HAL
open science

Tian's invariant of the Grassmann manifold

Julien Grivaux

To cite this version:

Julien Grivaux. Tian's invariant of the Grassmann manifold. Journal of Geometric Analysis, 2006, 16, pp.523 - 533. 10.1007/BF02922065 . hal-01398457

HAL Id: hal-01398457

https://hal.science/hal-01398457

Submitted on 17 Nov 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

TIAN'S INVARIANT OF THE GRASSMANN MANIFOLD

JULIEN GRIVAUX

Abstract. — We prove that Tian's invariant on the complex Grassmann manifold $G_{p,q}(\mathbb{C})$ is equal to $1/(p+q)$. The method introduced here uses a Lie group of holomorphic isometries which operates transitively on the considered manifolds and a natural imbedding of $\big(\mathbb{P}^1(\mathbb{C})\big)^p$ in $G_{p,q}(\mathbb{C})$.

Résumé. — On prouve que l'invariant de Tian sur la grassmannienne $G_{p,q}(\mathbb{C})$ est $1/(p+q)$. La méthode présentée dans cet article utilise un groupe de Lie d'isométries holomorphes qui opère transitivement sur les variétés considérées ainsi qu'un plongement naturel de $\big(\mathbb{P}^1(\mathbb{C})\big)^p$ dans $G_{p,q}(\mathbb{C})$.

1. Introduction

On a complex manifold, an hermitian metric h is characterized by the *1-1 symplectic* form ω defined by $\omega = i\, g_{\lambda\bar{\mu}}\, dz^\lambda \wedge d\bar{z}^\mu$, where $g_{\lambda\bar{\mu}} = h_{\lambda\bar{\mu}}/2$.
The metric is a *Kähler metric* if ω is closed, i. e. $d\omega = 0$; then M is a *Kähler manifold.*
On a Kähler manifold, we can define the *Ricci form* by $R = i\, R_{\lambda\bar{\mu}}\, dz^\lambda \wedge d\bar{z}^\mu$, where $R_{\lambda\bar{\mu}} = -\partial_{\lambda\bar{\mu}} \log|g|$.
A Kähler manifold is *Einstein with factor* k if $R = k\omega$. For instance, choosing a local coordinate system $Z = (z_1, \dots, z_m)$, the projective space $\mathbb{P}_m(\mathbb{C})$ with the Fubini-Study metric $\omega = i\partial\bar{\partial}\log\big(1 + ||Z||^2\big)$ is Einstein with factor $m+1$.
On a Kähler manifold M, the *first Chern class* $C^1(M)$ is the cohomology class of the Ricci tensor, that is the set of the forms $R + i\partial\bar{\partial}\varphi$, where φ is C^∞ on M. If there is a form in $C^1(M)$ which is positive (resp. negative, zero), then $C^1(M)$ is *positive* (resp. *negative*, *zero*). If a Kähler manifold is Einstein, then $C^1(M)$ and k are both positive (resp. negative, zero). In the negative case, it was proved by Aubin ([Au1], see also [Au4]), that there exists a unique Einstein-Kähler metric (E.K. metric) on M . It is so for the zero case too ([Au1], [Ya]). The question for the positive case is still open: some manifolds, such as the complex projective space blown up at one point, do not admit an E.K. metric (for obstructions, see [Li] and [Fu]). Aubin [Au2] and Tian [Ti] have shown that for suitable values of holomorphic invariants of the metric, there exists an E.K. metric on M.

1991 *Mathematics Subject Classification.* 53C55, 32M10.

Key words and phrases. Kähler manifold, Einstein-Kähler metric, first Chern class, admissible functions, Tian's invariant, grassmannian.

For $\omega/2\pi$ in $C^1(M)$, *Tian's invariant* $\alpha(M)$ is the supremum of the set of the real numbers α satisfying the following: there exists a constant C such that the inequality $\int_M e^{-\alpha\varphi} \leq C$ holds for all the C^∞ functions φ with $\omega + i\partial\bar{\partial}\varphi > 0$ and $\sup\varphi \geq 0$, where $\omega = i\, g_{\lambda\bar{\mu}}\, dz^\lambda \wedge d\bar{z}^\mu$ is the metric form. Such functions φ are said *ω-admissible*.
In [Ti], Tian established that if $\alpha(M) > m/(m+1)$, m being the dimension of M, there exists an E.K. metric on M. This condition is not necessary: it does not hold on the projective space, where Tian's invariant is $1/(m+1)$.
In the same paper, Tian introduces a more restrictive invariant $\alpha_G(M)$, considering only the admissible functions φ invariant by the action of a compact group G of holomorphic isometries. The sufficient condition for the existence of an E.K. metric on M remains $\alpha_G(M) > m/(m+1)$; it is more easily satisfied if the group G is rich enough.
In many cases, the group G is a non-discrete Lie group. The invariant $\alpha_G(M)$ can be computed using subharmonic functions methods and the maximum principle (for effective examples, see [Be1], [Be2], [Be-Ch1], [Be-Ch2], [Re]).
In this paper, we prove the following theorem:

Theorem 1.1. *Tian's invariant on $G_{p,q}(\mathbb{C})$ is given by $\alpha\left(G_{p,q}(\mathbb{C})\right) = 1/(p+q)$.*

This generalizes the known result on $\mathbb{P}^m(\mathbb{C})$ ([Ti], see also [Au3]). Let us also mention that Tian's invariant has been computed on $\mathbb{P}^m(\mathbb{C})$ blown up at one point and on certain Fermat hypersurfaces using Hörmander L^2 estimates for the $\overline{\partial}$-equation ([Ti]).
We first compute the volume element of the metric $\mathscr{G}_{p,q}$; then we will establish some general preliminary results concerning Tian's invariant as well as imbeddings of $\left\{\mathbb{P}^1(\mathbb{C})\right\}^p$ in $G_{p,q}(\mathbb{C})$ which allow us to deduce $\alpha\left(G_{p,q}(\mathbb{C})\right)$ from $\alpha\left(\mathbb{P}^1(\mathbb{C})\right)$.

2. Basic properties of the Grassmann manifold

We propose here a short survey of the properties of Grassmann manifold (for more details, see [Ko-No]). We denote by $G_{p,q}(\mathbb{C})$ the set of the subspaces of dimension p in $\mathbb{C}^{p+q}$; in particular, $G_{1,m}(\mathbb{C})$ is the complex projective space of dimension m. It is known (see [Au3]) that on $\mathbb{P}_m(\mathbb{C})$, the Fubini-Study metric is Einstein with factor $m+1$ and that Tian's invariant is $1/(m+1)$. Now, let $M^*(p+q,p)$ be the set of the matrices of rank p in $M_{p+q,p}(\mathbb{C})$. The group $Gl_p(\mathbb{C})$ acts by multiplication on the right on $M^*(p+q,p)$. More precisely $\left(M^*(p+q,p),\ \pi,\ G_{p,q}(\mathbb{C})\right)$ is a principal fiber bundle with group $Gl_p(\mathbb{C})$. The group $Gl_{p+q}(\mathbb{C})$ acts by multiplication on the left on $M^*(p+q,p)$ and induces an action on $G_{p,q}(\mathbb{C})$; so does the unitary group $U(p+q)$. These groups act transitively on $G_{p,q}(\mathbb{C})$, which shows that $G_{p,q}(\mathbb{C})$ is compact.

We denote by $\mathcal{I}$ the set of all increasing-ordered subsets of p elements in $\{1,\dots,p+q\}$. Let P be an element of $M^*(p+q,p)$, $P=\big(p_{ij}\big)_{\substack{1\le i\le p+q\\ 1\le j\le p}}$. By Cauchy-Binet formula we get: $\det\big({}^tP\overline{P}\big)=\sum_{I\in\mathcal{I}}|\det m_{_I}(P)|^2$, where $m_I(P)$ is the matrix $\big(p_{ij}\big)_{\substack{i\in I\\ 1\le j\le p}}$. The form ω, where $\omega=i\,\partial\overline{\partial}\log\det\big({}^tP\overline{P}\big)$, is invariant by the action of $Gl_p(\mathbb{C})$ on $M^*(p+q,p)$, and so it projects onto a form $\mathscr{G}_{p,q}$. The metric $\mathscr{G}_{p,q}$ is a Kähler metric form on $G_{p,q}(\mathbb{C})$. For $p=1$, this metric on $G_{1,m}(\mathbb{C})$ is the Fubini-Study metric on the complex projective space. The action of the unitary group $U(p+q)$ on $G_{p,q}(\mathbb{C})$ preserves the metric $\mathscr{G}_{p,q}$ so that $U(p+q)$ is a group of holomorphic isometries which operates transitively on $G_{p,q}(\mathbb{C})$. For I in $\mathcal{I}$, let U_I be the set of the matrices P in $M^*(p+q,p)$ such that $\det(m_I(P))$ is non-zero. Then $\pi(U_I)$ is a coordinate open set on $G_{p,q}(\mathbb{C})$, the matrix Z_I in $M_{q,p}(\mathbb{C})$ is the coordinate, the inverse of the chart φ_I sends $M^*(p+q,p)$ onto $\pi(U_I)$ and we have $m_I\big(\varphi_I^{-1}(Z_I)\big)=I^{(p)}$ where $I^{(p)}$ is the $p\times p$ identity matrix, and $m_{I^c}\big(\varphi_I^{-1}(Z_I)\big)=Z_I$.

Lemma 2.1. *For I in $\mathcal{I}$, let λ_I be the map from $\pi(U_I)$ to $\mathbb{R}_+$ defined by*

$$\lambda_I(Z_I)=\big|\det(\mathrm{Id}+{}^tZ_I\overline{Z}_I)\big|^{-(p+q)}.$$

Then $\big(\lambda_I\big)_{I\in\mathcal{I}}$ are the components of a maximal differential form η on $G_{p,q}(\mathbb{C})$, namely:

$$\eta=\lambda_I\big(i/2\big)^{pq}\big(dZ\wedge d\overline{Z}\big)_I.$$

Proof. It suffices to show that the following transformation rule holds:

$$\text{for every } I,\ \widetilde{I} \text{ in } \mathcal{I},\ \lambda_I \text{ is equal to } \lambda_{\widetilde{I}}\times\left|\det\ \frac{\partial\,Z_{\widetilde{I}}}{\partial\,Z_I}\right|^2 \text{ on } \pi(U_I)\cap\pi(U_{\widetilde{I}}).$$

Let P_I be the matrix $\varphi_I^{-1}(Z_I)$. Then $P_I\big\{m_{\widetilde{I}}(P_I)\big\}^{-1}=P_{\widetilde{I}}$, so $Z_{\widetilde{I}}=m_{\,\widetilde{I}^c}(P_I)\,\big\{m_{\widetilde{I}}(P_I)\big\}^{-1}$. The differential of the map which sends Z_I on P_I is the map which sends H on $\breve{H}$, where $m_{\,I^c}(\breve{H})=H$ and $m_I(\breve{H})=0$. The change of charts sending Z_I on $Z_{\widetilde{I}}$, we obtain

$$\begin{aligned}D\,Z_{\widetilde{I}}(H)&=m_{\,\widetilde{I}^c}(\breve{H})\,\big\{m_{\widetilde{I}}(P_I)\big\}^{-1}-m_{\,\widetilde{I}^c}(P_I)\,\big\{m_{\widetilde{I}}(P_I)\big\}^{-1}\,m_{\widetilde{I}}(\breve{H})\,\big\{m_{\widetilde{I}}(P_I)\big\}^{-1}\\&=\big(m_{\,\widetilde{I}^c}(\breve{H})-\gamma\,m_{\widetilde{I}}(\breve{H})\big)\,\alpha^{-1},\\ \text{where}\quad \alpha&=m_{\widetilde{I}}(P_I),\ \beta=m_{\,\widetilde{I}^c}(P_I)\text{ and }\gamma=\beta\,\alpha^{-1}.\end{aligned}$$

Let us define a map u from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ by $u(H)=m_{\,\widetilde{I}^c}(\breve{H})-\gamma\,m_{\widetilde{I}}(\breve{H})$. We can choose $I=\{q+1,\dots,q+p\}$ and $\widetilde{I}=\{1,\dots,r\}\cup\{q+1+r,\dots,q+p\}$, where $0\le r\le\inf(p,q)$. We define the $k\times l$ matrix $E_{i,j}^{(k\times l)}$ by $\big(E_{i,j}^{(k\times l)}\big)_{\lambda\mu}=\delta_{i\lambda}\,\delta_{j\mu}$. We have

$$\begin{aligned}m_{\widetilde{I}}\big(\breve{E}_{i,j}^{(q\times p)}\big)&=E_{i,j}^{(p\times p)}\quad\text{if } i\le r,\ \text{and } 0 \text{ if } i>r,\\ \text{and}\quad m_{\,\widetilde{I}^c}\big(\breve{E}_{i,j}^{(q\times p)}\big)&=E_{i-r,j}^{(q\times p)}\quad\text{if } i>r,\ \text{and } 0 \text{ if } i\le r.\ \text{Hence}\\ \big(\gamma\,m_{\widetilde{I}}(\breve{E}_{i,j}^{(q\times p)})\big)_{\alpha\beta}&=\gamma_{\alpha i}\,m_{\widetilde{I}}(\breve{E}_{i,j}^{(q\times p)})_{ij}\,\delta_{j\beta}=\gamma_{\alpha i}\,\delta_{j\beta}\quad\text{if } i\le r,\ \text{and } 0 \text{ elsewhere.}\end{aligned}$$

Now the map which sends H to $\gamma\, m_{\widetilde{I}}(\breve{H})$ can be restricted if $1 \leq j \leq p$ to the span B_j of the $\big(E_{i,j}\big)_{1\leq i\leq q}$. The r first columns of its matrix are those of γ, the others are 0. The map which sends H to $\gamma\, m_{\widetilde{I}^c}(\breve{H})$ maps also B_j into itself. The right upper block of its matrix is $I^{(q-r)}$, the other elements are 0. This allows us to compute the matrix of the restriction of u to B_j, whose determinant is $(-1)^{r\times(q-r)}\det\big(\gamma_{ij}\big)_{\substack{q-r+1\leq i\leq q\\ 1\leq j\leq r}}$. So $\det u = (-1)^{p\times r\times(q-r)}\left[\det\big(\gamma_{i\,j}\big)_{\substack{q-r+1\leq i\leq q\\ 1\leq j\leq r}}\right]^p$. For $1\leq i\leq q$, let C_i be the span of the $(E_{i,j})_{1\leq j\leq p}$. Each C_i is stable by the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to $H\,\alpha^{-1}$. The matrix of the restriction is α^{-1}, so the determinant of the map is $(\det\alpha)^{-q}$. Hence

$$\big|\det DZ_{\widetilde{I}}(H)\big|^2 = \big|\det\left(\gamma_{i,j}\right)_{\substack{q-r+1\leq i\leq q\\ 1\leq j\leq r}}\big|^{2p}\times\big|\det\alpha\big|^{-2q}.$$

Let A be the right $r\times r$ upper block of α. The left $(p-r)\times(p-r)$ lower block of α is $I^{(p-r)}$ and the right $(p-r)\times r$ lower block is 0, so $\det\alpha = (-1)^{r(p-r)}\det A$. The left $r\times(p-r)$ lower block of β is 0, the right $r\times r$ block is $I^{(r)}$ so that the left $r\times r$ lower block of γ is A^{-1}.
From this we deduce $\big|\det\, DZ_{\widetilde{I}}(H)\big|^2 = \big|\det\alpha\big|^{-2(p+q)}$. Since $P_I\,\alpha^{-1} = P_{\widetilde{I}}$, we have

$$\lambda_{\widetilde{I}} = \Big|\det\big({}^tP_{\widetilde{I}}\overline{P}_{\widetilde{I}}\big)\Big|^{-(p+q)} = \big|\det\alpha\big|^{2(p+q)}\lambda_I = \left|\det\frac{\partial Z_{\widetilde{I}}}{\partial Z_I}\right|^{-2}\lambda_I.$$

□

Lemma 2.2. *The unitary group $U(p+q)$ preserves η.*

Proof. We call I the set $\{q+1,\dots,q+p\}$. We define P_I in $\pi(U_I)$ by $P_I = \varphi_I^{-1}(Z_I)$. Let U be an element in $U(p+q)$ such that $m_I(UP_I)$ is invertible. Let $\tilde{P}_I = UP_I\left\{m_I(UP_I)\right\}^{-1}$ and $\tilde{Z}_I = m_{I^c}(\tilde{P}_I)$. We have $\tilde{Z}_I = m_{I^c}(U)\,P_I\left\{m_I(U)P_I\right\}^{-1}$. So

$$D\tilde{Z}_I(H) = m_{I^c}(U)\left[\breve{H}\big\{m_I(U)P_I\big\}^{-1} - P_I\big\{m_I(U)P_I\big\}^{-1}m_I(U)\,\breve{H}\left\{m_I(U)P_I\right\}^{-1}\right].$$

Thus $D\tilde{Z}_I(H) = X\breve{H}\delta^{-1}$, where $\delta = m_I(U)P_I$ and $X = m_{I^c}(U)\big[I^{(p+q)} - P_I\,\delta^{-1}\,m_I(U)\big]$. Let X_1 be the $q\times q$ matrix of the q first columns of X. Then, $X\breve{H} = X_1H$ and we get $D\tilde{Z}_I(H) = X_1H\delta^{-1}$. The determinant of the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to $H\delta^{-1}$ is $(\det\delta)^{-q}$. The determinant of the map from $M_{q,p}(\mathbb{C})$ to $M_{q,p}(\mathbb{C})$ which sends H to X_1H is $(\det X_1)^p$, so $\det D\tilde{Z}_I = (\det X_1)^p\,(\det\delta)^{-q}$. We divide U into four blocks:

$$U = \begin{pmatrix} U_q & U_{q,p}\\ U_{p,q} & U_p\end{pmatrix},\quad U_q\in M_q(\mathbb{C}),\ U_p\in M_p(\mathbb{C}),\ U_{p,q}\in M_{p,q}(\mathbb{C}),\ U_{q,p}\in M_{q,p}(\mathbb{C}).$$

Then $\delta = U_{p,q}\, Z_I + U_p$, so $X_1 = U_q - \big(U_q\, Z_I + U_{q,p}\big)\,\big(U_{p,q}\, Z_I + U_p\big)^{-1}\, U_{q,p}$. Let Z in $M_{p+q,p+q}(\mathbb{C})$ be the matrix with blocks $Z_q = I^{(q)}$, $Z_{p,q} = 0$, $Z_{q,p} = Z_I$, $Z_p = I^{(p)}$, the notations being the same as above. Writing $\det U = \det(UZ)$ and using the column transformation $C_1 \leftarrow C_1 - C_2\,\big(U_{p,q}\, Z_I + U_p\big)^{-1}\, U_{p,q}$ where C_1 is made of the first q columns and C_2 of the remaining ones, we get

$$\det U = \det\Big[U_q - \big(U_q\, Z_I + U_{q,p}\big)\,\big(U_{p,q}\, Z_I + U_p\big)^{-1}\, U_{p,q}\Big] \times \det\big(U_{p,q} Z_I + U_p\big).$$

Hence $\big|\det\, D\tilde{Z}_I\big|^2 = \big|\det\delta\big|^{-2(p+q)}$. We have $\tilde{P}_I = AP_I\delta^{-1}$, so

$$\lambda_{\tilde{I}} = \det\big({}^t\tilde{P}_I\overline{\tilde{P}}_I\big)^{-(p+q)} = \det\big({}^tP_I\overline{P}_I\big)^{-(p+q)} \times \big|\det\delta\big|^{2(p+q)} = \lambda_I\,\big|\det D\tilde{Z}_I\big|^{-2},$$

which proves the result. □

Proposition 2.3. *1.* $dV\left(\mathscr{G}_{p,q}\right) = \eta$.

2. If $I \in \mathcal{I}$, $\left|\mathscr{G}_{p,q}\right|_I = \Big\{\det\big(I^{(p)} + {}^tZ_I\overline{Z}_I\big)\Big\}^{-(p+q)}$.

3. $\mathcal{R}\left(\mathscr{G}_{p,q}\right) = (p+q)\mathscr{G}_{p,q}$.

Proof. 1. Let I in $\mathcal{I}$. It is easy to compute $\mathscr{G}_{p,q}$ at the point $Z_I = 0$: $\mathscr{G}_{p,q}(H,K) = Tr(H\overline{K})$. Then $dV(\mathscr{G}_{p,q})_{\big|Z_I=0} = \big(i/2\big)^{pq}\,\big(dZ\wedge d\overline{Z}\big)_I = \eta_{\big|Z_I=0}$. Since $dV(\mathscr{G}_{p,q})$ and η are invariant by the transitive action of $U(p+q)$, we have $dV(\mathscr{G}_{p,q}) = \eta$.

2. Since $dV(\mathscr{G}_{p,q}) = \big|\mathscr{G}_{p,q}\big|_I\,(i/2)^{pq}\,\big(dZ\wedge d\overline{Z}\big)_I$, property 1 gives the result.

3. Remark that $\mathscr{G}_{p,q} = i\,\partial\overline{\partial}\,\log\big\{\det(I^{(p)} + {}^tZ_I\overline{Z}_I)\big\}$. Since $\mathcal{R}\left(\mathscr{G}_{p,q}\right) = -i\,\partial\overline{\partial}\,\log\left|\mathscr{G}_{p,q}\right|_I$, we obtain $\mathcal{R}\left(\mathscr{G}_{p,q}\right) = (p+q)\mathscr{G}_{p,q}$, which expresses that $\mathscr{G}_{p,q}$ is Einstein, with factor $p+q$. □

3. Some general results about Tian's invariant

3.1. **Tian's invariant with a normalization on a finite set.** If X is a manifold, we will denote by μ_X a measure on X compatible with the manifold structure.

Theorem 3.1. *Let M be a compact Kähler manifold. We suppose that there exists a compact Lie group G of holomorphic isometries. Let $\Delta_n = \{P_1,\dots,P_n\}$ be a finite subset of M. Let $\alpha(\omega)$ (resp. $\alpha_{\Delta_n}(\omega)$) be the supremum of the set of the nonnegative real numbers α satisfying the condition: there exists a constant C such that the inequality $\displaystyle\int_M e^{-\alpha\varphi} \le C$ holds for all the ω-admissible functions φ with $\sup\varphi \ge 0$ (resp. with $\varphi(P_i) \ge 0$ for $1 \le i \le n$). Suppose in addition that the orbit of each P_i under the action of G has positive measure. Then $\alpha(\omega) = \alpha_{\Delta_n}(\omega)$.*

We first establish a few lemmas which will be useful for the proof.

Lemma 3.2. *Let $\left(\varphi_n\right)_{n\geq 0}$ be a sequence of admissible functions with nonnegative maxima. Then there exists a subset Ω of M, with $\mu_{_M}(\Omega)=\mu_{_M}(M)$, and a subsequence φ_{n_k} of φ_n, such that for every p in Ω, the sequence $\left(\varphi_{n_k}(p)\right)_{k\geq 0}$ has a finite lower bound (depending on p).*

Proof. It is sufficient to assume that φ_n has null maxima. Let Q_n be a point such that $\varphi_n(Q_n)$ vanishes. Green's formula runs as follows:

$$\varphi_n(Q_n)=\frac{1}{V}\int_M \varphi_n+\int_M G(Q_n,R)\,\Delta\varphi_n(R)\,dV(R),$$

with $G(Q,R)\geq 0$ and $\displaystyle\int_M G(Q,R)\,dV(R)=C$, where C is a positive constant (see [Au4]). Since φ_n is admissible, $\Delta\varphi_n$ is less than m, m being the dimension of M. Thus $\displaystyle\int_M|\varphi_n|\leq C\,m\,V$. Furthermore, $\displaystyle\int_M\Delta\varphi_n=0$, so $\displaystyle\int_M|\Delta\varphi_n|=2\int_{\{\Delta\varphi_n>0\}}\Delta\varphi_n\leq 2mV$. For every Q in M, we have $\displaystyle\nabla\varphi_n(Q)=\int_M\nabla_Q G(Q,R)\Delta\varphi_n(R)dv(R)$, so that

$$\int_M|\nabla\varphi_n|\leq\int_M\Big[\int_M|\nabla_Q G(Q,R)|dv(Q)\Big]|\Delta\varphi_n(R)|dv(R)\leq 2m\widetilde{C}V,$$

since $\displaystyle\int_M|\nabla_Q G(Q,R)|dv(Q)$ is a continuous, hence a bounded function on M. Thus $(\varphi_n)_{n\geq 0}$ is bounded in the Sobolev space $H^{1,1}(M)$. By Kondrakov's theorem, we can extract from $(\varphi_n)_{n\geq 0}$ a subsequence which converges in $L^1(M)$, and after an other extraction we can suppose that this sequence converges almost everywhere to a function φ of $L^1(M)$. Since φ is finite almost everywhere, we get the result. □

Lemma 3.3. *Let $\left(\varphi_n\right)_{n\geq 0}$ be a sequence of admissible functions with nonnegative maxima and suppose that there exists a compact group G of holomorphic isometries of M such that the orbit of each P_i has positive measure. Let $\Phi:G\to\mathbb{R}\cup\{-\infty\}$ be the map defined by $\Phi(g)=\inf\limits_{\Delta_n}\inf\limits_{k\geq 0}(\varphi_k\circ g)$. Then there exists g in G such that $\Phi(g)$ is finite.*

Proof. Suppose that $\Phi\equiv-\infty$. For $i=1,\ldots,n$, let A_i be the set of the g in G such that $\inf\limits_{k\geq 0}(\varphi_k\circ g)(P_i)=-\infty$. The sets A_i are measurable and $\cup_{i=1}^n A_i=G$, so there exists i such that A_i has positive measure. From Lemma (3.2), $A_i.P_i$ is a subset of Ω^c. Since Ω and M have the same measure, the measure of $A_i.P_i$ vanishes. Let u_i be the map from G to M which sends g to $g(P_i)$. Then u_i has constant rank on G. Indeed, $u_i\circ L(g)=\sigma_g\circ u_i$, where $L(g)$ is the left translation by g and σ_g the map from M to M which sends x to $g.x$. Since $G.P_i$ has positive measure, u_i is a submersion on G, so that $u_i(A_i)$ has positive measure. This is a contradiction since $u_i(A_i)=A_i.P_i$. □

We can now prove Theorem (3.1).

Proof. It is clear that $\alpha(\omega) \leq \alpha_{\Delta_n}(\omega)$. Conversely, let $\varepsilon > 0$. There exists a sequence $\big(\varphi_n\big)_{n\geq 0}$ of admissible functions with positive maxima such that $\displaystyle\int_M e^{-(\alpha(\omega)+\varepsilon)\varphi_k}$ goes to infinity as k goes to infinity. Replacing φ_n by $\varphi_n - \sup\varphi_n$, we can take $\sup\varphi_n = 0$. First we apply Lemma(3.2). For the sake of simplicity, we take $\varphi_{n_k} = \varphi_k$. From Lemma (3.3), there exists an element g in G such that $\Phi(g)$ is finite; we define Ψ_k by $\Psi_k = \varphi_k \circ g - \Phi(g)$. Since g is an isometry, Ψ_k is ω-admissible, and from the very definition of Φ, $\Psi_k(P_i)$ is nonnegative. Furthermore, $\displaystyle\int_M e^{-(\alpha(\omega)+\varepsilon)\Psi_k} = e^{(\alpha(\omega)+\varepsilon)\Phi(g)} \int_M e^{-(\alpha(\omega)+\varepsilon)\varphi_k}$. This proves that $\displaystyle\int_M e^{-(\alpha(\omega)+\varepsilon)\Psi_k}$ goes to infinity as k goes to infinity. Then, $\alpha_{\Delta_n}(\omega) \leq \alpha(\omega) + \varepsilon$. This inequality holds for every positive ε, and so $\alpha_{\Delta_n}(\omega) \leq \alpha(\omega)$. □

3.2. **Tian's invariant on a product.** For a Kähler form ω on a compact Kähler manifold M, $\alpha(\omega)$ is defined as in Theorem (3.1).

Proposition 3.4. *Let $\big(M_i\big)_{1\leq i\leq n}$ be compact Kähler manifolds with metric forms $\big(\omega_i\big)_{1\leq i\leq n}$. We endow the product $M_1\times\cdots\times M_n$ with the metric $\omega_1\oplus\cdots\oplus\omega_n$. Then $\alpha(\omega_1\oplus\cdots\oplus\omega_n) = \inf\limits_{1\leq i\leq n}\alpha(\omega_i)$.*

Proof. It suffices to make the proof when $n = 2$, the general result will follow by induction.

(1) Suppose that $\alpha(\omega_1) \leq \alpha(\omega_2)$, and let $\varepsilon > 0$. There exists a sequence $\big(\varphi_n\big)_{n\geq 0}$ of ω_1-admissible functions on M_1 with positive maxima such that $\displaystyle\int_{M_1} e^{-\big(\alpha(\omega_1)+\varepsilon\big)\varphi_n}$ goes to infinity when n goes to infinity. We define ψ_n on $M_1\times M_2$ by $\psi_n(m_1, m_2) = \varphi_n(m_1)$. Thus ψ_n is $(\omega_1\oplus\omega_2)$-admissible on $M_1\times M_2$, with positive maximum, and $\displaystyle\int_{M_1\times M_2} e^{-\big(\alpha(\omega_1)+\varepsilon\big)\psi_n} = V(M_2)\int_{M_1} e^{-\big(\alpha(\omega_1)+\varepsilon\big)\varphi_n}$, so that $\displaystyle\int_{M_1\times M_2} e^{-\big(\alpha(\omega_1)+\varepsilon\big)\psi_n}$ goes to infinity when n goes to infinity. We have therefore $\alpha(\omega_1\oplus\omega_2) \leq \alpha(\omega_1) + \varepsilon$. This yields $\alpha(\omega_1\oplus\omega_2) \leq \alpha(\omega_1)$.

(2) Let us now prove the opposite inequality. Let α be a real number such that $\alpha < \inf\big(\alpha(\omega_1), \alpha(\omega_2)\big)$ and φ an $(\omega_1\oplus\omega_2)$-admissible function on $M_1\times M_2$. If m_2 is in M_2, the function which sends m_1 to $\varphi(m_1, m_2)$ is ω_1-admissible. The same holds for M_1. Let (u, v) in $M_1\times M_2$ be such that $\varphi(u, v) \geq 0$. Then

$$\begin{aligned}\int_{M_1\times M_2} e^{-\alpha\varphi(m_1,m_2)} dV_1\, dV_2 &= \int_{M_1} e^{-\alpha\varphi(m_1,v)}\Big(\int_{M_2} e^{-\alpha\big[\varphi(m_1,m_2)-\varphi(m_1,v)\big]} dV_2\Big) dV_1 \\ &\leq C_2\int_{M_1} e^{-\alpha\varphi(m_1,v)} dV_1 \leq C_1\, C_2.\end{aligned}$$

Thus, $\alpha \leq \alpha\big(\omega_1 \oplus \omega_2\big)$ and we get $\inf\big(\alpha(\omega_1), \alpha(\omega_2)\big) \leq \alpha\big(\omega_1 \oplus \omega_2\big)$. $\square$

3.3. **Tian's invariant on $G_{p,q}(\mathbb{C})$.** Since there is a natural duality isomorphism between $G_{p,q}(\mathbb{C})$ and $G_{q,p}(\mathbb{C})$, we can assume that $p \leq q$ without loss of generality.

3.3.1. *Imbedding of $\big\{\mathbb{P}^1(\mathbb{C})\big\}^p$ into $G_{p,q}(\mathbb{C})$ when $p \leq q$.* For w in $\mathbb{C}^{p(q-1)}$, $w = \big(w_{i,j}\big)_{\substack{1\leq i\leq q\\ 1\leq j\leq p\\ i\neq j}}$, we define the map $\tilde{\rho}_w$ from $\Big\{\mathbb{C}^2 \setminus (0,0)\Big\}^p$ to $M_{p+q,p}(\mathbb{C})$ by

$$\tilde{\rho}_w\Big((\lambda_i, \mu_i)_{1\leq i\leq p}\Big) = \begin{cases} \lambda_i\, \delta_{ij} & \text{if} \quad i \leq p \\ w_{i-p,j}\, \lambda_j & \text{if} \quad i > p \text{ and } i \neq j+p \\ \mu_i & \text{if} \quad i > p \text{ and } i = j+p \end{cases}$$

We make, for $p+1 \leq i \leq p+q$, the following row transformations: $L_i \leftarrow L_i - \sum\limits_{\substack{1\leq j\leq p\\ i\neq j+p}} w_{i-p,j}\, L_j$. We get a matrix $\big(c_{ij}\big)_{\substack{1\leq i\leq p+q\\ 1\leq j\leq p}}$ with $c_{ij} = \delta_{ij}\, \lambda_i$ if $1 \leq i \leq p$ and $c_{ij} = \delta_{i-p,j}\, \mu_j$ if $p+1 \leq i \leq p+q$, which has rank p. $\tilde{\rho}_w$ induces a map from $\big\{\mathbb{P}^1(\mathbb{C})\big\}^p$ into $G_{p,q}(\mathbb{C})$ as shown on the following diagram, where γ is the projection of the principal fiber bundle $\Big\{\mathbb{C}^2\setminus(0,0)\Big\}^p$ onto $\big\{\mathbb{P}^1(\mathbb{C})\big\}^p$. Remark that $\tilde{\rho}_w$ sends $[0,1]\times\cdots\times[0,1]$ onto $\pi(A)$, where $m_{\{p+1,\ldots,2p\}}(A) = I^{(p)}$ and $m_{\{p+1,\ldots,2p\}^c}(A) = 0^{(q\times p)}$.

$$\begin{array}{ccc} \Big\{\mathbb{C}^2 \setminus (0,0)\Big\}^p & \xrightarrow{\tilde{\rho}_w} & M^*(p+q,p) \\ \gamma \Big\downarrow & & \Big\downarrow \pi \\ \big\{\mathbb{P}^1(\mathbb{C})\big\}^p & \xrightarrow[\rho_w]{} & G_{p,q}(\mathbb{C}) \end{array}$$

We have

$$\begin{aligned} \big(\pi\circ\tilde{\rho}_w\big)^*\big(\mathscr{G}_{p,q}\big) &= i\,\partial\overline{\partial}\,\log\Big(\det\big({}^t\tilde{\rho}_w\bar{\tilde{\rho}}_w\big)\Big) \\ &= i\,\partial\overline{\partial}\,\log\left(\frac{\det\big({}^t\tilde{\rho}_w\bar{\tilde{\rho}}_w\big)}{\prod\limits_{k=1}^{p}\big(|\lambda_k|^2+|\mu_k|^2\big)}\right) + \sum_{k=1}^{p} i\,\partial\overline{\partial}\,\log\big(|\lambda_k|^2+|\mu_k|^2\big) \\ &= i\,\partial\overline{\partial}\,\log\tilde{\Phi} + \gamma^*\big(FS_1\oplus\cdots\oplus FS_1\big), \end{aligned}$$

where FS_1 is the Fubini-Study metric on $\mathbb{P}^1(\mathbb{C})$. $\tilde{\Phi}$ is invariant by the action of the structural group $\mathbb{C}^*\times\cdots\times\mathbb{C}^*$, so it induces a map Φ from $\big\{\mathbb{P}^1(\mathbb{C})\big\}^p$ into $\mathbb{C}$. Note that $\Phi\big([0,1]\times\cdots\times[0,1]\big) = 1$. Then $\big(\pi\circ\tilde{\rho}_w\big)^*\big(\mathscr{G}_{p,q}\big) = \pi^*\big(i\,\partial\overline{\partial}\,\log\Phi + FS_1\oplus\cdots\oplus FS_1\big)$, so that $\rho_w^*\big(\mathscr{G}_{p,q}\big) = i\,\partial\overline{\partial}\,\log\Phi + FS_1\oplus\cdots\oplus FS_1$.

3.3.2. *Lower bound of* $\alpha(\mathscr{G}_{p,q})$. For I in $\mathcal{I}$, we define P_I by $m_I(P_I) = I^{(p)}$ and $m_{I^c}(P_I) = 0^{(q\times p)}$. If $n = \binom{p+q}{p}$, we set $\Delta_n = \{P_I\}_{I\in\mathcal{I}}$. Since $U(p+q)$ is a transitive group of holomorphic isometries of $G_{p,q}(\mathbb{C})$, we know from proposition (3.1), that $\alpha\big(\mathscr{G}_{p,q}\big) = \alpha_{\Delta_n}\big(\mathscr{G}_{p,q}\big)$. We set $I = \{p+1,\dots,2p\}$. Let φ be an admissible function on $G_{p,q}(\mathbb{C})$, nonnegative on Δ_n. The last equality of the precedent section shows that the function $\varphi\circ\rho_w + \log\Phi$ is $\big(FS_1\oplus\cdots\oplus FS_1\big)$-admissible for every w in $\mathbb{C}^{p(q-1)}$. Furthermore, $\big(\varphi\circ\rho_w + \log\Phi\big)$ sends $[0,1]\times\cdots\times[0,1]$ to the nonnegative number $\varphi(P_I)$. It is known that $\alpha(FS_1) = 1$ (see [Au3]). Proposition (3.4) yields $\alpha\big(FS_1\oplus\cdots\oplus FS_1\big) = 1$.
Let α be a real number such that $\alpha < 1$. There exists a constant C, independent of φ, such that $\displaystyle\int_{\{\mathbb{P}^1(\mathbb{C})\}^p} e^{-\alpha\varphi\circ\rho_w}\,\Phi^{-\alpha} \le C$. We define the map F_I from $\pi(U_I)$ to $\mathbb{R}_+$ by $F_I(Z_I) = \det\big(Id + {}^tZ_I\overline{Z}_I\big)$. On $\{\mathbb{P}^1(\mathbb{C})\}^p$, we work with the coordinates $\mu_1,\dots,\mu_p$ in the chart $\lambda_1 = \cdots = \lambda_p = 1$. Thus

$$\Phi(\mu) = \frac{F_I\circ\rho_w(\mu)}{\displaystyle\prod_{k=1}^{p}\big(1+|\mu_k|^2\big)}, \quad \text{so that} \quad \int_{\mu\in\mathbb{C}^p} e^{-\alpha\varphi\circ\rho_w(\mu)}\,\frac{dV_\mu\big(\mathbb{C}^p\big)}{\displaystyle\prod_{k=1}^{p}\big(1+|\mu_k|^2\big)^{2-\alpha}\,\big(F_I\circ\rho_w(\mu)\big)^{\alpha}} \le C.$$

We have the inequality $\displaystyle\sum_{i=1}^{q}\sum_{j=1}^{p}\big|Z_{ij}\big|^2 \le F_I\big(P_I\big)$. In particular, for every k in $\{1,\dots,p\}$, $1+|\mu_k|^2 \le F_I\circ\rho_w(\mu)$, and $f_I\circ\rho_w(\mu) \ge 1 + \displaystyle\sum_{\substack{1\le i\le q\\ 1\le j\le p\\ i\ne j}}\big|w_{ij}\big|^2$. Thus, for $\kappa > 0$ and $w\in\mathbb{C}^{p(q-1)}$,

$$\frac{\displaystyle\prod_{k=1}^{p}\big(1+|\mu_k|^2\big)^{2-\alpha}}{\big(F_I\circ\rho_w(\mu)\big)^{\kappa+p+q-\alpha}} \le \frac{1}{\big(F_I\circ\rho_w(\mu)\big)^{\kappa-p+q+\alpha(p-1)}} \le \frac{1}{\Big(1+\displaystyle\sum_{\substack{1\le i\le q\\ 1\le j\le p\\ i\ne j}}\big|w_{ij}\big|^2\Big)^{\kappa}} = \frac{1}{\Big(1+\big\|w\big\|^2\Big)^{\kappa}}.$$

We have, according to Proposition (2.3),

$$\int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^\kappa} = \int_{w\in\mathbb{C}^{p(q-1)}} \int_{\mu\in\mathbb{C}^p} \frac{e^{-\alpha\varphi\circ\rho_w(\mu)}}{\big(F_I\circ\rho_w(\mu)\big)^{\kappa+p+q}}\, dV_\mu\big(\mathbb{C}^p\big)\, dV_w\big(\mathbb{C}^{p(q-1)}\big)$$

$$= \int_{w\in\mathbb{C}^{p(q-1)}} \int_{\mu\in\mathbb{C}^p} \left(\frac{e^{-\alpha\varphi\circ\rho_w(\mu)}}{\prod_{k=1}^p \big(1+|\mu_k|^2\big)^{2-\alpha}\, \big(F_I\circ\rho_w(\mu)\big)^\alpha} \right) \times \frac{\prod_{k=1}^p \big(1+|\mu_k|^2\big)^{2-\alpha}}{\big(F_I\circ\rho_w(\mu)\big)^{\kappa+p+q-\alpha}}\, dV_\mu\big(\mathbb{C}^p\big)\, dV_w\big(\mathbb{C}^{p(q-1)}\big)$$

$$= \int_{w\in\mathbb{C}^{p(q-1)}} \left(\int_{\mu\in\mathbb{C}^p} \frac{e^{-\alpha\varphi\circ\rho_w(\mu)}}{\prod_{k=1}^p \big(1+|\mu_k|^2\big)^{2-\alpha}\, \big(F_I\circ\rho_w(\mu)\big)^\alpha}\, dV_\mu\big(\mathbb{C}^p\big) \right) \times \frac{dV_w\big(\mathbb{C}^{p(q-1)}\big)}{\big(1+\|w\|^2\big)^\kappa}$$

$$\leq C \int_{w\in\mathbb{C}^{p(q-1)}} \frac{dV_w\big(\mathbb{C}^{p(q-1)}\big)}{\big(1+\|w\|^2\big)^\kappa} \leq C' \qquad \text{if } \kappa > p(q-1).$$

Thus, we obtain that for all I in $\mathcal{I}$, $\displaystyle\int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^\kappa} \leq C$, where C is independent of φ.

Since $G_{p,q}(\mathbb{C})$ is compact, there exists a family $\big(V_I\big)_{I\in\mathcal{I}}$ of open sets of $G_{p,q}(\mathbb{C})$ such that V_I is relatively compact in $\pi(U_I)$ for every $I\in\mathcal{I}$, and $\bigcup_{I\in\mathcal{I}} V_I = G_{p,q}(\mathbb{C})$. There exists $M>0$ such that $F_I \leq M$ on V_I for every $I\in\mathcal{I}$. Thus

$$\int_{G_{p,q}(\mathbb{C})} e^{-\alpha\varphi} \leq \sum_{I\in\mathcal{I}} \int_{V_I} e^{-\alpha\varphi} \leq \sum_{I\in\mathcal{I}} M^\kappa \int_{V_I} \frac{e^{-\alpha\varphi}}{F_I^\kappa} \leq M^\kappa \sum_{I\in\mathcal{I}} \int_{\pi(U_I)} \frac{e^{-\alpha\varphi}}{F_I^\kappa} \leq C\, M^\kappa \binom{p+q}{p}.$$

We deduce that $\alpha\big(\mathscr{G}_{p,q}\big) \geq 1$.

3.3.3. *Upper bound of* $\alpha(\mathscr{G}_{p,q})$. We use here a method which can be found in [Re] for the complex projective space. Let I in $\mathcal{I}$. We define $\tilde{K}$ from $M^*(p+q,p)$ to $\mathbb{P}^1(\mathbb{R})$ by the relation $\tilde{K}(M) = \big[\,|\det m_I(H)|^2, \det {}^tM\overline{M}\,\big]$. $\tilde{K}$ is invariant by the action of the structural group $G_p(\mathbb{C})$, so it induces a C^∞ map K from $G_{p,q}(\mathbb{C})$ to $\mathbb{P}^1(\mathbb{R})$. Remark that $\psi = \log K$ is a Kähler potential on U_I for the metric $\mathscr{G}_{p,q}$.

Lemma 3.5. *There exists a decreasing sequence* $\big(\varphi_n\big)_{n\geq 0}$ *of admissible functions with positive maxima which converges pointwise to* $-\psi$ *on* $\pi\big(U_I\big)$.

Proof. We construct a decreasing sequence $\big(f_n\big)_{n\geq 0}$ of C^∞ convex functions on $\mathbb{R}_+$ satisfying the conditions $1+f_n' > 0$, $f_n(x) = -\big(1-1/n\big)x$ for x in $[0,n]$ and $f_n(x) = -n$ for

$x \geq 2n$. Let y be an element of $\pi(U_I)^c$ and Ω_n the set of the elements x in $\pi(U_I)$ such that $\psi(x) > 2n$. Since $F_I(y) = [0,1]$, there exists a neighborhood V of y such that the inequality $z > e^{2n}$ holds for every point $[1,z]$ in $F_I(V)$. Thus $V \cap \pi\big(U_I\big)$ is included in Ω_n. We have proved that $W_n = \Omega_n \cup \pi\big(U_I\big)^c$, so that W_n is an open neighborhood of $\pi\big(U_I\big)^c$. We define φ_n by $\varphi_n = f_n \circ \psi$ on $\pi\big(U_I\big)$ and $\varphi_n = -n$ on W_n. Thus φ_n is well defined and $\varphi_n(0) = 0$. It remains to show that φ_n is admissible on $\pi\big(U_I\big)$. We have

$$\Big(\mathscr{G}_{p,q} + i\,\partial\overline{\partial}\,\varphi_n\Big)_{\lambda\bar{\mu}} = \partial_{\lambda\bar{\mu}}\psi + \partial_\lambda\big(f_n' \circ \psi\big)\partial_{\bar{\mu}}\psi = \big(1 + f_n' \circ \psi\big)\partial_{\lambda\bar{\mu}}\psi + f_n'' \circ \psi\,\partial_\lambda\psi\,\partial_{\bar{\mu}}\psi.$$

Hence the matrix of the metric $\mathscr{G}_{p,q} + i\,\partial\overline{\partial}\,\varphi_n$ is of the form $A + T$ where A is positive definite and T has rank one and positive trace. So $A + T$ is positive definite and we get the result. □

Lemma 3.6. *Let n in $\mathbb{N}^*$ and r a positive real number. Then*

$$\int_{||X||\leq r} \frac{dV_X(M_n(\mathbb{C}))}{\big|\det X\big|^2} = +\infty.$$

Proof. We can write

$$\int_{||X||\leq r} \frac{dV_X(M_n(\mathbb{C}))}{\big|\det X\big|^2} = \sum_{k=0}^{\infty} \int_{r/2^{k+1}\leq||X||\leq r/2^k} \frac{dV_X(M_n(\mathbb{C}))}{\big|\det X\big|^2}.$$

We put $Y = 2^k X$, so

$$\int_{r/2^{k+1}\leq||X||\leq r/2^k} \frac{dV_X(M_n(\mathbb{C}))}{\big|\det X\big|^2} = \int_{1/2\leq||Y||\leq 1} \frac{dV_Y(M_n(\mathbb{C}))}{\big|\det Y\big|^2}.$$

The terms in the series are strictly positive and independent of k. The sum is therefore infinite. □

We can now prove that $\alpha\big(\mathscr{G}_{p,q}\big)$ is upper bounded by 1. Suppose that $\alpha\big(\mathscr{G}_{p,q}\big) > 1$. Then there exists a positive C such that for every integer n, $\displaystyle\int_{\pi(U_I)} e^{-\varphi_n} \leq C$. Using Lemma (3.5) and monotonous convergence, $\displaystyle\int_{\pi(U_I)} F_I \leq C$. Since $\pi(U_I)^c$ has zero measure, $\displaystyle\int_{G_{p,q}(\mathbb{C})} F_I \leq C$. Let $\tilde{I}$ in $\mathcal{I}$ be such that $I \cap \tilde{I} = \emptyset$ (this is possible since $p \leq q$). We have $P_{\tilde{I}}\big\{m_I\left(P_{\tilde{I}}\right)\big\}^{-1} = P_I$. Remark that $m_I\big(P_{\tilde{I}}\big) = m_I\big(Z_{\tilde{I}}\big)$. Thus $\det\big(Id + {}^t Z_I \overline{Z}_I\big) = \det\big({}^t P_{\tilde{I}} \overline{P}_{\tilde{I}}\big)\,\big|\det m_I\big(Z_{\tilde{I}}\big)\big|^{-2}$. For $\big\|Z_{\tilde{I}}\big\| \leq r$, $\det\big({}^t P_{\tilde{I}} \overline{P}_{\tilde{I}}\big) \leq M$, so that $\displaystyle\int_{||Z_{\tilde{I}}||\leq r} \frac{dV_{Z_{\tilde{I}}}(M_{q,p}(\mathbb{C}))}{\big|\det m_I\big(Z_{\tilde{I}}\big)\big|^2} < +\infty$. Integrating over the remaining variables $\big(Z_{ij}\big)_{i\in\tilde{I}^c\cap I^c}$ yields

$\displaystyle\int_{||Z||\leq r} \frac{dV_Z(M_p(\mathbb{C}))}{\left|\det Z\right|^2} < +\infty$, which is in contradiction with the result of Lemma (3.6). Thus we obtain $\alpha\big(\mathscr{G}_{p,q}\big) \leq 1$.

References

[Au1] T. Aubin, *Équations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sci. Math. **102** (1978), 63-95.

[Au2] T. Aubin, *Réduction du cas positif de l'équation de Monge-Ampère sur les variétés Kählériennes compactes à la démonstration d'une inégalité*, J. Funct. Anal. **57** (1984), 143-153.

[Au3] T. Aubin, *Métriques d'Einstein-Kähler et exponentiel de fonctions admissibles*, J. Func. Anal. **88** (1990), 385-394.

[Au4] T. Aubin, *Some Non-linear Problems in Riemannian Geometry*, Springer-Verlag, Berlin, 1998.

[Be1] A. Ben Abdesselem, *Lower bound of admissible functions on sphere*, Bull. Sci. Math **126** (2002), 675-680.

[Be2] A. Ben Abdesselem, *Enveloppes inférieures de fonctions admissibles sur l'espace projectif complexe. Cas symétrique*, To appear in Diff. Geom. and its App.

[Be-Ch1] A. Ben Abdesselem, P. Cherrier, *Einstein-Kähler metrics on a class of bundles involving integral weights*, J. Math. Pures Appl. **3**, vol 81, (2002), 259-281.

[Be-Ch2] A. Ben Abdesselem, P. Cherrier, *Estimations of Ricci tensor on certain Fano manifolds*, Math. Z. **233**, 2000, 481-505.

[Fu] A. Futaki, *An obstruction to the existence of Kähler-Einstein metrics*, Invent. Math. **73**, 1983, 437-443.

[Ko-No] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry*, Volume II John Wiley (1969).

[Li] A. Lichnerowicz, *Sur les transformations analytiques des variétés kählériennes*, Cr. Acac. Sci **244**, 1957, 3011-3014.

[Ma] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne*, Nagoya Math. J. **11** (1957), 145-150.

[Re] C. Real, *Métriques d'Einstein-Kähler sur des variétés à première classe de Chern positive*, J. Func. Anal. **106** (1992), 145-188.

[Ti] G. Tian, *On Kähler-Einstein metrics on certain Kähler manifolds with $C^1(M) > 0$*, Invent. Math. **89** (1987), 225-246.

[Ya] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equations*, I. Comm. Pure Appl. Math. **31** (1978), 339-411.

E-mail address: `julien.grivaux@free.fr`

Université Pierre et Marie Curie