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# NON-ABELIAN SYMMETRIES OF THE HALF-INFINITE XXZ SPIN CHAIN

PASCAL BASEILHAC AND SAMUEL BELLIARD

ABSTRACT. The non-Abelian symmetries of the half-infinite XXZ spin chain for all possible types of integrable boundary conditions are classified. For each type of boundary conditions, an analog of the Drinfeld-Jimbo presentation is given for the corresponding symmetry algebra. In particular, two new algebras arise that are, respectively, generated by the symmetry operators of the model with triangular and special  $U_q(\mathfrak{gl}_2)$ -invariant integrable boundary conditions.

MSC: 81R50; 81R10; 81U15.

**Keywords:** Open XXZ spin chain, Coideal subalgebras, Non-Abelian symmetry, Quantum integrable models.

## 1. INTRODUCTION

Identifying non-Abelian infinite dimensional symmetries of quantum integrable systems provides a very efficient starting point for transferring the Hamiltonian and eigenstates formulation into the language of operator algebra, representation theory and related special functions, on which the non-perturbative analysis is thus based. Besides the restricted class of models with conformal symmetry for which the Virasoro algebra and its representation theory play a central role, in the thermodynamic limit of lattice systems it is known that quantum groups, related current algebras and infinite dimensional  $q$ -vertex operators representations provide an appropriate mathematical framework. As an important example, in the thermodynamic limit  $N \rightarrow \infty$  of the finite XXZ spin chain with  $N$  sites and periodic boundary conditions, the  $U_q(\widehat{\mathfrak{sl}}_2)$  algebra emerges as a hidden non-Abelian symmetry of the Hamiltonian [FM92, J92, DFJMN92]. Based on the representation theory of the  $U_q(\widehat{\mathfrak{sl}}_2)$  quantum affine algebra at level one and its current algebra, an explicit characterization of the Hamiltonian's spectrum, corresponding eigenstates, as well as multiple integral representations of correlation functions and form factors of local operators has been given [DFJMN92]. This approach has been later on extended to lattice systems with periodic boundary conditions associated with higher spins [IJMNT92] or higher rank affine Lie algebras [Koy93], as well as for certain class of boundary conditions, see for instance [JKKKMW95, BB13, BKo13, BKo14, Koj10].

For lattice models with general integrable boundary conditions, identifying the non-Abelian symmetries in the thermodynamic limits has remained essentially unexplored although coideal subalgebras of quantum affine Lie algebras are natural candidates. For instance, in the thermodynamic limit  $N \rightarrow \infty$  of the *finite* open XXZ spin chain it was expected that non-Abelian infinite dimensional symmetries emerge, that are associated with certain coideal subalgebras of  $U_q(\widehat{\mathfrak{sl}}_2)$ . Recall that the Hamiltonian of the half-infinite XXZ spin chain is formally defined as (see also [JKKKMW95, BB13, BKo13, BKo14]):

$$(1.1) \quad H_{\frac{1}{2}XXZ} = -\frac{1}{2} \sum_{k=1}^{\infty} \left( \sigma_1^{k+1} \sigma_1^k + \sigma_2^{k+1} \sigma_2^k + \Delta \sigma_3^{k+1} \sigma_3^k \right) + h_B$$

with boundary interaction

$$h_B = -\frac{(q - q^{-1})(\epsilon_+ - \epsilon_-)}{4(\epsilon_+ + \epsilon_-)} \sigma_3^1 - \frac{1}{(\epsilon_+ + \epsilon_-)} (k_+ \sigma_+^1 + k_- \sigma_-^1).$$

Here  $\sigma_{1,2,3}$  and  $\sigma_{\pm} = (\sigma_1 \pm i\sigma_2)/2$  - see Appendix A - are usual Pauli matrices,  $\Delta = (q + q^{-1})/2$  denotes the anisotropy parameter and  $\epsilon_{\pm}, k_{\pm}$  are scalar parameters associated with the right boundary field. Formally, the Hamiltonian acts on an infinite tensor product of 2-dimensional vector spaces. Note that the ordering of the tensor components in (1.1) is such that:

$$(1.2) \quad \mathcal{V} = \dots \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 .$$

Given the Hamiltonian of the half-infinite XXZ spin chain (1.1), until recently the hidden non-Abelian symmetry for  $q$  generic and any type of boundary conditions<sup>1</sup>  $k_{\pm}, \epsilon_{\pm}$  has remained unknown. However, at the end of the article [BB13], it was pointed out that for the generic case ( $k_{\pm} \neq 0$  and  $\epsilon_{\pm} \neq 0$ ) and the diagonal case ( $k_{\pm} = 0$  and  $\epsilon_{\pm} \neq 0$ ) the respective hidden non-Abelian symmetries are associated with two different coideal subalgebras of  $U_q(\widehat{sl_2})$ : the  $q$ -Onsager algebra [T03, B04] and the augmented  $q$ -Onsager algebra [BC12, BB13] (see also [IT09]).

In this letter, we present a unified picture that complete the preliminary observations of [BB13]. Namely, for each type of boundary conditions, the non-Abelian symmetry algebra of the half-infinite XXZ spin chain is identified and characterized through generators and relations. In particular, two new coideal subalgebras are obtained.

The text is organized as follows. In Section 2, four different types of coideal subalgebras of  $U_q(\widehat{sl_2})$  are defined through generators and relations in a Drinfeld-Jimbo's type of presentation. The coaction and counit maps are given in each case. Then, it is shown that these algebras are symmetry algebras of the Hamiltonian (1.1) for generic ( $\epsilon_{\pm} \neq 0, k_{\pm} \neq 0$ ), triangular ( $\epsilon_{\pm} \neq 0, k_+ = 0, k_- \neq 0$ ), diagonal ( $\epsilon_{\pm} \neq 0, k_{\pm} = 0$ ) and special<sup>2</sup> boundary conditions ( $\epsilon_+ = 1, \epsilon_- = 0, k_{\pm} \neq 0$ ), respectively. Note that since the case of upper triangular ( $k_- = 0, k_+ \neq 0$ ) and lower triangular ( $k_+ = 0, k_- \neq 0$ ) boundary conditions are related through conjugation of the Hamiltonian (1.1) by the spin-reversal operator  $\hat{\nu} = \prod_{j=1}^{\infty} \sigma_j^x$ , it is sufficient for our purpose to restrict our attention to the case of lower boundary conditions. In Section 3, we propose an alternative and simpler derivation of the symmetry operators which is based on the remarkable connection between the infinite dimensional algebra  $\mathcal{A}_q$  introduced in [BK05] and the  $q$ -Onsager algebra. Concluding remarks follow in Section 4.

**Notation .** The  $q$ -commutator  $[X, Y]_q = qXY - q^{-1}YX$  is introduced, where  $q$  is the deformation parameter assumed not to be a root of unity.

## 2. FOUR DIFFERENT TYPES OF $q$ -ONsAGER SYMMETRY ALGEBRAS

In the first part of this Section, four different types of coideal subalgebras with central extension are introduced through basic generators and relations. The coaction and counit maps are given. Note that two of these algebras have already appeared in the literature: the  $q$ -Onsager algebra [T03, B04] and the augmented  $q$ -Onsager algebra [IT09, BC12, BB13]. The two others, the so-called triangular  $q$ -Onsager algebra and  $U_q(gl_2)$  invariant  $q$ -Onsager algebra<sup>3</sup> are new. In a second part, depending on the choice of boundary conditions, by analogy with [J92, eq. (3.2)] it is shown that the Hamiltonian (1.1) commutes with all generators of one of these four algebras.

**2.1. The  $q$ -Onsager algebra.** The  $q$ -Onsager algebra  $O_q(\widehat{sl_2})$  is an example of tridiagonal algebra [T03]. Including a central extension, it is generated by two elements  $W_0, W_1$ , a central element  $\Gamma$  and unit. The defining relations are:

$$(2.1) \quad \begin{aligned} [W_0, [W_0, [W_0, W_1]_q]_{q^{-1}}] &= \rho[W_0, W_1], \\ [W_1, [W_1, [W_1, W_0]_q]_{q^{-1}}] &= \rho[W_1, W_0], \\ [W_0, \Gamma] &= [W_1, \Gamma] = 0, \end{aligned}$$

where  $\rho$  is a scalar. Without loss of generality, let us define

$$(2.2) \quad \rho = (q + q^{-1})^2 k_+ k_-$$

where  $k_{\pm}$  are nonzero scalars. By analogy with the situation for Hopf algebras, one endows the  $q$ -Onsager algebra with the coaction map  $\delta : O_q(\widehat{sl_2}) \rightarrow U_q(\widehat{sl_2}) \otimes O_q(\widehat{sl_2})$  defined by:

$$(2.3) \quad \begin{aligned} \delta(W_0) &= (k_+ e_1 + k_- q^{-1} f_1 q^{h_1}) \otimes 1 + q^{h_1} \otimes W_0, \\ \delta(W_1) &= (k_- e_0 + k_+ q^{-1} f_0 q^{h_0}) \otimes 1 + q^{h_0} \otimes W_1, \\ \delta(\Gamma) &= q^c \otimes \Gamma \end{aligned}$$

<sup>1</sup>In (1.1), we implicitly assume that  $\epsilon_+ + \epsilon_- \neq 0$ .

<sup>2</sup>The corresponding Hamiltonian can be understood as the thermodynamic limit of the  $U_q(sl_2)$  invariant spin chain studied in [PS90].

<sup>3</sup>However, note that the generators of the  $U_q(gl_2)$  invariant  $q$ -Onsager algebra have already appeared in [R12, Koll12].

and counit  $\mathcal{E} : O_q(\widehat{sl_2}) \rightarrow \mathbb{C}$ :

$$(2.4) \quad \mathcal{E}(W_0) = \epsilon_+, \quad \mathcal{E}(W_1) = \epsilon_-, \quad \mathcal{E}(\Gamma) = \mathcal{E}(1) = 1.$$

This induces an homomorphism  $\psi = (id \times \mathcal{E}) \circ \delta : O_q(\widehat{sl_2}) \rightarrow U_q(\widehat{sl_2})$  from the  $q$ -Onsager algebra to a subalgebra of  $U_q(\widehat{sl_2})$ :

$$(2.5) \quad \begin{aligned} \psi(W_0) &= k_+e_1 + k_-q^{-1}f_1q^{h_1} + \epsilon_+q^{h_1}, \\ \psi(W_1) &= k_-e_0 + k_+q^{-1}f_0q^{h_0} + \epsilon_-q^{h_0}, \\ \psi(\Gamma) &= q^c. \end{aligned}$$

**2.2. The triangular  $q$ -Onsager algebra.** The triangular  $q$ -Onsager algebra  $O_q^t(\widehat{sl_2})$  is generated by three elements  $T_0, T_1, \tilde{P}_1$ , a central element  $\Gamma$  and unit. The defining relations are:

$$(2.6) \quad \begin{aligned} [T_0, [T_0, \tilde{P}_1]_{q^{-1}}] &= \rho_t [T_0, T_1], & [T_1, [T_1, \tilde{P}_1]_q] &= \rho_t [T_0, T_1], \\ [T_1, T_0]_{q^{-1}} &= \tilde{\rho}_t \Gamma, & [T_0, \Gamma] = [T_1, \Gamma] = [\tilde{P}_1, \Gamma] &= 0. \end{aligned}$$

Let us define:

$$(2.7) \quad \rho_t = k_-(q + q^{-1})^2 \quad \text{and} \quad \tilde{\rho}_t = -\epsilon_+\epsilon_-(q - q^{-1}),$$

where  $k_-, \epsilon_{\pm}$  are nonzero scalars. We endow this algebra with the coaction map  $\delta_t : O_q^t(\widehat{sl_2}) \rightarrow U_q(\widehat{sl_2}) \otimes O_q^t(\widehat{sl_2})$  defined by:

$$(2.8) \quad \begin{aligned} \delta_t(T_0) &= k_-q^{-1}f_1q^{h_1} \otimes 1 + q^{h_1} \otimes T_0, \\ \delta_t(T_1) &= k_-e_0 \otimes 1 + q^{h_0} \otimes T_1, \\ \delta_t(\tilde{P}_1) &= k_-([e_1, e_0]_q + q^c[f_1, f_0]_q) \otimes 1 + q^c \otimes \tilde{P}_1 + (q^2 - q^{-2})(q^{-1}e_1q^{h_0} \otimes T_1 + q^cf_0q^c \otimes T_0), \\ \delta_t(\Gamma) &= q^c \otimes \Gamma \end{aligned}$$

and counit  $\mathcal{E}_t : O_q^t(\widehat{sl_2}) \rightarrow \mathbb{C}$ :

$$(2.9) \quad \mathcal{E}_t(T_0) = \epsilon_+, \quad \mathcal{E}_t(T_1) = \epsilon_-, \quad \mathcal{E}_t(\tilde{P}_1) = \tilde{p}, \quad \mathcal{E}_t(\Gamma) = \mathcal{E}_t(1) = 1.$$

This induces an homomorphism  $\psi_t = (id \times \mathcal{E}_t) \circ \delta_t$  from the triangular  $q$ -Onsager algebra to a subalgebra of  $U_q(\widehat{sl_2})$ :

$$(2.10) \quad \begin{aligned} \psi_t(T_0) &= k_-q^{-1}f_1q^{h_1} + \epsilon_+q^{h_1}, \\ \psi_t(T_1) &= k_-e_0 + \epsilon_-q^{h_0}, \\ \psi_t(\tilde{P}_1) &= (q^2 - q^{-2})(\epsilon_-q^{-1}e_1q^{h_0} + \epsilon_+f_0q^{h_1+h_0}) + k_-([e_1, e_0]_q + q^c[f_1, f_0]_q) + \tilde{p}q^c, \\ \psi_t(\Gamma) &= q^c. \end{aligned}$$

**2.3. The augmented  $q$ -Onsager algebra.** The augmented  $q$ -Onsager algebra  $O_q^d(\widehat{sl_2})$  has been introduced in [BB13], as a generalization<sup>4</sup> of the augmented tridiagonal algebra introduced in [IT09]. It is generated by four elements  $K_0, K_1, Z_1, \tilde{Z}_1$ ,

<sup>4</sup>In [IT09], the special case  $\Gamma = 1$  is considered. In [BB13], the central element is not explicitly introduced and the first relation is replaced by  $[K_0, K_1] = 0$ . Note that for the level one  $q$ -vertex operators representations constructed in [BB13], one has  $\Gamma = q$ .

a central element  $\Gamma$  and unit. The defining relations are:

$$(2.11) \quad \begin{aligned} \mathbf{K}_0 \mathbf{K}_1 &= \mathbf{K}_1 \mathbf{K}_0 = \epsilon_- \epsilon_+ \Gamma, \\ \mathbf{K}_0 \mathbf{Z}_1 &= q^{-2} \mathbf{Z}_1 \mathbf{K}_0, & \mathbf{K}_0 \tilde{\mathbf{Z}}_1 &= q^2 \tilde{\mathbf{Z}}_1 \mathbf{K}_0, \\ \mathbf{K}_1 \mathbf{Z}_1 &= q^2 \mathbf{Z}_1 \mathbf{K}_1, & \mathbf{K}_1 \tilde{\mathbf{Z}}_1 &= q^{-2} \tilde{\mathbf{Z}}_1 \mathbf{K}_1, \\ [\mathbf{Z}_1, [\mathbf{Z}_1, [\mathbf{Z}_1, \tilde{\mathbf{Z}}_1]_q]_{q^{-1}}] &= \rho_d \mathbf{Z}_1 (\mathbf{K}_1 \mathbf{K}_1 - \mathbf{K}_0 \mathbf{K}_0) \mathbf{Z}_1, \\ [\tilde{\mathbf{Z}}_1, [\tilde{\mathbf{Z}}_1, [\tilde{\mathbf{Z}}_1, \mathbf{Z}_1]_q]_{q^{-1}}] &= \rho_d \tilde{\mathbf{Z}}_1 (\mathbf{K}_0 \mathbf{K}_0 - \mathbf{K}_1 \mathbf{K}_1) \tilde{\mathbf{Z}}_1, \\ [\mathbf{Z}_1, \Gamma] = [\tilde{\mathbf{Z}}_1, \Gamma] &= [\mathbf{K}_0, \Gamma] = [\mathbf{K}_1, \Gamma] = 0 \end{aligned}$$

with the identification:

$$(2.12) \quad \rho_d = \frac{(q^3 - q^{-3})(q^2 - q^{-2})^3}{q - q^{-1}}.$$

We endow this algebra with the coaction map  $\delta_d : O_q^d(\widehat{sl_2}) \rightarrow U_q(\widehat{sl_2}) \otimes O_q^d(\widehat{sl_2})$  defined by:

$$(2.13) \quad \begin{aligned} \delta_d(\mathbf{K}_0) &= q^{h_1} \otimes \mathbf{K}_0, & \delta_d(\mathbf{K}_1) &= q^{h_0} \otimes \mathbf{K}_1, \\ \delta_d(\mathbf{Z}_1) &= q^c \otimes \mathbf{Z}_1 + (q^2 - q^{-2})(q^{-1} e_0 q^{h_1} \otimes \mathbf{K}_0 + f_1 q^c \otimes \mathbf{K}_1), \\ \delta_d(\tilde{\mathbf{Z}}_1) &= q^c \otimes \tilde{\mathbf{Z}}_1 + (q^2 - q^{-2})(f_0 q^c \otimes \mathbf{K}_0 + q^{-1} e_1 q^{h_0} \otimes \mathbf{K}_1), \\ \delta_d(\Gamma) &= q^c \otimes \Gamma \end{aligned}$$

and counit  $\mathcal{E}_d : O_q^d(\widehat{sl_2}) \rightarrow \mathbb{C}$ :

$$(2.14) \quad \mathcal{E}_d(\mathbf{K}_0) = \epsilon_+, \quad \mathcal{E}_d(\mathbf{K}_1) = \epsilon_-, \quad \mathcal{E}_d(\mathbf{Z}_1) = \mathcal{E}_d(\tilde{\mathbf{Z}}_1) = 0, \quad \mathcal{E}_d(\Gamma) = \epsilon_- \epsilon_+, \quad \mathcal{E}_d(1) = 1.$$

This induces an homomorphism  $\psi_d$  from the augmented  $q$ -Onsager algebra to a subalgebra of  $U_q(\widehat{sl_2})$ :

$$(2.15) \quad \begin{aligned} \psi_d(\mathbf{K}_0) &= \epsilon_+ q^{h_1}, & \psi_d(\mathbf{K}_1) &= \epsilon_- q^{h_0}, \\ \psi_d(\mathbf{Z}_1) &= (q^2 - q^{-2})(\epsilon_+ q^{-1} e_0 q^{h_1} + \epsilon_- f_1 q^{h_1+h_0}), \\ \psi_d(\tilde{\mathbf{Z}}_1) &= (q^2 - q^{-2})(\epsilon_- q^{-1} e_1 q^{h_0} + \epsilon_+ f_0 q^{h_1+h_0}), \\ \psi_d(\Gamma) &= \epsilon_- \epsilon_+ q^c. \end{aligned}$$

**2.4. The  $U_q(gl_2)$  invariant  $q$ -Onsager algebra:** The  $U_q(gl_2)$  invariant  $q$ -Onsager algebra  $O_q^i(\widehat{sl_2})$  is generated by six elements  $e, f, q^h, \mathbf{X}, \mathbf{Y}, \tilde{\mathbf{Y}}$ , a central element  $\Gamma$  and unit. The defining relations are:

$$(2.16) \quad \begin{aligned} [\mathbf{e}, \mathbf{f}] &= \frac{q^h - q^{-h}}{q - q^{-1}}, & [\mathbf{e}, q^h]_q &= [q^h, \mathbf{f}]_q = 0, \\ [q^h, \mathbf{X}] &= 0, & [\mathbf{X}, \mathbf{e}] &= \mathbf{Y}, & [\mathbf{f}, \mathbf{X}] &= \tilde{\mathbf{Y}}, \\ [q^h, \tilde{\mathbf{Y}}]_q &= 0, & [\mathbf{e}, \mathbf{Y}]_q &= 0, & [q^h, \mathbf{Y}]_{q^{-1}} &= 0, & [\mathbf{f}, \tilde{\mathbf{Y}}]_{q^{-1}} &= 0, \\ [\mathbf{e}, \tilde{\mathbf{Y}}] &= (q + q^{-1}) \mathbf{X} q^h, & [\mathbf{Y}, \mathbf{f}] &= (q + q^{-1}) \mathbf{X} q^h, \\ [\mathbf{X}, [\mathbf{Y}, \tilde{\mathbf{Y}}]] &= (q^2 - q^{-2})^3 (q \tilde{\mathbf{Y}} \mathbf{f} \mathbf{e} - q^{-1} \mathbf{Y} \mathbf{e} \mathbf{f} \mathbf{f} + (q + q^{-1})^2 \mathbf{X} \mathbf{e} \mathbf{f} q^h) q^h \\ &+ q^2 (q^2 - q^{-2}) (q + q^{-1})^2 \left( (q^2 - q^{-2} - 1) \tilde{\mathbf{Y}} \mathbf{e} + (q^2 - q^{-2} + 1) \mathbf{Y} \mathbf{f} - (q + q^{-1}) \mathbf{X} q^h \right) q^{2h} \\ &+ (q^2 - q^{-2}) (q + q^{-1})^2 \left( q^{-2} \tilde{\mathbf{Y}} \mathbf{e} - q^2 \mathbf{Y} \mathbf{f} + q^2 (q + q^{-1}) \mathbf{X} q^h \right), \\ [\mathbf{Y}, \Gamma] &= [\tilde{\mathbf{Y}}, \Gamma] = [\mathbf{X}, \Gamma] = [\mathbf{e}, \Gamma] = [\mathbf{f}, \Gamma] = [q^h, \Gamma] = 0. \end{aligned}$$

We endow this algebra with the coaction map  $\delta_i : O_q^i(\widehat{sl_2}) \rightarrow U_q(\widehat{sl_2}) \otimes O_q^i(\widehat{sl_2})$  defined by:

$$\begin{aligned}
(2.17) \quad \delta_i(\mathbf{e}) &= e_0 \otimes 1 + q^{h_0} \otimes \mathbf{e}, & \delta_i(\mathbf{f}) &= f_0 \otimes q^{-h} + 1 \otimes \mathbf{f}, & \delta_i(q^h) &= q^{h_0} \otimes q^h, \\
\delta_i(\mathbf{X}) &= ([e_1, e_0]_q - [f_1, f_0]_{q^{-1}} q^c) \otimes 1 + q^c \otimes \mathbf{X} + (q^2 - q^{-2})(q^{h_0+1} e_1 \otimes \mathbf{e} - q^{c-1} f_1 \otimes \mathbf{f} q^h), \\
(2.18) \quad \delta_i(\mathbf{Y}) &= ([e_1, e_0], e_0]_q - q^{c+1}(q + q^{-1})f_1 q^{h_0}) \otimes 1 + q^{h_0+c} \otimes \mathbf{Y} \\
&\quad + (q^2 - q^{-2}) \left( q^{h_0} [e_1, e_0]_q \otimes \mathbf{e} + (q^2 - 1) q^{2h_0} e_1 \otimes \mathbf{e}^2 + q^{c+h_0} f_1 \otimes \frac{q^{2h} - 1}{q^2 - 1} \right), \\
(2.19) \quad \delta_i(\tilde{\mathbf{Y}}) &= q^c ([f_1, f_0], f_0]_{q^{-1}} - q^{-1}(q + q^{-1})e_1 q^{h_0}) \otimes q^{-h} + q^c \otimes \tilde{\mathbf{Y}} - (q^2 - q^{-2}) \left( q^c [f_0, f_1]_q \otimes \mathbf{f} \right. \\
&\quad \left. + q^{1+h_0} e_1 \otimes \frac{q^h - q^{-h}}{q - q^{-1}} + (q - q^{-1}) q^c f_1 \otimes \mathbf{f}^2 q^{-h} \right), \\
\delta_i(\Gamma) &= q^c \otimes \Gamma
\end{aligned}$$

and counit  $\mathcal{E}_i : O_q^i(\widehat{sl_2}) \rightarrow \mathbb{C}$ :

$$(2.20) \quad \mathcal{E}_i(e) = \mathcal{E}_i(f) = \mathcal{E}_i(\mathbf{X}) = \mathcal{E}_i(\mathbf{Y}) = \mathcal{E}_i(\tilde{\mathbf{Y}}) = 0, \quad \mathcal{E}_i(q^h) = \mathcal{E}_i(\Gamma) = \mathcal{E}_i(1) = 1.$$

This induces an homomorphism  $\psi_i = (id \otimes \mathcal{E}_i) \circ \delta_i$  from the  $U_q(gl_2)$  invariant  $q$ -Onsager algebra to a certain subalgebra of  $U_q(\widehat{sl_2})$ :

$$\begin{aligned}
(2.21) \quad \psi_i(\mathbf{e}) &= e_0, & \psi_i(\mathbf{f}) &= f_0, & \psi_i(q^h) &= q^{h_0}, \\
\psi_i(\mathbf{X}) &= [e_1, e_0]_q - [f_1, f_0]_{q^{-1}} q^c, \\
\psi_i(\mathbf{Y}) &= [[e_1, e_0], e_0]_q - q^{c+1}(q + q^{-1})f_1 q^{h_0}, \\
\psi_i(\tilde{\mathbf{Y}}) &= q^c ([f_1, f_0], f_0]_{q^{-1}} - q^{-1}(q + q^{-1})e_1 q^{h_0}), \\
\psi_i(\Gamma) &= q^c.
\end{aligned}$$

**2.5. Non-Abelian symmetry algebras of the Hamiltonians.** In the thermodynamic limit of the XXZ spin chain Hamiltonian, it is well-known that the Hamiltonian commutes with the generators of the quantum affine algebra  $U_q(\widehat{sl_2})$  acting on an infinite tensor product representation [FM92, J92, DFJMN92]. In this subsection, we basically use similar arguments in order to characterize the hidden non-Abelian symmetries of the open XXZ spin chain for four different types of boundary conditions. As an example, let us consider the Hamiltonian (1.1). On the semi-infinite tensor product vector space (1.2), the generators (2.5) of the  $q$ -Onsager algebra act as (see also [BB13]):

$$\begin{aligned}
(2.22) \quad \mathcal{W}_0^{(\infty)} &= \sum_{j=1}^{\infty} \left( \cdots \otimes q^{\sigma_3} \otimes q^{\sigma_3} \otimes \underbrace{(k_+ \sigma_+ + k_- \sigma_-)}_{sitej} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \right) + \epsilon_+ \left( \cdots \otimes q^{\sigma_3} \otimes q^{\sigma_3} \right), \\
\mathcal{W}_1^{(\infty)} &= \sum_{j=1}^{\infty} \left( \cdots \otimes q^{-\sigma_3} \otimes q^{-\sigma_3} \otimes \underbrace{(k_+ \sigma_+ + k_- \sigma_-)}_{sitej} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \right) + \epsilon_- \left( \cdots \otimes q^{-\sigma_3} \otimes q^{-\sigma_3} \right),
\end{aligned}$$

where the coaction map (2.3) is iterated repeatedly and (2.5) is used. The action of the operators (2.22) on (1.1) is easy to compute. On one hand, introduce the local density  $h_i = \sigma_1^{i+1} \sigma_1^i + \sigma_2^{i+1} \sigma_2^i + \Delta \sigma_3^{i+1} \sigma_3^i$ . By straightforward calculations, one first observes:

$$(2.23) \quad [h_i, \mathcal{W}_0^{(\infty)}] = \cdots \otimes q^{\sigma_3} \otimes q^{\sigma_3} \otimes \Gamma_{i+1, i} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}$$

where

$$\Gamma_{i+1, i} = (2k_+ \sigma_+ - 2k_- \sigma_-) \otimes (\Delta - q^{\sigma_3}) \sigma_3 + (\Delta q^{\sigma_3} - 1) \sigma_3 \otimes (2k_+ \sigma_+ - 2k_- \sigma_-).$$

Summing up the local densities, after some simplifications one ends up with:

$$\left[ \sum_{i=1}^{\infty} h_i, \mathcal{W}_0^{(\infty)} \right] = -\frac{1}{2}(q - q^{-1})(\cdots \otimes q^{\sigma_3} \otimes q^{\sigma_3}) \otimes (k_+ \sigma_+ - k_- \sigma_-).$$

On the other hand, the contribution from the boundary term in (1.1) is non-vanishing. It gives:

$$[h_B, \mathcal{W}_0^{(\infty)}] = \frac{1}{2}(q - q^{-1})(\cdots \otimes q^{\sigma_3} \otimes q^{\sigma_3}) \otimes (k_+ \sigma_+ - k_- \sigma_-).$$

Combining the above expressions together and repeating the analysis for  $\mathcal{W}_1^{(\infty)}$ , we conclude that the  $q$ -Onsager algebra is a symmetry algebra of the Hamiltonian with generic boundary conditions:

$$(2.24) \quad [H_{\frac{1}{2}XXZ}, \mathcal{W}_0^{(\infty)}] = 0 \quad \text{and} \quad [H_{\frac{1}{2}XXZ}, \mathcal{W}_1^{(\infty)}] = 0.$$

Although the calculations become quickly more involved, the same technique is then extended to the Hamiltonian (1.1) with triangular ( $\epsilon_{\pm} \neq 0, k_+ = 0, k_- \neq 0$ ), diagonal ( $\epsilon_{\pm} \neq 0, k_{\pm} = 0$ ) and special boundary conditions ( $\epsilon_+ = 1, \epsilon_- = 0, k_{\pm} \neq 0$ ), respectively. With respect to the boundary conditions chosen, it is found that the corresponding Hamiltonian is commuting with all generators of the triangular  $q$ -Onsager, augmented  $q$ -Onsager and  $U_q(gl_2)$  invariant  $q$ -Onsager algebras acting on (1.2), respectively. In the next Section, we present an alternative and much simpler derivation of the symmetry operators generating the four different types of  $q$ -Onsager algebras.

### 3. AN ALTERNATIVE DERIVATION OF THE SYMMETRY OPERATORS

In the  $q$ -Onsager approach of the *finite* open XXZ spin chain [BK07] with left and right *generic non-diagonal* boundary conditions, it is known that all *local* or *non-local* conserved quantities are generated from elements in a certain quotient<sup>5</sup> of the infinite dimensional algebra  $\mathcal{A}_q$  [BK07] (see also [BK05]). In the literature, the generators of  $\mathcal{A}_q$  are usually denoted  $\{\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1} | k \in \mathbb{Z}_+\}$ , and  $\mathcal{W}_0, \mathcal{W}_1$  are called the fundamental generators. Importantly,  $\mathcal{W}_0, \mathcal{W}_1$  satisfy the defining relations of the  $q$ -Onsager algebra (2.1) [B04, B05].

By analogy with Onsager's [Ons44], Dolan-Grady's [DG82] and Davies' [D90] works on the Onsager algebra, generalizing the results of [BK05, Subsection 2.4.2] it was shown in [BB10, Example 2] through a brute force calculation that the first few 'descendants' generators  $\mathcal{W}_{-1}, \mathcal{W}_2, \mathcal{G}_1, \tilde{\mathcal{G}}_1, \mathcal{G}_2, \tilde{\mathcal{G}}_2$  can be written as polynomials in the two fundamental generators  $\mathcal{W}_0, \mathcal{W}_1$  only. Including a central element  $\gamma$ , for instance one finds:

$$(3.1) \quad \begin{aligned} \mathcal{G}_1 &= [\mathcal{W}_1, \mathcal{W}_0]_q - (q - q^{-1})\epsilon_+ \epsilon_- \gamma, \\ \mathcal{W}_{-1} &= \frac{1}{\rho} ((q^2 + q^{-2})\mathcal{W}_0 \mathcal{W}_1 \mathcal{W}_0 - \mathcal{W}_0^2 \mathcal{W}_1 - \mathcal{W}_1 \mathcal{W}_0^2) + \mathcal{W}_1 - \frac{(q - q^{-1})^2}{\rho} \epsilon_+ \epsilon_- \gamma \mathcal{W}_0, \\ \mathcal{G}_2 &= \frac{1}{\rho(q^2 + q^{-2})} \left( (q^{-3} + q^{-1})\mathcal{W}_0^2 \mathcal{W}_1^2 - (q^3 + q)\mathcal{W}_1^2 \mathcal{W}_0^2 + (q^{-3} - q^3)(\mathcal{W}_0 \mathcal{W}_1^2 \mathcal{W}_0 + \mathcal{W}_1 \mathcal{W}_0^2 \mathcal{W}_1) \right. \\ &\quad \left. - (q^{-5} + q^{-3} + 2q^{-1})\mathcal{W}_0 \mathcal{W}_1 \mathcal{W}_0 \mathcal{W}_1 + (q^5 + q^3 + 2q)\mathcal{W}_1 \mathcal{W}_0 \mathcal{W}_1 \mathcal{W}_0 \right. \\ &\quad \left. + \rho(q - q^{-1})(\mathcal{W}_0^2 + \mathcal{W}_1^2) - (q^2 + q^{-2})(q - q^{-1})^2 \epsilon_+ \epsilon_- \gamma (q\mathcal{W}_1 \mathcal{W}_0 - q^{-1}\mathcal{W}_0 \mathcal{W}_1) \right) \\ &\quad - \left( (\epsilon_+^2 + \epsilon_-^2) \gamma^2 \frac{q - q^{-1}}{q^2 + q^{-2}} - \epsilon_+^2 \epsilon_-^2 \gamma^2 \frac{(q - q^{-1})^3}{\rho(q^2 + q^{-2})} - \rho q^2 \frac{(\gamma^2 - 1)}{(q + q^{-1})^3 (q^2 + q^{-2})} \right). \end{aligned}$$

The expressions for the elements  $\tilde{\mathcal{G}}_1, \mathcal{W}_2, \tilde{\mathcal{G}}_2$  are obtained from  $\mathcal{G}_1, \mathcal{W}_{-1}, \mathcal{G}_2$  by exchanging  $\mathcal{W}_0 \leftrightarrow \mathcal{W}_1$  in the above formulae.

<sup>5</sup>Because the generators of  $\mathcal{A}_q$  act on a finite dimensional vector space  $\mathcal{V}^{(N)}$  of dimension  $2^N$ , they satisfy additional relations that are  $q$ -deformed analogs of Davies' relations. Compare the first reference of [BK07, eqs. (17),(18)] to the first reference of [D90, eq. (2.6a),(2.6b)] for details.

Actually, under certain assumptions it is possible to show that *all* descendant generators of the algebra  $\mathcal{A}_q$  admit a unique explicit polynomial formulae in terms of the two fundamental generators  $\mathcal{W}_0, \mathcal{W}_1$  [BB10]. Roughly speaking, they take the form:

$$(3.2) \quad \begin{aligned} \mathcal{W}_{-k} &= \mathcal{F}_{-k}(\mathcal{W}_0, \mathcal{W}_1), & \mathcal{G}_{k+1} &= \mathcal{F}_{k+1}(\mathcal{W}_0, \mathcal{W}_1), \\ \mathcal{W}_{k+1} &= \mathcal{F}_{-k}(\mathcal{W}_1, \mathcal{W}_0), & \tilde{\mathcal{G}}_{k+1} &= \mathcal{F}_{k+1}(\mathcal{W}_1, \mathcal{W}_0) \quad \text{for all } k \in \mathbb{N}, \end{aligned}$$

where  $\{\mathcal{F}_k(X, Y), k \in \mathbb{Z}\}$  is a family of two-variable polynomials in  $X, Y$ . Note that the polynomial formulae that are obtained in [BB10] can be independently derived using the connection between the algebra  $\mathcal{A}_q$  and the reflection equation algebra [BS09]. Details will be reported elsewhere.

We now show how each realization of the four different types of  $q$ -Onsager algebras in terms of  $U_q(\widehat{sl_2})$  Chevalley generators (2.5), (2.10), (2.15), (2.21) can be recovered in a straightforward manner starting from the polynomial formulae (3.1) for the first few descendant generators. Define:

$$(3.3) \quad \mathcal{W}_0 = \psi(\mathbf{W}_0), \quad \mathcal{W}_1 = \psi(\mathbf{W}_1), \quad \gamma = \psi(\Gamma),$$

with (2.5). Observe that the polynomials (3.2) can be systematically expanded in terms of the Chevalley generators and parameters  $k_{\pm}, \epsilon_{\pm}$ . The explicit expressions for the first few examples (3.1), sufficient for our purpose, are reported in Appendix B. Then, we specialize case by case the boundary parameters into these expressions. According to the specialization chosen, the corresponding ‘reduced’ descendant generators do produce the symmetry operators of each of the  $q$ -Onsager algebras exhibited in the previous Section. Explicitly, we obtain:

- **Triangular boundary conditions**  $k_- \neq 0, k_+ = 0, \epsilon_{\pm} \neq 0$ .

$$\mathcal{W}_0|_{k_+ \rightarrow 0} = \psi_t(\mathbf{T}_0), \quad \mathcal{W}_1|_{k_+ \rightarrow 0} = \psi_t(\mathbf{T}_1), \quad \frac{\tilde{\mathcal{G}}_1}{k_+}|_{k_+ \rightarrow 0} = \psi_t(\tilde{\mathbf{P}}_1).$$

- **Diagonal generic boundary conditions**  $k_{\pm} = 0, \epsilon_{\pm} \neq 0$ .

$$\mathcal{W}_0|_{k_{\pm} \rightarrow 0} = \psi_d(\mathbf{K}_0), \quad \mathcal{W}_1|_{k_{\pm} \rightarrow 0} = \psi_d(\mathbf{K}_1), \quad \frac{\mathcal{G}_1}{k_{\pm}}|_{k_{\pm} \rightarrow 0} = \psi_d(\mathbf{Z}_1), \quad \frac{\tilde{\mathcal{G}}_1}{k_{\pm}}|_{k_{\pm} \rightarrow 0} = \psi_d(\tilde{\mathbf{Z}}_1).$$

- **Special diagonal boundary conditions**  $k_{\pm} = 0, \epsilon_- = 0, \epsilon_+ = 1$ .

$$\begin{aligned} \mathcal{W}_1|_{k_{\pm} \rightarrow 0, \epsilon_+ \rightarrow 1, \epsilon_- \rightarrow 0} &= 0, & \mathbf{W}_0|_{k_{\pm} \rightarrow 0, \epsilon_+ \rightarrow 1, \epsilon_- \rightarrow 0} &= \psi_i(\Gamma q^{-h}), \\ \frac{\mathcal{G}_1}{k_-}|_{k_{\pm} \rightarrow 0, \epsilon_+ \rightarrow 1, \epsilon_- \rightarrow 0} &= \psi_i((q^2 - q^{-2})\Gamma q^{-1} \mathbf{e} q^{-h}), & \frac{\tilde{\mathcal{G}}_1}{k_+}|_{k_{\pm} \rightarrow 0, \epsilon_+ \rightarrow 1, \epsilon_- \rightarrow 0} &= \psi_i((q^2 - q^{-2})\Gamma \mathbf{f}), \\ \mathcal{W}_2|_{k_{\pm} \rightarrow 0, \epsilon_+ \rightarrow 1, \epsilon_- \rightarrow 0} &= \psi_i\left(\frac{\Gamma}{q + q^{-1}}(q^{1+h} + q^{-1-h} + (q - q^{-1})^2 \mathbf{f} \mathbf{e})\right), \\ \mathcal{W}_{-1}|_{k_{\pm} \rightarrow 0, \epsilon_+ \rightarrow 1, \epsilon_- \rightarrow 0} &= \psi_i\left(\frac{q - q^{-1}}{(q + q^{-1})^2} \mathbf{X} q^{-h} \Gamma\right), \\ \frac{\mathcal{G}_2}{k_-}|_{k_{\pm} \rightarrow 0, \epsilon_+ \rightarrow 1, \epsilon_- \rightarrow 0} &= \psi_i\left(-q^{-2} \frac{(q - q^{-1})}{(q + q^{-1})} \Gamma \mathbf{Y} q^{-h} + q^{-1} \frac{(q - q^{-1})^2}{(q + q^{-1})} \Gamma \mathbf{e} \mathbf{X} q^h\right), \\ \frac{\tilde{\mathcal{G}}_2}{k_+}|_{k_{\pm} \rightarrow 0, \epsilon_+ \rightarrow 1, \epsilon_- \rightarrow 0} &= \psi_i\left(-q \frac{(q - q^{-1})}{(q + q^{-1})} \Gamma \tilde{\mathbf{Y}} + \frac{(q - q^{-1})^2}{(q + q^{-1})} \mathbf{f} \mathbf{X}\right). \end{aligned}$$

Finally, as an alternative check of the analysis of the previous Section, we now show that according to the choice of boundary conditions, the corresponding above set of operators acting on the vector space (1.2) commute with the associated Hamiltonian. For generic boundary conditions  $k_{\pm}, \epsilon_{\pm}$ , from (2.24) recall that the two operators  $\mathcal{W}_0, \mathcal{W}_1$  acting on (1.2) commute with the Hamiltonian. As a corollary of (2.24) and (3.2), for generic boundary conditions it implies that any descendant generator is commuting with (1.1):

$$(3.4) \quad \left[H_{\frac{1}{2}XXZ}, \mathcal{F}_k(\mathcal{W}_0^{(\infty)}, \mathcal{W}_1^{(\infty)})\right] = 0, \quad \left[H_{\frac{1}{2}XXZ}, \mathcal{F}_k(\mathcal{W}_1^{(\infty)}, \mathcal{W}_0^{(\infty)})\right] = 0 \quad \text{for all } k \in \mathbb{Z}.$$



Specializing the boundary parameters accordingly, it implies that the four different types of  $q$ -Onsager algebras are symmetry algebras of the Hamiltonian.

Let us remark that the defining relations of the current algebra associated with the  $q$ -Onsager algebra are given in [BS09]. For the triangular, augmented and  $U_q(gl_2)$  invariant  $q$ -Onsager algebras, the defining relations of the corresponding current algebras can be derived, respectively, by taking appropriate limits of the relations in  $\mathcal{A}_q$ , given by [BS09, Definition 3.1] and [BS09, Proposition 3.1].

#### 4. CONCLUDING REMARKS

It is instructive to consider the limit  $q \rightarrow 1$  of the four different types of  $q$ -Onsager algebras described in Section 2. In particular, in this limit the  $q$ -Onsager, the augmented  $q$ -Onsager and the  $U_q(gl_2)$  invariant  $q$ -Onsager algebras specialize to three different invariant fixed-point subalgebras of  $U(\widehat{sl_2})$ . For the  $q$ -Onsager algebra with  $q \rightarrow 1$ , the corresponding automorphism is the Chevalley involution of  $U(\widehat{sl_2})$ , given by  $\theta(e_i) = f_i$ ,  $\theta(h_i) = -h_i$ . For the augmented  $q$ -Onsager algebra with  $q \rightarrow 1$ , one considers the composition of the Chevalley involution with the outer automorphism of  $U(\widehat{sl_2})$ . It reads:  $\theta_d(e_0) = f_1$ ,  $\theta_d(e_1) = f_0$ ,  $\theta_d(f_0) = e_1$ ,  $\theta_d(f_1) = e_0$  and  $\theta_d(h_0) = -h_1$ . For the  $U_q(gl_2)$  invariant  $q$ -Onsager algebra at  $q \rightarrow 1$ , one considers the composition of the Chevalley involution with the Lusztig's automorphism of  $U(\widehat{sl_2})$  [L93]. It reads:  $\theta_i(e_0) = e_0$ ,  $\theta_i(f_0) = f_0$ ,  $\theta_i(h_0) = h_0$ ,  $\theta_d(f_1) = [[e_1, e_0], e_0]/2$ ,  $\theta_d(e_1) = [[f_1, f_0], f_0]/2$  and  $\theta_i(h_1) = -h_1 - 2h_0$ . From that point of view, three of the algebras introduced in Section 2 can be understood as a  $q$ -deformation of these invariant fixed-point subalgebras of  $U(\widehat{sl_2})$  [Kol12]. For the  $q$ -Onsager and augmented  $q$ -Onsager algebras, see [BC12] for details. Note that it is a simple exercise to apply the technique of [BC12] to the  $U_q(gl_2)$  invariant  $q$ -Onsager algebra. Besides these three algebras, let us point out that the triangular  $q$ -Onsager algebra gives an example of coideal subalgebra that does not correspond to an invariant fixed-point subalgebra of  $U_q(\widehat{sl_2})$  even at  $q \rightarrow 1$ .

From the point of view of physics, in the thermodynamic limit of the open XXZ spin chain with Hamiltonian (1.1), the solution of the model (spectrum, eigenstates, multiple integral representations of correlation functions and form factors) can be derived using infinite dimensional ( $q$ -vertex operators) representations of the symmetry algebras here considered. Clearly, the  $q$ -vertex operators for each symmetry algebra follow from the representation theory of  $U_q(\widehat{sl_2})$ . Note that the solution for special, diagonal and triangular boundary conditions is given in [JKKKMW95, BB13, BKo13, BKo14]. For the Hamiltonian with generic boundary conditions, although  $q$ -vertex operators for the  $q$ -Onsager algebra are known [BB13], the solution remains an open problem.

Finally, an interesting problem would be to extend the analysis here presented to integrable models with higher rank symmetries of  $q$ -Onsager's type (see for instance [BB09]). The higher rank infinite dimensional algebra extending  $\mathcal{A}_q$ , yet unknown, could be a starting point for the identification of the symmetry algebras.

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#### APPENDIX A: Drinfeld-Jimbo presentation of $U_q(\widehat{sl_2})$

Define the extended Cartan matrix  $\{a_{ij}\}$  ( $a_{ii} = 2$ ,  $a_{ij} = -2$  for  $i \neq j$ ). The quantum affine algebra  $U_q(\widehat{sl_2})$  is generated by the elements  $\{h_j, e_j, f_j\}$ ,  $j \in \{0, 1\}$  which satisfy the defining relations

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij}e_j, \quad [h_i, f_j] = -a_{ij}f_j, \quad [e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}$$

together with the  $q$ -Serre relations

$$(4.1) \quad [e_i, [e_i, [e_i, e_j]_q]_{q^{-1}}] = 0, \quad \text{and} \quad [f_i, [f_i, [f_i, f_j]_q]_{q^{-1}}] = 0.$$

The sum  $c = h_0 + h_1$  is the central element of the algebra. The Hopf algebra structure is ensured by the existence of a comultiplication  $\Delta : U_q(\widehat{sl_2}) \mapsto U_q(\widehat{sl_2}) \otimes U_q(\widehat{sl_2})$ , antipode  $\mathcal{S} : U_q(\widehat{sl_2}) \mapsto U_q(\widehat{sl_2})$  and a counit  $\mathcal{E} : U_q(\widehat{sl_2}) \mapsto \mathbb{C}$  with

$$(4.2) \quad \begin{aligned} \Delta(e_i) &= e_i \otimes 1 + q^{h_i} \otimes e_i, \\ \Delta(f_i) &= f_i \otimes q^{-h_i} + 1 \otimes f_i, \\ \Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i, \end{aligned}$$

$$\mathcal{S}(e_i) = -q^{-h_i} e_i, \quad \mathcal{S}(f_i) = -f_i q^{h_i}, \quad \mathcal{S}(h_i) = -h_i, \quad \mathcal{S}(1) = 1$$

and

$$\mathcal{E}(e_i) = \mathcal{E}(f_i) = \mathcal{E}(h_i) = 0, \quad \mathcal{E}(1) = 1.$$

Note that the opposite coproduct  $\Delta'$  can be similarly defined with  $\Delta' \equiv \sigma \circ \Delta$  where the permutation map  $\sigma(x \otimes y) = y \otimes x$  for all  $x, y \in U_q(\widehat{sl_2})$  is used.

The (evaluation in the principal gradation) endomorphism  $\pi_\zeta : U_q(\widehat{sl_2}) \mapsto \text{End}(\mathcal{V}_\zeta)$  is chosen such that ( $\mathcal{V} \equiv \mathbb{C}^2$ )

$$(4.3) \quad \begin{aligned} \pi_\zeta[e_1] &= \zeta \sigma_+, & \pi_\zeta[e_0] &= \zeta \sigma_-, \\ \pi_\zeta[f_1] &= \zeta^{-1} \sigma_-, & \pi_\zeta[f_0] &= \zeta^{-1} \sigma_+, \\ \pi_\zeta[q^{h_1}] &= q^{\sigma_3}, & \pi_\zeta[q^{h_0}] &= q^{-\sigma_3}, \end{aligned}$$

in terms of the Pauli matrices  $\sigma_\pm, \sigma_3$ :

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

## APPENDIX B: Descendant generators in the Drinfeld-Jimbo presentation of $U_q(\widehat{sl_2})$

$$(4.4) \quad \begin{aligned} \frac{\mathcal{G}_1}{k_-} &= (q^2 - q^{-2}) (\epsilon_- q^c f_1 + \epsilon_+ q^{-1} e_0 q^{h_1} + k_- f_1 q^{h_1} e_0) \\ &\quad + k_+ (q e_1 e_0 - q^{-1} e_0 e_1 + (q f_0 f_1 - q^{-1} f_1 f_0) q^c), \end{aligned}$$

$$(4.5) \quad \begin{aligned} \mathcal{W}_{-1} &= \frac{\epsilon_-}{q + q^{-1}} (q^{1+h_1} + q^{-1-h_1} + (q - q^{-1})^2 f_1 e_1) q^c \\ &\quad + \epsilon_+ \frac{(q - q^{-1})}{(q + q^{-1})^2} (q e_1 e_0 - q^{-1} e_0 e_1 + (q f_0 f_1 - q^{-1} f_1 f_0) q^c) q^{h_1} \\ &\quad + \frac{k_+}{(q + q^{-1})^2} \left( (q - q^{-1}) (q f_0 f_1 - q^{-1} f_1 f_0) q^c e_1 + q^{-2} (q + q^{-1}) f_0 q^{h_0} \right. \\ &\quad \quad \left. - (e_1)^2 e_0 + (q^2 + q^{-2}) e_1 e_0 e_1 - e_0 (e_1)^2 \right) \\ &\quad + \frac{k_-}{(q + q^{-1})^2} \left( q^{-1} (q + q^{-1}) e_0 + q^{-1} (q - q^{-1}) f_1 q^{h_1} (q e_1 e_0 - q^{-1} e_0 e_1) \right. \\ &\quad \quad \left. - q^{c-1} ((f_1)^2 f_0 - (q^2 + q^{-2}) f_1 f_0 f_1 + f_0 (f_1)^2) \right), \end{aligned}$$

$$\begin{aligned}
(4.6) \quad \frac{\mathcal{G}_2}{k_-} &= \epsilon_- \frac{q - q^{-1}}{q + q^{-1}} \left( (q + q^{-1}) q^c e_0 q^{h_1} + \frac{q^2 - q^{-2}}{q + q^{-1}} q^c f_1 (q e_0 e_1 - q^{-1} e_1 e_0) \right. \\
&\quad \left. - q^{2c} ((f_1)^2 f_0 - (q^2 + q^{-2}) f_1 f_0 f_1 + f_0 (f_1)^2) \right) \\
&\quad + \epsilon_+ \frac{q - q^{-1}}{q + q^{-1}} \left( (q + q^{-1}) q^{2c+1} f_1 + \frac{q^2 - q^{-2}}{q + q^{-1}} q^{c-1} (q f_0 f_1 - q^{-1} f_1 f_0) e_0 q^{h_1} \right. \\
&\quad \left. - q^{-1} ((e_0)^2 e_1 - (q^2 + q^{-2}) e_0 e_1 e_0 + e_1 (e_0)^2) \right) + O(k_+) + O(k_-).
\end{aligned}$$

Note that for simplicity, the terms of order  $k_+$  and  $k_-$  in  $\frac{\mathcal{G}_2}{k_-}$  are not explicitly written, as they do not contribute in the limit  $k_{\pm} = 0$ . Also, the descendant generators  $\frac{\tilde{\mathcal{G}}_1}{k_+}$ ,  $\mathcal{W}_2$  and  $\frac{\tilde{\mathcal{G}}_2}{k_+}$  can be derived in terms of the generators of the Drinfeld-Jimbo presentation using the map  $x_0 \rightarrow x_1$ ,  $x_1 \rightarrow x_0$ , with  $x \in \{h, e, f\}$ ,  $k_{\pm} \rightarrow k_{\mp}$  and  $\epsilon_{\pm} \rightarrow \epsilon_{\mp}$  on the expressions (4.4), (4.5) and (4.6) respectively.

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