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THE FARRELL–TATE AND BREDON HOMOLOGY FOR $\mathrm{PSL}_4(\mathbb{Z})$ AND OTHER ARITHMETIC GROUPS

ANH TUAN BUI, ALEXANDER D. RAHM AND MATTHIAS WENDT

ABSTRACT. We provide some new computations of Farrell–Tate and Bredon (co)homology for arithmetic groups. For calculations of Farrell–Tate or Bredon homology, one needs cell complexes where cell stabilizers fix their cells pointwise. We provide an algorithm computing an efficient subdivision of a complex to achieve this rigidity property. Applying this algorithm to available cell complexes for $\mathrm{PSL}_4(\mathbb{Z})$ and $\mathrm{PGL}_3(\mathbb{Z}[i])$ provides computations of Farrell–Tate cohomology for small primes as well as the Bredon homology for the classifying spaces of proper actions with coefficients in the complex representation ring. On the other hand, in order to check correctness of the computer calculations, we describe the Farrell–Tate cohomology in some rank-one cases, using Brown’s complex and a number-theoretic description of the conjugacy classification of cyclic subgroups.

1. INTRODUCTION

Understanding the structure of the cohomology of arithmetic groups is a very important problem with relations to number theory and various K-theoretic areas. Explicit cohomology computations usually proceed via the study of the actions of the arithmetic groups on their associated symmetric spaces, and recent years have seen several advances in algorithmic computation of equivariant cell structures for these actions. To approach computations of Farrell–Tate and Bredon (co)homology of arithmetic groups, one needs cell complexes having a *rigidity property*: cell stabilizers must fix their cells pointwise. The known algorithms (using Voronoi decompositions and such techniques, cf. e.g. [6, 7]) do not provide complexes with this rigidity property, and this leads to a significant bottleneck, both for the computation of Farrell–Tate cohomology (resp. the torsion at small prime numbers in group cohomology) of arithmetic groups as well as for the computation of Bredon homology.

In theory, it is always possible to obtain this rigidity property via the barycentric subdivision. However, the barycentric subdivision of an n -dimensional cell complex can multiply the number of cells by $(n + 1)!$ and thus easily let the memory stack overflow. We provide an algorithm, called *rigid facets subdivision*, cf. Section 3, which subdivides cell complexes for arithmetic groups such that stabilisers fix their cells pointwise, but only leads to a controlled increase (in terms of sizes of stabilizer groups) in the number of cells, avoiding an explosion of the data volume. An implementation of the algorithm, cf. [4], shows that cases like $\mathrm{PSL}_4(\mathbb{Z})$ or $\mathrm{PGL}_3(\mathbb{Z}[i])$ can effectively be treated with it, using commonly available machine resources. For the sake of comparison, barycentric subdivision and rigid facets subdivision applied to the cell complex for $\mathrm{PSL}_4(\mathbb{Z})$ from [5] leads to the following numbers of cells in the individual dimensions:

- (2832, 14160, 56640, 169920, 339840, 339840) using barycentric subdivision,
- (1632, 6000, 7776, 3840, 1152, 96) using rigid facets subdivision.

1.1. Computations of Farrell–Tate cohomology. Farrell–Tate cohomology is a modification of cohomology of arithmetic groups which is particularly suitable to investigate torsion related to finite subgroups (in particular, the torsion in cohomological degrees above the virtual cohomological dimension). While the known cell complexes for arithmetic groups can deal very well with the rational cohomology and torsion at primes which do not divide orders of finite subgroups, computations with these complexes run into serious trouble for small prime numbers because the differentials in the relevant spectral sequence are too complicated to evaluate. There is a suitable new technique called *torsion subcomplex reduction*, cf. [10], which produces significantly smaller cell complexes and therefore simplifies the equivariant spectral sequence calculations. To apply this simplification, however,

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one needs cell complexes with the abovementioned rigidity property. Using the rigid facets subdivision, applied to cell complexes for $\mathrm{PSL}_4(\mathbb{Z})$ and $\mathrm{PGL}_3(\mathbb{Z}[i])$, we have computed the Farrell–Tate cohomology of these groups, at the primes 3 and 5 for $\mathrm{PSL}_4(\mathbb{Z})$ and at the prime 3 for $\mathrm{PGL}_3(\mathbb{Z}[i])$. These results can be found in Theorem 12, Proposition 27 and Section 6.2. Since the computation proceeds through a complete description of the reduced torsion subcomplex, we can compute the torsion above the virtual cohomological dimension in all degrees.

In the cases which are effectively of rank one (5-torsion in $\mathrm{PSL}_4(\mathbb{Z})$, 3-torsion in PGL_3 over imaginary quadratic integers), we can check the results of the cohomology computation using torsion subcomplex reduction by comparing to a computation using Brown’s formula. For this, we outline a generalization of a theorem of Reiner [13], giving a description of conjugacy classes of cyclic subgroups and the group structure of their normalizers. These results are proved in Sections 5 and 6 and provide generalizations of the computations in [12].

1.2. Computations of Bredon homology. For any group G , Baum and Connes introduced a map from the equivariant K -homology of G to the K -theory of the reduced C^* -algebra of G , called the assembly map. For many classes of groups, it has been proven that the assembly map is an isomorphism; and the Baum–Connes conjecture claims that it is an isomorphism for all finitely presented groups G (counter-examples have been found only for stronger versions of the Baum–Connes conjecture). The assembly map is known to be injective for arithmetic groups. For an overview on the conjecture, see the monograph [9].

The geometric-topological side of Baum and Connes’ assembly map, namely the equivariant K -homology, can be determined using an Atiyah–Hirzebruch spectral sequence with E_2 -page given by the Bredon homology $H_n^{\mathfrak{F}\mathrm{in}}(\underline{E}G; R_{\mathbb{C}})$ of the classifying space $\underline{E}G$ for proper actions with coefficients in the complex representation ring $R_{\mathbb{C}}$ and with respect to the system $\mathfrak{F}\mathrm{in}$ of finite subgroups of G . This Bredon homology can be computed explicitly, as described by Sanchez-Garcia [14, 15].

While for Coxeter groups with a small system of generators [15] and arithmetic groups of rank 2 [11], general formulae for the equivariant K -homology have been established, the only known higher-rank case to date is the example $\mathrm{SL}_3(\mathbb{Z})$ in [14]. Although there are by now considerably more arithmetic groups for which cell complexes have been worked out [5–7], no further computations of Bredon homology $H_n^{\mathfrak{F}\mathrm{in}}(\underline{E}G; R_{\mathbb{C}})$ have been done since 2008 because the relevant cell complexes fail to have the rigidity property required for Sanchez-Garcia’s method. We discuss an explicit example, cf. Section 2, demonstrating that the rigidity property is essential for the computation of Bredon homology and cannot be circumvented by a different method.

The application of the rigid facets subdivision to cell complexes for $\mathrm{PSL}_4(\mathbb{Z})$ and $\mathrm{PGL}_3(\mathbb{Z}[i])$ leads to the following computations:

- Applying rigid facets subdivision to the cell complex for $\underline{\mathrm{E}}\mathrm{PSL}_4(\mathbb{Z})$ from [5], we obtain

$$H_n^{\mathfrak{F}\mathrm{in}}(\underline{\mathrm{E}}\mathrm{PSL}_4(\mathbb{Z}); R_{\mathbb{C}}) \cong \begin{cases} 0, & n \geq 5, \\ \mathbb{Z}^{10}, & n = 4, \\ \mathbb{Z}, & n = 3, \\ 0, & n = 2, \\ \mathbb{Z}^4, & n = 1, \\ \mathbb{Z}^{25} \oplus \mathbb{Z}/2, & n = 0. \end{cases}$$

- Applying rigid facets subdivision to the cell complex for $\underline{\mathrm{E}}\mathrm{GL}_3(\mathbb{Z}[i])$ from [1, 16], we obtain

$$H_n^{\mathfrak{F}\mathrm{in}}(\underline{\mathrm{E}}\mathrm{GL}_3(\mathbb{Z}[i]); R_{\mathbb{C}}) \cong \begin{cases} 0, & n \geq 5, \\ \mathbb{Z}^{20}, & n = 4, \\ \mathbb{Z}^4 \oplus (\mathbb{Z}/8)^4 \oplus (\mathbb{Z}/3)^4, & n = 3, \\ \mathbb{Z}^{20}, & n = 2, \\ \mathbb{Z}^{36}, & n = 1, \\ (\mathbb{Z}^{36})^3 \oplus (\mathbb{Z}/4)^8, & n = 0. \end{cases}$$

- Applying rigid facets subdivision to the cell complex for $\underline{\mathrm{EGL}}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})})$ from [1, 16], we obtain

$$H_n^{\tilde{\delta}^{\mathrm{in}}}(\underline{\mathrm{EGL}}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}); R_{\mathbb{C}}) \cong \begin{cases} 0, & n \geq 5, \\ \mathbb{Z}^{38}, & n = 4, \\ \mathbb{Z}^{12} \oplus (\mathbb{Z}/4)^4 \oplus (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/2)^2, & n = 3, \\ \mathbb{Z}^{14}, & n = 2, \\ \mathbb{Z}^{24}, & n = 1, \\ \mathbb{Z}^{30} \oplus (\mathbb{Z}/2)^{10}, & n = 0. \end{cases}$$

The correctness of our results depends of course heavily on the cell complexes for $\underline{\mathrm{E}G}$ that we take as input for the rigid facets subdivision algorithm and the subsequent calculations. Therefore, for $\underline{\mathrm{EGL}}_3(\mathbb{Z}[i])$, we have compared two independent implementations, namely Sebastian Schönnebeck's [1, 16] and Mathieu Dutour Sikirić's [5]. They produce the same group homology for $\mathrm{PGL}_3(\mathbb{Z}[i])$. We further have checked that the homology of the cell complex remains unchanged under our implementation of rigid facets subdivision, and that the equivariant Euler characteristic of $\underline{\mathrm{E}G}$ vanishes before and after subdividing.

Organization of the paper. In Section 2, we provide a counterexample in order to contradict the possibility to compute the Bredon homology from an arbitrary non-rigid cell complex. In Section 3, we provide the rigid facets subdivision algorithm. In Section 4, we apply the rigid facets subdivision algorithm to a $\mathrm{PSL}_4(\mathbb{Z})$, and compare the result with a computation using Brown's conjugacy classes cell complex. Concerning Brown's conjugacy classes cell complex, in Section 5, we provide a slight modification of a partial conjugacy classification of cyclic subgroups in general linear groups over S -integer rings. In Section 6, we apply this modification in example computations on $\mathrm{PSL}_4(\mathbb{Z})$ and $\mathrm{PGL}_3(\mathbb{Z}[i])$, and compare with the results that we obtain using the rigid facets subdivision algorithm.

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2. NECESSITY OF RIGIDITY FOR BREDON HOMOLOGY

From a non-rigid cell complex, i.e., a cell complex where cell stabilisers do not necessarily fix the corresponding cell pointwise, one can compute classical group homology via the equivariant spectral sequence with coefficients in the orientation module. Such an orientation module, where elements of the stabilizer group act by multiplication with 1 or -1 , depending on whether they preserve or reverse the orientation of the cell, cannot exist for Bredon homology. We make this precise in the following statement:

Proposition 1. *There is no module-wise variation of the Bredon module with coefficients in the complex representation ring and with respect to the system of finite subgroups such that Bredon homology can be computed from a non-rigid cell complex.*

Proof. We provide a counterexample in order to contradict the possibility to compute the Bredon homology from an arbitrary non-rigid cell complex.

Consider the classical modular group $\mathrm{PSL}_2(\mathbb{Z})$. A model for $\underline{\mathrm{E}}\mathrm{PSL}_2(\mathbb{Z})$ is given by the modular tree [17]. There is a rigid cell complex structure T_1 on it, given as follows. By [17], the modular tree admits a strict fundamental domain for $\mathrm{PSL}_2(\mathbb{Z})$, of the shape

$$\mathbb{Z}/3\mathbb{Z} \bullet \text{---} \bullet \mathbb{Z}/2\mathbb{Z}$$

with vertex stabilisers as indicated and trivial edge stabiliser. We obtain the cell complex T_1 by tessellating the modular tree with the $\mathrm{PSL}_2(\mathbb{Z})$ -images of this fundamental domain. Obviously, T_1 is rigid, and it yields the Bredon chain complex

$$0 \rightarrow R_{\mathbb{C}}(\langle 1 \rangle) \xrightarrow{d} R_{\mathbb{C}}(\mathbb{Z}/2\mathbb{Z}) \oplus R_{\mathbb{C}}(\mathbb{Z}/3\mathbb{Z}) \rightarrow 0.$$

The map d in the above Bredon chain complex is injective, and as $R_{\mathbb{C}}(\mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}^n$, we read off

$$H_1^{\mathfrak{F}^{\text{in}}}(\underline{\text{E}}\text{PSL}_2(\mathbb{Z}); R_{\mathbb{C}}) = 0, \quad H_0^{\mathfrak{F}^{\text{in}}}(\underline{\text{E}}\text{PSL}_2(\mathbb{Z}); R_{\mathbb{C}}) \cong \mathbb{Z}^4.$$

Now we equip the modular tree with an alternative equivariant cell structure T_2 , induced by the non-strict fundamental domain

$$\mathbb{Z}/3\mathbb{Z} \bullet \text{---} \bullet \mathbb{Z}/3\mathbb{Z}$$

where the (set-wise) edge stabilizer is $\mathbb{Z}/2\mathbb{Z}$, flipping the edge onto itself. It can be seen as a ramified double cover of the fundamental domain for T_1 discussed above. A system of representative cells for T_2 is given by the edge of double length, and one vertex of stabiliser type $\mathbb{Z}/3\mathbb{Z}$. This yields a chain complex

$$0 \rightarrow R_{\mathbb{C}}(\widetilde{\mathbb{Z}/2\mathbb{Z}}) \rightarrow R_{\mathbb{C}}(\mathbb{Z}/3\mathbb{Z}) \rightarrow 0,$$

where the tilde could be any construction which takes the non-trivial $\mathbb{Z}/2\mathbb{Z}$ -action on the edge of double length into account (similar to the coefficients in the orientation module for group homology computed from non-rigid cell complexes). But no matter how this construction is done, from $R_{\mathbb{C}}(\mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}^3$, we can never reach $H_0^{\mathfrak{F}^{\text{in}}}(\underline{\text{E}}\text{PSL}_2(\mathbb{Z}); R_{\mathbb{C}}) \cong \mathbb{Z}^4$. Hence T_2 is our desired counterexample. \square

Remark 2. We could of course drop the condition ‘‘module-wise’’ in the above proposition, and investigate whether there is a reasonable construction which maps the representation ring to a complex of modules and yields a quasi-isomorphism from the total complex to the Bredon complex for the subdivided tree. But with such a construction, one would only superficially hide the fact that one needs to know how to subdivide in order to get the constructed complexes right. This means that it will not be practicable to compute Bredon homology with respect to the system of finite subgroups and coefficients in the complex representation ring without subdividing the cell complex under consideration to make it rigid.

3. THE RIGID FACETS SUBDIVISION ALGORITHM

In this section, we discuss the rigid facets subdivision algorithm which rigidifies equivariant cell complexes. The core of the method is Algorithm 2, which is expected to run in reasonable time for input coming from cell complexes for arithmetic groups. The key fact which guarantees that rigid facets subdivision works, is Lemma 9.

Definition 3. Following the notation in [3], we use the term Γ -equivariant CW-complex, or simply Γ -cell complex, to mean a CW-complex X on which a discrete group Γ acts cellularly, i.e., in such a way that the action induces a permutation of the cells of X . We say the cell complex is *rigid* if each element in the stabilizer of any cell fixes the cell pointwise.

Remark 4. The algorithm producing the subdivision of the Γ -equivariant CW-complex X only modifies combinatorial data, based on the barycentric subdivision of individual cells. We require X to come with a geometric realization, equipped with a metric such that each of the cells of X is convex, the restriction of the metric to each cell is CAT(0) and the cell stabilizers act by CAT(0)-isometries. We are not requiring that the metric is CAT(0) on the whole CW-complex. However, the examples we are most interested in are those where Γ is an arithmetic group and the geometric realization of X is the associated symmetric space.

Definition 5. A *rigidification* \hat{X} of a Γ -cell complex X is a rigid Γ -cell complex \hat{X} with the same underlying topological space as X . The map passing through the underlying topological space is then a Γ -equivariant homeomorphism $\hat{X} \rightarrow X$ of Γ -spaces. Note that a Γ -equivariant homeomorphism of Γ -spaces does not need to preserve existing cell structure, so \hat{X} is allowed to have more cells than X .

The outer shell of the rigid facets subdivision is Algorithm 1, which subdivides (whenever necessary) representatives of cell orbits using Algorithm 2.

Proposition 6. *Let Γ be a discrete group, and let X be a Γ -equivariant CW-complex having finitely many Γ -orbits and finite cell stabilizers. Assume furthermore that X is equipped with a metric as in Remark 4. Then Algorithm 1 finds a rigidification of X (with respect to the Γ -action). It terminates in finite time.*

Algorithm 1 Subdivide to get stabilisers which fix their cells pointwise

Input: An n -dimensional Γ -equivariant CW-complex X with finite cell stabilizers and a metric as in Remark 4.

Output: A rigidification of X .

```

for  $m$  running from 0 to  $n$  do
  for  $\sigma$  running through lifts of  $m$ -cells in  $\Gamma \backslash X^{(m)}$  do
    if  $\sigma$  is not rigid then
      Use Algorithm 2 to subdivide  $\sigma$  into a partition  $P$ , which is a union of rigid  $m$ -cells,
      disjoint up to boundaries, with a fundamental domain  $F$  for the  $\Gamma_\sigma$ -action on  $P$ .
      Run through all the  $(m + 1)$ -cells; if their boundaries contain  $\sigma$ ,
      then replace  $\sigma$  by its partition  $P$ .
      Replace the cell  $\sigma$  by  $F$  in  $\Gamma \backslash X^{(m)}$ .
    end if
  end for
end for

```

Proof. The key step of the algorithm is proved by Lemma 9 below; the rest is a routine induction. Lemma 9 is the point where the convexity and isometry requirements are needed. By the finiteness assumptions for orbits and cell stabilizers, the loops are all deterministic over finite index sets. Each operation inside them takes finite time, cf. Corollary 10, whence the claim. \square

Observation 7. The outer shell Algorithm 1 can be used with any subdivision algorithm for the cells. In particular, replacing the use of Algorithm 2 by the barycentric subdivision in Algorithm 1, the claims of Proposition 6 still hold. However, as mentioned in the introduction, the reason for developing Algorithm 2 is to reduce the blow-up in the number of cells, so as to make the algorithm practically applicable to cell complexes for higher-rank arithmetic groups.

We now discuss the actual subdivision to rigidify cells, Algorithm 2.

Algorithm 2 — Rigid Facets Subdivision

Input: An m -cell σ with stabilizer group Γ_σ , with rigid faces, equipped with a metric as in Remark 4.

Output: A Γ_σ -equivariant set of rigid m -cells, disjoint up to boundaries, constituting a partition P of σ , together with a fundamental domain F for the action of Γ_σ on P .

- Sort the $(m - 1)$ -faces of σ into orbits $\{\{g_{jt}\sigma_j\}_t\}_j$ under the action of Γ_σ , where j is indexing the orbits and t is indexing the cells inside each orbit.
 - Let T be the list containing the element $g_{11}\sigma_1$.
 - In each orbit $\{g_{jt}\sigma_j\}$ with $j \geq 2$, choose one cell such that its union with the cells in T is connected. Add the chosen cell to T .
 - Use Algorithm 3 to construct the m -cell $F := \bigcup_{\tau \in T} \text{convex envelope}(\tau, \text{barycenter}(\sigma))$.
 - Let Γ_σ^{pw} be the subgroup of Γ_σ which fixes the cell σ pointwise.
 - Then $P := \sum_{1 \leq t \leq |\Gamma_\sigma / \Gamma_\sigma^{pw}|} g_{1t}F$ is the desired partition of σ .
 - Return P and F .
-

Remark 8. Essentially, Algorithm 2 produces a convex union of cells of the barycentric subdivision which is a fundamental domain for the Γ_σ -action. The slight complications arise from the fact that we don't actually want to compute the full barycentric subdivision, to gain computational feasibility.

Lemma 9 (Rigid Facets Lemma). *Let σ be a cell (with stabilizer Γ_σ), whose faces are all rigid and which is equipped with a metric as in Remark 4. Let Γ_σ^{pw} be the subgroup of Γ_σ which fixes the cell σ pointwise. Then there is a fundamental domain F for the action of $\Gamma_\sigma / \Gamma_\sigma^{pw}$ on σ such that σ is tessellated by $|\Gamma_\sigma / \Gamma_\sigma^{pw}|$ copies of F .*

Proof. First we have to check that the statement is well defined in the sense that $\Gamma_\sigma/\Gamma_\sigma^{pw}$ is a group. This is the case because for all g in Γ_σ , for all γ in Γ_σ^{pw} , for all x in σ we have $(g^{-1}\gamma g)x = g^{-1}(\gamma(gx)) = g^{-1}(gx) = x$. Therefore, as the kernel of the action of Γ_σ on σ , Γ_σ^{pw} is a normal subgroup; and there is a short exact sequence of groups,

$$1 \rightarrow \Gamma_\sigma^{pw} \rightarrow \Gamma_\sigma \rightarrow \Gamma_\sigma/\Gamma_\sigma^{pw} \rightarrow 1,$$

which makes our statement well defined.

Suppose that α is one of the facets of σ . We are going to prove that the size of the orbit of α , under the action of Γ_σ on the set of facets of σ , is $|\Gamma_\sigma/\Gamma_\sigma^{pw}|$. Let Γ_α be the stabilizer of α . We claim that $\Gamma_\alpha \cap \Gamma_\sigma = \Gamma_\sigma^{pw}$. The action of Γ_σ on the compact set $\text{closure}(\sigma)$ is by homeomorphisms; therefore, any element of Γ_σ fixing σ pointwise also fixes the boundary $\partial\sigma$ pointwise. Hence $\Gamma_\alpha \cap \Gamma_\sigma \supset \Gamma_\sigma^{pw}$. On the other hand, let $g \in \Gamma_\alpha \cap \Gamma_\sigma$. Then by assumption on the rigidity of the facets, g fixes the cell α pointwise. Since the cell σ is convex and the group acts by CAT(0)-isometries, the barycenter of σ preserves its distances to the boundary $\partial\sigma$ under the action of Γ_σ on σ , and hence remains fixed. As a further consequence of the CAT(0) isometry, the fixed point set of g extends, by preservation of the distances, from the convex envelope of α and the barycenter of σ to the whole cell σ . Hence, g is an element of Γ_σ^{pw} . Thus, we can conclude that $\Gamma_\alpha \cap \Gamma_\sigma = \Gamma_\sigma^{pw}$. Whence, the size of the orbit of α under the action of Γ_σ is $|\Gamma_\sigma/\Gamma_\sigma^{pw}|$.

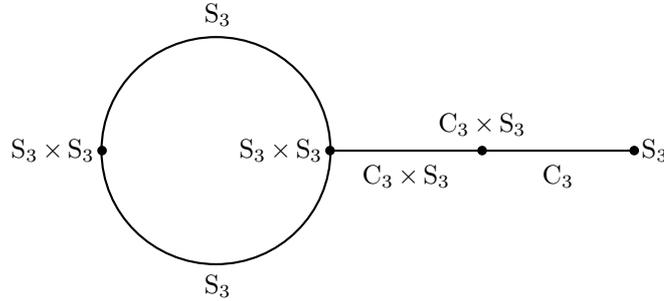
Furthermore, from $\Gamma_\alpha \cap \Gamma_\sigma = \Gamma_\sigma^{pw}$, we see that $\Gamma_\sigma/\Gamma_\sigma^{pw}$ acts freely on the set of facets of σ . So, we can take one arbitrary representative α_k for each orbit of facets, to unite to a fundamental domain for $\Gamma_\sigma/\Gamma_\sigma^{pw}$ on the set of facets of σ . Taking the convex envelope e_k of α_k and the barycenter of σ , we get a fundamental domain $F := \bigcup_k e_k$ for $\Gamma_\sigma/\Gamma_\sigma^{pw}$ on σ . By the above orbit size calculation, it yields the desired tessellation. \square

Corollary 10. *Algorithm 2 terminates after finitely many steps and produces a rigid subdivision of the cell σ .*

Remark 11. The worst-case complexity of the above subdivision algorithm is not better than that of the barycentric subdivision. If X is a single n -simplex with the natural permutation action of the symmetric group Σ_{n+1} acting on the vertices, then any rigidification will need to produce at least $(n+1)! = \#\Sigma_{n+1}$ top cells for X . However, the point is that the average cell complex for interesting arithmetic groups has most of its cells rigid and only very few with maximally possible stabilizer. Therefore, the average case complexity for the envisioned applications is significantly better than that of the barycentric subdivision, as evidenced by the discussion in the introduction.

4. EXAMPLE: FARRELL–TATE COHOMOLOGY OF $\text{PSL}_4(\mathbb{Z})$ AT THE PRIME 3

Applying the rigid facets subdivision algorithm to the $\text{PSL}_4(\mathbb{Z})$ -equivariant cell complex from [5], extracting the 3-torsion subcomplex, and reducing it using the methods of [10], we get the following graph of groups \mathcal{T} , decorated with the groups stabilizing the cells that are the pre-images of the projection to the quotient space.



The machine computation provided the following system of morphisms among the above cell stabilizers. The $C_3 \times S_3$ edge stabilizer admits an isomorphism of groups (not the identity, though) to the $C_3 \times S_3$ vertex stabilizer and an inclusion into the $S_3 \times S_3$ vertex stabilizer. Of the two S_3 edge stabilizers, one has maps $\text{diag}(1, 1)$ and $\text{diag}(1, 0)$ to the two $S_3 \times S_3$ vertex stabilizers, and the other one has maps $\text{diag}(1, -1)$ and $\text{diag}(-1, -1)$ to the two $S_3 \times S_3$ vertex stabilizers. The C_3 edge stabilizer admits an inclusion into the second factor of the $C_3 \times S_3$ vertex stabilizer, and an inclusion into the S_3 vertex stabilizer.

Algorithm 3 Constructing the union of convex envelopes

Input: A list T of j -cells and an m -cell σ such that all $\tau \in T$ are faces of σ .

Output: A $(j + 1)$ -cell F which has the same underlying topological space as

$$\bigcup_{\tau \in T} \text{convex envelope}(\tau, \text{barycenter}(\sigma)).$$

Record the barycenter of σ as a new vertex, with stabilizer Γ_σ .

Enumerate the finite set $S := \{\rho \text{ is } (j - 1)\text{-cell} \mid \exists! \tau \in T : \rho \in \partial\tau\}$.

Then S contains all the $(j - 1)$ -faces ρ of all $\tau \in T$ such that ρ is not a common face of any two j -cells $\tau_1, \tau_2 \in T$.

- For each $s \in S$, take the convex envelope $e(s)$ of s and the barycenter of σ .
- Record $e(s)$ as an oriented j -cell, with boundary

$$\partial e(s) = \{s\} \cup \bigcup_{\varepsilon \in \partial s} \{\text{convex envelope}(\varepsilon, \text{barycenter}(\sigma))\}.$$

For the stabilizers, we record $\Gamma_{e(s)} = \Gamma_s \cap \Gamma_\sigma$ and $\Gamma_{\text{convex envelope}(\varepsilon, \text{barycenter}(\sigma))} = \Gamma_\varepsilon \cap \Gamma_\sigma$. The $(j + 1)$ -cell F is determined by all the j -faces $\tau \in T$ and $e(s)$ (for all $s \in S$). Here, we have to take care of which of the newly constructed cells $e(s)$ are on the same Γ_σ -orbit. In order to decide this, we make use of their common vertex, the barycenter of σ :

- Identify the orbits $e(s_1)$ and $e(s_2)$ if and only if $\exists \gamma \in \Gamma_\sigma : \gamma s_1 = s_2$.
- Attribute arbitrary orientations to Γ_σ -representatives of the new cells $e(s)$, and spread them on their Γ_σ -orbit using the above identifications.
- Return the $(j + 1)$ -cell F with boundary

$$\partial F = \bigcup_{\tau \in T} \tau \cup \bigcup_{s \in S \text{ subject to the above identifications}} e(s)$$

and stabilizer $\Gamma_\sigma \cap \bigcap_{\tau \in T} \Gamma_\tau$.

By the properties of torsion subcomplex reduction, the $\text{PSL}_4(\mathbb{Z})$ -equivariant cohomology of the 3-torsion subcomplex is isomorphic to the $\text{PSL}_4(\mathbb{Z})$ -equivariant cohomology of the above graph of groups \mathcal{T} . Similarly, the Farrell–Tate cohomology of $\text{PSL}_4(\mathbb{Z})$ at the prime 3 is isomorphic to the Farrell–Tate cohomology of the above graph of groups. In the following, we evaluate the isotropy spectral sequence

$$E_1^{p,q} = \bigoplus_{\sigma \in \mathcal{T}_p} H^q(\text{Stab}(\sigma); \mathbb{F}_3) \Rightarrow \widehat{H}^{p+q}(\text{PSL}_4(\mathbb{Z}); \mathbb{F}_3)$$

converging to Farrell–Tate cohomology. As we only consider a graph, the spectral sequence is concentrated in the two columns $p = 0, 1$. The differential d_1 is induced from the inclusions of subgroups, up to the sign coming from the choice of orientation of the graph. Since the spectral sequence is only concentrated in the first two columns, we will have $E_2 = E_\infty$. Since we are interested in field coefficients, there are no extension problems to solve at the E_∞ -page.

The relevant cohomology groups of the finite groups are:

- $H^\bullet(C_3; \mathbb{F}_3) \cong \mathbb{F}_3[x](a)$ with $\deg a = 1$ and $\deg x = 2$.
- $H^\bullet(S_3; \mathbb{F}_3) \cong \mathbb{F}_3[y](b)$ with $\deg b = 3$ and $\deg y = 4$.
- By the Künneth formula,

$$H^\bullet(S_3 \times S_3; \mathbb{F}_3) \cong H^\bullet(S_3; \mathbb{F}_3)^{\otimes 2}, \quad H^\bullet(C_3 \times S_3; \mathbb{F}_3) \cong H^\bullet(C_3; \mathbb{F}_3) \otimes H^\bullet(S_3; \mathbb{F}_3).$$

To describe the d_1 -differential, it is enough to note that the restriction map associated to the inclusion $C_3 \hookrightarrow S_3$ is the inclusion of $\mathbb{Z}/2\mathbb{Z}$ -invariants.

Now, for the evaluation of the spectral sequence, we first deal with the edges attached to the loop.

(1) The restriction map

$$\begin{aligned} \phi \oplus (\text{Res}_{C_3}^{S_3} \circ \text{pr}_2^*) : \mathbb{F}_3[x_2](a_1) \otimes \mathbb{F}_3[y_4](b_3) &\cong H^\bullet(C_3 \times S_3; \mathbb{F}_3) \\ &\rightarrow H^\bullet(C_3 \times S_3; \mathbb{F}_3) \oplus H^\bullet(C_3; \mathbb{F}_3) \\ &\cong \mathbb{F}_3[z_2](c_1) \otimes \mathbb{F}_3[w_4](d_3) \oplus \mathbb{F}_3[u_2](e_1) \end{aligned}$$

is injective with cokernel isomorphic to $H^\bullet(C_3; \mathbb{F}_3)$. Here ϕ denotes the isomorphism $C_3 \times S_3 \cong C_3 \times S_3$ appearing as stabilizer inclusion in the reduced torsion subcomplex.

(2) The inclusion of the dihedral vertex group into the cyclic edge group is an injection

$$\text{Res}_{C_3}^{S_3} : H^\bullet(S_3; \mathbb{F}_3) \hookrightarrow H^\bullet(C_3; \mathbb{F}_3)$$

given by the inclusion of the invariant elements for the C_2 -action by -1 . Therefore, the cokernel is concentrated in degrees $1, 2 \pmod{4}$ (except for the degree 0).

Therefore, we can reduce the E_1 -page of the spectral sequence as follows: from (1), we find that we can remove the two summands for $C_3 \times S_3$ from the columns $p = 0$ and $p = 1$, respectively; but in turn, we have to replace the restriction map for $C_3 \times S_3 \hookrightarrow S_3 \times S_3$ by a map from the cohomology of $S_3 \times S_3$ to the cokernel of the restriction map in (2). However, since the latter is concentrated in degrees 1 and 2, the induced restriction map is trivial in the generating degrees 3 and 4, hence it is trivial. Therefore, the E_1 -page decomposes: one contribution comes from the cokernel of the restriction map $H^\bullet(S_3; \mathbb{F}_3) \hookrightarrow H^\bullet(C_3; \mathbb{F}_3)$, the other contribution comes from the loop connecting the two copies of $S_3 \times S_3$ via the S_3 -edges.

The cohomology of the loop is computed as follows:

(3) The restriction maps $\text{diag}(1, 1), \text{diag}(1, 0), \text{diag}(1, -1)$ and $\text{diag}(-1, -1)$:

$$\mathbb{F}_3[x_4, y_4](a_3, b_3) \cong H^\bullet(S_3 \times S_3; \mathbb{F}_3) \rightarrow H^\bullet(S_3; \mathbb{F}_3) \cong \mathbb{F}_3[z_4](c_3)$$

are surjective. On the kernel of $\text{diag}(1, 1)$, the restriction of $\text{diag}(1, -1)$ is still surjective. On the kernel of $\text{diag}(-1, -1)$, the restriction of $\text{diag}(1, 0)$ is still surjective.

Therefore, the cohomology of one edge of the loop is killed already by the restriction map from any one of the vertex groups. Number the $S_3 \times S_3$ -vertices by 1 and 2, and the S_3 -edges by a and b . The restriction from the vertex group 1 to the edge a is surjective. Removing this part from the spectral sequence, the restriction from the vertex group 2 to the edge a is trivial, but we still have the restriction to the edge b . This kills the edge cohomology b , showing that the differential d_1 is surjective in the loop part of the E_1 -page. The kernel of the differential consists then exactly of two copies of the kernel of a restriction map $\text{Res}_{S_3}^{S_3 \times S_3}$.

Theorem 12. *The Farrell–Tate cohomology of the group $\text{PSL}_4(\mathbb{Z})$ (with coefficients in \mathbb{F}_3) in degrees ≥ 2 is given as follows:*

$$\widehat{H}^\bullet(\text{PSL}_4(\mathbb{Z}); \mathbb{F}_3) \cong \left(\ker \text{Res}_{S_3}^{S_3 \times S_3} \right)^{\oplus 2} \oplus \text{coker} \left(H^{\bullet-1}(S_3; \mathbb{F}_3) \rightarrow H^{\bullet-1}(C_3; \mathbb{F}_3) \right).$$

This, in particular also computes the 3-torsion group cohomology of $\text{PSL}_4(\mathbb{Z})$ above the virtual cohomological dimension.

The cokernel of the d_1 -differential in degree 0 and hence the first cohomology $\widehat{H}^1(\text{PSL}_4(\mathbb{Z}); \mathbb{F}_3)$ is of \mathbb{F}_3 -rank 1, coming from the loop of the 3-torsion graph.

Remark 13. The kernel comes from the $p = 0$ column of the $E_2 = E_\infty$ -page. The cokernel comes from the $p = 1$ column and consequently has a shift.

Inspired by Grunewald, we consider the Hilbert–Poincaré series of the Farrell–Tate cohomology of $\text{PSL}_4(\mathbb{Z})$ with coefficients in \mathbb{F}_ℓ :

$$\text{HP}_{\text{PSL}_4(\mathbb{Z})}(T; \ell) := \sum_{q=1}^{\infty} \dim \widehat{H}^q(\text{PSL}_4(\mathbb{Z}); \mathbb{F}_\ell) \cdot T^q.$$

Corollary 14. *The Hilbert–Poincaré series of the 3-torsion Farrell–Tate cohomology of $\text{PSL}_4(\mathbb{Z})$ (for degrees ≥ 1) is then*

$$\text{HP}_{\text{PSL}_4(\mathbb{Z})}(T; 3) = T + \frac{2(T^3 + T^4 + T^6 + T^7)}{(1 - T^4)^2} + \frac{T^2 + T^3}{1 - T^4}.$$

Remark 15. Note that the above calculation describes the Farrell–Tate cohomology in all degrees, not just some small ones. Essentially, the computer calculation produces the reduced torsion subcomplex (which encodes the cohomology for all degrees). The spectral sequence is evaluated using the cup-product structure. Note that the finiteness results for group homology imply that the cup-product structure for both group and Farrell–Tate cohomology is finitely generated. Using suitable commutative algebra packages, such computations of the ring structure (and therefore additive computations for all cohomological degrees) could probably also be automated.

We make the following consideration on the compatibility of our result for Farrell–Tate cohomology with the result of Dutour–Ellis–Schürmann [5] for group homology in low degrees. The isomorphism types computed in the latter article are to correspond as follows to the evaluation of our above Hilbert–Poincaré series in those degrees.

$$H_q(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{Z}) \cong \begin{cases} 0, & q = 1, \\ (\mathbb{Z}/2)^3, & q = 2, \\ \mathbb{Z} \oplus (\mathbb{Z}/4)^2 \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/5, & q = 3, \\ (\mathbb{Z}/2)^4 \oplus \mathbb{Z}/5, & q = 4, \\ (\mathbb{Z}/2)^{13}, & q = 5, \end{cases} \quad \text{---} \quad \dim \widehat{H}^q(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{F}_3) = \begin{cases} 1, & q = 1, \\ 1, & q = 2, \\ 3, & q = 3, \\ 2, & q = 4, \\ 0, & q = 5. \end{cases}$$

For this to be consistent, the Farrell–Tate cohomology groups in degrees 1 and 2 need to vanish in group homology; so, these should be annihilated by differentials from the orbit space. We have evidence for this in degree 1, since the loop in the graph becomes contractible in the orbit space of the full locally symmetric space. In degree 3, one of the summands in $H_3(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{Z})$ is rationally non-trivial and must come from the orbit space. This means that only the submodule $(\mathbb{Z}/3)^2$ can come from Farrell–Tate cohomology, and the third dimension that we observe in degree 3 Farrell–Tate cohomology must belong to the degree 2 stabilizer cohomology class that is annihilated by the above mentioned differentials from the orbit space.

From Theorem 12, we deduce that the degree 2 Farrell–Tate class can only come from

$$\mathrm{coker}(\mathrm{H}^{\bullet-1}(S_3; \mathbb{F}_3) \rightarrow \mathrm{H}^{\bullet-1}(C_3; \mathbb{F}_3)).$$

Then, this class and its group homology counterpart sit at position $p = 1$, $q = 1$ in the respective equivariant spectral sequence, and hence the annihilating differential, emanating from the orbit space homology module $\mathbb{Z} \subset H_3(\mathrm{PSL}_4(\mathbb{Z}); \mathbb{Z})$ sitting at position $p = 3$, $q = 0$, must be of second degree.

In degrees 4 and 5, the dimensions already agree via the Universal Coefficient Theorem, so here we infer that the submodule $(\mathbb{Z}/3)^2$ in degree 3 should actually come from Farrell–Tate cohomology, so it should be stabilizer cohomology that is not hit by higher degree differentials.

5. CONJUGACY CLASSIFICATION OF CYCLIC SUBGROUPS

In this section, we will provide a slight modification of a partial conjugacy classification of cyclic subgroups in general linear groups over S -integer rings. Most of what follows is essentially based on Reiner’s article [13] on the isomorphism classification of modules over the integral group ring $\mathbb{Z}[C_\ell]$.

Denote by $\mathcal{O}_{K,S}$ a ring of S -integers in a global field K , and let C_ℓ be the cyclic group of order ℓ . The goal is the conjugacy classification of embeddings $C_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$. As first step, the classical argument, cf. [8], provides a relation between the conjugacy classification and isomorphism classification of modules over group rings.

Proposition 16. *There is an injection from conjugacy classes of embeddings $C_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ to isomorphism classes of $\mathcal{O}_{K,S}[C_\ell]$ -modules whose underlying $\mathcal{O}_{K,S}$ -module is free of rank n . The only isomorphism class not in the image is the one where the C_ℓ -action is trivial.*

Proof. (i) Assume we have a subgroup $C_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$. In particular, we have a non-trivial action of C_ℓ on $M = \mathcal{O}_{K,S}^{\oplus n}$. We use this action to turn M into an $\mathcal{O}_{K,S}[C_\ell]$ -module by letting the element $[g]$ for $g \in C_\ell$ act via the embedding $C_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$.

(ii) Assume we have two subgroups $\phi, \phi' : C_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ which are conjugate. Then any conjugating matrix A gives rise to commutative diagrams

$$\begin{array}{ccc} \mathcal{O}_{K,S}^{\oplus n} & \xrightarrow{\phi(g)} & \mathcal{O}_{K,S}^{\oplus n} \\ A \downarrow & & \downarrow A \\ \mathcal{O}_{K,S}^{\oplus n} & \xrightarrow{\phi'(g)} & \mathcal{O}_{K,S}^{\oplus n} \end{array}$$

showing that the two $\mathcal{O}_{K,S}[C_\ell]$ -modules associated to ϕ and ϕ' are isomorphic via A .

(iii) Conversely, assume we have an $\mathcal{O}_{K,S}[C_\ell]$ -module M whose underlying $\mathcal{O}_{K,S}$ -module is free of rank n . We choose an $\mathcal{O}_{K,S}$ -basis for M . The representing matrices for the automorphisms $[g]$ for $g \in C_\ell$ provide an embedding $C_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ since by assumption the action of $[g] \in C_\ell$ is non-trivial. Different choices of basis will give rise to subgroups which are conjugate via change-of-basis matrices.

(iv) Assume we have an isomorphism $f : M \cong M'$ of $\mathcal{O}_{K,S}[C_\ell]$ -modules as in (iii). Then a choice of basis for M induces a choice of basis for M' via f . With these choices of bases, the modules M and M' give rise to the same subgroup of $\mathrm{GL}_n(\mathcal{O}_{K,S})$. The independence-of-basis statement in (iii) implies that the subgroups associated to M and M' (for arbitrary choices of bases) are conjugate. \square

Let $\phi : C_\ell \rightarrow C_\ell$ be an automorphism of the cyclic group. Then ϕ induces an $\mathcal{O}_{K,S}$ -linear automorphism of $\mathcal{O}_{K,S}[C_\ell]$ in the obvious way. For the purposes of the next result, we call such automorphisms *special*.

Corollary 17. *Under the bijection of Proposition 16, the centralizer of a subgroup $C_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ is isomorphic to the $\mathcal{O}_{K,S}[C_\ell]$ -automorphism group of the corresponding module M . The normalizer is isomorphic to the group of $\mathcal{O}_{K,S}$ -automorphisms which are semilinear with respect to a special automorphism of $\mathcal{O}_{K,S}[C_\ell]$.*

Proof. We consider a fixed subgroup (as opposed to a conjugacy class), and consider the associated module M , equipped with the corresponding choice of basis. Then a matrix A in the centralizer of $\iota : C_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ provides commutative diagrams for all $g \in C_\ell$:

$$\begin{array}{ccc} \mathcal{O}_{K,S}^{\oplus n} & \xrightarrow{\iota(g)} & \mathcal{O}_{K,S}^{\oplus n} \\ A \downarrow & & \downarrow A \\ \mathcal{O}_{K,S}^{\oplus n} & \xrightarrow{\iota(g)} & \mathcal{O}_{K,S}^{\oplus n}. \end{array}$$

As in the proof of Proposition 16, this provides an automorphism of the $\mathcal{O}_{K,S}[C_\ell]$ -module M . This construction is obviously compatible with composition.

Conversely, an $\mathcal{O}_{K,S}[C_\ell]$ -automorphism of the module M corresponding to ι provides a change-of-basis matrix which is in the centralizer of ι . Again, this is obviously compatible with composition.

The two constructions above are inverses, proving the claim for the centralizer. The claims for the normalizer are proved in the same way, changing the lower morphism in the commutative diagram from $\iota(g)$ to $\phi \circ \iota(g)$. \square

To compute the relevant examples of Farrell–Tate cohomology of linear groups, we will use Brown’s formula for ℓ -rank 1, cf. [2, Corollary X.7.4]. For this, we need to determine conjugacy classes of cyclic subgroups as well as the structure of their normalizers. The above statements translate these questions to an isomorphism classification of modules over group rings, and the question of automorphism groups of such modules. For cyclic groups, these questions can be approached using the classical work of Reiner, cf. [13].

5.1. Relative integral bases. Reiner’s analysis of the modules over the group ring $\mathbb{Z}[C_\ell]$ is essentially based on the class group theory for cyclotomic integers. In the generalization to rings of S -integers, we will therefore need some assumption on the situation.

As usual, denote by $\Phi_\ell(T)$ the cyclotomic polynomial. If $\Phi_\ell(T)$ is not K -irreducible, then the degree of ζ_ℓ over K is a strict divisor of the degree of $\Phi_\ell(T)$. In this case, we have $\mathcal{O}_{K,S}[\zeta_\ell] = \mathcal{O}_{K,S}[T]/(\Psi_\ell(T))$ where $\Psi_\ell(T)$ is the minimal polynomial of ζ_ℓ over K .

To get a full analogue of Reiner’s result, we assume that the ring $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T))$ is a Dedekind domain. Some results will work under the weaker hypothesis that $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind domain. We will make these cases explicit.

Note that even if $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind domain, $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T))$ need not be a Dedekind domain. If $\Phi_\ell(T)$ is not K -irreducible, then the total ring of fractions is $K[T]/(\Phi_\ell(T))$ which is a direct sum of copies of $K(\zeta_\ell)$, corresponding to the number of K -factors of $\Phi_\ell(T)$.

Example 18. In the case $K = \mathbb{Q}(\sqrt{-7})$ and $\ell = 7$, denote by N_7 the norm element in $\mathbb{Z}[C_7]$. Then $\mathcal{O}_K[C_\ell]/(N_7)$ is a fiber product of two copies of $\mathcal{O}_K[\zeta_7]$ over the quotient $\mathcal{O}_K[\zeta_7]/(\sqrt{-7}^3)$ where $\sqrt{-7}^3$ is the resultant of the two K -factors of $\Phi_7(T)$. \square

The Dedekind domain requirement is crucial because it provides a bijection between finitely generated torsion-free modules of fixed rank n and the class group. The ring $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind ring precisely when the relevant powers of ζ_ℓ form a relative integral basis of $K(\zeta_\ell)/K$. For most of our purposes, the following statement will be sufficient.

Lemma 19. *Let K/\mathbb{Q} be Galois extension such that $(\ell, d_K) = 1$. Then*

$$\mathcal{O}_{K(\zeta_\ell)} = \mathcal{O}_K[\zeta_\ell] \cong \mathcal{O}_K[T]/(\Phi_\ell(T)).$$

Proof. The discriminant of $\mathbb{Q}(\zeta_\ell)/\mathbb{Q}$ is a power of ℓ so that by assumption the discriminants of K and $\mathbb{Q}(\zeta_\ell)$ are coprime. Then the product of the integral bases of K/\mathbb{Q} and $\mathbb{Q}(\zeta_\ell)/\mathbb{Q}$ is an integral basis of $K(\zeta_\ell)/\mathbb{Q}$. In particular, any element of $\mathcal{O}_{K(\zeta_\ell)}$ is an \mathcal{O}_K -linear combination of $1, \zeta_\ell, \dots, \zeta_\ell^{\ell-1}$, hence these form a relative integral basis of $K(\zeta_\ell)/K$. \square

5.2. Conjugacy classification. In this section, we provide a recollection and slight extension of Reiner’s study of isomorphism classification of modules over group rings for cyclic groups.

Our situation is the following: let K be a number field, let S be a finite set of places containing the infinite ones, and denote by $\mathcal{O}_{K,S}$ the ring of S -integers in K . Denote by C_ℓ the cyclic group of order ℓ where ℓ is a prime. In some cases relevant to the Farrell–Tate cohomology computation, we will give a classification of $\mathcal{O}_{K,S}[C_\ell]$ -modules which are $\mathcal{O}_{K,S}$ -free.

We now proceed with the analysis of the finitely generated $\mathcal{O}_{K,S}$ -free $\mathcal{O}_{K,S}[C_\ell]$ -modules under the assumption that $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T))$ is a Dedekind domain. Note that in this case we actually have $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T)) \cong \mathcal{O}_{K,S}[\zeta_\ell]$. The argument essentially follows [13].

Let M be an $\mathcal{O}_{K,S}$ -free $\mathcal{O}_{K,S}[C_\ell]$ -module. Denote by $N = \sum_{g \in C_\ell} [g]$ the norm element. The set $M_N = \{m \in M \mid N \cdot m = 0\}$ of elements of M annihilated by the norm element has a natural module structure over the quotient ring $\mathcal{O}_{K,S}[C_\ell]/(N)$. The kernel of the natural surjective morphism

$$\mathcal{O}_{K,S}[C_\ell] \rightarrow \mathcal{O}_{K,S}[T]/(\Phi_\ell(T)) : [1] \mapsto T$$

is generated by $\Phi_\ell([1]) = N$. In particular, we get an induced isomorphism

$$\mathcal{O}_{K,S}[C_\ell]/(N) \cong \mathcal{O}_{K,S}[T]/(\Phi_\ell(T)) \cong \mathcal{O}_{K,S}[\zeta_\ell],$$

From the above, the module M_N embeds into a direct sum of copies of $K(\zeta_\ell)$ and hence is finitely generated and torsion-free over $\mathcal{O}_{K,S}[\zeta_\ell]$. By assumption, $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind ring, hence finitely generated and torsion-free implies projective and the general theory states that M_N is of the form $\mathcal{O}_{K,S}[\zeta_\ell]^r \oplus \mathfrak{a}$ with \mathfrak{a} a fractional ideal of $\mathcal{O}_{K,S}[\zeta_\ell]$. The $\mathcal{O}_{K,S}[\zeta_\ell]$ -module (and the restricted $\mathcal{O}_{K,S}[C_\ell]$ -module) M_N is completely determined by r and the ideal class of \mathfrak{a} .

There is an inclusion of $\mathcal{O}_{K,S}[\zeta_\ell]$ -modules

$$M_N \supset ([1] - 1)M \supset (\zeta_\ell - 1)M_N,$$

where $[1]$ denotes the element of $\mathcal{O}_{K,S}[\zeta_\ell]$ corresponding to a (choice of) generator of C_ℓ . From standard results on Dedekind rings (as in Reiner’s paper), we find that the quotient $([1] - 1)M/(\zeta_\ell - 1)M_N$ is a free module over the quotient ring $\mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1)$. It should be noted at this point that $\mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1) \cong \mathcal{O}_{K,S}/(\ell)$ (because the same is true over \mathbb{Z}).

The quotient M/M_N is a finitely generated torsion-free $\mathcal{O}_{K,S}$ -module. Hence it is projective and the sequence $0 \rightarrow M_N \rightarrow M \rightarrow M/M_N \rightarrow 0$ splits (as $\mathcal{O}_{K,S}$ -modules). The module M is $\mathcal{O}_{K,S}$ -free by assumption. Therefore, as $\mathcal{O}_{K,S}$ -modules, we have $M_N \cong \mathcal{O}_{K,S}^a \oplus \mathfrak{b}$ and $M/M_N \cong \mathcal{O}_{K,S}^b \oplus \mathfrak{b}^{-1}$ for some fractional $\mathcal{O}_{K,S}$ -ideal \mathfrak{b} . The module M/M_N (both as $\mathcal{O}_{K,S}$ -module and as $\mathcal{O}_{K,S}[C_\ell]$ -module) is determined up to isomorphism by b and the ideal class of \mathfrak{b} . Since the ideal \mathfrak{b} is equivalent to the norm of the ideal \mathfrak{a} in the extension $\mathcal{O}_{K,S}[\zeta_\ell]/\mathcal{O}_{K,S}$, its ideal class is determined by the one of \mathfrak{a} .

It remains to identify the $\mathcal{O}_{K,S}[C_\ell]$ -module structure of M in terms of the module structures of M_N and M/M_N . For this, it suffices to determine the action of $[1]$. We noted above that $M_N \cong \mathcal{O}_{K,S}[\zeta_\ell]^r \oplus \mathfrak{a}$ and there is a surjection $M_N \twoheadrightarrow (\mathcal{O}_{K,S}[\zeta_\ell]/(\zeta_\ell - 1))^s =: B$ compatible with the above decomposition. Choose β_1, \dots, β_s preimages of 1: in the summands $\mathcal{O}_{K,S}[\zeta_\ell]$ we can just choose 1, in the \mathfrak{a} -summand we can choose any element not contained in $(\zeta_\ell - 1)\mathfrak{a}$. As in Reiner’s paper, we have

$$([1] - 1)M = (\zeta_\ell - 1)M_N + ([1] - 1)X,$$

where X is a choice of $\mathcal{O}_{K,S}$ -complement of M_N lifting M/M_N . Therefore, any element of the form $(g - 1)x$ for $x \in M/M_N$ is congruent module $(\zeta_\ell - 1)M_N$ to an $\mathcal{O}_{K,S}/(\ell)$ -linear combination of the β_i . The normalization of the action for the $\mathcal{O}_{K,S}$ -free part is done as in [13, Lemma 4].

To deal with the non-free part \mathfrak{b} of M/M_N , denote by β the choice of lift of $1 \in \mathcal{O}_{K,S}/(\ell) \cong \mathfrak{a}/(\zeta_\ell - 1)\mathfrak{a}$ to \mathfrak{a} . The norm $\text{Nm}_{K(\zeta_\ell)/K}(\beta)$ is an element of $\text{Nm}(\mathfrak{a}) \cong \mathfrak{b}$. Then $([1] - 1)x$ for $x \in \mathfrak{b}$ is congruent to $\beta \cdot \text{ev}(\text{Nm}(\beta), x)$, by an appropriate version of [13, Lemma 4]. Here $\text{ev} : \mathfrak{b} \otimes \mathfrak{b}^{-1} \rightarrow \mathcal{O}_{K,S}$ is the evaluation pairing.

To sum up, this shows the following

Theorem 20. *Let $\mathcal{O}_{K,S}$ be a ring of S -integers in a number field, let ℓ be a prime and assume that $\mathcal{O}_{K,S}[T]/(\Phi_\ell(T))$ is a Dedekind domain. Then the isomorphism classes of finitely generated $\mathcal{O}_{K,S}$ -free $\mathcal{O}_{K,S}[\mathbb{C}_\ell]$ -modules are parametrized by*

- (1) the $\mathcal{O}_{K,S}[\zeta_\ell]$ -rank of M_N ,
- (2) the ideal class of the determinant of the $\mathcal{O}_{K,S}[\zeta_\ell]$ -module M_N ,
- (3) the $\mathcal{O}_{K,S}$ -rank of M/M_N ,
- (4) the $\mathcal{O}_{K,S}/(\ell)$ -rank of the quotient $([1] - 1)M/(\zeta_\ell - 1)M_N$.

In the above, any integer $n \geq 0$ is possible in (i) and (iii), but the integer in (iv) is bounded above by $\min(i, iii)$. Any ideal class is possible.

We now outline a very special case of the classification which works under the weaker assumption that $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind ring but in which $\Phi_\ell(T)$ need not be K -irreducible. We restrict to the case where M_N has $\mathcal{O}_{K,S}[\zeta_\ell]$ -rank 1. In this case, base-change to K results in one of the irreducible K -representations of \mathbb{C}_ℓ . The $\mathcal{O}_{K,S}[T]/(\Phi_\ell)$ -module structure factors through a projection $\mathcal{O}_{K,S}[T]/(\Phi_\ell) \twoheadrightarrow \mathcal{O}_{K,S}[\zeta_\ell]$ and is completely determined by this. Again, the $\mathcal{O}_{K,S}[\mathbb{C}_\ell]$ -module structure of M_N is completely determined by a fractional ideal \mathfrak{a} in $\mathcal{O}_{K,S}[\zeta_\ell]$. The rest of the analysis goes through to show the following

Proposition 21. *Let $\mathcal{O}_{K,S}$ be a ring of S -integers in a number field, let ℓ be a prime and assume that $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind domain. Then the isomorphism classes of finitely generated $\mathcal{O}_{K,S}$ -free $\mathcal{O}_{K,S}[\mathbb{C}_\ell]$ -modules M where M_N has $\mathcal{O}_{K,S}[\zeta_\ell]$ -rank 1 are parametrized by*

- (1) the ideal class of the $\mathcal{O}_{K,S}[\zeta_\ell]$ -module M_N ,
- (2) the $\mathcal{O}_{K,S}$ -rank of M/M_N ,
- (3) the $\mathcal{O}_{K,S}/(\ell)$ -rank of the quotient $([1] - 1)M/(\zeta_\ell - 1)M_N$.

In the above, any integer $n \geq 0$ is possible in (ii), but the integer in (iii) can only be 0 or 1. Any ideal class is possible.

Remark 22. Pulling back modules along the two projections $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}[T]/(\Phi_7(T)) \rightarrow \mathcal{O}_{\mathbb{Q}(\zeta_7)}$ results in non-isomorphic modules, belonging to non-isomorphic $\mathbb{Q}(\sqrt{-7})$ -representations of \mathbb{C}_7 . However, this effectively only amounts to different choices of generators of conjugate subgroups. If we are only interested in counting subgroups, this doesn't affect the end result.

5.3. Centralizers and normalizers. We now need to describe centralizers and normalizers of the corresponding \mathbb{C}_ℓ -subgroups of $\mathrm{GL}_n(\mathcal{O}_{K,S})$. For the purpose of the following section, fix a subgroup $\iota : \mathbb{C}_\ell \hookrightarrow \mathrm{GL}_n(\mathcal{O}_{K,S})$ and the corresponding $\mathcal{O}_{K,S}[\mathbb{C}_\ell]$ -module M . Since our intended application is to essential rank one cases, most notably $\mathrm{GL}_3(\mathcal{O}_{K,S})$, we assume throughout the section that the associated module M is such that its associated representation over K is of the form $K \times K(\zeta_\ell)$. We also assume in the following section that $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind domain.

First, we can embed $\mathrm{GL}_n(\mathcal{O}_{K,S}) \hookrightarrow \mathrm{GL}_n(K)$. The centralizer of $\mathbb{C}_\ell \hookrightarrow \mathrm{GL}_n(K)$ is the automorphism group of the representation $M \otimes_{\mathcal{O}_{K,S}} K \cong K \times K(\zeta_\ell)$ of \mathbb{C}_ℓ over K . Under our assumptions $\zeta_\ell \notin K$ the \mathbb{C}_ℓ -representation $K(\zeta_\ell)$ is K -irreducible. In particular,

$$\mathrm{Hom}_{K[\mathbb{C}_\ell]}(K(\zeta_\ell), K) \cong \mathrm{Hom}_{K[\mathbb{C}_\ell]}(K, K(\zeta_\ell)) \cong 0.$$

From this, any $K[\mathbb{C}_\ell]$ -automorphism ϕ of $K \oplus K(\zeta_\ell)$ must be of the form $\phi_K \oplus \phi_{K(\zeta_\ell)}$ where ϕ_K and $\phi_{K(\zeta_\ell)}$ are $K[\mathbb{C}_\ell]$ -automorphisms of K and $K(\zeta_\ell)$, respectively. Via the embedding $\mathrm{GL}_n(\mathcal{O}_{K,S}) \hookrightarrow \mathrm{GL}_n(K)$, the same must be true for automorphisms of the $\mathcal{O}_{K,S}[\mathbb{C}_\ell]$ -modules. In terms of the centralizer as a subgroup of $\mathrm{GL}_n(\mathcal{O}_{K,S})$, this means that the centralizer must be conjugate to a block-diagonal matrix. For the normalizer, similar statements apply. The only additional elements in the normalizer would come from K -linear automorphisms of $K(\zeta_\ell)$ which are accounted for by the Galois group $\mathrm{Gal}(K(\zeta_\ell)/K)$.

Now we need some induction-type theorems to determine the automorphism groups of the individual almost-direct summands of the module M .

Lemma 23. *Let M be an $\mathcal{O}_{K,S}[\mathbb{C}_\ell]$ -module such that multiplication with the norm element N is the zero map and assume that the $\mathcal{O}_{K,S}[\zeta_\ell]$ -rank of M is 1. Then*

$$\mathrm{Aut}_{\mathcal{O}_{K,S}[\mathbb{C}_\ell]}(M) \cong \mathrm{Aut}_{\mathcal{O}_{K,S}[\zeta_\ell]}(M) \cong \mathcal{O}_{K,S}[\zeta_\ell]^\times.$$

Proof. Since the norm element N annihilates M , it has an induced module structure for

$$\mathcal{O}_{K,S}[\mathbb{C}_\ell]/(N) \cong \mathcal{O}_{K,S}[\zeta_\ell].$$

This yields a homomorphism $\text{Aut}_{\mathcal{O}_{K,S}[\mathbb{C}_\ell]}(M) \rightarrow \text{Aut}_{\mathcal{O}_{K,S}[\zeta_\ell]}(M)$. This homomorphism is injective, since both automorphism groups embed into $\text{Aut}_{\mathcal{O}_{K,S}}(M)$. The natural restriction map along the homomorphism $\mathcal{O}_{K,S}[\mathbb{C}_\ell] \rightarrow \mathcal{O}_{K,S}[\zeta_\ell]$ provides an inverse, establishing the first isomorphism.

For the second isomorphism, we know by Reiner's classification result that M is a finitely generated projective $\mathcal{O}_{K,S}[\zeta_\ell]$ -module, and our additional assumption is that its rank is 1. Since local units can be patched to global units, the automorphism group of a finitely generated projective $\mathcal{O}_{K,S}[\zeta_\ell]$ -module of rank 1 is isomorphic to $\mathcal{O}_{K,S}[\zeta_\ell]^\times$. \square

Lemma 24. *Let M be an $\mathcal{O}_{K,S}[\mathbb{C}_\ell]$ -module such that multiplication with the norm element N is injective and M is $\mathcal{O}_{K,S}$ -free of rank 1. Then*

$$\text{Aut}_{\mathcal{O}_{K,S}[\mathbb{C}_\ell]}(M) \cong \text{Aut}_{\mathcal{O}_{K,S}}(M) \cong \mathcal{O}_{K,S}^\times.$$

Proof. Injectivity of multiplication with the norm implies that the action of \mathbb{C}_ℓ is trivial, by Reiner's classification result [13].

By assumption we have $M \cong \mathcal{O}_{K,S}$ (as $\mathcal{O}_{K,S}$ -modules), and therefore the second isomorphism $\text{Aut}_{\mathcal{O}_{K,S}}(M) \cong \mathcal{O}_{K,S}^\times$ follows immediately.

An $\mathcal{O}_{K,S}[\mathbb{C}_\ell]$ -automorphism of M is in particular an $\mathcal{O}_{K,S}$ -automorphism, giving rise to an injective restriction map $\text{Aut}_{\mathcal{O}_{K,S}[\mathbb{C}_\ell]}(M) \rightarrow \text{Aut}_{\mathcal{O}_{K,S}}(M)$. Since any $\mathcal{O}_{K,S}$ -automorphism of M automatically commutes with the trivial \mathbb{C}_ℓ -action we get the first isomorphism. \square

The above results now imply that we have an induced morphism

$$\text{Aut}_{\mathcal{O}_{K,S}[\mathbb{C}_\ell]}(M) \rightarrow \mathcal{O}_{K,S}[\mathbb{C}_\ell]^\times \times \mathcal{O}_{K,S}^\times,$$

where M is a module corresponding to a \mathbb{C}_ℓ -subgroup of $\text{GL}_3(\mathcal{O}_{K,S})$. For the split module, this actually describes the full centralizer. For the non-split module where there is an additional unipotent action, we have morphisms $\mathcal{O}_{K,S}[\zeta_\ell] \rightarrow \mathcal{O}_{K,S}/(\ell)$ and $\mathcal{O}_{K,S} \rightarrow \mathcal{O}_{K,S}/(\ell)$ given by reduction mod ℓ . These ring homomorphisms induce maps on the unit groups.

Lemma 25. *Assume M is the $\mathcal{O}_{K,S}[\mathbb{C}_\ell]$ -module associated to a \mathbb{C}_ℓ -subgroup of $\text{GL}_3(\mathcal{O}_{K,S})$ where $[K(\zeta_\ell) : K] = 2$. The induced morphism from the automorphism group above factors through an isomorphism*

$$\text{Aut}_{\mathcal{O}_{K,S}[\mathbb{C}_\ell]}(M) \rightarrow \mathcal{O}_{K,S}[\mathbb{C}_\ell]^\times \times_{(\mathcal{O}_{K,S}/(\ell))^\times} \mathcal{O}_{K,S}^\times.$$

Proof. It remains to identify the image of the induced morphism. Let

$$(\phi, \psi) \in \mathcal{O}_{K,S}[\mathbb{C}_\ell]^\times \times_{(\mathcal{O}_{K,S}/(\ell))^\times} \mathcal{O}_{K,S}^\times.$$

To set up notation, let $M = \mathfrak{a} \oplus \mathcal{O}_{K,S}$ with the action specified as in Reiner's results. We need to check when (ϕ, ψ) commutes with the action on the summand $\mathcal{O}_{K,S}$. This action sends a generator y to (β, y) where $\beta \in \mathfrak{a}$ is a choice of preimage of 1 in $\mathcal{O}_{K,S}/(\ell)$. Formulated differently, the action on $x \in \mathcal{O}_{K,S}$ adds a specific choice of lift $\tilde{x} \in \mathfrak{a}$ of the reduction \bar{x} of x mod ℓ . For notational purposes, we denote \tilde{x} by $\beta(x)$.

Now we want to determine when the action commutes with the automorphism (ϕ, ψ) . If we first apply the action and then the automorphism, then we get $\phi(\beta(y))$ in the component \mathfrak{a} . If, on the other hand, we first apply the automorphism and then the action, we get $\beta(\psi(y))$ in the component \mathfrak{a} . For $\phi(\beta(y)) = \beta(\psi(y))$, it is necessary and sufficient that the reduction of ϕ and ψ to $\mathcal{O}_{K,S}/(\ell)$ are the same. This is precisely the claim. \square

Lemma 26. *Assume M is the $\mathcal{O}_{K,S}[\mathbb{C}_\ell]$ -module associated to a \mathbb{C}_ℓ -subgroup of $\text{GL}_3(\mathcal{O}_{K,S})$ with $[K(\zeta_\ell) : K] = 2$. In particular, $M \cong \mathfrak{a} \oplus \mathcal{O}_{K,S}$ for an ideal class \mathfrak{a} of $\mathcal{O}_{K,S}[\zeta_\ell]$. The group of special semilinear automorphisms of M is of the form*

$$(\text{Aut}_{\mathcal{O}_{K,S}[\mathbb{C}_\ell]}(M)) \rtimes \text{Stab}(\mathfrak{a}, \text{Gal}(K(\zeta_\ell)/K)).$$

The action is the natural Galois action on the automorphism group, viewed as fiber product of unit groups as in Lemma 25.

Proof. By embedding $\mathcal{O}_{K,S}[\mathbb{C}_\ell] \hookrightarrow K[\mathbb{C}_\ell]$, we already know that the only semilinear automorphisms that are not in the automorphism group come from the Galois-action of $\text{Gal}(K(\zeta_\ell)/K)$. However, the Galois group does not need to stabilize the isomorphism class of the module; this happens whenever we have a non-trivial Galois action on the class group of $K(\zeta_\ell)$. The semilinear automorphisms modulo the linear ones are exactly identified with the stabilizer of the ideal class \mathfrak{a} in the Galois group, as claimed. \square

6. EXAMPLE CASES

Now we discuss a couple of example cases to compare them to the computer calculations as sanity check. Actually, the following examples can be generalized to computations of Farrell–Tate cohomology of groups $\mathrm{PGL}_3(\mathcal{O}_{K,S})$ provided $\zeta_\ell \notin K$ and $\mathcal{O}_{K,S}[\zeta_\ell]$ is a Dedekind ring.

6.1. Homological 3-torsion in PGL_3 over quadratic imaginary integers. We applied the rigid facets subdivision algorithm to the $\mathrm{PGL}_3(\mathbb{Z}[i])$ cell complex of Mathieu Dutour Sikiric [5] and the $\mathrm{GL}_3(\mathbb{Z}[i])$ cell complex of Sebastian Schoennenbeck [16], extracted the 3-torsion subcomplex, and reduced it using the methods of [10], in both cases obtaining the following graph \mathcal{T} consisting of two equivalent connected components.

$$\mathrm{D}_3 \bullet \xrightarrow{\mathrm{C}_3} \mathrm{D}_3 \quad \mathrm{D}_3 \bullet \xrightarrow{\mathrm{C}_3} \mathrm{D}_3$$

Because there is up to conjugating isomorphism just one inclusion $\mathrm{C}_3 \rightarrow \mathrm{D}_3$, the $d_1^{p,q}$ -differentials of the equivariant spectral sequence with \mathbb{F}_3 -coefficients on \mathcal{T} have the maximal possible ranks, i.e., they are surjective whenever both originating domain and target contain 3-torsion. Then the $E_2 = E_\infty$ page yields

$$\dim_{\mathbb{F}_3} \widehat{\mathrm{H}}^{p+q}(\mathrm{PGL}_3(\mathbb{Z}[i]); \mathbb{F}_3) = \begin{cases} 0, & p+q \equiv 1 \pmod{4}, \\ 2, & p+q \equiv 2 \pmod{4}, \\ 4, & p+q \equiv 3 \pmod{4}, \\ 2, & p+q \equiv 4 \pmod{4}. \end{cases}$$

In order to check the above result, we compute now the Farrell–Tate cohomology of $\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-d})})$ with \mathbb{F}_3 -coefficients, for positive square-free d , using the Brown complex. To be able to use Lemma 19 to get relative integral bases we are excluding the problematic cases where $3 \mid d$. Restricting further to the short list of those d where $\mathbb{Q}(\sqrt{-d})$ has class number 1, we can then apply Reiner’s result, cf. Theorem 20. These results tell us that there are $2h_{\mathbb{Q}(\sqrt{-d}, \zeta_3)}$ conjugacy classes of embeddings $\mathrm{C}_3 \hookrightarrow \mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-d})})$; for each element of the class group of $\mathbb{Q}(\sqrt{-d}, \zeta_3)$, we have the two possible choices of either the split module or the fiber product over $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}/(3)$. However, for the imaginary quadratic fields with class number 1 and discriminant coprime to 3, i.e., the fields $\mathbb{Q}(\sqrt{-d})$ with $d \in \{1, 2, 7, 11, 19, 43, 67, 163\}$, the extension fields $\mathbb{Q}(\sqrt{-d}, \zeta_3)$ also all have class number 1. Therefore, for these d , the group $\mathrm{GL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-d})})$ has exactly 2 conjugacy classes of cyclic subgroups of order 3. The corresponding centralizers are of the form

$$\mathcal{O}_{\mathbb{Q}(\sqrt{-d}, \zeta_3)}^\times \times \mathcal{O}_{\mathbb{Q}(\sqrt{-d})}^\times \quad \text{and} \quad \mathcal{O}_{\mathbb{Q}(\sqrt{-d}, \zeta_3)}^\times \times_{(\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}/(3))^\times} \mathcal{O}_{\mathbb{Q}(\sqrt{-d})}^\times,$$

respectively, and the normalizers will be extensions of these by the group

$$\mathrm{Gal}(\mathbb{Q}(\sqrt{-d}, \zeta_3)/\mathbb{Q}(\sqrt{-d})) \cong \mathbb{Z}/2\mathbb{Z}$$

acting as multiplication by -1 on the first factor and trivially on the second. Note that these actions are actually compatible via the reduction to $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}/(3)$ because the extension $\mathbb{Q}(\sqrt{-d}, \zeta_3)/\mathbb{Q}(\sqrt{-d})$ is completely ramified over (3) .

By Dirichlet’s unit theorem, we have

$$\left(\mathcal{O}_{\mathbb{Q}(\sqrt{-d}, \zeta_3)}^\times \times \mathcal{O}_{\mathbb{Q}(\sqrt{-d})}^\times \right) \rtimes \mathbb{Z}/2\mathbb{Z} \cong ((\mathbb{Z} \times \mu_3) \rtimes \mathbb{Z}/2\mathbb{Z}) \times \mu_n^{\times 2}$$

where $n = 2$ except in the case $K = \mathbb{Q}(i)$ where $n = 4$.

For the normalizers of the non-split representation, denote by $E = \mathcal{O}_{\mathbb{Q}(\sqrt{-d})}/(3)$. We have two cases: $E \cong \mathbb{F}_3 \times \mathbb{F}_3$ if $-d$ is a square mod 3 (i.e. for $d \in \{2, 11\}$) and $E \cong \mathbb{F}_9$ if not (i.e. for $d \in \{1, 7, 19, 43, 67, 163\}$). We have $(\mathbb{F}_3 \times \mathbb{F}_3)^\times \cong (\mathbb{Z}/2\mathbb{Z})^{\times 2}$ and $\mathbb{F}_9^\times \cong \mathbb{Z}/8\mathbb{Z}$. The reduction map $\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}^\times \rightarrow E^\times$ is injective for any d . For $\mathcal{O}_{\mathbb{Q}(\sqrt{-d}, \zeta_3)}^\times \cong \mathbb{Z} \times \mu_{3n}$, the reduction map $\mathcal{O}_{\mathbb{Q}(\sqrt{-d}, \zeta_3)}^\times \rightarrow E^\times$ is injective on μ_n and the zero map on μ_3 . In particular, we have

$$\left(\mathcal{O}_{\mathbb{Q}(\sqrt{-d}, \zeta_3)}^\times \times_{E^\times} \mathcal{O}_{\mathbb{Q}(\sqrt{-d})}^\times \right) \rtimes \mathbb{Z}/2\mathbb{Z} \cong ((\mathbb{Z} \times \mu_3) \rtimes \mathbb{Z}/2\mathbb{Z}) \times \mu_n$$

We can now state the computation of the Farrell–Tate cohomology of $\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-d})})$.

Proposition 27. *Let $d \in \{1, 2, 7, 11, 19, 43, 67, 163\}$. Then we have*

$$\widehat{\mathrm{H}}^\bullet(\mathrm{PGL}_3(\mathcal{O}_{\mathbb{Q}(\sqrt{-d})}); \mathbb{F}_3) \cong \widehat{\mathrm{H}}^\bullet((\mathbb{Z} \times \mu_3) \rtimes \mathbb{Z}/2\mathbb{Z}; \mathbb{F}_3)^{\oplus 2}$$

More explicit information on the Farrell–Tate cohomology of such groups can now be obtained via the following computation included in [12]:

Proposition 28. *Let $A = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}^r$, and let ℓ be an odd prime with $\ell \mid n$. Then, with b_1, x_1, \dots, x_r denoting classes in degree 1 and a_2 a class of degree 2, we have*

$$\widehat{H}^\bullet(A; \mathbb{F}_\ell) \cong \widehat{H}^\bullet(\mathbb{Z}/n\mathbb{Z}; \mathbb{F}_\ell) \otimes_{\mathbb{F}_\ell} \bigwedge^r \mathbb{F}_\ell \cong \mathbb{F}_\ell[a_2, a_2^{-1}](b_1, x_1, \dots, x_r).$$

The Hochschild–Serre spectral sequence associated to the semi-direct product $A \rtimes \mathbb{Z}/2\mathbb{Z}$ (where $\mathbb{Z}/2\mathbb{Z}$ acts as -1 on A) degenerates and yields an isomorphism

$$\widehat{H}^\bullet(A \rtimes \mathbb{Z}/2\mathbb{Z}; \mathbb{F}_\ell) \cong \widehat{H}^\bullet(A; \mathbb{F}_\ell)^{\mathbb{Z}/2\mathbb{Z}}.$$

The invariant classes are then given by $a_2^{\otimes 2i}$ tensor the even exterior powers plus $a_2^{\otimes (2i+1)}$ tensor the odd exterior powers.

As a direct application, the Farrell–Tate cohomology of a group like

$$\mathcal{O}_K[\zeta_\ell]^\times \rtimes \text{Gal}(K(\zeta_\ell)/K) \cong (\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$$

with $K = \mathbb{Q}(\sqrt{-d})$ (where the action in the semidirect product on the right is consequently given by multiplication with -1) looks like the direct sum of two copies of the cohomology of the dihedral group with $2n$ elements, with one copy shifted by one.

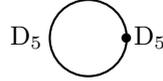
The algebra in Proposition 27 is given by the $\mathbb{Z}/2\mathbb{Z}$ -invariant elements in $\mathbb{F}_3[a_2^{\pm 1}](b_1, x_1)$, where the action of $\mathbb{Z}/2\mathbb{Z}$ is by multiplication with -1 on all the generators. The invariant subalgebra is then generated by the classes $b_1 x_1$ in degree 2, $b_1 a_2$ and $x_1 a_2$ in degree 3, and a_2^2 in degree 4. Consequently, the Hilbert–Poincaré series for the positive degrees is

$$2 \frac{T^2 + 2T^3 + T^4}{1 - T^4} = 2 \frac{T^2(1 + T)^2}{1 - T^4}.$$

Actually, similar results are true for real quadratic fields of class number one with discriminant coprime to 3. There are two conjugacy classes of order 3 subgroups. Their normalizers, however, are of the form $(\mu_3 \rtimes \mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}^2$. The Farrell–Tate cohomology algebra for this is

$$\mathbb{F}_3[a_2^{\pm 2}](b_1^3, x_1, y_1)^{\oplus 2}.$$

6.2. Homological 5-torsion in $\text{PSL}_4(\mathbb{Z})$. We applied the rigid facets subdivision algorithm to the $\text{PSL}_4(\mathbb{Z})$ -equivariant cell complex of [5], extracted the 5-torsion subcomplex, and reduced it using the methods of [10] to the following graph \mathcal{T} .



The d^1 -differential of the equivariant spectral sequence on \mathcal{T} is zero, because the isomorphisms at edge end and edge origin cancel each other. Then the $E_1 = E_\infty$ page is concentrated in the columns $p = 0$ and 1, with dimensions over \mathbb{F}_5 being 1 in rows q congruent to 3 or 4 mod 4, and zero otherwise. This yields

$$\dim_{\mathbb{F}_5} \widehat{H}^{p+q}(\text{PSL}_4(\mathbb{Z}); \mathbb{F}_5) = \begin{cases} 1, & p+q \equiv 1 \pmod{4}, \\ 0, & p+q \equiv 2 \pmod{4}, \\ 1, & p+q \equiv 3 \pmod{4}, \\ 2, & p+q \equiv 4 \pmod{4}. \end{cases}$$

We check this result with a computation of $\widehat{H}^\bullet(\text{PSL}_4(\mathbb{Z}); \mathbb{F}_5)$ using Brown’s complex [2, last chapters]. In this case, it is standard that the set $\{1, \zeta_5, \zeta_2^3, \zeta_5^3\}$ is an integral basis of $\mathcal{O}_{\mathbb{Q}(\zeta_5)}$ and in particular $\mathbb{Z}[\zeta_5] = \mathcal{O}_{\mathbb{Q}(\zeta_5)}$ is a Dedekind ring.

We can therefore use Reiner’s result to determine conjugacy classes of C_5 -subgroups in $\text{GL}_4(\mathbb{Z})$. Since both \mathbb{Z} and $\mathbb{Z}[\zeta_5]$ have trivial class group, there is only one isomorphism class of $\mathbb{Z}[C_5]$ -module with nontrivial action and \mathbb{Z} -rank 4. Hence, there is a unique conjugacy class of cyclic order 5 subgroup in $\text{GL}_4(\mathbb{Z})$. Since the center of $\text{GL}_4(\mathbb{Z})$ is of order 2, the same is true for $\text{PGL}_4(\mathbb{Z})$.

Now there is a necessary modification to deal with the case $\text{SL}_4(\mathbb{Z})$, along the lines of the discussion in [12]. While conjugacy classes of C_5 -subgroups in $\text{GL}_4(\mathbb{Z})$ correspond to isomorphism classes of $\mathbb{Z}[C_5]$ -modules, the conjugacy classes of C_5 -subgroups in $\text{SL}_4(\mathbb{Z})$ correspond to such modules *equipped with an additional orientation*, i.e., a choice of isomorphism $\det M \cong \bigwedge_{\mathbb{Z}}^4 M \cong \mathbb{Z}$. The conjugacy class in $\text{GL}_4(\mathbb{Z})$ lifts to $\text{SL}_4(\mathbb{Z})$, and the corresponding module has two different choices of orientation. The Galois group $\text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q}) \cong \mathbb{Z}/4\mathbb{Z}$ acts on the set of oriented modules. The action exchanges the orientations. Therefore, there is one conjugacy class of C_5 -subgroup in $\text{SL}_4(\mathbb{Z})$ stabilized by $\mathbb{Z}/2\mathbb{Z} \hookrightarrow \text{Gal}(\mathbb{Q}(\zeta_5)/\mathbb{Q})$.

The centralizer of this C_5 -subgroup is the group of norm-1 units of $\mathbb{Z}[\zeta_5]$, which by Dirichlet's unit theorem is isomorphic to

$$\ker(\mathbb{Z}[\zeta_5]^\times \rightarrow \mathbb{Z}^\times) \cong \mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}.$$

As before, the normalizer is an extension of the centralizer by an action of the stabilizer of the corresponding oriented module in the Galois group. We noted above that the Galois group $\mathbb{Z}/4\mathbb{Z}$ exchanges the two orientations of the trivial module, hence the stabilizer is the subgroup $\mathbb{Z}/2\mathbb{Z} \subset \mathbb{Z}/4\mathbb{Z}$. The normalizer therefore is of the form $(\mathbb{Z}/10\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$. The action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/10\mathbb{Z}$ is by multiplication with -1 because the action of the Galois group is via the identification $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/5\mathbb{Z}^\times$. The action of $\mathbb{Z}/2\mathbb{Z}$ on \mathbb{Z} is trivial: the full Galois group acts on \mathbb{Z} via a surjective homomorphism $\mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}^\times \cong \mathbb{Z}/2\mathbb{Z}$. The stabilizer of the oriented module in the Galois group lies in the kernel of the above action, as claimed. Therefore, the normalizer is in fact of the form $D_{10} \times \mathbb{Z}$.

By Proposition 28, the Farrell–Tate cohomology of the normalizer is of the form $\mathbb{F}_5[a_2^{\pm 2}](b_1^3)^{\oplus 2} \oplus \mathbb{F}_5[a_2^{\pm 2}](b_1^3)_{-1}^{\oplus 2}$ where the lower subscript -1 indicates a degree shift by -1 . The Hilbert–Poincaré series for the positive degrees is

$$\frac{T^3 + 2T^4 + T^5}{1 - T^4} = \frac{T^3(1 + T)^2}{1 - T^4}.$$

The computations in [5] show that the 5-torsion in integral homology of $\mathrm{PSL}_4(\mathbb{Z})$ of dimension 1 in degrees $0, 3 \bmod 4$ and trivial otherwise. By the universal coefficient theorem, this agrees with the above computation.

REFERENCES

- [1] Oliver Braun, Renaud Coulangéon, Gabriele Nebe, and Sebastian Schönnenbeck, *Computing in arithmetic groups with Voronoi's algorithm*, J. Algebra **435** (2015), 263–285. Zbl 1323.16014
- [2] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. MR1324339
- [3] Anh Tuan Bui and Graham J. Ellis, *Computing Bredon homology of groups*, accepted for publication in Journal of Homotopy and Related Structures, 2016.
- [4] Anh Tuan Bui and Alexander D. Rahm, *Torsion Subcomplexes Subpackage*, accepted sub-package in HAP (Homological Algebra Programming) in the computer algebra system GAP., 2016, Source code available at <http://math.uni.lu/~rahm/subpackage-documentation/TorsionSubcomplexesSubpackage.tar.gz>.
- [5] Graham J. Ellis, Achill Schuermann, and Mathieu Dutour Sikirić, *On the integral homology of $\mathrm{PSL}_4(\mathbb{Z})$ and other arithmetic groups*, J. Number Theory **131** (2011), no. 12, 2368–2375. Zbl 1255.11028
- [6] Herbert Gangl, Paul E. Gunnells, Jonathan Hanke, Achill Schuermann, Mathieu Dutour Sikirić, and Dan Yasaki, *On the cohomology of linear groups over imaginary quadratic fields*, to appear in Journal of Pure and Applied Algebra, arXiv: 1307.1165 (2016).
- [7] Philippe Elbaz-Vincent and Herbert Gangl and Christophe Soulé, *Perfect forms, K-theory and the cohomology of modular groups*, Adv. Math. **245** (2013), 587–624, DOI 10.1016/j.aim.2013.06.014. Zbl 1290.11104
- [8] Claiborne G. Latimer and C. C. MacDuffee, *A correspondence between classes of ideals and classes of matrices*, Ann. of Math. (2) **34** (1933), no. 2, 313–316, DOI 10.2307/1968204. MR1503108
- [9] Guido Mislin and Alain Valette, *Proper group actions and the Baum–Connes conjecture*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2003. MR2027168 (2005d:19007), Zbl 1028.46001
- [10] Alexander D. Rahm, *Accessing the cohomology of discrete groups above their virtual cohomological dimension*, J. Algebra **404** (2014), 152–175. MR3177890
- [11] ———, *On the equivariant K-homology of PSL_2 of the imaginary quadratic integers*, Annales de l'Institut Fourier **66** (2016), no. 4, 1667–1689.
- [12] Alexander D. Rahm and Matthias Wendt, *On Farrell–Tate cohomology of SL_2 over S -integers*, arXiv:1411.3542v2.
- [13] Irving Reiner, *Integral representations of cyclic groups of prime order*, Proc. Amer. Math. Soc. **8** (1957), 142–146. MR0083493
- [14] Rubén J. Sánchez-García, *Bredon homology and equivariant K-homology of $\mathrm{SL}(3, \mathbb{Z})$* , J. Pure Appl. Algebra **212** (2008), no. 5, 1046–1059. MR2387584 (2009b:19007)
- [15] Rubén J. Sánchez-García, *Equivariant K-homology for some Coxeter groups*, J. Lond. Math. Soc. (2) **75** (2007), no. 3, 773–790, DOI 10.1112/jlms/jdm035. MR2352735 (2009b:19006)
- [16] Sebastian Schoenbeck, *Resolutions for unit groups of orders*, ArXiv e-prints, math.KT 1609.08835 (2016).
- [17] Jean-Pierre Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003. Translated from the French original by John Stillwell; Corrected 2nd printing of the 1980 English translation. MR1954121

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