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# About Jarník's-type relation in higher dimension

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## Abstract

Using the Parametric Geometry of Numbers introduced recently by W.M. Schmidt and L. Summerer [17, 18] and results by D. Roy [13, 14], we show that German's transference inequalities between the two most classical exponents of uniform Diophantine approximation are optimal. Further, we establish that the  $n$  uniform exponents of Diophantine approximation in dimension  $n$  are algebraically independent. Thus, no Jarník's-type relation holds between them.

## 1 Introduction

Throughout this paper, the integer  $n \geq 1$  denotes the dimension of the ambient space,  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$  denotes an  $n$ -tuple of real numbers such that  $1, \theta_1, \dots, \theta_n$  are  $\mathbb{Q}$ -linearly independent.

Let  $d$  be an integer with  $0 \leq d \leq n - 1$ . We define the exponent  $\omega_d(\boldsymbol{\theta})$  (resp. the uniform exponent  $\hat{\omega}_d(\boldsymbol{\theta})$ ) as the supremum of the real numbers  $\omega$  for which there exist rational affine subspaces  $L \subset \mathbb{R}^n$  such that

$$\dim(L) = d, \quad H(L) \leq H \quad \text{and} \quad H(L)d(\boldsymbol{\theta}, L) \leq H^{-\omega}$$

for arbitrarily large real numbers  $H$  (resp. for every sufficiently large real number  $H$ ). Here  $H(L)$  denotes the height of  $L$  (see [16] for more details), and  $d(\boldsymbol{\theta}, L) = \min_{P \in L} d(\boldsymbol{\theta}, P)$  is the minimal distance between  $\boldsymbol{\theta}$  and a point of  $L$ .

These exponents were introduced originally by M. Laurent [11]. They interpolate between the classical exponents  $\omega(\boldsymbol{\theta}) = \omega_{n-1}(\boldsymbol{\theta})$  and  $\lambda(\boldsymbol{\theta}) = \omega_0(\boldsymbol{\theta})$  (resp.  $\hat{\omega}(\boldsymbol{\theta}) = \hat{\omega}_{n-1}(\boldsymbol{\theta})$  and  $\hat{\lambda}(\boldsymbol{\theta}) = \hat{\omega}_0(\boldsymbol{\theta})$ ) that were introduced by A. Khinchin [7, 8], V. Jarník [6] and Y. Bugeaud and M. Laurent [1, 2].

We have the relations

$$\omega_0(\boldsymbol{\theta}) \leq \omega_1(\boldsymbol{\theta}) \leq \dots \leq \omega_{n-1}(\boldsymbol{\theta}),$$

$$\hat{\omega}_0(\boldsymbol{\theta}) \leq \hat{\omega}_1(\boldsymbol{\theta}) \leq \cdots \leq \hat{\omega}_{n-1}(\boldsymbol{\theta}),$$

and Minkowski's First Convex Body Theorem [12] and Mahler's compound convex bodies theory provide the lower bounds

$$\omega_d(\boldsymbol{\theta}) \geq \hat{\omega}_d(\boldsymbol{\theta}) \geq \frac{d+1}{n-d}, \quad \text{for } 0 \leq d \leq n-1.$$

These exponents happen to be related, as was first noticed by Khinchin with his transference theorem [8]. The study of these transferences has two aspects. First, establishing transference inequalities valid for every suitable point  $\boldsymbol{\theta}$ . Then, there is the reverse problem, that consists in constructing points  $\boldsymbol{\theta}$  to show that these inequalities are sharp. For this, one can prove that there exists points  $\boldsymbol{\theta}$  whose exponents satisfy the equality in the transference inequalities. In this case, we say that the inequalities are *best possible*. A stronger result is to prove that given  $k$  exponents  $e_1, \dots, e_k$ , the transference inequalities between these  $k$  exponents define a subset of  $\mathbb{R}^k$  that is exactly the set of all  $k$ -uples  $(e_1(\boldsymbol{\theta}), \dots, e_k(\boldsymbol{\theta}))$  as  $\boldsymbol{\theta}$  runs through all points  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$  such that  $1, \theta_1, \dots, \theta_n$  are  $\mathbb{Q}$ -linearly independent. The latter set is called the *spectrum* of the exponents  $(e_1, \dots, e_k)$ .

When the dimension is  $n = 1$ , we have the equality  $\hat{\omega}_0(\boldsymbol{\theta}) = \hat{\omega}(\boldsymbol{\theta}) = \hat{\lambda}(\boldsymbol{\theta}) = 1$ . In [6], V. Jarník showed that in dimension  $n = 2$ , we have the following algebraic relation between  $\hat{\omega}_1(\boldsymbol{\theta})$  and  $\hat{\omega}_0(\boldsymbol{\theta})$ :

$$\hat{\omega}_0(\boldsymbol{\theta}) + \frac{1}{\hat{\omega}_1(\boldsymbol{\theta})} = 1. \quad (*)$$

Furthermore, V. Jarník noted that, in higher dimension  $n \geq 3$ , no algebraic relation holds anymore. He proved [6, Satz 3] that for  $n \geq 2$ , there exist two  $n$ -tuples of real numbers  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$  and  $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$  such that

$$\hat{\omega}_{n-1}(\boldsymbol{\theta}) = \hat{\omega}_{n-1}(\boldsymbol{\nu}) = +\infty, \quad \hat{\omega}_0(\boldsymbol{\theta}) = 1 \quad \text{and} \quad \hat{\omega}_0(\boldsymbol{\nu}) = \frac{1}{n-1}.$$

V. Jarník also proved the following transference theorem:

**Theorem 1** (Jarník, 1938). *Let  $n \geq 2$ . For any  $n$ -tuples of real number  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$  such that  $1, \theta_1, \dots, \theta_n$  are  $\mathbb{Q}$ -linearly independent, we have*

$$\frac{\hat{\omega}_{n-1}(\boldsymbol{\theta})}{(n-1)\hat{\omega}_{n-1}(\boldsymbol{\theta}) + n} \leq \hat{\omega}_0(\boldsymbol{\theta}) \leq \frac{\hat{\omega}_{n-1}(\boldsymbol{\theta}) - n + 1}{n}.$$

If  $\hat{\omega}_{n-1}(\boldsymbol{\theta}) = n$ , the interval reduces to the single point  $\hat{\omega}_0(\boldsymbol{\theta}) = \frac{1}{n}$ .

*Remark.* O. German [5] and A. Khinchin [9] claim that V. Jarník [6] proved the existence of  $n$ -tuples  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$  with  $\hat{\omega}_{n-1}(\boldsymbol{\theta}) = +\infty$  and  $\hat{\omega}_0(\boldsymbol{\theta})$  anywhere in the interval  $\left[\frac{1}{n-1}, 1\right]$ . It appears to the author that this is not written explicitly in [6].

Recently, O. German [5] improved Theorem 1:

**Theorem 2** (German, 2012). *With the notation of Theorem 1, we have*

$$\frac{\hat{\omega}_{n-1}(\boldsymbol{\theta}) - 1}{(n-1)\hat{\omega}_{n-1}(\boldsymbol{\theta})} \leq \hat{\omega}_0(\boldsymbol{\theta}) \leq \frac{\hat{\omega}_{n-1}(\boldsymbol{\theta}) - (n-1)}{\hat{\omega}_{n-1}(\boldsymbol{\theta})}. \quad (**)$$

Note that the interval reduces to a single point if  $n = 2$ , and that in this case we recover Jarník's relation (\*).

The first goal of this paper is to prove that German's inequalities describe the spectrum of the two exponents  $(\hat{\omega}_0, \hat{\omega}_{n-1})$ .

**Theorem 3.** *Let  $n \geq 2$  be an integer, let  $\hat{\omega} \in [n, +\infty]$  and let  $\hat{\lambda} \in \left[\frac{\hat{\omega} - 1}{(n-1)\hat{\omega}}, \frac{\hat{\omega} - n + 1}{\hat{\omega}}\right]$ ,*

*where we understand that the interval for  $\hat{\lambda}$  is  $\left[\frac{1}{n-1}, 1\right]$  when  $\hat{\omega} = +\infty$ . Then there exist uncountably many  $n$ -tuples of real numbers  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ , with  $1, \theta_1, \dots, \theta_n$   $\mathbb{Q}$ -linearly independent, such that  $\hat{\omega}_{n-1}(\boldsymbol{\theta}) = \hat{\omega}$  and  $\hat{\omega}_0(\boldsymbol{\theta}) = \hat{\lambda}$ .*

In [19], W. Schmidt and L. Summerer obtained independently a similar result, proving that the inequalities (\*\*) of German are best possible.

One can wonder if in higher dimension ( $n \geq 3$ ), there exists a Jarník's-type relation between the  $n$  uniform exponents  $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}$ . The next theorem states that no such algebraic relation holds.

**Theorem 4.** *For every integer  $n \geq 3$ , the  $n$  uniform exponents  $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}$  are algebraically independent.*

Thus, the spectrum of the  $n$  uniform exponents  $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}$  is a subset of  $\mathbb{R}^n$  with nonempty interior.

We also know the spectrum of other families of exponents. M. Laurent [10] described the spectrum of the four exponents  $\omega_0, \hat{\omega}_0, \omega_{n-1}, \hat{\omega}_{n-1}$  in dimension  $n = 2$ . In his PhD thesis, the author gives an alternative proof of this result. However, for  $n \geq 3$  this spectrum is still unknown.

D. Roy showed in [13] that the going-up and going-down transference inequalities of M. Laurent [11] describe the spectrum of the  $n$  exponents  $\omega_0, \dots, \omega_{n-1}$ .

In section 2, we introduce Parametric Geometry of Numbers, which is the main tool to prove Theorem 3 (section 3) and Theorem 4 (section 5), and to give an alternative proof of Theorem 2 (section 4).

## 2 Parametric Geometry of Numbers

The parametric geometry of numbers answers a question of W. M. Schmidt [15]. Given a convex body and a lattice, we deform either of them with a one parameter diagonal map. We study the behavior of the successive minima in terms of this parameter. It was developed by W. M. Schmidt and L. Summerer [17, 18], and further by D. Roy [13, 14]. Independently, I. Cheung [3, 4] also developed a similar theory.

In this paper, we use the notation introduced by D. Roy in [13, 14] which is essentially dual to the one of W. M. Schmidt and L. Summerer [17, 18]. We refer the reader to these papers for further details. Here  $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_n y_n$  is the usual scalar product of vectors  $\mathbf{x}$  and  $\mathbf{y}$ , and  $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}}$  is the usual Euclidean norm.

Let  $\mathbf{u} = (u_0, \dots, u_n)$  be a vector in  $\mathbb{R}^{n+1}$ , with Euclidean norm  $\|\mathbf{u}\|_2 = 1$ . For a real parameter  $Q \geq 1$  we consider the convex body

$$\mathcal{C}_{\mathbf{u}}(Q) = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\|_2 \leq 1, |\mathbf{x} \cdot \mathbf{u}| \leq Q^{-1} \right\}.$$

For  $1 \leq d \leq n+1$  we denote by  $\lambda_d(\mathcal{C}_{\mathbf{u}}(Q))$  the  $d$ -th minimum of  $\mathcal{C}_{\mathbf{u}}(Q)$  relatively to the lattice  $\mathbb{Z}^{n+1}$ . For  $q \geq 0$  and  $1 \leq d \leq n+1$  we set

$$L_{\mathbf{u},d}(q) = \log \lambda_d(\mathcal{C}_{\mathbf{u}}(e^q)).$$

Finally, we define the following map associated with  $\mathbf{u}$ :

$$\begin{aligned} \mathbf{L}_{\mathbf{u}} : [0, \infty) &\rightarrow \mathbb{R}^{n+1} \\ q &\mapsto (L_{\mathbf{u},1}(q), \dots, L_{\mathbf{u},n+1}(q)). \end{aligned}$$

The lattice  $\mathbb{Z}^{n+1}$  is invariant under permutation of coordinates. Hence,  $\mathbf{L}_{\mathbf{u}}$  remains the same if we permute the coordinates in  $\mathbf{u}$ . Since  $\|\mathbf{u}\|_2 = 1$  we can thus assume that  $u_0 \neq 0$ .

The following proposition links the exponents of Diophantine approximation associated with  $\boldsymbol{\theta} = \left(\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0}\right)$  to the behavior of the map  $\mathbf{L}_{\mathbf{u}}$ , assuming  $u_0 \neq 0$ . It was first stated by W.M. Schmidt and L. Summerer in [17] (Theorem 1.4). It also appears as Relations (1.8) and (1.9) in [18]. In the notation of D. Roy [13] (Proposition 3.1), it reads as follows.

**Proposition 1** (Schmidt, Summerer, 2009). *Let  $\mathbf{u} = (u_0, \dots, u_n) \in \mathbb{R}^{n+1}$ , with Euclidean norm  $\|\mathbf{u}\|_2 = 1$  and  $u_0 \neq 0$ . Set  $\boldsymbol{\theta} = (\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0})$ . For  $1 \leq k \leq n$ , we have the following relations:*

$$\begin{aligned} \liminf_{q \rightarrow +\infty} \frac{L_{\mathbf{u},1}(q) + \dots + L_{\mathbf{u},k}(q)}{q} &= \frac{1}{1 + \omega_{n-k}(\boldsymbol{\theta})}, \\ \limsup_{q \rightarrow +\infty} \frac{L_{\mathbf{u},1}(q) + \dots + L_{\mathbf{u},k}(q)}{q} &= \frac{1}{1 + \hat{\omega}_{n-k}(\boldsymbol{\theta})}. \end{aligned}$$

Thus, if we know an explicit map  $\mathbf{P} = (P_1, \dots, P_{n+1}) : [0, \infty) \rightarrow \mathbb{R}^{n+1}$ , such that  $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$  is bounded, then we can compute the  $2n$  exponents  $\hat{\omega}_0(\boldsymbol{\theta}), \dots, \hat{\omega}_{n-1}(\boldsymbol{\theta}), \omega_0(\boldsymbol{\theta}), \dots, \omega_{n-1}(\boldsymbol{\theta})$  for the above point  $\boldsymbol{\theta}$  upon replacing  $L_{\mathbf{u},i}$  by  $P_i$  in the above formulas for  $1 \leq i \leq n$ . For this purpose, we consider the following family of maps, introduced by D. Roy in [13].

**Definition** (Roy, 2014). Let  $I$  be a subinterval of  $[0, \infty)$  with non-empty interior. A generalized  $(n+1)$ -system on  $I$  is a continuous piecewise linear map  $\mathbf{P} = (P_1, \dots, P_{n+1}) : I \rightarrow \mathbb{R}^{n+1}$  with the following three properties.

- (S1) For each  $q \in I$ , we have  $0 \leq P_1(q) \leq \dots \leq P_{n+1}(q)$  and  $P_1(q) + \dots + P_{n+1}(q) = q$ .
- (S2) If  $H$  is a non empty open subinterval of  $I$  on which  $\mathbf{P}$  is differentiable, then there are integers  $\underline{r}, \bar{r}$  with  $1 \leq \underline{r} \leq \bar{r} \leq n+1$  such that  $P_{\underline{r}}, P_{\underline{r}+1}, \dots, P_{\bar{r}}$  coincide on the whole interval  $H$  and have slope  $1/(\bar{r} - \underline{r} + 1)$  while any other component  $P_k$  of  $\mathbf{P}$  is constant on  $H$ .
- (S3) If  $q$  is an interior point of  $I$  at which  $\mathbf{P}$  is not differentiable, if  $\underline{r}, \bar{r}, \underline{s}, \bar{s}$  are the integers for which

$$P'_k(q^-) = \frac{1}{\bar{r} - \underline{r} + 1} \quad (\underline{r} \leq k \leq \bar{r}) \quad \text{and} \quad P'_k(q^+) = \frac{1}{\bar{s} - \underline{s} + 1} \quad (\underline{s} \leq k \leq \bar{s}),$$

and if  $\underline{r} < \bar{s}$ , then we have  $P_{\underline{r}}(q) = P_{\underline{r}+1}(q) = \dots = P_{\bar{s}}(q)$ .

Here  $P'_k(q^-)$  (resp.  $P'_k(q^+)$ ) denotes the left (resp. right) derivative of  $P_k$  at  $q$ . The next result combines Theorem 4.2 and Corollary 4.7 of [13].

**Theorem 5** (Roy, 2014). *For each non-zero point  $\mathbf{u} \in \mathbb{R}^{n+1}$ , there exists  $q_0 \geq 0$  and a generalized  $(n+1)$ -system  $\mathbf{P}$  on  $[q_0, \infty)$  such that  $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$  is bounded on  $[q_0, \infty)$ . Conversely, for each generalized  $(n+1)$ -system  $\mathbf{P}$  on an interval  $[q_0, \infty)$  with  $q_0 \geq 0$ , there exists a non-zero point  $\mathbf{u} \in \mathbb{R}^{n+1}$  such that  $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$  is bounded on  $[q_0, \infty)$ .*

In view of the remark following Proposition 1, this result reduces the determination of the joint spectrum of Diophantine approximation exponents to a combinatorial study of generalized  $(n + 1)$ -systems.

Although the definition of a generalized  $(n + 1)$ -system  $\mathbf{P} = (P_1, \dots, P_{n+1})$  may look complicated, it is easy to understand in terms of the *combined graph* of  $\mathbf{P}$ , that is the union of the graphs of  $P_1, \dots, P_{n+1}$  over the interval of definition  $I$  of  $\mathbf{P}$ . We explain this below.

A *division point* of  $\mathbf{P}$  is an endpoint of  $I$  contained in  $I$  or an interior point of  $I$  at which  $\mathbf{P}$  is not differentiable. Such points form a discrete subset of  $I$ . Between two consecutive division points  $q^* < q$  of  $I$ , the graph of each component of  $\mathbf{P}$  is a line segment. All these line segments have slope 0 except for one line segment of positive slope  $1/t$  where  $t$  is the number of components of  $\mathbf{P}$  whose graph over  $[q^*, q]$  is that line segment. In view of the condition  $P_1 \leq P_2 \leq \dots \leq P_{n+1}$ , there must be consecutive components  $P_{\underline{r}}, \dots, P_{\bar{r}}$  of  $\mathbf{P}$  with  $\bar{r} - \underline{r} + 1 = t$ . If  $q$  is also an interior point of  $I$  and if  $P_{\underline{s}}, \dots, P_{\bar{s}}$  are the components of  $\mathbf{P}$  whose graph has positive slope  $\frac{1}{\bar{s} - \underline{s} + 1}$  to the right of  $q$ , then there are two cases.

- 1) If  $\underline{r} < \bar{s}$ , we say that  $q$  is an *ordinary division point*. In this case, we have  $P_{\underline{r}}(q) = \dots = P_{\bar{s}}(q)$  according to **(S3)**. This implies that  $\underline{r} \leq \underline{s}$  and  $\bar{r} \leq \bar{s}$ . Among  $P_{\underline{r}}, \dots, P_{\bar{s}}$ , the components  $P_j$  with  $\underline{s} \leq j \leq \bar{r}$  (if any) change slope from  $\frac{1}{\bar{r} - \underline{r} + 1}$  to  $\frac{1}{\bar{s} - \underline{s} + 1}$ . Those with  $j \leq \min(\bar{r}, \underline{s} - 1)$  change slope from  $\frac{1}{\bar{r} - \underline{r} + 1}$  to 0. The remaining components  $P_j$  with  $\bar{r} + 1 \leq j \leq \underline{s} - 1$  (if any) have constant slope 0 in a neighborhood of  $q$ . The reader is invited to draw a picture for himself or to look at those in [13, §4].
- 2) Otherwise, we have  $\underline{r} > \bar{s}$  because it cannot happen that  $\underline{r} = \bar{s}$  (or  $\mathbf{P}$  is differentiable at  $q$ ). Then, we say that  $q$  is a *switch point*. In this case, we have  $P_{\underline{r}}(q) = \dots = P_{\bar{r}}(q) > P_{\underline{s}}(q) = \dots = P_{\bar{s}}(q)$  which mean that the end point of the line segment of slope  $\frac{1}{\bar{r} - \underline{r} + 1}$  at the left of  $q$  lies above the initial point of the line segment of slope  $\frac{1}{\bar{s} - \underline{s} + 1}$  at the right of  $q$ .

It can be shown that the combined graph of a generalized  $(n + 1)$ -system  $\mathbf{P}$  uniquely determines the map  $\mathbf{P}$  provided that we know the value of  $\mathbf{P}$  at one point of its interval of definition. An example of this is shown in [13, §4]. We will see two other examples in the sections 3 and 5.

In [17, 18] W. M. Schmidt and L. Summerer introduce the following exponents for an

integer  $1 \leq d \leq n + 1$ :

$$\varphi_d = \liminf_{q \rightarrow \infty} \frac{L_{\mathbf{u},d}(q)}{q},$$

$$\bar{\varphi}_d = \limsup_{q \rightarrow \infty} \frac{L_{\mathbf{u},d}(q)}{q}.$$

For these exponents, we have the following analogue of Theorem 4:

**Theorem 6.** *For every integer  $n \geq 3$ , the exponents  $\bar{\varphi}_1, \dots, \bar{\varphi}_n$  are algebraically independent.*

### 3 Proof of Theorem 3

In this section, we construct a family of generalized  $(n+1)$ -systems. Then, via Theorem 5, we get a family of  $n$ -tuples having the requested properties stated in Theorem 3. We first treat the case where  $\hat{\omega}_{n-1}$  is finite and  $n \geq 3$ . We will explain later how to adapt the construction if  $n = 2$  or  $\hat{\omega}_{n-1}$  is infinite.

First, note that a generalized  $(n+1)$ -system with all components equal to  $\frac{q}{n+1}$  provides via Theorem 5 a point  $\boldsymbol{\theta}$  with  $\hat{\omega}_{n-1}(\boldsymbol{\theta}) = n$  and  $\hat{\omega}_0(\boldsymbol{\theta}) = \frac{1}{n}$ . Thus, we can exclude this case in the next construction.

Let  $q_0$  be a positive real number, fix a real number  $\hat{\omega} > n \geq 2$  and set a parameter  $a$  with  $\frac{1}{n-1} \leq a \leq 1$ . We define the sequence  $(q_{6m})_{m \geq 0}$  by:

$$q_{6m} = (1 + a(\hat{\omega} - n))q_{6(m-1)}, \text{ for } m \geq 1.$$

Since  $\hat{\omega} > n$ ,  $q_{6m}$  goes to infinity.

We construct a generalized  $(n+1)$ -system  $\mathbf{P}$  whose graph is invariant under the dilation of factor  $(1 + a(\hat{\omega} - n)) > 1$  on the interval  $[q_0, +\infty)$ . Thus, we only need to define  $\mathbf{P}$  on a generic interval  $[q_{6m}, q_{6(m+1)}]$ . Figure 1 shows the pattern of the combined graph of  $\mathbf{P}$ .

For every integer  $m \geq 0$ , we define  $\mathbf{P}$  at  $q_{6m}$  as follows:

$$P_1(q_{6m}) = P_2(q_{6m}) = \frac{q_{6m}}{\hat{\omega} + 1},$$

$$P_3(q_{6m}) = \dots = P_n(q_{6m}) = \frac{1 + \left(\frac{1-a}{n-2}\right)(\hat{\omega} - n)}{\hat{\omega} + 1} q_{6m},$$

$$P_{n+1}(q_{6m}) = \frac{1 + a(\hat{\omega} - n)}{\hat{\omega} + 1} q_{6m}.$$

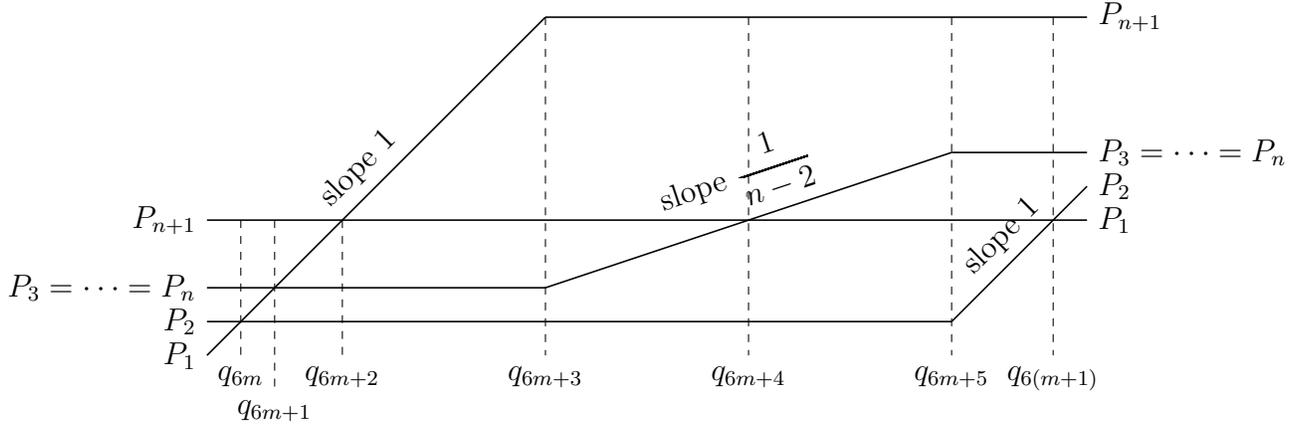


Figure 1: Combined graph of  $\mathbf{P}$  on a generic interval  $[q_{6m}, q_{6(m+1)}]$

Here the parameter  $a$  says how large  $P_{n+1}$  is at each point  $q_{6m}$ . The condition  $a \geq \frac{1}{n-1}$  imposes the condition  $P_{n+1}(q_{6m}) \geq P_n(q_{6m})$ , and the condition  $a \leq 1$  imposes that  $P_3(q_{6m}) \geq P_2(q_{6m})$ . We have the dilation condition  $\mathbf{P}(q_{6(m+1)}) = \mathbf{P}((1+a(\hat{\omega}-n))q_{6m}) = (1+a(\hat{\omega}-n))\mathbf{P}(q_{6m})$  by the definition of the sequence  $(q_{6m})_{m \geq 0}$ .

For  $k = 0, \dots, 5$  the graph has only one line segment of positive slope on the interval  $[q_{6m+k}, q_{6m+k+1}]$ . The graph is clearly the combined graph of a generalized  $(n+1)$ -system with seven division points  $q_{6m}, \dots, q_{6m+6}$ . The points  $q_{6m+3}$  and  $q_{6m+5}$  are switch points while the others are ordinary division points. Furthermore it is uniquely defined since we know the value of  $\mathbf{P}$  at the point  $q_{6m}$ , where as requested

$$P_1(q_{6m}) + \dots + P_{n+1}(q_{6m}) = q_{6m}.$$

Easy computation gives

$$\begin{aligned} q_{6m} &= (1+a(\hat{\omega}-n))q_{6(m-1)}, & q_{6m+1} &= \frac{(n-2)(\hat{\omega}+1) + (1-a)(\hat{\omega}-n)}{(n-2)(\hat{\omega}+1)}q_{6m}, \\ q_{6m+2} &= \frac{(n+1) + (1+a)(\hat{\omega}-n)}{\hat{\omega}+1}q_{6m}, & q_{6m+3} &= \frac{\hat{\omega} + (1+a(\hat{\omega}-n))^2}{\hat{\omega}+1}q_{6m}, \\ q_{6m+4} &= \frac{1 + (1+a(\hat{\omega}-n))(n+a(\hat{\omega}-n))}{\hat{\omega}+1}q_{6m}, & q_{6m+5} &= \frac{1 + 2a(\hat{\omega}-n) + \hat{\omega}(1+a(\hat{\omega}-n))}{\hat{\omega}+1}q_{6m}. \end{aligned}$$

We now compute its associated exponents with Proposition 1. One can notice that the local extrema of the functions  $q \rightarrow q^{-1}P_k(q)$ ,  $1 \leq k \leq n+1$  are located at division points where  $P_k$  changes slope.

Since  $\mathbf{P}$  is invariant under dilation of factor  $C = (1 + a(\hat{\omega} - n))$  we have for every  $m \geq 0$ , every  $1 \leq k \leq n+1$ , and every  $q$  in  $[q_{6m}, q_{6m+6})$  the relation

$$q^{-1}P_k(q) = q^{-1}C^m P_k(qC^{-m}),$$

where  $C^{-m}q$  lies in the fundamental interval  $[q_0, q_6]$ .

Thus,

$$\begin{aligned} \limsup_{q \rightarrow +\infty} \frac{P_1(q)}{q} &= \max_{q_0 \leq q \leq q_6} \frac{P_1(q)}{q} = \frac{P_1(q_0)}{q_0} = \frac{1}{\hat{\omega} + 1}, \\ \liminf_{q \rightarrow +\infty} \frac{P_{n+1}(q)}{q} &= \min_{q_0 \leq q \leq q_6} \frac{P_{n+1}(q)}{q} = \frac{P_{n+1}(q_2)}{q_2} = \frac{1 + a(\hat{\omega} - n)}{n + 1 + (1 + a)(\hat{\omega} - n)}, \end{aligned}$$

because the component  $P_{n+1}$  changes slope from zero to some positive value only at  $q_{6m+2}$ .

Then, according to Proposition 1, Theorem 5 provides an  $n$ -tuple  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$  such that

$$\begin{aligned} \frac{1}{\hat{\omega}_{n-1}(\boldsymbol{\theta}) + 1} &= \limsup_{q \rightarrow +\infty} \frac{P_1(q)}{q} = \frac{1}{\hat{\omega} + 1}, \\ \frac{\hat{\omega}_0(\boldsymbol{\theta})}{\hat{\omega}_0(\boldsymbol{\theta}) + 1} &= \liminf_{q \rightarrow +\infty} \frac{P_{n+1}(q)}{q} = \frac{1 + a(\hat{\omega} - n)}{n + 1 + (1 + a)(\hat{\omega} - n)}. \end{aligned}$$

Thus, this  $\boldsymbol{\theta}$  satisfies

$$\hat{\omega}_{n-1}(\boldsymbol{\theta}) = \hat{\omega} \quad \text{and} \quad \hat{\omega}_0(\boldsymbol{\theta}) = \frac{1 + a(\hat{\omega} - n)}{\hat{\omega}}.$$

When  $a$  runs through the interval  $[\frac{1}{n-1}, 1]$ , then  $\hat{\omega}_0(\boldsymbol{\theta})$  runs through the interval

$$\left[ \frac{\hat{\omega} - 1}{(n-1)\hat{\omega}}, \frac{\hat{\omega} - (n-1)}{\hat{\omega}} \right].$$

If  $n = 2$ , we remove the line  $P_3 = \dots = P_n$  and the interval  $[q_{6m+3}, q_{6m+5}]$  from the generic graph on the interval  $[q_{6m}, q_{6(m+1)}]$ , the parameter  $a$  is then forced to be equal to 1. Thus, we construct  $\boldsymbol{\theta}$  with

$$\hat{\omega}_1(\boldsymbol{\theta}) = \hat{\omega} \quad \text{and} \quad \hat{\omega}_0(\boldsymbol{\theta}) = 1 - \frac{1}{\hat{\omega}},$$

which agrees with Jarník's relation (\*).

If  $\hat{\omega}$  is infinite, we replace  $\hat{\omega}$  by  $m + n + 1$  in our construction. For a given real number  $q_0$  we consider the sequence  $(q_{6m})_{m \geq 1}$  defined by

$$q_{6m} = (m + 1)q_{6(m-1)}.$$

Figure 1 still represents the combined graph from  $\mathbf{P}$  on a generic interval  $[q_{6m}, q_{6m+6}]$ , with the following settings at  $q_{6m}$ :

$$\begin{aligned} P_1(q_{6m}) &= P_2(q_{6m}) = \frac{q_{6m}}{m + n + 2}, \\ P_3(q_{6m}) &= \dots = P_n(q_{6m}) = \frac{1 + \left(\frac{1-a}{n-2}\right)(m+1)}{m + n + 2} q_{6m}, \\ P_{n+1}(q_{6m}) &= \frac{1 + a(m+1)}{m + n + 2} q_{6m}. \end{aligned}$$

Note that the combined graph is not invariant under dilation anymore. We have

$$\begin{aligned} \limsup_{q \rightarrow +\infty} \frac{P_1(q)}{q} &= \limsup_{m \rightarrow +\infty} \max_{q_{6m} \leq q \leq q_{6(m+1)}} \frac{P_1(q)}{q} = \limsup_{m \rightarrow +\infty} \frac{P_1(q_{6m})}{q_{6m}} = \limsup_{m \rightarrow +\infty} \frac{1}{m + n + 2} = 0, \\ \liminf_{q \rightarrow +\infty} \frac{P_{n+1}(q)}{q} &= \liminf_{m \rightarrow +\infty} \min_{q_{6m} \leq q \leq q_{6(m+1)}} \frac{P_{n+1}(q)}{q} = \liminf_{m \rightarrow +\infty} \frac{P_{n+1}(q_{6m+2})}{q_{6m+2}} \\ &= \liminf_{m \rightarrow +\infty} \frac{1 + a(m+1)}{n + 1 + (1+a)(m+1)} = \frac{a}{a+1}. \end{aligned}$$

Again, Theorem 5 provides us with an  $n$ -tuple  $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$  such that

$$\hat{\omega}_{n-1}(\boldsymbol{\theta}) = +\infty \quad \text{and} \quad \hat{\omega}_0(\boldsymbol{\theta}) = a,$$

where  $a$  runs through the interval  $[\frac{1}{n-1}, 1]$ .

Note that if  $1, \theta_1, \dots, \theta_n$  are  $\mathbb{Q}$ -linearly dependent, then there exists an integer point  $\mathbf{x} \in \mathbb{Z}^n$  such that  $|\mathbf{x} \cdot \mathbf{u}| = 0$ . This implies that  $L_{\mathbf{u},1}(q)$  is bounded above by  $\log(\|\mathbf{x}\|_2)$ . In our construction by dilatation  $P_1$  is not bounded, hence the independence by contradiction.

To complete the proof of Theorem 3, we have to check that we can construct uncountably many  $n$ -tuples with given exponents. Let  $\hat{\omega}$  and  $\hat{\lambda}$  as in Theorem 3, and  $a$  the parameter such that Theorem 5 provides an  $n$ -tuple  $\theta$  whose exponents satisfy

$$\hat{\omega}_{n-1}(\theta) = \hat{\omega}, \text{ and } \hat{\omega}_0(\theta) = \hat{\lambda} = \frac{1 + a(\hat{\omega} - n)}{\hat{\omega}}.$$

Fix  $q_0$  a real number to start the construction from  $\mathbf{P}$  as above with parameter  $a$ . For every  $\rho_1$  and  $\rho_2$  such that  $q_0 \leq \rho_1 < \rho_2 \leq q_5$ , we denote by  $\mathbf{P}_{\rho_1}$  and  $\mathbf{P}_{\rho_2}$  the  $(n+1)$ -generalized system with parameter  $a$  starting in  $\rho_1$  and  $\rho_2$ . We have  $\mathbf{P}_{\rho_1}(q_6) \neq \mathbf{P}_{\rho_2}(q_6)$  and

$$\|\mathbf{P}_{\rho_1}(q_{6m}) - \mathbf{P}_{\rho_2}(q_{6m})\|_{\infty} = \frac{q_{6m}}{q_6} \|\mathbf{P}_{\rho_1}(q_6) - \mathbf{P}_{\rho_2}(q_6)\|_{\infty} \rightarrow_{n \rightarrow \infty} \infty,$$

where  $\|(x_1, \dots, x_n)\|_{\infty} = \max_{1 \leq k \leq n} |x_k|$ .

Thus, their difference is unbounded, and they cannot correspond to the same  $\theta$  via Theorem 5. This ends the proof of Theorem 3. □

## 4 An alternative proof of Theorem 2

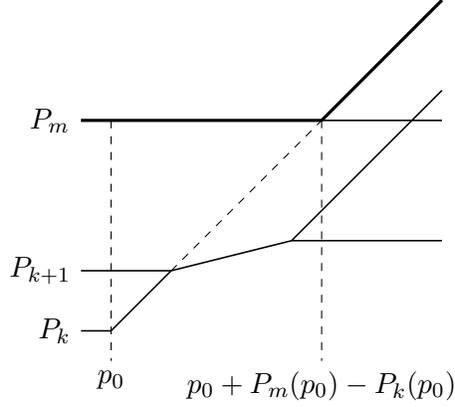
In this section, we give an alternative proof of Theorem 2 using arguments from Parametric Geometry of Numbers.

One can notice that the extremal values of the components of  $\mathbf{P}$  are reached at the division points. The condition **(S3)** translates into the fact that for every division point  $q$ , the right endpoint of the segment with non-zero slope ending at  $q$  lies above the left endpoint of the one starting at  $q$ . A first consequence is that when  $P_1$  is non constant, it increases until reaching  $P_2(q)$ . A second consequence is the following proposition.

**Proposition 2.** *For every  $1 \leq k < m \leq n+1$ , if  $p_0$  is a division point such that  $P_k$  has slope changing from 0 to 1 at  $p_0$ , then we have for every  $p > p_0$*

$$P_m(p) \leq \max(P_m(p_0), P_k(p_0) + p - p_0).$$

In particular,  $P_m$  is constant on the interval  $[p_0, p_0 + P_m(p_0) - P_k(p_0)]$ .



**Upper bound:** Let  $\mathbf{P}$  be a generalized  $(n+1)$ -system, and  $\hat{\omega}$  the real number such that

$$\limsup_{q \rightarrow +\infty} \frac{P_1(q)}{q} = \frac{1}{\hat{\omega} + 1}.$$

If this limit is zero, we set  $\hat{\omega} = +\infty$ . We treat this case after.

Let  $\varepsilon > 0$ . There exist arbitrarily large division points  $p_0$  where  $q^{-1}P_1(q)$  has a local maximum and

$$\frac{1 - \varepsilon}{\hat{\omega} + 1} \leq \frac{P_1(p_0)}{p_0} \leq \frac{1 + \varepsilon}{\hat{\omega} + 1}.$$

Since  $q_0$  is a local maximum, we have  $P_1(q_0) = P_2(q_0)$ . Furthermore,  $P_1(q) \leq P_2(q) \leq \dots \leq P_{n+1}(q)$  and  $P_1(q) + \dots + P_{n+1}(q) = q$  provide

$$P_{n+1}(p_0) \leq p_0 - nP_1(p_0) \leq \frac{\hat{\omega} + 1 - n - n\varepsilon}{\hat{\omega} + 1} p_0.$$

At the point  $p = p_0 + \frac{\hat{\omega} - n - n\varepsilon}{\hat{\omega} + 1} p_0$ , according to Proposition 2, we have the upper bound

$$P_{n+1}(p) \leq \max(P_{n+1}(p_0), P_1(p_0) + p - p_0) \leq \frac{1 + \varepsilon + \hat{\omega} - n - n\varepsilon}{\hat{\omega} + 1} p_0.$$

Thus, for arbitrarily large real numbers  $p$ , we have

$$\frac{P_{n+1}(p)}{p} \leq \frac{\hat{\omega} + 1 - n - (n-1)\varepsilon}{2\hat{\omega} - n + 1 - n\varepsilon},$$

thus

$$\frac{\hat{\lambda}}{\hat{\lambda} + 1} = \liminf_{q \rightarrow +\infty} \frac{P_{n+1}(q)}{q} \leq \frac{\hat{\omega} + 1 - n}{2\hat{\omega} - n + 1},$$

giving that

$$\hat{\lambda} \leq \frac{\hat{\omega} - (n-1)}{\hat{\omega}}.$$

If  $\hat{\omega}$  is infinite, then we have

$$0 \leq \frac{P_1(p_0)}{p_0} \leq \varepsilon, \quad P_1(p_0) = P_2(p_0).$$

Using Proposition 2 at the point  $p = 2p_0 - n\varepsilon$  we get

$$P_{n+1}(p) \leq p_0 - (n+1)\varepsilon.$$

Thus, for arbitrarily large real numbers  $p$ , we have

$$\frac{P_{n+1}(p)}{p} \leq \frac{p_0 - (n+1)\varepsilon}{2p_0 - n\varepsilon}.$$

This implies that

$$\hat{\lambda} \leq 1.$$

Thus, we have proved the upper bound in Theorem 2.

**Lower bound:** Let  $\mathbf{P}$  be a generalized  $(n+1)$ -system, and  $\hat{\omega}, \hat{\lambda}$  the real numbers such that

$$\limsup_{q \rightarrow +\infty} \frac{P_1(q)}{q} = \frac{1}{\hat{\omega} + 1}, \quad \liminf_{q \rightarrow +\infty} \frac{P_{n+1}(q)}{q} = \frac{\hat{\lambda}}{\hat{\lambda} + 1},$$

where we understand that  $\hat{\omega} = +\infty$  if the corresponding lim sup is zero. Again, we treat this case after.

Let  $\varepsilon_1 > 0$ . There exists a real number  $q_0$  such that  $q \geq q_0$  implies

$$\frac{P_1(q)}{q} \leq \frac{1 + \varepsilon_1}{\hat{\omega} + 1}.$$

Let  $\varepsilon_2 > 0$ . There exist arbitrarily large division points  $p \geq q_0$  where  $q^{-1}P_1(q)$  has a local minimum and

$$\left| \frac{P_{n+1}(p)}{p} - \frac{\hat{\lambda}}{\hat{\lambda} + 1} \right| \leq \varepsilon_2.$$

Let  $p_0 = \max \{q \leq p \mid P_1(q) = P_2(q)\}$ . At the point  $p_0$  we have

$$\begin{aligned} P_1(p_0) &= P_2(p_0) \leq \frac{1 + \varepsilon_1}{\hat{\omega} + 1} p_0, \\ P_{n+1}(p_0) &\geq \frac{p_0 - 2P_1(p_0)}{n-1}, \end{aligned}$$

since  $p_0 = P_1(p_0) + \cdots + P_{n+1}(p_0) \leq 2P_1(p_0) + (n-1)P_{n+1}(p_0)$ .

We first show that  $q \rightarrow P_1(q)$  is constant on the interval  $[p_0, p]$ . If not, there exists a real number  $p_0 < p_1 < p$  where  $P_1$  has slope 1. Since  $p$  is a local minimum from  $q^{-1}P_1(q)$ , we have  $P_1'(p^-) = 0$ . Thus, there exists a point in  $(p_1, p)$  where  $P_1$  changes slope from 1 to 0. At this point  $P_1 = P_2$ , which contradicts the definition of  $p_0$ . Thus,

$$P_1(p_0) = P_1(p).$$

We can write

$$p = \sum_{k=1}^{n+1} P_k(p) \leq nP_{n+1}(p) + P_1(p_0).$$

Thus,

$$\frac{P_{n+1}(p)}{p} \geq \frac{P_{n+1}(p)}{nP_{n+1}(p) + P_1(p_0)},$$

where the right hand side is an increasing function of  $P_{n+1}(p)$ . Since

$$P_{n+1}(p) \geq P_{n+1}(p_0) \geq \frac{p_0 - 2P_1(p_0)}{n-1},$$

we have

$$\frac{P_{n+1}(p)}{p} \geq \frac{p_0 - 2P_1(p_0)}{np_0 - (n+1)P_1(p_0)},$$

where the right hand side is a decreasing function of  $P_1(p_0)$ . Since

$$P_1(p_0) \leq \frac{1 + \varepsilon_1}{\hat{\omega} + 1} p_0,$$

we have

$$\frac{P_{n+1}(p)}{p} \geq \frac{\hat{\omega} - 1 - 2\varepsilon_1}{n\hat{\omega} - 1 - (n+1)\varepsilon_1}.$$

Thus,

$$\frac{\hat{\lambda}}{\hat{\lambda} + 1} \geq \frac{\hat{\omega} - 1 - 2\varepsilon_1}{n\hat{\omega} - 1 - (n+1)\varepsilon_1} - \varepsilon_2.$$

This gives the expected bound

$$\hat{\lambda} \geq \frac{\hat{\omega} - 1}{(n-1)\hat{\omega}}.$$

If  $\hat{\omega}$  is infinite, we consider a real number  $q_0$  such that  $q \geq q_0$  implies

$$0 \leq \frac{P_1(q)}{q} \leq \varepsilon_1.$$

This provides the following estimates at  $p$ :

$$\frac{P_{n+1}(p)}{p} \geq \frac{1 - 2\varepsilon_1}{n - (n+1)\varepsilon_1}.$$

Thus, we get

$$\frac{\hat{\lambda}}{\hat{\lambda} + 1} \geq \frac{1 - 2\varepsilon_1}{n - (n+1)\varepsilon_1} - \varepsilon_2.$$

This gives the expected bound

$$\hat{\lambda} \geq \frac{1}{n-1}.$$

Thus, we have proved the lower bound. This ends the proof of Theorem 2. □

## 5 Proof of Theorems 4 and 6

In this section, we construct a family of generalized  $(n+1)$ -systems depending on  $n$  parameters which via Theorem 5 provides us with a family of  $n$ -tuples whose uniform exponents are expressed as a function of these  $n$  parameters. Then, we show that these functions are algebraically independent.

Fix the dimension  $n \geq 3$ . Choose  $n+2$  parameters  $A_1, A_2, \dots, A_{n+1}, C$  satisfying

$$\begin{aligned} 0 < A_1 = A_2 < A_3 < A_4 < \dots < A_{n+1}, \\ 1 &= A_1 + A_2 + \dots + A_{n+1}, \\ \frac{A_{k+1}}{A_k} < C < \frac{A_{k+2}}{A_k} \text{ for } 2 \leq k \leq n-1, \\ 1 < \frac{A_{n+1}}{A_n} < C. \end{aligned} \tag{0}$$

We consider the generalized  $(n+1)$ -system  $\mathbf{P}$  on the interval  $[1, C]$  whose combined graph is given by Figure 2, where

$$P_k(1) = A_k \text{ and } P_k(C) = CA_k \text{ for } 1 \leq k \leq n+1.$$

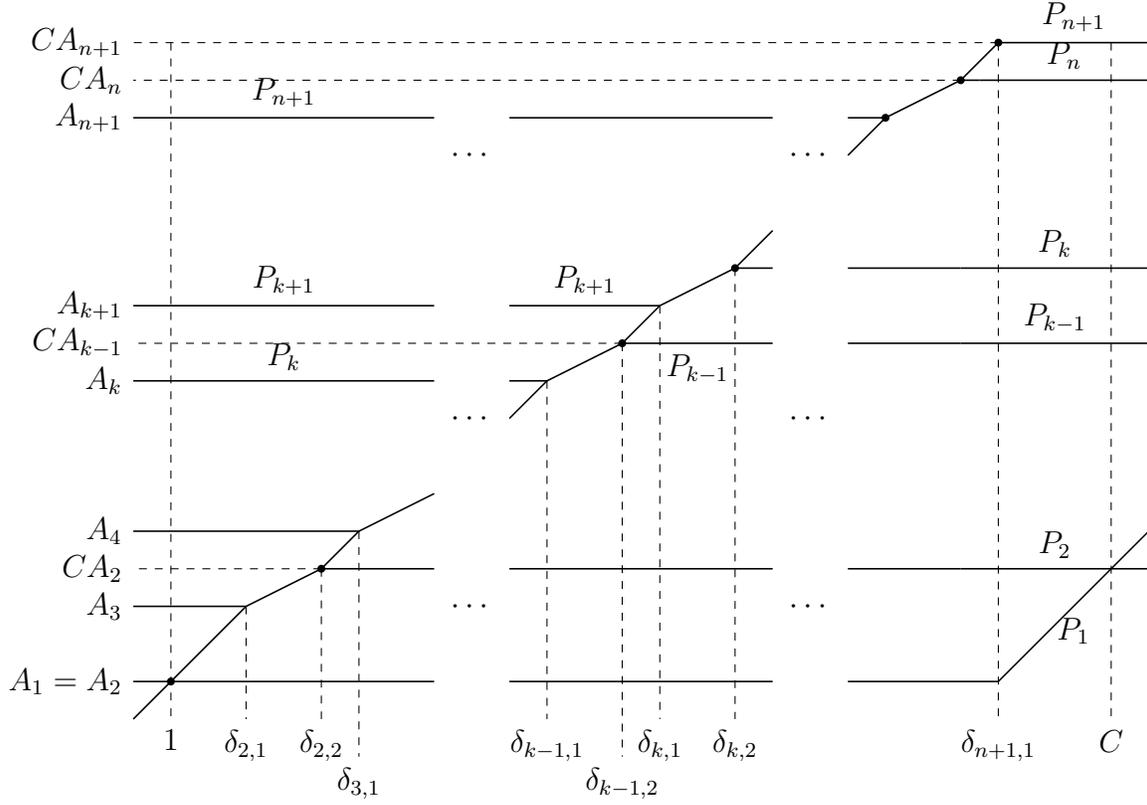


Figure 2: Pattern of the combined graph of  $\mathbf{P}$  on the fundamental interval  $[1, C]$

On each interval between two consecutive division points, there is only one line segment with nonzero slope. This line segment has slope 1 on the intervals  $[1, \delta_{2,1}]$ ,  $[\delta_{n+1,1}, C]$  and  $[\delta_{k-1,2}, \delta_{k,1}]$  for  $3 \leq k \leq n+1$ , and has slope  $1/2$  on the interval  $[\delta_{k,1}, \delta_{k,2}]$ , for  $3 \leq k \leq n$ , where the two components  $P_k$  and  $P_{k+1}$  coincide. We have  $2n+1$  division points  $1, C, \delta_{k,1}$  and  $\delta_{l,2}$  for  $2 \leq k \leq n+1$  and  $2 \leq l \leq n$ . They are all ordinary division points except  $\delta_{n+1,1}$  which is a switch point. Note that the conditions (0) are consistent with the graph. The points which will be most relevant for the proofs are labeled with black dots.

We extend  $\mathbf{P}$  to the interval  $[1, \infty)$  by self-similarity, that is  $\mathbf{P}(q) = C^m \mathbf{P}(qC^{-m})$  for every positive integer  $m$ . In view of the value of  $\mathbf{P}$  and its derivative at 1 and  $C$ , one sees that this extension provides a generalized  $(n+1)$ -system on  $[1, \infty)$ .

Proposition 1 suggests to define quantities  $\hat{W}_{n-1}, \dots, \hat{W}_0$  by

$$\frac{1}{1 + \hat{W}_{n-k}} := \limsup_{q \rightarrow +\infty} \frac{P_1(q) + \dots + P_k(q)}{q}, \quad 1 \leq k \leq n. \quad (1)$$

Since  $\mathbf{P}$  is invariant under dilation of factor  $C$ , we can replace  $\limsup_{q \rightarrow \infty}$  by  $\max_{[1, C]}$  in the above formulae.

We observe that for  $1 \leq k \leq n$ , the function  $P_1 + \dots + P_k$  has slope 1 on the intervals  $[1, \delta_{k,1}]$  and  $[\delta_{n+1,1}, C]$ , slope 1/2 on the interval  $[\delta_{k,1}, \delta_{k,2}]$  and is constant on the interval  $[\delta_{k,2}, \delta_{n+1,1}]$ . Thus the maximum on  $[1, C]$  of the function  $q \rightarrow q^{-1}(P_1(q) + \dots + P_k(q))$  is reached either at  $\delta_{k,1}$  or at  $\delta_{k,2}$ , when slope changes from 1 to 1/2 or from 1/2 to 0. Namely, the maximum is reached at  $\delta_{k,1}$  if

$$\frac{P_1(\delta_{k,1}) + \dots + P_k(\delta_{k,1})}{\delta_{k,1}} \geq \frac{1}{2} \quad (2)$$

and at  $\delta_{k,2}$  if the lefthand side is  $\leq 1/2$ . We deduce that for  $1 \leq k \leq n$ ,

$$\hat{W}_{n-k} = \frac{P_{k+1}(q) + \dots + P_{n+1}(q)}{P_1(q) + \dots + P_k(q)} \text{ where } q = \begin{cases} \delta_{k,1} & \text{if (2) is satisfied} \\ \delta_{k,2} & \text{otherwise} \end{cases}.$$

For  $2 \leq k \leq n+1$ , we have the following values at  $\delta_{k,1}$  and  $\delta_{k,2}$ :

$$P_i(\delta_{k,1}) = \begin{cases} A_1 & \text{if } i = 1 \\ CA_i & \text{if } 2 \leq i \leq k-1 \\ A_{k+1} & \text{if } i = k \\ A_i & \text{if } k+1 \leq i \leq n+1 \end{cases}, \quad P_i(\delta_{k,2}) = \begin{cases} A_1 & \text{if } i = 1 \\ CA_i & \text{if } 2 \leq i \leq k \\ CA_k & \text{if } i = k+1 \\ A_i & \text{if } k+2 \leq i \leq n+1 \end{cases}$$

It is easy to check that the parameters

$$C = 3, A_1 = A_2 = 2^{-n}, A_k = 2^{-n+k-2} \text{ for } 3 \leq k \leq n+1 \quad (3)$$

satisfy the conditions (0). For this choice of parameters, the lefthand side of inequality (2) is  $> 1/2$  for  $1 \leq k \leq n-1$  and  $< 1/2$  for  $k = n$ . This property remains true for  $(C, A_2, \dots, A_n)$  in an open neighborhood of  $(3, 2^{-n}, \dots, 2^{-2})$  provided that we set  $A_1 = A_2$  and  $A_{n+1} = 1 - (A_1 + \dots + A_n)$ . In this neighborhood, the quantities  $\hat{W}_0, \dots, \hat{W}_{n-1}$  are given by the following rational fractions in  $\mathbb{Q}(C, A_2, A_3, \dots, A_n)$  :

$$\begin{aligned} \hat{W}_{n-1} &= \frac{1}{A_2} - 1, \\ \hat{W}_{n-k} &= \frac{1 - (2A_2 + A_3 + A_4 + \dots + A_{k+1}) + CA_k}{A_2 + C(A_2 + \dots + A_k)}, \quad 2 \leq k \leq n-1 \\ \hat{W}_0 &= \frac{1 - (2A_2 + A_3 + A_4 + \dots + A_n)}{A_2 + C(A_2 + \dots + A_{n-1})}. \end{aligned} \quad (4)$$

Since  $\hat{W}_0, \dots, \hat{W}_{n-1}$  come from a generalized  $(n+1)$ -system  $\mathbf{P}$ , Theorem 5 provides a point  $\boldsymbol{\theta}$  in  $\mathbb{R}^n$  such that  $\hat{\omega}_k(\boldsymbol{\theta}) = \hat{W}_k$  for every  $0 \leq k \leq n-1$ . Thus, to prove Theorem 4, it is sufficient to show that the rational fractions  $\hat{W}_0, \dots, \hat{W}_{n-1} \in \mathbb{Q}(C, A_2, A_3, \dots, A_n)$  are algebraically independent.

Suppose on the contrary that there exists an irreducible polynomial  $R \in \mathbb{Q}(X_1, \dots, X_n)$  such that

$$R(\hat{W}_0, \hat{W}_1, \dots, \hat{W}_{n-1}) = 0.$$

Specializing  $C$  in 0, we obtain

$$R\left(\frac{1 - A_2 - A_2 - \dots - A_n}{A_2}, \frac{1 - A_2 - A_2 - \dots - A_n}{A_2}, \dots, \frac{1 - A_2 - A_2 - A_3}{A_2}, \frac{1 - A_2}{A_2}\right) = 0.$$

Here, the first two rational fractions are the same, and the last  $n-1$  rational fractions generate the field  $\mathbb{Q}(A_2, A_3, \dots, A_n)$ . Therefore the latter are algebraically independent, and  $R = \alpha(X_2 - X_1)$  for a nonzero constant  $\alpha \in \mathbb{Q}$ . This is impossible since  $\hat{W}_0 \neq \hat{W}_1$ .  $\square$

### *Proof of Theorem 6*

We consider the same generalized  $(n+1)$ -system as above. Notice that for  $1 \leq k \leq n$  we have  $P_k \leq P_{n+1}$  and therefore

$$0 \leq \frac{P_k(q)}{q} \leq 1/2.$$

Since all nonzero slopes of the combined graph are at least  $1/2$ , the maxima of the functions  $q \mapsto q^{-1}P_k(q)$  are reached at points where  $P_k$  changes slope from 1 or  $1/2$  to 0. It happens that for each component there is only one such point on the interval  $[1, C]$ .

The definition of the exponents  $\bar{\varphi}_k$  leads to define quantities  $F_k$  by

$$F_k := \limsup_{q \rightarrow \infty} \frac{P_k(q)}{q} = \max_{[1, C]} \frac{P_k(q)}{q} = \frac{P_k(p)}{p} \text{ where } p = \begin{cases} 1 & \text{if } k = 1, \\ \delta_{k,2} & \text{if } 2 \leq k \leq n. \end{cases}$$

We express the quantities  $F_1, \dots, F_n$  as rational fractions in  $\mathbb{Q}(C, A_2, \dots, A_n)$ , using the relations  $A_1 = A_2$  and  $A_{n+1} = 1 - A_1 - A_2 - \dots - A_n$  :

$$\begin{aligned} F_1 &= A_1, \\ F_k &= \frac{CA_k}{A_1 + C(A_2 + \dots + A_k) + CA_k + 1 - (2A_2 + A_3 + \dots + A_{k+1})}, \end{aligned}$$

Since  $F_1, \dots, F_n$  come from a generalized  $(n+1)$ -system  $\mathbf{P}$ , by Theorem 5 there exists a point  $\boldsymbol{\theta}$  in  $\mathbb{R}^n$  such that  $\bar{\varphi}_k(\boldsymbol{\theta}) = F_k$  for every  $1 \leq k \leq n$ . To prove Theorem 6 it is sufficient

to show that the rational fractions  $F_1, \dots, F_n \in \mathbb{Q}(C, A_2, A_3, \dots, A_n)$  are algebraically independent.

Suppose that there exists an irreducible polynomial  $R \in \mathbb{Q}(X_1, \dots, X_n)$  such that

$$R(F_1, \dots, F_n) = 0.$$

Specializing  $C$  in infinity, we obtain

$$R\left(A_2, \frac{1}{2}, \frac{A_3}{(A_2 + A_3) + A_3}, \dots, \frac{A_n}{(A_2 + \dots + A_n) + A_n}\right) = 0$$

where all coordinates except  $1/2$  are algebraically independent. Thus,  $R$  is a constant multiple of  $2X_2 - 1$ , which contradicts  $F_2 \neq 1/2$ . □

We are not able to prove Theorem 6 for the  $n + 1$  exponents  $\bar{\varphi}_1, \dots, \bar{\varphi}_{n+1}$  with this construction. However with some extra work, we can show that the theorem holds for any  $n$  exponents among them.

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## References

- [1] Yann Bugeaud and Michel Laurent. On exponents of homogeneous and inhomogeneous diophantine approximation. *Moscow Math. J.*, 5:747–766, 2005.
- [2] Yann Bugeaud and Michel Laurent. Exponents of diophantine approximation. *Diophantine Geometry Proceedings*, 4:101–121, 2007.
- [3] Y. Cheung. Special divergent trajectories for a homogeneous flow. 2008. [Seminar of Geometry, Chicago University](#).
- [4] Y. Cheung. Prescriptions for a diagonal flow on the space of lattices. 2015. [Diophantine Approximation and Related Topics, Aarhus](#).
- [5] Oleg N. German. On Diophantine exponents and Khintchine’s transference principle. *Mosc. J. Comb. Number Theory*, 2(2):22–51, 2012.
- [6] Vojtěch Jarník. Zum khintchineschen "Übertragungssatz". *Trav. Inst. Math. Tbilissi*, 3:193–212, 1938.

- [7] Alexandre Ya. Khintchine. Über eine klasse linearer diophantischer approximationen. *Rend. Circ. Mat. Palermo* 50, pages 170–195, 1926.
- [8] Alexandre Ya. Khintchine. Zur metrischen theorie der diophantischen approximationen. *Math.Z.*, 24:706–714, 1926.
- [9] Alexandre Ya. Khintchine. On some applications of the method of the additional variable. *Amer. Math. Soc. Translation*, 1950(18):14, 1950.
- [10] Michel Laurent. Exponents of diophantine approximation in dimension two. *Canad. J. Math.*, 61:165–189, 2009.
- [11] Michel Laurent. On transfer inequalities in Diophantine approximation. In *Analytic number theory*, pages 306–314. Cambridge Univ. Press, Cambridge, 2009.
- [12] Hermann Minkowski. *Geometrie der Zahlen*. Bibliotheca Mathematica Teubneriana, Band 40. Johnson Reprint Corp., New York-London, 1968.
- [13] Damien Roy. Spectrum of the exponents of best rational approximation. *Math. Z.*, 13 pages, à paraître.
- [14] Damien Roy. On Schmidt and Summerer parametric geometry of numbers. *Ann. of Math.*, 182:739–786, 2015.
- [15] W. M. Schmidt. Open problems in Diophantine approximation. In *Diophantine approximations and transcendental numbers (Luminy, 1982)*, volume 31 of *Progr. Math.*, pages 271–287. Birkhäuser Boston, Boston, MA, 1983.
- [16] Wolfgang M. Schmidt. On heights of algebraic subspaces and diophantine approximations. *Ann. of Math. (2)*, 85:430–472, 1967.
- [17] Wolfgang M. Schmidt and Leonhard Summerer. Parametric geometry of numbers and applications. *Acta Arithmetica*, 140(1):67–91, 2009.
- [18] Wolfgang M. Schmidt and Leonhard Summerer. Diophantine approximation and parametric geometry of numbers. *Monatsch. Math.*, 169(1):51–104, 2013.
- [19] Wolfgang M. Schmidt and Leonhard Summerer. The generalization of jarnik’s identity. *Acta Arithmetica*, to appear.