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# Equivalent modal parameters identification using the free decay of a weakly nonlinear structure

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## Abstract

This paper describes a method for the identification and the tracking of poles of a weakly nonlinear structure from its free responses. This method is based on a model of multi-channel damped sines whose parameters evolve over time. Their variations are approximated in discrete time by a nonlinear state space model. States are recursively estimated by a new method which couples a two-pass Bayesian estimator with an Expectation-Maximization (EM) algorithm. An iterative procedure between them allows an accurate and robust tracking of poles. As a result, equivalent modal parameters such as frequency and damping are obtained as a function of amplitudes. The method is applied on numerical and experimental cases and show promising results.

*Keywords:* modal testing, frequency, damping, identification, Unscented Kalman Filter, Expectation-Maximization algorithm

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## 1. Introduction

2 According to the linear framework in experimental modal analysis, for  
3 a given frequency range a structure has a finite set of invariant poles and  
4 corresponding shapes. They can be used to model its whole dynamics in  
5 this frequency range, taking also into account residual effects due to out-of-  
6 frequency-band modes [1].

7 In reality, invariant poles are unlikely to be observed on experimental re-  
8 sults. Most of structural dynamics show a certain degree of nonlinearity, due  
9 to materials nature, geometries, joints, friction, impacts, etc. In severe forms  
10 of nonlinearities, typical phenomena such as internal resonances, unstable  
11 modes may occur and lead the relevancy of modal analysis to questioning.  
12 Hence the detection and the identification of nonlinearity is more and more  
13 considered as a main step by test engineers [2, 3].

14 As a general rule, the success of an identification method highly depends  
15 on a chosen model and on a specialized algorithm for parameter estimation  
16 [4, 5]. Since usual tools of modal analysis aim to get linear modal models,  
17 testing the linearity assumption is essential to study their appropriateness on  
18 experimental data. Any deviation from expected results that cannot be ex-  
19 plained with measurement errors (such as signal noise, reduced observation,  
20 ...) is attributed to a lack of good linear behaviour, and then to non-linear  
21 dynamics influence. For instance, identifying Frequency Response Functions  
22 (FRFs) at different force levels is a simple and efficient way to test the ho-  
23 mogeneity (a weak form of linearity) [6]. Another indicator is the coherence  
24 function, which gives a scaled degree of linearity over frequency, although it  
25 might be misled by some kinds of nonlinearities [2].

26 However, for many structures, nonlinear forces are often weak compared  
27 to linear ones (predominantly mass and stiffness). As a consequence, the  
28 linear theory remains relevant for most of industrial structures, since the be-  
29 haviour of a structure submitted to an external force is close to an equivalent  
30 linear one, as long as no bifurcation occurs and nonlinear forces are smooth  
31 [7].

32 Equivalency means here that dynamics analysis can still be reduced to  
33 the identification of modes, but these ones are no longer invariant. They  
34 depend on the nature of external force and on the structure movement. More  
35 specifically, only poles are mainly affected by nonlinearities; except in the  
36 case of high energy level, mode shapes barely change and can be regarded as  
37 invariant [8].

38 In the field of signal processing, the problem of identifying variations of  
39 poles has often been addressed. Many techniques have been proposed, start-  
40 ing from the Teager energy operator [9, 10]. Feldman proposed the FREEVIB  
41 method that uses a Hilbert transform on the free decay and hence obtains  
42 the instantaneous characteristics of a mechanical system [11, 12]. When iden-  
43 tifying signals coming from free decay, the modulations have to be slower  
44 than the carrier as demonstrated by Bedrosian [13], Nuttall [14] and more

45 recently by Brown[15]. The Empirical Mode Decomposition has been de-  
46 signed by Huang et al. [16, 17, 18] to extract signal components satisfying  
47 this requirement before applying the Hilbert transform.

48 The free decay could also be processed by time-frequency analysis like  
49 wavelets [19, 20, 21]. Their high frequency resolution allows an accurate  
50 characterization of poles evolution, even though the first periods of free decay  
51 are often affected by edge effects. Moreover, when considering a set of several  
52 sensors, each signal is processed independently, without using the redundancy  
53 of information shared by all sensors. Despite these drawbacks, wavelets were  
54 successfully used to extend normal mode appropriation testing to nonlinear  
55 systems [7, 22].

56 This paper is devoted to the analysis of free decay time histories of weakly  
57 nonlinear structures. A new method for processing data from multiple sensors  
58 will be presented. The response is modelled by multi-channel damped sines  
59 whose parameters are recursively estimated over time by a two-pass Bayesian  
60 smoothing algorithm based on a Kalman filter. As accurate values are wished  
61 from the beginning of signals, an iterative algorithm based on maximization-  
62 expectation technique is proposed for obtaining relevant inputs to Kalman  
63 filter.

64 The outline of this paper is as follows. First, a state space in discrete time  
65 is introduced for modelling damped waves in section 2. Then instantaneous  
66 values of state vector are estimated by a nonlinear version of Kalman filter  
67 and an iterative algorithm is proposed in section 3. In section 4, the pro-  
68 posed method is first applied on numerical simulations to assess its efficiency.  
69 Finally, an experimental case is studied in section 5.

## 70 2. Development of state space model

71 This part is devoted to the development of a state space model for mod-  
72 elling the temporal evolution and the observation of damped sines. The tran-  
73 sition function will be developed in section 2.1 and the observation function  
74 in section 2.2.

### 75 2.1. Transition function

76 The free response of a structure, generally produced by a hammer hit or  
77 a stepped sine excitation, can be modelled by the sum of  $S$  damped sines:

$$x(t) = \sum_{s=1}^S A_s^r \exp(\sigma_s t) \cos(\omega_s t + \phi_s) \quad (1)$$

78 with, for each sine  $s$ ,  $A_s^r$  its corresponding maximum amplitude,  $\phi_s$  its initial  
 79 phase,  $\omega_s$  its damped angular frequency and  $\sigma_s$  its damping. For practical  
 80 reasons, the corresponding analytical signal is more used:

$$x(t) = \sum_{s=1}^S A_s \exp((\sigma_s + j\omega_s)t) \quad (2)$$

81 with each complex amplitude  $A_s$  composed of  $A_s^r$  and  $\phi_s$ . Equivalent eigen-  
 82 frequencies and eigendampings are computed by:

$$\omega_{es}^2 = \sigma_s^2 + \omega_s^2 \quad (3)$$

$$\xi_{es} = -\sigma_s/\omega_{es} \quad (4)$$

83 In discrete time, the response is given by:

$$x(n\Delta t) = \sum_{s=1}^S A_s \exp((\sigma_s + j\omega_s)n\Delta t) = \sum_{s=1}^S c_{s,n} \quad (5)$$

84 with  $\Delta t$  the time sampling. One can express  $x_n = x(n\Delta t)$  as a function of  
 85 the previous step:

$$x_n = \sum_{s=1}^S A_s \exp((\sigma_s + j\omega_s)(n-1)\Delta t) \exp((\sigma_s + j\omega_s)\Delta t) \quad (6)$$

86 The transition between two consecutive amplitudes is given by:

$$c_{s,n} = c_{s,n-1} \exp((\sigma_s + j\omega_s)\Delta t) \quad (7)$$

87 Each complex amplitude is made of real and imaginary parts as  $c_{s,n} = a_{s,n} +$   
 88  $jb_{s,n}$ . Then the previous relation can be written as:

$$\begin{bmatrix} a_{s,n} \\ b_{s,n} \end{bmatrix} = \mathbf{T}_s \begin{bmatrix} a_{s,n-1} \\ b_{s,n-1} \end{bmatrix} \quad (8)$$

89 with  $\mathbf{T}_s$  the transition matrix of damped sine  $s$ :

$$\mathbf{T}_s = \exp(\sigma_s\Delta t) \begin{bmatrix} \cos(\omega_s\Delta t) & -\sin(\omega_s\Delta t) \\ \sin(\omega_s\Delta t) & \cos(\omega_s\Delta t) \end{bmatrix} \quad (9)$$

90 It is simply obtained by expanding and combining all terms of equations  
 91 7 and 8. This transition matrix  $\mathbf{T}_s$  expresses that a wave is damped and  
 92 rotated between two consecutive time samples [23].

93 For the general case of  $M$  responses, the vector of the complex amplitudes  
 94 of a damped sine  $s$  is:

$$\mathbf{x}_{s,n}^a = [a_{s,1,n} \ b_{s,1,n} \ a_{s,2,n} \ b_{s,2,n} \ \dots \ a_{s,M,n} \ b_{s,M,n}]^T \quad (10)$$

95 where  $[a_{s,i,n} \ b_{s,i,n}]^T$  is the complex amplitude of a damped sine  $s$  at time  
 96 sample  $n$  and observation point  $i$ . Here the superscript  $a$  of  $\mathbf{x}_{n,s}^a$  denotes the  
 97 amplitude. Then the function transition for  $M$  observations is given by:

$$\mathbf{x}_{s,n}^a = (\mathbf{I}_M \otimes \mathbf{T}_s) \mathbf{x}_{s,n-1}^a \quad (11)$$

98 since they share the same transition matrix  $\mathbf{T}_s$ . Here the symbol  $\otimes$  denotes  
 99 the Kronecker product and  $\mathbf{I}_M$  is the identity matrix of size  $M$ .

100 As a free response of a structure generally shows several modes, a vector  
 101 of  $S$  damped sine amplitudes should be considered:

$$\mathbf{x}_n^a = \begin{bmatrix} \mathbf{x}_{1,n}^a \\ \mathbf{x}_{2,n}^a \\ \vdots \\ \mathbf{x}_{S,n}^a \end{bmatrix} \quad (12)$$

102 In fact, it only concatenates the  $2M$  previous  $\mathbf{x}_{s,n}^a$  amplitude terms per  
 103 damped sine.

104 By combining equations 11 and 12, a relation between two consecutive  
 105 amplitude vectors is given by:

$$\mathbf{x}_n^a = \mathbf{\Gamma} \mathbf{x}_{n-1}^a \quad (13)$$

106 with the transition matrix of complex amplitudes:

$$\mathbf{\Gamma} = \begin{bmatrix} \mathbf{I}_M \otimes \mathbf{T}_1 & \mathbf{0}_{2M} & \dots & \mathbf{0}_{2M} \\ \mathbf{0}_{2M} & \mathbf{I}_M \otimes \mathbf{T}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0}_{2M} \\ \mathbf{0}_{2M} & \dots & \mathbf{0}_{2M} & \mathbf{I}_{2M} \otimes \mathbf{T}_S \end{bmatrix} \quad (14)$$

107 This transition matrix  $\mathbf{\Gamma}$  is made of parameters (angular frequencies  $\{\omega_s\}$   
 108 and dampings  $\{\sigma_s\}$ ) which are unknown and have also to be identified. The  
 109 vector of amplitudes  $\mathbf{x}_n^a$  is concatenated to a vector of parameters which  
 110 assembles itself all the pulsations and dampings:

$$\mathbf{x}_n = \begin{bmatrix} \mathbf{x}_n^p \\ \mathbf{x}_n^a \end{bmatrix} \quad \text{with} \quad \mathbf{x}_n^p = [\omega_{1,n} \ \sigma_{1,n} \ \omega_{2,n} \ \sigma_{2,n} \ \dots \ \omega_{S,n} \ \sigma_{S,n}]^T \times \Delta t \quad (15)$$

111 Parameters are assumed to be almost constant between two samples. As  
 112 there is no deterministic law for their temporal evolution, here a stochastic  
 113 framework is chosen to model their variations:

$$\mathbf{x}_n^p = \mathbf{x}_{n-1}^p + \mathbf{w}_{n-1}^p \quad (16)$$

114 with  $\mathbf{w}_n^p$  a random noise which allows parameters to evolve slowly over time.

115 In conclusion, the dynamical evolution in discrete time is given by

$$\mathbf{x}_n = f(\mathbf{x}_{n-1}) + \mathbf{w}_{n-1} \quad (17)$$

116 with the deterministic part given by:

$$f\left(\begin{bmatrix} \mathbf{x}_{n-1}^p \\ \mathbf{x}_{n-1}^a \end{bmatrix}\right) = \begin{bmatrix} \mathbf{I}_{2S} & \mathbf{0}_{2MS} \\ \mathbf{0}_{2S} & \mathbf{\Gamma}_{n-1} \end{bmatrix} \begin{bmatrix} \mathbf{x}_{n-1}^p \\ \mathbf{x}_{n-1}^a \end{bmatrix} \quad (18)$$

117 where  $\mathbf{I}_{2S}$  is the identity matrix of size  $2S$ . This transition function is nonlin-  
 118 ear since, as parameters are allowed to evolve from sample to sample thanks  
 119 to  $\mathbf{w}_n^p$ , the matrix  $\mathbf{\Gamma}_{n-1}$  depends now on values of parameters given by  $\mathbf{x}_{n-1}^p$ .

120 Finally, the random part

$$\mathbf{w}_n = \begin{bmatrix} \mathbf{w}_n^p \\ \mathbf{w}_n^a \end{bmatrix} \quad (19)$$

121 is modelled by a Gaussian law of zero mean and variance  $\mathbf{Q}$ . Practical rules  
 122 for determining relevant value of  $\mathbf{Q}$  will be provided in paragraph 3.2.

## 123 2.2. Observation function

124 Let  $\mathbf{y}_n$  be the measurement vector of  $M$  sensors at sample time  $n$ :

$$\mathbf{y}_n = [y_{1,n} \ y_{2,n} \ \dots \ y_{M,n}]^T \quad (20)$$

125 Three types of observation function can be enumerated, depending on the  
 126 nature of measured quantities : displacement, speed or acceleration. In the  
 127 simplest case, each sensor gives a displacement at an observation point, with  
 128 an unavoidable random noise  $v_i$  of measurement. Here they are given by the  
 129 real parts of the complex sines:

$$y_{i,n} = \sum_{s=1}^S a_{s,i,n} + v_{i,n} \quad (21)$$

130 or in a matrix form

$$\mathbf{y}_n = \mathbf{H}\mathbf{x}_n + \mathbf{v}_n \quad (22)$$

131 The matrix  $\mathbf{H}$  is only composed of 0 and 1 necessary to capture all the  
 132  $a_{s,i,n}$ . The observation noise vector  $\mathbf{v}_n$  is assumed to be a zero-mean random  
 133 process of variance  $\mathbf{R}$ . It could be noticed that, for a displacement sensor,  
 134 this observation function is linear.

135 If speed sensors are used, thus the observation function depends on discrete  
 136 values of  $\dot{x}(t)$ . For a damped sine in continuous time,

$$\dot{x}(t) = \sum_{s=1}^S (\sigma_s + j\omega_s) A_s \exp((\sigma_s + j\omega_s)t) \quad (23)$$

137 Then in discrete time,

$$\dot{x}_{i,n} = \sum_{s=1}^S (\sigma_{s,n} + j\omega_{s,n}) c_{s,i,n} \quad (24)$$

$$\dot{x}_{i,n} = \sum_{s=1}^S (\sigma_{s,n} a_{s,i,n} - \omega_{s,n} b_{s,i,n}) + j(\sigma_{s,n} b_{s,i,n} + \omega_{s,n} a_{s,i,n}) \quad (25)$$

138 Hence the observation function of a speed sensor is given by

$$y_{i,n} = \mathbf{Re}(\dot{x}_{i,n}) + v_{i,n} \quad (26)$$

$$y_{i,n} = \sum_{s=1}^S (\sigma_{s,n} a_{s,i,n} - \omega_{s,n} b_{s,i,n}) + v_{i,n} \quad (27)$$

139 The last case deals with acceleration. Following the same reasoning as  
 140 for speed sensor, discrete values of  $\ddot{x}(t)$  are given by:

$$\ddot{x}_{i,n} = \sum_{s=1}^S (\sigma_{s,n} + j\omega_{s,n})^2 c_{s,i,n} \quad (28)$$

141 and, skipping all the intermediate steps, for an accelerometer the observation  
 142 function is:

$$y_{i,n} = (\sigma_{s,n}^2 - \omega_{s,n}^2) a_{s,i,n} - 2\sigma_{s,n}\omega_{s,n} b_{s,i,n} + v_{i,n} \quad (29)$$

143 It is also possible to combine these three kinds of observation function if  
 144 heterogeneous sensors are used.

### 145 3. Tracking of poles by identification of a state space model

146 This part is devoted to the use of Kalman filter for the tracking of damped  
147 sines based on the state model presented in section 2. The choice of the Un-  
148 scented Kalman Filter and its corresponding Unscented Rauch-Tung-Striebel  
149 Smoother (URTSS) will be presented in section 3.1. Ad-hoc rules of param-  
150 eterization for the initialization of filtering will be given in section 3.2. Since  
151 the first state values and noise covariance matrices have a strong impact on  
152 Kalman filtering, a technique for improving their estimation will be proposed  
153 in section 3.3. Finally, an iterative algorithm resulting from the previous  
154 ideas will be introduced in section 3.4.

#### 155 3.1. Unscented Kalman Filtering and URTSS

156 Kalman filtering refers to a family of algorithms which are devoted to the  
157 tracking and the estimation of first-order dynamical systems [24]. For known  
158 system matrices  $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ , measured observations  $\{\mathbf{y}_n\}$  and inputs  $\{\mathbf{u}_n\}$ ,  
159 it estimates a series of state  $\{\mathbf{x}_n\}$  which are governed by:

$$\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1} + \mathbf{B}\mathbf{u}_{n-1} + \mathbf{w}_{n-1} \quad (30)$$

$$\mathbf{y}_n = \mathbf{C}\mathbf{x}_n + \mathbf{D}\mathbf{u}_n + \mathbf{v}_n \quad (31)$$

160 where  $\mathbf{w}_{n-1}$  is the process noise and  $\mathbf{v}_n$  is the measurement noise. It can  
161 be proved that the series thus obtained  $\{\hat{\mathbf{x}}_n\}$  is the best approximate in the  
162 least-squares sense [25].

163 Although the first version only handles linear system of equations, non-  
164 linear ones were also developed to generalize the range of applications [25].  
165 One of the latest versions, named Unscented Kalman Filtering (UKF), is  
166 particularly promising since it can provide satisfying estimates of solutions  
167 for an acceptable algorithmic cost. The UKF is basically the combination  
168 of the Unscented Transform (UT) with the Kalman filtering [26, 27]. For a  
169 system of equations such as:

$$\mathbf{x}_n = f(\mathbf{x}_{n-1}, \mathbf{u}_{n-1}, \mathbf{w}_{n-1}) \quad (32)$$

$$\mathbf{y}_n = g(\mathbf{x}_n, \mathbf{u}_n, \mathbf{v}_n) \quad (33)$$

170 the UT allows estimating the first and second moments of a stochastic vari-  
171 able, even if it is transformed by a nonlinear transition function  $f(\cdot)$  or a  
172 nonlinear observation function  $g(\cdot)$  (see appendix 7.1 for details of compu-  
173 tation). Contrary to the linear version, the estimated series  $\{\hat{\mathbf{x}}_n\}$  is only  
174 optimal at the first order of approximation.

175 As Kalman filtering was originally used for real-time applications, it only  
 176 uses past observations. It recursively computes an estimate at time sample  
 177  $n$  based on the previous estimate  $\hat{\mathbf{x}}_{n-1}$  and the new observation  $\mathbf{y}_n$ . Hence  
 178 for each time sample, Kalman filtering computes the likeliest distribution,  
 179 assumed here to be Gaussian, based on current and past observations up to  
 180 time sample  $n$ :

$$p(\mathbf{x}_n|\mathbf{y}_{1 \rightarrow n}) \approx N(\mathbf{x}_n|\hat{\mathbf{x}}_n, \hat{\mathbf{P}}_n) \quad (34)$$

181 where  $\mathbf{y}_{1 \rightarrow n}$  denotes the reduced set of observations and  $N(\mathbf{x}_n|\hat{\mathbf{x}}_n, \hat{\mathbf{P}}_n)$  is the  
 182 Gaussian law of mean  $\hat{\mathbf{x}}_n$  and variance  $\hat{\mathbf{P}}_n$ .

183 Although it can already provide good approximations, better results could  
 184 be obtained by considering future observations too, since they are available  
 185 in post-processing analysis. An algorithm named Unscented Rauch-Tung-  
 186 Striebel Smoother (URTSS) was selected for this task [28]. It involves a  
 187 separate backward smoothing pass that computes corrections to the first  
 188 forward filtered data. Therefore the smoothed distribution of states depends,  
 189 for each time sample, on the whole observations set:

$$p(\mathbf{x}_n|\mathbf{y}_{1 \rightarrow T}) \approx N(\mathbf{x}_n|\hat{\mathbf{x}}_n^{\dagger}, \hat{\mathbf{P}}_n^{\dagger}) \quad (35)$$

190 where  $T$  is the number of time samples and  $\mathbf{y}_{1 \rightarrow T}$  denotes all measured data.  
 191  $\hat{\mathbf{x}}_n^{\dagger}$  and  $\hat{\mathbf{P}}_n^{\dagger}$  are the smoothed mean vector and covariance matrix at time  
 192 sample  $n$ . In practice, the URTSS needs the estimated series  $\{\hat{\mathbf{x}}_n, \hat{\mathbf{P}}_n\}$  of  
 193 the first pass and recursively computes  $\{\hat{\mathbf{x}}_n^{\dagger}, \hat{\mathbf{P}}_n^{\dagger}\}$  from  $n = T$  to  $n = 0$  in a  
 194 backward manner. For the sake of conciseness, all steps of computation are  
 195 postponed in appendix 7.2.

196 In conclusion, the identification of states  $\{\mathbf{x}_n\}$  is performed with two  
 197 successive steps:

- 198 1. the UKF computes a first series of estimate  $\{\hat{\mathbf{x}}_n\}$  based on past obser-  
 199 vations  $\mathbf{y}_{1 \rightarrow n}$ ,
- 200 2. the URTSS computes a corrected series of estimate  $\{\hat{\mathbf{x}}_n^{\dagger}\}$  based on the  
 201 whole set of observations  $\mathbf{y}_{1 \rightarrow T}$ .

202 Both steps form the two-pass Bayesian smoothing algorithm.

### 203 3.2. Initialization of parameters

204 The recursive computation starts with the initial estimation of  $\{\hat{\mathbf{x}}_0, \hat{\mathbf{P}}_0\}$ .  
 205 As previously detailed in paragraph 2.1, the state vector  $\mathbf{x}$  combines both  
 206 parameter values and real-imaginary amplitudes  $\mathbf{x} = [(\mathbf{x}^p)^T \quad (\mathbf{x}^a)^T]^T$ .

207 First of all, the number of damped sines should be equal to the number  
 208 of major peaks in the Power Spectral Densities (PSDs) of signals. It gives  
 209 the model size directly.

210 Each damped sine has two terms in the parameter vector:

$$\mathbf{x}^p = \begin{bmatrix} \omega_d \Delta t \\ \sigma \Delta t \end{bmatrix}, \quad \text{with} \quad \begin{cases} \omega_d = 2\pi f_e \sqrt{1 - \xi_e^2} \\ \sigma = -2\pi f_e \xi_e \end{cases} \quad (36)$$

211 Here the case of a single damped sine has been chosen for the sake of clarity.  
 212 A first estimation of  $f_e$  is given by picking the major peak in the PSDs.  
 213 Moreover, an arbitrary value of 1% is used to initialize  $\xi_e$ . Although they  
 214 could seem to be rough estimates for  $f_{e,0}$  and  $\xi_e$ , they are quite good enough  
 215 as first values.

216 Amplitudes stored in vector  $\mathbf{x}^a$  are initialized by taking for each measured  
 217 signal at each selected frequency the corresponding square root amplitudes  
 218 in the PSDs.

219 The state covariance matrix  $P$  can be initialized by the expected lack of  
 220 knowledge on states. As for the state vector  $x$ , parameters and amplitudes  
 221 are considered separately:

$$\hat{\mathbf{P}}_0 = \begin{bmatrix} \hat{\mathbf{P}}_0^p & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{P}}_0^a \end{bmatrix} \quad (37)$$

222 Uncertainties on the initial frequency  $\delta_f$  and the initial damping  $\delta_\xi$  are in-  
 223 troduced in the state covariance matrix by:

$$\hat{\mathbf{P}}_0^p = \begin{bmatrix} (2\pi\delta_f\Delta t)^2 & 0 \\ 0 & (2\pi\hat{f}_{e,0}\delta_\xi\Delta t)^2 \end{bmatrix} \quad (38)$$

224 where  $\hat{f}_{e,0}$  is an initial value of frequency. Typical values of uncertainties are  
 225  $\delta_f = 1\text{Hz}$  and  $\delta_\xi = 1\%$ .

226 The sub-matrix  $\hat{\mathbf{P}}_0^a$  contains initial uncertainties on amplitudes. It could  
 227 be approximated by a fraction of  $\mathbf{x}_0^a$ .

$$\mathbf{P}_0^a = \text{diag} \left[ \left( \frac{\mathbf{x}_0^a}{\delta_x} \right)^2 \right] \quad (39)$$

228 where  $\delta_x = 10$  for instance means an initial uncertainty of 10% on amplitudes.

229 Kalman filtering also relies on two covariance matrices whose values can-  
 230 not be directly measured :  $\mathbf{Q}$  and  $\mathbf{R}$ . As for the state covariance matrix  $\mathbf{P}$ ,  
 231 the noise covariance matrix  $\mathbf{Q}$  separates parameters and amplitudes compo-  
 232 nents:

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^p & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^a \end{bmatrix} \quad (40)$$

233 The matrix  $\mathbf{Q}^p$  is a diagonal matrix whose terms are variances of angular  
 234 frequencies and dampings. A relevant  $\mathbf{Q}^p$  is extremely important for the  
 235 success of Kalman filtering. If the value is too low, the filter is not able to  
 236 adapt to observations and cannot converge to relevant states. But in the  
 237 other side, if the value is too high, identified parameters may evolve too  
 238 fast and are not representative of an expected evolution. Then finding good  
 239 values for  $\mathbf{Q}^p$  is often one of the main points of Kalman filtering [25].

240 It will always be assumed that a diagonal matrix form is suitable for  $\mathbf{Q}^p$   
 241 without lack of generality. For instance, for  $S = 1$ , i.e. one damped sine

$$\mathbf{Q}^p = \begin{bmatrix} q_\omega & 0 \\ 0 & q_\sigma \end{bmatrix} \quad (41)$$

242 The first state, namely the angular frequency  $\omega_n$  times the time step  $\Delta t$ , is  
 243 able to evolve by a random walk:

$$\omega_n \Delta t = \omega_{n-1} \Delta t + \epsilon_n \quad (42)$$

244 where  $\epsilon_n = N(0, q_\omega)$  allows for a slight evolution of  $\omega_n$  over a time step.  
 245 During the whole response, the final value  $\omega_N$  is given by  $N$  successive steps:

$$\omega_N = \hat{\omega}_0 + \frac{1}{\Delta t} \sum_{n=1}^N \epsilon_n \quad (43)$$

246 As a relevant value for  $q_\omega$  could be obtained by assuming a regular evolution  
 247 over  $N$  steps with the same standard deviation  $\sqrt{q_\omega}$ , then:

$$\omega_N = \hat{\omega}_0 + \frac{1}{\Delta t} \sum_{n=1}^N \sqrt{q_\omega} \quad (44)$$

$$\omega_N = \hat{\omega}_0 (1 + V_f) \quad (45)$$

248 where  $V_f$  is the frequency variation of a structure response during a free  
 249 decay. Hence

$$q_\omega = \left( \frac{V_f \hat{\omega}_{e,0} \Delta t}{N} \right)^2 \quad (46)$$

250 For most structures which have nonlinear joints, frequency variations such  
 251 as  $V_f = 5\%$  are common.

252 By analogy, a similar expression is given for  $\sigma$ :

$$q_\sigma = \left( \frac{V_\xi \hat{\xi}_0 \hat{\omega}_{e,0} \Delta t}{N} \right)^2 \quad (47)$$

253 Contrary to frequency, a damping ratio  $\xi$  is known to experience significant  
 254 variations in function of amplitude. As typical eigendamping ranges in value  
 255 from 1 to 5%,  $V_\xi = 5$  enables the Kalman filtering to start at 1% and finishes  
 256 at 5%.

257 In the case of a free response,  $\mathbf{Q}^a$  should be a null matrix as there is  
 258 no external excitation. But giving a non-zero value to  $\mathbf{Q}^a$ , even very small,  
 259 ensures flexibility and stability in Kalman filtering. Hence a diagonal matrix  
 260  $\mathbf{Q}^a = \mathbf{I}\epsilon$  is chosen for this purpose, with  $\epsilon$  is a small value which depends on  
 261 the accuracy of the used computer.

262 The matrix of observation noise variances  $\mathbf{R}$  can be estimated by knowing  
 263 the characteristics of used sensors. If they are not available, rough estimates  
 264 can be given by a percentage of signals variances. For instance, if  $\mathbf{R} =$   
 265  $(0.01)^2 * \text{var}(\mathbf{y})$  for a single sensor, then it is assumed that the Signal on  
 266 Noise Ratio is 1%.

### 267 3.3. Expectation-Maximization algorithm for noise covariances estimation

268 Although formulae introduced in paragraph 3.2 often give satisfying val-  
 269 ues for  $\{\mathbf{Q}, \mathbf{R}, \mathbf{x}_0, \mathbf{P}_0\}$ , it is advisable to improve their estimation as well as  
 270 possible. Indeed they are directly used as inputs for the two-pass Bayesian  
 271 smoothing algorithm, and therefore states estimates strongly depend on  
 272 them. In this work their values are optimized thanks to the Expectation-  
 273 Maximization (EM) algorithm [29]. This technique was initially designed for  
 274 the linear Kalman filter, and then adapted to the Extended Kalman filter  
 275 [30]; here it is proposed to extend it to the Unscented Kalman Filter.

276 Let  $\Theta = \{\mathbf{x}_0, \mathbf{P}_0, \mathbf{Q}, \mathbf{R}\}$  be the set of parameters that are unknown and  
 277  $\{\mathbf{X}_N, \mathbf{Y}_N\} = (\mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{y}_1, \dots, \mathbf{y}_N)$  contain the true states and observa-  
 278 tions set of the system. The set of parameters  $\Theta$  will be estimated under the  
 279 maximum likelihood framework by maximising the log-likelihood function:

$$\max_{\Theta} \log L(\Theta | \mathbf{X}_N, \mathbf{Y}_N) = \log p(\mathbf{X}_N, \mathbf{Y}_N | \Theta) \quad (48)$$

280 where  $p(\mathbf{X}_N, \mathbf{Y}_N | \Theta)$  is the joint probability density function:

$$\begin{aligned}
p(\mathbf{X}_N, \mathbf{Y}_N | \Theta) &= \frac{1}{2\pi |\mathbf{P}_0|^{1/2}} \exp\left(-\frac{(\mathbf{x}_0 - \mu_0)^T \mathbf{P}_0^{-1} (\mathbf{x}_0 - \mu_0)}{2}\right) \\
&\times \prod_{n=1}^N \frac{1}{(2\pi)^{S(M+1)} |\mathbf{Q}|^{1/2}} \exp\left(-\frac{\mathbf{e}_n^T \mathbf{Q}^{-1} \mathbf{e}_n}{2}\right) \\
&\times \prod_{n=0}^N \frac{1}{(2\pi)^{M/2} |\mathbf{R}|^{1/2}} \exp\left(-\frac{\mathbf{v}_n^T \mathbf{R}^{-1} \mathbf{v}_n}{2}\right) \quad (49)
\end{aligned}$$

281 where  $|\cdot|$  is the determinant of a matrix. As the true set  $\{\mathbf{X}_N, \mathbf{Y}_N\}$  is not  
282 available, Shumway and Stoffer proposed an iterative approach based on an  
283 EM technique which works on the conditional expectation of  $\log L(\Theta | \mathbf{X}_N, \mathbf{Y}_N)$   
284 (see [29]). Although this method was initially designed for the linear Kalman  
285 filter, its nonlinear version is derived here thanks to the Unscented Trans-  
286 form.

287 Given the current value of  $\Theta^{(j)} = \{\mathbf{x}_0^{(j)}, \mathbf{P}_0^{(j)}, \mathbf{Q}^{(j)}, \mathbf{R}^{(j)}\}$  and the obser-  
288 vation set  $\mathbf{Y}_N$ , a data set  $\hat{\mathbf{X}}_N^{(j)} = (\hat{\mathbf{x}}_{0|N}^{(j)}, \dots, \hat{\mathbf{x}}_{N|N}^{(j)})$  is generated by the two-  
289 pass Bayesian smoothing algorithm. At the  $j$ th iteration, the expectation of  
290  $\log L(\Theta | \mathbf{X}_N, \mathbf{Y}_N)$  conditioned to  $\mathbf{Y}_N$  and  $\Theta^{(j)}$  is given as:

$$\begin{aligned}
E[\log L(\Theta | \mathbf{X}_N, \mathbf{Y}_N)] &= -\log |\mathbf{P}_0| - N \log |\mathbf{Q}| \\
&\quad -\text{tr} \left[ \mathbf{P}_0^{-1} \left( \mathbf{P}_{0|N}^{(j-1)} - (\mathbf{x}_{0|N}^{(j)} - \mathbf{x}_0) (\mathbf{x}_{0|N}^{(j)} - \mathbf{x}_0)^T \right) \right] \\
&\quad -\text{tr} \left[ \mathbf{Q}^{-1} \left( \mathbf{B}_1^{(j)} - \mathbf{B}_2^{(j)} - (\mathbf{B}_2^{(j)})^T + \mathbf{B}_3^{(j)} \right) \right] \\
&\quad -\text{tr} \left[ \mathbf{R}^{-1} \left( \mathbf{B}_4^{(j)} - \mathbf{B}_5^{(j)} - (\mathbf{B}_5^{(j)})^T + \mathbf{B}_6^{(j)} \right) \right] \quad (50)
\end{aligned}$$

291 where matrices  $\{\mathbf{B}_i\}$ ,  $i = 1..6$  are given in appendix 7.4.

292 Secondly, the likelihood  $E[\log L(\Theta | \mathbf{X}_N, \mathbf{Y}_N)]$  is maximized by taking the  
293 partial derivatives with respect to  $\mathbf{Q}$  and  $\mathbf{R}$  and setting them to zero:

$$\mathbf{Q}^{(j+1)} = \frac{1}{N} \left( \mathbf{B}_1^{(j)} - \mathbf{B}_2^{(j)} - (\mathbf{B}_2^{(j)})^T + \mathbf{B}_3^{(j)} \right) \quad (51)$$

294 In part 3.2, a splitting in two diagonal parts has been assumed for the noise  
295 covariance matrix  $\mathbf{Q}$ :

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}^p & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}^a \end{bmatrix} \quad (52)$$

296 In practice, it is better to update the diagonal terms of the sub-matrix  $\mathbf{Q}^p$   
 297 only and to keep the sub-matrix  $\mathbf{Q}^a$  to a constant value. If we denote with  
 298 a superscript  $p$  the corresponding left-upper sub-matrices of  $\mathbf{B}_1, \mathbf{B}_2$  and  $\mathbf{B}_3$ ,  
 299 the update of  $\mathbf{Q}$  can be restricted to:

$$\mathbf{Q}^{p,(j+1)} = \frac{1}{N} \text{diag} \left[ \left( \mathbf{B}_1^{p,(j)} - \mathbf{B}_2^{p,(j)} - \left( \mathbf{B}_2^{p,(j)} \right)^T + \mathbf{B}_3^{p,(j)} \right) \right] \quad (53)$$

300 and the matrix  $\mathbf{Q}^a$  is unchanged. The update of the noise measurement  
 301 matrix  $\mathbf{R}$  is performed with:

$$\mathbf{R}^{(j+1)} = \frac{1}{N} \left( \mathbf{B}_4^{(j)} - \mathbf{B}_5^{(j)} - \left( \mathbf{B}_5^{(j)} \right)^T + \mathbf{B}_6^{(j)} \right) \quad (54)$$

302 Contrary to the noise covariance matrix  $\mathbf{Q}$ , extra-diagonal terms may ap-  
 303 pear in  $\mathbf{R}$ , allowing then the potential cross-terms of noise between different  
 304 sensors.

305 Since it is not possible to isolate the initial mean  $\mathbf{x}_0$  and covariance ma-  
 306 trice  $\mathbf{P}_0$  in equation 50, the updates are simply made by taking their last  
 307 estimated values provided by the URTSS at the previous iteration:

$$\mathbf{x}_0^{(j+1)} = \hat{\mathbf{x}}_{0|N}^{(j)} \quad \text{and} \quad \mathbf{P}_0^{(j+1)} = \hat{\mathbf{P}}_{0|N}^{(j)} \quad (55)$$

### 308 3.4. Final algorithm

309 The resulting method is an iterative process which alternates between the  
 310 two-pass Bayesian algorithm and the maximum likelihood estimator with the  
 311 EM technique as follows.

- 312 1. Initialize the set of parameters  $\Theta^{(0)} = \{\mathbf{x}_0, \mathbf{P}_0, \mathbf{Q}, \mathbf{R}\}$  by using formulae  
 313 of paragraph 3.2.
- 314 2. Using the set  $\Theta^{(j)}$ , compute a set of states  $\hat{\mathbf{X}}_N$  by applying the UKF.
- 315 3. Improve the results by applying the URTSS.
- 316 4. Expectation step : the smoothed state values are used to compute the  
 317 matrices of maximum likelihood thanks to the UT  $\{\mathbf{B}_1^{(j)}, \dots, \mathbf{B}_6^{(j)}\}$ .
- 318 5. Maximization step : the estimates of the parameters  $\{\mathbf{x}_0, \mathbf{P}_0, \mathbf{Q}, \mathbf{R}\}$  are  
 319 updated to get  $\Theta^{(j+1)}$ .
- 320 6. Repeat Steps 2-4 until convergence.

321 The iterations are ended if a maximum number of iterations is reached or if  
 322 the following criterion is satisfied

$$\frac{\|\mathbf{p}^{(j)} - \mathbf{p}^{(j-1)}\|}{\|\mathbf{p}^{(j-1)}\|} < \epsilon \quad (56)$$

323 where  $\mathbf{p}$  denotes a poles series ( $\{-\xi\omega + i\omega\sqrt{1 - \xi^2}\}$ ) and  $\epsilon$  is a small positive  
 324 number.

## 325 4. Simulation results

### 326 4.1. Numerical model

327 The numerical case is an one-degree-of-freedom system

$$m\ddot{x} + c\dot{x} + kx + r(t) = f_e(t) \quad (57)$$

328 where  $m = 1$ ,  $c = 2m\xi_0\omega_0$ ,  $k = m\omega_\infty$  and  $r(t)$  is the restoring force due to  
 329 an Iwan model (see figure 1). The Iwan model is a parallel association of  $N_J$   
 330 Jenkins elements, each of them being a series association of a linear spring  
 331 of stiffness  $\sigma/N_J$  and a Coulomb slider of breaking force  $F_j$ . The restoring  
 332 force of each Jenkins element is simulated by a modified Dahl formulation  
 333 (see [31])

$$\dot{r}_j = \frac{\sigma}{N_J} \left| 1 - \frac{r_j}{F_j} \operatorname{sgn}(\dot{x}) \right|^\alpha \operatorname{sgn} \left( 1 - \frac{r_j}{F_j} \operatorname{sgn}(\dot{x}) \right) \dot{x} \quad (58)$$

334 where  $\alpha$  is a small exponent.

335 The system is assumed to be observed with four accelerometers

$$\ddot{y} = \Phi\ddot{x} + v, \quad \Phi = \begin{bmatrix} 3 & 2 & 8 & 6 \end{bmatrix} \quad (59)$$

336 where  $v$  is the noise observation vector such as 1% of random noise is added  
 337 to signals.

338 The values of  $\{r_m, \sigma, \xi_0, \omega_0, \omega_\infty\}$  are chosen in order to have an evolution  
 339 of equivalent modal parameters as a function of accelerometer 1 amplitude  
 340 such as presented on figure 2.

### 341 4.2. Application of the methodology

342 The system is excited by a Morlet wavelet whose central frequency is 15  
 343 Hz. The accelerations generated by this excitation force are simulated on 10  
 344 seconds and depicted on figure 3. A white noise is added to signals in order  
 345 to obtain realistic measured signals. The ratio of standard deviation between  
 346 the added noise and the free-noise signals is about 1%.

347 A clarification should be made on the transient part due to the wavelet  
 348 (see figure 4). In fact, a wavelet has a short but not negligible time of fading  
 349 down. Although signals used for identification may begin at the maximum

350 of amplitude, the free response starts approximately one period later, when  
351 the excitation force is close to nil. Then there is one period of signals which  
352 could be used for post-processing, but irrelevant for final results presentation  
353 since the system is not exactly free during this period.

354 Results of the proposed algorithm are depicted on figure 5. Equivalent  
355 modal parameters are identified as a function of time for the whole system,  
356 and amplitudes are computed for each sensor. It can be firstly noticed that  
357 strong variations occur in a short temporal frame (one second in this example)  
358 compared to the whole range of time. It emphasizes the need to identify the  
359 first periods as well as possible.

360 The speed of convergence is studied by depicting the first and last results  
361 on figure 6. Two curves show the results of the first iteration : the first pass  
362 (UKF) and the second pass (URTSS). The third curve gives the results of  
363 the 7th iteration (last two-pass Bayesian smoothing). Intermediate results  
364 are not shown since most significant variations are observed during the first  
365 pass of Kalman filtering and the second pass of Bayesian smoothing.

366 By representing the equivalent modal parameters as a function of an  
367 accelerometer amplitude (numbered 1 here), a comparison with the reference  
368 curves is plotted on figure 7. These reference curves were obtained by using  
369 the harmonic balance method on the system [32]. Very accurate matching  
370 is obtained for both frequency and damping variations. Maximum error  
371 deviations are about 0.01 Hz on frequency and 0.1% on damping.

### 372 *4.3. Discussion*

373 Three main reasons could be enumerated to explain the small deviations.  
374 Firstly, the reference curves are not the theoretical evolutions for this exci-  
375 tation since no closed form solutions are known. Instead we use evolutions  
376 obtained for a harmonic evolution, and equivalent modal parameters calcu-  
377 lated for a sine excitation are not exactly equal to the ones of a free response.

378 Secondly, although the Runge-Kutta solver is very accurate, a trade-off  
379 between instantaneous frequency and damping is made at each time step and  
380 resulting numerical errors are unavoidable during simulation.

381 Finally, the identification algorithm has also its own limitations and ac-  
382 curacy. Nevertheless, such deviations are very low and could be considered  
383 as acceptable.

384 An additional check of performance is given by synthesizing the accelera-  
385 tions based on identified states (from the final iteration) and comparing them  
386 to noise-free and noisy accelerations signals. For example, on figure 8 these

387 three cases are plotted for the accelerometer 1. At the beginning (left part),  
388 all amplitudes match well, since the amplitude on noise ratio is high and the  
389 first states have been correctly estimated by the algorithm. Furthermore,  
390 even if noise amplitude is more significant at low amplitude (right part of  
391 the figure), the method still provides accurate states since the synthesized  
392 acceleration matches the noise-free acceleration well.

## 393 5. Experimental application

### 394 5.1. *H Bench*

395 The experimental rig consists of two beams linked by a lap-joint and two  
396 seismic masses (see figure 9). All of them are made of steel. The seismic  
397 masses and the beam are glued together and tightened with one M10 bolt  
398 on each side. The glue associated with a strong clamping of the M10 bolts  
399 ensures a very weak damping. The joint itself is assembled with two M6  
400 bolts in the center of the overlap and four M4 bolts close to the edges.

401 This bench is designed to load the joint either on a torsional or on a  
402 bending motion thanks to the two first modes. A FE model gives these two  
403 modes in free boundary conditions at 50 and 90 Hz (see figure 10). The third  
404 mode, namely the second bending mode, has a natural frequency above 200  
405 Hz.

406 According to the FE model, the torsion and bending modes have two com-  
407 mon nodes where their nodal lines intersect. The free boundary conditions  
408 are obtained by hanging the structure with nylon cables at these points.

409 The torsion and bending modes have very low amplitude at joints between  
410 the lap-joint and the masses (around the M10 bolts). So it is expected that  
411 the M10 bolts have very low influence on the vibratory dissipation of these  
412 modes. On the other hand, the M6 and M4 bolts located on the lap-joint  
413 should have a significant impact on the poles.

414 For each mode, the hit position is at the intersection of the maximum  
415 amplitude of the studied mode and of the nodal line of the other mode  
416 (named H1 and H2 points on figure 10). The hammer mass and its flexibility  
417 tip are chosen to have a cutting frequency just over the second mode ( $> 100$   
418 Hz). Then hitting the structure at these two points enabled us to highlight  
419 as well as possible the bending and torsion modes separately.

420 Accelerations are measured at six points along the beam (Acc1 to Acc6)  
421 and at one point on the seismic mass (Acc7). Residual participations of low

422 dynamic rigid motions and higher-order modes are removed from signals by  
423 applying a FIR band-pass filter.

## 424 *5.2. Results*

425 A hammer hit is applied on the H2 point of the structure to excite the  
426 bending mode and measured signals are processed by the proposed methodol-  
427 ogy. Then equivalent modal parameters values are obtained per time sample,  
428 together with amplitude level per damped sine and per signal (see figure 11).  
429 As for the numerical case in section 4, most variations occur during a short  
430 period of time at the beginning of the decay. In fact, the most useful in-  
431 formation is located in the first 0.5 s. Then the fast convergence of the  
432 UKF and the corrections due to the URTSS allow identifying the parameters  
433 accurately, even in a short time frame.

434 As a better presentation, the previous equivalent modal parameters are  
435 plotted as a function of Acc1 displacement amplitude on figure 12. The fre-  
436 quency lowers as a function of amplitude level, while damping ratio grows.  
437 Such variations are typical of the influence of dry friction on modal param-  
438 eters.

439 Four hits of hammer were applied on the H1 point and all sets of measured  
440 signals were processed according to the same methodology (see figure 13).  
441 Curves match very well, which shows the good repeatability of the tests,  
442 even if the force level is not constant. It can be noticed that the level of  
443 hammer hit impulse has a slight effect on the equivalent modal parameters.  
444 For example, the hit 4, which is the strongest one since it gives the maximum  
445 amplitude of displacement, has always lower frequency and higher damping  
446 values compared to the three other cases. Anyway these deviations are very  
447 low and can be considered as acceptable.

448 The influence of the M4 bolts torque is studied by estimating the equiv-  
449 alent parameters of the torsion mode for three tightening cases of the four  
450 M10 bolts: 0.5, 1 and 2 Nm. For each case, several hits were performed  
451 for evaluating the repeatability of tests (see figure 14). For each tightening  
452 case, all curves of equivalent parameters match very well, since the deviations  
453 are about 0.3 Hz for frequency and 0.5% for damping, considering that they  
454 evolve strongly in the range 39-45 Hz for the frequency and 0.5-5.7 % for the  
455 damping ratio. As a result, the influence of tightening is easily identified.

456 At 0.5Nm, the friction due to lap-joint movement has a maximum impact  
457 at 0.25 mm of Acc7 displacement. At 1 and 2 Nm, the frequency lowers  
458 and the damping ratio grows continuously as a function of amplitude. As a

459 rule, the stronger the torque is, the less influential it is on equivalent modal  
 460 parameters. In the extreme case of tightening, these ones should be constant  
 461 and the structure dynamics should be linear, as if the whole structure was  
 462 only one block of steel.

## 463 6. Conclusion

464 The aim of this study was to present a new method for the identification  
 465 of equivalent modal parameters from free responses of structures. A state  
 466 space model was expressed in order to model the discrete time evolution of  
 467 damped sines. Frequency and damping were integrated in the state vector,  
 468 so that their instantaneous values can be identified. Different observation  
 469 functions were proposed depending on the type of used sensors. The track-  
 470 ing of state vector was performed by an Unscented Kalman Filter followed its  
 471 corresponding Unscented Rauch-Tung-Striebel Smoother. As accurate identi-  
 472 fied states are required from the beginning of signals decay, an iterative  
 473 algorithm was proposed for estimating first values of state vector and noise  
 474 covariances matrices.

475 The proposed method was applied and assessed on numerical and experi-  
 476 mental cases. Results showed that equivalent modal parameters as a function  
 477 of amplitude level can be obtained accurately. More specifically, satisfying  
 478 states are estimated since the beginning of signals, thus providing equivalent  
 479 modal parameters even at high level amplitude.

## 480 7. Appendixes

### 481 7.1. The Unscented Transform

482 The Unscented Transform (UT) is a deterministic sampling technique  
 483 which computes a minimal set of sigma points  $\{\mathbf{x}^{(i)}\}$  around a density prob-  
 484 ability function of mean  $\mathbf{x}$  and covariance  $\mathbf{P}$  :

$$\{\mathbf{x}^{(i)}\} = [\mathbf{x} \quad \mathbf{x} + \gamma(\mathbf{P})^{1/2} \quad \mathbf{x} - \gamma(\mathbf{P})^{1/2}] \quad i = 0 \dots 2L \quad (60)$$

485 where  $(\mathbf{P})^{1/2}$  denotes the Cholesky decomposition of  $\mathbf{P}$ . Rules for obtaining  
 486 a relevant value of  $\gamma$  are given in [27]. It can be easily proved that the mean  
 487 and covariance of this computed set of samples match the initial mean  $\mathbf{x}$  and  
 488 covariance  $\mathbf{P}$ . The UT is extensively used in the Unscented Kalman Filter  
 489 and its related Unscented Rauch-Tung-Striebel smoother.

490 *7.2. Unscented Kalman filter*

491 The Kalman filter estimates the dynamic evolution of a probability den-  
 492 sity function (mean  $\hat{\mathbf{x}}_n$  and covariance  $\hat{\mathbf{P}}_n$ ) at step  $n$  based on previous step  
 493  $n - 1$ . Knowing a first couple  $\{\hat{\mathbf{x}}_0, \hat{\mathbf{P}}_0\}$ , it is applied from  $n = 1$  up to  $n = T$   
 494 in two major phases : prediction and update.

495

496 *Prediction phase*

- 497 1. Creation of a sigma points set  $\{\mathbf{x}_{n|n-1}^{(i)}\}$  from  $\hat{\mathbf{x}}_{n-1}$  and  $\hat{\mathbf{P}}_{n-1}$  by formula  
 498 60  
 499 2. Each point is propagated through the transition function :

$$\mathbf{x}_{n|n-1}^{(i)} = f\left(\mathbf{x}_{n-1}^{(i)}\right), \quad i = 0 \dots 2L \quad (61)$$

- 500 3. The predicted state estimate is computed as :

$$\hat{\mathbf{x}}_n^- = \sum_{i=0}^{2L} W_i^{(m)} \mathbf{x}_{n|n-1}^{(i)} \quad (62)$$

- 501 4. The predicted covariance estimate is given by :

$$\hat{\mathbf{P}}_n^- = \sum_{i=0}^{2L} W_i^{(c)} \left(\mathbf{x}_{n|n-1}^{(i)} - \hat{\mathbf{x}}_n^-\right) \left(\mathbf{x}_{n|n-1}^{(i)} - \hat{\mathbf{x}}_n^-\right)^T + \mathbf{Q} \quad (63)$$

502

503 *Update phase*

504

- 505 5. Creation of a new set of sigma points  $\{\mathbf{x}_{n-1}^{(i)}\}$  from  $\hat{\mathbf{x}}_{n-1}^-$  and  $\hat{\mathbf{P}}_{n-1}^-$  by  
 506 formula 60

507

- 508 6. Application of the observation function on each point :

$$\mathbf{y}_{n|n-1}^{(i)} = g\left(\mathbf{x}_{n|n-1}^{(i)}\right), \quad i = 0 \dots 2L \quad (64)$$

- 509 7. The predicted observation is computed as :

$$\hat{\mathbf{y}}_n^- = \sum_{i=0}^{2L} W_i^{(m)} \mathbf{y}_{n|n-1}^{(i)} \quad (65)$$

510 8. The innovation covariance is calculated by :

$$\hat{\mathbf{P}}_{\mathbf{y}\mathbf{y},\mathbf{n}} = \sum_{i=0}^{2L} W_i^{(c)} [\mathbf{y}_{\mathbf{n}|\mathbf{n}-1}^{(i)} - \hat{\mathbf{y}}_{\mathbf{n}}^-] [\mathbf{y}_{\mathbf{n}|\mathbf{n}-1}^{(i)} - \hat{\mathbf{y}}_{\mathbf{n}}^-]^T + \mathbf{R} \quad (66)$$

511 9. The cross covariance matrix is given by :

$$\hat{\mathbf{P}}_{\mathbf{x}\mathbf{y},\mathbf{n}} = \sum_{i=0}^{2L} W_i^{(c)} [\mathbf{x}_{\mathbf{n}|\mathbf{n}-1}^{(i)} - \hat{\mathbf{x}}_{\mathbf{n}}^-] [\mathbf{y}_{\mathbf{n}|\mathbf{n}-1}^{(i)} - \hat{\mathbf{y}}_{\mathbf{n}}^-]^T \quad (67)$$

512 10. The Kalman gain is computed as :

$$\mathbf{K}_{\mathbf{n}} = \hat{\mathbf{P}}_{\mathbf{x}\mathbf{y},\mathbf{n}} \hat{\mathbf{P}}_{\mathbf{y}\mathbf{y},\mathbf{n}}^{-1} \quad (68)$$

513 11. The updated state estimate is calculated by :

$$\hat{\mathbf{x}}_{\mathbf{n}} = \hat{\mathbf{x}}_{\mathbf{n}}^- + \mathbf{K}_{\mathbf{n}} (\mathbf{y}_{\mathbf{n}} - \hat{\mathbf{y}}_{\mathbf{n}}^-) \quad (69)$$

514 12. The updated covariance matrix is computed as :

$$\hat{\mathbf{P}}_{\mathbf{n}} = \hat{\mathbf{P}}_{\mathbf{n}}^- - \mathbf{K}_{\mathbf{n}} \hat{\mathbf{P}}_{\mathbf{y}\mathbf{y},\mathbf{n}} \mathbf{K}_{\mathbf{n}}^T \quad (70)$$

515 Rules for computing  $W_i^{(c)}$  and  $W_i^{(m)}$  are given in [27].

### 516 7.3. Rauch-Tung-Striebel smoother

517 Contrary to Kalman filter, the Rauch-Tung-Striebel smoother processes  
518 data in a recursive manner. The followings steps are performed on  $\{\hat{\mathbf{x}}_n, \hat{\mathbf{P}}_n\}$ ,  
519 starting at  $n = T - 1$  and ending at  $n = 0$ .

- 520 1. Creation of a set of sigma points  $\{\mathbf{x}_n^{(i)}\}$  from  $\hat{\mathbf{x}}_n$  and  $\hat{\mathbf{P}}_n$  by formula 60
- 521 2. Propagate the sigma points through the transition function :

$$\hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{(i)} = f(\hat{\mathbf{x}}_{\mathbf{n}}^{(i)}), \quad i = 0 \dots 2L \quad (71)$$

- 522 3. The predicted mean is computed as :

$$\hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{\natural} = \sum_{i=0}^{2L} W_i^{(m)} \hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{(i)} \quad (72)$$

- 523 4. The predicted covariance matrix is calculated by :

$$\hat{\mathbf{P}}_{\mathbf{n}+1|\mathbf{n}}^{\natural} = \sum_{i=0}^{2L} W_i^{(c)} (\hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{(i)} - \hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{\natural}) (\hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{(i)} - \hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{\natural})^T \quad (73)$$

524 5. The cross covariance matrix is computed as :

$$\hat{\mathbf{S}}_{\mathbf{n}+1|\mathbf{n}}^{\mathfrak{h}} = \sum_{i=0}^{2L} W_i^{(c)} \left( \hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{(i)} - \hat{\mathbf{x}}_{\mathbf{n}} \right) \left( \hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{(i)} - \hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{\mathfrak{h}} \right)^T \quad (74)$$

525 6. The smoother gain is given by :

$$\mathbf{K}_{\mathbf{n}}^{\mathfrak{h}} = \hat{\mathbf{S}}_{\mathbf{n}+1|\mathbf{n}}^{\mathfrak{h}} \left( \hat{\mathbf{P}}_{\mathbf{n}+1|\mathbf{n}}^{\mathfrak{h}} \right)^{-1} \quad (75)$$

526 7. The smoothed mean is computed as :

$$\hat{\mathbf{x}}_{\mathbf{n}}^{\mathfrak{h}} = \hat{\mathbf{x}}_{\mathbf{n}} + \mathbf{K}_{\mathbf{n}}^{\mathfrak{h}} \left( \hat{\mathbf{x}}_{\mathbf{n}+1}^{\mathfrak{h}} - \hat{\mathbf{x}}_{\mathbf{n}+1|\mathbf{n}}^{\mathfrak{h}} \right) \quad (76)$$

527 8. The smoothed covariance matrix is calculated by :

$$\hat{\mathbf{P}}_{\mathbf{n}}^{\mathfrak{h}} = \hat{\mathbf{P}}_{\mathbf{n}} + \mathbf{K}_{\mathbf{n}}^{\mathfrak{h}} \left( \hat{\mathbf{P}}_{\mathbf{n}+1}^{\mathfrak{h}} - \hat{\mathbf{P}}_{\mathbf{n}+1|\mathbf{n}}^{\mathfrak{h}} \right) \left( \mathbf{K}_{\mathbf{n}}^{\mathfrak{h}} \right)^T \quad (77)$$

#### 528 7.4. Unscented EM-algorithm

529 The derivation of matrices  $\{\mathbf{B}_i\}, i = 1..6$  follows the same reasoning as  
 530 for the Extended Kalman Filter (see [30] for details), apart from the UT that  
 531 handles the nonlinear functions  $f(\cdot)$  and  $g(\cdot)$ .

532 Creation of a sigma points set  $\{\mathbf{x}_{\mathbf{n}-1}^{(i)}\}$  from  $\hat{\mathbf{x}}_{\mathbf{n}-1}$  and  $\hat{\mathbf{P}}_{\mathbf{n}-1}$  by formula 60

$$\mathbf{B}_1 = \sum_{n=1}^N \left( \hat{\mathbf{x}}_n \hat{\mathbf{x}}_n^T + \hat{\mathbf{P}}_n \right) \quad (78)$$

533

$$\mathbf{B}_2 = \sum_{n=1}^N \sum_{i=0}^{2L} W_i^{(c)} \hat{\mathbf{x}}_n^{(i)} f \left( \hat{\mathbf{x}}_n^{(i)} \right)^T \quad (79)$$

534

$$\mathbf{B}_3 = \sum_{n=1}^N \sum_{i=0}^{2L} W_i^{(c)} f \left( \hat{\mathbf{x}}_{n-1}^{(i)} \right) f \left( \hat{\mathbf{x}}_{n-1}^{(i)} \right)^T \quad (80)$$

535

$$\mathbf{B}_4 = \sum_{n=1}^N \mathbf{y}_n \mathbf{y}_n^T \quad (81)$$

536

$$\mathbf{B}_5 = \sum_{n=1}^N \sum_{i=0}^{2L} W_i^{(c)} \mathbf{y}_n g \left( \hat{\mathbf{x}}_n^{(i)} \right) \quad (82)$$

537

$$\mathbf{B}_6 = \sum_{n=1}^N \sum_{i=0}^{2L} W_i^{(c)} g \left( \hat{\mathbf{x}}_n^{(i)} \right) g \left( \hat{\mathbf{x}}_n^{(i)} \right)^T \quad (83)$$

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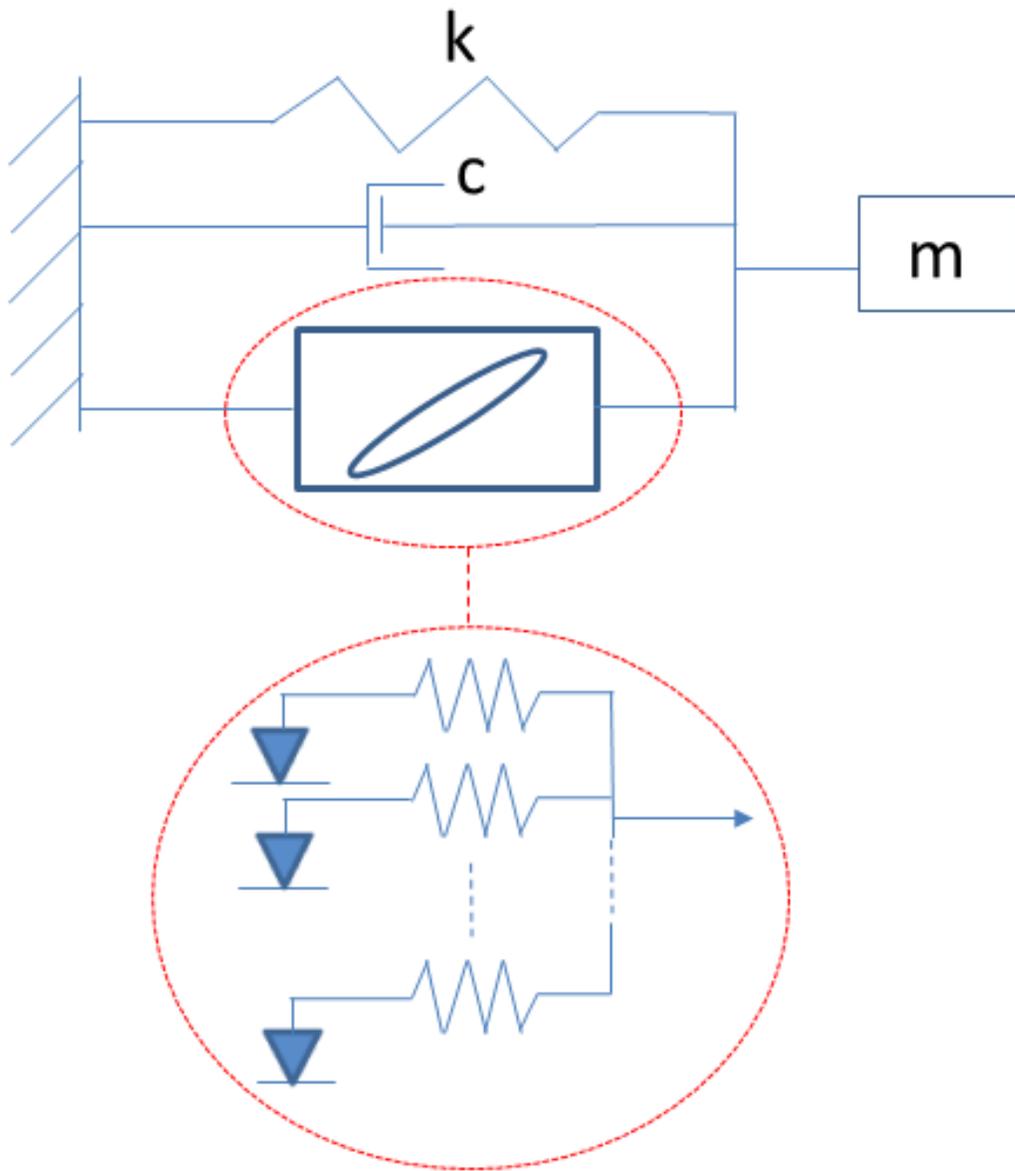


Figure 1: Iwan model

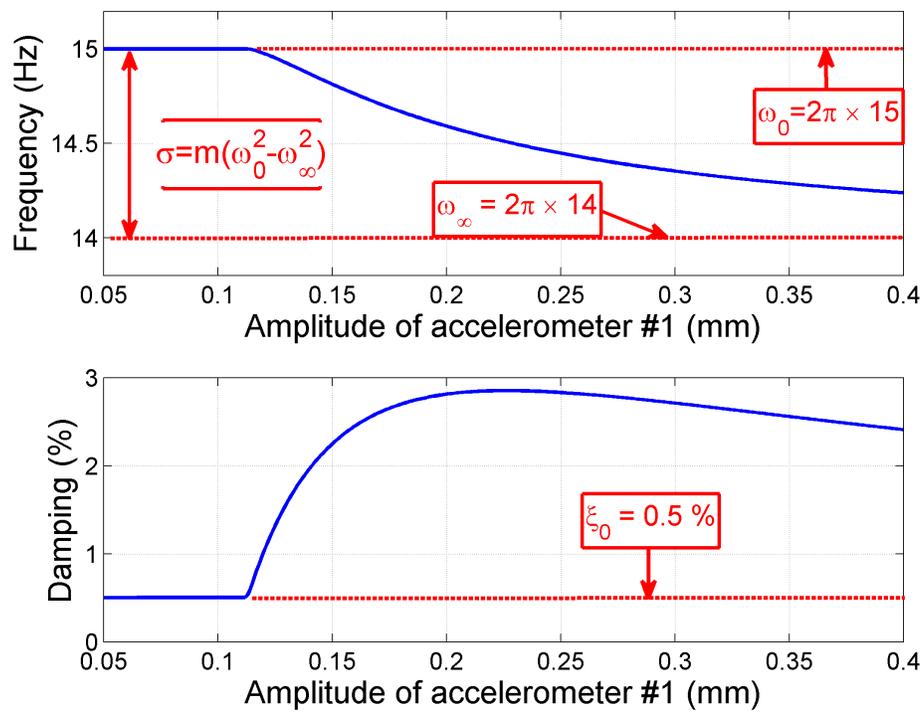


Figure 2: Equivalent modal parameters for the model example

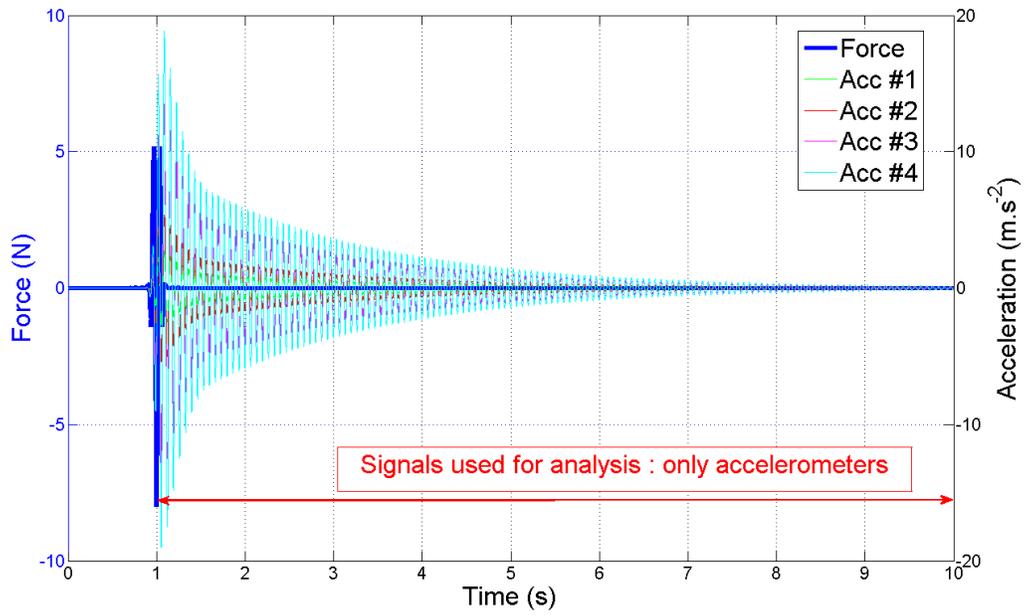


Figure 3: Response due to a wavelet excitation

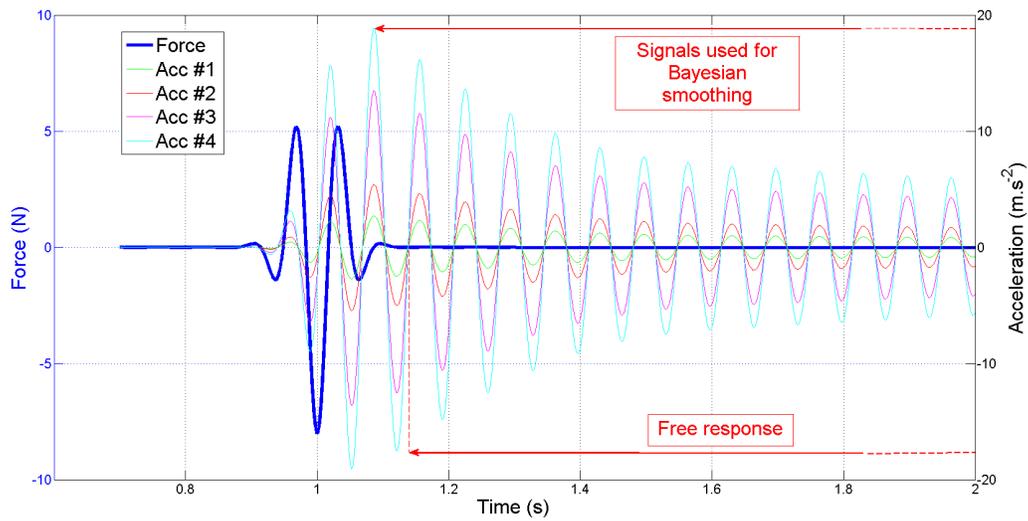


Figure 4: Zoom on the transient part due to excitation

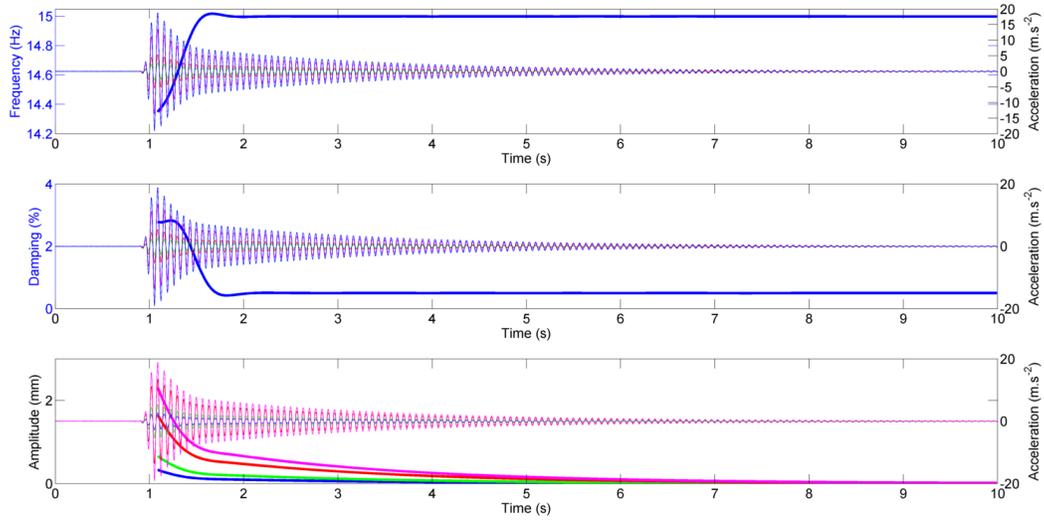


Figure 5: Equivalent modal parameters and amplitudes as a function of time

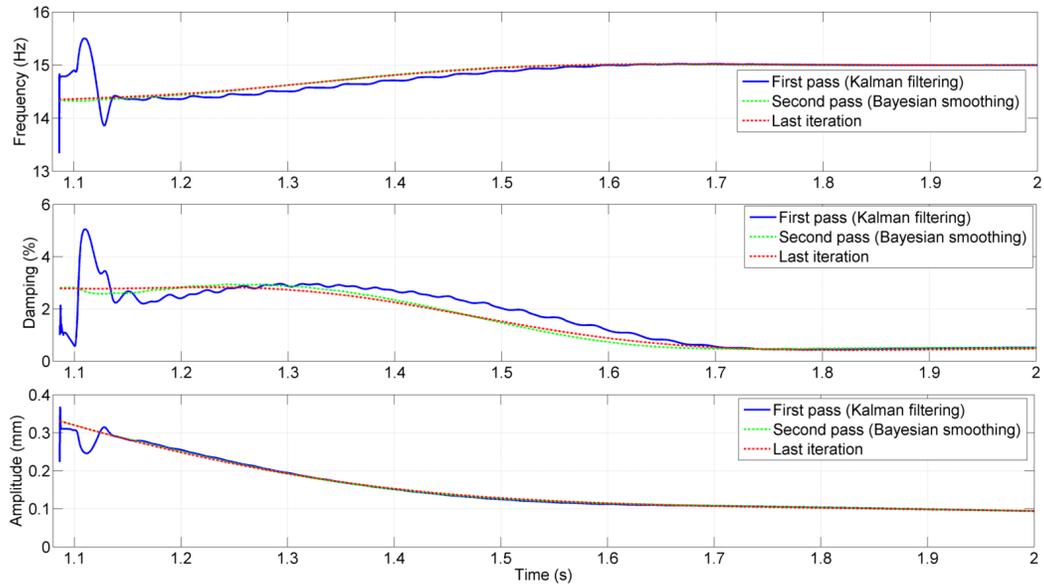


Figure 6: Equivalent modal parameters and amplitudes as a function of time : frequency, damping and amplitude of accelerometer 1

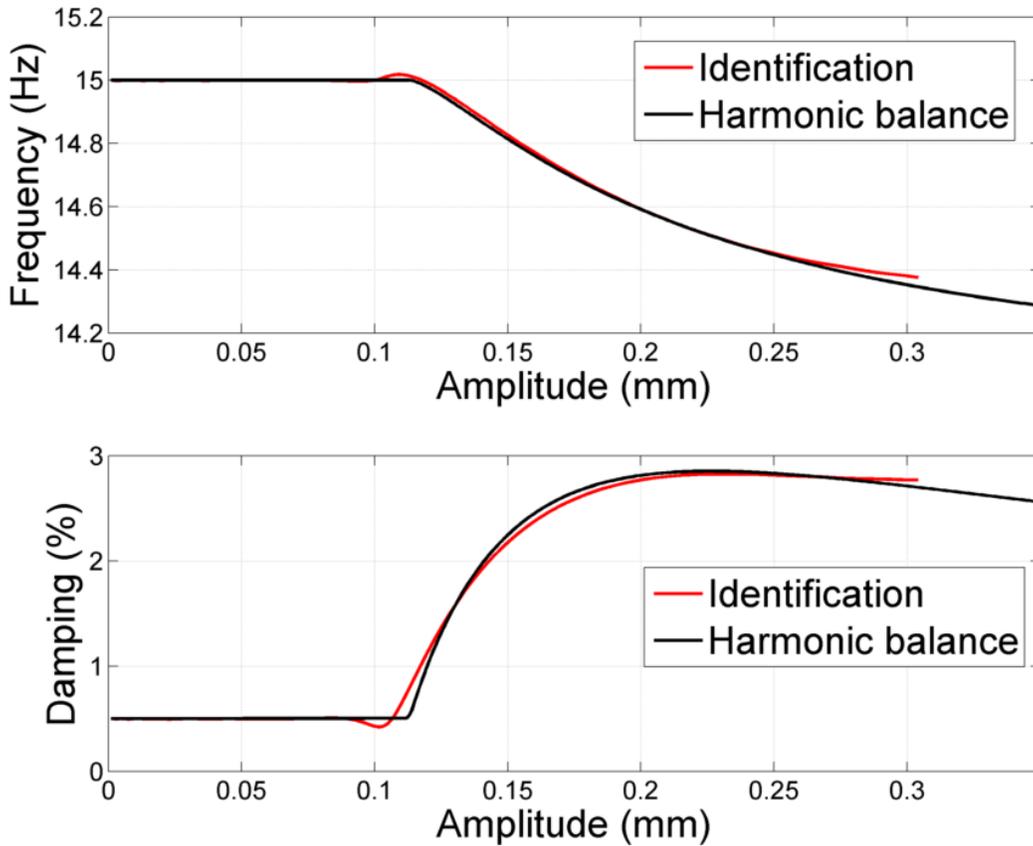


Figure 7: Equivalent modal parameters as a function of accelerometer 1 amplitude

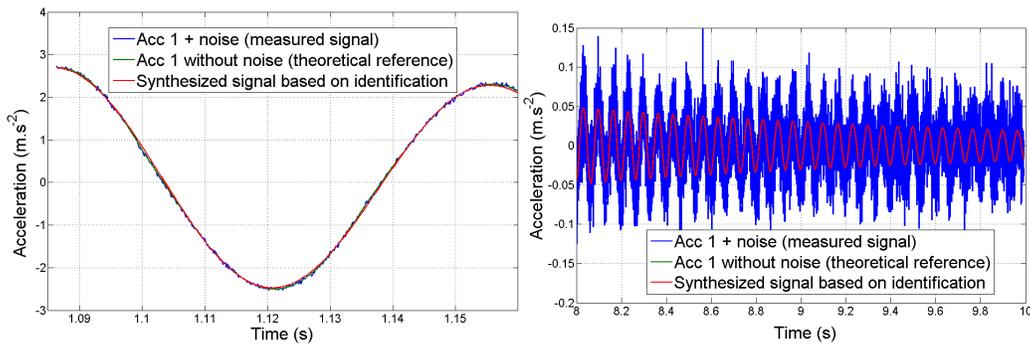


Figure 8: Measured, theoretical and synthesized acceleration signals : accurate matching of theoretical and synthesized accelerations, at beginning and end

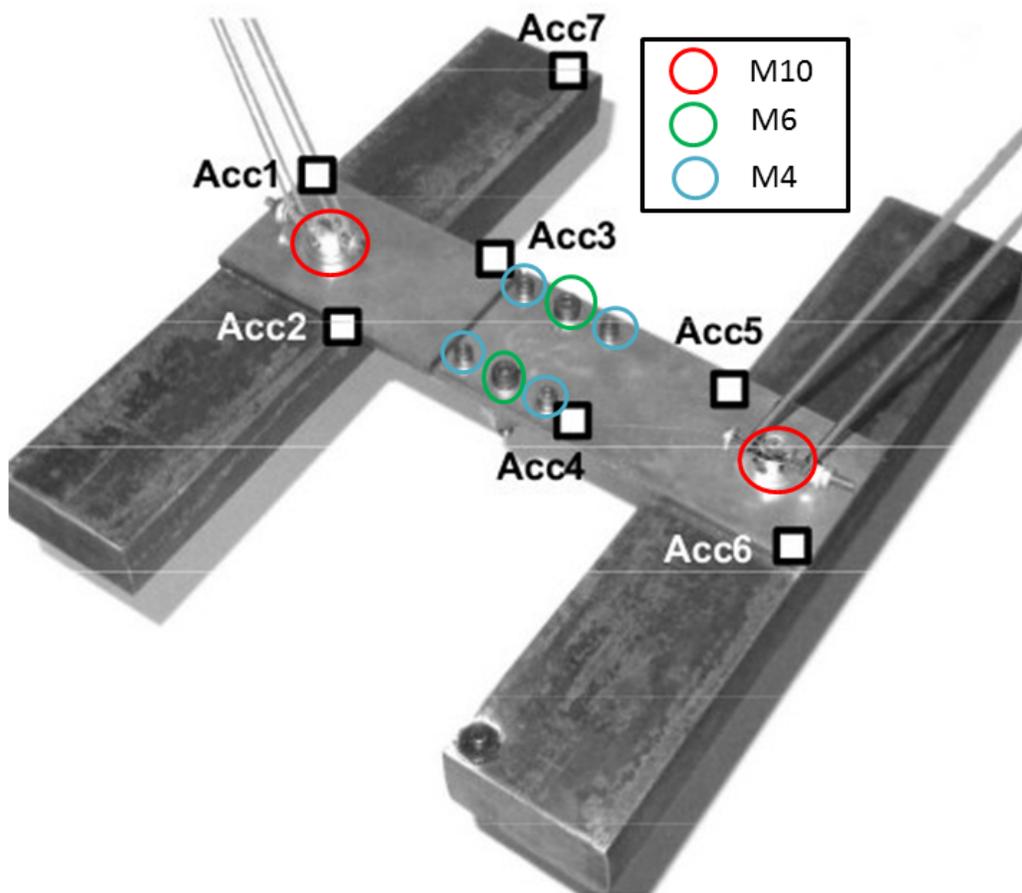


Figure 9: H Testbench

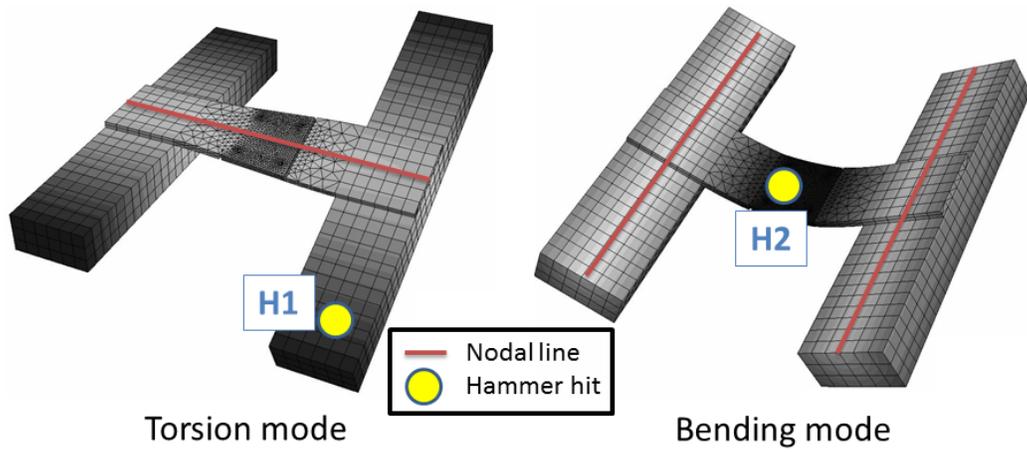


Figure 10: Torsion and bending modes

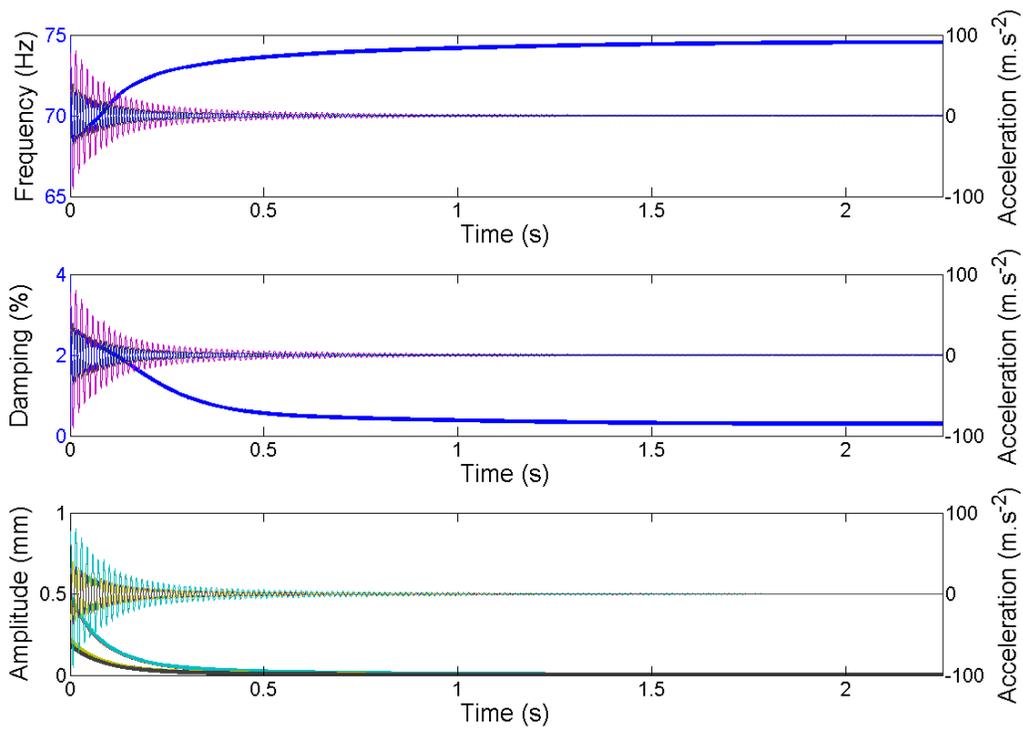


Figure 11: Equivalent modal parameters and amplitudes of the bending mode as a function of time (torque of 2 Nm)

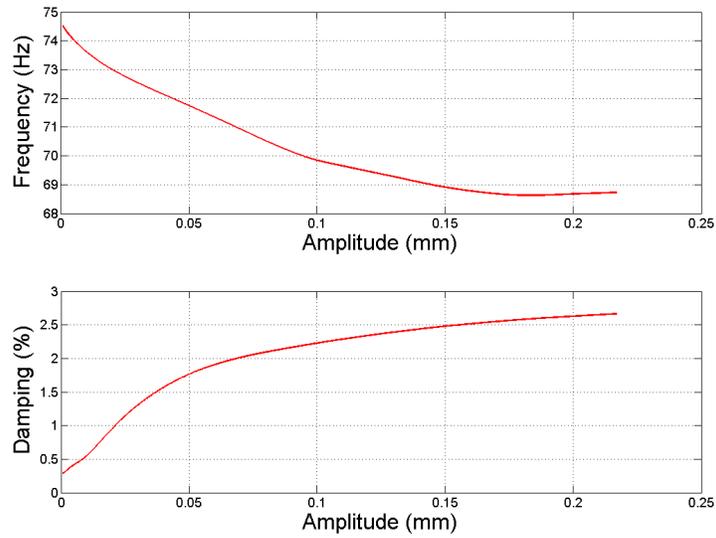


Figure 12: Equivalent modal parameters of the bending mode as a function of Acc1 displacement (torque of 2Nm)

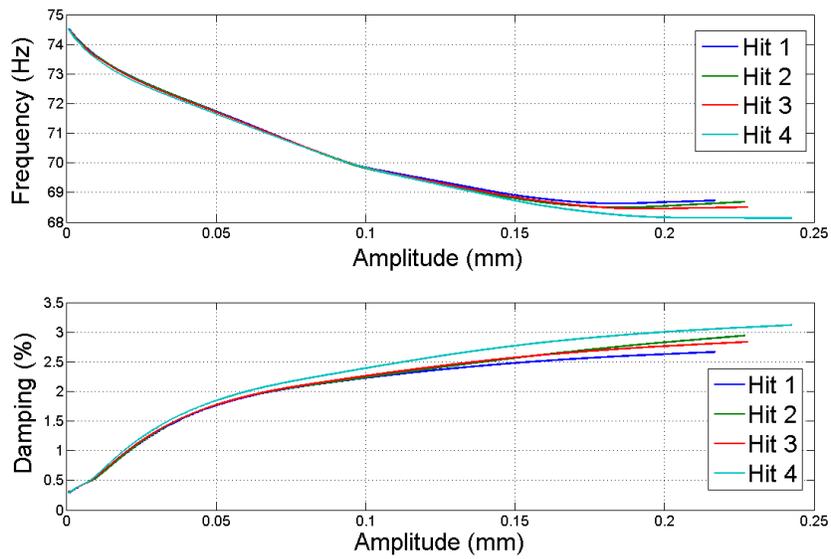


Figure 13: Repeatability of tests

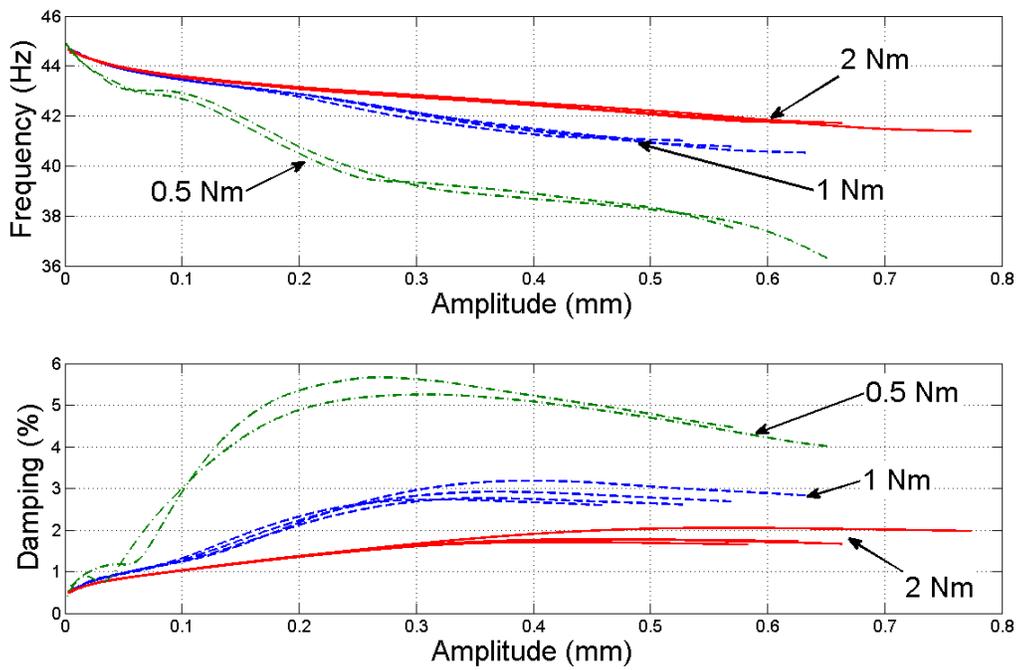


Figure 14: Equivalent modal parameters of the torsion mode as a function of Acc7 displacement