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UNIVERSAL DYNAMICS FOR THE DEFOCUSING LOGARITHMIC SCHRÖDINGER EQUATION

RÉMI CARLES AND ISABELLE GALLAGHER

Abstract. We consider the nonlinear Schrödinger equation with a logarithmic nonlinearity in a dispersive regime. We show that the presence of the nonlinearity affects the large time behavior of the solution: the dispersion is faster than usual by a logarithmic factor in time and the positive Sobolev norms of the solution grow logarithmically in time. Moreover, after rescaling in space by the dispersion rate, the modulus of the solution converges to a universal Gaussian profile. These properties are suggested by explicit computations in the case of Gaussian initial data, and remain when an extra power-like nonlinearity is present in the equation. One of the key steps of the proof consists in working in hydrodynamical variables to reduce the equation to a variant of the isothermal compressible Euler equation, whose large time behavior turns out to be governed by a parabolic equation involving a Fokker–Planck operator.

1. Introduction

1.1. **Setting.** We are interested in the following equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0\,, \tag{1.1}$$

with $x \in \mathbb{R}^d$, $d \geqslant 1$, $\lambda \in \mathbb{R} \setminus \{0\}$. It was introduced as a model of nonlinear wave mechanics and in nonlinear optics ([9], see also [11, 41, 42, 43, 48]). The mathematical study of this equation goes back to [16, 14] (see also [15]). The sign $\lambda < 0$ seems to be the more interesting from a physical point of view, and this case has been studied formally and rigorously (see [19, 42] for instance). On the other hand, the case $\lambda > 0$ seems to have been little studied mathematically, except as far as the Cauchy problem is concerned (see [16, 35]). In this article, we address the large time properties of the solution in the case $\lambda > 0$, revealing several new features in the context of Schrödinger equations, and more generally Hamiltonian dispersive equations.

Key words and phrases. Nonlinear Schrödinger equation, logarithmic nonlinearity, global attractor, Sobolev norms, Euler equation, Fokker-Planck operator.

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We recall that the mass, angular momentum and energy are (formally) conserved, in the sense that defining

$$\begin{aligned}
&M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^d)}\,,\\
&J(u(t)) := \operatorname{Im}\int_{\mathbb{R}^d} \bar{u}(t,x)\nabla u(t,x)dx\,,\\
&E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda\int_{\mathbb{R}^d} |u(t,x)|^2 \ln|u(t,x)|^2 dx\,,
\end{aligned}$$

then formally,

$$\frac{d}{dt}M(u(t)) = \frac{d}{dt}J(u(t)) = \frac{d}{dt}E(u(t)) = 0\,.$$

The last identity reveals the Hamiltonian structure of (1.1).

Remark 1.1 (Effect of scaling factors). Unlike what happens in the case of an homogeneous nonlinearity (classically of the form $|u|^p u$), replacing u with κu ($\kappa > 0$) in (1.1) has only little effect, since we have

$$i\partial_t(\kappa u) + \frac{1}{2}\Delta(\kappa u) = \lambda \ln\left(|\kappa u|^2\right)\kappa u - 2\lambda(\ln\kappa)\kappa u\,.$$

The scaling factor thus corresponds to a purely time-dependent gauge transform:

$$\kappa u(t,x)e^{2it\lambda\ln\kappa}$$

solves (1.1) (with initial datum κu_0). In particular, the L^2-norm of the initial datum does not influence the dynamics of the solution. This suggests that the presence of the nonlinearity has a rather weak influence on the dynamics, since it shares a feature with linear equations. We will see however that the effects of the nonlinear term are, from various perspectives, much stronger than the effects of a defocusing power-like nonlinearity. Nevertheless, if we denote by u_κ the above solution, we readily compute

$$\frac{du_\kappa}{d\kappa} = (1+2it\lambda)\,u_1 e^{2it\lambda\ln\kappa}, \quad \frac{d^2u_\kappa}{d\kappa^2} = \frac{2it\lambda}{\kappa}\,(1+2it\lambda)\,u_1 e^{2it\lambda\ln\kappa}.$$

This shows that for any $t \neq 0$, the flow map $u_0 \mapsto u(t)$ fails to be C^2 at the origin, whichever the functional spaces considered to measure this regularity.

Note that whichever the sign of λ, the energy E has no definite sign. The distinction between focusing or defocusing nonlinearity is thus a priori ambiguous. We shall see however that in the case $\lambda < 0$, no solution is dispersive, while for $\lambda > 0$, solutions have a dispersive behavior (with a non-standard rate of dispersion). This is why we choose to call *defocusing* the case $\lambda > 0$.

1.2. **The focusing case.** In [16] (see also [15]), the Cauchy problem is studied in the case $\lambda < 0$. Define

$$W := \left\{u \in H^1(\mathbb{R}^d)\,,\, x \mapsto |u(x)|^2\ln|u(x)|^2 \in L^1(\mathbb{R}^d)\right\}.$$

Theorem 1.2 (Théorème 2.1 from [16], see also Theorem 9.3.4 from [15])**.** *Let the initial data u_0 belong to W. In the case when $\lambda < 0$, there exists a unique, global solution $u \in C(\mathbb{R}; W)$ to (1.1). In particular, for all $t \in \mathbb{R}$, $|u(t,\cdot)|^2\ln|u(t,\cdot)|^2$ belongs to $L^1(\mathbb{R}^d)$, and the mass $M(u)$ and the energy $E(u)$ are independent of time.*

In the case when $\lambda < 0$, it can be proved that there is no dispersion for large times. Indeed the following result holds.

Lemma 1.3 (Lemma 3.3 from [14])**.** *Let $\lambda < 0$ and $k < \infty$ such that*

$$L_k := \Big\{ u \in W,\ \|u\|_{L^2(\mathbb{R}^d)} = 1,\ E(u) \leqslant k \Big\} \neq \emptyset\,.$$

Then

$$\inf_{\substack{u \in L_k \\ 1 \leqslant p \leqslant \infty}} \|u\|_{L^p(\mathbb{R}^d)} > 0\,.$$

This lemma, along with the conservation of the energy for (1.1), indicates that in the case $\lambda < 0$, the solution to (1.1) is not dispersive: typically, its L^∞ norm is bounded from below. Actually in the case of Gaussian initial data, some solutions are even known to be periodic in time, as proved in [19] (and already noticed in [9]).

Proposition 1.4 ([19])**.** *In the case $\lambda < 0$, the* Gausson

$$\exp\left(-2i\lambda\omega t + \omega + d/2 + \lambda|x|^2\right)$$

is a solution to (1.1) *for any period $\omega \in \mathbb{R}$.*

We emphasize that several results address the existence of stationary solutions to (1.1) in the case $\lambda < 0$, and the orbital stability of the Gausson; see e.g. [9, 14, 19, 6]. We also note that when $\lambda > 0$, the above formula yields a solution to (1.1) which is C^∞, time periodic, but not even a tempered distribution.

1.3. **Main results.** Throughout the rest of this paper, we assume $\lambda > 0$.

1.3.1. *The Cauchy problem.* For $0 < \alpha \leqslant 1$, we have

$$\mathcal{F}(H^\alpha) := \Big\{ u \in L^2(\mathbb{R}^d)\,,\ x \mapsto \langle x \rangle^\alpha u(x) \in L^2(\mathbb{R}^d) \Big\}\,,$$

where $\langle x \rangle := \sqrt{1 + |x|^2}$ and $\mathcal{F}$ denotes the Fourier transform (whose normalization is irrelevant here), with norm

$$\|u\|_{\mathcal{F}(H^\alpha)} := \|\langle x \rangle^\alpha u(x)\|_{L^2(\mathbb{R}^d)}\,.$$

Note that for any $\alpha > 0$, $\mathcal{F}(H^\alpha) \cap H^1 \subset W$. The Cauchy problem for (1.1) is investigated in [35], where in three space dimensions, the existence of a unique solution in $L^\infty(\mathbb{R}; H^1(\mathbb{R}^3)) \cap C(\mathbb{R}; L^2(\mathbb{R}^3))$ is proved as soon as the initial data belongs to $\mathcal{F}(H^{1/2}) \cap H^1(\mathbb{R}^3)$. Actually it is possible to improve slightly that result into the following theorem.

Theorem 1.5. *Let the initial data u_0 belong to $\mathcal{F}(H^\alpha) \cap H^1(\mathbb{R}^d)$ with $0 < \alpha \leqslant 1$. In the case when $\lambda > 0$, there exists a unique, global solution $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}; \mathcal{F}(H^\alpha) \cap H^1)$ to* (1.1)*. Moreover the mass $M(u)$, the angular momentum $J(u)$, and the energy $E(u)$ are independent of time. If in addition $u_0 \in H^2(\mathbb{R}^d)$, then $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}; H^2)$.*

The main focus of this paper concerns large time asymptotics of the solution. The situation is very different from the $\lambda < 0$ case described above (see Proposition 1.4). Indeed we can prove that (some) solutions tend to zero in L^∞ for large time, while the H^s norm is always unbounded for $s > 0$.

As often in the context of nonlinear Schrödinger equations, we denote by

$$\Sigma = H^1 \cap \mathcal{F}(H^1)(\mathbb{R}^d) = \{ f \in H^1(\mathbb{R}^d),\ x \mapsto |x| f(x) \in L^2(\mathbb{R}^d) \}.$$

1.3.2. *Long time behavior.* We show that three new features characterize the dynamics associated to (1.1):

- The standard dispersion of the Schrödinger equation, in $t^{-d/2}$, is altered by a logarithmic factor, in $\left(t\sqrt{\ln t}\right)^{-d/2}$. This acceleration is of course an effect of the nonlinearity.
- All the positive Sobolev norms of the solution grow logarithmically in time (explicit rate), due to some discrepancy between the dispersive rate and a universal quadratic oscillation in space.
- Up to a rescaling, the modulus of the solution converges for large time to a *universal* Gaussian profile.

Before stating the general result, let us introduce the universal dispersion rate τ through the following lemma. We define from now on the function

$$\ell(t) := \frac{\ln\ln t}{\ln t}\,. \tag{1.2}$$

Lemma 1.6 (Universal dispersion)**.** *Consider the ordinary differential equation*

$$\ddot\tau = \frac{2\lambda}{\tau}\,, \quad \tau(0)=1\,, \quad \dot\tau(0)=0\,. \tag{1.3}$$

It has a unique solution $\tau\in C^2(0,\infty)$, and it satisfies, as $t\to\infty$,

$$\tau(t) = 2t\sqrt{\lambda\ln t}\Big(1+\mathcal{O}\big(\ell(t)\big)\Big)\,, \quad \dot\tau(t) = 2\sqrt{\lambda\ln t}\Big(1+\mathcal{O}\big(\ell(t)\big)\Big)\,.$$

We now state the main result of this paper. Denote by

$$\gamma(x) := e^{-|x|^2/2}.$$

Theorem 1.7. *Let $u_0\in\Sigma$, and rescale the solution provided by Theorem* 1.5 *to $v=v(t,y)$ by setting*

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t,\frac{x}{\tau(t)}\right)\frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}}\exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right). \tag{1.4}$$

There exists C such that for all $t\geqslant 0$,

$$\int_{\mathbb{R}^d}\left(1+|y|^2+\left|\ln|v(t,y)|^2\right|\right)|v(t,y)|^2dy + \frac{1}{\tau(t)^2}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)} \leqslant C\,. \tag{1.5}$$

We have moreover

$$\int_{\mathbb{R}^d}\begin{pmatrix}1\\ y\\ |y|^2\end{pmatrix}|v(t,y)|^2dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d}\begin{pmatrix}1\\ y\\ |y|^2\end{pmatrix}\gamma^2(y)dy\,. \tag{1.6}$$

Finally,

$$|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \textit{weakly in } L^1(\mathbb{R}^d)\,. \tag{1.7}$$

Remark 1.8. The scaling factor in (1.4) is here to normalize the L^2-norm of v to be the same as the L^2-norm of γ, and is of no influence regarding the statement, in view of Remark 1.1. To the best of our knowledge, this is the first time that a universal profile is observed for the large time behavior of solutions to a dispersive, Hamiltonian equation. This profile is reached in a weak sense only in Theorem 1.7, as far as the convergence is concerned, but also because the modulus of the solution only is captured. This indicates that a lot of information remains encoded in the oscillations of the solution. See Section 1.4 for more on the large time asymptotics.

Remark 1.9. As a straightforward consequence, we infer the slightly weaker property that $|v(t,\cdot)|^2$ converges to γ^2 in Wasserstein distance:

$$W_2\left(\frac{|v(t,\cdot)|^2}{\pi^{d/2}}, \frac{\gamma^2}{\pi^{d/2}}\right) \underset{t\to\infty}{\longrightarrow} 0,$$

where we recall that the Wasserstein distance is defined, for ν_1 and ν_2 probability measures, by

$$W_p(\nu_1,\nu_2) = \inf\left\{\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\mu(x,y)\right)^{1/p}; \quad (\pi_j)_\sharp\mu = \nu_j\right\},$$

where μ varies among all probability measures on $\mathbb{R}^d\times\mathbb{R}^d$, and $\pi_j : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$ denotes the canonical projection onto the j-th factor (see e.g. [54]).

In the context of nonlinear Hamiltonian partial differential equations, a general question is the evolution of Sobolev norms, as emphasized in [10]. In [37], the existence of unbounded, in H^s with $s>1$, solutions to the cubic defocusing Schrödinger equation was established for the first time, in the case where the equation is considered on the domain $\mathbb{R}\times\mathbb{T}^d$, $d\geqslant 2$: for *some* solutions, the Sobolev norms grow logarithmically along a sequence of times. See also [17, 33, 32], in the space periodic case. For other equations (cubic Szegő equation or half-wave equation), with *specific initial data*, a growth rate can be exhibited, possibly along a sequence of time; see [27, 28, 52]. In particular, for the cubic Szegő equation, the generic (in the sense of Baire) growth of Sobolev norms with superpolynomial rates is established in [29]. Combining the approaches of [37] and [29], it was proved in [55] that a system of half-wave–Schrödinger on the cylinder $\mathbb{R}_x\times\mathbb{T}_y$ possesses unbounded solutions in $L^2_xH^s_y$ for $s>1/2$, where the growth rate is a superpolynomial function of $\ln t$. We show that in the case of (1.1), the Sobolev norms of *all* solutions grow, regardless of the data, and we give a sharp rate.

Corollary 1.10. *Let $u_0\in\Sigma\setminus\{0\}$, and $0<s\leqslant 1$. The solution to* (1.1) *satisfies, as $t\to\infty$,*

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} 2\lambda d\|u_0\|^2_{L^2(\mathbb{R}^d)}\ln t,$$

and

$$(\ln t)^{s/2} \lesssim \|u(t)\|_{\dot H^s(\mathbb{R}^d)} \lesssim (\ln t)^{s/2},$$

where $\dot H^s(\mathbb{R}^d)$ denotes the standard homogeneous Sobolev space.

The weak convergence in the last point of Theorem 1.7 may seem puzzling. In the case of Gaussian initial data, we can prove that the convergence is strong.

Corollary 1.11 (Strong convergence in the Gaussian case)**.** *Suppose that the initial data u_0 is a Gaussian,*

$$u_0(x) = b_0\exp\Big(-\frac{1}{2}\sum_{j=1}^d a_{0j}(x_j-x_{0j})^2\Big)$$

with $b_0, a_{0j}\in\mathbb{C}$, $\operatorname{Re} a_{0j}>0$, and $x_{0j}\in\mathbb{R}$. Then, with v given by (1.4)*, the relative entropy of $|v|^2$ goes to zero for large time:*

$$\int_{\mathbb{R}^d} |v(t,y)|^2 \ln\left|\frac{v(t,y)}{\gamma(y)}\right|^2 dy \underset{t\to\infty}{\longrightarrow} 0\,,$$

and the convergence of $|v|^2$ *to* γ^2 *is strong in* L^1*:*

$$\left\| |v(t,\cdot)|^2 - \gamma^2 \right\|_{L^1(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0\,.$$

1.4. **Comments and further result.** In the linear case

$$i\partial_t u_{\text{free}} + \frac{1}{2}\Delta u_{\text{free}} = 0\,, \quad u_{\text{free}|t=0} = u_0\,, \tag{1.8}$$

the integral representation

$$u_{\text{free}}(t,x) = \frac{1}{(2i\pi t)^{d/2}} \int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}} u_0(y)dy$$

readily yields the well-known dispersive estimate

$$\|u_{\text{free}}(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}} \|u_0\|_{L^1(\mathbb{R}^d)}\,.$$

Moreover, defining the Fourier transform as

$$\mathcal{F}f(\xi) := \hat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx\,,$$

we have the standard asymptotics (see e.g. [53]),

$$\|u_{\text{free}}(t) - A(t)u_0\|_{L^2(\mathbb{R}^d)} \underset{t\to\pm\infty}{\longrightarrow} 0, \quad A(t)u_0(x) := \frac{1}{(it)^{d/2}} \hat{u}_0\left(\frac{x}{t}\right) e^{i\frac{|x|^2}{2t}}\,, \tag{1.9}$$

a formula which has proven very useful in the nonlinear (long range) scattering theory (see e.g. [30, 39]).

In the case of the defocusing nonlinear Schrödinger equation with power-like nonlinearity,

$$i\partial_t u + \frac{1}{2}\Delta u = |u|^{2\sigma} u\,, \quad u_{|t=0} = u_0\,, \tag{1.10}$$

if σ is sufficiently large (say $\sigma > 2/d$ if $u_0 \in H^1(\mathbb{R}^d)$, even though this bound can be lowered if in addition $\hat{u}_0 \in H^1(\mathbb{R}^d)$), then there exists $u_+ \in H^1(\mathbb{R}^d)$ such that in $L^2(\mathbb{R}^d)$,

$$u(t,x) \underset{t\to\infty}{\sim} e^{i\frac{t}{2}\Delta} u_+(x) \underset{t\to\infty}{\sim} \frac{1}{(it)^{d/2}} \hat{u}_+\left(\frac{x}{t}\right) e^{i\frac{|x|^2}{2t}}\,,$$

where the last relation stems from (1.9). Therefore, Theorem 1.7 shows that unlike in the free case (1.8) or in the above nonlinear case (1.10), the dispersion is modified (it is even enhanced the larger the λ), and the asymptotic profile $\hat{u}_+$ (with $u_+ = u_0$ in the free case), which depends on the initial profile, is replaced by a universal one (up to a normalizing factor),

$$\frac{\|u_0\|_{L^2}}{\pi^{d/4}} e^{-|x|^2/2}\,.$$

As already pointed out, the nonlinearity is responsible for the new dispersive rate, as it introduces a logarithmic factor. In particular, no scattering result relating the dynamics of (1.1) to the free dynamics $e^{i\frac{t}{2}\Delta}$ must be expected. This situation can be compared with the more familiar one with low power nonlinearity, where a long range scattering theory is (sometimes) available. If $\sigma \leqslant 1/d$ in (1.10), then u cannot be be compared with a free evolution for large time, in the sense that if for some $u_+ \in L^2(\mathbb{R}^d)$,

$$\|u(t) - e^{i\frac{t}{2}\Delta} u_+\|_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0\,,$$

then $u = u_+ = 0$ ([8]). In the case $\sigma = 1/d$, $d = 1, 2, 3$, a nonlinear phase modification of $e^{i\frac{t}{2}\Delta}$ must be incorporated in order to describe the asymptotic behavior of u ([51, 39]). The same is true when (1.10) is replaced with the Hartree equation [31, 49, 50]. In all these cases, as well as for some quadratic nonlinearities in dimension 3 ([38]), the dispersive rate of the solution remains the same as in the free case, of order $t^{-d/2}$. Note however that a similar logarithmic perturbation of the dispersive rate was observed in [40], for the equation

$$i\partial_t u + \frac{1}{2}\partial_x^2 u = i\lambda u^3 + |u|^2 u\,, \quad x \in \mathbb{R}\,, \tag{1.11}$$

with $\lambda \in \mathbb{R}$, $0 < |\lambda| < \sqrt{3}$. More precisely, the authors construct small solutions satisfying the bounds

$$\frac{1}{\sqrt{t}(\ln t)^{1/4}} \lesssim \sup_{|x|\leqslant\sqrt{t}} |u(t,x)| \lesssim \frac{1}{\sqrt{t}(\ln t)^{1/4}}\,, \quad \text{as } t \to \infty\,.$$

An important difference with (1.1) though is that the L^2-norm of the solution of (1.11) is not preserved by the flow, and that (1.11) has no Hamiltonian structure.

On the other hand, (1.4) brings out a universal spatial oscillation, namely the term $\exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right)$. In view of Lemma 1.6, we have

$$\frac{\dot\tau(t)}{\tau(t)} - \frac{1}{t} = \frac{\mathcal{O}\left(\ell(t)\right)}{t\left(1+\mathcal{O}\left(\ell(t)\right)\right)}, \quad \text{hence } \exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right) \underset{t\to\infty}{\sim} e^{i\frac{|x|^2}{2t}},$$

the same universal oscillation as for the linear Schrödinger equation (1.8), see (1.9).

Finally let us remark that the convergence to a universal profile is reminiscent of what happens for the linear heat equation on $\mathbb{R}^d$. Indeed, let u solve

$$\partial_t u = \frac{1}{2}\Delta u, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d, \quad u_{|t=0} = u_0. \tag{1.12}$$

This equation can be solved thanks to Fourier analysis, $\hat u(t,\xi) = \hat u_0(\xi) e^{-\frac{t}{2}|\xi|^2}$. This allows to show the following standard asymptotics (see e.g. [53]):

$$u(t,x) \underset{t\to\infty}{\sim} \frac{m}{(2\pi t)^{d/2}} e^{-|x|^2/(2t)} \quad \text{in } L^2 \cap L^\infty(\mathbb{R}^d), \text{ provided } m := \int_{\mathbb{R}^d} u_0(x)dx \neq 0.$$

We note that at leading order, the only role played by the initial data is the presence of the total mass m. The asymptotic profile is *universal*, and corresponds to the Gaussian γ.

In a nonlinear setting, our result is reminiscent of the works [23, 24, 25] on the Navier-Stokes equations: there it is proved that up to a rescaling which corresponds to the natural scaling of the equations, the vorticity converges strongly to a Gaussian which is known as the Oseen vortex. The main argument, as in the present case, is the reduction to a Fokker-Planck equation.

In view of the above discussion, one may ask if the phenomena stated in Theorem 1.7 and Corollary 1.10 are bound to the very special structure of the nonlinearity. Our final result shows that it is not the case, inasmuch as these results remain (possibly up to a uniqueness issue) when an energy-subcritical defocusing power-like nonlinearity is added,

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda u \ln(|u|^2) + \mu |u|^{2\sigma} u, \quad u_{|t=0} = u_0. \tag{1.13}$$

The mass and angular momentum of u are the same as before, and they are formally conserved, as well as the energy

$$E(u(t)) = \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \ln |u(t,x)|^2 dx + \frac{\mu}{\sigma+1}\int_{\mathbb{R}^d} |u(t,x)|^{2\sigma+2} dx.$$

Theorem 1.12. *Let $\lambda, \mu > 0$ and $0 < \sigma < 2/(d-2)_+$. For $u_0 \in \Sigma$, (1.13) has a solution $u \in L^\infty_{\mathrm{loc}}(\mathbb{R};\Sigma)$. It is unique (at least) if $d = 1$. Its mass, angular momentum and energy are independent of time. Setting*

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}} \exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right),$$

we have

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) dy\,,$$

and

$$|v(t,\cdot)|^2 \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \textit{weakly in } L^1(\mathbb{R}^d)\,.$$

Finally, for $0 < s \leqslant 1$, u satisfies, as $t \to \infty$,

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} 2\lambda d \|u_0\|^2_{L^2(\mathbb{R}^d)} \ln t,$$

and

$$(\ln t)^{s/2} \lesssim \|u(t)\|_{\dot H^s(\mathbb{R}^d)} \lesssim (\ln t)^{s/2}\,.$$

The above result shows that the logarithmic nonlinearity, far from being a weak nonlinearity, dictates the dynamics even in the presence of an energy-subcritical defocusing power-like nonlinearity, in sharp contrast with the scattering results of the case $\lambda = 0$ mentioned above. The dynamics due to the logarithmic nonlinearity is thus rather stable.

1.5. **Outline of the paper.** In Section 2, we give the proof of Theorem 1.5. Section 3 contains explicit computations in the Gaussian case, paving the way for the general case. The main step consists in a reduction to ordinary differential equations, and the computations concerning the main ODEs are gathered in this section, including the proof of Lemma 1.6. The first part of Theorem 1.7, that is everything except (1.7), is proved in Section 4 and relies on energy estimates with appropriate weights in space. The weak limit (1.7) is proved in Section 5. The main idea consists in using a Madelung transform, which leads to the study of a hyperbolic system which is a variant of the isothermal, compressible Euler equation. A rescaling in the time variable reduces the study to a non autonomous perturbation of the Fokker-Planck equation, and the a priori estimates obtained in Section 4 imply that a weak limit of the solution satisfies the Fokker-Planck equation for large times. Since it is known that the large time behavior of the solution to the Fokker-Planck equation is the centered Gaussian, a tightness argument on the rescaled solution concludes the proof. The proofs of the corollaries are given in Section 6, and the main arguments leading to Theorem 1.12 are given in Section 7.

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2. Cauchy problem: proof of Theorem 1.5

In this section we sketch the proof of the existence of a unique weak solution, which follows very standard ideas (see [15, 16]). We first prove, in Section 2.1, the existence of weak solutions by solving an approximate system and passing to the limit in the approximation parameter. This produces a weak solution, whose uniqueness is proved in Section 2.2. The proof of the propagation of higher regularity is given in Section 2.3.

2.1. **Existence.** To prove the existence of a weak solution we proceed by approximating the equation as follows: consider for all $\varepsilon \in (0,1)$ the equation

$$i\partial_t u_\varepsilon + \frac{1}{2}\Delta u_\varepsilon = \lambda \ln\left(\varepsilon + |u_\varepsilon|^2\right) u_\varepsilon\,, \quad u_{\varepsilon|t=0} = u_0\,. \tag{2.1}$$

Equation (2.1) is easily solved in $C(\mathbb{R}; L^2(\mathbb{R}^d))$ since it is subcritical in L^2 (see [15]). It remains therefore to prove uniform bounds for $u_\varepsilon(t)$ in $\mathcal{F}(H^\alpha) \cap H^1(\mathbb{R}^d)$, which will provide compactness in space for the sequence u_ε. Since time compactness (in $H^{-2}(\mathbb{R}^d)$) is a direct consequence of the equation, the Ascoli theorem will then give the result. Actually once a bound in $L^\infty_{\rm loc}(\mathbb{R}; H^1(\mathbb{R}^d))$ is derived, then the $L^\infty_{\rm loc}(\mathbb{R}; \mathcal{F}(H^\alpha))$ bound can be obtained directly thanks to the following computation: define

$$I_{\varepsilon,\alpha}(t) := \int_{\mathbb{R}^d} \langle x\rangle^{2\alpha} |u_\varepsilon|^2(t,x)\,dx\,.$$

Then multiplying the equation by $\langle x\rangle^{2\alpha}\overline{u}_\varepsilon$ and integrating in space provides

$$\begin{aligned}\frac{d}{dt} I_{\varepsilon,\alpha}(t) &= 2\alpha\,\mathrm{Im}\int \frac{x\cdot\nabla u_\varepsilon}{\langle x\rangle^{2-2\alpha}}\,\overline{u}_\varepsilon(t)\,dx \\ &\leqslant 2\alpha \|\,\langle x\rangle^{2\alpha-1}\,u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)} \|\nabla u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)} \\ &\leqslant 2\alpha \|\,\langle x\rangle^{\alpha}\,u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)} \|\nabla u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\,,\end{aligned}$$

where the last estimate stems from the property $\alpha \leqslant 1$. Therefore,

$$\|u_\varepsilon(t)\|^2_{\mathcal{F}(H^\alpha)} \leqslant \|u_0\|^2_{\mathcal{F}(H^\alpha)} + 2\alpha \int_0^t \|u_\varepsilon(t')\|_{\mathcal{F}(H^\alpha)} \|\nabla u_\varepsilon(t')\|_{L^2(\mathbb{R}^d)}\,dt'\,.$$

So it remains to compute the $H^1(\mathbb{R}^d)$ norm of $u_\varepsilon(t)$. This is quite easy since the problem becomes linear in ∇u_ε. Indeed for any $1 \leqslant j \leqslant d$ one has

$$i\partial_t\partial_j u_\varepsilon + \frac{1}{2}\Delta\partial_j u_\varepsilon = \lambda \ln\left(\varepsilon + |u_\varepsilon|^2\right)\partial_j u_\varepsilon + 2\lambda \frac{1}{\varepsilon + |u_\varepsilon|^2}\,\mathrm{Re}(\bar{u}_\varepsilon \partial_j u_\varepsilon) u_\varepsilon, \tag{2.2}$$

and we note that $\left|\frac{1}{\varepsilon + |u_\varepsilon|^2} 2\,\mathrm{Re}(\bar{u}_\varepsilon\partial_j u_\varepsilon)u_\varepsilon\right| \leqslant 2|\partial_j u_\varepsilon|$. We therefore conclude that u_ε belongs to $L^\infty_{\rm loc}(\mathbb{R}; H^1(\mathbb{R}^d))$, uniformly in ε.

Passing to the limit to obtain a solution conserving mass, angular momentum, and energy is established in the same way as in [16] (see also [15]). The existence part of Theorem 1.5 follows.

For the proof of uniqueness below it is useful to prove that any solution in the class $L^\infty_{\rm loc}(\mathbb{R}; \mathcal{F}(H^\alpha) \cap H^1)$ belongs actually to $C(\mathbb{R}; (L^2 \cap H^1_{\rm weak})(\mathbb{R}^d))$, which allows to make sense of the initial data in $H^1(\mathbb{R}^d)$. The method of proof follows the idea of [16], and is in fact easier due to our functional setting. Let u be such a solution,

then clearly Δu belongs to $L^\infty_{\rm loc}(\mathbb{R};H^{-1}(\mathbb{R}^d))$ and we claim that $u\ln|u|^2$ belongs to $L^\infty_{\rm loc}(\mathbb{R};L^2(\mathbb{R}^d))$. Indeed there holds

$$\int |u|^2(\ln|u|^2)^2 \lesssim \int |u|^{2-\epsilon} + \int |u|^{2+\epsilon}$$

for all $\epsilon>0$, and moreover we have the estimate

$$\int_{\mathbb{R}^d} |u|^{2-\epsilon} \lesssim \|u\|_{L^2}^{2-\epsilon-\frac{d\epsilon}{2\alpha}} \|x^\alpha u\|_{L^2}^{\frac{d\epsilon}{2\alpha}}\,, \tag{2.3}$$

for $0<\epsilon<\frac{4\alpha}{d+2\alpha}$, which can be readily proved by an interpolation method (cutting the integral into $|y|<R$ and $|y|>R$, using Hölder inequality and optimizing over R; see e.g. [13]). This, along with Sobolev embeddings, implies that

$$\int |u|^2(\ln|u|^2)^2 \lesssim \|u\|_{L^2}^{2-\epsilon-\frac{d\epsilon}{2\alpha}} \|x^\alpha u\|_{L^2}^{\frac{d\epsilon}{2\alpha}} + \|u\|_{H^1}^{2+\epsilon}$$

so finally $\partial_t u$ belongs to $L^\infty_{\rm loc}(\mathbb{R};H^{-1}(\mathbb{R}^d))$ and the result follows.

2.2. **Uniqueness.** The uniqueness of the solution constructed above is a consequence of the following lemma.

Lemma 2.1 (Lemma 9.3.5 from [15])**.** *We have*

$$\left|\mathrm{Im}\left(\left(z_2\ln|z_2|^2 - z_1\ln|z_1|^2\right)(\bar z_2-\bar z_1)\right)\right| \leqslant 4|z_2-z_1|^2\,,\quad \forall z_1,z_2\in\mathbb{C}\,.$$

Consider indeed u_1 and u_2 two solutions of (1.1) as constructed in the previous section. Then the function $u:=u_1-u_2$ satisfies

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda\big(\ln\left(|u_1|^2\right)u_1 - \ln\left(|u_2|^2\right)u_2\big)$$

and the regularity of u_1 and u_2 enables one to write an energy estimate in L^2 on this equation. We get directly

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{L^2(\mathbb{R}^d)}^2 &= \lambda\,\mathrm{Im}\int_{\mathbb{R}^d}\big(\ln\left(|u_1|^2\right)u_1 - \ln\left(|u_2|^2\right)u_2\big)(\bar u_1-\bar u_2)(t)\,dx\\ &\leqslant 4\lambda\|u(t)\|_{L^2(\mathbb{R}^d)}^2\end{aligned} \tag{2.4}$$

thanks to Lemma 2.1. Uniqueness (and in fact stability in L^2) follows directly, by integration in time.

2.3. **Higher regularity.** As in [16], the idea is to consider time derivatives. This fairly general idea in the context of nonlinear Schrödinger equations (see [15]) is all the more precious in the present framework that the logarithmic nonlinearity is very little regular. In particular, we emphasize that if $u_0\in H^k(\mathbb{R}^d)$, $k\geqslant 3$, we cannot guarantee in general that this higher regularity is propagated.

To complete the proof of Theorem 1.5, assume that $u_0\in\mathcal{F}(H^\alpha)\cap H^2$, for some $\alpha>0$. We already know that a unique, global, weak solution $u\in L^\infty_{\rm loc}(\mathbb{R};\mathcal{F}(H^\alpha)\cap H^1)$ is obtained by the procedure described in the previous subsection, that is, as the limit of u_ε solution to (2.1). The idea is that for all $T>0$, there exists $C=C(T)$ independent of $\varepsilon\in(0,1)$ such that

$$\sup_{-T\leqslant t\leqslant T}\|\partial_t u_\varepsilon(t)\|_{L^2(\mathbb{R}^d)}\leqslant C\,.$$

Indeed, we know directly from (2.1) that

$$\partial_t u_{\varepsilon|t=0} = \frac{i}{2}\Delta u_0 - i\lambda \ln\left(\varepsilon + |u_0|^2\right) u_0 \in L^2(\mathbb{R}^d)\,,$$

uniformly in ε, in view of the pointwise estimate

$$\left|\ln\left(\varepsilon + |u_0|^2\right) u_0\right| \leqslant C\left(|u_0|^{1+\eta} + |u_0|^{1-\eta}\right)\,,$$

where $\eta > 0$ can be chosen arbitrarily small, and C is independent of $\varepsilon \in (0,1)$. Then we can replace the spatial derivative ∂_j in (2.2) with the time derivative ∂_t, and infer that $\partial_t u_\varepsilon \in L^\infty_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^d))$, uniformly in ε: by passing to the limit (up to a subsequence), $\partial_t u \in L^\infty_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^d))$. Using the equation (1.1), we infer that $\Delta u \in L^\infty_{\mathrm{loc}}(\mathbb{R}; L^2(\mathbb{R}^d))$. This concludes the proof of Theorem 1.5. □

3. Propagation of Gaussian data

As noticed already in [9], an important feature of (1.1) is that the evolution of initial Gaussian data remains Gaussian. Since (1.1) is invariant by translation in space, we may consider centered Gaussian initial data. The following result is a crucial guide for the general case.

Theorem 3.1. *Let $\lambda > 0$, and consider the initial data*

$$u_0(x) = b_0 \exp\Big(-\frac{1}{2}\sum_{j=1}^d a_{0j} x_j^2\Big) \tag{3.1}$$

with $b_0, a_{0j} \in \mathbb{C}$, $\alpha_{0j} = \operatorname{Re} a_{0j} > 0$. Then the solution u to (1.1) is given by

$$u(t,x) = b_0 \prod_{j=1}^d \frac{1}{\sqrt{r_j(t)}} \exp\Big(i\phi_j(t) - \alpha_{0j}\frac{x_j^2}{2r_j^2(t)} + i\frac{\dot r_j(t)}{r_j(t)}\frac{x_j^2}{2}\Big)$$

for some real-valued functions ϕ_j, r_j depending on time only, such that, as $t \to \infty$,

$$r_j(t) = 2t\sqrt{\lambda\alpha_{0j}\ln t}\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big)\,, \quad \dot r_j(t) = 2\sqrt{\lambda\alpha_{0j}\ln t}\Big(1 + \mathcal{O}\big(\ell(t)\big)\Big)\,, \tag{3.2}$$

where ℓ is defined in (1.2). In particular, as $t \to \infty$,

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \sim \frac{1}{\Big(t\sqrt{\ln t}\Big)^{d/2}} \frac{\|u_0\|_{L^2}}{\big(2\lambda\sqrt{2\pi}\big)^{d/2}}\,.$$

On the other hand u belongs to $L^\infty_{\mathrm{loc}}(\mathbb{R}; H^1(\mathbb{R}^d))$ and as $t \to \infty$

$$\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} \underset{t\to\infty}{\sim} 2\lambda d \|u_0\|^2_{L^2(\mathbb{R}^d)} \ln t.$$

3.1. From (1.1) to ordinary differential equations.

3.1.1. *The Gaussian structure.* As noticed in [9], the flow of (1.1) preserves any initial Gaussian structure. We consider the data given by (3.1), and we seek the solution u to (1.1) under the form

$$u(t,x) = b(t)\exp\Big(-\frac{1}{2}\sum_{j=1}^d a_j(t) x_j^2\Big)\,, \tag{3.3}$$

with $\operatorname{Re} a_j(t) > 0$. With u of this form, (1.1) becomes equivalent to

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda\Big(\ln|b(t)|^2 - \sum_{j=1}^d \operatorname{Re} a_j(t) x_j^2\Big)u\,, \quad u_{|t=0} = u_0\,.$$

This is a linear Schrödinger equation with a time-dependent harmonic potential, and an initial Gaussian. It is well-known in the context of the propagation of coherent states (see [36, 18]) that the evolution of a Gaussian wave packet under a time-dependent harmonic oscillator is a Gaussian wave packet. Therefore, it is consistent to look for a solution to (1.1) of this form. Notice in particular that

$$\|u(t)\|_{L^p(\mathbb{R}^d)} = \left(\frac{2\pi}{p}\right)^{d/(2p)} \frac{|b(t)|}{\Big(\prod_{j=1}^d \operatorname{Re} a_j(t)\Big)^{1/(2p)}}\,, \quad 1 \leqslant p \leqslant \infty\,. \tag{3.4}$$

To prove Theorem 3.1 we therefore need to find the asymptotic behavior in time of $b(t)$ and $a_j(t)$.

3.1.2. *The ODEs.* Plugging (3.3) into (1.1), we obtain

$$i\dot b - i\sum_{j=1}^d \dot a_j \frac{x_j^2}{2} b - \sum_{j=1}^d \frac{a_j b}{2} + \sum_{j=1}^d a_j^2 \frac{x_j^2}{2} b = \lambda\Big(\ln\left(|b|^2\right) - \sum_{j=1}^d \left(\operatorname{Re} a_j\right) x_j^2\Big) b\,.$$

Equating the constant in x and the factors of x_j^2, we get

$$i\dot a_j - a_j^2 = 2\lambda \operatorname{Re} a_j\,, \quad a_{j|t=0} = a_{0j}\,, \tag{3.5}$$

$$i\dot b - \sum_{j=1}^d \frac{a_j b}{2} = \lambda b \ln\left(|b|^2\right), \quad b_{|t=0} = b_0\,. \tag{3.6}$$

We can express the solution to (3.6) directly as a function of the a_j's: indeed

$$b(t) = b_0 \exp\Big(-i\lambda t \ln\left(|b_0|^2\right) - \frac{i}{2}\sum_{j=1}^d A_j(t) - i\lambda \sum_{j=1}^d \operatorname{Im}\int_0^t A_j(s)ds\Big)\,,$$

where we have set

$$A_j(t) := \int_0^t a_j(s)ds\,.$$

We also infer from (3.5) that $y := \operatorname{Re} a_j$ solves $\dot y = 2y \operatorname{Im} a_j$, hence

$$\operatorname{Re} a_j(t) = \operatorname{Re} a_{0j} \exp\Big(2\int_0^t \operatorname{Im} a_j(s)ds\Big)\,.$$

Since the equations (3.5) are decoupled as j varies, we simply consider from now on

$$i\dot a - a^2 = 2\lambda \operatorname{Re} a\,, \quad a_{|t=0} = a_0 = \alpha_0 + i\beta_0\,, \tag{3.7}$$

which amounts to assuming $d = 1$ in (1.1). Following [45], we seek a of the form

$$a = -i\frac{\dot\omega}{\omega}\cdot$$

Then (3.7) becomes

$$\ddot\omega = 2\lambda\omega \operatorname{Im}\frac{\dot\omega}{\omega}\cdot$$

Introducing the polar decomposition $\omega = re^{i\theta}$, we get

$$\ddot{r} - (\dot{\theta})^2 r = 2\lambda r\dot{\theta}, \quad \ddot{\theta} r + 2\dot{\theta}\dot{r} = 0\,.$$

Notice that

$$\dot{\theta}_{|t=0} = \alpha_0\,, \quad \left(\frac{\dot{r}}{r}\right)_{|t=0} = -\beta_0\,.$$

We therefore have a degree of freedom to set $r(0)$, and we decide $r(0) = 1$ so

$$\dot{\theta}(0) = \operatorname{Re} a_0 = \alpha_0\,, \quad \dot{r}(0) = -\operatorname{Im} a_0 = -\beta_0\,.$$

The second equation yields

$$\frac{d}{dt}\left(r^2\dot{\theta}\right) = r\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right) = 0\,,$$

so $r^2\dot{\theta}$ is constant and we can express the problem in terms of r only: we write

$$a(t) = \frac{\alpha_0}{r(t)^2} - i\frac{\dot{r}(t)}{r(t)}\,, \tag{3.8}$$

with

$$\ddot{r} = \frac{\alpha_0^2}{r^3} + 2\lambda\frac{\alpha_0}{r}\,, \quad r(0) = 1\,,\ \dot{r}(0) = -\beta_0\,. \tag{3.9}$$

Multiplying by $\dot{r}$ and integrating, we infer

$$(\dot{r})^2 = \beta_0^2 + \alpha_0^2\left(1 - \frac{1}{r^2}\right) + 4\lambda\alpha_0 \ln r\,. \tag{3.10}$$

Back to the solution u, in the case when $d = 1$ then writing in view of (3.4) and (3.8)

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} = |b(t)| = |b_0| \exp\left(\frac{1}{2}\int_0^t \operatorname{Im} a(s)ds\right) = \frac{|b_0|}{\sqrt{r(t)}}$$

we find that the study of $r(t)$ is enough to find the dispersion rate of $u(t)$. Once the rate in one space dimension is known, the result in d space dimensions follows directly. Moreover, we compute directly

$$\begin{aligned}
\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} &= \frac{1}{2}\pi^{d/2}\frac{|b(t)|^2}{\left(\prod_{j=1}^d \operatorname{Re} a_j(t)\right)^{1/2}}\sum_{j=1}^d \frac{|a_j(t)|^2}{(\operatorname{Re} a_j(t))} \\
&= \frac{\pi^{d/2}|b_0|^2}{2\left(\prod_{j=1}^d r_j(t)\right)\left(\prod_{j=1}^d \operatorname{Re} a_j(t)\right)^{1/2}}\sum_{j=1}^d \frac{|a_j(t)|^2}{(\operatorname{Re} a_j(t))} \\
&= \frac{\pi^{d/2}|b_0|^2}{2\sqrt{\prod_{j=1}^d \alpha_{0j}}}\sum_{j=1}^d \left((\dot{r}_j)^2 + \frac{\alpha_0^2}{r_j^2}\right)\frac{1}{\alpha_{0j}} \\
&= c + \lambda\frac{\pi^{d/2}|b_0|^2}{\sqrt{\prod_{j=1}^d \alpha_{0j}}}\sum_{j=1}^d \ln r_j(t)\,.
\end{aligned}$$

As soon as $r_j(t) \to \infty$ when $|t| \to \infty$, the H^1 norm therefore becomes unbounded. This is proved to be the case below (with an explicit rate): actually it can be seen from the rate provided in Lemma 3.9 below that the energy remains bounded because the unbounded contributions of both parts of the energy cancel exactly.

3.2. **Study of $r(t)$.** The aim of this paragraph is to prove the following result. Recall notation (1.2).

Lemma 3.2. *Let r solve* (3.9)*. Then as $t \to \infty$, there holds*

$$r(t) = 2t\sqrt{\lambda\alpha_0 \ln t}\left(1 + \mathcal{O}\big(\ell(t)\big)\right).$$

The proof of the lemma is achieved in three steps: first we prove, in Paragraph 3.2.1, that $r(t) \to \infty$ as $t \to \infty$. In view of that result it is natural to approximate the solution to (3.9) by

$$\ddot{r}_{\rm eff} = 2\lambda\frac{\alpha_0}{r_{\rm eff}}\,, \quad r_{\rm eff}(T) = r(T)\,,\ \dot{r}_{\rm eff}(T) = \dot{r}(T)\,, \tag{3.11}$$

for $T \gg 1$. This is proved in Paragraph 3.2.2, along with a first estimate on the large time behavior of $r_{\rm eff}$. The conclusion of the proof is achieved in Paragraph 3.2.3, by proving Lemma 1.6.

3.2.1. *First step: $r(t) \to \infty$.* We readily see from (3.10) that r is bounded from below:

$$r(t) \geqslant \exp\left(-\frac{\beta_0^2 + \alpha_0^2}{4\lambda\alpha_0}\right) > 0\,, \quad \forall t \in \mathbb{R}\,.$$

Now let us prove that $r(t) \to +\infty$ as $t \to +\infty$. Assume first that $\dot{r}(0) > 0$. Then (3.9) yields $\ddot{r} \geqslant 0$, hence $\dot{r}(t) \geqslant \dot{r}(0)$ for all $t \geqslant 0$, and

$$r(t) \geqslant \dot{r}(0)t + 1 \underset{t\to+\infty}{\longrightarrow} +\infty\,. \tag{3.12}$$

On the other hand, for $\dot{r}(0) \leqslant 0$, assume that r is bounded, $r(t) \leqslant M$. Then (3.9) yields

$$\ddot{r}(t) \geqslant \frac{\alpha_0^2}{M^3} + 2\lambda\frac{\alpha_0}{M}\,,$$

hence a contradiction for t large enough. We infer that for T sufficiently large, there holds $r(T) \geqslant 1$ and $\dot{r}(T) > 0$. The first case then implies $r(t) \to +\infty$.

Note that we have proved in particular that

$$\exists T \geqslant 1\,, \quad \dot{r}(T) > 0 \quad \text{and} \quad \forall t \geqslant T\,, \quad r(t) \geqslant \dot{r}(T)(t-T) + 1\,. \tag{3.13}$$

3.2.2. *Second step: $r(t) \sim r_{\rm eff}(t)$ with a rough bound.*

Lemma 3.3. *There is T large enough so that defining $r_{\rm eff}$ the solution of* (3.11) *then as $t \to \infty$, there holds*

$$|r_{\rm eff}(t)| = 2t\sqrt{\lambda\alpha_0 \ln t} + \epsilon(t\sqrt{\ln t})\,, \quad \textit{and} \quad |r(t) - r_{\rm eff}(t)| \leqslant C(T)t\,, \quad \forall t \geqslant T\,,$$

where $\epsilon(t)/t$ goes to zero as t goes to infinity.

Proof. Let us start by studying $r_{\rm eff}$. Multiplying (3.11) by $\dot{r}_{\rm eff}$ and integrating, we get

$$\begin{aligned}(\dot{r}_{\rm eff}(t))^2 &= (\dot{r}(T))^2 + 4\lambda\alpha_0 \ln r_{\rm eff}(t) - 4\lambda\alpha_0 \ln r(T)\\ &= 4\lambda\alpha_0 \ln r_{\rm eff}(t) + \beta_0^2 + \alpha_0^2\left(1 - \frac{1}{r(T)^2}\right)\,,\end{aligned}$$

where we have used (3.10) at time $t = T$. Denote by

$$C_0 := \beta_0^2 + \alpha_0^2\left(1 - \frac{1}{r(T)^2}\right) \approx \beta_0^2 + \alpha_0^2 = |a_0|^2\,,$$

since $T \gg 1$. By similar arguments as in the proof of (3.13) in Paragraph 3.2.1, we have $\dot r_{\rm eff}(t) > 0$ for all $t \geqslant T$, and

$$r_{\rm eff}(t) \geqslant \dot r(T)(t-T)+1$$

hence

$$\dot r_{\rm eff}(t) = \sqrt{4\lambda\alpha_0 \ln r_{\rm eff}(t) + C_0}\,.$$

Separating the variables,

$$\frac{dr_{\rm eff}}{\sqrt{4\lambda\alpha_0 \ln r_{\rm eff} + C_0}} = dt\,,$$

so we naturally consider the anti-derivative

$$I := \int \frac{dr}{\sqrt{4\lambda\alpha_0 \ln r + C_0}}\,.$$

The change of variable

$$y := \sqrt{4\lambda\alpha_0 \ln r + C_0}$$

yields

$$I = \frac{1}{2\lambda\alpha_0}\int e^{(y^2-C_0)/(4\lambda\alpha_0)}dy\,.$$

Since for x large (Dawson function, see e.g. [1]),

$$\int e^{x^2}dx \sim \frac{1}{2x}e^{x^2}\,,$$

we infer

$$I \sim \frac{r}{\sqrt{4\lambda\alpha_0 \ln r + C_0}}\,.$$

In particular,

$$\frac{r_{\rm eff}(t)}{\sqrt{4\lambda\alpha_0 \ln r_{\rm eff}(t) + C_0}} \underset{t\to+\infty}{\sim} t\,,$$

hence

$$\frac{r_{\rm eff}(t)}{\sqrt{\ln r_{\rm eff}(t)}} \underset{t\to+\infty}{\sim} 2t\sqrt{\lambda\alpha_0}\,.$$

We conclude that

$$r_{\rm eff}(t) \underset{t\to+\infty}{\sim} 2t\sqrt{\lambda\alpha_0 \ln t}\,.$$

Now let us prove that r can be well approximated by $r_{\rm eff}$. We define $h := r - r_{\rm eff}$ and we want to prove that if T is chosen large enough, then $h(t) \lesssim t$ when $t \to \infty$. We have

$$\begin{aligned}\dot h(t) &= \sqrt{4\lambda\alpha_0 \ln r(t) + \beta_0^2 + \alpha_0^2\Big(1-\frac{1}{r(t)^2}\Big)}\\ &\quad - \sqrt{4\lambda\alpha_0 \ln r_{\rm eff}(t) + \beta_0^2 + \alpha_0^2\Big(1-\frac{1}{r(T)^2}\Big)}\\ &\leqslant \sqrt{4\lambda\alpha_0\Big|\ln\frac{r(t)}{r_{\rm eff}(t)}\Big| + \alpha_0^2\Big(\frac{1}{r(T)^2}-\frac{1}{r(t)^2}\Big)}\,.\end{aligned}$$

Given $\varepsilon \in (0,1/2)$, let $T \geqslant 1$ be large enough so that for all $t \geqslant T$

$$r_{\rm eff}(t) \geqslant t\sqrt{\lambda\alpha_0 \ln t} \tag{3.14}$$

and

$$\alpha_0^2\Big(\frac{1}{r(T)^2}-\frac{1}{r(t)^2}\Big)\leqslant \varepsilon^2\,. \tag{3.15}$$

We shall also need that

$$\left(2\frac{(\lambda\alpha_0)^{1/4}}{\sqrt{\ln T}}+\varepsilon\right)\leqslant\frac{1}{2}\,. \tag{3.16}$$

Then noticing that

$$\Big|\ln\frac{r(t)}{r_{\rm eff}(t)}\Big|=\Big|\ln\Big(1+\frac{h(t)}{r_{\rm eff}(t)}\Big)\Big|\leqslant\frac{|h(t)|}{r_{\rm eff}(t)}\leqslant\frac{|h(t)|}{t\sqrt{\lambda\alpha_0\ln t}}\leqslant\frac{|h(t)|}{t\sqrt{\lambda\alpha_0\ln T}}$$

as soon as $t\geqslant T$ thanks to (3.14), we infer that

$$\forall t\geqslant T\,,\quad \dot h(t)\leqslant\varepsilon+2\sqrt{\lambda\alpha_0}\left(\frac{|h(t)|}{t\sqrt{\lambda\alpha_0\ln T}}\right)^{1/2}\,,\quad \text{with}\quad h(T)=0\,.$$

Our goal is to prove that the function $t\mapsto h(t)/t$ is bounded for large t, so let $T^*>T$ be the maximal time such that

$$\forall t\in[T,T^*)\,,\quad |h(t)|\leqslant t\,.$$

Then for $t\in[T,T^*)$,

$$\dot h(t)\leqslant\varepsilon+2(\lambda\alpha_0)^{1/4}\frac{1}{\sqrt{\ln T}}$$

so thanks to (3.16)

$$h(t)\leqslant\Big(\varepsilon+2(\lambda\alpha_0)^{1/4}\frac{1}{\sqrt{\ln T}}\Big)(t-T)\leqslant\frac{t}{2}\,,$$

which contradicts the maximality of T^*. The result follows, and Lemma 3.3 is proved. □

3.2.3. *Third step: $r(t)\sim r_{\rm eff}(t)$ with improved bound.* Let us end the proof of Lemma 3.2. By (3.10) and as in the previous paragraph, we have for T sufficiently large so that $\dot r(t)\geqslant\dot r(T)>0$ for $t\geqslant T$:

$$\dot r=\sqrt{C_0+\alpha_0^2\left(\frac{1}{r(T)^2}-\frac{1}{r^2}\right)+4\lambda\alpha_0\ln r}\,,$$

with the same constant C_0 as above: recall that

$$\dot r_{\rm eff}=\sqrt{C_0+4\lambda\alpha_0\ln r_{\rm eff}}\,.$$

To lighten notation let us recall that $h:=r-r_{\rm eff}$ and let us define

$$R_{\rm eff}:=C_0+4\lambda\alpha_0\ln r_{\rm eff}\,.$$

Then using a Taylor expansion for $\dot r$, we have:

$$\begin{aligned}\dot r&=\sqrt{R_{\rm eff}+\alpha_0^2\big(\frac{1}{r(T)^2}-\frac{1}{r^2}\big)+4\lambda\alpha_0\ln\big(1+\frac{h}{r_{\rm eff}}\big)}\\&=\sqrt{R_{\rm eff}}\sqrt{1+\frac{1}{R_{\rm eff}}\alpha_0^2\big(\frac{1}{r(T)^2}-\frac{1}{r^2}\big)+4\frac{\lambda\alpha_0}{R_{\rm eff}}\ln\big(1+\frac{h}{r_{\rm eff}}\big)}\,.\end{aligned}$$

On the one hand we know that $R_{\rm eff} \to \infty$ and by Lemma 3.3 we have $h \lesssim t$ and $r_{\rm eff} \underset{t\to\infty}{\sim} t\sqrt{\ln t}$ so we infer that

$$\dot r \underset{t\to\infty}{\sim} \sqrt{R_{\rm eff}}\Big(1+\frac{1}{2R_{\rm eff}}\Big(\alpha_0^2\big(\frac{1}{r(T)^2}-\frac{1}{r^2}\big)+4\lambda\alpha_0\ln\big(1+\frac{h}{r_{\rm eff}}\big)\Big)\Big)\,.$$

As a consequence

$$\dot r-\dot r_{\rm eff} \underset{t\to\infty}{\sim} \frac{1}{2\sqrt{R_{\rm eff}}}\Big(\alpha_0^2\big(\frac{1}{r(T)^2}-\frac{1}{r^2}\big)+4\lambda\alpha_0\ln\big(1+\frac{h}{r_{\rm eff}}\big)\Big)$$

and since $h/r_{\rm eff}=\mathcal{O}(1/\sqrt{\ln t})$ we infer that

$$\dot r-\dot r_{\rm eff} \underset{t\to\infty}{\sim} \frac{C(T)}{\sqrt{\lambda\ln t}}\,.$$

By integration, and comparison of diverging integrals, we find

$$h(t) \underset{t\to\infty}{\sim} C_1\frac{t}{\sqrt{\ln t}}\,,$$

hence

$$r(t)=2t\sqrt{\lambda\alpha_0\ln t}\,\Big(1+\mathcal{O}\big(\ell(t)\big)\Big)\,,$$

as soon as we know that this holds for $r_{\rm eff}$. Lemma 3.2 is therefore proved, up to the study of the universal dispersion τ.

3.3. **Study of the universal dispersion $\tau(t)$: proof of Lemma 1.6.** It remains to prove Lemma 1.6. By scaling, we may assume $\lambda=1$, to lighten the notations. Introduce the approximate solution

$$\tau_{\rm eff}(t):=2t\sqrt{\ln t}.$$

We have clearly

$$\sqrt{\ln t}=\sqrt{\ln\tau_{\rm eff}}\left(1+\mathcal{O}\left(\frac{\ln\ln t}{\ln\tau_{\rm eff}}\right)\right)\,.$$

In view of a comparison with (1.3), which reads

$$\dot\tau=2\sqrt{\ln\tau}\,,$$

write

$$\dot\tau_{\rm eff}=2\sqrt{\ln t}+\frac{1}{\sqrt{\ln t}}=2\sqrt{\ln\tau_{\rm eff}}\left(1+\mathcal{O}\left(\frac{\ln\ln t}{\ln\tau_{\rm eff}}\right)\right)=2\sqrt{\ln\tau_{\rm eff}}+\mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right)\,.$$

Thus,

$$\begin{aligned}\dot\tau-\dot\tau_{\rm eff}&=2\left(\sqrt{\ln\tau}-\sqrt{\ln\tau_{\rm eff}}\right)+\mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right)\\&=2\sqrt{\ln\tau_{\rm eff}+\ln\frac{\tau}{\tau_{\rm eff}}}-2\sqrt{\ln\tau_{\rm eff}}+\mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right)\,.\end{aligned}$$

Since we already know from Lemma 3.3 that $\tau/\tau_{\rm eff}\to 1$, we obtain

$$\dot\tau-\dot\tau_{\rm eff}=\mathcal{O}\left(\frac{\ln\ln t}{\sqrt{\ln t}}\right)\,,\quad\text{and}\quad \tau-\tau_{\rm eff}=\mathcal{O}\left(t\frac{\ln\ln t}{\sqrt{\ln t}}\right)\,,$$

by integration. This proves Lemma 1.6. □

Back to the previous section, we simply note that

$$\dot r_{\rm eff} - \sqrt{\alpha_0}\dot\tau = \sqrt{C_0 + 4\lambda\alpha_0 \ln r_{\rm eff}} - \sqrt{4\lambda\alpha_0 \ln \tau}\,,$$

with $C_0 \neq 0$ in general, so the same computation as above yields

$$\dot r_{\rm eff} - \sqrt{\alpha_0}\dot\tau = \mathcal{O}\left(\frac{1}{\sqrt{\ln r_{\rm eff}}}\right) = \mathcal{O}\left(\frac{1}{\sqrt{\ln t}}\right),$$

hence

$$r_{\rm eff} - \sqrt{\alpha_0}\tau = \mathcal{O}\left(\frac{t}{\sqrt{\ln t}}\right),$$

by integration. This completes the proof of Lemma 3.2. □

4. General case: preparation for the proof of Theorem 1.7

In this section we prove (1.5) and (1.6) of Theorem 1.7.

4.1. **First a priori estimates.** Recall that by definition, v is related to u through the relation

$$u(t,x) = \frac{1}{\tau(t)^{d/2}} v\left(t, \frac{x}{\tau(t)}\right) \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}} \exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right), \tag{4.1}$$

where τ is the solution to

$$\ddot\tau = \frac{2\lambda}{\tau}\,, \quad \tau(0) = 1\,, \quad \dot\tau(0) = 0\,.$$

Then v solves

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta_y v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2 - \lambda d v \ln\tau + 2\lambda v \ln\left(\frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}}\right),$$

where we recall that $\gamma(y) = e^{-|y|^2/2}$, and the initial datum for v is

$$v_{|t=0} = v_0 := \frac{\|\gamma\|_{L^2(\mathbb{R}^d)}}{\|u_0\|_{L^2(\mathbb{R}^d)}} u_0.$$

Using the gauge transform consisting in replacing v with $ve^{-i\theta(t)}$ for

$$\theta(t) = \lambda d \int_0^t \ln\tau(s)ds - 2\lambda t \ln(\|u_0\|_{L^2}/\|\gamma\|_{L^2}),$$

we may assume that the last two terms are absent, and we focus our attention on

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta_y v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2, \quad v_{|t=0} = v_0\,. \tag{4.2}$$

We compute

$$\mathcal{E}(t) := \mathrm{Im}\int_{\mathbb{R}^d} \bar v(t,y)\partial_t v(t,y)dy = \mathcal{E}_{\rm kin}(t) + \lambda\mathcal{E}_{\rm ent}(t)\,,$$

where

$$\mathcal{E}_{\rm kin}(t) := \frac{1}{2\tau(t)^2}\|\nabla_y v(t)\|_{L^2}^2$$

is the (modified) kinetic energy and

$$\mathcal{E}_{\rm ent}(t) := \int_{\mathbb{R}^d} |v(t,y)|^2 \ln\left|\frac{v(t,y)}{\gamma(y)}\right|^2 dy$$

is a relative entropy. The transform (4.1) is unitary on $L^2(\mathbb{R}^d)$ so the conservation of mass for u trivially corresponds to the conservation of mass for v:

$$\|v(t)\|_{L^2} = \|v_0\|_{L^2} = \|\gamma\|_{L^2}\,. \tag{4.3}$$

The Csiszár-Kullback inequality reads (see e.g. [3, Th. 8.2.7]), for $f, g \geqslant 0$ with $\int f = \int g$,

$$\|f-g\|^2_{L^1(\mathbb{R}^d)} \leqslant 2\|f\|_{L^1(\mathbb{R}^d)} \int f(x) \ln\left(\frac{f(x)}{g(x)}\right) dx.$$

Thanks to (4.3), this inequality yields

$$\mathcal{E}_{\rm ent}(t) \geqslant \frac{1}{2\|\gamma^2\|_{L^1}} \left\| |v(t)|^2 - \gamma^2 \right\|^2_{L^1(\mathbb{R}^d)}\,, \tag{4.4}$$

hence in particular $\mathcal{E}_{\rm ent} \geqslant 0$. We easily compute

$$\dot{\mathcal{E}} = -2\frac{\dot\tau}{\tau}\mathcal{E}_{\rm kin}\,. \tag{4.5}$$

Ideally, we would like to prove directly $\mathcal{E}(t) \underset{t\to\infty}{\longrightarrow} 0$. The property $\mathcal{E}(t) \to 0$ can be understood as follows:

- $\mathcal{E}_{\rm kin} \to 0$ means that v oscillates in space more slowly than τ, hence that the main spatial oscillations of u have been taken into account in (4.1) (as a matter of fact, the boundedness of $\mathcal{E}_{\rm kin}$ suffices to reach this conclusion).
- $\mathcal{E}_{\rm ent} \to 0$ implies $|v(t)|^2 \to \gamma^2$ strongly in $L^1(\mathbb{R}^d)$.

It turns out than in the case of Gaussian initial data, we can infer from Section 3 that indeed $\mathcal{E}(t) \to 0$, each term going to zero logarithmically in time (see Section 6 for the case of $\mathcal{E}_{\rm ent}$). In the general case, we cannot reach this conclusion. Note however that if we had $\mathcal{E}_{\rm kin} \gtrsim 1$, then integrating (4.5) we would get $\mathcal{E}(t) \to -\infty$ as $t \to \infty$, hence a contradiction. Therefore,

$$\exists t_k \to \infty\,, \quad \mathcal{E}_{\rm kin}(t_k) \to 0\,.$$

We now prove the first part of Theorem 1.7, that is, (1.5) which is recast and complemented in the next lemma.

Lemma 4.1. *Under the assumptions of Theorem* 1.7*, there holds*

$$\sup_{t\geqslant 0}\left(\int_{\mathbb{R}^d} \left(1+|y|^2+\left|\ln|v(t,y)|^2\right|\right)|v(t,y)|^2 dy + \frac{1}{\tau(t)^2}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)}\right) < \infty$$

and

$$\int_0^\infty \frac{\dot\tau(t)}{\tau^3(t)}\|\nabla_y v(t)\|^2_{L^2(\mathbb{R}^d)} dt < \infty. \tag{4.6}$$

Proof. Write

$$\mathcal{E}_{\rm ent} = \int_{\mathbb{R}^d} |v|^2 \ln|v|^2 + \int_{\mathbb{R}^d} |y|^2|v|^2\,,$$

and

$$\int_{\mathbb{R}^d} |v|^2\ln|v|^2 = \int_{|v|>1} |v|^2\ln|v|^2 + \int_{|v|<1} |v|^2\ln|v|^2.$$

We have

$$\mathcal{E}_+ := \mathcal{E}_{\rm kin} + \lambda\int_{|v|>1}|v|^2\ln|v|^2 + \lambda\int_{\mathbb{R}^d}|y|^2|v|^2 \leqslant \mathcal{E}(0) + \lambda\int_{|v|<1}|v|^2\ln\frac{1}{|v|^2}\,\cdot$$

The last term is controlled by

$$\int_{|v|<1} |v|^2 \ln \frac{1}{|v|^2} \lesssim \int_{\mathbb{R}^d} |v|^{2-\epsilon}\,,$$

for all $\epsilon > 0$. We conclude thanks to the estimate (2.3) with $\alpha = 1$:

$$\int_{\mathbb{R}^d} |v|^{2-\epsilon} \lesssim \|v\|_{L^2}^{2-(1+d/2)\epsilon} \|yv\|_{L^2}^{d\epsilon/2}\,,$$

for $0 < \epsilon < 4/(d+2)$. This implies

$$\mathcal{E}_+ \lesssim 1 + \mathcal{E}_+^{d\epsilon/4}\,,$$

and thus $\mathcal{E}_+ \in L^\infty(\mathbb{R})$.

Finally, (4.6) follows from (4.5), since $\mathcal{E}(t) \geqslant 0$ for all $t \geqslant 0$. □

4.2. **Convergence of some quadratic quantities.** Let us prove (1.6), as stated in the next lemma.

Lemma 4.2. *Under the assumptions of Theorem* 1.7*, there holds*

$$\int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^d} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) dy\,.$$

Proof. Introduce

$$I_1(t) := \operatorname{Im} \int_{\mathbb{R}^d} \overline{v}(t,y) \nabla_y v(t,y) dy\,, \quad I_2(t) := \int_{\mathbb{R}^d} y |v(t,y)|^2 dy\,.$$

We compute:

$$\dot I_1 = -2\lambda I_2\,, \qquad \dot I_2 = \frac{1}{\tau^2(t)} I_1\,. \tag{4.7}$$

Set $\tilde I_2 := \tau I_2$: we have $\ddot{\tilde I}_2 = 0$, hence (unless the data are well prepared in the sense that $I_1(0) = 0$)

$$I_2(t) = \frac{1}{\tau(t)} \left(\dot{\tilde I}_2(0) t + \tilde I_2(0) \right) = \frac{1}{\tau(t)} \left(-I_1(0) t + I_2(0) \right) \underset{t\to\infty}{\sim} \frac{c}{\sqrt{\ln t}}\,,$$

and

$$I_1(t) \underset{t\to\infty}{\sim} \tilde c \frac{t}{\sqrt{\ln t}}\,.$$

In particular,

$$\int_{\mathbb{R}^d} y |v(t,y)|^2 dy \underset{t\to\infty}{\longrightarrow} 0 = \int_{\mathbb{R}^d} y \gamma(y)^2 dy\,.$$

In order to obtain estimates for higher order quadratic observables, we observe that Cauchy-Schwarz inequality and Lemma 4.1 yield

$$\left| \operatorname{Im} \int v(t,y) y \cdot \nabla_y \bar v(t,y) dy \right| \lesssim \tau(t).$$

We now go back to the conservation of energy for u,

$$\frac{d}{dt} \left(\frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \ln |u(t,x)|^2 dx \right) = 0\,.$$

and translate this property into estimates on v, recalling the mass conservation for v stated in (4.3).

$$\frac{d}{dt}\Bigg(\mathcal{E}_{\rm kin}+\frac{(\dot\tau)^2}{2}\int|y|^2|v|^2-\frac{\dot\tau}{\tau}\,{\rm Im}\int v(t,y)y\cdot\nabla_y\bar v(t,y)dy+\lambda\int|v|^2\ln|v|^2$$
$$-\lambda d\ln\tau\int|v|^2+2\lambda\|\gamma\|_{L^2}^2\ln\left(\frac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}}\right)\Bigg)=0\,.$$

In the above expression, all the terms are bounded functions of time, but possibly three: from the above estimate, the third term is $\mathcal{O}(\dot\tau)=\mathcal{O}(\sqrt{\ln t})$, while the terms

$$\frac{(\dot\tau)^2}{2}\int|y|^2|v|^2\quad\text{and}\quad-\lambda d\ln\tau\int|v|^2$$

are of the same order, $\mathcal{O}(\ln t)$. We infer

$$\frac{(\dot\tau)^2}{2}\int|y|^2|v|^2-\lambda d\ln\tau\int|v|^2=\mathcal{O}\left(\sqrt{\ln t}\right)\,.$$

Integrating (1.3), we find

$$\frac{(\dot\tau)^2}{2}=2\lambda\ln\tau\,,$$

hence

$$\int|y|^2|v|^2-\frac{d}{2}\|v\|_{L^2}^2=\mathcal{O}\left(\frac{1}{\sqrt{\ln t}}\right)\,,$$

and the lemma is proved. □

At this stage, we therefore have proved Theorem 1.7, up to the final point regarding the asymptotic profile for $|v|^2$.

5. End of the proof of Theorem 1.7

In this section we conclude the proof of Theorem 1.7 by obtaining the weak convergence to a universal profile as stated in (1.7).

5.1. **Hydrodynamical approach.** We recall that the Madelung transform is a classical tool (see e.g. [47, 44, 26], or the survey [12]) to relate the (nonlinear) Schrödinger equation to fluid dynamics equations, via the change of unknown

$$v(t,y)=a(t,y)e^{i\phi(t,y)}\,,\quad a,\phi\in\mathbb{R}\,. \tag{5.1}$$

Formally one obtains in our case the system of equations

$$\begin{cases}\partial_t\phi+\dfrac{1}{2\tau^2}|\nabla_y\phi|^2+\lambda\ln\left|\dfrac{a}{\gamma}\right|^2=\dfrac{1}{2\tau^2}\dfrac{\Delta_y a}{a}\\ \partial_t a+\dfrac{1}{\tau^2}\nabla_y\phi\cdot\nabla_y a+\dfrac{1}{2\tau^2}a\Delta_y\phi=0\,,\end{cases} \tag{5.2}$$

which is easily related to the compressible Euler equations by using the change of unknown

$$\rho:=a^2\,,\quad\Lambda:=a\nabla\phi\,,\quad J:=a\Lambda\,. \tag{5.3}$$

Note that in the explicit case of Gaussian initial data studied in Section 3, ρ is bounded in Sobolev spaces uniformly in time, whereas Λ and J are unbounded as $t \to \infty$. In terms of these hydrodynamical variables, the above system becomes

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2} \nabla \cdot J = 0 \\ \partial_t J + \dfrac{1}{\tau^2} \nabla \cdot (\Lambda \otimes \Lambda) + \lambda \nabla \rho + 2\lambda y \rho = \dfrac{1}{4\tau^2} \Delta \nabla \rho - \dfrac{1}{\tau^2} \nabla \cdot (\nabla \sqrt{\rho} \otimes \nabla \sqrt{\rho}) \\ \partial_j J^k - \partial_k J^j = 2\Lambda^k \partial_j \sqrt{\rho} - 2\Lambda^j \partial_k \sqrt{\rho}\,, \quad j,k \in \{1,\dots,d\}\,. \end{cases} \tag{5.4}$$

Note that in the case where the initial data for (5.4) are well prepared, in the sense that they stem from the polar decomposition of an initial wave function as in (5.1)–(5.3), then the approach from [4, 5] (see also [12, Section 5]) can readily be adapted to show that (5.4) holds true in the distributional sense, thanks to a suitable polar factorization technique. However, we will see below that the Madelung transform can be by-passed thanks to a more direct approach, where (5.1) is not invoked. We shall prove that

$$\rho(t) \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d)\,.$$

This will stem from the fact that the weak limit of ρ evolves according to a Fokker–Planck operator. We note that a formal link between the hydrodynamical formulation of (1.1) and the Fokker–Planck equation can be found in [46, 34].

5.2. **Heuristics.** Let us explain the heuristics of the proof, which will be made rigorous in the next section. Formally only retaining the higher order terms (in terms of growth in time) in (5.4) we are led to studying the following simple model

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2} \nabla \cdot J = 0 \\ \partial_t J + \lambda \nabla \rho + 2\lambda y \rho = 0\,. \end{cases} \tag{5.5}$$

Note that in the explicit case of the evolution of a Gaussian (recall the computations of Section 3), we can check that in the above simplification, we have indeed eliminated negligible terms. By elimination of J, (5.5) implies that

$$\partial_t \left(\tau^2 \partial_t \rho\right) = \lambda \nabla \cdot (\nabla + 2y)\, \rho = \lambda L \rho\,,$$

where

$$L := \Delta_y + \nabla_y \cdot (2y\,\cdot)$$

is a Fokker–Planck operator. On the other hand,

$$\partial_t \left(\tau^2 \partial_t \rho\right) = \tau^2 \partial_t^2 \rho + 2\dot\tau \tau \partial_t \rho\,,$$

so since $\tau^2 \ll (\dot\tau\tau)^2$, it is natural to change scales in time and define s such that

$$\frac{\dot\tau \tau}{\lambda} \partial_t = \partial_s\,,$$

or in other words define the following change of variables:

$$s = \int \frac{\lambda}{\dot\tau\tau} = \int \frac{\ddot\tau}{2\dot\tau} = \frac{1}{2} \ln \dot\tau(t)\,. \tag{5.6}$$

Notice that

$$s \sim \frac{1}{4} \ln\ln t\,, \quad t \to \infty\,. \tag{5.7}$$

Then again discarding formally lower order terms we find

$$\partial_s \rho = L\rho\,,$$

for which it is well-known (see for instance [24]) that in large times the solution converges strongly to an element of the kernel of L, hence a Gaussian. Notice that the convergence is exponentially fast in s variables, so returning to t variables produces a logarithmic decay due to (5.7): we recover the logarithmic convergence rate observed in the Gaussian case (Section 3).

The difficulty to make this argument rigorous is the justification that the lower order terms may indeed be discarded, since we have very little control on higher norms on v to guarantee compactness in space of the solution: we have more precisely a sharp control of the momenta of v, but rather poor estimates in H^1. More precisely, we do expect v to oscillate rapidly in time (in view of the Gaussian case), but $\sqrt{\rho}$ should be bounded in H^1, a property that does not seem easy to prove (because of the prefactor $1/\tau^2$ in the equation). This is the main obstacle to proving strong convergence to a Gaussian in the general case, and explains why in the end we only obtain a weak convergence result in L^1. This is made precise in the next section.

5.3. **End of the proof.** Let us follow the steps of the previous paragraph, this time neglecting no term. First, we rewrite (5.4) by using a more direct, and rigorous, derivation of a hydrodynamical system. Instead of the definition (5.3), relying on (5.1) (which in turn demands a rigorous justification of the polar decomposition), we set

$$\rho := |v|^2, \quad J := \mathrm{Im}\,(\bar{v}\nabla v)\,. \tag{5.8}$$

When (5.1) is available, this definition is equivalent to (5.3), but it has the advantage of being completely rigorous for free. We then have

$$\begin{cases} \partial_t \rho + \dfrac{1}{\tau^2}\nabla\cdot J = 0 \\ \partial_t J + \lambda\nabla\rho + 2\lambda y\rho = \dfrac{1}{4\tau^2}\Delta\nabla\rho - \dfrac{1}{\tau^2}\nabla\cdot \mathrm{Re}\,(\nabla v\otimes\nabla\bar{v})\ . \end{cases} \tag{5.9}$$

By elimination of J,

$$\partial_t\left(\tau^2\partial_t\rho\right) = -\partial_t\nabla\cdot J = \lambda L\rho - \frac{1}{4\tau^2}\Delta^2\rho - \frac{1}{\tau^2}\nabla\cdot\left(\nabla\cdot\mathrm{Re}\,(\nabla v\otimes\nabla\bar{v})\right)\,,$$

with again $L := \Delta + \nabla\cdot(2y\,\cdot)$. With the change of variable (5.6) we introduce the notation $\tilde{\rho}(s(t),y) := \rho(t,y)$, and we find for $\tilde{\rho}$ the following equation:

$$\partial_s\tilde{\rho} - \frac{2\lambda}{(\dot{\tau})^2}\partial_s\tilde{\rho} + \frac{\lambda}{(\dot{\tau})^2}\partial_s^2\tilde{\rho} = L\tilde{\rho} - \frac{1}{4\lambda\tau^2}\Delta^2\tilde{\rho} - \frac{1}{\lambda\tau^2}\nabla\cdot\left(\nabla\cdot\mathrm{Re}\,(\nabla\tilde{v}\otimes\nabla\bar{\tilde{v}})\right)\,, \tag{5.10}$$

where one should keep in mind that the functions τ and $\dot{\tau}$ also have undergone the change of time variable. In terms of s, Lemma 1.6 yields

$$\dot{\tau}(s) \underset{s\to\infty}{\sim} 2\sqrt{\lambda}e^{2s}\,, \quad \tau(s) \underset{s\to\infty}{\sim} 2\sqrt{\lambda}e^{2s+e^{4s}}\,.$$

To make the discussion at the end of the previous subsection more precise, and explain why we prove a weak convergence only, we comment on the various terms in (5.10):

- The term $\frac{2\lambda}{(\dot{\tau})^2}\partial_s\tilde{\rho}$ is essentially harmless in the large time limit, for it could be handled by a slight modification of the time variable, for instance.

- The term $\frac{\lambda}{(\dot\tau)^2}\partial_s^2\tilde\rho$ is expected to be negligible in the large time limit. However, it makes (5.10) second order in time: one would like to take advantage of the smoothing properties of e^{sL}, by using Duhamel's formula typically, but this approach is delicate in this context. Note that it has been established before that in similar situations, the parabolic behavior gives the leading order large time dynamics, even if the coefficient of $\partial_s^2\tilde\rho$ is not asymptotically vanishing ([22]): the proof of this fact relies on energy estimates whose analogue in the case of (5.10) we could not establish.
- By using the Fourier transform in space, it is easy to compute the fundamental solution of
$$\partial_s\tilde\rho = L\tilde\rho - \frac{1}{4\lambda\tau^2}\Delta^2\tilde\rho\,,$$
in the same way as [23] (without the last term).
- A possible idea to prove that the solution to (5.10) converges (strongly) to the Gaussian γ^2 as $s\to\infty$ would be to use the spectral decomposition of L, as given for instance in [23]. The main issue is then that we can control the last term in (5.10) in L^1-based spaces, as opposed to L^2 where the spectral decomposition is available. Note that this term is the only one that prevents (5.10) from being a linear, homogeneous, equation.

In terms of s, the time integrability property of $\mathcal{E}_{\rm kin}$ provided in (4.6) becomes
$$\int_0^\infty \left(\frac{\dot\tau(s)}{\tau(s)}\right)^2 \|\nabla\tilde v(s)\|_{L^2}^2 ds < \infty\,. \tag{5.11}$$
On the other hand, Lemma 4.1 yields
$$\sup_{s\geqslant 0}\int_{\mathbb{R}^d}\tilde\rho(s,y)\left(1+|y|^2+|\ln\tilde\rho(s,y)|\right)dy<\infty\,. \tag{5.12}$$
Mimicking the general approach of e.g. [20, 21], for $s\in[-1,2]$ and $s_n\to\infty$, set
$$\tilde\rho_n(s,y) := \tilde\rho(s+s_n,y)\,.$$
From (5.12) along with the de la Vallée-Poussin and Dunford–Pettis Theorems, we get up to extracting a subsequence
$$\tilde\rho_n \rightharpoonup \tilde\rho_\infty \quad \text{in } L^p_s(-1,2;L^1_y)\,,$$
for all $p\in[1,\infty)$. Up to another subsequence,
$$\tilde\rho_n(0)\rightharpoonup\tilde\rho_{0,\infty}\quad\text{in } L^1_y\,.$$
Moreover thanks to (5.12) the family $(\tilde\rho(s_n,\cdot))_n$ is tight and hence
$$\int\tilde\rho_{0,\infty}(y)\,dy = \int\gamma(y)^2\,dy\,,$$
and there holds also
$$\int_{\mathbb{R}^d}\tilde\rho_{0,\infty}(y)\left(1+|y|^2+|\ln\tilde\rho_{0,\infty}(y)|\right)dy<\infty\,.$$
Property (5.11) implies that
$$\frac{1}{\lambda\tau_n^2}\nabla\cdot(\nabla\cdot\mathrm{Re}\,(\nabla\tilde v_n\otimes\nabla\bar{\tilde v}_n))\rightharpoonup 0 \text{ in } L^1_sW^{-2,1},\quad \text{where } \tau_n(s)=\tau(s+s_n).$$

In addition, in (5.10), all the other terms but two obviously go weakly to zero, which yields

$$\begin{cases} \partial_s \tilde{\rho}_\infty = L\tilde{\rho}_\infty \text{ in } \mathcal{D}'\left((-1,2)\times\mathbb{R}^d\right), \\ \tilde{\rho}_{\infty|s=0} = \tilde{\rho}_{0,\infty} \in L^1 . \end{cases} \tag{5.13}$$

Thanks to the above bounds on $\tilde{\rho}_{0,\infty}$ it is known (see for instance [7, Corollary 2.17]) that the solution $\tilde{\rho}_\infty$ to (5.13) is actually defined for all $s \geqslant 0$ and satisfies

$$\lim_{s\to\infty} \|\tilde{\rho}_\infty(s) - \gamma^2\|_{L^1(\mathbb{R}^d)} = 0 . \tag{5.14}$$

We now go back to (5.9) and show that $\tilde{\rho}_\infty$ is independent of s. In the s variable, we have

$$\begin{cases} \partial_s \tilde{\rho} + \dfrac{\dot{\tau}}{\lambda\tau}\nabla\cdot\tilde{J} = 0 \\ \partial_s \tilde{J} + \tau\dot{\tau}\left(\nabla + 2y\right)\tilde{\rho} - \dfrac{\dot{\tau}}{4\lambda\tau}\nabla\Delta\tilde{\rho} = -\dfrac{\dot{\tau}}{\lambda\tau}\nabla\cdot\mathrm{Re}\left(\nabla\tilde{v}\otimes\nabla\bar{\tilde{v}}\right) . \end{cases} \tag{5.15}$$

Since $J = \mathrm{Im}\,\bar{v}\nabla_y v$, (5.11) implies

$$\frac{\dot{\tau}}{\tau}\tilde{J} \in L^2_s L^1_y .$$

With $\tilde{J}_n(s) := \tilde{J}(s+s_n)$, we have

$$\frac{\dot{\tau}}{\tau}\nabla\cdot\tilde{J}_n \underset{n\to\infty}{\longrightarrow} 0 \quad \text{in } L^2(-1,2;W^{-1,1}) ,$$

hence

$$\partial_s\tilde{\rho}_\infty = 0 . \tag{5.16}$$

Putting (5.14) and (5.16) together, we infer that $\tilde{\rho}_\infty = \gamma^2$. The limit being unique, no extraction of a subsequence is needed, and we conclude

$$\tilde{\rho}(s) \underset{s\to\infty}{\rightharpoonup} \gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^d) .$$

Theorem 1.7 is proved. □

6. Proof of the corollaries

6.1. **Proof of Corollary 1.10.** In the energy for u, write the potential energy in terms of v.

$$\begin{aligned} \int_{\mathbb{R}^d} |u(t,x)|^2 \ln|u(t,x)|^2 dx &= -d\ln\tau(t)\frac{\|u_0\|_{L^2}^2}{\|\gamma\|_{L^2}^2}\int_{\mathbb{R}^d}|v(t,y)|^2 dy \\ &\quad + \frac{\|u_0\|_{L^2}^2}{\|\gamma\|_{L^2}^2}\int_{\mathbb{R}^d}|v(t,y)|^2\ln\left(\frac{\|u_0\|_{L^2}^2}{\|\gamma\|_{L^2}^2}|v(t,y)|^2\right)dy \\ &= -d\|u_0\|_{L^2}^2\ln\tau(t) + \mathcal{O}(1), \end{aligned}$$

from (1.5). Therefore, the conservation of the energy for u yields

$$\|\nabla u(t)\|_{L^2}^2 = 2E_0 + 2\lambda d\|u_0\|_{L^2}^2\ln\tau(t) + \mathcal{O}(1) \underset{t\to\infty}{\sim} 2\lambda d\|u_0\|_{L^2}^2\ln t,$$

hence the first point of Corollary 1.10. Now fix $0 < s < 1$. By interpolation, we readily have

$$\|u(t)\|_{\dot{H}^s} \lesssim \|u(t)\|_{L^2}^{1-s}\|u(t)\|_{\dot{H}^1}^s \lesssim (\ln t)^{s/2} ,$$

where we have used the conservation of mass and the above asymptotics.

The convergence in the Wasserstein distance W_2 (Remark 1.9) implies (see e.g. [54, Theorem 7.12])

$$\int |y|^{2s}|v(t,y)|^2dy \underset{t\to\infty}{\longrightarrow} \int |y|^{2s}\gamma^2(y)dy. \tag{6.1}$$

The idea is then to apply a fractional derivative to (1.4), that is

$$u(t,x)=\frac{1}{\tau(t)^{d/2}}v\left(t,\frac{x}{\tau(t)}\right)\frac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}}\exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right).$$

In order to shortcut this step, we recall a lemma employed in a somehow similar situation, even though in the context of semi-classical limit. We therefore simplify the initial statement and leave out the dependence on the semi-classical parameter:

Lemma 6.1 (Lemma 5.1 from [2])**.** *There exists C such that if $u\in H^1(\mathbb{R}^d)$ and w is such that $\nabla w\in L^\infty(\mathbb{R}^d)$,*

$$\||w|^su\|_{L^2}\leqslant \|u\|_{\dot H^s}+\|(\nabla-iw)u\|_{L^2}^s\|u\|_{L^2}^{1-s}+C\left(1+\|\nabla w\|_{L^\infty}\right)\|u\|_{L^2}.$$

In [2], w corresponds to the gradient of rapid oscillations carried by an exponential, so we naturally introduce

$$w(t,x)=\frac{\dot\tau(t)}{\tau(t)}x.$$

In the present framework, Lemma 6.1 yields:

$$(\dot\tau)^s\||y|^sv(t)\|_{L^2}\frac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}}\leqslant \|u(t)\|_{\dot H^s}+\left\|\frac{1}{\tau}\nabla v(t)\right\|_{L^2}^s\frac{\|u_0\|_{L^2}}{\|\gamma\|_{L^2}^s}+C\left(1+\frac{\dot\tau}{\tau}\right)\|u_0\|_{L^2}.$$

The result follows readily: the behavior of the left hand side is given by Lemma 1.6 and (6.1), and all the terms of the right hand side are bounded, but the first one.

6.2. **Proof of Corollary 1.11.** In view of the tensorization in Theorem 3.1, we prove Corollary 1.11 in the case $d=1$ to lighten the notations, and we assume

$$u_0(x)=b_0\exp\left(-a_0(x-x_0)^2/2\right),$$

with $b_0,a_0\in\mathbb{C}$, $\operatorname{Re}a_0=\alpha_0>0$. We start with an initial center x_0 to show that in terms of v, the center is eventually zero (like in [24]). Recall that we have

$$u(t,x)=b_0\frac{1}{\sqrt{r(t)}}e^{i\phi(t)}\exp\left(-\alpha_0\frac{(x-x_0)^2}{2r^2(t)}+i\frac{\dot r(t)}{r(t)}\frac{(x-x_0)^2}{2}\right),$$

with r solution to (3.9), $r(0)=1$, $\dot r(0)=-\operatorname{Im}a_0$. We thus have, since

$$\|u_0\|_{L^2}=|b_0|\left(\frac{\pi}{\alpha_0}\right)^{1/4}\quad\text{and}\quad\|\gamma\|_{L^2}=\pi^{1/4},$$

$$\begin{aligned}v(t,y)=\frac{b_0}{|b_0|}\alpha_0^{1/4}\sqrt{\frac{\tau(t)}{r(t)}}e^{i\phi(t)}\exp\left(-\alpha_0\frac{\tau^2}{r^2}\frac{y^2}{2}+\alpha_0\frac{\tau}{r^2}yx_0-\alpha_0\frac{x_0^2}{2r^2}\right)\\ \times\exp\left(i\left(\frac{\dot r}{r}-\frac{\dot\tau}{\tau}\right)\tau^2\frac{y^2}{2}-i\frac{\dot r}{r}\tau yx_0+i\frac{\dot r}{r}\frac{x_0^2}{2}\right).\end{aligned}$$

In particular,

$$|v(t,y)|^2=\frac{\tau(t)}{r(t)}\sqrt{\alpha_0}\exp\left(-\alpha_0\frac{\tau^2}{r^2}y^2+2\alpha_0\frac{\tau}{r^2}yx_0-\alpha_0\frac{x_0^2}{r^2}\right).$$

Therefore, the relative entropy is

$$\begin{aligned}\mathcal{E}_{\rm ent}(t) &= \int_{\mathbb{R}} |v(t,y)|^2 \ln\left(\frac{|v(t,y)|^2}{\gamma^2(y)}\right) dy \\ &= \ln\left(\sqrt{\alpha_0}\frac{\tau(t)}{r(t)}\right)\|v_0\|_{L^2}^2 - \left(\alpha_0\frac{\tau(t)^2}{r(t)^2}-1\right)\int_{\mathbb{R}} y^2|v(t,y)|^2dy \\ &\quad + 2\alpha_0 x_0\frac{\tau(t)}{r^2(t)}\int_{\mathbb{R}} y|v(t,y)|^2dy - \alpha_0\frac{x_0^2}{r^2(t)}\|v_0\|_{L^2}^2 \underset{t\to\infty}{\longrightarrow} 0\,,\end{aligned}$$

where we have used the properties of the solutions to (3.9) and (1.3), established in Section 3. The end of the corollary simply stems from the Csiszár-Kullback inequality (4.4).

7. Stability of the dynamics under a power-like perturbation

In this section, we give the main arguments to adapt the previous proofs in the case of (1.13),

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda u\ln(|u|^2) + \mu|u|^{2\sigma}u\,, \quad u_{|t=0} = u_0\,.$$

7.1. **Construction of the solution.** To show that (1.13) has a solution $u \in L^\infty_{\rm loc}(\mathbb{R};\Sigma)$ whose mass, angular momentum and energy as conserved, we can follow exactly the same strategy as in Subsection 2.1, and consider the family of functions u_ε solution to

$$i\partial_t u_\varepsilon + \frac{1}{2}\Delta u_\varepsilon = \lambda u_\varepsilon\ln(\varepsilon+|u_\varepsilon|^2) + \mu|u_\varepsilon|^{2\sigma}u_\varepsilon, \quad u_{\varepsilon|t=0} = u_0.$$

For fixed $\varepsilon>0$, the above equation has a unique, local, solution $u\in C([-T,T];\Sigma)\cap L^{\frac{4\sigma+4}{d\sigma}}([-T,T];L^{2\sigma+2})$, thanks to Strichartz estimates (see e.g. [15, 30]). To see that the solution is global in time for fixed $\varepsilon>0$ and satisfies estimates which are uniform in ε, we notice that the momentum $I_{\varepsilon,1} = \int \langle x\rangle^2 |u_\varepsilon(t,x)|^2dx$ is controlled exactly like in Subsection 2.1. On the other hand, we compute

$$\begin{aligned}&\frac{d}{dt}\left(\frac{1}{2}\|\nabla u_\varepsilon\|_{L^2}^2 + \lambda\int |u_\varepsilon|^2\ln\left(\varepsilon+|u_\varepsilon|^2\right) + \frac{\mu}{\sigma+1}\int |u_\varepsilon|^{2\sigma+2}\right) \\ &\qquad = \lambda\int \frac{|u_\varepsilon|^2}{\varepsilon+|u_\varepsilon|^2}\partial_t\left(|u_\varepsilon|^2\right),\end{aligned}$$

and since $\partial_t\left(|u_\varepsilon|^2\right) = -\operatorname{Im}(\bar u\Delta u)$, Gronwall's lemma yields estimates which are uniform in $\varepsilon>0$, as desired, and we conclude like in Subsection 2.1.

We note that the uniqueness argument used in the case $\mu=0$ can be repeated provided that $u(t,\cdot)\in L^\infty(\mathbb{R}^d)$, which is granted if $d=1$ by Sobolev embedding. We cannot conclude "as usual" thanks to Strichartz estimates (case $\lambda=0$), since the logarithmic nonlinearity is not Lipschitz, and so uniqueness is not clear for $d\geqslant 2$.

7.2. **Estimates for v.** We note that in the presence of a power-like nonlinearity, we can no longer rely on explicit computations in the case of Gaussian initial data. This is not a problem, since the main role of the explicit computations was to suggest the change of unknown function

$$u(t,x) = \frac{1}{\tau(t)^{d/2}}v\left(t,\frac{x}{\tau(t)}\right)\frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}}\exp\left(i\frac{\dot\tau(t)}{\tau(t)}\frac{|x|^2}{2}\right).$$

Now v solves

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta_y v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2 - \lambda d v \ln\tau + 2\lambda v \ln\left(\frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}}\right) + \frac{\tilde\mu}{\tau^{d\sigma}}|v|^{2\sigma}v\,,$$

with

$$\tilde\mu := \left(\frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{\|\gamma\|_{L^2(\mathbb{R}^d)}}\right)^{2\sigma}\mu\,.$$

We simply note that $\tilde\mu > 0$, and the numerical value of μ is unimportant. Like before, the previous two terms can be absorbed by a gauge transform, and we consider

$$i\partial_t v + \frac{1}{2\tau(t)^2}\Delta_y v = \lambda v \ln\left|\frac{v}{\gamma}\right|^2 + \frac{\tilde\mu}{\tau^{d\sigma}}|v|^{2\sigma}v\,, \quad v_{|t=0} = v_0\,.$$

Noticing that the pseudo-energy

$$\mathcal{E}(t) = \frac{1}{2\tau(t)^2}\|\nabla_y v(t)\|_{L^2}^2 + \lambda\int |v|^2 \ln\left|\frac{v(t,y)}{\gamma(y)}\right|^2 dy + \frac{\tilde\mu}{(\sigma+1)\tau(t)^{d\sigma}}\int |v(t,y)|^{2\sigma+2}dy$$

satisfies

$$\dot{\mathcal{E}} = -2\frac{\dot\tau}{\tau}\times\frac{1}{2\tau(t)^2}\|\nabla_y v(t)\|_{L^2}^2 - d\sigma\frac{\dot\tau}{\tau}\times\frac{\tilde\mu}{(\sigma+1)\tau(t)^{d\sigma}}\|v(t)\|_{L^{2\sigma+2}}^{2\sigma+2}\,,$$

Lemma 4.1 is readily generalized to this case, with in addition

$$\sup_{t\geqslant 0}\frac{1}{\tau(t)^{d\sigma}}\|v(t)\|_{L^{2\sigma+2}}^{2\sigma+2} < \infty\,,$$

and property (4.6) is enriched with the extra estimate

$$\int_0^\infty \frac{\dot\tau(t)}{\tau(t)^{d\sigma+1}}\|v(t)\|_{L^{2\sigma+2}}^{2\sigma+2}dt < \infty\,. \tag{7.1}$$

The convergence of the momenta (Lemma 4.2) is adapted in a straightforward manner as well: the system (4.7) which yields the convergence of the first momentum remains the same, and the convergence of the momentum of order two follows from the same argument.

7.3. **Hydrodynamics.** To prove the weak convergence of $|v|^2$ to γ^2, we mimic the proof presented in Section 5. Essentially, one new term appears in (5.9),

$$\begin{cases}\partial_t\rho + \dfrac{1}{\tau^2}\nabla\cdot J = 0\\ \partial_t J + \lambda\nabla\rho + 2\lambda y\rho = \dfrac{1}{4\tau^2}\Delta\nabla\rho - \dfrac{1}{\tau^2} = \nabla\cdot\mathrm{Re}(\nabla v\otimes\nabla\bar v) = -\dfrac{\tilde\mu}{\tau^{d\sigma}}\dfrac{\sigma}{\sigma+1}\nabla\left(\rho^{\sigma+1}\right)\,.\end{cases}$$

Working with the time variable s defined in (5.6), the last two terms vanish in the large time weak limit: the term in $|\nabla v|^2$ thanks to (4.6) (like before), and the last term thanks to (7.1) (in the same fashion).

7.4. **Growth of Sobolev norms.** To prove the final part of Theorem 1.12, we can resume exactly the same arguments as in the proof of Corollary 1.10.

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CNRS, IMAG, Montpellier, France
E-mail address: Remi.Carles@math.cnrs.fr

Département de mathématiques et applications, École normale supérieure, CNRS, PSL Research University, 75005 Paris, France, & UFR de mathématiques, Université Paris-Diderot, Sorbonne Paris-Cité, 75013 Paris, France
E-mail address: Isabelle.Gallagher@ens.fr