approx \mathbf{I}_M$.

Using (43) and (44) into (41) and the fact that as $M \rightarrow \infty$ the products of uncorrelated quantities can be ignored [5], we have:

$$\frac{\mathbf{r}_l}{\sqrt{p_u} M} \xrightarrow{M \rightarrow \infty} \mathbf{D}_{ll} \mathbf{D}_l^{-1} \left(\sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \right)^H \frac{\mathbf{A}^H \mathbf{A}}{M} \left(\sum_{j=1}^L \mathbf{H}_{j,l} \mathbf{D}_{j,l}^{1/2} \mathbf{x}_j \right) \quad (45)$$

Then, using: $\lim_{M \rightarrow \infty} \frac{1}{M} \mathbf{A}^H \mathbf{A} = \frac{1}{P} \mathbf{I}_P$, it is straightforward to obtain :

$$\frac{\mathbf{r}_l}{\sqrt{p_u} M} \xrightarrow{M \rightarrow \infty} \frac{1}{P} \mathbf{D}_{ll} \mathbf{D}_l^{-1} \left(\sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \right)^H \left(\sum_{j=1}^L \mathbf{H}_{j,l} \mathbf{D}_{j,l}^{1/2} \mathbf{x}_j \right) \quad (46)$$

The interpretation of (46) is quite simple: since the elements of $\mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2}$ are independent, the above result reveals that the performance of the system under the finite-dimensional channel model with P angular bins and with an unlimited number of base station antennas is the same as the performance under an uncorrelated channel model with P antennas.

4.2. Analysis of the Pilot Contamination effect:

To gain further insight into the pilot contamination effect, we derive lower bounds on the achievable rate for a finite and an infinite number of base station antennas (the analysis requires that $M > P$). To obtain these lower bounds we use the techniques of [3,4,6]. We assume an uniform array at the base station, and the elements of $\mathbf{H}_{i,l}$ are i.i.d Gaussian random variables. Let r_{ln} be the n th element of the received $K \times 1$ vector \mathbf{r}_l and denote $\tilde{r}_{ln} \equiv r_{ln} \sqrt{p_p} \sum_{i=1}^L \beta_{i ln} / \beta_{l ln}$, we deduce from (41) that:

$$\tilde{r}_{ln} = \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{g}_{i ln} + \mathbf{w}_{ln} \right)^H \mathbf{\Pi}_A \left(\sqrt{p_u} \mathbf{A} \sum_{j=1}^L \mathbf{G}_{j,l} \mathbf{x}_j + \mathbf{n}_l \right) \quad (47)$$

Remember that we have: $\mathbf{G}_{j,l} \equiv [\mathbf{g}_{j l 1}, \dots, \mathbf{g}_{j l K}]$ which is a $P \times K$ matrix with: $\mathbf{g}_{j l k} \equiv [g_{j l k 1}, \dots, g_{j l k P}]^T$ which is a vector that contains the path gains from the k th user in the i th cell to the l th base station. Equation (47) can be rewritten as:

$$\begin{aligned} \tilde{r}_{ln} &= \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{g}_{i ln} + \mathbf{w}_{ln} \right)^H \mathbf{\Pi}_A \left(\sqrt{p_u} \mathbf{A} \sum_{j=1}^L \mathbf{G}_{j,l} \mathbf{x}_j + \mathbf{n}_l \right) \\ &= \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{g}_{i ln}^H \mathbf{A}^H + \mathbf{w}_{ln}^H \right) \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \left(\sqrt{p_u} \mathbf{A} \sum_{j=1}^L \mathbf{G}_{j,l} \mathbf{x}_j + \mathbf{n}_l \right) \\ &= \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{g}_{i ln}^H \mathbf{A}^H + \mathbf{w}_{ln}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right) \left(\sqrt{p_u} \mathbf{A} \sum_{j=1}^L \mathbf{G}_{j,l} \mathbf{x}_j + \mathbf{n}_l \right) \quad (48) \\ &= \sqrt{p_p p_u} \sum_{i=1}^L \mathbf{g}_{i ln}^H \mathbf{A}^H \mathbf{A} \sum_{j=1}^L \mathbf{G}_{j,l} \mathbf{x}_j + \sqrt{p_u} \mathbf{w}_{ln}^H \mathbf{A} \sum_{j=1}^L \mathbf{G}_{j,l} \mathbf{x}_j + \sqrt{p_p} \sum_{i=1}^L \mathbf{g}_{i ln}^H \mathbf{A}^H \mathbf{n}_l + \mathbf{w}_{ln}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{n}_l \end{aligned}$$

Equation (48) is eventually equivalent to:

$$\tilde{r}_{ln} = \boldsymbol{\alpha}_{ln}^T \mathbf{x}_l + \sum_{j \neq l}^L \boldsymbol{\alpha}_{jn}^T \mathbf{x}_j + z_{ln} \quad (49)$$

With: $\boldsymbol{\alpha}_{jn}^T \equiv \sqrt{p_p p_u} \sum_{i=1}^L \mathbf{g}_{i ln}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} + \sqrt{p_u} \mathbf{w}_{ln}^H \mathbf{A} \mathbf{G}_{j,l}$ and $z_{ln} = (\sqrt{p_p} \sum_{i=1}^L \mathbf{g}_{i ln}^H \mathbf{A}^H + \mathbf{w}_{ln}^H) \mathbf{\Pi}_A \mathbf{n}_l$

We have:

$$\mathcal{E} \{ \boldsymbol{\alpha}_{ln}^T \} \equiv \sqrt{p_p p_u} \mathcal{E} \{ \mathbf{g}_{i ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{l ln} \} \mathbf{e}_n^T \quad (50)$$

Where \mathbf{e}_n is the n th column of the $K \times K$ identity matrix. By adding and subtracting $\mathcal{E} \{ \boldsymbol{\alpha}_{ln}^T \}$ into (49) and using (50) we obtain:

$$\begin{aligned}
\tilde{r}_{ln} &= \mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\} \mathbf{x}_l + (\boldsymbol{\alpha}_{ln}^T - \mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}) \mathbf{x}_l + \sum_{j \neq l}^L \boldsymbol{\alpha}_{jn}^T \mathbf{x}_j + z_{ln} \\
&= \sqrt{p_p p_u} \mathcal{E}\{\mathbf{g}_{ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{ln}\} x_{ln} + (\boldsymbol{\alpha}_{ln}^T - \mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}) \mathbf{x}_l + \sum_{j \neq l}^L \boldsymbol{\alpha}_{jn}^T \mathbf{x}_j + z_{ln}
\end{aligned} \tag{51}$$

Let define the function $C(x)$ as: $C(x) = \log_2(1+x)$, the n th user in the l th cell can achieve an uplink rate of at least:

$$R_{ln} = C \left(\frac{p_p p_u \left| \mathcal{E}\{\mathbf{g}_{ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{ln}\} \right|^2}{\sum_{j=1}^L \mathcal{E}\{\|\boldsymbol{\alpha}_{jn}^T\|^2\} - \|\mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}\|^2 + \mathcal{E}\{|z_{ln}|^2\}} \right) \tag{52}$$

In this case, the achievable uplink rate of the n th user in the l th cell in (52) can be lower bounded by (60) since we have:

$$\begin{aligned}
\|\mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}\|^2 &= \|\sqrt{p_p p_u} \mathcal{E}\{\mathbf{g}_{ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{ln}\} \mathbf{e}_n^T\|^2 \\
&= p_p p_u \text{Trace}[\mathcal{E}\{\mathbf{g}_{ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{ln}\} \mathbf{e}_n^T \mathbf{e}_n \mathcal{E}\{\mathbf{g}_{ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{ln}\}] \\
&= p_p p_u \left[\mathcal{E}\{\mathbf{g}_{ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{ln}\} \right]^2 \text{Trace}[\mathbf{e}_n^T \mathbf{e}_n] \\
&= p_p p_u \left[\mathcal{E}\{\mathbf{g}_{ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{ln}\} \right]^2
\end{aligned}$$

We study the product $\mathbf{g}_{ln}^H \mathbf{A}^H$ by considering only one column of matrix: \mathbf{A}^H , we have :

$$\mathbf{A} \equiv [\mathbf{a}(\phi_1), \dots, \mathbf{a}(\phi_p)] = \frac{1}{\sqrt{P}} \begin{bmatrix} e^{-j\phi_1} & e^{-j\phi_2} & \dots & e^{-j\phi_p} \\ e^{-j\phi_2} & e^{-j\phi_2} & \dots & e^{-j\phi_p} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-j\phi_M} & e^{-j\phi_2} & \dots & e^{-j\phi_p} \end{bmatrix}$$

Hence, we obtain:

$$\mathbf{g}_{ln}^H \mathbf{A}^H(:,k) = \frac{1}{\sqrt{P}} [g_{ln1}^*, \dots, g_{lnP}^*] \begin{bmatrix} e^{j\phi_1} \\ e^{j\phi_2} \\ \vdots \\ e^{j\phi_p} \end{bmatrix} = \frac{1}{\sqrt{P}} \sum_{j=1}^P g_{lnj}^* e^{j\phi_j}$$

In this case, we obtain:

$$\mathbf{g}_{l\ln}^H \mathbf{A}^H = \frac{1}{\sqrt{P}} \left[\sum_{j=1}^P \mathbf{g}_{l\ln j}^* e^{jf_1(\phi_j)}, \dots, \sum_{j=1}^P \mathbf{g}_{l\ln j}^* e^{jf_k(\phi_j)}, \dots, \sum_{j=1}^P \mathbf{g}_{l\ln j}^* e^{jf_M(\phi_j)} \right]_{1 \times M}$$

And it is straightforward to calculate:

$$\begin{aligned} & \mathcal{E} \left\{ \mathbf{g}_{l\ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{l\ln} \right\} \\ &= \frac{1}{P} \mathcal{E} \left\{ \left(\sum_{j=1}^P \mathbf{g}_{l\ln j}^* e^{jf_1(\phi_j)} \right) \left(\sum_{j=1}^P \mathbf{g}_{l\ln j} e^{-jf_1(\phi_j)} \right) + \dots + \left(\sum_{j=1}^P \mathbf{g}_{l\ln j}^* e^{jf_M(\phi_j)} \right) \left(\sum_{j=1}^P \mathbf{g}_{l\ln j} e^{-jf_M(\phi_j)} \right) \right\} \quad (53) \\ &= \frac{M}{P} \sum_{j=1}^P \mathcal{E} \left[\left| \mathbf{g}_{l\ln j} \right|^2 \right] = \frac{M}{P} \sum_{j=1}^P \mathcal{E} \left[\beta_{l\ln} \left| h_{l\ln j} \right|^2 \right] = \frac{M}{P} \beta_{l\ln} \sum_{j=1}^P \mathcal{E} \left[\left| h_{l\ln j} \right|^2 \right] = \frac{M}{P} \beta_{l\ln} \end{aligned}$$

Hence, we obtain:

$$\left[\mathcal{E} \left\{ \mathbf{g}_{l\ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{l\ln} \right\} \right]^2 = \frac{M^2}{P^2} \beta_{l\ln}^2 \quad (54)$$

- Concerning: $\mathcal{E} \left\{ \left| z_{ln} \right|^2 \right\}$, we have :

$$\begin{aligned} \mathcal{E} \left\{ \left| z_{ln} \right|^2 \right\} &= \mathcal{E} \left\{ \left| \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{g}_{i\ln}^H \mathbf{A}^H + \mathbf{w}_{ln}^H \right) \Pi_A \mathbf{n}_l \right|^2 \right\} \\ &= \mathcal{E} \left\{ \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{g}_{i\ln}^H \mathbf{A}^H + \mathbf{w}_{ln}^H \right) \Pi_A \mathbf{n}_l \mathbf{n}_l^H \Pi_A^H \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{A} \mathbf{g}_{i\ln} + \mathbf{w}_{ln}^H \right) \right\} \\ &= \mathcal{E} \left\{ \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{g}_{i\ln}^H \mathbf{A}^H + \mathbf{w}_{ln}^H \right) \Pi_A \mathcal{E} \left[\mathbf{n}_l \mathbf{n}_l^H \right] \Pi_A^H \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{A} \mathbf{g}_{i\ln} + \mathbf{w}_{ln}^H \right) \right\} \\ &= \mathcal{E} \left\{ \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{g}_{i\ln}^H \mathbf{A}^H + \mathbf{w}_{ln}^H \right) \Pi_A \sigma_{n_l}^2 \mathbf{I}_M \Pi_A^H \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{A} \mathbf{g}_{i\ln} + \mathbf{w}_{ln}^H \right) \right\} \\ &= \sigma_{n_l}^2 \mathcal{E} \left\{ \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{g}_{i\ln}^H \mathbf{A}^H + \mathbf{w}_{ln}^H \right) \Pi_A \Pi_A^H \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{A} \mathbf{g}_{i\ln} + \mathbf{w}_{ln}^H \right) \right\} \\ &= \sigma_{n_l}^2 \mathcal{E} \left\{ \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{g}_{i\ln}^H \mathbf{A}^H + \mathbf{w}_{ln}^H \right) \Pi_A \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{A} \mathbf{g}_{i\ln} + \mathbf{w}_{ln}^H \right) \right\} \end{aligned}$$

Remark :

$$\begin{aligned} \Pi_A \Pi_A^H &= \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \left[\mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right]^H \\ &= \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \\ &= \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H = \Pi_A \end{aligned}$$

We use the independency of columns $\mathbf{g}_{i ln}$ and $\mathbf{g}_{j ln}$, i.e.: $\mathcal{E}[\mathbf{g}_{i ln}^H \mathbf{g}_{j ln}] = 0$, for $i \neq j$.

$$\begin{aligned} \mathcal{E}\{|z_{ln}|^2\} &= \sigma_{n_l}^2 \left\{ \mathcal{E}\left[p_p \sum_{i=1}^L \mathbf{g}_{i ln}^H \mathbf{A}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{g}_{i ln} \right] + \mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{\Pi}_A \mathbf{\Pi}_A^H \mathbf{w}_{ln} \right] \right\} \\ &= \sigma_{n_l}^2 \left\{ \mathcal{E}\left[p_p \sum_{i=1}^L \mathbf{g}_{i ln}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{g}_{i ln} \right] + \mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{\Pi}_A \mathbf{w}_{ln} \right] \right\} \\ &= \sigma_{n_l}^2 \left\{ \mathcal{E}\left[p_p \sum_{i=1}^L \mathbf{g}_{i ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{i ln} \right] + \mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{w}_{ln} \right] \right\} \end{aligned}$$

Due to (53), we have :

$$\mathcal{E}\{\mathbf{g}_{i ln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{i ln}\} = \frac{M}{P} \beta_{ln}$$

The singular value decomposition of matrix $(\mathbf{A}^H \mathbf{A})^{-1}$ yields :

$$(\mathbf{A}^H \mathbf{A})^{-1} = \mathbf{U} \mathbf{D} \mathbf{U}^H$$

\mathbf{D} is a diagonal matrix which can be written as: $\mathbf{D} = \text{diag}[\lambda_1, \lambda_2, \dots, \lambda_k, \dots, \lambda_p]$ with $\lambda_i = 1/\beta_i$, $i = 1, \dots, P$ designing the eigenvalues of matrix $\mathbf{A}^H \mathbf{A}$. Thus, we have:

$$\mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{w}_{ln} \right] = \mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{A} \mathbf{U} \mathbf{D} \mathbf{U}^H \mathbf{A}^H \mathbf{w}_{ln} \right] = \mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{A} \mathbf{U} \mathbf{D} \left[\mathbf{w}_{ln}^H \mathbf{A} \mathbf{U} \right]^H \right]$$

We have:

$$\begin{aligned} \mathbf{w}_{ln}^H \mathbf{A} &= \frac{1}{\sqrt{P}} \left[w_{ln}^*(1), w_{ln}^*(2), \dots, w_{ln}^*(M) \right] \begin{bmatrix} e^{-j\phi_1} & e^{-j\phi_2} & \dots & e^{-j\phi_p} \\ e^{-j2\phi_1} & e^{-j2\phi_2} & \dots & e^{-j2\phi_p} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-jM\phi_1} & e^{-jM\phi_2} & \dots & e^{-jM\phi_p} \end{bmatrix} \\ &= \frac{1}{\sqrt{P}} \left[w_{ln}^*(1) e^{-j\phi_1} + \dots + w_{ln}^*(M) e^{-jM\phi_1}, \dots, w_{ln}^*(1) e^{-j\phi_p} + \dots + w_{ln}^*(M) e^{-jM\phi_p} \right] \end{aligned}$$

Hence, the k th column of matrix $\mathbf{w}_{ln}^H \mathbf{A} \mathbf{U}$ is thus written as:

$$\begin{aligned} &(\mathbf{w}_{ln}^H \mathbf{A} \mathbf{U})(:, k) \\ &= \frac{1}{\sqrt{P}} \left[\sum_{m=1}^P U(m, k) \sum_{i=1}^M w_{ln}^*(i) e^{-j\phi_m} \right] \end{aligned}$$

In these conditions we have:

$$\begin{aligned}
& \mathcal{E} \left[\left(\mathbf{w}_{ln}^H \mathbf{A} \mathbf{U} \right) (:, k) \mathbf{D} \left(\mathbf{w}_{ln}^H \mathbf{A} \mathbf{U} \right)^H (:, k) \right] \\
&= \frac{1}{P} \mathcal{E} \left[\sum_{m=1}^P U(m, k) \sum_{i=1}^M w_{ln}^*(i) e^{-j f_i(\phi_m)} \lambda_k \sum_{p=1}^P U^*(p, k) \sum_{j=1}^M w_{ln}(j) e^{j f_j(\phi_p)} \right] \\
&= \frac{1}{P} \left[\lambda_k \sum_{m=1}^P U(m, k) \sum_{p=1}^P U^*(p, k) \mathcal{E} \left[\sum_{i=1}^M w_{ln}^*(i) e^{-j f_i(\phi_m)} \sum_{j=1}^M w_{ln}(j) e^{j f_j(\phi_p)} \right] \right] \\
&= \frac{1}{P} \left[\lambda_k \sum_{m=1}^P U(m, k) \sum_{p=1}^P U^*(p, k) \sum_{j=1}^M \sum_{i=1}^M \mathcal{E} \left[w_{ln}^*(i) w_{ln}(j) \right] e^{j f_j(\phi_p)} e^{-j f_i(\phi_m)} \right] \\
&= \frac{1}{P} \left[\lambda_k \sum_{m=1}^P U(m, k) \sum_{p=1}^P U^*(p, k) \sum_{i=1}^M \sigma_{n_l}^2 e^{j [f_i(\phi_p) - f_i(\phi_m)]} \right] \\
&= \frac{\lambda_k \sigma_{n_l}^2}{P} \sum_{m=1}^P U(m, k) \sum_{p=1}^P U^*(p, k) \sum_{i=1}^M e^{j [f_i(\phi_p) - f_i(\phi_m)]}
\end{aligned}$$

And we eventually obtain:

$$\mathcal{E} \left[\mathbf{w}_{ln}^H \mathbf{A} \mathbf{U} \mathbf{D} \left[\mathbf{w}_{ln}^H \mathbf{A} \mathbf{U} \right]^H \right] = \frac{\sigma_{n_l}^2}{P} \sum_{k=1}^P \sum_{p=1}^P \sum_{m=1}^P \sum_{i=1}^M \lambda_k U(m, k) U^*(p, k) e^{j [f_i(\phi_p) - f_i(\phi_m)]} \quad (55)$$

Concerning: $\mathcal{E} \left\{ |z_{ln}|^2 \right\}$, this yields :

$$\begin{aligned}
\mathcal{E} \left\{ |z_{ln}|^2 \right\} &= \frac{\sigma_{n_l}^2 M}{P} p_p \sum_{i=1}^L \beta_{iln} + \frac{\sigma_{n_l}^2}{P} \sum_{k=1}^P \sum_{p=1}^P \sum_{m=1}^P \sum_{i=1}^M \lambda_k U(m, k) U^*(p, k) e^{j [f_i(\phi_p) - f_i(\phi_m)]} \\
\mathcal{E} \left\{ |z_{ln}|^2 \right\} &= \frac{M \sigma_{n_l}^2}{P} \left(p_p \sum_{i=1}^L \beta_{iln} + \frac{\sigma_{n_l}^2}{M} \sum_{k=1}^P \sum_{p=1}^P \sum_{m=1}^P \sum_{i=1}^M \lambda_k U(m, k) U^*(p, k) e^{j [f_i(\phi_p) - f_i(\phi_m)]} \right) \quad (56) \\
\mathcal{E} \left\{ |z_{ln}|^2 \right\} &= \frac{M \sigma_{n_l}^2}{P} \left(p_p \bar{\beta}_{ln} + \frac{\sigma_{n_l}^2}{M} \sum_{k=1}^P \sum_{p=1}^P \sum_{m=1}^P \sum_{i=1}^M \lambda_k U(m, k) U^*(p, k) e^{j [f_i(\phi_p) - f_i(\phi_m)]} \right)
\end{aligned}$$

Using the notation: $S_{U^2(:,k)}^{f_i} \equiv \sum_{p=1}^P \sum_{m=1}^P \sum_{i=1}^M U(m, k) U^*(p, k) e^{j [f_i(\phi_p) - f_i(\phi_m)]}$, we obtain :

$$\mathcal{E} \left\{ |z_{ln}|^2 \right\} = \frac{M \sigma_{n_l}^2}{P} \left(p_p \bar{\beta}_{ln} + \frac{\sigma_{n_l}^2}{M} \sum_{k=1}^P S_{U^2(:,k)}^{f_i} \right) \quad (57)$$

Remark: in the case of a uniform array, we have:

$$f_i(\phi_p) = \frac{2\pi(i-1)d}{\lambda} \sin(\phi_p)$$

And we obtain:

$$\mathcal{E}\left\{|z_{ln}|^2\right\} = \frac{M\sigma_{n_l}^2}{P} \left(p_p \bar{\beta}_{ln} + \frac{\sigma_{n_l}^2}{M} \sum_{k=1}^P \sum_{p=1}^P \sum_{m=1}^P \sum_{i=1}^M \lambda_k U(m,k) U^*(p,k) e^{\frac{j2\pi(i-1)d}{\lambda} [\sin(\phi_p) - \sin(\phi_m)]} \right) \quad (58)$$

Remark: Using Lemma 1 (Eq. (59)), it is possible to obtain a simpler expression. We have:

$$\begin{aligned} & \mathcal{E}\left[\mathbf{w}_{ln}^H \Pi_A \Pi_A^H \mathbf{w}_{ln}\right] \\ &= \frac{1}{M} \text{Trace}\left[\Pi_A \Pi_A^H\right] \mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{w}_{ln}\right] \\ &= \frac{1}{M} \text{Trace}\left[\Pi_A \Pi_A^H\right] M \sigma_{n_l}^2 \quad (\text{see (59)}) \\ &= \sigma_{n_l}^2 \text{Trace}\left[\Pi_A\right] \end{aligned}$$

To proceed further on with this expression, we need to calculate:

$$\begin{aligned} \text{Trace}\left[\Pi_A\right] &= \text{Trace}\left[\mathbf{A}(\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H\right] \\ &\approx_{M \rightarrow \infty} \frac{P}{M} \text{Trace}\left[\mathbf{A} \mathbf{A}^H\right] = \frac{P}{M} M = P \end{aligned}$$

And, hence we obtain:

$$\mathcal{E}\left\{|z_{ln}|^2\right\}_{M \rightarrow \infty} = \sigma_{n_l}^2 \left(p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_l}^2 P \right) \quad (58)$$

- Concerning: $\mathcal{E}\left(\left\|\boldsymbol{\alpha}_{jn}^T\right\|^2\right)$, we have:

$$\begin{aligned} & \mathcal{E}\left(\left\|\boldsymbol{\alpha}_{jn}^T\right\|^2\right) = \mathcal{E}\left(\boldsymbol{\alpha}_{jn}^T \boldsymbol{\alpha}_{jn}^*\right) \\ &= \mathcal{E}\left(\left(\sqrt{p_p p_u} \sum_{i=1}^L \mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} + \sqrt{p_u} \mathbf{w}_{ln}^H \mathbf{A} \mathbf{G}_{j,l}\right) \left(\sqrt{p_p p_u} \sum_{m=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{mln} + \sqrt{p_u} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{w}_{ln}\right)\right) \\ &= p_p p_u \mathcal{E}\left(\sum_{i=1}^L \mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \sum_{m=1}^L \mathbf{A}^H \mathbf{A} \mathbf{g}_{mln}\right) + p_u \mathcal{E}\left(\mathbf{w}_{ln}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{w}_{ln}\right) \\ &= p_p p_u \mathcal{E}\left(\sum_{i=1, i \neq j}^L \mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{A} \mathcal{E}\left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H\right] \sum_{m=1}^L \mathbf{A}^H \mathbf{A} \mathbf{g}_{mln}\right) + p_p p_u \mathcal{E}\left(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln}\right) \\ &\quad + p_u \mathcal{E}\left(\mathbf{w}_{ln}^H \mathbf{A} \mathcal{E}\left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H\right] \mathbf{A}^H \mathbf{w}_{ln}\right) \end{aligned}$$

Concerning the product: $\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H$, we have:

$$\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H = \begin{bmatrix} g_{j\Omega 1} & g_{j\Omega 2} & \cdots & g_{j\Omega k} & \cdots & g_{j\Omega P} \\ g_{j\Omega 2} & g_{j\Omega 2} & \cdots & g_{j\Omega k} & \cdots & g_{j\Omega P} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{j\Omega m} & g_{j\Omega m} & \cdots & g_{j\Omega k} & \cdots & g_{j\Omega P} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{j\Omega K} & g_{j\Omega K} & \cdots & g_{j\Omega k} & \cdots & g_{j\Omega P} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^K |g_{j\Omega k}|^2 & \sum_{k=1}^K g_{j\Omega k} g_{j\Omega k}^* & \cdots & \sum_{k=1}^K g_{j\Omega k} g_{j\Omega m}^* & \cdots & \sum_{k=1}^K g_{j\Omega k} g_{j\Omega P}^* \\ \sum_{k=1}^K g_{j\Omega k} g_{j\Omega k}^* & \sum_{k=1}^K |g_{j\Omega k}|^2 & \cdots & \sum_{k=1}^K g_{j\Omega k} g_{j\Omega m}^* & \cdots & \sum_{k=1}^K g_{j\Omega k} g_{j\Omega P}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^K g_{j\Omega m} g_{j\Omega k}^* & \sum_{k=1}^K g_{j\Omega m} g_{j\Omega k}^* & \cdots & \sum_{k=1}^K |g_{j\Omega m}|^2 & \cdots & \sum_{k=1}^K g_{j\Omega m} g_{j\Omega P}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^K g_{j\Omega P} g_{j\Omega k}^* & \sum_{k=1}^K g_{j\Omega P} g_{j\Omega k}^* & \cdots & \sum_{k=1}^K g_{j\Omega P} g_{j\Omega m}^* & \cdots & \sum_{k=1}^K |g_{j\Omega P}|^2 \end{bmatrix}$$

So, we obtain:

$$\mathcal{E}[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H] = \text{diag} \left[\mathcal{E} \left[\sum_{k=1}^K |g_{j\Omega k}|^2 \right], \mathcal{E} \left[\sum_{k=1}^K |g_{j\Omega k}|^2 \right], \dots, \mathcal{E} \left[\sum_{k=1}^K |g_{j\Omega m}|^2 \right], \dots, \mathcal{E} \left[\sum_{k=1}^K |g_{j\Omega P}|^2 \right] \right]$$

With:

$$\mathcal{E} \left[\sum_{k=1}^K |g_{j\Omega k}|^2 \right] = \sum_{k=1}^K \beta_{j\Omega k} \mathcal{E} \left[|h_{j\Omega k}|^2 \right]$$

Thus :

$$\mathcal{E}[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H] = \text{diag} \left[\sum_{k=1}^K \beta_{j\Omega k} \mathcal{E} \left[|h_{j\Omega k}|^2 \right], \sum_{k=1}^K \beta_{j\Omega k} \mathcal{E} \left[|h_{j\Omega k}|^2 \right], \dots, \sum_{k=1}^K \beta_{j\Omega k} \mathcal{E} \left[|h_{j\Omega m}|^2 \right], \dots, \sum_{k=1}^K \beta_{j\Omega k} \mathcal{E} \left[|h_{j\Omega P}|^2 \right] \right]$$

Then, we use the following lemma:

Lemma 1: If we consider a given complex random vector \mathbf{V} of size $K \times 1$ with i.i.d random variables of variance σ^2 and a complex deterministic matrix \mathbf{W} of size $K \times M$, then we have:

$$\begin{aligned} & \mathcal{E}[\mathbf{V}^H \mathbf{W} \mathbf{W}^H \mathbf{V}] \\ &= \frac{1}{K} \mathcal{E}[\mathbf{V}^H \mathbf{V}] \text{Trace}(\mathbf{W} \mathbf{W}^H) \end{aligned} \quad (59)$$

Proof:

We take \mathbf{V} as: $\mathbf{V} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_K \end{pmatrix}$ and \mathbf{W} as: $\mathbf{W} = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} & \cdots & w_{1M} \\ w_{21} & w_{22} & \vdots & w_{2n} & \cdots & w_{2M} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{p1} & w_{p2} & \cdots & w_{pn} & \cdots & w_{pM} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{K1} & w_{K2} & \cdots & w_{Kn} & \cdots & w_{KM} \end{pmatrix}$, in these

conditions, we have :

$$\mathbf{V}^H \mathbf{W}$$

$$\begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} & \cdots & w_{1M} \\ w_{21} & w_{22} & \vdots & w_{2n} & \cdots & w_{2M} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{p1} & w_{p2} & \cdots & w_{pn} & \cdots & w_{pM} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{K1} & w_{K2} & \cdots & w_{Kn} & \cdots & w_{KM} \end{pmatrix}$$
$$= (d_1^* \quad d_2^* \quad \cdots \quad d_m^* \quad \cdots \quad d_K^*) = \left(\sum_{k=1}^K w_{k1} d_k^* \quad \sum_{k=1}^K w_{k2} d_k^* \quad \cdots \quad \sum_{k=1}^K w_{km} d_k^* \quad \cdots \quad \sum_{k=1}^K w_{kM} d_k^* \right)$$

And, we have:

$$\mathcal{E}[\mathbf{V}^H \mathbf{W} \mathbf{W}^H \mathbf{V}]$$
$$= \mathcal{E} \left[\left| \sum_{k=1}^K w_{k1} d_k^* \right|^2 + \left| \sum_{k=1}^K w_{k2} d_k^* \right|^2 + \cdots + \left| \sum_{k=1}^K w_{km} d_k^* \right|^2 + \cdots + \left| \sum_{k=1}^K w_{kM} d_k^* \right|^2 \right]$$
$$= \sum_{k=1}^K |w_{k1}|^2 \mathcal{E} \left[|d_k^*|^2 \right] + \sum_{k=1}^K |w_{k2}|^2 \mathcal{E} \left[|d_k^*|^2 \right] + \cdots + \sum_{k=1}^K |w_{km}|^2 \mathcal{E} \left[|d_k^*|^2 \right] + \cdots + \sum_{k=1}^K |w_{kM}|^2 \mathcal{E} \left[|d_k^*|^2 \right]$$
$$= \sigma^2 \left[\sum_{k=1}^K |w_{k1}|^2 + |w_{k2}|^2 + \cdots + |w_{km}|^2 + \cdots + |w_{kM}|^2 \right]$$
$$= \frac{1}{K} \mathcal{E}[\mathbf{V}^H \mathbf{V}] \text{Trace}(\mathbf{W} \mathbf{W}^H)$$

Remark: It is possible to generalize this lemma to the case of a random matrix \mathbf{V} of size $K \times M$, and with a matrix \mathbf{W} of size $K \times P$. We still have:

$$\begin{aligned} & \mathcal{E}[\mathbf{V}^H \mathbf{W} \mathbf{W}^H \mathbf{V}] \\ &= \frac{1}{K} \mathcal{E}[\mathbf{V}^H \mathbf{V}] \text{Trace}(\mathbf{W} \mathbf{W}^H) \end{aligned}$$

We take \mathbf{V} as:
$$\mathbf{V} = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1m} & \cdots & d_{1M} \\ d_{21} & d_{22} & \cdots & d_{2m} & \cdots & d_{2M} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{k1} & d_{k2} & \cdots & d_{km} & \cdots & d_{kM} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{K1} & d_{K2} & \cdots & d_{Km} & \cdots & d_{KM} \end{pmatrix}$$
 and \mathbf{W} as:

$$\mathbf{W} = \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} & \cdots & w_{1P} \\ w_{21} & w_{22} & \cdots & w_{2n} & \cdots & w_{2P} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} & \cdots & w_{nP} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{K1} & w_{K2} & \cdots & w_{Kn} & \cdots & w_{KP} \end{pmatrix}, \text{ in these conditions we have:}$$

$\mathbf{V}^H \mathbf{W}$

$$\begin{aligned} & \begin{pmatrix} w_{11} & w_{12} & \cdots & w_{1n} & \cdots & w_{1P} \\ w_{21} & w_{22} & \cdots & w_{2n} & \cdots & w_{2P} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} & \cdots & w_{nP} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{K1} & w_{K2} & \cdots & w_{Kn} & \cdots & w_{KP} \end{pmatrix} \\ &= \begin{pmatrix} d_{11}^* & d_{21}^* & \cdots & d_{k1}^* & \cdots & d_{K1}^* \\ d_{12}^* & d_{22}^* & \cdots & d_{k2}^* & \cdots & d_{K2}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{1m}^* & d_{2m}^* & \cdots & d_{km}^* & \cdots & d_{Km}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{1M}^* & d_{2M}^* & \cdots & d_{kM}^* & \cdots & d_{KM}^* \end{pmatrix} \Rightarrow \mathbf{V}^H \mathbf{W}(m,n) = \sum_{k=1}^K d_{km}^* w_{kn} = a_{m,n} \end{aligned}$$

The general term of matrix $\mathbf{V}^H \mathbf{W} \mathbf{W}^H \mathbf{V}$ is thus written as:

$$\mathbf{V}^H \mathbf{W} \mathbf{W}^H \mathbf{V}(m,n) = \sum_{k=1}^M a_{m,k} b_{k,n} = \sum_{k=1}^M a_{m,k} a_{n,k}^* = \sum_{k=1}^M \left(\sum_{j=1}^K d_{jm}^* w_{jk} \right) \left(\sum_{i=1}^K d_{in} w_{ik}^* \right)$$

Hence, we obtain :

$$\mathcal{E}[\mathbf{V}^H \mathbf{W} \mathbf{W}^H \mathbf{V}(m, n)] = \sum_{k=1}^M \mathcal{E} \left[\left(\sum_{j=1}^K d_{jm}^* w_{jk} \right) \left(\sum_{i=1}^K d_{in} w_{ik}^* \right) \right] = \sum_{k=1}^M \sum_i \sum_j \mathcal{E} [d_{jm}^* d_{in}] w_{jk} w_{ik}^*$$

Clearly, we have: $\mathcal{E}[d_{jm}^* d_{in}] = \sigma^2$ if $i = j$ and $m = n$, otherwise $\mathcal{E}[d_{jm}^* d_{in}] = 0$. Hence we obtain:

$$\mathcal{E}[\mathbf{V}^H \mathbf{W} \mathbf{W}^H \mathbf{V}] = \text{Diag}[\sigma^2 \text{Trace}(\mathbf{W} \mathbf{W}^H), \dots, \sigma^2 \text{Trace}(\mathbf{W} \mathbf{W}^H)] = \frac{1}{K} \mathcal{E}[\mathbf{V}^H \mathbf{V}] \text{Trace}(\mathbf{W} \mathbf{W}^H)$$

$$\mathbf{V}^H \mathbf{V}$$

$$= \begin{pmatrix} d_{11}^* & d_{12}^* & \cdots & d_{1k_1}^* & \cdots & d_{1K_1}^* \\ d_{12}^* & d_{22}^* & \cdots & d_{k_2}^* & \cdots & d_{K_2}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{1m}^* & d_{2m}^* & \cdots & d_{km}^* & \cdots & d_{Km}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{1M}^* & d_{2M}^* & \cdots & d_{kM}^* & \cdots & d_{KM}^* \end{pmatrix} \Rightarrow \mathbf{V}^H \mathbf{V}(n, n) = \sum_{k=1}^K d_{kn}^* d_{kn}$$

Hence:

$$\mathcal{E}[\mathbf{V}^H \mathbf{V}(n, n)] = K \sigma^2$$

And : $\mathcal{E}[\mathbf{V}^H \mathbf{V}] = K \sigma^2 \mathbf{I}_K$ End of proof.

Using this lemma here, we obtain:

$$\begin{aligned}
& \mathcal{E} \left(\sum_{i=1, i \neq j}^L \mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{A} \mathcal{E} [\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H] \sum_{m=1}^L \mathbf{A}^H \mathbf{A} \mathbf{g}_{mln} \right) \\
&= \frac{1}{P} \text{Trace} [\mathcal{E} [\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H]] \sum_{i=1, i \neq j}^L \mathcal{E} [\mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{A} \mathbf{A}^H \mathbf{A} \mathbf{g}_{iln}] \\
&= \frac{1}{P^2} \left(\sum_{i=1, i \neq j}^L \mathcal{E} [\mathbf{g}_{iln}^H \mathbf{g}_{iln}] \right) \text{Trace} [(\mathbf{A}^H \mathbf{A})(\mathbf{A}^H \mathbf{A})] \text{Trace} [\mathcal{E} [\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H]] \\
&= \frac{1}{P^2} \left(\sum_{i=1, i \neq j}^L \mathcal{E} [\mathbf{g}_{iln}^H \mathbf{g}_{iln}] \right) \text{Trace} [(\mathbf{A}^H \mathbf{A})^2] \text{Trace} [\mathcal{E} [\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H]]
\end{aligned}$$

$$\begin{aligned}
& \text{Trace} [\mathcal{E} [\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H]] \\
&= \text{Trace} \left[\text{diag} \left[\sum_{k=1}^K \beta_{jlk} \mathcal{E} [|h_{jlk1}|^2], \sum_{k=1}^K \beta_{jlk} \mathcal{E} [|h_{jlk2}|^2], \dots, \sum_{k=1}^K \beta_{jlk} \mathcal{E} [|h_{jlkP}|^2] \right] \right] \\
&= \sum_{k=1}^K \beta_{jlk} \mathcal{E} \left[\sum_{m=1}^P |h_{jlk m}|^2 \right] \\
&= \sum_{k=1}^K \beta_{jlk} = \beta_{jl}
\end{aligned}$$

Hence:

$$\begin{aligned}
& \mathcal{E} \left(\sum_{i=1, i \neq j}^L \mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{A} \mathcal{E} [\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H] \sum_{m=1}^L \mathbf{A}^H \mathbf{A} \mathbf{g}_{mln} \right) \\
&= \frac{1}{P^2} \left(\sum_{i=1, i \neq j}^L \mathcal{E} [\mathbf{g}_{iln}^H \mathbf{g}_{iln}] \right) \text{Trace} [(\mathbf{A}^H \mathbf{A})^2] \beta_{jl}
\end{aligned}$$

And, using:

$$\sum_{i=1, i \neq j}^L \mathcal{E} [\mathbf{g}_{iln}^H \mathbf{g}_{iln}] = \sum_{i=1, i \neq j}^L \sum_{m=1}^P \mathcal{E} [|g_{ilm}|^2] = \sum_{i=1, i \neq j}^L \beta_{iln} \sum_{m=1}^P \mathcal{E} [|h_{ilm}|^2] = \sum_{i=1, i \neq j}^L \beta_{iln}$$

We eventually obtain:

$$\begin{aligned}
& \mathcal{E} \left(\sum_{i=1, i \neq j}^L \mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{A} \mathcal{E} [\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H] \sum_{m=1}^L \mathbf{A}^H \mathbf{A} \mathbf{g}_{mln} \right) \\
&= \frac{1}{P^2} \text{Trace} [(\mathbf{A}^H \mathbf{A})^2] \beta_{jl} \sum_{i=1, i \neq j}^L \beta_{iln}
\end{aligned} \tag{60}$$

Using the approximation: $\lim_{M \rightarrow \infty} \frac{1}{M} \mathbf{A}^H \mathbf{A} = \frac{1}{P} \mathbf{I}_P$, we obtain:

$$\begin{aligned}
& \lim_{M \rightarrow \infty} \mathcal{E} \left(\sum_{i=1, i \neq j}^L \mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{A} \mathcal{E} [\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H] \sum_{m=1}^L \mathbf{A}^H \mathbf{A} \mathbf{g}_{mln} \right) \\
&= \frac{M^2}{P^3} \beta_{jl} \sum_{i=1, i \neq j}^L \beta_{iln} = \frac{M^2 \beta_{jl} \sum_{i=1, i \neq j}^L \beta_{iln}}{P^3}
\end{aligned} \tag{61}$$

Concerning the term: $\mathcal{E}(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln})$, we have:

$$\begin{aligned}
& \mathcal{E}(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln}) \\
&= \mathcal{E} \left(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \sum_{p=1}^K \mathbf{G}_{j,l}(\cdot, p) \mathbf{G}_{j,l}^H(\cdot, p) \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln} \right) \\
&= \mathcal{E} \left(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \sum_{p=1}^K \mathbf{g}_{jlp} \mathbf{g}_{jlp}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln} \right) \\
&= \mathcal{E} \left(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \sum_{p=1, p \neq n}^K \mathbf{g}_{jlp} \mathbf{g}_{jlp}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln} \right) + \mathcal{E}(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln} \mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln})
\end{aligned} \tag{62}$$

For $p \neq n$ we have:

$$\begin{aligned}
& \mathcal{E}(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jlp} \mathbf{g}_{jlp}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln}) \\
&= \mathcal{E}(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathcal{E}[\mathbf{g}_{jlp} \mathbf{g}_{jlp}^H] \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln}) \\
&= \frac{1}{P} \text{Trace}[\mathcal{E}[\mathbf{g}_{jlp} \mathbf{g}_{jlp}^H]] \mathcal{E}[\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln}] \\
&= \frac{1}{P^2} \mathcal{E}[\mathbf{g}_{jln}^H \mathbf{g}_{jln}] \text{Trace}[(\mathbf{A}^H \mathbf{A})(\mathbf{A}^H \mathbf{A})] \text{Trace}[\mathcal{E}[\mathbf{g}_{jlp} \mathbf{g}_{jlp}^H]] \\
&= \frac{1}{P^2} \mathcal{E}[\mathbf{g}_{jln}^H \mathbf{g}_{jln}] \text{Trace}[(\mathbf{A}^H \mathbf{A})^2] \text{Trace}[\mathcal{E}[\mathbf{g}_{jlp} \mathbf{g}_{jlp}^H]]
\end{aligned}$$

With:

$$\text{Trace}[\mathcal{E}[\mathbf{g}_{jlp} \mathbf{g}_{jlp}^H]] = \mathcal{E}[|\mathbf{g}_{jlp1}|^2 + |\mathbf{g}_{jlp2}|^2 + \dots + |\mathbf{g}_{jlpP}|^2] = \beta_{jlp}$$

So:

$$\begin{aligned}
& \mathcal{E}(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jlp} \mathbf{g}_{jlp}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln}) \\
&= \frac{1}{P^2} \mathcal{E}[\mathbf{g}_{jln}^H \mathbf{g}_{jln}] \text{Trace}[(\mathbf{A}^H \mathbf{A})^2] \beta_{jlp} \tag{63} \\
&= \frac{1}{P^2} \text{Trace}[(\mathbf{A}^H \mathbf{A})^2] \beta_{jlp} \beta_{jln}
\end{aligned}$$

And:

$$\begin{aligned}
& \mathcal{E} \left(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \sum_{p=1, p \neq n}^K \mathbf{g}_{jlp} \mathbf{g}_{jlp}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln} \right) \\
&= \frac{1}{P^2} \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jln} \sum_{p=1, p \neq n}^K \beta_{jlp} \\
&= \frac{1}{P^2} \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jln} \sum_{p=1}^K \beta_{jlp} - \frac{1}{P^2} \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jln}^2 \quad (64) \\
&= \frac{1}{P^2} \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jln} \beta_{jl} - \frac{1}{P^2} \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jln}^2 \\
&= \frac{1}{P^2} \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \left(\beta_{jln} \beta_{jl} - \beta_{jln}^2 \right)
\end{aligned}$$

The only term we have to evaluate now is:

$$\begin{aligned}
& \mathcal{E} \left(\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln} \mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln} \right) \\
&= \mathcal{E} \left(\left[\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln} \right]^2 \right)
\end{aligned}$$

We have:

$$\begin{aligned}
& \mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln} \\
&= \frac{1}{P} \left\{ \left(\sum_{m=1}^P \mathbf{g}_{jlnm}^* e^{j f_1(\phi_m)} \right) \left(\sum_{m=1}^P \mathbf{g}_{jlnk} e^{-j f_1(\phi_k)} \right) + \dots + \left(\sum_{m=1}^P \mathbf{g}_{jlnm}^* e^{j f_M(\phi_m)} \right) \left(\sum_{m=1}^P \mathbf{g}_{jlnk} e^{-j f_M(\phi_k)} \right) \right\} \\
&= \frac{1}{P} \left\{ \sum_k \sum_m \mathbf{g}_{jlnm}^* \mathbf{g}_{jlnk} e^{j[f_1(\phi_m) - f_1(\phi_k)]} + \dots + \sum_k \sum_m \mathbf{g}_{jlnm}^* \mathbf{g}_{jlnk} e^{j[f_M(\phi_m) - f_M(\phi_k)]} \right\} \\
&= \frac{1}{P} \left\{ A_{m,k}^1 + A_{m,k}^2 + \dots + A_{m,k}^M \right\}
\end{aligned}$$

$$\text{With : } A_{m,k}^r = \sum_k \sum_m \mathbf{g}_{jlnm}^* \mathbf{g}_{jlnk} e^{j[f_r(\phi_m) - f_r(\phi_k)]}$$

And:

$$\begin{aligned}
& \left[\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln} \right]^2 \\
&= \frac{1}{P^2} \left[\left(A_{m,k}^1 \right)^2 + \left(A_{m,k}^2 \right)^2 + \dots + \left(A_{m,k}^M \right)^2 + \sum_r \sum_t A_{m,k}^r A_{m,k}^t \right]
\end{aligned}$$

We have:

$$\begin{aligned}
\mathcal{E}\left[\left(A_{m,k}^n\right)^2\right] &= \mathcal{E}\left[\left(\sum_k \sum_m \mathbf{g}_{jlnm}^* \mathbf{g}_{jlnk} e^{j[l f_n(\phi_m) - f_n(\phi_k)]}\right) \left(\sum_s \sum_t \mathbf{g}_{jlns}^* \mathbf{g}_{jlnt} e^{j[l f_n(\phi_s) - f_n(\phi_t)]}\right)\right] \\
&= \mathcal{E}\left[\left(\sum_k \sum_m \sum_s \sum_t \mathbf{g}_{jlnm}^* \mathbf{g}_{jlnk} \mathbf{g}_{jlns}^* \mathbf{g}_{jlnt} e^{j[l f_n(\phi_m) - f_n(\phi_k) + f_n(\phi_s) - f_n(\phi_t)]}\right)\right] \\
&= \sum_k \sum_m \sum_s \sum_t \mathcal{E}\left[\mathbf{g}_{jlnm}^* \mathbf{g}_{jlnk} \mathbf{g}_{jlns}^* \mathbf{g}_{jlnt}\right] e^{j[l f_n(\phi_m) - f_n(\phi_k) + f_n(\phi_s) - f_n(\phi_t)]} \\
&= \sum_k \sum_s \mathcal{E}\left[|\mathbf{g}_{jlnk}|^2\right] \mathcal{E}\left[|\mathbf{g}_{jlns}|^2\right] \\
&= \beta_{jn}^2
\end{aligned}$$

Then, we have to calculate:

$$\begin{aligned}
\mathcal{E}\left[A_{m,k}^r A_{m,k}^t\right] &= \mathcal{E}\left[\left(\sum_k \sum_m \mathbf{g}_{jlnm}^* \mathbf{g}_{jlnk} e^{j[l f_r(\phi_m) - f_r(\phi_k)]}\right) \left(\sum_s \sum_p \mathbf{g}_{jlns}^* \mathbf{g}_{jlnp} e^{j[l f_t(\phi_s) - f_t(\phi_p)]}\right)\right] \\
&= \sum_k \sum_m \sum_s \sum_p \mathcal{E}\left[\mathbf{g}_{jlnm}^* \mathbf{g}_{jlnk} \mathbf{g}_{jlns}^* \mathbf{g}_{jlnp}\right] e^{j[l f_r(\phi_m) - f_r(\phi_k) + f_t(\phi_s) - f_t(\phi_p)]} \\
&= \sum_k \sum_p \mathcal{E}\left[\mathbf{g}_{jlnm}^* \mathbf{g}_{jlnk}\right] \mathcal{E}\left[\mathbf{g}_{jlns}^* \mathbf{g}_{jlnp}\right] \\
&= \beta_{jn}^2
\end{aligned}$$

With these approximations, we obtain:

$$\begin{aligned}
&\mathcal{E}\left[\mathbf{g}_{jln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{jln}\right]^2 \\
&\approx \frac{1}{P^2} \left[M \beta_{jn}^2 + M^2 \beta_{jn}^2 \right] \quad (65) \\
&= \frac{M(M+1) \beta_{jn}^2}{P^2}
\end{aligned}$$

In the same manner we obtain:

$$\begin{aligned}
&\mathcal{E}\left(\mathbf{w}_{ln}^H \mathbf{A} \mathcal{E}\left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H\right] \mathbf{A}^H \mathbf{w}_{ln}\right) \\
&= \frac{1}{P} \text{Trace}\left[\mathcal{E}\left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H\right] \mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{A} \mathbf{A}^H \mathbf{w}_{ln}\right]\right] \\
&= \frac{\beta_{jl}}{P} \frac{1}{P} \sum P \mathcal{E}\left[|w_{ln}(1)|^2 + |w_{ln}(1)|^2 + |w_{ln}(M)|^2\right] \quad (66) \text{ (see (68))} \\
&= \frac{M \sigma_n^2 \beta_{jl}}{P}
\end{aligned}$$

We deduce:

$$\mathcal{E}\left(\|\boldsymbol{\alpha}_{jn}^T\|^2\right) = \frac{p_p p_u}{P^2} \text{Trace}\left[\left(\mathbf{A}^H \mathbf{A}\right)^2\right] \left(\bar{\beta}_{ln} \beta_{jl} - \beta_{jn}^2\right) + \frac{p_p p_u M(M+1)}{P^2} \beta_{jn}^2 + \frac{p_u M \sigma_{n_l}^2 \beta_{jl}}{P} \quad (67)$$

The calculation of $\mathbf{w}_{ln}^H \mathbf{A}$ yields:

$$\begin{aligned} \mathbf{w}_{ln}^H \mathbf{A} &= \frac{1}{\sqrt{P}} \left[w_{ln}^*(1), w_{ln}^*(2), \dots, w_{ln}^*(M) \right] \begin{bmatrix} e^{-jf_1(\phi_1)} & e^{-jf_1(\phi_2)} & \dots & e^{-jf_1(\phi_P)} \\ e^{-jf_2(\phi_1)} & e^{-jf_2(\phi_2)} & \dots & e^{-jf_2(\phi_P)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-jf_M(\phi_1)} & e^{-jf_M(\phi_2)} & \dots & e^{-jf_M(\phi_P)} \end{bmatrix} \\ &= \frac{1}{\sqrt{P}} \left[w_{ln}^*(1) e^{-jf_1(\phi_1)} + \dots + w_{ln}^*(M) e^{-jf_M(\phi_1)}, \dots, w_{ln}^*(1) e^{-jf_1(\phi_P)} + \dots + w_{ln}^*(M) e^{-jf_M(\phi_P)} \right] \end{aligned}$$

And, in these conditions, we have:

$$\begin{aligned} &\mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{A} \mathbf{A}^H \mathbf{w}_{ln}\right] \\ &= \frac{1}{P} \sum P \mathcal{E}\left[\left|w_{ln}(1)\right|^2 + \left|w_{ln}(2)\right|^2 + \dots + \left|w_{ln}(M)\right|^2\right] \quad (68) \\ &= M \sigma_{n_l}^2 \end{aligned}$$

Remark : Using the lemma of appendix 2, we can directly obtain:

$$\mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{A} \mathbf{A}^H \mathbf{w}_{ln}\right] = \frac{1}{M} \mathcal{E}\left[\mathbf{w}_{ln}^H \mathbf{w}_{ln}\right] \text{Trace}\left[\mathbf{A} \mathbf{A}^H\right] = \frac{1}{M} M \sigma_{n_l}^2 M = M \sigma_{n_l}^2$$

We obtain the uplink rate of the n th user in the l th cell as:

$$R_{ln} = C \left(\frac{p_p p_u M^2 / P^2 \beta_{ln}^2}{\sum_{j=1}^L \left(p_p p_u / P^2 \text{Trace}\left[\left(\mathbf{A}^H \mathbf{A}\right)^2\right] \left(\bar{\beta}_{ln} \beta_{jl} - \beta_{jn}^2\right) + p_p p_u M(M+1) \beta_{jn}^2 / P^2 + p_u M \sigma_{n_l}^2 \beta_{jl} / P \right) - p_p p_u M^2 / P^2 \beta_{ln}^2 + M \sigma_{n_l}^2 / P \left(p_p \bar{\beta}_{ln} + \frac{\sigma_{n_l}^2}{M} \sum_{k=1}^P S_{U^2(\cdot, k)}^{f_i} \right)} \right)$$

This can be simplified as:

$$R_{ln} = C \left(\frac{\beta_{ln}^2}{\sum_{j=1}^L \left(\text{Trace}\left[\left(\mathbf{A}^H \mathbf{A}\right)^2\right] \left(\bar{\beta}_{ln} \beta_{jl} - \beta_{jn}^2\right) / M^2 + M(M+1) \beta_{jn}^2 / M^2 + \sigma_{n_l}^2 \beta_{jl} P / M p_p \right) - \beta_{ln}^2 + \sigma_{n_l}^2 P / M \left(\bar{\beta}_{ln} / p_u + \frac{\sigma_{n_l}^2}{p_p p_u M} \sum_{k=1}^P S_{U^2(\cdot, k)}^{f_i} \right)} \right)$$

With: $C(x) = \log_2(1+x)$ and $S_{U^2(\cdot, k)}^{f_i}$ which has been defined in equation (57).

If we examine the behaviour of the former expression for an unlimited number of base station antennas, we have clearly:

$$\lim_{M \rightarrow \infty} \frac{1}{M^2} \text{Trace} [\mathbf{A}^H \mathbf{A}]^2 = \frac{1}{P^2} \text{Trace} [\mathbf{I}_P] = \frac{1}{P}$$

Concerning the term at the extreme side of the denominator we have to resort to the original definition:

$$\begin{aligned} \mathcal{E} \left\{ |z_{ln}|^2 \right\} &= \sigma_{n_l}^2 \left\{ \mathcal{E} \left[p_p \sum_{i=1}^L \mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{g}_{iln} \right] + \mathcal{E} \left[\mathbf{w}_{ln}^H \mathbf{\Pi}_A \mathbf{\Pi}_A^H \mathbf{w}_{ln} \right] \right\} \\ &= \sigma_{n_l}^2 \left\{ \mathcal{E} \left[p_p \sum_{i=1}^L \mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{g}_{iln} \right] + \mathcal{E} \left[\mathbf{w}_{ln}^H \mathbf{\Pi}_A \mathbf{w}_{ln} \right] \right\} \\ &= \sigma_{n_l}^2 \left\{ \mathcal{E} \left[p_p \sum_{i=1}^L \mathbf{g}_{iln}^H \mathbf{A}^H \mathbf{A} \mathbf{g}_{iln} \right] + \mathcal{E} \left[\mathbf{w}_{ln}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{w}_{ln} \right] \right\} \end{aligned}$$

We have:

$$\mathcal{E} \left[\mathbf{w}_{ln}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{w}_{ln} \right] \approx_{M \rightarrow \infty} \mathcal{E} \left[\mathbf{w}_{ln}^H \mathbf{A} \frac{P}{M} \mathbf{I}_P \mathbf{A}^H \mathbf{w}_{ln} \right] = \frac{P}{M} \mathcal{E} \left[\mathbf{w}_{ln}^H \mathbf{A} \mathbf{A}^H \mathbf{w}_{ln} \right]$$

We have (see (68)): $\mathcal{E} \left[\mathbf{w}_{ln}^H \mathbf{A} \mathbf{A}^H \mathbf{w}_{ln} \right] = M \sigma_{n_l}^2$, so we obtain :

$$\mathcal{E} \left[\mathbf{w}_{ln}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{w}_{ln} \right] \approx_{M \rightarrow \infty} P \sigma_{n_l}^2$$

Reporting all these obtained equivalent values into the former expression, we obtain:

$$R_{ln} \approx_{M \rightarrow \infty} C \left(\frac{\beta_{ln}^2}{\sum_{j=1}^L \left((\bar{\beta}_{ln} \beta_{jl} - \beta_{jln}^2) / P + \beta_{jln}^2 \right) - \beta_{ln}^2} \right) \quad (69)$$

Which can be rewritten as:

$$R_{ln} \approx_{M \rightarrow \infty} C \left(\frac{\beta_{ln}^2}{\bar{\beta}_{ln} \sum_{j=1}^L \beta_{jl} / P + \sum_{j=1}^L \beta_{jln}^2 (1 - 1/P) - \beta_{ln}^2} \right) \quad (70)$$

If P also goes to infinity, the lower bound on the achievable rate becomes:

$$R_{ln} \approx_{M \rightarrow \infty} C \left(\frac{\beta_{lln}^2}{\sum_{j=1, j \neq l}^L \beta_{jln}^2} \right) \quad (71)$$

4.3. Zero forcing Receiver:

In this case the l th base station processes the received signal by multiplying it by the pseudo-invert of the channel estimate. From (6) and (14), we obtain:

$$\mathbf{r}_l = \left(\hat{\Gamma}_{ll}^H \hat{\Gamma}_{ll} \right)^{-1} \hat{\Gamma}_{ll}^H \mathbf{y}_l \quad (72)$$

With: $\hat{\Gamma}_{ll} = \mathbf{A}_{M \times P} \left[\hat{\mathbf{H}}_{ll} \mathbf{D}_{ll}^{1/2} \right]_{P \times K} = p_p^{-1/2} \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll}$. $\mathbf{\Pi}_A$ is of size $M \times M$, $\tilde{\mathbf{Y}}_{p,l}$ is of size $M \times K$. The product $\hat{\Gamma}_{ll}^H \hat{\Gamma}_{ll}$ can be written as:

$$\begin{aligned} \hat{\Gamma}_{ll}^H \hat{\Gamma}_{ll} &= p_p^{-1/2} \left[\mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll} \right]^H p_p^{-1/2} \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll} \\ &= p_p^{-1} \mathbf{D}_{ll}^H \left(\mathbf{D}_l^{-1} \right)^H \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll} \end{aligned} \quad (73)$$

Of course, we have: $\mathbf{\Pi}_A^H \mathbf{\Pi}_A = \mathbf{\Pi}_A$, so (73) reduces to :

$$\begin{aligned} \hat{\Gamma}_{ll}^H \hat{\Gamma}_{ll} &= p_p^{-1} \mathbf{D}_{ll}^H \left(\mathbf{D}_l^{-1} \right)^H \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll} \\ &= \frac{1}{p_p} \mathbf{D}_{ll} \mathbf{D}_l^{-1} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll} \end{aligned} \quad (74)$$

In these conditions, we have:

$$\begin{aligned} \left[\hat{\Gamma}_{ll}^H \hat{\Gamma}_{ll} \right]^{-1} &= p_p \left[\mathbf{D}_{ll} \left(\mathbf{D}_l^{-1} \right)^H \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll} \right]^{-1} \\ &= p_p \mathbf{D}_{ll}^{-1} \mathbf{D}_l \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \mathbf{D}_l \mathbf{D}_{ll}^{-1} \end{aligned} \quad (75)$$

Then, we obtain:

$$\begin{aligned} &\left[\hat{\Gamma}_{ll}^H \hat{\Gamma}_{ll} \right]^{-1} \hat{\Gamma}_{ll}^H \\ &= p_p \mathbf{D}_{ll}^{-1} \mathbf{D}_l \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \mathbf{D}_l \mathbf{D}_{ll}^{-1} p_p^{-1/2} \left[\mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll} \right]^H \\ &= p_p^{1/2} \mathbf{D}_{ll}^{-1} \mathbf{D}_l \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \mathbf{D}_l \mathbf{D}_{ll}^{-1} \mathbf{D}_{ll} \mathbf{D}_l^{-1} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A^H \\ &= p_p^{1/2} \mathbf{D}_{ll}^{-1} \mathbf{D}_l \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A^H \end{aligned} \quad (76)$$

And:

$$\begin{aligned}
\mathbf{r}_l &= \left(\hat{\Gamma}_l^H \hat{\Gamma}_l \right)^{-1} \hat{\Gamma}_l^H \mathbf{y}_l \\
&= p_p^{1/2} \mathbf{D}_l^{-1} \mathbf{D}_l \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{y}_l \quad (77)
\end{aligned}$$

Let $\tilde{\mathbf{r}}_l$ the received $K \times 1$ vector: $\tilde{\mathbf{r}}_l \equiv \frac{1}{\sqrt{p_p}} \mathbf{D}_l^{-1} \mathbf{D}_l \mathbf{r}_l$, we deduce from (77) that :

$$\tilde{\mathbf{r}}_l = \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{y}_l \quad (78)$$

4.4. Analysis of the Pilot Contamination effect:

We denote as $\mathbf{M}_{p,l}^A$ the matrix: $\left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]_{K \times K}^{-1}$. In this case, if we look at the n th component of vector $\tilde{\mathbf{r}}_l$, we have :

$$\begin{aligned}
\tilde{\mathbf{r}}_l(n) &= \tilde{r}_{ln} \\
&= \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{y}_l \\
&= \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \left[\sqrt{p_u} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \mathbf{x}_i + \mathbf{n}_l \right] \\
&= \sqrt{p_u} \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \mathbf{x}_l + \sqrt{p_u} \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \sum_{i \neq l}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \mathbf{x}_i + \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{n}_l \\
&= \boldsymbol{\alpha}_{ln}^T \mathbf{x}_l + \sum_{j \neq l}^L \boldsymbol{\alpha}_{jn}^T \mathbf{x}_j + z_{ln} \quad (79)
\end{aligned}$$

With: $\boldsymbol{\alpha}_{jn}^T \equiv \sqrt{p_u} \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{H}_{j,l} \mathbf{D}_{j,l}^{1/2}$ and $z_{ln} \equiv \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{n}_l$. We have :

$$\mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\} = \mathcal{E}\left[\sqrt{p_u} \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \right] \mathbf{e}_n \quad (80)$$

Where \mathbf{e}_n is the n th column of the $K \times K$ identity matrix. By adding and subtracting $\mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}$ from $\boldsymbol{\alpha}_{ln}^T$ in (79), we obtain:

$$\tilde{r}_{ln} = \mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\} \mathbf{x}_l + (\boldsymbol{\alpha}_{ln}^T - \mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}) \mathbf{x}_l + \sum_{j \neq l}^L \boldsymbol{\alpha}_{jn}^T \mathbf{x}_j + z_{ln} \quad (81)$$

We obtain again the capacity of the n th user in the l th cell:

$$R_{ln} = C \left(\frac{\mathcal{E}\{\|\boldsymbol{\alpha}_{ln}^T\|^2\}}{\sum_{j=1}^L \mathcal{E}\{\|\boldsymbol{\alpha}_{jn}^T\|^2\} - \mathcal{E}\{\|\boldsymbol{\alpha}_{ln}^T\|^2\} + \mathcal{E}\{|z_{ln}|^2\}} \right) \quad (82)$$

To develop (82) we detail first the calculation of the matrix: $(\mathbf{M}_{p,l}^A)^{-1} = \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l}$, we have :

$$\begin{aligned} (\mathbf{M}_{p,l}^A)^{-1} &= \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \\ &= \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right) \\ &= \left(\sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H + \mathbf{W}_l^H \right) \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right) \\ &= p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \dots \\ &\quad + \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \\ &= p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \end{aligned}$$

Clearly, this reduces to:

$$\begin{aligned} &\mathcal{E}[\tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l}] \\ &= p_p \mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{j=1}^L \mathbf{G}_{j,l} \right] + \mathcal{E}[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l] \\ &= p_p \sum_{i,j} \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}] + \mathcal{E}[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l] \end{aligned}$$

To further proceed this quantity, we have to calculate:

$$\mathbf{G}_{j,l}^H \mathbf{A}^H = \begin{bmatrix} g_{j11}^* & g_{j12}^* & \cdots & g_{j1k}^* & \cdots & g_{j1P}^* \\ g_{j21}^* & g_{j22}^* & \cdots & g_{j2k}^* & \cdots & g_{j2P}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{jlm1}^* & g_{jlm2}^* & \cdots & g_{jlmk}^* & \cdots & g_{jlmP}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ g_{jK1}^* & g_{jK2}^* & \cdots & g_{jKk}^* & \cdots & g_{jKP}^* \end{bmatrix} = \frac{1}{\sqrt{P}} \begin{bmatrix} e^{jf_1(\phi_1)} & e^{jf_2(\phi_1)} & \cdots & e^{jf_M(\phi_1)} \\ e^{jf_1(\phi_2)} & e^{jf_2(\phi_2)} & \cdots & e^{jf_M(\phi_2)} \\ \vdots & \vdots & \ddots & \vdots \\ e^{jf_1(\phi_P)} & e^{jf_2(\phi_P)} & \cdots & e^{jf_M(\phi_P)} \end{bmatrix} \begin{bmatrix} \sum_{k=1}^P g_{j1k}^* e^{jf_1(\phi_k)} & \sum_{k=1}^P g_{j1k}^* e^{jf_2(\phi_k)} & \cdots & \sum_{k=1}^P g_{j1k}^* e^{jf_M(\phi_k)} \\ \sum_{k=1}^P g_{j2k}^* e^{jf_1(\phi_k)} & \sum_{k=1}^P g_{j2k}^* e^{jf_2(\phi_k)} & \cdots & \sum_{k=1}^P g_{j2k}^* e^{jf_M(\phi_k)} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^P g_{jKk}^* e^{jf_1(\phi_k)} & \sum_{k=1}^P g_{jKk}^* e^{jf_2(\phi_k)} & \cdots & \sum_{k=1}^P g_{jKk}^* e^{jf_M(\phi_k)} \end{bmatrix}$$

The (m,n) th element of $\mathbf{G}_{i,l}^H \mathbf{A}^H$ is thus written as :

$$\left[\mathbf{G}_{i,l}^H \mathbf{A}^H(m,n) \right]_{K \times M} = a_{m,n} = \frac{1}{\sqrt{P}} \sum_{k=1}^P g_{ilmk}^* e^{jf_n(\phi_k)} \quad (83)$$

Concerning the matrix: $\mathbf{A} \mathbf{G}_{j,l}$, we obtain :

$$\mathbf{A} \mathbf{G}_{j,l} = \frac{1}{\sqrt{P}} \begin{bmatrix} \sum_{n=1}^P e^{-jf_1(\phi_n)} g_{j1n} & \sum_{n=1}^P e^{-jf_1(\phi_n)} g_{j2n} & \cdots & \sum_{n=1}^P e^{-jf_1(\phi_n)} g_{jkn} & \cdots & \sum_{n=1}^P e^{-jf_1(\phi_n)} g_{jKn} \\ \sum_{n=1}^P e^{-jf_2(\phi_n)} g_{j1n} & \sum_{n=1}^P e^{-jf_2(\phi_n)} g_{j2n} & \cdots & \sum_{n=1}^P e^{-jf_2(\phi_n)} g_{jkn} & \cdots & \sum_{n=1}^P e^{-jf_2(\phi_n)} g_{jKn} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \sum_{n=1}^P e^{-jf_m(\phi_n)} g_{j1n} & \sum_{n=1}^P e^{-jf_m(\phi_n)} g_{j2n} & \cdots & \sum_{n=1}^P e^{-jf_m(\phi_n)} g_{jkn} & \cdots & \sum_{n=1}^P e^{-jf_m(\phi_n)} g_{jKn} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \sum_{n=1}^P e^{-jf_M(\phi_n)} g_{j1n} & \sum_{n=1}^P e^{-jf_M(\phi_n)} g_{j2n} & \cdots & \sum_{n=1}^P e^{-jf_M(\phi_n)} g_{jkn} & \cdots & \sum_{n=1}^P e^{-jf_M(\phi_n)} g_{jKn} \end{bmatrix}$$

Thus, the (m,k) th element of $\mathbf{A} \mathbf{G}_{j,l}$ is thus written as :

$$\left[\mathbf{A} \mathbf{G}_{j,l}(m,k) \right]_{M \times K} = \frac{1}{\sqrt{P}} \sum_{n=1}^P g_{jkn} e^{-jf_m(\phi_n)} \quad (84)$$

And the (m,n) th element of matrix $\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}$ is equal to:

$$\begin{aligned}
\left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] (m,n) &= \sum_{k=1}^M a_{m,k} b_{k,n} \\
&= \frac{1}{P} \sum_{k=1}^M \left(\sum_{r=1}^P g_{ilmr}^* e^{j f_k(\phi_r)} \right) \left(\sum_{q=1}^P g_{jlnq} e^{-j f_k(\phi_q)} \right) \\
&= \frac{1}{P} \sum_{k=1}^M \left(\sum_{r=1}^P \sum_{q=1}^P g_{ilmr}^* g_{jlnq} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \\
&= \frac{1}{P} \sum_{k=1}^M \left(\sum_{r=1}^P \sum_{q=1}^P h_{ilmr}^* h_{jlnq} \sqrt{\beta_{ilm} \beta_{jln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \tag{85}
\end{aligned}$$

Hence, we have:

$$\mathcal{E} \left[\left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] (m,n) \right] = \frac{1}{P} \sum_{k=1}^M \left(\sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[h_{ilmr}^* h_{jlnq} \right] \sqrt{\beta_{ilm} \beta_{jln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \tag{86}$$

For $i = j$, we obtain:

$$\mathcal{E} \left[\left[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] (m,n) \right] = \frac{1}{P} \sum_{k=1}^M \left(\sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[h_{jlmr}^* h_{jlnq} \right] \sqrt{\beta_{jlm} \beta_{jln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right)$$

The quantity: $\mathcal{E} \left[h_{ilmr}^* h_{jlnq} \right]$ is equal to zero for $i \neq j$ and for $i = j$, due to the independency between columns, we have $\mathcal{E} \left[h_{ilmr}^* h_{jlnq} \right] = 0$ if $m \neq n$. For $i = j$ and $m = n$ we have:

$$\begin{aligned}
\mathcal{E} \left[\left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] (n,n) \right] &= \frac{1}{P} \sum_{k=1}^M \left(\sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[h_{ilmr}^* h_{ilnq} \right] \beta_{iln} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \\
\mathcal{E} \left[\left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] (n,n) \right] &= \frac{1}{P} \sum_{k=1}^M \left(\sum_{r=1}^P \mathcal{E} \left[|h_{ilmr}|^2 \beta_{iln} \right] \right) = \frac{M \beta_{iln}}{P} \tag{87}
\end{aligned}$$

Concerning the quantity: $\mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right]$, we have already developed the calculations in the former section,

$$\begin{aligned}
&\mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= \frac{1}{M} \mathcal{E} \left[\mathbf{W}_l^H \mathbf{W}_l \right] \text{Trace} \left(\mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right) \tag{88} \\
&= \frac{1}{M} M \sigma_{n_l}^2 \mathbf{I}_K \text{Trace} \left[\mathbf{\Pi}_A \right] \approx_{M \rightarrow \infty} \sigma_{n_l}^2 \mathbf{P} \mathbf{I}_K
\end{aligned}$$

We eventually obtain:

$$\mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] = \text{diag} \left(\rho_{1,l}, \rho_{2,l}, \dots, \rho_{K,l} \right) \quad (89)$$

With:

$$\begin{aligned} \rho_{n,l} &= p_p \frac{M}{P} \sum_{i=1}^L \beta_{in} + \sigma_{n_i}^2 P \\ \rho_{n,l} &= p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_i}^2 P \end{aligned}$$

We come back to the calculation of: $\mathcal{E} \{ \boldsymbol{\alpha}_{ln}^T \}$, remember we have :

$$\begin{aligned} \mathcal{E} \{ \boldsymbol{\alpha}_{ln}^T \} &= \mathcal{E} \left[\sqrt{p_u} \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \right] \mathbf{e}_n \\ \mathcal{E} \{ \boldsymbol{\alpha}_{ln}^T \} &\approx \sqrt{p_u} \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \right] \mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{G}_{l,l} \right] \mathbf{e}_n \\ &= \sqrt{p_u} \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \right] \mathcal{E} \left[\left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{G}_{l,l} \right] \mathbf{e}_n \\ &= \sqrt{p_u} \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \right] \mathcal{E} \left[\sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{G}_{l,l} \right] \mathbf{e}_n \\ &= \sqrt{p_p p_u} \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \right] \mathcal{E} \left[\mathbf{G}_{l,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{l,l} \right] \mathbf{e}_n \end{aligned}$$

We have already found that (see equation (87)):

$$\mathcal{E} \left[\left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] (n,n) \right] = \frac{M \beta_{in}}{P}$$

Using: $\mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \right] \approx 1 / \rho_{n,l}$, we eventually obtain:

$$\begin{aligned} \mathcal{E} \{ \boldsymbol{\alpha}_{ln}^T \} &\approx \sqrt{p_p p_u} \frac{M \beta_{ln}}{P \rho_{n,l}} \mathbf{e}_n \\ &= \sqrt{p_p p_u} \frac{M \beta_{ln}}{P \left[p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_i}^2 P \right]} \mathbf{e}_n \quad (90) \\ &= \sqrt{p_p p_u} \frac{M \beta_{ln}}{p_p M \bar{\beta}_{ln} + P^2 \sigma_{n_i}^2} \mathbf{e}_n \\ &= \sqrt{p_p p_u} \frac{\beta_{ln}}{p_p \bar{\beta}_{ln} + P^2 \sigma_{n_i}^2 / M} \mathbf{e}_n \end{aligned}$$

Concerning the quantity: $\mathcal{E} \left\{ \left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right\}$, we have:

$$\begin{aligned}
& \mathcal{E} \left\{ \left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right\} = \mathcal{E} \left(\boldsymbol{\alpha}_{jn}^T \boldsymbol{\alpha}_{jn}^* \right) \\
& = p_u \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:) \right]^H \\
& = p_u \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:) \right]^H
\end{aligned}$$

We have:

$$\begin{aligned}
& \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \\
& = \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \left(\sqrt{p_p} \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} + \mathbf{W}_l \right) \\
& = \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \left(\sqrt{p_p} \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} + \mathbf{W}_l \right)
\end{aligned}$$

And this yields:

$$\begin{aligned}
& \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:) \right]^H \\
& = \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \left(\sqrt{p_p} \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} + \mathbf{W}_l \right) \mathbf{M}_{p,l}^A(n,:) \right]^H \\
& = \mathcal{E} \left[p_p \mathbf{M}_{p,l}^A(n,:) \sum_{i=1, i \neq j}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1, m \neq j}^L \mathbf{G}_{m,l} \mathbf{M}_{p,l}^A(n,:) \right]^H + \dots \\
& \dots + \mathcal{E} \left[p_p \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{M}_{p,l}^A(n,:) \right]^H + \dots \\
& \dots + \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{M}_{p,l}^A(n,:) \right]^H
\end{aligned}$$

And, we have:

$$\begin{aligned}
& \mathcal{E} \left[p_p \mathbf{M}_{p,l}^A(n,:) \sum_{i=1, i \neq j}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1, m \neq j}^L \mathbf{G}_{m,l} \mathbf{M}_{p,l}^A(n,:) \right]^H \\
& = \frac{p_p}{P} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \sum_{i=1, i \neq j}^L \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{M}_{p,l}^A(n,:) \right]^H \\
& = \frac{p_p}{P^2} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \text{Trace} \left[(\mathbf{A}^H \mathbf{A})^2 \right] \sum_{i=1, i \neq j}^L \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \mathbf{M}_{p,l}^A(n,:) \right]^H \\
& = \frac{p_p}{P^2 K} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \text{Trace} \left[(\mathbf{A}^H \mathbf{A})^2 \right] \text{Trace} \left[\mathcal{E} \left[\sum_{i=1, i \neq j}^L \mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \right] \right] \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:) \right]^H
\end{aligned}$$

$$\begin{aligned}
&= \frac{P_p}{P^2 K} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \left[\sum_{i=1, i \neq j}^L \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \right] \right] \right] \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:)^H \right] \\
&= \frac{p_p \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:)^H \right]}{P^2 K} \\
&= \frac{p_p \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) \gamma_{n,l}}{P^2 K}
\end{aligned}$$

With : $\mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:)^H \right] \approx \gamma_{n,l}$

We have (already demonstrated in the former paragraph):

$$\mathcal{E} \left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] = \text{diag} \left[\sum_{k=1}^K \beta_{jlk} \mathcal{E} \left[|h_{jlk1}|^2 \right], \sum_{k=1}^K \beta_{jlk} \mathcal{E} \left[|h_{jlk2}|^2 \right], \dots, \sum_{k=1}^K \beta_{jlk} \mathcal{E} \left[|h_{jlk m}|^2 \right], \dots, \sum_{k=1}^K \beta_{jlk} \mathcal{E} \left[|h_{jlk P}|^2 \right] \right]$$

And : $\text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] = \sum_{k=1}^K \beta_{jlk} = \beta_{jl}$. Concerning the quantity $\mathcal{E} \left[\mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \right]$, we have :

$$\mathbf{G}_{j,l}^H \mathbf{G}_{j,l} = \begin{bmatrix} \mathcal{G}_{j11} & \mathcal{G}_{j12} & \cdots & \mathcal{G}_{j1k} & \cdots & \mathcal{G}_{j1K} \\ \mathcal{G}_{j12} & \mathcal{G}_{j22} & \cdots & \mathcal{G}_{j2k} & \cdots & \mathcal{G}_{j2K} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{G}_{j1m} & \mathcal{G}_{j2m} & \cdots & \mathcal{G}_{jkm} & \cdots & \mathcal{G}_{jKm} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{G}_{j1P} & \mathcal{G}_{j2P} & \cdots & \mathcal{G}_{jKP} & \cdots & \mathcal{G}_{jKP} \end{bmatrix} = \begin{bmatrix} \sum_{k=1}^P |\mathcal{G}_{j1k}|^2 & \sum_{k=1}^P \mathcal{G}_{j1k}^* \mathcal{G}_{j2k} & \cdots & \sum_{k=1}^P \mathcal{G}_{j1k}^* \mathcal{G}_{jmk} & \cdots & \sum_{k=1}^P \mathcal{G}_{j1k}^* \mathcal{G}_{jKk} \\ \sum_{k=1}^P \mathcal{G}_{j2k} \mathcal{G}_{j1k}^* & \sum_{k=1}^P |\mathcal{G}_{j2k}|^2 & \cdots & \sum_{k=1}^P \mathcal{G}_{j2k} \mathcal{G}_{jmk}^* & \cdots & \sum_{k=1}^P \mathcal{G}_{j2k} \mathcal{G}_{jKk}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^P \mathcal{G}_{jmk} \mathcal{G}_{j1k}^* & \sum_{k=1}^P \mathcal{G}_{jmk} \mathcal{G}_{j2k}^* & \cdots & \sum_{k=1}^P |\mathcal{G}_{jmk}|^2 & \cdots & \sum_{k=1}^P \mathcal{G}_{jmk} \mathcal{G}_{jKk}^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{k=1}^P \mathcal{G}_{jKk} \mathcal{G}_{j1k}^* & \sum_{k=1}^P \mathcal{G}_{jKk} \mathcal{G}_{j2k}^* & \cdots & \sum_{k=1}^P \mathcal{G}_{jKk} \mathcal{G}_{jmk}^* & \cdots & \sum_{k=1}^P |\mathcal{G}_{jKk}|^2 \end{bmatrix}$$

$$\begin{aligned}
\mathcal{E}[\mathbf{G}_{j,l}^H \mathbf{G}_{j,l}] &= \text{diag} \left[\mathcal{E} \left[\sum_{k=1}^P |g_{jl1k}|^2 \right], \mathcal{E} \left[\sum_{k=1}^P |g_{jl2k}|^2 \right], \dots, \mathcal{E} \left[\sum_{k=1}^P |g_{jlmk}|^2 \right], \dots, \mathcal{E} \left[\sum_{k=1}^P |g_{jlKk}|^2 \right] \right] \\
&= \text{diag} \left[\beta_{jl1} \mathcal{E} \left[\sum_{k=1}^P |h_{jl1k}|^2 \right], \beta_{jl2} \mathcal{E} \left[\sum_{k=1}^P |h_{jl2k}|^2 \right], \dots, \beta_{jlm} \mathcal{E} \left[\sum_{k=1}^P |h_{jlmk}|^2 \right], \dots, \beta_{jlK} \mathcal{E} \left[\sum_{k=1}^P |h_{jlKk}|^2 \right] \right] \\
&= \text{diag} [\beta_{jl1}, \beta_{jl2}, \dots, \beta_{jlm}, \dots, \beta_{jlK}]
\end{aligned}$$

And: $\text{Trace} \left[\mathcal{E}[\mathbf{G}_{j,l}^H \mathbf{G}_{j,l}] \right] = \sum_{k=1}^K \beta_{jlk} = \beta_{jl}.$

Concerning the term: $\mathcal{E} \left[p_p \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{M}_{p,l}^A(n,:) \right]$, we have:

$$\begin{aligned}
&\mathcal{E} \left[p_p \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{M}_{p,l}^A(n,:) \right] \\
&= \frac{p_p}{K} \mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:) \right] \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \right] \\
&= \frac{p_p \gamma_{n,l}}{K} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right]^2 \right]
\end{aligned}$$

We use the following approximation (the demonstration is given later):

$$\begin{aligned}
&\mathcal{E} \left[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \\
&= \frac{M^2 \beta_{jl}}{P^2} \text{diag}(\beta_{jl1}, \beta_{jl2}, \dots, \beta_{jlK})
\end{aligned}$$

Hence this yields:

$$\begin{aligned}
&\mathcal{E} \left[p_p \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{M}_{p,l}^A(n,:) \right] \\
&= \frac{p_p}{K} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right]^2 \right] \gamma_{n,l} \\
&= \frac{p_p}{K} \frac{M^2 \beta_{jl}^2}{P^2} \gamma_{n,l}
\end{aligned}$$

Concerning the last quantity, we have :

$$\begin{aligned}
&\mathcal{E} \left[\mathbf{M}_{p,l}^A(n,:) \mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{M}_{p,l}^A(n,:) \right] \\
&= \frac{\gamma_{n,l}}{K} \text{Trace} \left[\mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \right] \right] \\
&= \frac{\gamma_{n,l}}{K} \frac{M^2 \sigma_{n_l}^2}{P} \beta_{jl} \\
&= \frac{M^2 \sigma_{n_l}^2}{KP} \beta_{jl} \gamma_{n,l}
\end{aligned}$$

We have:

$$\begin{aligned}
& \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \right] \\
&= \frac{1}{K} \mathcal{E} \left[\mathbf{W}_l^H \mathbf{W}_l \right] \text{Trace} \left(\mathcal{E} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] \right) \\
&= \frac{M \sigma_{n_l}^2}{K} \frac{M}{P} \beta_{jl} \mathbf{I}_K = \frac{M^2 \sigma_{n_l}^2}{KP} \beta_{jl} \mathbf{I}_K
\end{aligned}$$

Concerning the quantity: $\mathcal{E} \left(\left[\mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:) \right]^H \right)$, we have:

$$\begin{aligned}
& \mathcal{E} \left(\left[\mathbf{M}_{p,l}^A \mathbf{M}_{p,l}^A \right]^H \right) = \mathcal{E} \left(\left[\left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right)^{-1} \left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right)^{-1} \right]^H \right) \\
&= \mathcal{E} \left(\left[\left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right)^H \left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right) \right]^{-1} \right) \\
&= \mathcal{E} \left(\left[\left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right) \left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right) \right]^{-1} \right) \\
&= \mathcal{E} \left(\left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^{-1} \right) = \mathcal{E} \left(\left[\left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right)^2 \right]^{-1} \right)
\end{aligned}$$

We have:

$$\begin{aligned}
& \left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right)^2 \\
&= \left[p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right]^2 \\
&= p_p^2 \left(\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left(\sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) + \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \dots \\
&\quad \dots + p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \sum_{m=1}^L \mathbf{G}_{m,l}^H \mathbf{A}^H \mathbf{W}_l + p_p \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} + p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \dots \\
&\quad \dots + p_p^{3/2} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l + p_p^{3/2} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \dots \\
&\quad \dots + p_p \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l + \dots \\
&\quad \dots + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + p_p^{3/2} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \dots \\
&\quad \dots + \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \dots \\
&\quad \dots + p_p^{3/2} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l + \dots \\
&\quad \dots + p_p \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l
\end{aligned}$$

When we take the expectation of the different terms in the former expression we make the

following assumptions, due to the independency between channel parameters and additive noise we neglect the corresponding expectation term each time the stochastic variable appears an odd number of times in the calculation. Using this rule, it is not difficult to demonstrate that the following expectation terms are equal to zero:

$$\begin{aligned}
\mathcal{E}\left(\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l\right) &\approx \mathbf{0}, \quad \mathcal{E}\left(\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l}\right) \approx \mathbf{0} \\
\mathcal{E}\left(\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l\right) &\approx \mathbf{0}, \quad \mathcal{E}\left(\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l}\right) \approx \mathbf{0} \\
\mathcal{E}\left(\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l}\right) &\approx \mathbf{0}, \quad \mathcal{E}\left(\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l\right) \approx \mathbf{0} \\
\mathcal{E}\left(\mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l}\right) &\approx \mathbf{0}, \quad \mathcal{E}\left(\mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l\right) \approx \mathbf{0}
\end{aligned}$$

This yields:

$$\begin{aligned}
&\mathcal{E}\left(\left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l}\right]^2\right) \\
&\approx p_p^2 \mathcal{E}\left[\left(\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l}\right)\left(\sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l}\right)\right] \\
&\dots + \mathcal{E}\left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l\right] + p_p \mathcal{E}\left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \sum_{m=1}^L \mathbf{G}_{m,l}^H \mathbf{A}^H \mathbf{W}_l\right] + \dots \\
&\dots + p_p \mathcal{E}\left[\mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l}\right] + p_p \mathcal{E}\left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l\right] + \dots \\
&\dots + p_p \mathcal{E}\left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l}\right] + p_p \mathcal{E}\left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l}\right] + \dots \\
&\dots + p_p \mathcal{E}\left[\mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l\right]
\end{aligned}$$

Due to the independency between channel parameters and additive noise, we can further simplify using:

$$\mathcal{E}\left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \sum_{m=1}^L \mathbf{G}_{m,l}^H \mathbf{A}^H \mathbf{W}_l\right] \approx \mathcal{E}\left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \sum_{m=1}^L \mathcal{E}\left[\mathbf{W}_l \mathbf{G}_{m,l}^H\right] \mathbf{A}^H \mathbf{W}_l\right] \approx \mathbf{0}$$

And, for the same reason:

$$\mathcal{E}\left[\mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l}\right] \approx \mathbf{0}$$

This reduces the calculation of $\mathcal{E}\left(\left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l}\right]^2\right)$ and this gives:

$$\begin{aligned}
& \mathcal{E} \left(\left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^2 \right) \\
& \approx p_p^2 \mathcal{E} \left[\left(\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left(\sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) \right] + \dots \\
& \dots + \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + \dots \\
& \dots + p_p \mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + \dots \\
& \dots + p_p \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] + \dots \\
& \dots + p_p \mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] + p_p \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \right]
\end{aligned}$$

$$\begin{aligned}
& \mathcal{E} \left(\left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^2 \right) \\
& \approx p_p^2 \mathcal{E} \left[\left(\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left(\sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) \right] + \dots \\
& \dots + \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + 2p_p \sum_{i=1}^L \mathcal{E} \left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + \dots \\
& \dots + p_p \mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[\mathbf{W}_l \mathbf{W}_l^H \right] \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] + p_p \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathcal{E} \left[\mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \right] \mathbf{A}^H \mathbf{W}_l \right]
\end{aligned}$$

Concerning the first term in the expectation, we take a particular case with $L = 3$ before a complete generalization. We use the notation: $\mathbf{a}_i = \mathbf{A} \mathbf{G}_{i,l}$, then, we obtain :

$$\begin{aligned}
& \mathcal{E} \left[\left(\sum_{i=1}^3 \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left(\sum_{j=1}^3 \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) \right] \\
& = \mathcal{E} \left[\left(\sum_{j=1}^3 \mathbf{a}_j^H \sum_{i=1}^3 \mathbf{a}_i \right) \left(\sum_{m=1}^3 \mathbf{a}_m^H \sum_{k=1}^3 \mathbf{a}_k \right) \right] \\
& = \mathcal{E} \left[\left(\mathbf{a}_1^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_2 + \mathbf{a}_3^H \mathbf{a}_3 + \mathbf{a}_1^H \mathbf{a}_2 + \mathbf{a}_1^H \mathbf{a}_3 + \mathbf{a}_2^H \mathbf{a}_1 + \mathbf{a}_2^H \mathbf{a}_3 + \mathbf{a}_3^H \mathbf{a}_1 + \mathbf{a}_3^H \mathbf{a}_2 \right)^2 \right]
\end{aligned}$$

$$\begin{aligned}
&= \mathcal{E}[a_1^H a_1 a_1^H a_1 + a_2^H a_2 a_2^H a_2 + a_3^H a_3 a_3^H a_3 + a_1^H a_1 a_2^H a_2 + a_1^H a_1 a_3^H a_3 + a_1^H a_1 a_1^H a_2 + a_1^H a_1 a_1^H a_3 + a_1^H a_1 a_2^H a_1 + \dots \\
&\dots + a_1^H a_1 a_2^H a_3 + a_1^H a_1 a_3^H a_1 + a_1^H a_1 a_3^H a_2 + a_2^H a_2 a_1^H a_1 + a_2^H a_2 a_3^H a_3 + a_2^H a_2 a_1^H a_2 + a_2^H a_2 a_1^H a_3 + a_2^H a_2 a_2^H a_1 + \dots \\
&\dots + a_2^H a_2 a_2^H a_3 + a_2^H a_2 a_3^H a_1 + a_2^H a_2 a_3^H a_2 + a_3^H a_3 a_1^H a_1 + a_3^H a_3 a_2^H a_2 + a_3^H a_3 a_1^H a_2 + a_3^H a_3 a_1^H a_3 + a_3^H a_3 a_2^H a_1 + \dots \\
&\dots + a_3^H a_3 a_2^H a_3 + a_3^H a_3 a_3^H a_1 + a_3^H a_3 a_3^H a_2 + a_1^H a_2 a_1^H a_1 + a_1^H a_2 a_2^H a_2 + a_1^H a_2 a_3^H a_3 + a_1^H a_2 a_1^H a_2 + a_1^H a_2 a_1^H a_3 + \dots \\
&\dots + a_1^H a_2 a_2^H a_1 + a_1^H a_2 a_2^H a_3 + a_1^H a_2 a_3^H a_1 + a_1^H a_2 a_3^H a_2 + a_1^H a_3 a_1^H a_1 + a_1^H a_3 a_2^H a_2 + a_1^H a_3 a_3^H a_3 + a_1^H a_3 a_1^H a_2 + \dots \\
&\dots + a_1^H a_3 a_1^H a_3 + a_1^H a_3 a_2^H a_1 + a_1^H a_3 a_2^H a_3 + a_1^H a_3 a_3^H a_1 + a_1^H a_3 a_3^H a_2 + a_2^H a_1 a_1^H a_1 + a_2^H a_1 a_2^H a_2 + a_2^H a_1 a_3^H a_3 + \dots \\
&\dots + a_2^H a_1 a_1^H a_2 + a_2^H a_1 a_1^H a_3 + a_2^H a_1 a_2^H a_1 + a_2^H a_1 a_2^H a_3 + a_2^H a_1 a_3^H a_1 + a_2^H a_1 a_3^H a_2 + a_2^H a_3 a_1^H a_1 + a_2^H a_3 a_2^H a_2 + \dots \\
&\dots + a_2^H a_3 a_3^H a_3 + a_2^H a_3 a_1^H a_2 + a_2^H a_3 a_1^H a_3 + a_2^H a_3 a_2^H a_1 + a_2^H a_3 a_2^H a_3 + a_2^H a_3 a_3^H a_1 + a_2^H a_3 a_3^H a_2 + a_3^H a_1 a_1^H a_1 + \dots \\
&\dots + a_3^H a_1 a_2^H a_2 + a_3^H a_1 a_3^H a_3 + a_3^H a_1 a_1^H a_2 + a_3^H a_1 a_1^H a_3 + a_3^H a_1 a_2^H a_1 + a_3^H a_1 a_2^H a_3 + a_3^H a_1 a_3^H a_1 + a_3^H a_1 a_3^H a_2 + \dots \\
&\dots + a_3^H a_2 a_1^H a_1 + a_3^H a_2 a_2^H a_2 + a_3^H a_2 a_3^H a_3 + a_3^H a_2 a_1^H a_2 + a_3^H a_2 a_1^H a_3 + a_3^H a_2 a_2^H a_1 + a_3^H a_2 a_2^H a_3 + a_3^H a_2 a_3^H a_1 + a_3^H a_2 a_3^H a_2] \\
&= \mathcal{E}[a_1^H a_1 a_1^H a_1 + a_2^H a_2 a_2^H a_2 + a_3^H a_3 a_3^H a_3 + a_1^H a_1 a_2^H a_2 + a_1^H a_1 a_3^H a_3 + a_2^H a_2 a_1^H a_1 + a_2^H a_2 a_3^H a_3 + a_3^H a_3 a_1^H a_1 + \dots \\
&\dots + a_3^H a_3 a_2^H a_2 + a_1^H a_2 a_2^H a_1 + a_1^H a_3 a_3^H a_1 + a_2^H a_1 a_1^H a_2 + a_2^H a_3 a_3^H a_2 + a_3^H a_1 a_1^H a_3 + a_3^H a_2 a_2^H a_3]
\end{aligned}$$

In fact, due to the independency between different channels, we have the following results:

$$\mathcal{E}[a_i^H a_i a_i^H a_j] = 0, \mathcal{E}[a_i^H a_j a_j^H a_j] = 0, \mathcal{E}[a_i^H a_j a_j^H a_k] = 0, \mathcal{E}[a_i^H a_j a_i^H a_j] = 0$$

We have to calculate three kinds of terms:

-1) The first one is:

$$\mathcal{E}[a_i^H a_i a_i^H a_i] = \mathcal{E}[G_{i,l}^H A^H A G_{i,l} G_{i,l}^H A^H A G_{i,l}]$$

Concerning first matrix: $G_{i,l}^H A^H A G_{i,l}$, the general term of matrix is:

$$\begin{aligned}
&\left[G_{i,l}^H A^H A G_{i,l} \right] (m, n) = a_{m,n} \\
&= \frac{1}{P} \sum_{k=1}^M \left(\sum_{r=1}^P \sum_{q=1}^P h_{ilmr}^* h_{ilnq} \sqrt{\beta_{ilm} \beta_{iln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right) \\
&= \frac{1}{P} \sum_{k=1}^M A_{r,q}^{i,k} (m, n)
\end{aligned}$$

With:

$$A_{r,q}^{i,k} (m, n) = \sum_{r=1}^P \sum_{q=1}^P h_{ilmr}^* h_{ilnq} \sqrt{\beta_{ilm} \beta_{iln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]}$$

We have, in this case:

$$\mathcal{E}\left(\left[G_{i,l}^H A^H A G_{i,l} \right] (m, n)\right) = \frac{1}{P} \sum_{k=1}^M \left(\sum_{r=1}^P \sum_{q=1}^P \mathcal{E}\left[h_{ilmr}^* h_{ilnq}\right] \sqrt{\beta_{ilm} \beta_{iln}} e^{j[f_k(\phi_r) - f_k(\phi_q)]} \right)$$

Of course, this quantity is not equal to zero only and only if $n = m$ and $r = q$, this yields :

$$\begin{aligned}\mathcal{E}\left(\left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}\right](n,n)\right) &= \frac{1}{P} \sum_{k=1}^M \left(\sum_{r=1}^P \mathcal{E}\left[|h_{ilnr}|^2\right] \beta_{iln} \right) \\ &= \frac{1}{P} \sum_{k=1}^M \beta_{iln} \\ &= \frac{M}{P} \beta_{iln}\end{aligned}$$

Then, we obtain:

$$\mathcal{E}\left(\left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}\right]\right) = \frac{M}{P} \text{diag}(\beta_{il1}, \beta_{il2}, \dots, \beta_{ilK})$$

In these conditions, the general term of matrix $\left(\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}\right)^2$ is:

$$\begin{aligned}\left(\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}\right)^2(m,n) &= \sum_{p=1}^K a_{m,p} a_{p,n} \\ &= \sum_{p=1}^K \left(\frac{1}{P} \sum_{k=1}^M A_{r,q}^{j,k}(m,p) \right) \left(\frac{1}{P} \sum_{u=1}^M A_{s,t}^{j,u}(p,n) \right) \\ &= \frac{1}{P^2} \sum_{p=1}^K \sum_{k=1}^M \sum_{u=1}^M A_{r,q}^{j,k}(m,p) A_{s,t}^{j,u}(p,n)\end{aligned}$$

We have:

$$\begin{aligned}&\mathcal{E}\left[A_{r,q}^{j,k}(m,p) A_{s,t}^{j,u}(p,n)\right] \\ &= \mathcal{E}\left[\left(\sum_{r=1}^P \sum_{q=1}^P h_{jlmr}^* h_{jlpq} \sqrt{\beta_{jlm} \beta_{jlp}} e^{j[f_k(\phi_r) - f_k(\phi_q)]}\right) \left(\sum_{s=1}^P \sum_{t=1}^P h_{jlpq}^* h_{jlns} \sqrt{\beta_{jlp} \beta_{jln}} e^{j[f_u(\phi_s) - f_u(\phi_t)]}\right)\right] \\ &= \sum_{r=1}^P \sum_{q=1}^P \sum_{s=1}^P \sum_{t=1}^P \mathcal{E}\left[h_{jlmr}^* h_{jlpq} h_{jlpq}^* h_{jlns}\right] \beta_{jlp} \sqrt{\beta_{jlm} \beta_{jln}} e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_s) - f_u(\phi_t)]}\end{aligned}$$

The term $\mathcal{E}\left[h_{jlmr}^* h_{jlpq} h_{jlpq}^* h_{jlns}\right]$ is not equal to zero when: $n = m$ and $s = q$ and $r = t$. In these conditions we have:

$$\begin{aligned}
& \mathcal{E} \left[A_{r,q}^{j,k}(n,p) A_{s,t}^{j,u}(p,n) \right] \\
&= \sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[h_{jlnr}^* h_{jlpq} h_{jlpq}^* h_{jlnr} \right] \beta_{jlp} \beta_{jln} e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \\
&= \sum_{r=1}^P \sum_{q=1}^P \mathcal{E} \left[|h_{jlnr}|^2 \right] \mathcal{E} \left[|h_{jlpq}|^2 \right] \beta_{jlp} \beta_{jln} e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \\
&= \beta_{jlp} \beta_{jln} \sum_{q=1}^P \mathcal{E} \left[|h_{jlpq}|^2 \right] e^{j[f_u(\phi_q) - f_k(\phi_q)]} \sum_{r=1}^P \mathcal{E} \left[|h_{jlnr}|^2 \right] e^{j[f_k(\phi_r) - f_u(\phi_r)]}
\end{aligned}$$

We have:

$$\begin{aligned}
e^{j[f_u(\phi_q) - f_k(\phi_q)]} &= e^{j \left[2\pi \frac{d}{\lambda} (u-1) \sin(\phi_q) - 2\pi \frac{d}{\lambda} (k-1) \sin(\phi_q) \right]} = e^{j \left[2\pi \frac{d}{\lambda} (u-k) \sin(\phi_q) \right]} \\
e^{j[f_k(\phi_r) - f_u(\phi_r)]} &= e^{j \left[2\pi \frac{d}{\lambda} (k-1) \sin(\phi_r) - 2\pi \frac{d}{\lambda} (u-1) \sin(\phi_r) \right]} = e^{j \left[2\pi \frac{d}{\lambda} (k-u) \sin(\phi_r) \right]}
\end{aligned}$$

This yields:

$$\begin{aligned}
S &= \sum_{k=1}^M \sum_{u=1, u \neq k}^M e^{j \left[2\pi \frac{d}{\lambda} (u-k) (\sin(\phi_q) - \sin(\phi_r)) \right]} = \left[\sum_{k=1}^M e^{-j 2\pi \frac{d}{\lambda} k (\sin(\phi_q) - \sin(\phi_r))} \right] \left[\sum_{u=1, u \neq k}^M e^{j 2\pi \frac{d}{\lambda} u (\sin(\phi_q) - \sin(\phi_r))} \right] \\
&= \left[\sum_{k=1}^M e^{-j 2\pi \frac{d}{\lambda} k (\sin(\phi_q) - \sin(\phi_r))} \right] \left[\sum_{u=1}^M e^{j 2\pi \frac{d}{\lambda} u (\sin(\phi_q) - \sin(\phi_r))} - e^{j 2\pi \frac{d}{\lambda} k (\sin(\phi_q) - \sin(\phi_r))} \right] \\
&= \left(\sum_{k=1}^M e^{-j 2\pi \frac{d}{\lambda} k (\sin(\phi_q) - \sin(\phi_r))} \right) \left(\sum_{u=1}^M e^{j 2\pi \frac{d}{\lambda} u (\sin(\phi_q) - \sin(\phi_r))} \right) - M \\
&= \left(e^{-j 2\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))} \frac{1 - e^{-j 2\pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))}}{1 - e^{-j 2\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))}} \right) \left(e^{j 2\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))} \frac{1 - e^{j 2\pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))}}{1 - e^{j 2\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r))}} \right) - M \\
&= \left(\frac{\sin \left[\pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]}{\sin \left[\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right)^2 - M \\
&= M^2 \left(\frac{\sin \left[\pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right)^2 - M
\end{aligned}$$

Using this result, and supposing $\mathcal{E}\left[|h_{jlpq}|^2\right] = \mathcal{E}\left[|h_{jlnr}|^2\right] = 1/P \quad \forall r, \forall q$, (this means that the contributions of each multipath component is the same) we obtain:

$$\begin{aligned}
& \mathcal{E}\left[\left(\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}\right)^2(n,n)\right] \\
&= \frac{1}{P^2} \sum_{p=1}^K \sum_{k=1}^M \sum_{u=1}^M \mathcal{E}\left[A_{r,q}^{j,k}(n,p) A_{s,t}^{j,u}(p,n)\right] \\
&= \frac{1}{P^2} \sum_{p=1}^K \sum_{k=1}^M \sum_{u=1}^M \beta_{jlp} \beta_{jln} \sum_{r=1}^P \mathcal{E}\left[|h_{jlnr}|^2\right] \sum_{q=1}^P \mathcal{E}\left[|h_{jlpq}|^2\right] e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \\
&= \frac{1}{P^2} \beta_{jln} \sum_{p=1}^K \beta_{jlp} \sum_{k=1}^M \left[\sum_{r=1}^P \mathcal{E}\left[|h_{jlnr}|^2\right] \sum_{q=1}^P \mathcal{E}\left[|h_{jlpq}|^2\right] \right] + \dots \\
&\dots + \frac{1}{P^2} \beta_{jln} \sum_{p=1}^K \beta_{jlp} \sum_{k=1}^M \sum_{u=1, u \neq k}^M \left[\sum_{r=1}^P \mathcal{E}\left[|h_{jlnr}|^2\right] \sum_{q=1}^P \mathcal{E}\left[|h_{jlpq}|^2\right] e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \right] \\
&= \frac{M}{P^2} \beta_{jln} \beta_{jl} + \frac{1}{P^4} \beta_{jln} \beta_{jl} \sum_{k=1}^M \sum_{u=1, u \neq k}^M \left[\sum_{r=1}^P \sum_{q=1}^P e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \right] \\
&= \frac{M}{P^2} \beta_{jln} \beta_{jl} + \frac{1}{P^4} \beta_{jln} \beta_{jl} \sum_{r=1}^P \sum_{q=1}^P \left[\sum_{k=1}^M \sum_{u=1, u \neq k}^M e^{j[f_k(\phi_r) - f_k(\phi_q) + f_u(\phi_q) - f_u(\phi_r)]} \right] \\
&= \frac{M}{P^2} \beta_{jln} \beta_{jl} + \frac{1}{P^4} \beta_{jln} \beta_{jl} \sum_{r=1}^P \sum_{q=1}^P \left[\frac{\left(\sin \left[\pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right] \right)^2}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right] - M \\
&= \frac{M}{P^2} \beta_{jln} \beta_{jl} + \frac{M^2}{P^4} \beta_{jln} \beta_{jl} \sum_{r=1}^P \sum_{q=1}^P \left[\frac{\left(\sin \left[\pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right] \right)^2}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right] - \frac{1}{M} \\
&= \frac{M^2}{P^4} \beta_{jln} \beta_{jl} \sum_{r=1}^P \sum_{q=1}^P \left(\frac{\sin \left[\pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right)^2
\end{aligned}$$

Using the notation: $S_{r,q} = \sum_{r=1}^P \sum_{q=1}^P \left(\frac{\sin \left[\pi M \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_q) - \sin(\phi_r)) \right]} \right)^2$, we obtain eventually :

$$\begin{aligned} & \mathcal{E} \left[\left(\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right)^2 (n,n) \right] \\ & \approx \frac{M^2}{P^4} \beta_{jln} \beta_{jl} S_{r,q} \end{aligned}$$

This yields:

$$\begin{aligned} & \mathcal{E} \left[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \\ & = \frac{M^2 \beta_{jl} S_{r,q}}{P^4} \text{diag}(\beta_{j11}, \beta_{j12}, \dots, \beta_{j1K}) \end{aligned}$$

And:

$$\begin{aligned} & \mathcal{E} \left[\sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \\ & = \frac{M^2 S_{r,q}}{P^4} \sum_{j=1}^L \beta_{jl} \text{diag}(\beta_{j11}, \beta_{j12}, \dots, \beta_{j1K}) \end{aligned}$$

-2) The second one is:

$$\mathcal{E}[\mathbf{a}_i^H \mathbf{a}_i \mathbf{a}_j^H \mathbf{a}_j] = \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}]$$

If we suppose that the channels are independent, we can write:

$$\begin{aligned} & \mathcal{E}[\mathbf{a}_i^H \mathbf{a}_i \mathbf{a}_j^H \mathbf{a}_j] = \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}] = \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}] \mathcal{E}[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}] \\ & = \frac{M^2}{P^2} \text{diag}(\beta_{i11}, \beta_{i12}, \dots, \beta_{i1K}) \text{diag}(\beta_{j11}, \beta_{j12}, \dots, \beta_{j1K}) \\ & = \frac{M^2}{P^2} \text{diag}(\beta_{i11} \beta_{j11}, \beta_{i12} \beta_{j12}, \dots, \beta_{i1K} \beta_{j1K}) \end{aligned}$$

-3) The third one is:

$$\mathcal{E}[\mathbf{a}_i^H \mathbf{a}_j \mathbf{a}_j^H \mathbf{a}_i] = \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}] = \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{E}[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H] \mathbf{A} \mathbf{G}_{i,l}]$$

The general term of matrix $\mathbf{A} \mathbf{G}_{j,l}$ is written as :

$$\left[\mathbf{A} \mathbf{G}_{j,l}(m,k) \right]_{M \times K} = \frac{1}{\sqrt{P}} \sum_{s=1}^P g_{jllks} e^{-j f_m(\phi_s)} = a_{m,k}$$

And we have :

$$\left[\mathbf{G}_{j,l}^H \mathbf{A}^H(n, p) \right]_{K \times M} = \frac{1}{\sqrt{P}} \sum_{r=1}^P \mathbf{g}_{jlnr}^* e^{j\phi_r(\phi_r)} = \mathbf{b}_{n,p}$$

The general term of matrix $\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H$ is thus :

$$\begin{aligned} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right](m, p) &= \sum a_{m,k} b_{k,p} \\ &= \sum_{k=1}^K \left(\frac{1}{\sqrt{P}} \sum_{s=1}^P \mathbf{g}_{jlk s} e^{-j\phi_s(\phi_s)} \right) \left(\frac{1}{\sqrt{P}} \sum_{r=1}^P \mathbf{g}_{jlk r}^* e^{j\phi_r(\phi_r)} \right) \\ &= \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \sum_{s=1}^P \mathbf{g}_{jlk r}^* \mathbf{g}_{jlk s} e^{j\phi_r(\phi_r) - j\phi_s(\phi_s)} \\ &= \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \sum_{s=1}^P h_{jlk r}^* h_{jlk s} \sqrt{\beta_{jlk} \beta_{jlk}} e^{j\phi_r(\phi_r) - j\phi_s(\phi_s)} \end{aligned}$$

Hence, we have :

$$\mathcal{E} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right](m, p) = \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \sum_{s=1}^P \mathcal{E} \left[h_{jlk r}^* h_{jlk s} \right] \beta_{jlk} e^{j\phi_r(\phi_r) - j\phi_s(\phi_s)}$$

For $m=p$, we obtain:

$$\mathcal{E} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right](m, m) = \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \mathcal{E} \left[|h_{jlk r}|^2 \right] \beta_{jlk} = \frac{1}{P} \sum_{k=1}^K \beta_{jlk} = \frac{1}{P} \beta_{jl}$$

However, if $m \neq p$, we have:

$$\begin{aligned} \mathcal{E} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right](m, p) &= \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \mathcal{E} \left[|h_{jlk r}|^2 \right] \beta_{jlk} e^{j\phi_r(\phi_r) - j\phi_m(\phi_r)} \\ &= \frac{1}{P} \sum_{k=1}^K \sum_{r=1}^P \mathcal{E} \left[|h_{jlk r}|^2 \right] \beta_{jlk} e^{j2\pi \frac{d}{\lambda} (p-m) \sin(\phi_r)} \\ &= \frac{1}{P^2} \sum_{k=1}^K \beta_{jlk} \sum_{r=1}^P e^{j2\pi \frac{d}{\lambda} (p-m) \sin(\phi_r)} \\ &= \frac{\beta_{jl}}{P^2} \sum_{r=1}^P e^{j2\pi \frac{d}{\lambda} (p-m) \sin(\phi_r)} \end{aligned}$$

And this quantity is not necessarily equal to zero. We can approximate $\mathcal{E} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right]$ as :

$$\mathcal{E} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] = \frac{1}{P} \beta_{jl} \mathbf{I}_M$$

The (m,n) th element of $\mathbf{G}_{i,l}^H \mathbf{A}^H$ is written as :

$$\left[\mathbf{G}_{i,l}^H \mathbf{A}^H(m,n) \right]_{K \times M} = a_{m,n} = \frac{1}{\sqrt{P}} \sum_{k=1}^P g_{ilmk}^* e^{jf_n(\phi_k)}$$

And we obtain:

$$\begin{aligned} \mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] (m,p) &= \sum_{r=1}^M a_{m,r} b_{r,p} \\ &= \sum_{r=1}^M \frac{1}{\sqrt{P}} \left[\sum_{k=1}^P g_{ilmk}^* e^{jf_r(\phi_k)} \right] \frac{\beta_{jl}}{P^2} \sum_{q=1}^P e^{j2\pi \frac{d}{\lambda} (p-r) \sin(\phi_q)} \\ &= \sum_{r=1}^M \frac{1}{\sqrt{P}} \left[\sum_{k=1}^P g_{ilmk}^* e^{j2\pi \frac{d}{\lambda} (r-1) \sin(\phi_k)} \right] \frac{\beta_{jl}}{P^2} \sum_{q=1}^P e^{j2\pi \frac{d}{\lambda} (p-r) \sin(\phi_q)} \\ &= \frac{\beta_{jl}}{P^{5/2}} \sum_{r=1}^M \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j2\pi \frac{d}{\lambda} (r-1) \sin(\phi_k)} e^{j2\pi \frac{d}{\lambda} (p-r) \sin(\phi_q)} \\ &= \frac{\beta_{jl}}{P^{5/2}} \sum_{r=1}^M \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j2\pi \frac{d}{\lambda} r (\sin(\phi_k) - \sin(\phi_q))} e^{j2\pi \frac{d}{\lambda} (p \sin(\phi_q) - \sin(\phi_k))} \\ &= \frac{\beta_{jl}}{P^{5/2}} \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j2\pi \frac{d}{\lambda} (p \sin(\phi_q) - \sin(\phi_k))} \sum_{r=1}^M e^{j2\pi \frac{d}{\lambda} r (\sin(\phi_k) - \sin(\phi_q))} \\ &= \frac{\beta_{jl}}{P^{5/2}} \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j2\pi \frac{d}{\lambda} (p \sin(\phi_q) - \sin(\phi_k))} e^{j2\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q))} \frac{1 - e^{j2\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q))}}{1 - e^{j2\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q))}} \\ &= \frac{M \beta_{jl}}{P^{5/2}} \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j2\pi \frac{d}{\lambda} (p-1) \sin(\phi_q)} e^{j\pi \frac{d}{\lambda} (M-1) (\sin(\phi_k) - \sin(\phi_q))} \frac{\sin \left[\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \\ &= \frac{M \beta_{jl}}{P^{5/2}} \sum_{k=1}^P g_{ilmk}^* e^{j\pi \frac{d}{\lambda} (M-1) \sin(\phi_k)} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \\ &= \frac{M \beta_{jl}}{P^{5/2}} \sum_{k=1}^P \sum_{q=1}^P g_{ilmk}^* e^{j\pi \frac{d}{\lambda} [(M-1) \sin(\phi_k) + (2p-M-1) \sin(\phi_q)]} \frac{\sin \left[\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \end{aligned}$$

The general term of matrix $\mathbf{A} \mathbf{G}_{i,l}$ is written as :

$$\left[\mathbf{A} \mathbf{G}_{i,l}(p,n) \right]_{M \times K} = \frac{1}{\sqrt{P}} \sum_{s=1}^P g_{ilns} e^{-jf_p(\phi_s)} = a_{p,n}$$

We have :

$$\begin{aligned}
& \left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] \mathbf{A} \mathbf{G}_{i,l} \right] (m, n) \\
&= \sum_{p=1}^M \frac{M \beta_{jl}}{P^{5/2}} \sum_{k=1}^P \mathcal{E} \left[\mathbf{g}_{ilmk} \right]^* e^{j\pi \frac{d}{\lambda} (M-1) \sin(\phi_k)} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \frac{1}{\sqrt{P}} \sum_{s=1}^P \mathbf{g}_{ilns} e^{-j\pi \frac{d}{\lambda} \sin(\phi_s)} \\
&= \frac{M \beta_{jl}}{P^3} \sum_{p=1}^M \sum_{k=1}^P \sum_{s=1}^P \mathcal{E} \left[\mathbf{g}_{ilmk} \mathbf{g}_{ilns} \right]^* e^{j\pi \frac{d}{\lambda} (M-1) \sin(\phi_k)} e^{-j2\pi \frac{d}{\lambda} (p-1) \sin(\phi_s)} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]}
\end{aligned}$$

We obtain eventually :

$$\begin{aligned}
& \mathcal{E} \left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] \mathbf{A} \mathbf{G}_{i,l} \right] (m, n) \\
&= \frac{M \beta_{jl}}{P^3} \sum_{p=1}^M \sum_{k=1}^P \sum_{s=1}^P \mathcal{E} \left[\mathbf{g}_{ilmk} \mathbf{g}_{ilns} \right]^* e^{j\pi \frac{d}{\lambda} (M-1) \sin(\phi_k)} e^{-j2\pi \frac{d}{\lambda} (p-1) \sin(\phi_s)} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]}
\end{aligned}$$

The quantity is not equal to zero when $m = n$ and $k = s$, this entails:

$$\begin{aligned}
& \mathcal{E} \left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \right] \mathbf{A} \mathbf{G}_{i,l} \right] (n, n) \\
&= \frac{M \beta_{jl}}{P^3} \sum_{p=1}^M \sum_{k=1}^P \frac{\beta_{iln}}{P} e^{j\pi \frac{d}{\lambda} (M-1) \sin(\phi_k)} e^{-j2\pi \frac{d}{\lambda} (p-1) \sin(\phi_k)} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \\
&= \frac{M \beta_{jl} \beta_{iln}}{P^4} \sum_{p=1}^M \sum_{k=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_k) [M-2p+1]} \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} \sin(\phi_q) [2p-M-1]} \frac{\sin \left[\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \\
&= \frac{M \beta_{jl} \beta_{iln}}{P^4} \sum_{p=1}^M \sum_{k=1}^P \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} [M-2p+1] (\sin(\phi_k) - \sin(\phi_q))} \frac{\sin \left[\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]} \\
\text{Using the notation: } T_{p,k,q} &= \sum_{p=1}^M \sum_{k=1}^P \sum_{q=1}^P e^{j\pi \frac{d}{\lambda} [M-2p+1] (\sin(\phi_k) - \sin(\phi_q))} \frac{\sin \left[\pi \frac{d}{\lambda} M (\sin(\phi_k) - \sin(\phi_q)) \right]}{M \sin \left[\pi \frac{d}{\lambda} (\sin(\phi_k) - \sin(\phi_q)) \right]},
\end{aligned}$$

we obtain :

$$\begin{aligned}
& \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E}[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H] \mathbf{A} \mathbf{G}_{i,l}] \\
&= \frac{M}{P^4} \beta_{jl} T_{p,k,q} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK})
\end{aligned}$$

In fact, for the sake of simplicity we will use a lower bound for the different expectation forms due to the difficulty of obtaining asymptotic expressions. We will have in this case:

$$\begin{aligned}
& \mathcal{E} \left[\left(\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right)^2 (n, n) \right] \\
&> \frac{M}{P^2} \beta_{jl} \beta_{jl}
\end{aligned}$$

And:

$$\begin{aligned}
& \mathcal{E} \left[\sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \\
&> \frac{M}{P^2} \sum_{j=1}^L \beta_{jl} \text{diag}(\beta_{j1}, \beta_{j2}, \dots, \beta_{jK})
\end{aligned}$$

And, in the same way :

$$\begin{aligned}
& \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E}[\mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H] \mathbf{A} \mathbf{G}_{i,l}] \\
&> \frac{M}{P^2} \beta_{jl} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK})
\end{aligned}$$

Using these lower bounds we obtain in the general case:

$$\begin{aligned}
& \mathcal{E} \left[\left(\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left(\sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) \right] \\
&= \sum_{i=1}^L \mathcal{E}(\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}) + \sum_{i=1}^L \sum_{j=1, j \neq i}^L \mathcal{E}(\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l}) + \dots \\
&\dots + \sum_{i=1}^L \sum_{j=1, j \neq i}^L \mathcal{E}[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l}] \\
&> \sum_{i=1}^L \frac{M \beta_{il}}{P^2} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) + \sum_{i=1}^L \sum_{j=1, j \neq i}^L \frac{M^2}{P^2} \text{diag}(\beta_{i1} \beta_{j1}, \beta_{i2} \beta_{j2}, \dots, \beta_{iK} \beta_{jK}) + \\
&\dots + \sum_{i=1}^L \sum_{j=1, j \neq i}^L \frac{M}{P^2} \beta_{jl} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \\
&> \frac{M}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[\beta_{il} \mathbf{I}_K + M \sum_{j=1, j \neq i}^L \text{diag}(\beta_{j1}, \beta_{j2}, \dots, \beta_{jK}) + \sum_{j=1, j \neq i}^L \beta_{jl} \mathbf{I}_K \right]
\end{aligned}$$

$$\begin{aligned}
&> \frac{M}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[\sum_{j=1}^L \beta_{ji} \mathbf{I}_K + M \sum_{j=1, j \neq i}^L \text{diag}(\beta_{j1}, \beta_{j2}, \dots, \beta_{jK}) \right] \\
&> \frac{M}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[\hat{\beta}_i \mathbf{I}_K + M \sum_{j=1, j \neq i}^L \text{diag}(\beta_{j1}, \beta_{j2}, \dots, \beta_{jK}) \right] \\
&> \frac{M^2}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[\frac{\hat{\beta}_i}{M} \mathbf{I}_K + \sum_{j=1, j \neq i}^L \text{diag}(\beta_{j1}, \beta_{j2}, \dots, \beta_{jK}) \right] \\
&> \frac{M^2}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[\text{diag} \left(\sum_{j=1, j \neq i}^L \beta_{ji} + \frac{\hat{\beta}_i}{M}, \sum_{j=1, j \neq i}^L \beta_{j2} + \frac{\hat{\beta}_i}{M}, \dots, \sum_{j=1, j \neq i}^L \beta_{jK} + \frac{\hat{\beta}_i}{M} \right) \right]
\end{aligned}$$

We obtain a lower bound which is quite accurate.

The following term is: $\mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right]$, we have:

$$\begin{aligned}
&\mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&\approx \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathcal{E} \left[\mathbf{W}_l \mathbf{W}_l^H \right] \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= K \sigma_{n_l}^2 \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right]
\end{aligned}$$

$$\Pi_A \equiv \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \approx_{M \rightarrow \infty} \mathbf{A} \left(\frac{M}{P} \mathbf{I}_P \right)^{-1} \mathbf{A}^H = \frac{P}{M} \mathbf{A} \mathbf{A}^H$$

$$\begin{aligned}
&\mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= \frac{1}{M} \mathcal{E} \left[\mathbf{W}_l^H \mathbf{W}_l \right] \text{Trace} \left(\mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \right) \\
&= \frac{1}{M} M \sigma_{n_l}^2 \mathbf{I}_K \text{Trace} \left[\Pi_A \right] \approx_{M \rightarrow \infty} \sigma_{n_l}^2 \frac{P}{M} \text{Trace}(\mathbf{A} \mathbf{A}^H) \mathbf{I}_K
\end{aligned}$$

We eventually obtain:

$$\begin{aligned}
&\mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= \frac{KP}{M} \sigma_{n_l}^4 \text{Trace}(\mathbf{A} \mathbf{A}^H) \mathbf{I}_K
\end{aligned}$$

Then, we find :

$$\begin{aligned}
& \mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= \sum_{i=1}^L \mathcal{E} \left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \\
&= \sum_{i=1}^L \frac{M}{P} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \sigma_{n_i}^2 \frac{P}{M} \text{Trace}(\mathbf{A} \mathbf{A}^H) \mathbf{I}_K \\
&= \sigma_{n_i}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) \text{diag} \left(\sum_{i=1}^L \beta_{i1}, \sum_{i=1}^L \beta_{i2}, \dots, \sum_{i=1}^L \beta_{iK} \right) \\
&= \sigma_{n_i}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K})
\end{aligned}$$

$$\begin{aligned}
& \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] \\
&= \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] \sum_{i=1}^L \mathcal{E} \left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \\
&= \sigma_{n_i}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K})
\end{aligned}$$

The remaining terms are : $\mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right]$ and $\mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \right]$, we have :

$$\mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] = \mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[\mathbf{W}_l \mathbf{W}_l^H \right] \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right]$$

And using : $\mathcal{E} \left[\mathbf{W}_l \mathbf{W}_l^H \right] = K \sigma_{n_i}^2 \mathbf{I}_M$, this yields :

$$\begin{aligned}
& \mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] = K \sigma_{n_i}^2 \mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] = K \sigma_{n_i}^2 \sum_{i=1}^L \mathcal{E} \left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \\
&= K \sigma_{n_i}^2 \sum_{i=1}^L \frac{M}{P} \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) = \frac{K \sigma_{n_i}^2 M}{P} \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K})
\end{aligned}$$

And, for the last term :

$$\begin{aligned}
& \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathcal{E} \left[\mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \right] \mathbf{A}^H \mathbf{W}_l \right] = \frac{1}{P} \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \mathbf{A}^H \mathbf{W}_l \right] \text{Trace} \left[\sum_{i=1}^L \mathcal{E} \left[\mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \right] \right] \\
& = \frac{1}{P} \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \mathbf{A}^H \mathbf{W}_l \right] \sum_{i=1}^L \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \right] \right] \\
& = \frac{\text{Trace}(\mathbf{A} \mathbf{A}^H)}{MP} \mathcal{E} \left[\mathbf{W}_l^H \mathbf{W}_l \right] \sum_{i=1}^L \beta_{il} \\
& = M \sigma_{n_l}^2 \mathbf{I}_K \frac{\text{Trace}(\mathbf{A} \mathbf{A}^H)}{MP} \sum_{i=1}^L \beta_{il} \\
& = \sigma_{n_l}^2 \frac{\text{Trace}(\mathbf{A} \mathbf{A}^H)}{P} \sum_{i=1}^L \beta_{il} \mathbf{I}_K \\
& = \frac{\sigma_{n_l}^2 \hat{\beta}_l \text{Trace}(\mathbf{A} \mathbf{A}^H)}{P} \mathbf{I}_K
\end{aligned}$$

We eventually obtain :

$$\begin{aligned}
& p_p^2 \mathcal{E} \left[\left(\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{k=1}^L \mathbf{G}_{k,l} \right) \left(\sum_{j=1}^L \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} \right) \right] + \dots \\
& \dots + \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + 2p_p \sum_{i=1}^L \mathcal{E} \left[\mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \right] \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W}_l \right] + \dots \\
& \dots + p_p \mathcal{E} \left[\sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathcal{E} \left[\mathbf{W}_l \mathbf{W}_l^H \right] \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} \right] + p_p \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathcal{E} \left[\mathbf{G}_{i,l} \mathbf{G}_{i,l}^H \right] \mathbf{A}^H \mathbf{W}_l \right]
\end{aligned}$$

$$\begin{aligned}
& \mathcal{E} \left(\left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^2 \right) \\
& = p_p^2 \frac{M^2}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[\text{diag} \left(\sum_{j=1, j \neq i}^L \beta_{j1} + \frac{\hat{\beta}_1}{M}, \sum_{j=1, j \neq i}^L \beta_{j2} + \frac{\hat{\beta}_2}{M}, \dots, \sum_{j=1, j \neq i}^L \beta_{jK} + \frac{\hat{\beta}_K}{M} \right) \right] + \dots \\
& \dots + \frac{KP}{M} \sigma_{n_l}^4 \text{Trace}(\mathbf{A} \mathbf{A}^H) \mathbf{I}_K + 2p_p \sigma_{n_l}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K}) + p_p \frac{K \sigma_{n_l}^2 M}{P} \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K}) + \dots \\
& \dots + p_p \frac{\sigma_{n_l}^2 \hat{\beta}_l \text{Trace}(\mathbf{A} \mathbf{A}^H)}{P} \mathbf{I}_K \\
& = p_p^2 \frac{M^2}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[\text{diag} \left(\bar{\beta}_{11} - \beta_{i1} + \frac{\hat{\beta}_1}{M}, \bar{\beta}_{12} - \beta_{i2} + \frac{\hat{\beta}_2}{M}, \dots, \bar{\beta}_{1K} - \beta_{iK} + \frac{\hat{\beta}_K}{M} \right) \right] + \dots \\
& \dots + \left(2p_p \sigma_{n_l}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) + p_p \frac{K \sigma_{n_l}^2 M}{P} \right) \text{diag}(\bar{\beta}_{11}, \bar{\beta}_{12}, \dots, \bar{\beta}_{1K}) + \text{Trace}(\mathbf{A} \mathbf{A}^H) \left(\frac{KP}{M} \sigma_{n_l}^4 + p_p \frac{\sigma_{n_l}^2 \hat{\beta}_l}{P} \right) \mathbf{I}_K
\end{aligned}$$

We obtain in these conditions:

$$\mathcal{E} \left[\mathbf{M}_{p,l}^A(n, :) \mathbf{M}_{p,l}^A(n, :)^H \right] \approx \gamma_{n,l}$$

Considering the case where M is sufficiently high, we have $\mathbf{A}\mathbf{A}^H \approx \mathbf{I}_M$ and this yields:

$$\gamma_{n,l} = \frac{1}{p_p^2 \frac{M^2}{P^2} \left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] \right) + p_p M \sigma_{n_l}^2 \left(2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \sigma_{n_l}^2 M \left(p_p \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \sigma_{n_l}^2 \right)}$$

$$\gamma_{n,l} = \frac{P^2}{p_p^2 M^2 \left[\left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_l}^2}{Mp_p} \left(2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \sigma_{n_l}^2 \frac{P^2}{p_p M} \left(p_p \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \sigma_{n_l}^2 \right) \right]}$$

This can be simplified as:

$$\gamma_{n,l} = \frac{P^2}{p_p^2 M^2 \left[\left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_l}^2}{Mp_p} \left[\left(2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \frac{\sigma_{n_l}^2}{p_p} \right] \right]}$$

$$\gamma_{n,l} \approx_{M \rightarrow \infty} \frac{P^2}{p_p^2 M^2 \left[\left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} \right] \right) \right]}$$

And, we eventually obtain:

$$\mathcal{E} \left\{ \left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right\} = \mathcal{E} \left(\boldsymbol{\alpha}_{jn}^T \boldsymbol{\alpha}_{jn}^* \right)$$

$$= \frac{\gamma_{n,l} p_u p_p \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right)}{P^2 K} + \frac{\gamma_{n,l} p_u p_p M^2 \beta_{jl}^2}{KP^2} + \frac{\sigma_{n_l}^2 p_u M^2 \beta_{jl} \gamma_{n,l}}{KP}$$

$$= \frac{\gamma_{n,l} \left[p_u p_p \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_l}^2 p_u M^2 P \beta_{jl} \right]}{P^2 K} \quad (91)$$

$$= \frac{\left[p_u p_p \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_l}^2 p_u M^2 P \beta_{jl} \right]}{p_p^2 M^2 K \left[\left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_l}^2}{Mp_p} \left[\left(2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \frac{\sigma_{n_l}^2}{p_p} \right] \right]}$$

Considering the case where M goes towards infinity, we have:

$$\lim_{M \rightarrow \infty} \frac{1}{M} \mathbf{A}^H \mathbf{A} = \frac{1}{P} \mathbf{I}_P$$

This yields: $\lim_{M \rightarrow \infty} \mathbf{A}^H \mathbf{A} = \frac{M}{P} \mathbf{I}_P$ and $\lim_{M \rightarrow \infty} \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] = \frac{M^2}{P^2} P = \frac{M^2}{P}$ and we obtain:

$$\begin{aligned}
& \mathcal{E} \left\{ \left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right\} \\
&= \frac{\left[p_u p_p \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) M^2 / P + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_i}^2 p_u M^2 P \beta_{jl} \right]}{p_p^2 M^2 K \left[\left(\sum_{i=1}^L \beta_{in} \left[\bar{\beta}_{ln} - \beta_{in} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_i}^2}{M p_p} \left[\left(2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \frac{\hat{\beta}_l}{P} + \frac{K P}{M} \frac{\sigma_{n_i}^2}{p_p} \right] \right]} \\
&= \frac{\left[p_u p_p \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) / P + p_u p_p \beta_{jl}^2 + \sigma_{n_i}^2 p_u P \beta_{jl} \right]}{p_p^2 K \left[\left(\sum_{i=1}^L \beta_{in} \left[\bar{\beta}_{ln} - \beta_{in} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_i}^2}{M p_p} \left[\left(2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \frac{\hat{\beta}_l}{P} + \frac{K P}{M} \frac{\sigma_{n_i}^2}{p_p} \right] \right]} \\
&\approx_{M \rightarrow \infty} \frac{p_u p_p \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) / P + p_u p_p \beta_{jl}^2 + \sigma_{n_i}^2 p_u P \beta_{jl}}{p_p^2 K \sum_{i=1}^L \beta_{in} \left[\bar{\beta}_{ln} - \beta_{in} \right]} \\
&= \frac{p_u}{p_p} \beta_{jl} \frac{\left(\hat{\beta}_l - \beta_{jl} \right) / P + \beta_{jl} + \frac{\sigma_{n_i}^2}{P} P}{K \sum_{i=1}^L \beta_{in} \left[\bar{\beta}_{ln} - \beta_{in} \right]} \tag{92}
\end{aligned}$$

Concerning the last term: $\mathcal{E} \left[|z_{ln}|^2 \right] = \mathcal{E} \left[\left| \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{n}_l \right|^2 \right]$, we have :

$$\begin{aligned}
& \mathcal{E} \left\{ |z_{ln}|^2 \right\} = \mathcal{E} \left\{ \left| \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{n}_l \right|^2 \right\} \\
&= \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{n}_l \mathbf{n}_l^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathcal{E} \left[\mathbf{n}_l \mathbf{n}_l^H \right] \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \sigma_{n_i}^2 \mathbf{I}_M \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \sigma_{n_i}^2 \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \sigma_{n_i}^2 \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \sigma_{n_i}^2 \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \text{diag} \left(\rho_{1,l}, \rho_{2,l}, \dots, \rho_{K,l} \right) \mathbf{M}_{p,l}^A(n,:)^H \right\} \\
&= \frac{\sigma_{n_i}^2}{K} \sum_{i=1}^K \rho_{i,l} \mathcal{E} \left\{ \mathbf{M}_{p,l}^A(n,:) \mathbf{M}_{p,l}^A(n,:)^H \right\} \tag{93} \\
&\approx \frac{\sigma_{n_i}^2 \gamma_{n,l} \sum_{i=1}^K \rho_{i,l}}{K}
\end{aligned}$$

We have:
$$\frac{\sum_{i=1}^K \rho_{i,l}}{K} = \frac{1}{K} \sum_{n=1}^K \left[p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_l}^2 P \right] = \frac{p_p M}{KP} \sum_{n=1}^K \bar{\beta}_{ln} + \sigma_{n_l}^2 P$$

$$\mathcal{E} \left\{ |z_{ln}|^2 \right\} = \frac{\left(p_p \frac{M}{KP} \sum_{n=1}^K \bar{\beta}_{ln} + \sigma_{n_l}^2 P \right) \sigma_{n_l}^2 P^2}{p_p^2 M^2 \left[\left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] \right) + \frac{P^2 \sigma_{n_l}^2}{M p_p} \left[\left(2 + \frac{K}{P} \right) \bar{\beta}_{ln} + \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \frac{\sigma_{n_l}^2}{p_p} \right] \right]}$$

The final expression for the uplink capacity in the case of a ZF receiver is then equal to:

$$R_{ln} = C \left(\frac{\|\mathcal{E}(\boldsymbol{\alpha}_{ln}^T)\|^2}{\sum_{j=1}^L \mathcal{E}\{\|\boldsymbol{\alpha}_{jn}^T\|^2\} - \|\mathcal{E}(\boldsymbol{\alpha}_{ln}^T)\|^2 + \mathcal{E}(|z_{ln}|^2)} \right) \left(\frac{p_p p_u \beta_{ln}^2}{\left(p_p \bar{\beta}_{ln} + P^2 \sigma_{n_l}^2 / M \right)^2} \right. \\ \left. \frac{\frac{p_u}{p_p} \sum_{j=1}^L \beta_{jl} \frac{(\hat{\beta}_l - \beta_{jl}) / P + \beta_{jl} + \frac{\sigma_{n_l}^2}{p_p} P}{K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} - \frac{p_p p_u \beta_{ln}^2}{\left(p_p \bar{\beta}_{ln} + P^2 \sigma_{n_l}^2 / M \right)^2} + \frac{\sigma_{n_l}^2 \gamma_{n,l} \sum_{i=1}^K \rho_{i,l}}{K}}{\right)} \quad (94)$$

When M goes towards infinity, we obtain:

$$R_{ln} = C \left(\frac{\|\mathcal{E}(\boldsymbol{\alpha}_{ln}^T)\|^2}{\sum_{j=1}^L \mathcal{E}\{\|\boldsymbol{\alpha}_{jn}^T\|^2\} - \|\mathcal{E}(\boldsymbol{\alpha}_{ln}^T)\|^2 + \mathcal{E}(|z_{ln}|^2)} \right) \left(\frac{p_u \beta_{ln}^2}{p_p (\bar{\beta}_{ln})^2} \right. \\ \left. \frac{\frac{p_u}{p_p} \sum_{j=1}^L \beta_{jl} \frac{(\hat{\beta}_l - \beta_{jl}) / P + \beta_{jl} + \frac{\sigma_{n_l}^2}{p_p} P}{K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} - \frac{p_u \beta_{ln}^2}{p_p (\bar{\beta}_{ln})^2}}{\right)} \quad (95)$$

At high SNR we can neglect the term $\frac{\sigma_{n_l}^2}{P_p} P \beta_{jl}$ and, doing this, when we consider also the case where P goes towards infinity, we obtain:

$$\begin{aligned}
R_{ln} \approx_{M \rightarrow \infty} C & \left(\frac{\frac{P_u}{P_p} \frac{\beta_{lln}^2}{(\bar{\beta}_{ln})^2}}{\frac{P_u}{P_p} \frac{\sum_{j=1}^L \beta_{jl}^2}{K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} - \frac{P_u}{P_p} \frac{\beta_{lln}^2}{(\bar{\beta}_{ln})^2}} \right) \\
R_{ln} \approx_{M \rightarrow \infty} C & \left(\frac{\frac{\beta_{lln}^2}{(\bar{\beta}_{ln})^2}}{\frac{\sum_{j=1}^L \beta_{jl}^2}{K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} - \frac{\beta_{lln}^2}{(\bar{\beta}_{ln})^2}} \right) = \left(\frac{\beta_{lln}^2 \left[K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}] \right]}{(\bar{\beta}_{ln})^2 \sum_{j=1}^L \beta_{jl}^2 - \beta_{lln}^2 K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} \right) \quad (96)
\end{aligned}$$

4.5. MMSE Receiver:

In this case the l th base station processes the received signal by multiplying it by matrix:

$\left(\hat{\Gamma}_{ll}^H \hat{\Gamma}_{ll} + \frac{\sigma_{n_l}^2}{P_u} \mathbf{I}_K \right)^{-1}$. From (6) and (14), we obtain:

$$\mathbf{r}_l = \left(\hat{\Gamma}_{ll}^H \hat{\Gamma}_{ll} + \frac{\sigma_{n_l}^2}{P_u} \mathbf{I}_K \right)^{-1} \hat{\Gamma}_{ll}^H \mathbf{y}_l \quad (97)$$

With:

$$\hat{\Gamma}_{ll} = \mathbf{A}_{M \times P} \left[\hat{\mathbf{H}}_{ll} \mathbf{D}_{ll}^{1/2} \right]_{P \times K} = P_p^{-1/2} \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll}$$

$\mathbf{\Pi}_A$ is of size $M \times M$, $\tilde{\mathbf{Y}}_{p,l}$ is of size $M \times K$. The product $\hat{\Gamma}_{ll}^H \hat{\Gamma}_{ll}$ can be written as:

$$\hat{\Gamma}_{ll}^H \hat{\Gamma}_{ll} = P_p^{-1} \mathbf{D}_{ll}^H \left(\mathbf{D}_l^{-1} \right)^H \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_{ll} \quad (98)$$

Of course, we have:

$$\begin{aligned}
\Pi_A^H \Pi_A &= \left[A(A^H A)^{-1} A^H \right]^H A(A^H A)^{-1} A^H \\
&= A \left[(A^H A)^{-1} \right]^H A^H A(A^H A)^{-1} A^H \\
&= A \left[(A^H A)^{-1} \right]^H A^H = A \left[(A^H A)^{-1} \right] A^H \\
&= \Pi_A
\end{aligned}$$

So (98) reduces to:

$$\begin{aligned}
\hat{\Gamma}_u^H \hat{\Gamma}_u &= p_p^{-1} \mathbf{D}_u^H (\mathbf{D}_l^{-1})^H \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_u \\
&= \frac{1}{p_p} \mathbf{D}_u \mathbf{D}_l^{-1} \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_u \\
\left[\hat{\Gamma}_u^H \hat{\Gamma}_u + \frac{\sigma_{n_i}^2}{p_u} \mathbf{I}_K \right]^{-1} &= \left[\frac{1}{p_p} \mathbf{D}_u \mathbf{D}_l^{-1} \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_u + \frac{\sigma_{n_i}^2}{p_u} \mathbf{I}_K \right]^{-1} \\
&= \left[\frac{1}{p_p} \mathbf{D}_u \mathbf{D}_l^{-1} \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_u + \mathbf{D}_u \mathbf{D}_l^{-1} \mathbf{D}_l \mathbf{D}_u^{-1} \frac{\sigma_{n_i}^2}{p_u} \mathbf{I}_K \mathbf{D}_u^{-1} \mathbf{D}_l \mathbf{D}_l^{-1} \mathbf{D}_u \right]^{-1} \\
&= \left[\frac{1}{p_p} \mathbf{D}_u \mathbf{D}_l^{-1} \left(\tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} + \mathbf{D}_l \mathbf{D}_u^{-1} \frac{\sigma_{n_i}^2}{p_u} \mathbf{I}_K \mathbf{D}_u \mathbf{D}_l^{-1} \mathbf{D}_l \right) \mathbf{D}_l^{-1} \mathbf{D}_u \right]^{-1} \quad (99) \\
&= p_p (\mathbf{D}_l^{-1} \mathbf{D}_u)^{-1} \left(\tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} + \mathbf{D}_l \mathbf{D}_u^{-1} \frac{\sigma_{n_i}^2}{p_u} \mathbf{I}_K \mathbf{D}_u \mathbf{D}_l^{-1} \mathbf{D}_l \right)^{-1} (\mathbf{D}_u \mathbf{D}_l^{-1})^{-1} \\
&= p_p \mathbf{D}_u^{-1} \mathbf{D}_l \left(\tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_i}^2}{p_u} \mathbf{D}_l \mathbf{D}_u^{-1} \mathbf{D}_u \mathbf{D}_l^{-1} \mathbf{D}_l \right)^{-1} \mathbf{D}_l \mathbf{D}_u^{-1}
\end{aligned}$$

With: $[\mathbf{D}_u]_{k,k} = \beta_{lk}$ and $\mathbf{D}_l \square \sum_{i=1}^L \mathbf{D}_{il}$. We obtain:

$$\begin{aligned}
&\left(\hat{\Gamma}_u^H \hat{\Gamma}_u + \frac{\sigma_{n_i}^2}{p_u} \mathbf{I}_K \right)^{-1} \hat{\Gamma}_u^H \\
&= p_p \mathbf{D}_u^{-1} \mathbf{D}_l \left(\tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_i}^2}{p_u} \mathbf{D}_l \mathbf{D}_u^{-1} \mathbf{D}_u \mathbf{D}_l^{-1} \mathbf{D}_l \right)^{-1} \mathbf{D}_l \mathbf{D}_u^{-1} (p_p^{-1/2} \Pi_A \tilde{\mathbf{Y}}_{p,l} \mathbf{D}_l^{-1} \mathbf{D}_u)^H \\
&= p_p^{1/2} \mathbf{D}_u^{-1} \mathbf{D}_l \left(\tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_i}^2}{p_u} \mathbf{D}_l \mathbf{D}_u^{-1} \mathbf{D}_u \mathbf{D}_l^{-1} \mathbf{D}_l \right)^{-1} \mathbf{D}_l \mathbf{D}_u^{-1} \mathbf{D}_u \mathbf{D}_l^{-1} \tilde{\mathbf{Y}}_{p,l}^H \Pi_A \\
&= p_p^{1/2} \mathbf{D}_u^{-1} \mathbf{D}_l \left(\tilde{\mathbf{Y}}_{p,l}^H \Pi_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_i}^2}{p_u} \mathbf{D}_l \mathbf{D}_u^{-1} \mathbf{D}_u \mathbf{D}_l^{-1} \mathbf{D}_l \right)^{-1} \tilde{\mathbf{Y}}_{p,l}^H \Pi_A
\end{aligned} \quad (100)$$

We deduce from (100):

$$\begin{aligned}\hat{\mathbf{r}}_l &= \left[\hat{\mathbf{\Gamma}}_{ll}^H \hat{\mathbf{\Gamma}}_{ll} + \frac{\sigma_{n_l}^2}{p_u} \mathbf{I}_K \right]^{-1} \hat{\mathbf{\Gamma}}_{ll}^H \mathbf{y}_l \\ &= p_p^{1/2} \mathbf{D}_{ll}^{-1} \mathbf{D}_l \left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{p_u} \mathbf{D}_l \mathbf{D}_{ll}^{-1} \mathbf{D}_{ll}^{-1} \mathbf{D}_l \right)^{-1} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{y}_l\end{aligned}\quad (101)$$

Let $\tilde{\mathbf{r}}$ the received $K \times 1$ vector: $\tilde{\mathbf{r}}_l \square p_p^{-1/2} \sum_{i=1}^L \beta_{iln} / \beta_{lln} \hat{\mathbf{r}}_l$, we deduce from (101) that:

$$\tilde{\mathbf{r}}_l = \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{p_u} \mathbf{D}_l \mathbf{D}_{ll}^{-1} \mathbf{D}_{ll}^{-1} \mathbf{D}_l \right]^{-1} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{y}_l \quad (102)$$

4.6. Analysis of the Pilot Contamination effect:

We denote as $\mathbf{S}_{p,l}^A$ the matrix: $\left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{p_u} \mathbf{D}_l \mathbf{D}_{ll}^{-1} \mathbf{D}_{ll}^{-1} \mathbf{D}_l \right]_{K \times K}^{-1}$. In this case, if we look at the n th component of vector $\tilde{\mathbf{r}}_l$, we have :

$$\begin{aligned}\tilde{\mathbf{r}}_l(n) &= \tilde{r}_{ln} \\ &= \mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{y}_l \\ &= \mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \left[\sqrt{p_u} \mathbf{A} \sum_{i=1}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \mathbf{x}_i + \mathbf{n}_l \right] \\ &= \sqrt{p_u} \mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \mathbf{x}_l + \sqrt{p_u} \mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \sum_{i \neq l}^L \mathbf{H}_{i,l} \mathbf{D}_{i,l}^{1/2} \mathbf{x}_i + \mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{n}_l \\ &= \boldsymbol{\alpha}_{ln}^T \mathbf{x}_l + \sum_{j \neq l}^L \boldsymbol{\alpha}_{jn}^T \mathbf{x}_j + z_{ln} \quad (103)\end{aligned}$$

With: $\boldsymbol{\alpha}_{jn}^T \square \sqrt{p_u} \mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{H}_{j,l} \mathbf{D}_{j,l}^{1/2}$ and $z_{ln} = \mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{n}_l$. We have:

$$\mathcal{E}[\boldsymbol{\alpha}_{ln}^T] = \mathcal{E} \left[\sqrt{p_u} \mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2} \right] \mathbf{e}_n \quad (104)$$

Where \mathbf{e}_n is the n th column of the $K \times K$ identity matrix. By adding and subtracting $\mathcal{E}[\boldsymbol{\alpha}_{ln}^T]$ from $\boldsymbol{\alpha}_{ln}^T$ in (103), we obtain:

$$\tilde{r}_{ln} = \mathcal{E} \{ \boldsymbol{\alpha}_{ln}^T \} \mathbf{x}_l + (\boldsymbol{\alpha}_{ln}^T - \mathcal{E} \{ \boldsymbol{\alpha}_{ln}^T \}) \mathbf{x}_l + \sum_{j \neq l}^L \boldsymbol{\alpha}_{jn}^T \mathbf{x}_j + z_{ln} \quad (105)$$

We obtain again the capacity of the n th user in the l th cell :

$$R_{ln} = C \left(\frac{\|\mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}\|^2}{\sum_{j=1}^L \mathcal{E}\{\|\boldsymbol{\alpha}_{jn}^T\|^2\} - \|\mathcal{E}\{\boldsymbol{\alpha}_{ln}^T\}\|^2 + \mathcal{E}\{|z_{ln}|^2\}} \right) \quad (106)$$

To develop (106) we detail first the calculation of the matrix:

$$\left(\mathbf{S}_{p,l}^A\right)^{-1} = \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{p_u} \mathbf{D}_l \mathbf{D}_{ll}^{-1} \mathbf{D}_{ll}^{-1} \mathbf{D}_l, \text{ we have:}$$

$$\begin{aligned} \left(\mathbf{S}_{p,l}^A\right)^{-1} &= \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{p_u} \mathbf{D}_l \mathbf{D}_{ll}^{-1} \mathbf{D}_{ll}^{-1} \mathbf{D}_l \\ &= p_p \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{W} + \dots \\ &\dots + \sqrt{p_p} \mathbf{W}_l^H \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{W}_l + \frac{\sigma_{n_l}^2}{p_u} \mathbf{D}_l \mathbf{D}_{ll}^{-1} \mathbf{D}_{ll}^{-1} \mathbf{D}_l \end{aligned} \quad (107)$$

Re-using equations (83) up to (89) we deduce that the matrix $\mathcal{E}\left[\left(\mathbf{S}_{p,l}^A\right)^{-1}\right]$ is a diagonal matrix with the n th term equal to :

$$\begin{aligned} \mathcal{E}\left[\left(\mathbf{S}_{p,l}^A\right)^{-1}\right] &= \mathcal{E}\left[\tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{p_u} \mathbf{D}_l \mathbf{D}_{ll}^{-1} \mathbf{D}_{ll}^{-1} \mathbf{D}_l\right] \\ &= \text{diag}\left(\xi_{1,l}, \xi_{2,l}, \dots, \xi_{K,l}\right) \end{aligned} \quad (108)$$

With:

$$\begin{aligned} \xi_{n,l} &= p_p \frac{M}{P} \sum_{i=1}^L \beta_{in} + \sigma_{n_l}^2 P + \frac{\sigma_{n_l}^2}{p_u} \frac{\beta_{lln}^2}{\left(\sum_{i=1}^L \beta_{in}\right)^2} \\ \xi_{n,l} &= p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_l}^2 P + \frac{\sigma_{n_l}^2}{p_u} \frac{\beta_{lln}^2}{\bar{\beta}_{ln}^2} \end{aligned} \quad (109)$$

We come back to the calculation of $\mathcal{E}\left[\boldsymbol{\alpha}_{ln}^T\right]$, remember we have :

$$\begin{aligned}
\mathcal{E}[\boldsymbol{\alpha}_{ln}^T] &= \mathcal{E}\left[\sqrt{p_u} \mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{H}_{l,l} \mathbf{D}_{l,l}^{1/2}\right] \mathbf{e}_n \\
\mathcal{E}[\boldsymbol{\alpha}_{ln}^T] &\approx \sqrt{p_u} \mathcal{E}\left[\mathbf{S}_{p,l}^A(n,:)\right] \mathcal{E}\left[\tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{l,l}\right] \mathbf{e}_n \\
&= \sqrt{p_u} \mathcal{E}\left[\mathbf{S}_{p,l}^A(n,:)\right] \mathcal{E}\left[\left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l\right)^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{G}_{l,l}\right] \mathbf{e}_n \\
&= \sqrt{p_u} \mathcal{E}\left[\mathbf{S}_{p,l}^A(n,:)\right] \mathcal{E}\left[\sqrt{p_p} \sum_{i=1}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} (\mathbf{A}^H \mathbf{A})^{-1} \mathbf{A}^H \mathbf{A} \mathbf{G}_{l,l}\right] \mathbf{e}_n \\
&= \sqrt{p_u p_p} \mathcal{E}\left[\mathbf{S}_{p,l}^A(n,:)\right] \mathcal{E}\left[\mathbf{G}_{l,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{l,l}\right] \mathbf{e}_n
\end{aligned}$$

We have already found that (see equation (86)):

$$\mathcal{E}\left[\left[\mathbf{G}_{l,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{l,l}\right](n,n)\right] = \frac{M \beta_{ln}}{P}$$

We eventually obtain:

$$\begin{aligned}
&\mathcal{E}\left\{\left[\boldsymbol{\alpha}_n^T\right]\right\} \\
&= \sqrt{p_u p_p} \frac{M \beta_{ln}}{P \xi_{n,l}} \mathbf{e}_n \\
&= \sqrt{p_u p_p} \frac{M \beta_{ln}}{P \left[p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_l}^2 P + \frac{\sigma_{n_l}^2}{p_u} \frac{\beta_{ln}^2}{\bar{\beta}_{ln}^2}\right]} \mathbf{e}_n \quad (110) \\
&= \sqrt{\frac{p_u}{p_p}} \frac{\beta_{ln}}{\bar{\beta}_{ln} + \frac{P^2 \sigma_{n_l}^2}{M p_p} + \frac{P \sigma_{n_l}^2}{M p_p p_u} \frac{\beta_{ln}^2}{\bar{\beta}_{ln}^2}} \mathbf{e}_n
\end{aligned}$$

Then, we have to calculate $\mathcal{E}\left\{\left\|\boldsymbol{\alpha}_{jn}^T\right\|^2\right\}$, we have:

$$\begin{aligned}
\mathcal{E}\left\{\left\|\boldsymbol{\alpha}_{jn}^T\right\|^2\right\} &= \mathcal{E}\left(\boldsymbol{\alpha}_{jn}^T \boldsymbol{\alpha}_{jn}^*\right) \\
&= p_u \mathcal{E}\left[\mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{S}_{p,l}^A(n,:) \right] \\
&= p_u \mathcal{E}\left[\mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{S}_{p,l}^A(n,:) \right]
\end{aligned}$$

We have:

$$\begin{aligned}
&\tilde{\mathbf{Y}}_{p,l}^H \boldsymbol{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \boldsymbol{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \\
&= \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l\right)^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \left(\sqrt{p_p} \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} + \mathbf{W}_l\right)
\end{aligned}$$

And this yields:

$$\begin{aligned}
& \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{S}_{p,l}^A(n,:) \right] \\
&= \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \left(\sqrt{p_p} \mathbf{A} \sum_{i=1}^L \mathbf{G}_{i,l} + \mathbf{W}_l \right)^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \left(\sqrt{p_p} \mathbf{A} \sum_{m=1}^L \mathbf{G}_{m,l} + \mathbf{W}_l \right) \mathbf{S}_{p,l}^A(n,:) \right] \\
&= \mathcal{E} \left[p_p \mathbf{S}_{p,l}^A(n,:) \sum_{i=1, i \neq j}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1, m \neq j}^L \mathbf{G}_{m,l} \mathbf{S}_{p,l}^A(n,:) \right] + \dots \\
&\dots + \mathcal{E} \left[p_p \mathbf{S}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{S}_{p,l}^A(n,:) \right] + \dots \\
&\dots + \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{S}_{p,l}^A(n,:) \right]
\end{aligned}$$

And, we have:

$$\begin{aligned}
& \mathcal{E} \left[p_p \mathbf{S}_{p,l}^A(n,:) \sum_{i=1, i \neq j}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1, m \neq j}^L \mathbf{G}_{m,l} \mathbf{S}_{p,l}^A(n,:) \right] \\
&= \frac{p_p}{P} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \sum_{i=1, i \neq j}^L \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{A}^H \mathbf{A} \mathbf{G}_{i,l} \mathbf{S}_{p,l}^A(n,:) \right] \\
&= \frac{p_p}{P^2} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \sum_{i=1, i \neq j}^L \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \mathbf{S}_{p,l}^A(n,:) \right] \\
&= \frac{p_p}{P^2 K} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \text{Trace} \left[\mathcal{E} \left[\sum_{i=1, i \neq j}^L \mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \right] \right] \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \mathbf{S}_{p,l}^A(n,:) \right] \\
&= \frac{p_p}{P^2 K} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \right] \right] \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \left[\sum_{i=1, i \neq j}^L \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{i,l}^H \mathbf{G}_{i,l} \right] \right] \right] \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \mathbf{S}_{p,l}^A(n,:) \right] \\
&= \frac{p_p \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \mathbf{S}_{p,l}^A(n,:) \right]}{P^2 K}
\end{aligned}$$

Concerning the quantity: $\mathcal{E} \left(\left[\mathbf{S}_{p,l}^A(n,:) \mathbf{S}_{p,l}^A(n,:) \right] \right)$, we have:

$$\begin{aligned}
& \mathcal{E} \left(\left[\mathbf{S}_{p,l}^A \mathbf{S}_{p,l}^A \right] \right) \\
&= \mathcal{E} \left(\left[\left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_n^2}{p_u} \mathbf{D}_l \mathbf{D}_l^{-1} \mathbf{D}_l^{-1} \mathbf{D}_l \right)^H \left(\tilde{\mathbf{Y}}_{p,l} \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_n^2}{p_u} \mathbf{D}_l \mathbf{D}_l^{-1} \mathbf{D}_l^{-1} \mathbf{D}_l \right) \right]^{-1} \right)
\end{aligned}$$

We use the following notation: $\mathbf{A}_l = \mathbf{D}_l \mathbf{D}_l^{-1} \mathbf{D}_l^{-1} \mathbf{D}_l$, \mathbf{A}_l is a diagonal matrix with deterministic elements, $\mathbf{A}_l = \text{diag}(\eta_{1,l}, \eta_{2,l}, \dots, \eta_{n,l}, \dots, \eta_{K,l})$ and n th element equal to: $\eta_{n,l} = \frac{\beta_{ln}^2}{\left(\sum_{i=1}^L \beta_{iln} \right)^2} = \frac{\beta_{ln}^2}{\beta_{ln}^2}$. In

these conditions we have:

$$\begin{aligned}
& \mathcal{E} \left(\left[\mathbf{S}_{p,l}^A \mathbf{S}_{p,l}^{A^H} \right] \right) \\
&= \mathcal{E} \left(\left[\left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{P_u} \mathbf{A}_l \right)^H \left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{P_u} \mathbf{A}_l \right) \right]^{-1} \right) \\
&= \mathcal{E} \left(\left[\left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{P_u} \mathbf{A}_l \right) \left(\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{P_u} \mathbf{A}_l \right) \right]^{-1} \right) \\
&= \mathcal{E} \left(\left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^2}{P_u} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{A}_l + \frac{\sigma_{n_l}^2}{P_u} \mathbf{A}_l \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} + \frac{\sigma_{n_l}^4}{P_u^2} \mathbf{A}_l^2 \right]^{-1} \right) \\
&\approx \left(\mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] + \frac{\sigma_{n_l}^2}{P_u} \mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \right] \mathbf{A}_l + \frac{\sigma_{n_l}^2}{P_u} \mathbf{A}_l \mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] + \frac{\sigma_{n_l}^4}{P_u^2} \mathbf{A}_l^2 \right)^{-1} \\
&= \left(\mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] + \frac{\sigma_{n_l}^2}{P_u} \mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] \mathbf{A}_l + \frac{\sigma_{n_l}^2}{P_u} \mathbf{A}_l \mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] + \frac{\sigma_{n_l}^4}{P_u^2} \mathbf{A}_l^2 \right)^{-1} \\
&= \left(\mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^2 + \frac{\sigma_{n_l}^2}{P_u} \mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] \mathbf{A}_l + \frac{\sigma_{n_l}^2}{P_u} \mathbf{A}_l \mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] + \frac{\sigma_{n_l}^4}{P_u^2} \mathbf{A}_l^2 \right)^{-1}
\end{aligned}$$

We have already found that:

$$\begin{aligned}
& \mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right]^2 \\
&= p_p^2 \frac{M^2}{P^2} \sum_{i=1}^L \text{diag}(\beta_{i1}, \beta_{i2}, \dots, \beta_{iK}) \left[\text{diag} \left(\bar{\beta}_{i1} - \beta_{i1} + \frac{\hat{\beta}_l}{M}, \bar{\beta}_{i2} - \beta_{i2} + \frac{\hat{\beta}_l}{M}, \dots, \bar{\beta}_{iK} - \beta_{iK} + \frac{\hat{\beta}_l}{M} \right) \right] + \dots \\
&\dots + \left(2p_p \sigma_{n_l}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) + p_p \frac{K \sigma_{n_l}^2 M}{P} \right) \text{diag}(\bar{\beta}_{i1}, \bar{\beta}_{i2}, \dots, \bar{\beta}_{iK}) + \text{Trace}(\mathbf{A} \mathbf{A}^H) \left(\frac{KP}{M} \sigma_{n_l}^4 + p_p \frac{\sigma_{n_l}^2 \hat{\beta}_l}{P} \right) \mathbf{I}_K \\
&= \text{diag}(\mu_{l,1}, \mu_{l,2}, \dots, \mu_{l,n}, \dots, \mu_{l,K})
\end{aligned}$$

With:

$$\mu_{l,n} = p_p^2 \frac{M^2}{P^2} \sum_{i=1}^L \beta_{in} \left[\bar{\beta}_{in} - \beta_{in} + \frac{\hat{\beta}_l}{M} \right] + \left(2p_p M \sigma_{n_l}^2 + p_p K \sigma_{n_l}^2 \frac{M}{P} \right) \bar{\beta}_{in} + \sigma_{n_l}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) \left(p_p \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \sigma_{n_l}^2 \right)$$

We have:

$$\mathcal{E} \left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \right] = \text{diag}(\rho_{1,l}, \rho_{2,l}, \dots, \rho_{K,l})$$

With:

$$\rho_{n,l} = p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_l}^2 P$$

We obtain eventually :

$$\mathcal{E}\left(\left[\mathbf{S}_{p,l}^A \mathbf{S}_{p,l}^{A^H}\right]\right) = \text{diag}(\lambda_{1,l}, \lambda_{2,l}, \dots, \lambda_{n,l}, \dots, \lambda_{K,l})$$

With :

$$\lambda_{n,l} = \frac{1}{\mu_{n,l} + 2 \frac{\sigma_{n_l}^2}{p_u} \rho_{n,l} \eta_{n,l} + \frac{\sigma_{n_l}^4}{p_u^2} \eta_{n,l}^2} \quad (111)$$

We have :

$$\begin{aligned} \frac{1}{\lambda_{n,l}} &= \mu_{n,l} + 2 \frac{\sigma_{n_l}^2}{p_u} \rho_{n,l} \eta_{n,l} + \frac{\sigma_{n_l}^4}{p_u^2} \eta_{n,l}^2 \\ &= p_p^2 \frac{M^2}{P^2} \sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] + \left(2 p_p M \sigma_{n_l}^2 + p_p K \sigma_{n_l}^2 \frac{M}{P} \right) \bar{\beta}_{ln} + \sigma_{n_l}^2 \text{Trace}(\mathbf{A} \mathbf{A}^H) \left(p_p \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \sigma_{n_l}^2 \right) + \dots \\ &+ 2 \frac{\sigma_{n_l}^2}{p_u} \left(p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_l}^2 P \right) \frac{\beta_{lln}^2}{\bar{\beta}_{ln}^2} + \frac{\sigma_{n_l}^4}{p_u^2} \frac{\beta_{lln}^4}{\bar{\beta}_{ln}^4} \\ &\approx_{M \rightarrow \infty} p_p^2 \frac{M^2}{P^2} \sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] + \left(2 p_p M \sigma_{n_l}^2 + p_p K \sigma_{n_l}^2 \frac{M}{P} \right) \bar{\beta}_{ln} + \sigma_{n_l}^2 M \left(p_p \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \sigma_{n_l}^2 \right) + \dots \\ &+ 2 \frac{\sigma_{n_l}^2}{p_u} \left(p_p \frac{M}{P} \bar{\beta}_{ln} + \sigma_{n_l}^2 P \right) \frac{\beta_{lln}^2}{\bar{\beta}_{ln}^2} + \frac{\sigma_{n_l}^4}{p_u^2} \frac{\beta_{lln}^4}{\bar{\beta}_{ln}^4} \\ &= p_p^2 \frac{M^2}{P^2} \sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} + \frac{\hat{\beta}_l}{M} \right] + \dots \\ &+ M \left(\left[2 p_p \sigma_{n_l}^2 + p_p \frac{K}{P} \sigma_{n_l}^2 \right] \bar{\beta}_{ln} + \sigma_{n_l}^2 \left[p_p \frac{\hat{\beta}_l}{P} + \frac{KP}{M} \sigma_{n_l}^2 \right] \right) + 2 \frac{\sigma_{n_l}^2}{p_u} \left(p_p \frac{\bar{\beta}_{ln}}{P} + \sigma_{n_l}^2 \frac{P}{M} \right) \frac{\beta_{lln}^2}{\bar{\beta}_{ln}^2} + \frac{\sigma_{n_l}^4}{p_u^2} \frac{\beta_{lln}^4}{\bar{\beta}_{ln}^4} \\ &\approx_{M \rightarrow \infty} p_p^2 \frac{M^2}{P^2} \sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} \right] \end{aligned}$$

Then, with (111) it is possible to reuse the calculations of the ZF and we obtain:

$$\begin{aligned}
& \mathcal{E} \left[p_p \mathbf{S}_{p,l}^A(n,:) \sum_{i=1, i \neq j}^L \mathbf{G}_{i,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \sum_{m=1, m \neq j}^L \mathbf{G}_{m,l} \mathbf{S}_{p,l}^A(n,:) ^H \right] \\
&= \frac{p_p \text{Trace} \left[(\mathbf{A}^H \mathbf{A})^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) \lambda_{n,l}}{P^2 K} \\
&\approx_{M \rightarrow \infty} \frac{p_p M^2 \beta_{jl}}{P^3 K} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) \lambda_{n,l} \\
&= p_p \frac{M^2 \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right)}{P^3 K} \frac{P^2}{p_p^2 M^2 \left(\sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}] \right)} \\
&\approx_{M \rightarrow \infty} \frac{1}{p_p P K} \frac{\beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right)}{\sum_{i=1}^L \beta_{iln} [\beta_{il} + \bar{\beta}_{ln} - \beta_{iln}]}
\end{aligned}$$

Concerning the term: $\mathcal{E} \left[p_p \mathbf{S}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{S}_{p,l}^A(n,:) ^H \right]$, we have :

$$\begin{aligned}
& \mathcal{E} \left[p_p \mathbf{S}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{S}_{p,l}^A(n,:) ^H \right] \\
&= \frac{p_p}{K} \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \mathbf{S}_{p,l}^A(n,:) ^H \right] \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right] \right] \\
&= \frac{p_p \lambda_{n,l}}{K} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right]^2 \right]
\end{aligned}$$

Hence this yields:

$$\begin{aligned}
& \mathcal{E} \left[p_p \mathbf{M}_{p,l}^A(n,:) \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{M}_{p,l}^A(n,:) ^H \right] \\
&= \frac{p_p}{K} \text{Trace} \left[\mathcal{E} \left[\mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{A} \mathbf{G}_{j,l} \right]^2 \right] \lambda_{n,l} \\
&= \frac{p_p M^2 \beta_{jl}^2}{K P^2} \lambda_{n,l} \\
&\approx_{M \rightarrow \infty} \frac{p_p M^2 \beta_{jl}^2}{K P^2} \frac{P^2}{p_p^2 M^2 \left(\sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}] \right)} \\
&= \frac{\beta_{jl}^2}{K p_p \left(\sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}] \right)}
\end{aligned}$$

The remaining last term is: $\mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{S}_{p,l}^A(n,:) \right]^H$, we have:

$$\begin{aligned} & \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{S}_{p,l}^A(n,:) \right]^H \\ &= \frac{\lambda_{n,l}}{K} \text{Trace} \left[\mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \right] \right] \end{aligned}$$

With:

$$\begin{aligned} & \mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \right] \\ &= \frac{M^2 \sigma_{n_l}^2}{KP} \beta_{jl} \mathbf{I}_K \end{aligned}$$

And :

$$\begin{aligned} & \mathcal{E} \left[\mathbf{S}_{p,l}^A(n,:) \mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \mathbf{S}_{p,l}^A(n,:) \right]^H \\ &= \frac{\lambda_{n,l}}{K} \text{Trace} \left[\mathcal{E} \left[\mathbf{W}_l^H \mathbf{A} \mathbf{G}_{j,l} \mathbf{G}_{j,l}^H \mathbf{A}^H \mathbf{W}_l \right] \right] \\ &= \frac{\lambda_{n,l}}{K} \frac{M^2 \sigma_{n_l}^2}{P} \beta_{jl} \\ &= \frac{M^2 \sigma_{n_l}^2}{KP} \beta_{jl} \lambda_{n,l} \\ &\approx_{M \rightarrow \infty} \frac{\sigma_{n_l}^2 P}{p_p^2 K} \frac{\beta_{jl}}{\left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} \right] \right)} \end{aligned}$$

And, we eventually obtain:

$$\begin{aligned}
\mathcal{E} \left\{ \left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right\} &= \mathcal{E} \left(\boldsymbol{\alpha}_{jn}^T \boldsymbol{\alpha}_{jn}^* \right) \\
&= \frac{\lambda_{n,l} p_u p_p \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right)}{P^2 K} + \frac{\lambda_{n,l} p_u p_p M^2 \beta_{jl}^2}{K P^2} + \frac{\sigma_{n_l}^2 p_u M^2 \beta_{jl} \lambda_{n,l}}{K P} \\
&= \frac{\lambda_{n,l} \left[p_u p_p \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_l}^2 p_u M^2 P \beta_{jl} \right]}{P^2 K} \\
&= \frac{\left[p_u p_p \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_l}^2 p_u M^2 P \beta_{jl} \right]}{\mu_{n,l} + 2 \frac{\sigma_{n_l}^2}{p_u} \rho_{n,l} \eta_{n,l} + \frac{\sigma_{n_l}^4}{p_u^2} \eta_{n,l}^2} \\
&\approx_{M \rightarrow \infty} \frac{P^2 \left(p_u p_p \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_l}^2 p_u M^2 P \beta_{jl} \right)}{p_p^2 P^2 K M^2 \left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} \right] \right)} \quad (112)
\end{aligned}$$

Considering the case where M goes towards infinity, we have:

$$\lim_{M \rightarrow \infty} \frac{1}{M} \mathbf{A}^H \mathbf{A} = \frac{1}{P} \mathbf{I}_P$$

This yields: $\lim_{M \rightarrow \infty} \mathbf{A}^H \mathbf{A} = \frac{M}{P} \mathbf{I}_P$ and $\lim_{M \rightarrow \infty} \text{Trace} \left[\left(\mathbf{A}^H \mathbf{A} \right)^2 \right] = \frac{M^2}{P^2} P = \frac{M^2}{P}$ and we obtain:

$$\begin{aligned}
&\mathcal{E} \left\{ \left\| \boldsymbol{\alpha}_{jn}^T \right\|^2 \right\} \\
&\approx_{M \rightarrow \infty} \frac{P^2 \left(p_u p_p M^2 \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) / P + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_l}^2 p_u M^2 P \beta_{jl} \right)}{p_p^2 P^2 K M^2 \left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} \right] \right)} \\
&= \frac{p_u p_p \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) / P + p_u p_p \beta_{jl}^2 + \sigma_{n_l}^2 p_u P \beta_{jl}}{p_p^2 K \left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} \right] \right)} \quad (113) \\
&\approx_{M \rightarrow \infty} \frac{p_u}{p_p} \frac{\beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il} \right) / P + \beta_{jl}^2 + \frac{\sigma_{n_l}^2}{p_p} P \beta_{jl}}{K \left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln} \right] \right)}
\end{aligned}$$

Concerning the last term: $\mathcal{E}\left[|z_{ln}|^2\right] = \mathcal{E}\left[\left|\mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{n}_l\right|^2\right]$, we have :

$$\begin{aligned}
\mathcal{E}\left\{|z_{ln}|^2\right\} &= \mathcal{E}\left\{\left|\mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{n}_l\right|^2\right\} \\
&= \mathcal{E}\left\{\mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathbf{n}_l \mathbf{n}_l^H \mathbf{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{S}_{p,l}^A(n,:)^H\right\} \\
&= \mathcal{E}\left\{\mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \mathcal{E}\left[\mathbf{n}_l \mathbf{n}_l^H\right] \mathbf{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{S}_{p,l}^A(n,:)^H\right\} \\
&= \mathcal{E}\left\{\mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \sigma_{n_l}^2 \mathbf{I}_M \mathbf{\Pi}_A^H \tilde{\mathbf{Y}}_{p,l} \mathbf{S}_{p,l}^A(n,:)^H\right\} \\
&= \sigma_{n_l}^2 \mathcal{E}\left\{\mathbf{S}_{p,l}^A(n,:) \tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l} \mathbf{S}_{p,l}^A(n,:)^H\right\} \\
&= \sigma_{n_l}^2 \mathcal{E}\left\{\mathbf{S}_{p,l}^A(n,:) \mathcal{E}\left[\tilde{\mathbf{Y}}_{p,l}^H \mathbf{\Pi}_A \tilde{\mathbf{Y}}_{p,l}\right] \mathbf{S}_{p,l}^A(n,:)^H\right\} \\
&= \sigma_{n_l}^2 \mathcal{E}\left\{\mathbf{S}_{p,l}^A(n,:) \text{diag}\left(\rho_{1,l}, \rho_{2,l}, \dots, \rho_{K,l}\right) \mathbf{S}_{p,l}^A(n,:)^H\right\} \\
&= \frac{\sigma_{n_l}^2}{K} \sum_{i=1}^K \rho_{i,l} \mathcal{E}\left\{\mathbf{S}_{p,l}^A(n,:) \mathbf{S}_{p,l}^A(n,:)^H\right\} \tag{114} \\
&= \frac{\sigma_{n_l}^2 \lambda_{n,l} \sum_{i=1}^K \rho_{i,l}}{K}
\end{aligned}$$

$$\mathcal{E}\left\{|z_{ln}|^2\right\} \approx_{M \rightarrow \infty} \frac{\left(p_p \frac{M}{KP} \sum_{n=1}^K \bar{\beta}_{ln} + \sigma_{n_l}^2 P\right) \sigma_{n_l}^2 P^2}{p_p^2 M^2 \left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln}\right]\right)} \approx_{M \rightarrow \infty} \frac{\sum_{n=1}^K \bar{\beta}_{ln} \sigma_{n_l}^2 P}{p_p MK \left(\sum_{i=1}^L \beta_{iln} \left[\bar{\beta}_{ln} - \beta_{iln}\right]\right)} \tag{115}$$

The final expression for the uplink capacity in the case of a MMSE receiver is then equal to:

$$\begin{aligned}
R_{ln} &= C \left(\frac{\|\mathcal{E}(\boldsymbol{\alpha}_{ln}^T)\|^2}{\sum_{j=1}^L \mathcal{E}\left\{\|\boldsymbol{\alpha}_{jn}^T\|^2\right\} - \|\mathcal{E}(\boldsymbol{\alpha}_{ln}^T)\|^2 + \mathcal{E}\left(|z_{ln}|^2\right)} \right) \\
R_{ln} &= C \left(\frac{\frac{p_u}{p_p} \left[\frac{\beta_{ltn}}{\bar{\beta}_{ln} + \frac{P^2 \sigma_{n_l}^2}{Mp_p} + \frac{P \sigma_{n_l}^2 \beta_{ltn}^2}{Mp_p p_u \bar{\beta}_{ln}^2}} \right]^2}{\sum_{j=1}^L \left[\frac{p_u p_p \text{Trace}\left[\left(\mathbf{A}^H \mathbf{A}\right)^2\right] \beta_{jl} \left(\sum_{i=1, i \neq j}^L \beta_{il}\right) + p_u p_p M^2 \beta_{jl}^2 + \sigma_{n_l}^2 p_u M^2 P \beta_{jl}}{\mu_{n,l} + 2 \frac{\sigma_{n_l}^2}{p_u} \rho_{n,l} \eta_{n,l} + \frac{\sigma_{n_l}^4}{p_u^2} \eta_{n,l}^2} \right] - \frac{p_u}{p_p} \left[\frac{\beta_{ltn}}{\bar{\beta}_{ln} + \frac{P^2 \sigma_{n_l}^2}{Mp_p} + \frac{P \sigma_{n_l}^2 \beta_{ltn}^2}{Mp_p p_u \bar{\beta}_{ln}^2}} \right]^2} + \frac{\sigma_{n_l}^2 \lambda_{n,l} \sum_{i=1}^K \rho_{i,l}}{K}} \right)
\end{aligned}$$

$$\begin{aligned}
R_{ln} \approx_{M \rightarrow \infty} C & \left(\frac{\frac{\beta_{ln}^2}{(\bar{\beta}_{ln})^2}}{\frac{\sum_{j=1}^L \beta_{jl}^2}{K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} - \frac{\beta_{ln}^2}{(\bar{\beta}_{ln})^2}} \right) \\
& = \left(\frac{\beta_{ln}^2 \left[K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}] \right]}{(\bar{\beta}_{ln})^2 \sum_{j=1}^L \beta_{jl}^2 - \beta_{ln}^2 K \sum_{i=1}^L \beta_{iln} [\bar{\beta}_{ln} - \beta_{iln}]} \right)
\end{aligned} \tag{116}$$

We obtain the same expression as for the ZF.