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## To cite this version:

Jean-Philippe Anker. An elementary proof of the positivity of the intertwining operator in onedimensional trigonometric Dunkl theory. Proceedings of the American Mathematical Society, 2018, 10.1090/proc/13679 . hal-01399656v2

HAL Id: hal-01399656
https://hal.science/hal-01399656v2
Submitted on 2 Feb 2017

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# AN ELEMENTARY PROOF OF THE POSITIVITY OF THE INTERTWINING OPERATOR IN ONE-DIMENSIONAL TRIGONOMETRIC DUNKL THEORY 

JEAN-PHILIPPE ANKER


#### Abstract

This note is devoted to the intertwining operator in the one-dimensional trigonometric Dunkl setting. We obtain a simple integral expression of this operator and deduce its positivity.


To appear in Proc. Amer. Math. Soc.

## 1. Introduction

We use the lecture notes [6] as a general reference about trigonometric Dunkl theory. In dimension 1, this special function theory is a deformation of Fourier analysis on $\mathbb{R}$, depending on two complex parameters $k_{1}$ and $k_{2}$, where the classical derivative is replaced by the Cherednik operator

$$
\begin{aligned}
D f(x) & =\left(\frac{d}{d x}\right) f(x)+\left\{\frac{k_{1}}{1-e^{-x}}+\frac{2 k_{2}}{1-e^{-2 x}}\right\}\{f(x)-f(-x)\}-\left(\frac{k_{1}}{2}+k_{2}\right) f(x) \\
& =\left(\frac{d}{d x}\right) f(x)+\left\{\frac{k_{1}+k_{2}}{2} \operatorname{coth} \frac{x}{2}+\frac{k_{2}}{2} \tanh \frac{x}{2}\right\}\{f(x)-f(-x)\}-\left(\frac{k_{1}}{2}+k_{2}\right) f(-x),
\end{aligned}
$$

the Lebesgue measure by $A(x) d x$, where

$$
A(x)=\left|2 \sinh \frac{x}{2}\right|^{2 k_{1}}|2 \sinh x|^{2 k_{2}}
$$

and the exponential function $e^{i \lambda x}$ by the Opdam hypergeometric function

$$
\begin{gathered}
G_{i \lambda}(x)=\underbrace{\varphi_{2 \lambda}^{k_{1}+k_{2}-\frac{1}{2}, k_{2}-\frac{1}{2}}\left(\frac{x}{2}\right)}_{{ }_{2} \mathrm{~F}_{1}\left(\frac{k_{1}}{2}+k_{2}+i \lambda, \frac{k_{1}}{2}+k_{2}-i \lambda ; k_{1}+k_{2}+\frac{1}{2} ;-\sinh ^{2} \frac{x}{2}\right)}+\frac{\frac{k_{1}}{2}+k_{2}+i \lambda}{2 k_{1}+2 k_{2}+1}(\sinh x) \underbrace{\varphi_{2 \lambda}^{k_{1}+k_{2}+\frac{1}{2}, k_{2}+\frac{1}{2}}\left(\frac{x}{2}\right)}_{2 \lambda}
\end{gathered}
$$

Here $\varphi_{\lambda}^{\alpha, \beta}(x)$ denotes the Jacobi function and ${ }_{2} \mathrm{~F}_{1}(a, b ; c ; Z)$ the classical hypergeometric function.
In a series of papers ([2], [5], [7], [3], [8], [9], [10], [11], ... ), Trimèche and his collaborators have studied an intertwining operator $\mathcal{V}: C^{\infty}(\mathbb{R}) \longrightarrow C^{\infty}(\mathbb{R})$, which is characterized by

$$
\mathcal{V} \circ\left(\frac{d}{d x}\right)=D \circ \mathcal{V} \quad \text { and } \quad \delta_{0} \circ \mathcal{V}=\delta_{0},
$$

and the dual operator $\mathcal{V}^{t}: C_{c}^{\infty}(\mathbb{R}) \longrightarrow C_{c}^{\infty}(\mathbb{R})$, which satisfies

$$
\int_{-\infty}^{+\infty} \mathcal{V} f(x) g(x) A(x) d x=\int_{-\infty}^{+\infty} f(y) \mathcal{V}^{t} g(y) d y
$$

Let us mention in particular the following facts.

- Eigenfunctions. For every $\lambda \in \mathbb{C}$,

$$
\mathcal{V}\left(x \longmapsto e^{i \lambda x}\right)=G_{i \lambda}
$$

2010 Mathematics Subject Classification. 33C67.
Key words and phrases. Trigonometric Dunkl theory, intertwining operator.
Work partially supported by the regional project MADACA (Marches Aléatoires et processus de Dunkl - Approches Combinatoires et Algébriques, www.fdpoisson.fr/madaca).

- Explicit expression. An integral representation of $\mathcal{V}$ was computed in [2] (and independently in [1]), under the assumption that $k_{1} \geqslant 0, k_{2} \geqslant 0$ with $k_{1}+k_{2}>0$.
- Analytic continuation. It was shown in [3] that the intertwining operator $\mathcal{V}$ extends meromorphically with respect to $k \in \mathbb{C}^{2}$, with singularities in $\left\{k \in \mathbb{C}^{2} \left\lvert\, k_{1}+k_{2}+\frac{1}{2} \in-\mathbb{N}\right.\right\}$.
- Positivity. On the one hand, the positivity of $\mathcal{V}$ was disproved in [2], by using the above-mentioned expression of $\mathcal{V}$ in the case $k_{1}>0, k_{2}>0$. On the other hand, the positivity of $\mathcal{V}$ was investigated in [8], [9], [10], [11] by using the positivity of a heat type kernel in the case $k_{1} \geqslant 0, k_{2} \geqslant 0$.
In Section 2, we obtain an integral representation of $\mathcal{V}$ and $\mathcal{V}^{t}$ when $\operatorname{Re} k_{1}>0$ and $\operatorname{Re} k_{2}>0$. The expression is simpler and the proof is quicker than the previous ones in [2] or [1]. In Section 3, we deduce the positivity of $\mathcal{V}$ and $\mathcal{V}^{t}$ when $k_{1}>0, k_{2}>0$, and comment on the positivity issue.


## 2. Integral representation of the intertwining operator

In this section, we resume the computations in [2, Section 2] and prove the following result.

Theorem 2.1. Let $k=\left(k_{1}, k_{2}\right) \in \mathbb{C}^{2}$ with $\operatorname{Re} k_{1}>0$ and $\operatorname{Re} k_{2}>0$. Then

$$
\mathcal{V} f(x)=\int_{|y|<|x|} \mathcal{K}(x, y) f(y) d y \quad \forall x \in \mathbb{R}^{*}
$$

and

$$
\mathcal{V}^{t} g(y)=\int_{|x|>|y|} \mathcal{K}(x, y) g(x) A(x) d x
$$

where

$$
\begin{align*}
& \mathcal{K}(x, y)=\frac{c}{4} A(x)^{-1} \int_{|y|}^{|x|} \sigma(x, y, z)\left(\cosh \frac{z}{2}-\cosh \frac{y}{2}\right)^{k_{1}-1}(\cosh x-\cosh z)^{k_{2}-1}  \tag{1}\\
& \times\left(\sinh \frac{z}{2}\right) d z,
\end{align*}
$$

with

$$
\begin{equation*}
c=2^{3 k_{1}+3 k_{2}} \frac{\Gamma\left(k_{1}+k_{2}+\frac{1}{2}\right)}{\sqrt{\pi \Gamma\left(k_{1}\right) \Gamma\left(k_{2}\right)}} \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
\sigma(x, y, z)=(\operatorname{sign} x)\left\{e^{\frac{x}{2}}\left(2 \cosh \frac{x}{2}\right)-e^{-\frac{y}{2}}\left(2 \cosh \frac{z}{2}\right)\right\} . \tag{3}
\end{equation*}
$$

Proof. As observed in [2] and [5],

$$
\mathcal{V} f(x)=\int_{-|x|}^{+|x|} \mathcal{K}(x, y) f(y) d y
$$

is an integral operator, whose kernel

$$
\begin{equation*}
\mathcal{K}(x, y)=\frac{1}{4} K\left(\frac{x}{2}, \frac{y}{2}\right)+(\operatorname{sign} x)\left(\frac{k_{1}}{4}+\frac{k_{2}}{2}\right) A(x)^{-1} \widetilde{K}\left(\frac{x}{2}, \frac{y}{2}\right)-(\operatorname{sign} x) \frac{1}{2} A(x)^{-1} \frac{\partial}{\partial y} \widetilde{K}\left(\frac{x}{2}, \frac{y}{2}\right) \tag{4}
\end{equation*}
$$

can be expressed in terms of the kernel

$$
\begin{align*}
K(x, y) & =2 c A(2 x)^{-1}|\sinh 2 x| \\
& \times \int_{|y|}^{|x|}(\cosh z-\cosh y)^{k_{1}-1}(\cosh 2 x-\cosh 2 z)^{k_{2}-1}(\sinh z) d z \tag{5}
\end{align*}
$$

of the intertwining operator in the Jacobi setting (see [4, Subsection 5.3]) and of its integral

$$
\begin{align*}
\widetilde{K}(x, y) & =\int_{|y|}^{|x|} K(w, y) A(2 w) d w \\
& =\frac{c}{k_{2}} \int_{|y|}^{|x|}(\cosh z-\cosh y)^{k_{1}-1}(\cosh 2 x-\cosh 2 z)^{k_{2}}(\sinh z) d z \tag{6}
\end{align*}
$$

Let us integrate by parts (6) and differentiate the resulting expression with respect to $y$. This way, we obtain

$$
\begin{equation*}
\widetilde{K}(x, y)=\frac{4 c}{k_{1}} \int_{|y|}^{|x|}(\cosh z-\cosh y)^{k_{1}}(\cosh 2 x-\cosh 2 z)^{k_{2}-1}(\cosh z)(\sinh z) d z \tag{7}
\end{equation*}
$$

and

$$
\begin{gather*}
\frac{\partial}{\partial y} \widetilde{K}(x, y)=-4 c(\sinh y) \int_{|y|}^{|x|}(\cosh z-\cosh y)^{k_{1}-1}(\cosh 2 x-\cosh 2 z)^{k_{2}-1}  \tag{8}\\
\times(\cosh z)(\sinh z) d z
\end{gather*}
$$

We conclude by substituting (5), (6), (7), (8) in (4) and more precisely (6), respectively (7) in

$$
(\operatorname{sign} x) \frac{k_{2}}{2} A(x)^{-1} \widetilde{K}\left(\frac{x}{2}, \frac{y}{2}\right), \quad \text { respectively } \quad(\operatorname{sign} x) \frac{k_{1}}{4} A(x)^{-1} \widetilde{K}\left(\frac{x}{2}, \frac{y}{2}\right) .
$$

Remark 2.2. Let $x, y \in \mathbb{R}$ such that $|x|>|y|$. The expression (1) extends meromorphically with respect to $k \in \mathbb{C}^{2}$, with singularities in $\left\{k \in \mathbb{C}^{2} \left\lvert\, k_{1}+k_{2}+\frac{1}{2} \in-\mathbb{N}\right.\right\}$, produced by the factor $\Gamma\left(k_{1}+k_{2}+\frac{1}{2}\right)$ in (2). In the limit cases where either $k_{1}$ or $k_{2}$ vanishes, (1) reduces to the following expressions, already obtained in [2] and [1]:

- Assume that $k_{1}=0$ and $\operatorname{Re} k_{2}>0$. Then

$$
\begin{equation*}
\mathcal{K}(x, y)=2^{k_{2}-1} \frac{\Gamma\left(k_{2}+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(k_{2}\right)}|\sinh x|^{-2 k_{2}}(\cosh x-\cosh y)^{k_{2}-1}(\operatorname{sign} x)\left(e^{x}-e^{-y}\right) . \tag{9}
\end{equation*}
$$

- Assume that $k_{2}=0$ and $\operatorname{Re} k_{1}>0$. Then

$$
\begin{equation*}
\mathcal{K}(x, y)=2^{k_{1}-2} \frac{\Gamma\left(k_{1}+\frac{1}{2}\right)}{\sqrt{\pi} \Gamma\left(k_{1}\right)}\left|\sinh \frac{x}{2}\right|^{-2 k_{1}}\left(\cosh \frac{x}{2}-\cosh \frac{y}{2}\right)^{k_{1}-1}(\operatorname{sign} x)\left(e^{\frac{x}{2}}-e^{-\frac{y}{2}}\right) . \tag{10}
\end{equation*}
$$

## 3. Positivity of the intertwining operator

Corollary 3.1. Assume that $k_{1}>0$ and $k_{2}>0$. Then the kernel (1) is strictly positive, for every $x, y \in \mathbb{R}$ such that $|x|>|y|$. Hence the intertwining operator $\mathcal{V}$ and its dual $\mathcal{V}^{t}$ are positive.

Proof. Let us check the positivity of (3) when $x, y, z \in \mathbb{R}$ satisfy $|x|>z>|y|$. On the one hand, if $x>0$, then

$$
\begin{aligned}
\sigma(x, y, z) & =e^{\frac{x}{2}}\left(2 \cosh \frac{x}{2}\right)-e^{-\frac{y}{2}}\left(2 \cosh \frac{z}{2}\right) \\
& >\left(e^{\frac{x}{2}}-e^{-\frac{y}{2}}\right)\left(2 \cosh \frac{x}{2}\right)>0 .
\end{aligned}
$$

On the other hand, if $x<0$, then

$$
\begin{aligned}
\sigma(x, y, z) & =e^{-\frac{y}{2}}\left(2 \cosh \frac{z}{2}\right)-e^{\frac{x}{2}}\left(2 \cosh \frac{x}{2}\right) \\
& >e^{-\frac{y}{2}}\left(2 \cosh \frac{y}{2}\right)-e^{\frac{x}{2}}\left(2 \cosh \frac{x}{2}\right)=e^{-y}-e^{x}>0
\end{aligned}
$$

Remark 3.2. As already observed in [2], the positivity of (9), respectively (10) is immediate in the limit case where $k_{1}=0$ and $k_{2}>0$, respectively $k_{2}=0$ and $k_{1}>0$.

Remark 3.3. The positivity of $\mathcal{V}$ was mistakenly disproved in [2, Theorem 2.11] when $k_{1}>0$ and $k_{2}>0$. More precisely, by using a more complicated formula than (1), the density $\mathcal{K}(x, y)$ was shown to be negative, when $x>0$ and $y \searrow-x$. The error in the proof lies in the expression $A_{1}$, which is equal to $\frac{k}{k^{\prime}} \frac{\sinh (2 x)-\sinh (2|y|)}{E}$ and which tends to $+2 \frac{k}{k^{\prime}} \cosh (2 x)>0$.
Remark 3.4. A different approach, based on the positivity of a heat type kernel, was used in [8], [9], [10] and [11] in order to tackle the positivity of $\mathcal{V}$. While [8] may be right, the same flaw occurs in [9], [10], [11], namely the cut-off $\mathbf{1}_{Y_{\ell}}$ breaks down the differential-difference equations, which are not local.

In conclusion, the present note settles in a simple way the positivity issue in dimension 1 and hence in the product case. Otherwise, the positivity of the interwining operator $\mathcal{V}$ and its dual $\mathcal{V}^{t}$, when the multiplicity function $k$ is $\geqslant 0$, remains an open problem in higher dimensions.

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