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About Jarník's-type relation in higher dimension

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Abstract

Using the Parametric Geometry of Numbers introduced recently by W.M. Schmidt and L. Summerer [17, 18] and results by D. Roy [13, 14], we show that German's transference inequalities between the two most classical exponents of uniform Diophantine approximation are optimal. Further, we establish that the n uniform exponents of Diophantine approximation in dimension n are algebraically independent. Thus, no Jarník's-type relation holds between them.

1 Introduction

Throughout this paper, the integer $n \geq 1$ denotes the dimension of the ambient space, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ denotes an n -tuple of real numbers such that $1, \theta_1, \dots, \theta_n$ are \mathbb{Q} -linearly independent.

Let d be an integer with $0 \leq d \leq n - 1$. We define the exponent $\omega_d(\boldsymbol{\theta})$ (resp. the uniform exponent $\hat{\omega}_d(\boldsymbol{\theta})$) as the supremum of the real numbers ω for which there exist rational affine subspaces $L \subset \mathbb{R}^n$ such that

$$\dim(L) = d, \quad H(L) \leq H \quad \text{and} \quad H(L)d(\boldsymbol{\theta}, L) \leq H^{-\omega}$$

for arbitrarily large real numbers H (resp. for every sufficiently large real number H). Here $H(L)$ denotes the height of L (see [16] for more details), and $d(\boldsymbol{\theta}, L) = \min_{P \in L} d(\boldsymbol{\theta}, P)$ is the minimal distance between $\boldsymbol{\theta}$ and a point of L .

These exponents were introduced originally by M. Laurent [11]. They interpolate between the classical exponents $\omega(\boldsymbol{\theta}) = \omega_{n-1}(\boldsymbol{\theta})$ and $\lambda(\boldsymbol{\theta}) = \omega_0(\boldsymbol{\theta})$ (resp. $\hat{\omega}(\boldsymbol{\theta}) = \hat{\omega}_{n-1}(\boldsymbol{\theta})$ and $\hat{\lambda}(\boldsymbol{\theta}) = \hat{\omega}_0(\boldsymbol{\theta})$) that were introduced by A. Khinchin [7, 8], V. Jarník [6] and Y. Bugeaud and M. Laurent [1, 2].

We have the relations

$$\omega_0(\boldsymbol{\theta}) \leq \omega_1(\boldsymbol{\theta}) \leq \dots \leq \omega_{n-1}(\boldsymbol{\theta}),$$

$$\hat{\omega}_0(\boldsymbol{\theta}) \leq \hat{\omega}_1(\boldsymbol{\theta}) \leq \cdots \leq \hat{\omega}_{n-1}(\boldsymbol{\theta}),$$

and Minkowski's First Convex Body Theorem [12] and Mahler's compound convex bodies theory provide the lower bounds

$$\omega_d(\boldsymbol{\theta}) \geq \hat{\omega}_d(\boldsymbol{\theta}) \geq \frac{d+1}{n-d}, \quad \text{for } 0 \leq d \leq n-1.$$

These exponents happen to be related, as was first noticed by Khinchin with his transference theorem [8]. The study of these transferences has two aspects. First, establishing transference inequalities valid for every suitable point $\boldsymbol{\theta}$. Then, there is the reverse problem, that consists in constructing points $\boldsymbol{\theta}$ to show that these inequalities are sharp. For this, one can prove that there exists points $\boldsymbol{\theta}$ whose exponents satisfy the equality in the transference inequalities. In this case, we say that the inequalities are *best possible*. A stronger result is to prove that given k exponents e_1, \dots, e_k , the transference inequalities between these k exponents define a subset of \mathbb{R}^k that is exactly the set of all k -uples $(e_1(\boldsymbol{\theta}), \dots, e_k(\boldsymbol{\theta}))$ as $\boldsymbol{\theta}$ runs through all points $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n$ such that $1, \theta_1, \dots, \theta_n$ are \mathbb{Q} -linearly independent. The latter set is called the *spectrum* of the exponents (e_1, \dots, e_k) .

When the dimension is $n = 1$, we have the equality $\hat{\omega}_0(\boldsymbol{\theta}) = \hat{\omega}(\boldsymbol{\theta}) = \hat{\lambda}(\boldsymbol{\theta}) = 1$. In [6], V. Jarník showed that in dimension $n = 2$, we have the following algebraic relation between $\hat{\omega}_1(\boldsymbol{\theta})$ and $\hat{\omega}_0(\boldsymbol{\theta})$:

$$\hat{\omega}_0(\boldsymbol{\theta}) + \frac{1}{\hat{\omega}_1(\boldsymbol{\theta})} = 1. \quad (*)$$

Furthermore, V. Jarník noted that, in higher dimension $n \geq 3$, no algebraic relation holds anymore. He proved [6, Satz 3] that for $n \geq 2$, there exist two n -tuples of real numbers $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$ such that

$$\hat{\omega}_{n-1}(\boldsymbol{\theta}) = \hat{\omega}_{n-1}(\boldsymbol{\nu}) = +\infty, \quad \hat{\omega}_0(\boldsymbol{\theta}) = 1 \quad \text{and} \quad \hat{\omega}_0(\boldsymbol{\nu}) = \frac{1}{n-1}.$$

V. Jarník also proved the following transference theorem:

Theorem 1 (Jarník, 1938). *Let $n \geq 2$. For any n -tuples of real number $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ such that $1, \theta_1, \dots, \theta_n$ are \mathbb{Q} -linearly independent, we have*

$$\frac{\hat{\omega}_{n-1}(\boldsymbol{\theta})}{(n-1)\hat{\omega}_{n-1}(\boldsymbol{\theta}) + n} \leq \hat{\omega}_0(\boldsymbol{\theta}) \leq \frac{\hat{\omega}_{n-1}(\boldsymbol{\theta}) - n + 1}{n}.$$

If $\hat{\omega}_{n-1}(\boldsymbol{\theta}) = n$, the interval reduces to the single point $\hat{\omega}_0(\boldsymbol{\theta}) = \frac{1}{n}$.

Remark. O. German [5] and A. Khinchin [9] claim that V. Jarník [6] proved the existence of n -tuples $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ with $\hat{\omega}_{n-1}(\boldsymbol{\theta}) = +\infty$ and $\hat{\omega}_0(\boldsymbol{\theta})$ anywhere in the interval $\left[\frac{1}{n-1}, 1\right]$. It appears to the author that this is not written explicitly in [6].

Recently, O. German [5] improved Theorem 1:

Theorem 2 (German, 2012). *With the notation of Theorem 1, we have*

$$\frac{\hat{\omega}_{n-1}(\boldsymbol{\theta}) - 1}{(n-1)\hat{\omega}_{n-1}(\boldsymbol{\theta})} \leq \hat{\omega}_0(\boldsymbol{\theta}) \leq \frac{\hat{\omega}_{n-1}(\boldsymbol{\theta}) - (n-1)}{\hat{\omega}_{n-1}(\boldsymbol{\theta})}. \quad (**)$$

Note that the interval reduces to a single point if $n = 2$, and that in this case we recover Jarník's relation (*).

The first goal of this paper is to prove that German's inequalities describe the spectrum of the two exponents $(\hat{\omega}_0, \hat{\omega}_{n-1})$.

Theorem 3. *Let $n \geq 2$ be an integer, let $\hat{\omega} \in [n, +\infty]$ and let $\hat{\lambda} \in \left[\frac{\hat{\omega} - 1}{(n-1)\hat{\omega}}, \frac{\hat{\omega} - n + 1}{\hat{\omega}}\right]$,*

where we understand that the interval for $\hat{\lambda}$ is $\left[\frac{1}{n-1}, 1\right]$ when $\hat{\omega} = +\infty$. Then there exist uncountably many n -tuples of real numbers $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$, with $1, \theta_1, \dots, \theta_n$ \mathbb{Q} -linearly independent, such that $\hat{\omega}_{n-1}(\boldsymbol{\theta}) = \hat{\omega}$ and $\hat{\omega}_0(\boldsymbol{\theta}) = \hat{\lambda}$.

In [19], W. Schmidt and L. Summerer obtained independently a similar result, proving that the inequalities (**) of German are best possible.

One can wonder if in higher dimension ($n \geq 3$), there exists a Jarník's-type relation between the n uniform exponents $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}$. The next theorem states that no such algebraic relation holds.

Theorem 4. *For every integer $n \geq 3$, the n uniform exponents $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}$ are algebraically independent.*

Thus, the spectrum of the n uniform exponents $\hat{\omega}_0, \dots, \hat{\omega}_{n-1}$ is a subset of \mathbb{R}^n with nonempty interior.

We also know the spectrum of other families of exponents. M. Laurent [10] described the spectrum of the four exponents $\omega_0, \hat{\omega}_0, \omega_{n-1}, \hat{\omega}_{n-1}$ in dimension $n = 2$. In his PhD thesis, the author gives an alternative proof of this result. However, for $n \geq 3$ this spectrum is still unknown.

D. Roy showed in [13] that the going-up and going-down transference inequalities of M. Laurent [11] describe the spectrum of the n exponents $\omega_0, \dots, \omega_{n-1}$.

In section 2, we introduce Parametric Geometry of Numbers, which is the main tool to prove Theorem 3 (section 3) and Theorem 4 (section 5), and to give an alternative proof of Theorem 2 (section 4).

2 Parametric Geometry of Numbers

The parametric geometry of numbers answers a question of W. M. Schmidt [15]. Given a convex body and a lattice, we deform either of them with a one parameter diagonal map. We study the behavior of the successive minima in terms of this parameter. It was developed by W. M. Schmidt and L. Summerer [17, 18], and further by D. Roy [13, 14]. Independently, I. Cheung [3, 4] also developed a similar theory.

In this paper, we use the notation introduced by D. Roy in [13, 14] which is essentially dual to the one of W. M. Schmidt and L. Summerer [17, 18]. We refer the reader to these papers for further details. Here $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + \dots + x_n y_n$ is the usual scalar product of vectors \mathbf{x} and \mathbf{y} , and $\|\mathbf{x}\|_2 = \sqrt{\mathbf{x} \cdot \mathbf{x}}$ is the usual Euclidean norm.

Let $\mathbf{u} = (u_0, \dots, u_n)$ be a vector in \mathbb{R}^{n+1} , with Euclidean norm $\|\mathbf{u}\|_2 = 1$. For a real parameter $Q \geq 1$ we consider the convex body

$$\mathcal{C}_{\mathbf{u}}(Q) = \left\{ \mathbf{x} \in \mathbb{R}^{n+1} \mid \|\mathbf{x}\|_2 \leq 1, |\mathbf{x} \cdot \mathbf{u}| \leq Q^{-1} \right\}.$$

For $1 \leq d \leq n+1$ we denote by $\lambda_d(\mathcal{C}_{\mathbf{u}}(Q))$ the d -th minimum of $\mathcal{C}_{\mathbf{u}}(Q)$ relatively to the lattice \mathbb{Z}^{n+1} . For $q \geq 0$ and $1 \leq d \leq n+1$ we set

$$L_{\mathbf{u},d}(q) = \log \lambda_d(\mathcal{C}_{\mathbf{u}}(e^q)).$$

Finally, we define the following map associated with \mathbf{u} :

$$\begin{aligned} \mathbf{L}_{\mathbf{u}} : [0, \infty) &\rightarrow \mathbb{R}^{n+1} \\ q &\mapsto (L_{\mathbf{u},1}(q), \dots, L_{\mathbf{u},n+1}(q)). \end{aligned}$$

The lattice \mathbb{Z}^{n+1} is invariant under permutation of coordinates. Hence, $\mathbf{L}_{\mathbf{u}}$ remains the same if we permute the coordinates in \mathbf{u} . Since $\|\mathbf{u}\|_2 = 1$ we can thus assume that $u_0 \neq 0$.

The following proposition links the exponents of Diophantine approximation associated with $\boldsymbol{\theta} = (\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0})$ to the behavior of the map $\mathbf{L}_{\mathbf{u}}$, assuming $u_0 \neq 0$. It was first stated by W.M. Schmidt and L. Summerer in [17] (Theorem 1.4). It also appears as Relations (1.8) and (1.9) in [18]. In the notation of D. Roy [13] (Proposition 3.1), it reads as follows.

Proposition 1 (Schmidt, Summerer, 2009). *Let $\mathbf{u} = (u_0, \dots, u_n) \in \mathbb{R}^{n+1}$, with Euclidean norm $\|\mathbf{u}\|_2 = 1$ and $u_0 \neq 0$. Set $\boldsymbol{\theta} = (\frac{u_1}{u_0}, \dots, \frac{u_n}{u_0})$. For $1 \leq k \leq n$, we have the following relations:*

$$\begin{aligned} \liminf_{q \rightarrow +\infty} \frac{L_{\mathbf{u},1}(q) + \dots + L_{\mathbf{u},k}(q)}{q} &= \frac{1}{1 + \omega_{n-k}(\boldsymbol{\theta})}, \\ \limsup_{q \rightarrow +\infty} \frac{L_{\mathbf{u},1}(q) + \dots + L_{\mathbf{u},k}(q)}{q} &= \frac{1}{1 + \hat{\omega}_{n-k}(\boldsymbol{\theta})}. \end{aligned}$$

Thus, if we know an explicit map $\mathbf{P} = (P_1, \dots, P_{n+1}) : [0, \infty) \rightarrow \mathbb{R}^{n+1}$, such that $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded, then we can compute the $2n$ exponents $\hat{\omega}_0(\boldsymbol{\theta}), \dots, \hat{\omega}_{n-1}(\boldsymbol{\theta}), \omega_0(\boldsymbol{\theta}), \dots, \omega_{n-1}(\boldsymbol{\theta})$ for the above point $\boldsymbol{\theta}$ upon replacing $L_{\mathbf{u},i}$ by P_i in the above formulas for $1 \leq i \leq n$. For this purpose, we consider the following family of maps, introduced by D. Roy in [13].

Definition (Roy, 2014). Let I be a subinterval of $[0, \infty)$ with non-empty interior. A generalized $(n+1)$ -system on I is a continuous piecewise linear map $\mathbf{P} = (P_1, \dots, P_{n+1}) : I \rightarrow \mathbb{R}^{n+1}$ with the following three properties.

- (S1) For each $q \in I$, we have $0 \leq P_1(q) \leq \dots \leq P_{n+1}(q)$ and $P_1(q) + \dots + P_{n+1}(q) = q$.
- (S2) If H is a non empty open subinterval of I on which \mathbf{P} is differentiable, then there are integers \underline{r}, \bar{r} with $1 \leq \underline{r} \leq \bar{r} \leq n+1$ such that $P_{\underline{r}}, P_{\underline{r}+1}, \dots, P_{\bar{r}}$ coincide on the whole interval H and have slope $1/(\bar{r} - \underline{r} + 1)$ while any other component P_k of \mathbf{P} is constant on H .
- (S3) If q is an interior point of I at which \mathbf{P} is not differentiable, if $\underline{r}, \bar{r}, \underline{s}, \bar{s}$ are the integers for which

$$P'_k(q^-) = \frac{1}{\bar{r} - \underline{r} + 1} \quad (\underline{r} \leq k \leq \bar{r}) \quad \text{and} \quad P'_k(q^+) = \frac{1}{\bar{s} - \underline{s} + 1} \quad (\underline{s} \leq k \leq \bar{s}),$$

and if $\underline{r} < \bar{s}$, then we have $P_{\underline{r}}(q) = P_{\underline{r}+1}(q) = \dots = P_{\bar{s}}(q)$.

Here $P'_k(q^-)$ (resp. $P'_k(q^+)$) denotes the left (resp. right) derivative of P_k at q . The next result combines Theorem 4.2 and Corollary 4.7 of [13].

Theorem 5 (Roy, 2014). *For each non-zero point $\mathbf{u} \in \mathbb{R}^{n+1}$, there exists $q_0 \geq 0$ and a generalized $(n+1)$ -system \mathbf{P} on $[q_0, \infty)$ such that $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded on $[q_0, \infty)$. Conversely, for each generalized $(n+1)$ -system \mathbf{P} on an interval $[q_0, \infty)$ with $q_0 \geq 0$, there exists a non-zero point $\mathbf{u} \in \mathbb{R}^{n+1}$ such that $\mathbf{L}_{\mathbf{u}} - \mathbf{P}$ is bounded on $[q_0, \infty)$.*

In view of the remark following Proposition 1, this result reduces the determination of the joint spectrum of Diophantine approximation exponents to a combinatorial study of generalized $(n + 1)$ -systems.

Although the definition of a generalized $(n + 1)$ -system $\mathbf{P} = (P_1, \dots, P_{n+1})$ may look complicated, it is easy to understand in terms of the *combined graph* of \mathbf{P} , that is the union of the graphs of P_1, \dots, P_{n+1} over the interval of definition I of \mathbf{P} . We explain this below.

A *division point* of \mathbf{P} is an endpoint of I contained in I or an interior point of I at which \mathbf{P} is not differentiable. Such points form a discrete subset of I . Between two consecutive division points $q^* < q$ of I , the graph of each component of \mathbf{P} is a line segment. All these line segments have slope 0 except for one line segment of positive slope $1/t$ where t is the number of components of \mathbf{P} whose graph over $[q^*, q]$ is that line segment. In view of the condition $P_1 \leq P_2 \leq \dots \leq P_{n+1}$, there must be consecutive components $P_{\underline{r}}, \dots, P_{\bar{r}}$ of \mathbf{P} with $\bar{r} - \underline{r} + 1 = t$. If q is also an interior point of I and if $P_{\underline{s}}, \dots, P_{\bar{s}}$ are the components of \mathbf{P} whose graph has positive slope $\frac{1}{\bar{s} - \underline{s} + 1}$ to the right of q , then there are two cases.

- 1) If $\underline{r} < \bar{s}$, we say that q is an *ordinary division point*. In this case, we have $P_{\underline{r}}(q) = \dots = P_{\bar{s}}(q)$ according to **(S3)**. This implies that $\underline{r} \leq \underline{s}$ and $\bar{r} \leq \bar{s}$. Among $P_{\underline{r}}, \dots, P_{\bar{s}}$, the components P_j with $\underline{s} \leq j \leq \bar{r}$ (if any) change slope from $\frac{1}{\bar{r} - \underline{r} + 1}$ to $\frac{1}{\bar{s} - \underline{s} + 1}$. Those with $j \leq \min(\bar{r}, \underline{s} - 1)$ change slope from $\frac{1}{\bar{r} - \underline{r} + 1}$ to 0. The remaining components P_j with $\bar{r} + 1 \leq j \leq \underline{s} - 1$ (if any) have constant slope 0 in a neighborhood of q . The reader is invited to draw a picture for himself or to look at those in [13, §4].
- 2) Otherwise, we have $\underline{r} > \bar{s}$ because it cannot happen that $\underline{r} = \bar{s}$ (or \mathbf{P} is differentiable at q). Then, we say that q is a *switch point*. In this case, we have $P_{\underline{r}}(q) = \dots = P_{\bar{r}}(q) > P_{\underline{s}}(q) = \dots = P_{\bar{s}}(q)$ which mean that the end point of the line segment of slope $\frac{1}{\bar{r} - \underline{r} + 1}$ at the left of q lies above the initial point of the line segment of slope $\frac{1}{\bar{s} - \underline{s} + 1}$ at the right of q .

It can be shown that the combined graph of a generalized $(n + 1)$ -system \mathbf{P} uniquely determines the map \mathbf{P} provided that we know the value of \mathbf{P} at one point of its interval of definition. An example of this is shown in [13, §4]. We will see two other examples in the sections 3 and 5.

In [17, 18] W. M. Schmidt and L. Summerer introduce the following exponents for an

integer $1 \leq d \leq n + 1$:

$$\begin{aligned}\varphi_d &= \liminf_{q \rightarrow \infty} \frac{L_{\mathbf{u},d}(q)}{q}, \\ \bar{\varphi}_d &= \limsup_{q \rightarrow \infty} \frac{L_{\mathbf{u},d}(q)}{q}.\end{aligned}$$

For these exponents, we have the following analogue of Theorem 4:

Theorem 6. *For every integer $n \geq 3$, the exponents $\bar{\varphi}_1, \dots, \bar{\varphi}_n$ are algebraically independent.*

3 Proof of Theorem 3

In this section, we construct a family of generalized $(n+1)$ -systems. Then, via Theorem 5, we get a family of n -tuples having the requested properties stated in Theorem 3. We first treat the case where $\hat{\omega}_{n-1}$ is finite and $n \geq 3$. We will explain later how to adapt the construction if $n = 2$ or $\hat{\omega}_{n-1}$ is infinite.

First, note that a generalized $(n+1)$ -system with all components equal to $\frac{q}{n+1}$ provides via Theorem 5 a point $\boldsymbol{\theta}$ with $\hat{\omega}_{n-1}(\boldsymbol{\theta}) = n$ and $\hat{\omega}_0(\boldsymbol{\theta}) = \frac{1}{n}$. Thus, we can exclude this case in the next construction.

Let q_0 be a positive real number, fix a real number $\hat{\omega} > n \geq 2$ and set a parameter a with $\frac{1}{n-1} \leq a \leq 1$. We define the sequence $(q_{6m})_{m \geq 0}$ by:

$$q_{6m} = (1 + a(\hat{\omega} - n))q_{6(m-1)}, \text{ for } m \geq 1.$$

Since $\hat{\omega} > n$, q_{6m} goes to infinity.

We construct a generalized $(n+1)$ -system \mathbf{P} whose graph is invariant under the dilation of factor $(1 + a(\hat{\omega} - n)) > 1$ on the interval $[q_0, +\infty)$. Thus, we only need to define \mathbf{P} on a generic interval $[q_{6m}, q_{6(m+1)}]$. Figure 1 shows the pattern of the combined graph of \mathbf{P} .

For every integer $m \geq 0$, we define \mathbf{P} at q_{6m} as follows:

$$\begin{aligned}P_1(q_{6m}) &= P_2(q_{6m}) = \frac{q_{6m}}{\hat{\omega} + 1}, \\ P_3(q_{6m}) &= \dots = P_n(q_{6m}) = \frac{1 + \left(\frac{1-a}{n-2}\right)(\hat{\omega} - n)}{\hat{\omega} + 1} q_{6m}, \\ P_{n+1}(q_{6m}) &= \frac{1 + a(\hat{\omega} - n)}{\hat{\omega} + 1} q_{6m}.\end{aligned}$$

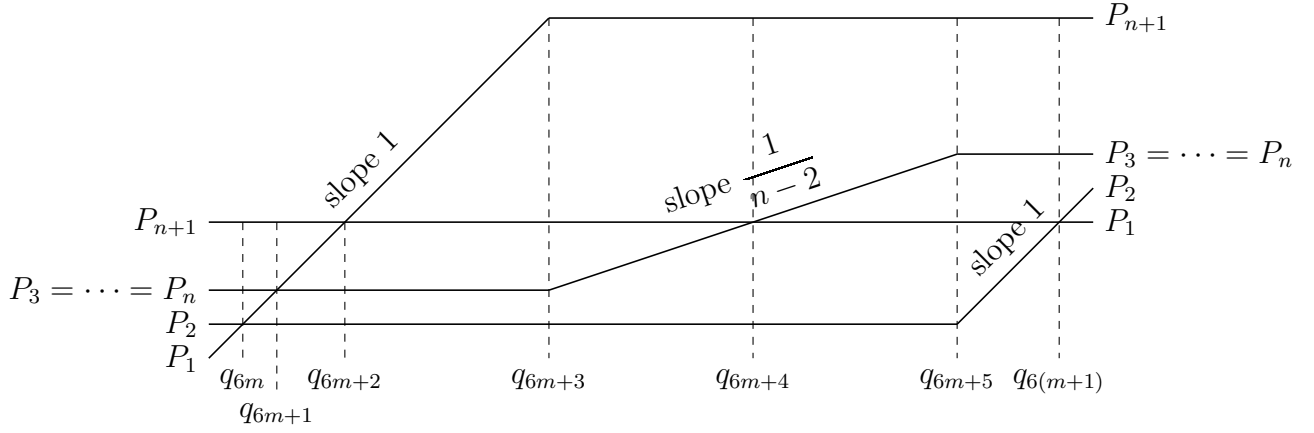


Figure 1: Combined graph of \mathbf{P} on a generic interval $[q_{6m}, q_{6(m+1)}]$

Here the parameter a says how large P_{n+1} is at each point q_{6m} . The condition $a \geq \frac{1}{n-1}$ imposes the condition $P_{n+1}(q_{6m}) \geq P_n(q_{6m})$, and the condition $a \leq 1$ imposes that $P_3(q_{6m}) \geq P_2(q_{6m})$. We have the dilation condition $\mathbf{P}(q_{6(m+1)}) = \mathbf{P}((1+a(\hat{\omega}-n))q_{6m}) = (1+a(\hat{\omega}-n))\mathbf{P}(q_{6m})$ by the definition of the sequence $(q_{6m})_{m \geq 0}$.

For $k = 0, \dots, 5$ the graph has only one line segment of positive slope on the interval $[q_{6m+k}, q_{6m+k+1}]$. The graph is clearly the combined graph of a generalized $(n+1)$ -system with seven division points q_{6m}, \dots, q_{6m+6} . The points q_{6m+3} and q_{6m+5} are switch points while the others are ordinary division points. Furthermore it is uniquely defined since we know the value of \mathbf{P} at the point q_{6m} , where as requested

$$P_1(q_{6m}) + \dots + P_{n+1}(q_{6m}) = q_{6m}.$$

Easy computation gives

$$\begin{aligned} q_{6m} &= (1+a(\hat{\omega}-n))q_{6(m-1)}, & q_{6m+1} &= \frac{(n-2)(\hat{\omega}+1) + (1-a)(\hat{\omega}-n)}{(n-2)(\hat{\omega}+1)}q_{6m}, \\ q_{6m+2} &= \frac{(n+1) + (1+a)(\hat{\omega}-n)}{\hat{\omega}+1}q_{6m}, & q_{6m+3} &= \frac{\hat{\omega} + (1+a(\hat{\omega}-n))^2}{\hat{\omega}+1}q_{6m}, \\ q_{6m+4} &= \frac{1 + (1+a(\hat{\omega}-n))(n+a(\hat{\omega}-n))}{\hat{\omega}+1}q_{6m}, & q_{6m+5} &= \frac{1 + 2a(\hat{\omega}-n) + \hat{\omega}(1+a(\hat{\omega}-n))}{\hat{\omega}+1}q_{6m}. \end{aligned}$$

We now compute its associated exponents with Proposition 1. One can notice that the local extrema of the functions $q \rightarrow q^{-1}P_k(q)$, $1 \leq k \leq n+1$ are located at division points where P_k changes slope.

Since \mathbf{P} is invariant under dilation of factor $C = (1 + a(\hat{\omega} - n))$ we have for every $m \geq 0$, every $1 \leq k \leq n+1$, and every q in $[q_{6m}, q_{6m+6})$ the relation

$$q^{-1}P_k(q) = q^{-1}C^m P_k(qC^{-m}),$$

where $C^{-m}q$ lies in the fundamental interval $[q_0, q_6]$.

Thus,

$$\begin{aligned} \limsup_{q \rightarrow +\infty} \frac{P_1(q)}{q} &= \max_{q_0 \leq q \leq q_6} \frac{P_1(q)}{q} = \frac{P_1(q_0)}{q_0} = \frac{1}{\hat{\omega} + 1}, \\ \liminf_{q \rightarrow +\infty} \frac{P_{n+1}(q)}{q} &= \min_{q_0 \leq q \leq q_6} \frac{P_{n+1}(q)}{q} = \frac{P_{n+1}(q_2)}{q_2} = \frac{1 + a(\hat{\omega} - n)}{n + 1 + (1 + a)(\hat{\omega} - n)}, \end{aligned}$$

because the component P_{n+1} changes slope from zero to some positive value only at q_{6m+2} .

Then, according to Proposition 1, Theorem 5 provides an n -tuple $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ such that

$$\begin{aligned} \frac{1}{\hat{\omega}_{n-1}(\boldsymbol{\theta}) + 1} &= \limsup_{q \rightarrow +\infty} \frac{P_1(q)}{q} = \frac{1}{\hat{\omega} + 1}, \\ \frac{\hat{\omega}_0(\boldsymbol{\theta})}{\hat{\omega}_0(\boldsymbol{\theta}) + 1} &= \liminf_{q \rightarrow +\infty} \frac{P_{n+1}(q)}{q} = \frac{1 + a(\hat{\omega} - n)}{n + 1 + (1 + a)(\hat{\omega} - n)}. \end{aligned}$$

Thus, this $\boldsymbol{\theta}$ satisfies

$$\hat{\omega}_{n-1}(\boldsymbol{\theta}) = \hat{\omega} \quad \text{and} \quad \hat{\omega}_0(\boldsymbol{\theta}) = \frac{1 + a(\hat{\omega} - n)}{\hat{\omega}}.$$

When a runs through the interval $[\frac{1}{n-1}, 1]$, then $\hat{\omega}_0(\boldsymbol{\theta})$ runs through the interval

$$\left[\frac{\hat{\omega} - 1}{(n-1)\hat{\omega}}, \frac{\hat{\omega} - (n-1)}{\hat{\omega}} \right].$$

If $n = 2$, we remove the line $P_3 = \dots = P_n$ and the interval $[q_{6m+3}, q_{6m+5}]$ from the generic graph on the interval $[q_{6m}, q_{6(m+1)}]$, the parameter a is then forced to be equal to 1. Thus, we construct $\boldsymbol{\theta}$ with

$$\hat{\omega}_1(\boldsymbol{\theta}) = \hat{\omega} \quad \text{and} \quad \hat{\omega}_0(\boldsymbol{\theta}) = 1 - \frac{1}{\hat{\omega}},$$

which agrees with Jarník's relation (*).

If $\hat{\omega}$ is infinite, we replace $\hat{\omega}$ by $m + n + 1$ in our construction. For a given real number q_0 we consider the sequence $(q_{6m})_{m \geq 1}$ defined by

$$q_{6m} = (m + 1)q_{6(m-1)}.$$

Figure 1 still represents the combined graph from \mathbf{P} on a generic interval $[q_{6m}, q_{6m+6}]$, with the following settings at q_{6m} :

$$\begin{aligned} P_1(q_{6m}) &= P_2(q_{6m}) = \frac{q_{6m}}{m + n + 2}, \\ P_3(q_{6m}) &= \dots = P_n(q_{6m}) = \frac{1 + \left(\frac{1-a}{n-2}\right)(m+1)}{m + n + 2} q_{6m}, \\ P_{n+1}(q_{6m}) &= \frac{1 + a(m+1)}{m + n + 2} q_{6m}. \end{aligned}$$

Note that the combined graph is not invariant under dilation anymore. We have

$$\begin{aligned} \limsup_{q \rightarrow +\infty} \frac{P_1(q)}{q} &= \limsup_{m \rightarrow +\infty} \max_{q_{6m} \leq q \leq q_{6(m+1)}} \frac{P_1(q)}{q} = \limsup_{m \rightarrow +\infty} \frac{P_1(q_{6m})}{q_{6m}} = \limsup_{m \rightarrow +\infty} \frac{1}{m + n + 2} = 0, \\ \liminf_{q \rightarrow +\infty} \frac{P_{n+1}(q)}{q} &= \liminf_{m \rightarrow +\infty} \min_{q_{6m} \leq q \leq q_{6(m+1)}} \frac{P_{n+1}(q)}{q} = \liminf_{m \rightarrow +\infty} \frac{P_{n+1}(q_{6m+2})}{q_{6m+2}} \\ &= \liminf_{m \rightarrow +\infty} \frac{1 + a(m+1)}{n + 1 + (1+a)(m+1)} = \frac{a}{a+1}. \end{aligned}$$

Again, Theorem 5 provides us with an n -tuple $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$ such that

$$\hat{\omega}_{n-1}(\boldsymbol{\theta}) = +\infty \quad \text{and} \quad \hat{\omega}_0(\boldsymbol{\theta}) = a,$$

where a runs through the interval $[\frac{1}{n-1}, 1]$.

Note that if $1, \theta_1, \dots, \theta_n$ are \mathbb{Q} -linearly dependent, then there exists an integer point $\mathbf{x} \in \mathbb{Z}^n$ such that $|\mathbf{x} \cdot \mathbf{u}| = 0$. This implies that $L_{\mathbf{u},1}(q)$ is bounded above by $\log(\|\mathbf{x}\|_2)$. In our construction by dilatation P_1 is not bounded, hence the independence by contradiction.

To complete the proof of Theorem 3, we have to check that we can construct uncountably many n -tuples with given exponents. Let $\hat{\omega}$ and $\hat{\lambda}$ as in Theorem 3, and a the parameter such that Theorem 5 provides an n -tuple θ whose exponents satisfy

$$\hat{\omega}_{n-1}(\theta) = \hat{\omega}, \text{ and } \hat{\omega}_0(\theta) = \hat{\lambda} = \frac{1 + a(\hat{\omega} - n)}{\hat{\omega}}.$$

Fix q_0 a real number to start the construction from \mathbf{P} as above with parameter a . For every ρ_1 and ρ_2 such that $q_0 \leq \rho_1 < \rho_2 \leq q_5$, we denote by \mathbf{P}_{ρ_1} and \mathbf{P}_{ρ_2} the $(n+1)$ -generalized system with parameter a starting in ρ_1 and ρ_2 . We have $\mathbf{P}_{\rho_1}(q_6) \neq \mathbf{P}_{\rho_2}(q_6)$ and

$$\|\mathbf{P}_{\rho_1}(q_{6m}) - \mathbf{P}_{\rho_2}(q_{6m})\|_{\infty} = \frac{q_{6m}}{q_6} \|\mathbf{P}_{\rho_1}(q_6) - \mathbf{P}_{\rho_2}(q_6)\|_{\infty} \rightarrow_{n \rightarrow \infty} \infty,$$

where $\|(x_1, \dots, x_n)\|_{\infty} = \max_{1 \leq k \leq n} |x_k|$.

Thus, their difference is unbounded, and they cannot correspond to the same θ via Theorem 5. This ends the proof of Theorem 3. □

4 An alternative proof of Theorem 2

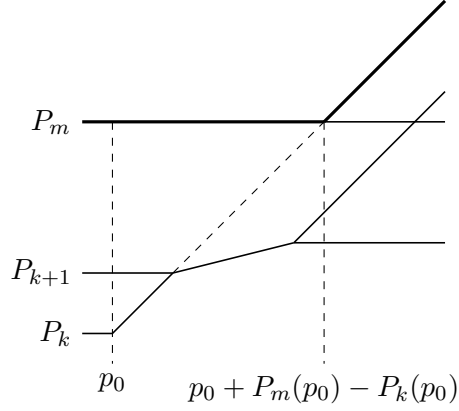
In this section, we give an alternative proof of Theorem 2 using arguments from Parametric Geometry of Numbers.

One can notice that the extremal values of the components of \mathbf{P} are reached at the division points. The condition **(S3)** translates into the fact that for every division point q , the right endpoint of the segment with non-zero slope ending at q lies above the left endpoint of the one starting at q . A first consequence is that when P_1 is non constant, it increases until reaching $P_2(q)$. A second consequence is the following proposition.

Proposition 2. *For every $1 \leq k < m \leq n+1$, if p_0 is a division point such that P_k has slope changing from 0 to 1 at p_0 , then we have for every $p > p_0$*

$$P_m(p) \leq \max(P_m(p_0), P_k(p_0) + p - p_0).$$

In particular, P_m is constant on the interval $[p_0, p_0 + P_m(p_0) - P_k(p_0)]$.



Upper bound: Let \mathbf{P} be a generalized $(n+1)$ -system, and $\hat{\omega}$ the real number such that

$$\limsup_{q \rightarrow +\infty} \frac{P_1(q)}{q} = \frac{1}{\hat{\omega} + 1}.$$

If this limit is zero, we set $\hat{\omega} = +\infty$. We treat this case after.

Let $\varepsilon > 0$. There exist arbitrarily large division points p_0 where $q^{-1}P_1(q)$ has a local maximum and

$$\frac{1 - \varepsilon}{\hat{\omega} + 1} \leq \frac{P_1(p_0)}{p_0} \leq \frac{1 + \varepsilon}{\hat{\omega} + 1}.$$

Since q_0 is a local maximum, we have $P_1(q_0) = P_2(q_0)$. Furthermore, $P_1(q) \leq P_2(q) \leq \dots \leq P_{n+1}(q)$ and $P_1(q) + \dots + P_{n+1}(q) = q$ provide

$$P_{n+1}(p_0) \leq p_0 - nP_1(p_0) \leq \frac{\hat{\omega} + 1 - n - n\varepsilon}{\hat{\omega} + 1} p_0.$$

At the point $p = p_0 + \frac{\hat{\omega} - n - n\varepsilon}{\hat{\omega} + 1} p_0$, according to Proposition 2, we have the upper bound

$$P_{n+1}(p) \leq \max(P_{n+1}(p_0), P_1(p_0) + p - p_0) \leq \frac{1 + \varepsilon + \hat{\omega} - n - n\varepsilon}{\hat{\omega} + 1} p_0.$$

Thus, for arbitrarily large real numbers p , we have

$$\frac{P_{n+1}(p)}{p} \leq \frac{\hat{\omega} + 1 - n - (n-1)\varepsilon}{2\hat{\omega} - n + 1 - n\varepsilon},$$

thus

$$\frac{\hat{\lambda}}{\hat{\lambda} + 1} = \liminf_{q \rightarrow +\infty} \frac{P_{n+1}(q)}{q} \leq \frac{\hat{\omega} + 1 - n}{2\hat{\omega} - n + 1},$$

giving that

$$\hat{\lambda} \leq \frac{\hat{\omega} - (n-1)}{\hat{\omega}}.$$

If $\hat{\omega}$ is infinite, then we have

$$0 \leq \frac{P_1(p_0)}{p_0} \leq \varepsilon, \quad P_1(p_0) = P_2(p_0).$$

Using Proposition 2 at the point $p = 2p_0 - n\varepsilon$ we get

$$P_{n+1}(p) \leq p_0 - (n+1)\varepsilon.$$

Thus, for arbitrarily large real numbers p , we have

$$\frac{P_{n+1}(p)}{p} \leq \frac{p_0 - (n+1)\varepsilon}{2p_0 - n\varepsilon}.$$

This implies that

$$\hat{\lambda} \leq 1.$$

Thus, we have proved the upper bound in Theorem 2.

Lower bound: Let \mathbf{P} be a generalized $(n+1)$ -system, and $\hat{\omega}, \hat{\lambda}$ the real numbers such that

$$\limsup_{q \rightarrow +\infty} \frac{P_1(q)}{q} = \frac{1}{\hat{\omega} + 1}, \quad \liminf_{q \rightarrow +\infty} \frac{P_{n+1}(q)}{q} = \frac{\hat{\lambda}}{\hat{\lambda} + 1},$$

where we understand that $\hat{\omega} = +\infty$ if the corresponding lim sup is zero. Again, we treat this case after.

Let $\varepsilon_1 > 0$. There exists a real number q_0 such that $q \geq q_0$ implies

$$\frac{P_1(q)}{q} \leq \frac{1 + \varepsilon_1}{\hat{\omega} + 1}.$$

Let $\varepsilon_2 > 0$. There exist arbitrarily large division points $p \geq q_0$ where $q^{-1}P_1(q)$ has a local minimum and

$$\left| \frac{P_{n+1}(p)}{p} - \frac{\hat{\lambda}}{\hat{\lambda} + 1} \right| \leq \varepsilon_2.$$

Let $p_0 = \max \{q \leq p \mid P_1(q) = P_2(q)\}$. At the point p_0 we have

$$\begin{aligned} P_1(p_0) &= P_2(p_0) \leq \frac{1 + \varepsilon_1}{\hat{\omega} + 1} p_0, \\ P_{n+1}(p_0) &\geq \frac{p_0 - 2P_1(p_0)}{n-1}, \end{aligned}$$

since $p_0 = P_1(p_0) + \cdots + P_{n+1}(p_0) \leq 2P_1(p_0) + (n-1)P_{n+1}(p_0)$.

We first show that $q \rightarrow P_1(q)$ is constant on the interval $[p_0, p]$. If not, there exists a real number $p_0 < p_1 < p$ where P_1 has slope 1. Since p is a local minimum from $q^{-1}P_1(q)$, we have $P_1'(p^-) = 0$. Thus, there exists a point in (p_1, p) where P_1 changes slope from 1 to 0. At this point $P_1 = P_2$, which contradicts the definition of p_0 . Thus,

$$P_1(p_0) = P_1(p).$$

We can write

$$p = \sum_{k=1}^{n+1} P_k(p) \leq nP_{n+1}(p) + P_1(p_0).$$

Thus,

$$\frac{P_{n+1}(p)}{p} \geq \frac{P_{n+1}(p)}{nP_{n+1}(p) + P_1(p_0)},$$

where the right hand side is an increasing function of $P_{n+1}(p)$. Since

$$P_{n+1}(p) \geq P_{n+1}(p_0) \geq \frac{p_0 - 2P_1(p_0)}{n-1},$$

we have

$$\frac{P_{n+1}(p)}{p} \geq \frac{p_0 - 2P_1(p_0)}{np_0 - (n+1)P_1(p_0)},$$

where the right hand side is a decreasing function of $P_1(p_0)$. Since

$$P_1(p_0) \leq \frac{1 + \varepsilon_1}{\hat{\omega} + 1} p_0,$$

we have

$$\frac{P_{n+1}(p)}{p} \geq \frac{\hat{\omega} - 1 - 2\varepsilon_1}{n\hat{\omega} - 1 - (n+1)\varepsilon_1}.$$

Thus,

$$\frac{\hat{\lambda}}{\hat{\lambda} + 1} \geq \frac{\hat{\omega} - 1 - 2\varepsilon_1}{n\hat{\omega} - 1 - (n+1)\varepsilon_1} - \varepsilon_2.$$

This gives the expected bound

$$\hat{\lambda} \geq \frac{\hat{\omega} - 1}{(n-1)\hat{\omega}}.$$

If $\hat{\omega}$ is infinite, we consider a real number q_0 such that $q \geq q_0$ implies

$$0 \leq \frac{P_1(q)}{q} \leq \varepsilon_1.$$

This provides the following estimates at p :

$$\frac{P_{n+1}(p)}{p} \geq \frac{1 - 2\varepsilon_1}{n - (n+1)\varepsilon_1}.$$

Thus, we get

$$\frac{\hat{\lambda}}{\hat{\lambda} + 1} \geq \frac{1 - 2\varepsilon_1}{n - (n+1)\varepsilon_1} - \varepsilon_2.$$

This gives the expected bound

$$\hat{\lambda} \geq \frac{1}{n-1}.$$

Thus, we have proved the lower bound. This ends the proof of Theorem 2. □

5 Proof of Theorems 4 and 6

In this section, we construct a family of generalized $(n+1)$ -systems depending on n parameters which via Theorem 5 provides us with a family of n -tuples whose uniform exponents are expressed as a function of these n parameters. Then, we show that these functions are algebraically independent.

Fix the dimension $n \geq 3$. Choose $n+2$ parameters $A_1, A_2, \dots, A_{n+1}, C$ satisfying

$$\begin{aligned} 0 < A_1 = A_2 < A_3 < A_4 < \dots < A_{n+1}, \\ 1 &= A_1 + A_2 + \dots + A_{n+1}, \\ \frac{A_{k+1}}{A_k} < C < \frac{A_{k+2}}{A_k} \text{ for } 2 \leq k \leq n-1, \\ 1 < \frac{A_{n+1}}{A_n} < C. \end{aligned} \tag{0}$$

We consider the generalized $(n+1)$ -system \mathbf{P} on the interval $[1, C]$ whose combined graph is given by Figure 2, where

$$P_k(1) = A_k \text{ and } P_k(C) = CA_k \text{ for } 1 \leq k \leq n+1.$$

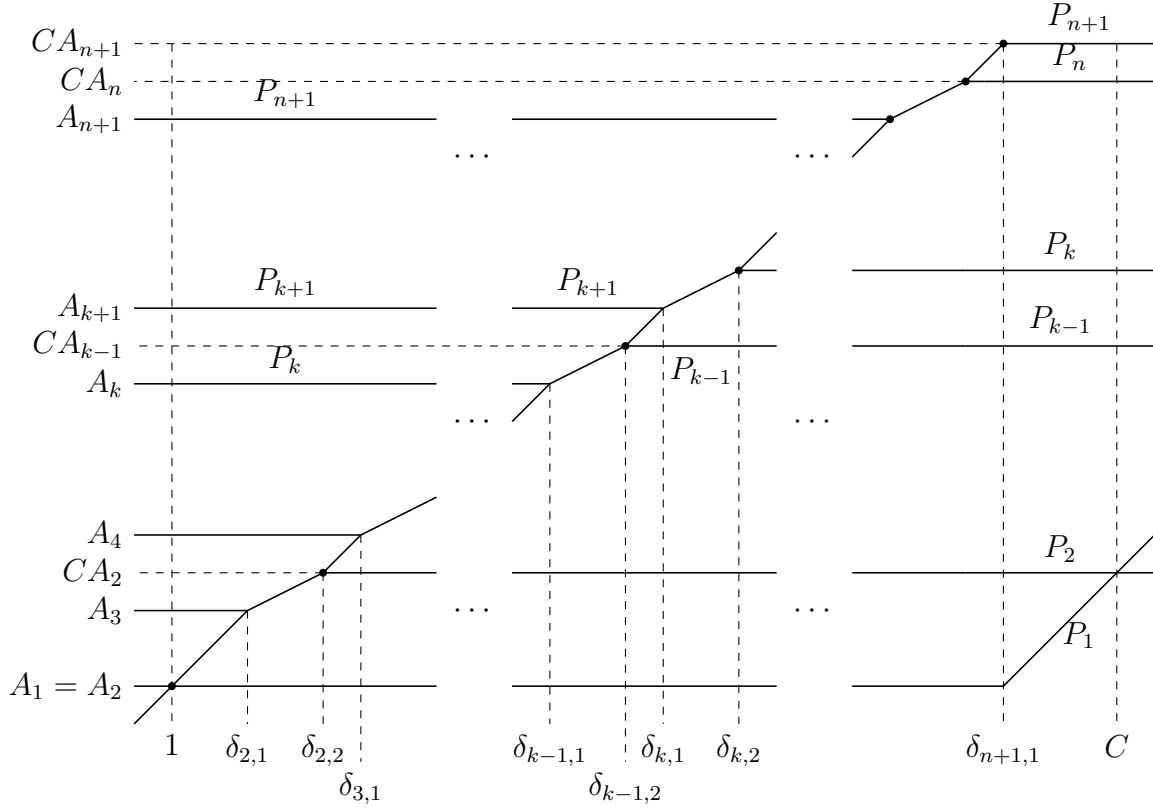


Figure 2: Pattern of the combined graph of \mathbf{P} on the fundamental interval $[1, C]$

On each interval between two consecutive division points, there is only one line segment with nonzero slope. This line segment has slope 1 on the intervals $[1, \delta_{2,1}]$, $[\delta_{n+1,1}, C]$ and $[\delta_{k-1,2}, \delta_{k,1}]$ for $3 \leq k \leq n+1$, and has slope $1/2$ on the interval $[\delta_{k,1}, \delta_{k,2}]$, for $3 \leq k \leq n$, where the two components P_k and P_{k+1} coincide. We have $2n+1$ division points $1, C, \delta_{k,1}$ and $\delta_{l,2}$ for $2 \leq k \leq n+1$ and $2 \leq l \leq n$. They are all ordinary division points except $\delta_{n+1,1}$ which is a switch point. Note that the conditions (0) are consistent with the graph. The points which will be most relevant for the proofs are labeled with black dots.

We extend \mathbf{P} to the interval $[1, \infty)$ by self-similarity, that is $\mathbf{P}(q) = C^m \mathbf{P}(qC^{-m})$ for every positive integer m . In view of the value of \mathbf{P} and its derivative at 1 and C , one sees that this extension provides a generalized $(n+1)$ -system on $[1, \infty)$.

Proposition 1 suggests to define quantities $\hat{W}_{n-1}, \dots, \hat{W}_0$ by

$$\frac{1}{1 + \hat{W}_{n-k}} := \limsup_{q \rightarrow +\infty} \frac{P_1(q) + \dots + P_k(q)}{q}, \quad 1 \leq k \leq n. \quad (1)$$

Since \mathbf{P} is invariant under dilation of factor C , we can replace $\limsup_{q \rightarrow \infty}$ by $\max_{[1, C]}$ in the above formulae.

We observe that for $1 \leq k \leq n$, the function $P_1 + \dots + P_k$ has slope 1 on the intervals $[1, \delta_{k,1}]$ and $[\delta_{n+1,1}, C]$, slope 1/2 on the interval $[\delta_{k,1}, \delta_{k,2}]$ and is constant on the interval $[\delta_{k,2}, \delta_{n+1,1}]$. Thus the maximum on $[1, C]$ of the function $q \rightarrow q^{-1}(P_1(q) + \dots + P_k(q))$ is reached either at $\delta_{k,1}$ or at $\delta_{k,2}$, when slope changes from 1 to 1/2 or from 1/2 to 0. Namely, the maximum is reached at $\delta_{k,1}$ if

$$\frac{P_1(\delta_{k,1}) + \dots + P_k(\delta_{k,1})}{\delta_{k,1}} \geq \frac{1}{2} \quad (2)$$

and at $\delta_{k,2}$ if the lefthand side is $\leq 1/2$. We deduce that for $1 \leq k \leq n$,

$$\hat{W}_{n-k} = \frac{P_{k+1}(q) + \dots + P_{n+1}(q)}{P_1(q) + \dots + P_k(q)} \text{ where } q = \begin{cases} \delta_{k,1} & \text{if (2) is satisfied} \\ \delta_{k,2} & \text{otherwise} \end{cases}.$$

For $2 \leq k \leq n+1$, we have the following values at $\delta_{k,1}$ and $\delta_{k,2}$:

$$P_i(\delta_{k,1}) = \begin{cases} A_1 & \text{if } i = 1 \\ CA_i & \text{if } 2 \leq i \leq k-1 \\ A_{k+1} & \text{if } i = k \\ A_i & \text{if } k+1 \leq i \leq n+1 \end{cases}, \quad P_i(\delta_{k,2}) = \begin{cases} A_1 & \text{if } i = 1 \\ CA_i & \text{if } 2 \leq i \leq k \\ CA_k & \text{if } i = k+1 \\ A_i & \text{if } k+2 \leq i \leq n+1 \end{cases}$$

It is easy to check that the parameters

$$C = 3, A_1 = A_2 = 2^{-n}, A_k = 2^{-n+k-2} \text{ for } 3 \leq k \leq n+1 \quad (3)$$

satisfy the conditions (0). For this choice of parameters, the lefthand side of inequality (2) is $> 1/2$ for $1 \leq k \leq n-1$ and $< 1/2$ for $k = n$. This property remains true for (C, A_2, \dots, A_n) in an open neighborhood of $(3, 2^{-n}, \dots, 2^{-2})$ provided that we set $A_1 = A_2$ and $A_{n+1} = 1 - (A_1 + \dots + A_n)$. In this neighborhood, the quantities $\hat{W}_0, \dots, \hat{W}_{n-1}$ are given by the following rational fractions in $\mathbb{Q}(C, A_2, A_3, \dots, A_n)$:

$$\begin{aligned} \hat{W}_{n-1} &= \frac{1}{A_2} - 1, \\ \hat{W}_{n-k} &= \frac{1 - (2A_2 + A_3 + A_4 + \dots + A_{k+1}) + CA_k}{A_2 + C(A_2 + \dots + A_k)}, \quad 2 \leq k \leq n-1 \\ \hat{W}_0 &= \frac{1 - (2A_2 + A_3 + A_4 + \dots + A_n)}{A_2 + C(A_2 + \dots + A_{n-1})}. \end{aligned} \quad (4)$$

Since $\hat{W}_0, \dots, \hat{W}_{n-1}$ come from a generalized $(n+1)$ -system \mathbf{P} , Theorem 5 provides a point $\boldsymbol{\theta}$ in \mathbb{R}^n such that $\hat{\omega}_k(\boldsymbol{\theta}) = \hat{W}_k$ for every $0 \leq k \leq n-1$. Thus, to prove Theorem 4, it is sufficient to show that the rational fractions $\hat{W}_0, \dots, \hat{W}_{n-1} \in \mathbb{Q}(C, A_2, A_3, \dots, A_n)$ are algebraically independent.

Suppose on the contrary that there exists an irreducible polynomial $R \in \mathbb{Q}(X_1, \dots, X_n)$ such that

$$R(\hat{W}_0, \hat{W}_1, \dots, \hat{W}_{n-1}) = 0.$$

Specializing C in 0, we obtain

$$R\left(\frac{1 - A_2 - A_2 - \dots - A_n}{A_2}, \frac{1 - A_2 - A_2 - \dots - A_n}{A_2}, \dots, \frac{1 - A_2 - A_2 - A_3}{A_2}, \frac{1 - A_2}{A_2}\right) = 0.$$

Here, the first two rational fractions are the same, and the last $n-1$ rational fractions generate the field $\mathbb{Q}(A_2, A_3, \dots, A_n)$. Therefore the latter are algebraically independent, and $R = \alpha(X_2 - X_1)$ for a nonzero constant $\alpha \in \mathbb{Q}$. This is impossible since $\hat{W}_0 \neq \hat{W}_1$. \square

Proof of Theorem 6

We consider the same generalized $(n+1)$ -system as above. Notice that for $1 \leq k \leq n$ we have $P_k \leq P_{n+1}$ and therefore

$$0 \leq \frac{P_k(q)}{q} \leq 1/2.$$

Since all nonzero slopes of the combined graph are at least $1/2$, the maxima of the functions $q \mapsto q^{-1}P_k(q)$ are reached at points where P_k changes slope from 1 or $1/2$ to 0. It happens that for each component there is only one such point on the interval $[1, C]$.

The definition of the exponents $\bar{\varphi}_k$ leads to define quantities F_k by

$$F_k := \limsup_{q \rightarrow \infty} \frac{P_k(q)}{q} = \max_{[1, C]} \frac{P_k(q)}{q} = \frac{P_k(p)}{p} \text{ where } p = \begin{cases} 1 & \text{if } k = 1, \\ \delta_{k,2} & \text{if } 2 \leq k \leq n. \end{cases}$$

We express the quantities F_1, \dots, F_n as rational fractions in $\mathbb{Q}(C, A_2, \dots, A_n)$, using the relations $A_1 = A_2$ and $A_{n+1} = 1 - A_1 - A_2 - \dots - A_n$:

$$\begin{aligned} F_1 &= A_1, \\ F_k &= \frac{CA_k}{A_1 + C(A_2 + \dots + A_k) + CA_k + 1 - (2A_2 + A_3 + \dots + A_{k+1})}, \end{aligned}$$

Since F_1, \dots, F_n come from a generalized $(n+1)$ -system \mathbf{P} , by Theorem 5 there exists a point $\boldsymbol{\theta}$ in \mathbb{R}^n such that $\bar{\varphi}_k(\boldsymbol{\theta}) = F_k$ for every $1 \leq k \leq n$. To prove Theorem 6 it is sufficient

to show that the rational fractions $F_1, \dots, F_n \in \mathbb{Q}(C, A_2, A_3, \dots, A_n)$ are algebraically independent.

Suppose that there exists an irreducible polynomial $R \in \mathbb{Q}(X_1, \dots, X_n)$ such that

$$R(F_1, \dots, F_n) = 0.$$

Specializing C in infinity, we obtain

$$R\left(A_2, \frac{1}{2}, \frac{A_3}{(A_2 + A_3) + A_3}, \dots, \frac{A_n}{(A_2 + \dots + A_n) + A_n}\right) = 0$$

where all coordinates except $1/2$ are algebraically independent. Thus, R is a constant multiple of $2X_2 - 1$, which contradicts $F_2 \neq 1/2$. □

We are not able to prove Theorem 6 for the $n + 1$ exponents $\bar{\varphi}_1, \dots, \bar{\varphi}_{n+1}$ with this construction. However with some extra work, we can show that the theorem holds for any n exponents among them.

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