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AN ELEMENTARY PROOF OF THE POSITIVITY OF THE INTERTWINING OPERATOR IN ONE-DIMENSIONAL TRIGONOMETRIC DUNKL THEORY

JEAN-PHILIPPE ANKER

ABSTRACT. This note is devoted to the intertwining operator in the one-dimensional trigonometric Dunkl setting. We obtain a simple integral expression of this operator and deduce its positivity.

1. INTRODUCTION

We use the lecture notes [6] as a general reference about trigonometric Dunkl theory. In dimension 1, this special function theory is a deformation of Fourier analysis on \mathbb{R} , depending on two complex parameters k_1 and k_2 , where the classical derivative is replaced by the Cherednik operator

$$\begin{aligned} Df(x) &= \left(\frac{d}{dx}\right)f(x) + \left\{\frac{k_1}{1-e^{-x}} + \frac{2k_2}{1-e^{-2x}}\right\} \{f(x) - f(-x)\} - \left(\frac{k_1}{2} + k_2\right)f(x) \\ &= \left(\frac{d}{dx}\right)f(x) + \left\{\frac{k_1+k_2}{2} \coth \frac{x}{2} + \frac{k_2}{2} \tanh \frac{x}{2}\right\} \{f(x) - f(-x)\} - \left(\frac{k_1}{2} + k_2\right)f(x), \end{aligned}$$

the Lebesgue measure by $A(x)dx$, where

$$A(x) = |2 \sinh \frac{x}{2}|^{2k_1} |2 \sinh x|^{2k_2},$$

and the exponential function $e^{i\lambda x}$ by the Opdam hypergeometric function

$$\begin{aligned} G_{i\lambda}(x) &= \underbrace{\varphi_{2\lambda}^{k_1+k_2-\frac{1}{2}, k_2-\frac{1}{2}}\left(\frac{x}{2}\right)} + \frac{k_1+k_2+i\lambda}{2k_1+2k_2+1} (\sinh x) \underbrace{\varphi_{2\lambda}^{k_1+k_2+\frac{1}{2}, k_2+\frac{1}{2}}\left(\frac{x}{2}\right)} \\ & {}_2F_1\left(\frac{k_1}{2}+k_2+i\lambda, \frac{k_1}{2}+k_2-i\lambda; k_1+k_2+\frac{1}{2}; -\sinh^2 \frac{x}{2}\right) {}_2F_1\left(\frac{k_1}{2}+k_2+1+i\lambda, \frac{k_1}{2}+k_2+1-i\lambda; k_1+k_2+\frac{3}{2}; -\sinh^2 \frac{x}{2}\right) \end{aligned}$$

Here $\varphi_\lambda^{\alpha,\beta}(x)$ denotes the Jacobi function and ${}_2F_1(a, b; c; Z)$ the classical hypergeometric function.

In a series of papers ([2], [5], [7], [3], [8], [9], ...), Trimèche and his collaborators have studied an intertwining operator $\mathcal{V}: C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R})$, which is characterized by

$$\mathcal{V} \circ \left(\frac{d}{dx}\right) = D \circ \mathcal{V} \quad \text{and} \quad \delta_0 \circ \mathcal{V} = \delta_0.$$

Let us mention in particular the following facts.

- *Eigenfunctions.* For every $\lambda \in \mathbb{C}$,

$$\mathcal{V}(x \mapsto e^{i\lambda x}) = G_{i\lambda}.$$

- *Explicit expression.* An integral representation of \mathcal{V} was computed in [2] (and independently in [1]), under the assumption that $k_1 \geq 0$, $k_2 \geq 0$ with $k_1 + k_2 > 0$.
- *Analytic continuation.* It was shown in [3] that the intertwining operator \mathcal{V} extends meromorphically with respect to $k \in \mathbb{C}^2$, with singularities in $\{k \in \mathbb{C}^2 \mid k_1 + k_2 + \frac{1}{2} \in -\mathbb{N}\}$.

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- *Positivity.* On the one hand, the positivity of \mathcal{V} was disproved in [2], by using the above-mentioned expression of \mathcal{V} in the case $k_1 > 0$, $k_2 > 0$. On the other hand, the positivity of \mathcal{V} was proved in [8], [9] by using the positivity of the heat kernel in the case $k_1 \geq 0$, $k_2 \geq 0$.

In Section 2, we obtain an integral representation of \mathcal{V} when $\operatorname{Re} k_1 > 0$ and $\operatorname{Re} k_2 > 0$. The expression is simpler and the proof is quicker than the previous ones in [2] or [1]. In Section 3, we deduce the positivity of \mathcal{V} when $k_1 > 0$, $k_2 > 0$, and comment on the positivity issue.

2. INTEGRAL REPRESENTATION OF THE INTERWINING OPERATOR

In this section, we resume the computations in [2, Section 2], [1, Ch. 4] and prove the following result.

Theorem 2.1. *Let $k = (k_1, k_2) \in \mathbb{C}^2$ with $\operatorname{Re} k_1 > 0$ and $\operatorname{Re} k_2 > 0$. Then*

$$\mathcal{V}f(x) = \int_{-|x|}^{+|x|} \mathcal{K}(x, y) f(y) dy,$$

where

$$\begin{aligned} \mathcal{K}(x, y) = \frac{c}{4} A(x)^{-1} \int_{|y|}^{|x|} \sigma(x, y, z) (\cosh \frac{z}{2} - \cosh \frac{y}{2})^{k_1-1} (\cosh x - \cosh z)^{k_2-1} \\ \times (\sinh \frac{z}{2}) dz, \end{aligned} \quad (1)$$

with

$$c = 2^{3k_1+3k_2} \frac{\Gamma(k_1+k_2+\frac{1}{2})}{\sqrt{\pi} \Gamma(k_1) \Gamma(k_2)} \quad (2)$$

and

$$\sigma(x, y, z) = (\operatorname{sign} x) \left\{ e^{\frac{x}{2}} (2 \cosh \frac{x}{2}) - e^{-\frac{y}{2}} (2 \cosh \frac{z}{2}) \right\}. \quad (3)$$

Proof. As observed in [2] and [5],

$$\mathcal{V}f(x) = \int_{-|x|}^{+|x|} \mathcal{K}(x, y) f(y) dy$$

is an integral operator, whose kernel

$$\mathcal{K}(x, y) = \frac{1}{4} K\left(\frac{x}{2}, \frac{y}{2}\right) + (\operatorname{sign} x) \left(\frac{k_1+k_2}{4}\right) A(x)^{-1} \tilde{K}\left(\frac{x}{2}, \frac{y}{2}\right) - (\operatorname{sign} x) \frac{1}{2} A(x)^{-1} \frac{\partial}{\partial y} \tilde{K}\left(\frac{x}{2}, \frac{y}{2}\right) \quad (4)$$

can be expressed in terms of the kernel

$$\begin{aligned} K(x, y) = 2c A(2x)^{-1} |\sinh 2x| \\ \times \int_{|y|}^{|x|} (\cosh z - \cosh y)^{k_1-1} (\cosh 2x - \cosh 2z)^{k_2-1} (\sinh z) dz \end{aligned} \quad (5)$$

of the intertwining operator in the Jacobi setting (see [4, Subsection 5.3]) and of its integral

$$\begin{aligned} \tilde{K}(x, y) &= \int_{|y|}^{|x|} K(w, y) A(2w) dw \\ &= \frac{c}{k_2} \int_{|y|}^{|x|} (\cosh z - \cosh y)^{k_1-1} (\cosh 2x - \cosh 2z)^{k_2} (\sinh z) dz. \end{aligned} \quad (6)$$

Let us integrate by parts (6) and differentiate the resulting expression with respect to y . This way, we obtain

$$\tilde{K}(x, y) = \frac{4c}{k_1} \int_{|y|}^{|x|} (\cosh z - \cosh y)^{k_1} (\cosh 2x - \cosh 2z)^{k_2-1} (\cosh z) (\sinh z) dz, \quad (7)$$

and

$$\begin{aligned} \frac{\partial}{\partial y} \tilde{K}(x, y) = & -4c(\sinh y) \int_{|y|}^{|x|} (\cosh z - \cosh y)^{k_1-1} (\cosh 2x - \cosh 2z)^{k_2-1} \\ & \times (\cosh z) (\sinh z) dz. \end{aligned} \quad (8)$$

We conclude by substituting (5), (6), (7), (8) in (4) and more precisely (6), respectively (7) in

$$(\text{sign } x) \frac{k_2}{2} A(x)^{-1} \tilde{K}\left(\frac{x}{2}, \frac{y}{2}\right), \quad \text{respectively} \quad (\text{sign } x) \frac{k_1}{4} A(x)^{-1} \tilde{K}\left(\frac{x}{2}, \frac{y}{2}\right).$$

□

Remark 2.2. Let $x, y \in \mathbb{R}$ such that $|x| > |y|$. The expression (1) extends meromorphically with respect to $k \in \mathbb{C}^2$, with singularities in $\{k \in \mathbb{C}^2 \mid k_1 + k_2 + \frac{1}{2} \in -\mathbb{N}\}$, produced by the factor $\Gamma(k_1 + k_2 + \frac{1}{2})$ in (2). In the limit cases where either k_1 or k_2 vanishes, (1) reduces to the following expressions, already obtained in [2] and [1]:

- Assume that $k_1 = 0$ and $\text{Re } k_2 > 0$. Then

$$\mathcal{K}(x, y) = 2^{k_2-1} \frac{\Gamma(k_2 + \frac{1}{2})}{\sqrt{\pi} \Gamma(k_2)} |\sinh x|^{-2k_2} (\cosh x - \cosh y)^{k_2-1} (\text{sign } x) (e^x - e^{-y}). \quad (9)$$

- Assume that $k_2 = 0$ and $\text{Re } k_1 > 0$. Then

$$\mathcal{K}(x, y) = 2^{k_1-2} \frac{\Gamma(k_1 + \frac{1}{2})}{\sqrt{\pi} \Gamma(k_1)} |\sinh \frac{x}{2}|^{-2k_1} (\cosh \frac{x}{2} - \cosh \frac{y}{2})^{k_1-1} (\text{sign } x) (e^{\frac{x}{2}} - e^{-\frac{y}{2}}). \quad (10)$$

3. POSITIVITY OF THE INTERWINING OPERATOR

Corollary 3.1. Assume that $k_1 > 0$ and $k_2 > 0$. Then the kernel (1) is strictly positive, for every $x, y \in \mathbb{R}$ such that $|x| > |y|$.

Proof. It is enough to check that $\sigma(x, y, z) > 0$, for every $x, y, z \in \mathbb{R}$ such that $|x| > z > |y|$. On the one hand, if $x > 0$, then

$$\begin{aligned} \sigma(x, y, z) &= e^{\frac{x}{2}} (2 \cosh \frac{x}{2}) - e^{-\frac{y}{2}} (2 \cosh \frac{z}{2}) \\ &> (e^{\frac{x}{2}} - e^{-\frac{y}{2}}) (2 \cosh \frac{x}{2}) > 0. \end{aligned}$$

On the other hand, if $x < 0$, then

$$\begin{aligned} \sigma(x, y, z) &= e^{-\frac{y}{2}} (2 \cosh \frac{z}{2}) - e^{\frac{x}{2}} (2 \cosh \frac{x}{2}) \\ &> e^{-\frac{y}{2}} (2 \cosh \frac{y}{2}) - e^{\frac{x}{2}} (2 \cosh \frac{x}{2}) = e^{-y} - e^x > 0. \end{aligned}$$

□

Remark 3.2. As already observed in [2], the positivity of (9), respectively (10) is immediate in the limit case where $k_1 = 0$ and $k_2 > 0$, respectively $k_2 = 0$ and $k_1 > 0$.

Remark 3.3. The positivity of \mathcal{V} was mistakenly disproved in [2, Theorem 2.11] when $k_1 > 0$ and $k_2 > 0$. More precisely, by using a more complicated formula than (1), the density $\mathcal{K}(x, y)$ was shown to be negative, when $x > 0$ and $y \searrow -x$. The error in the proof lies in the expression A_1 , which is equal to $\frac{k}{k'} \frac{\sinh(2x) - \sinh(2|y|)}{E}$ and which tends to $+2 \frac{k}{k'} \frac{\cosh(2x)}{\sinh(2x)} > 0$.

Remark 3.4. A different approach, based on the positivity of the heat kernel, was used in [8] and [9], in order to tackle the positivity of \mathcal{V} .

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