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► **To cite this version:**

Julien Baptiste, Julien Grepât, Emmanuel Lépinette. Approximation of non-Lipschitz SDEs by Picard iterations. 2016. hal-01397399v1

**HAL Id: hal-01397399**

**<https://hal.science/hal-01397399v1>**

Preprint submitted on 15 Nov 2016 (v1), last revised 28 Aug 2018 (v2)

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# Approximation of non-Lipschitz SDEs by Picard iterations.

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## Abstract

In this paper, we propose an approximation method based on Picard iterations deduced from the Doléans–Dade exponential formula. Our method allows to approximate trajectories of Markov processes in a large class, e.g. solutions to non-Lipchitz SDEs. An application to the pricing of Asian-style contingent claims in the CEV model is presented and compared to other methods of the literature.

*Keywords:* Pricing, European Options

*2000 MSC:* 60G44, G11-G13.

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## 1. Introduction

In frictionless models without arbitrage opportunities, it is possible to define the price of a contingent claim as the initial value of the unique portfolio process whose terminal value coincides with the payoff, see [4, Section 3.2.2]. The price of a replicable contingent claim may be deduced as an expectation as soon as the price process is a local martingale and the replicating portfolio is a martingale under some equivalent probability measure. If the price dynamics is simple enough, e.g. in the Black and Scholes model, it is possible to obtain an explicit expression for the Call or Put option prices. When sophisticated models are considered for a sake of realism [3, 6, 9], one can not expect to compute easily option prices which do not admit pricing

formulas in closed form. For this purpose, Monte-Carlo methods have been widely developed to simulate trajectories of diffusion processes and deduce approximations of European option prices [2, 11].

An alternative way is to study the density function of the terminal price value as in the Black and Scholes model [1]. This is also possible for the CEV model [12] where analytic expressions of European Call and Put option prices are deduced. Nevertheless, it is not possible to evaluate the price of an Asian option since a knowledge of the whole trajectory is required. Moreover, the Monte Carlo discretization method seems to be useless as the diffusion coefficient is not Lipschitz so that the Euler scheme should explode. In [8, 10], the asset price process is alternatively approximated through a Picard iteration scheme followed by approximations based on Wiener–Ito chaos expansions. Only the first three terms in the expansions are considered as significative and the associated density functions are approximated. The accuracy of the suggested numerical method illustrated by numerical implementations is highlighted even if it is not confirmed by mathematical arguments. A direct discretization scheme based on Doléans-Dade formula is also used (without mathematical proof of the convergence) in [14] to simulate trajectories in the CEV model. Inspired by these works, our goal is to study the method based on Picard iterations to approximate Asian option prices when the risky asset dynamics is a stochastic differential equation with non Lipschitz-coefficients. The outline of our work is as follows. By a truncation, we first approximate the stochastic process of interest by a diffusion process with Lipschitz coefficients. Second, we introduce a Picard iteration technique that approximates the process as a Doléans-Dade exponential. Finally, a Monte Carlo discretization scheme is implemented. Comparatively to the classical Euler scheme, we obtain a convergence order  $n$  instead of  $\sqrt{n}$  where  $n$  is the number of discrete dates. The method may be applied to the pricing of a large class of European and Asian options. We illustrate it by computing prices of average Put options in the CEV model where the diffusion coefficient is not Lipschitz. Comparing with the exact formulas and other techniques, our method appears to provide accurate results. Moreover, as soon as we con-

sider an average Put option with more than three dates, our method is less time-consuming than the method combining Monte Carlo simulations and analytic expressions. Nevertheless, in some cases, a simple truncated Euler scheme appears to be more efficient whereas the direct discretization scheme based on the Doléans-Dade formula without truncation does not converge.

## 2. Model

### 2.1. Notations

We denote by  $\mathbf{R}$  the set of all real numbers.

The smallest  $\sigma$ -algebra containing a family of subsets  $\mathcal{A}$  is denoted by  $\sigma(\mathcal{A})$ .

The set of all bounded real-valued functions on  $[0, T]$  is denoted by  $\mathcal{B}([0, T])$ .

Let  $(\Omega, \mathcal{F}, P)$  be a stochastic basis with a complete filtration  $\mathcal{F} = (\mathcal{F}_t)_{t \in [0, T]}$  satisfying the usual hypothesis and  $T > 0$  is fixed. We consider a standard Brownian motion  $W$  adapted to  $\mathcal{F}$ . For any predictable process  $H$ , we consider the stochastic integral of  $H$  with respect to  $W$  denoted by

$$H \cdot W_t := \int_0^t H_u dW_u, \quad t \in [0, T],$$

when existence holds. The Doleans-Dade exponential is denoted by

$$\mathcal{E}_t(H) = \exp \left( H \cdot W_t - \frac{1}{2} \int_0^t H_u^2 du \right), \quad t \in [0, T].$$

In the following,  $S_0$  is the deterministic initial value at time 0 of a process  $S$  modelling an asset price in a financial market model. A stochastic process  $r : u \mapsto r_u$  models an interest rate of the market. We define the price of the associated bond whose initial value is 1 by

$$F_t(r) := \exp \left( \int_0^t r_u du \right), \quad t \in [0, T]. \quad (2.1)$$

In the following, we consider uniform partitions  $(\tau_n)_{n \geq 0}$  of the interval  $[0, T]$ . The grid  $\tau_n$  is composed of  $n + 1$  dates where  $n$  is called the order of the discretization scheme. It is defined by

$$\tau_n = \{t_0 = 0, t_1 = T/n, t_2 = 2T/n, \dots, t_n = T\}.$$

We use the abuse of notation  $t_i^n = t_i = (T/n)i$ ,  $i = 0, \dots, n$ , for a sake of simplicity. If  $X$  is a stochastic process defined on  $[0, T]$ , we denote by  $X^{(n)}$  the piecewise constant process defined by

$$X_t^{(n)} = X_{t_i}, \quad t \in [t_i, t_{i+1}), \quad X_T^{(n)} = X_T.$$

## 2.2. The stochastic differential equation

Let  $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0, T]}, \mathbb{P})$  be a continuous-time stochastic basis satisfying the usual assumptions and supporting a standard Brownian motion  $W$ , i.e.  $\mathcal{F}_t := \sigma(W_s : s \leq t) \vee \mathcal{N}$  where  $\mathcal{N}$  designates the  $\mathbb{P}$ -negligible sets. In the case of a financial market model, there is a risk-free bond given by the interest rate  $r^0$  and a risky asset price stochastic process  $S$ .

By [7, Theorem 2.2 p 104], recall that we have the following.

**Proposition 2.1.** *Let  $\sigma : [0, T] \times \mathbf{R} \mapsto \mathbf{R}$  and  $r : [0, T] \times \mathbf{R} \mapsto \mathbf{R}$  be Lipschitz functions. The stochastic differential equation (s.d.e.)*

$$dS(t) = S(t)\sigma(t, S(t))dW_t + S(t)r(t, S(t))dt, \quad S(0) = S_0, \quad (2.2)$$

*admits a unique strong solution  $S$ . Moreover, it is positive.*

In the sequel, we suppose that  $S$  satisfies a more general dynamics than above, precisely:

$$dS(t) = \Sigma(t, S(t))dW_t + R(t, S(t))dt, \quad S(0) = S_0. \quad (2.3)$$

We suppose that  $S = 0$  on  $[\tau \wedge T, T]$  where  $\tau := \inf\{t : S_t = 0\}$ , meaning that 0 is an absorbing point. We define  $\sigma(t, x) := \Sigma(t, x)/x$ ,  $r(t, x) := R(t, x)/x$  on  $\mathbf{R} \setminus \{0\}$  where  $r$  is assumed to be bounded. The s.d.e. (2.3) admits a unique non negative solution under the conditions of the proposition above but we only suppose one of the following Conditions 2.2 or 2.3.

**Condition 2.2.** *There exists constants  $C_\kappa > 0$  for every  $\kappa > 0$ , such that*

$$\begin{aligned}
|\sigma(t, x \wedge \kappa) - \sigma(u, x \wedge \kappa)| &\leq C_\kappa |t - u|, & (t, u, x) &\in (\mathbf{R}_+)^2 \times (0, \infty), \\
|r(t, x \wedge \kappa) - r(u, x \wedge \kappa)| &\leq C_\kappa |t - u|, & (t, u, x) &\in (\mathbf{R}_+)^2 \times (0, \infty), \\
|\sigma(t, x \wedge \kappa) - \sigma(t, y \wedge \kappa)| &\leq C_\kappa |x - y|, & (t, x, y) &\in \mathbf{R}_+ \times (0, \infty)^2, \\
|r(t, x \wedge \kappa) - r(t, y \wedge \kappa)| &\leq C_\kappa |x - y|, & (t, x, y) &\in \mathbf{R}_+ \times (0, \infty)^2, \\
|r(t, x \wedge \kappa)| + |\sigma(t, x \wedge \kappa)| &\leq C_\kappa, & (t, x, y) &\in \mathbf{R}_+ \times (0, \infty)^2.
\end{aligned}$$

Under Condition 2.2,  $S$  admits moments at any order, see the proof in Section 5. Notice that this assumption is satisfied by the CEV model and, more generally, when  $\sigma$  is of the form  $\sigma(t, x) = \nu(t)x^{\beta(t)-1}$ ,  $x > 0$ , where  $\nu$  and  $\beta \geq 1$  are bounded functions and  $\nu$  is Lipschitz.

**Condition 2.3.** *There exists constants  $C_\kappa > 0$  for every  $\kappa > 0$ , such that*

$$\begin{aligned}
|\sigma(t, e^X \vee \kappa) - \sigma(u, e^X \vee \kappa)| &\leq C_\kappa |t - u|, & (t, u, Y) &\in (\mathbf{R}_+)^2 \times \mathbf{R}, \\
|r(t, e^X \vee \kappa) - r(u, e^X \vee \kappa)| &\leq C_\kappa |t - u|, & (t, u, Y) &\in (\mathbf{R}_+)^2 \times \mathbf{R}, \\
|\sigma(t, e^X \vee \kappa^{-1}) - \sigma(t, e^Y \vee \kappa^{-1})| &\leq C_\kappa |X - Y|, & (t, X, Y) &\in \mathbf{R} \times \mathbf{R}^2, \\
|r(t, e^X \vee \kappa^{-1}) - r(t, e^Y \vee \kappa^{-1})| &\leq C_\kappa |X - Y|, & (t, X, Y) &\in \mathbf{R} \times \mathbf{R}^2, \\
|r(t, e^X \vee \kappa)| + |\sigma(t, e^Y \vee \kappa)| &\leq C_\kappa, & (t, X, Y) &\in \mathbf{R}_+ \times \mathbf{R}^2.
\end{aligned}$$

Moreover, when  $P(\tau < T) > 0$ , we suppose that the mappings  $x \mapsto \sigma(t, x)$  and  $x \mapsto r(t, x)$  are differentiable on  $(0, \infty)$  for all  $t \in [0, T]$  and  $x \nabla_x \sigma(t, x)$ ,  $x \nabla_x r(t, x)$  are bounded on  $[\kappa^{-1}, \infty)$  by  $C_\kappa$ .

**Remark 2.4.** *Condition 2.3 is satisfied by the CEV model, e.g. when  $\sigma$  is of the form  $\sigma(t, x) = \nu(t)x^{\beta(t)-1}$  where  $\nu$  is Lipschitz and bounded and  $1 - \alpha \leq \beta \leq 1$ ,  $\alpha \in \mathbf{R}_+$ .*

In finance, under a risk-neutral probability measure for  $S$ , e.g. when the market is complete, we are interested in approximating the price of a contingent claim whose payoff  $G(S)1_{\tau > T}$  depends on the whole trajectory  $S = (S_t)_{t \in [0, T]}$ . Note that the delivered payoff  $G(S)1_{\tau > T}$  is 0 when  $S_T = 0$ , i.e. when the firm having issued the risky asset failed.

We suppose that the functional  $G : \mathcal{B}([0, T]) \rightarrow \mathbb{R}_+$  satisfies the Lipschitz condition:

$$|G(\alpha) - G(\beta)| \leq C \|\alpha - \beta\|, \quad \alpha, \beta \in \mathcal{B}([0, T]), \quad (2.4)$$

where  $C > 0$  and  $\|\cdot\|$  is the norm of uniform convergence on  $[0, T]$ , i.e.

$$\|f\| := \sup_{t \in [0, T]} |f(t)|, \quad f \in \mathcal{B}([0, T]).$$

If we assume that  $\mathbb{P}$  is the unique risk-neutral probability measure in a complete market, the smallest price  $v$  of the contingent claim  $G(S)1_{\tau > T}$  is evaluated under  $\mathbb{P}$  by  $v = \mathbf{E}e^{-r^0 T} G(S)1_{\tau > T}$  where  $\mathbf{E}$  is the expectation under  $\mathbb{P}$  and  $r^0$  is the constant risk-free interest rate.

We consider a class of payoff functions  $G$  adapted to the numerical scheme we propose. For a sake of simplicity, we make the abuse of notation  $r_u = r(u, S_u)$  as well as  $\sigma_u = \sigma(u, S_u)$  so that the notations  $F(r)$  and  $\mathcal{E}(\sigma)$  make sense.

**Definition 2.5.** *We say that the payoff function  $G$  is consistent of order  $\alpha(n) \geq 1$  with respect to the discretization scheme defined by the partition  $\tau_n$  if there exists  $p, q \in \mathbb{N}$  such that, for all Lipschitz functions  $\sigma$  and  $r$  satisfying*

$$\int_0^T \mathbf{E} F_u^{2p}(r) \mathcal{E}_u^{2p}(\sigma) du < \infty,$$

*we have, for some function  $f$ ,*

$$|\mathbf{E}G(S_{r,\sigma}) - \mathbf{E}G(S_{r,\sigma}^{(n)})| \leq \frac{C_G f(C(\sigma, r, p), q)}{\alpha(n)} \left( \int_0^T [1 + \mathbf{E} S_{r,\sigma}^{2p}(u) du] \right)^{1/q},$$

*where  $S_{r,\sigma} = F(r)\mathcal{E}(\sigma)$ ,  $C_G > 0$  depends on  $G$  and  $C(\sigma, r, p)$  is a constant which only depends on the quantities  $\max_x |\sigma(t, x)x|^p / (1 + |x|^{2p}) < \infty$  and  $\max_x |r(t, x)x|^p / (1 + |x|^{2p}) < \infty$ .*

Recall that  $F(r)$  is given by (2.1) and, in the definition above, the process  $S_{r,\sigma} = F(r)\mathcal{E}(\sigma)$  is interpreted as a price. Observe that the European call payoff function satisfies the consistency property with respect to the discretization at any order. Indeed, the payoff function of the European call

option with strike  $K$  is  $G(S)$  where  $G(\alpha) = (\alpha(T) - K)^+$ ,  $\alpha \in \mathcal{B}([0, T])$  and, since  $G$  only depends on the terminal value of  $\alpha$ , we have  $G(S_{r,\sigma}) = G(S_{r,\sigma}^{(n)})$ . The payoff of the average strike Asian call option with strike  $K$  is  $G(S)$  where

$$G(\alpha) = \left( \frac{1}{T} \int_0^T \alpha(t) dt - K \right)^+, \quad \alpha \in \mathcal{B}([0, T]).$$

**Lemma 2.6.** *The average strike Asian call option function  $G$  with strike  $K$  is consistent of order  $n$  with respect to discretization.*

All the proofs are postponed to Section 5.

### 3. Approximation of the stochastic process

The approximation scheme we propose is obtained in three steps. By a truncation, we first approximate the price process by a diffusion process  $S_\kappa$ ,  $\kappa > 0$ , with bounded and Lipschitz volatility. By Picard iterations, a recursive scheme is introduced to approximate  $S_\kappa$  by  $S_{\kappa,m}$ ,  $m \geq 1$ , Doléans–Dade solutions to stochastic differential equations defined by iterated and uniformly bounded volatilities. At last, we discretize  $S_{\kappa,m}$  in time.

#### 3.1. Volatility truncation

Let  $\kappa \in \mathbf{R}_+$  be such that  $\kappa \geq \max r$  and  $\kappa > S_0 \vee 1$  when Condition 2.2 holds and  $\kappa^{-1} < \min(\max r, S_0 \vee 1)$  when Condition 2.3 is satisfied. Consider  $Y_\kappa$  the unique solution to the s.d.e.

$$\begin{aligned} Y_\kappa(0) &= 0, \\ dY_\kappa(t) &= \sigma_\kappa(t, S_0 e^{Y_\kappa(t)}) dW_t + \gamma_\kappa(t, S_0 e^{Y_\kappa(t)}) dt, \end{aligned} \tag{3.5}$$

where  $\sigma_\kappa(u, x) := \sigma(u, x \wedge \kappa)$  and  $r_\kappa(u, x) := r(u, x \wedge \kappa)$  when Condition 2.2 holds. If Condition 2.3 holds, we define  $\sigma_\kappa(u, x) := \sigma(u, x \vee \kappa^{-1})$  and  $r_\kappa(u, x) := r(u, x \vee \kappa^{-1})$ . At last, we define  $\gamma_\kappa(t, x) = r_\kappa(t, x) - \frac{1}{2} \sigma_\kappa^2(t, x)$ . Observe that  $S_\kappa := S_0 e^{Y_\kappa}$  satisfies, for all  $t \in [0, T]$ ,

$$\begin{aligned} S_\kappa(0) &= S_0, \\ S_\kappa(t) &:= S_0 \exp \left[ \int_0^t \sigma_\kappa(u, S_\kappa(u)) dW_u + \int_0^t \gamma_\kappa(u, S_\kappa(u)) du \right]. \end{aligned}$$



**Lemma 3.1.** *Suppose that Conditions 2.2 and 2.3 hold respectively. Then,  $S_\kappa$  converges pointwise to  $S$  as  $\kappa \rightarrow \infty$  on the interval  $[0, T]$  (on the interval  $[0, \tau)$  respectively).*

In the following, we use the following assumption:

**Condition 3.2.** *There exists  $p > 0$  and a constant  $C_p$  such that*

$$\mathbf{E} \sup_{t \in [0, T]} S_t^{2p} + \sup_{\kappa} \mathbf{E} \sup_{t \in [0, T]} S_\kappa^{2p}(t) \leq C_p, \quad \forall \kappa > 0. \quad (3.6)$$

By Corollaries 5.6 and 5.7, we deduce that Condition 3.2 is satisfied for the large class of models satisfying the following Condition 3.3 above even if the coefficients of (2.2) are unbounded.

**Condition 3.3.** *There exists  $L > 0$  and  $\alpha \in (0, 1]$  such that the following conditions are satisfied under Conditions 2.2 and 2.3 respectively:*

$$(I): 0 < \sigma^2(t, x) \leq L(1 + |\ln(\ln(x))|1_{x>1}), \quad (t, x) \in \mathbf{R} \times \mathbf{R},$$

$$|r(t, x)| \leq L, \quad (t, x) \in \mathbf{R} \times \mathbf{R},$$

$$(II): |\sigma(t, x)| + |r(t, x)| \leq \frac{L}{|x|^\alpha} + L, \quad x > 0.$$

**Lemma 3.4.** *Suppose that Condition 3.2 is satisfied for some  $p > 0$ . Moreover, assume that Condition 3.3 (II) holds if  $\sigma_\kappa = \sigma(\cdot, \cdot \vee \kappa^{-1})$ . Then, for all  $l \geq 1$ ,  $l \in \mathbb{N}$ , there exists a constant  $C_{l,p}$  such that*

$$\begin{aligned} \sigma_\kappa = \sigma(\cdot, \cdot \wedge \kappa) &\Rightarrow \mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p \leq \frac{C_{l,p}}{\kappa^l}, \\ \sigma_\kappa = \sigma(\cdot, \cdot \vee \kappa^{-1}) &\Rightarrow \mathbf{E} \sup_{t \in [0, \tau \wedge T]} |S_\kappa(t) - S_t|^p \leq C_p \alpha(\tau, \kappa), \end{aligned}$$

where  $\alpha(\tau, \kappa) \rightarrow 0$ , as  $\kappa \rightarrow \infty$ .

This result shows that  $S$  may be approximated by  $S_\kappa$  in the  $L^p$  sense,  $p > 0$ , as  $\kappa \rightarrow \infty$ .

### 3.2. Picard iterations

In this section, we construct iterated Doleans-Dade processes  $S_{\kappa,m}$ ,  $m \in \mathbf{N}$ , such that  $S_{\kappa,m}$  converge to  $S_\kappa$  when  $m \rightarrow \infty$ . To do so, we introduce the following scheme:

$$\begin{aligned} S_{\kappa,0} &:= S_0, \\ S_{\kappa,m+1}(t) &:= F_t(r_{\kappa,m+1})\mathcal{E}_t(\sigma_{\kappa,m+1}), \quad t \in [0, T], \end{aligned}$$

where

$$\begin{aligned} \sigma_{\kappa,m+1}(t) &:= \sigma_\kappa(t, S_{\kappa,m}(t)), \quad m \geq 0, \quad t \in [0, T], \\ r_{\kappa,m+1}(t) &:= r_\kappa(t, S_{\kappa,m}(t)), \quad m \geq 0, \quad t \in [0, T]. \end{aligned}$$

Notice that we have

$$S_{\kappa,m+1}(t) = S_0 \exp \left[ \int_0^t \sigma_\kappa(u, S_{\kappa,m}(u)) dW_u + \int_0^t \gamma_\kappa(u, S_{\kappa,m}(u)) du \right].$$

Let us introduce

$$Y_{\kappa,m}(t) := \log S_{\kappa,m}(t) - \log S_0, \quad t \in [0, T].$$

This process satisfies the following s.d.e.

$$dY_{\kappa,m+1}(t) = \sigma_\kappa(t, S_{\kappa,m}(t)) dW_t + \gamma_\kappa(t, S_{\kappa,m}(t)) dt.$$

We shall prove the convergence of  $S_{\kappa,m}$  to  $S_\kappa$  in  $L^2$  as  $m \rightarrow \infty$ .

**Lemma 3.5.** *The sequence  $Y_{\kappa,m}$  converges in  $L^4$  to  $Y_{\kappa,\infty}$  and*

$$\left\| \sup_{u \leq T} |Y_{\kappa,m+1}(u) - Y_{\kappa,m}(u)| \right\|_2 \leq C(1 + S_0)(C_\kappa)^{m+1} \frac{\sqrt{T}^{m+1}}{\sqrt{(m+1)!}}, \quad (3.7)$$

$$\left\| \sup_{u \leq T} |Y_{\kappa,m+1}(u) - Y_{\kappa,m}(u)| \right\|_4 \leq C(1 + S_0)(C_\kappa)^{m+1} \frac{(T^{1/4})^{m+1}}{(m+1)!^{1/4}}, \quad (3.8)$$

where  $C$  is a constant which does not depend on  $\kappa$  and  $C_\kappa$  is a constant which depends on  $\kappa$ .

**Corollary 3.6.** *There exists constants  $C, C_\kappa$  independent of  $m$  such that*

$$\begin{aligned} \mathbf{E} \sup_{u \leq t} |Y_{\kappa, \infty}(u) - Y_{\kappa, m}(u)|^2 &\leq C(1 + S_0^2) \sum_{j=m+1}^{\infty} \frac{(C_\kappa)^j}{j!}, \\ \mathbf{E} \sup_{u \leq t} |Y_{\kappa, \infty}(u) - Y_{\kappa, m}(u)|^4 &\leq C(1 + S_0^4) \sum_{j=m+1}^{\infty} \frac{(C_\kappa)^j}{j!}. \end{aligned}$$

**Lemma 3.7.** *For every  $\kappa$ , there exists constants  $C, C_\kappa$  such that*

$$\begin{aligned} \mathbf{E} \sup_{t \in [0, T]} (S_{\kappa, m+1}(t) - S_{\kappa, m}(t))^2 &\leq C_\kappa \sqrt{\mathbf{E} \sup_{t \in [0, T]} (Y_{\kappa, m}(t) - Y_{\kappa, m-1}(t))^4} \\ &\leq C_\kappa \frac{(C_\kappa)^m}{\sqrt{m!}}. \end{aligned}$$

By the lemma above, we deduce that  $S_{\kappa, m}$  converges to  $S_{\kappa, \infty}$  in  $L^2$  with respect to the uniform convergence.

**Lemma 3.8.** *For every  $\kappa$ , there exists a constant  $C_\kappa$  such that*

$$\mathbf{E} \sup_{t \in [0, T]} (S_{\kappa, \infty}(t) - S_{\kappa, m}(t))^2 \leq C_\kappa \sum_{j=m}^{\infty} \frac{(C_\kappa)^j}{\sqrt{j!}}.$$

Note that the limit processes satisfy

$$dY_{\kappa, \infty}(t) = \sigma_\kappa(t, S_{\kappa, \infty}(t))dW_t + \gamma_\kappa(t, S_{\kappa, \infty}(t))dt, \quad (3.9)$$

$$dS_{\kappa, \infty}(t) = S_{\kappa, \infty}(t)\sigma_\kappa(t, S_{\kappa, \infty}(t))dW_t + S_{\kappa, \infty}(t)r_\kappa(t, S_{\kappa, \infty}(t))dt. \quad (3.10)$$

It follows that  $S_{\kappa, \infty} = S_\kappa$  and  $Y_{\kappa, \infty} = Y_\kappa$  since uniqueness holds.

### 3.3. Volatility discretization based on Picard iterations.

In the following, we suppose without loss of generality that  $T/n \leq 1$  if  $n$  is large enough. We recursively define the processes  $S_{\kappa, m, n}$ ,  $\sigma_{\kappa, m+1, n}$ ,  $m \geq 0$ , as follows:

$$\begin{aligned} S_{\kappa, 0, n} &:= S_0, \\ \sigma_{\kappa, 1, n}(t) &:= \sigma_\kappa(t_i, S_0), \quad r_{\kappa, 1, n}(t) := r_\kappa(t_i, S_0), \quad t_i \leq t < t_{i+1}, \\ S_{\kappa, m, n}(t) &:= F_t(r_{\kappa, m, n})\mathcal{E}_t(\sigma_{\kappa, m, n}), \quad m \geq 1, \quad t \in [0, T], \\ \sigma_{\kappa, m+1, n}(t) &= \sigma_\kappa(t^{(n)}, S_{\kappa, m, n}^{(n)}(t)), \quad m \geq 1, \quad t \in [0, T], \\ r_{\kappa, m+1, n}(t) &= r_\kappa(t^{(n)}, S_{\kappa, m, n}^{(n)}(t)), \quad m \geq 1, \quad t \in [0, T]. \end{aligned}$$

where we use the notation  $t^{(n)} = t_i$  if  $t_i \leq t < t_{i+1}$  and we recall that

$$\begin{aligned}\mathcal{E}_t(\sigma_{\kappa,m,n}) &= \exp \left[ \int_0^t \sigma_{\kappa}(u^{(n)}, S_{\kappa,m-1,n}^{(n)}(u)) dW_u - \frac{1}{2} \int_0^t \sigma_{\kappa}^2(u^{(n)}, S_{\kappa,m-1,n}^{(n)}(u)) du \right], \\ F_t(r_{\kappa,m,n}) &= S_0 \exp \left[ \int_0^t r_{\kappa}(u^{(n)}, S_{\kappa,m-1,n}^{(n)}(u)) du \right], \quad m \geq 1, \quad t \in [0, T].\end{aligned}$$

We also define  $Y_{\kappa,m,n} = \ln(S_{\kappa,m,n}/S_0)$ .

**Lemma 3.9.** *For all  $\kappa > 0$ , there exists constants  $\alpha_{\kappa}, C_{\kappa} > 0$  such that for all  $m, n \geq 1$ :*

$$\mathbf{E} \sup_{t \in [0, T]} |S_{\kappa,m}(t) - S_{\kappa,m,n}(t)|^2 \leq \frac{\alpha_{\kappa} C_{\kappa}^m}{n^2}.$$

### 3.4. Main result

In the following, we follow the convention that  $\tilde{\tau}^{\kappa,m,n} = +\infty$  if  $P(\tau \leq T) = 0$ . Otherwise,  $\tilde{\tau}^{\kappa,m,n}$  is defined as

$$\tilde{\tau}^{\kappa,m,n} := \inf\{t : S_{\kappa,m,n}^{(n)}(t) \leq \kappa^{-1}\}.$$

**Theorem 3.10.** *Assume that Conditions 2.2 or 2.3 hold and suppose that  $G$  is Lipschitz and consistent of order  $n$  with respect to discretization. Suppose that  $\tau = +\infty$  a.s. under Condition 2.2. Then, for  $\kappa > 0$  and  $m, n \in \mathbb{N}$ , there are some positive constants  $\alpha_{\kappa}, C_{\kappa}$  and sequences  $\epsilon(\kappa) > 0$ ,  $\epsilon_{\kappa,m} > 0$ , and  $\epsilon_{\kappa,m,n}$  such that*

$$|\mathbf{E}G(S)1_{\tau > T} - \mathbf{E}G(S_{\kappa,m,n}^{(n)})1_{\tilde{\tau}^{\kappa,m,n} > T}| \leq \epsilon_{\kappa} + \epsilon_{\kappa,m} + \frac{\alpha_{\kappa}(C_{\kappa})^m}{n} + \epsilon_{\kappa,m,n},$$

where  $\epsilon_{\kappa} \rightarrow 0$  as  $\kappa \rightarrow \infty$ ,  $\epsilon_{\kappa,m} \rightarrow 0$  as  $m \rightarrow \infty$ ,  $\epsilon_{\kappa,m,n} \rightarrow 0$  as  $n \rightarrow \infty$ . Moreover,  $\epsilon_{\kappa,m,n} = 0$  if  $P(\tau \leq T) = 0$ .

**Remark 3.11.** *Observe that  $Y_{\kappa,1,n}(t)$  is a Gaussian random variable whose mean and variance are explicit. This implies that there is no need of simulating trajectories by Monte Carlo technics in the case where  $G$  only depends on the terminal date and  $m = 1$ . In practice, we first fix  $\kappa$  large enough and we fix  $m$  large enough so that  $\epsilon_{\kappa,m}$  is sufficiently small. It remains to make  $n$  converged to  $\infty$ . The algorithm's complexity allowing to compute*

this scheme is then  $mn$ . Notice that in the case where  $G(S)$  only depends on  $S_T$  and  $G(0) = 0$ , then  $\mathbf{E}G(S) = \mathbf{E}G(S)1_{\tau > T}$  which is approximated by  $\mathbf{E}G(S_{\kappa, m, n}^{(n)})1_{\tilde{\tau}_{\kappa, m, n} > T}$ .

#### 4. Numerical example: the CEV model

The Constant Elasticity of Variance (CEV) model is defined by the dynamics

$$dS_t = \sigma S_t^\beta dW_t, \quad t \in [0, T], \quad (4.11)$$

where the parameter  $\beta$  is called the elasticity. We suppose that  $\sigma = \sigma_{LN} S_0^{1-\beta}$  where  $\sigma_{LN}$  is the effective lognormal volatility. This implies that the volatility depends on  $\beta$  as in [12], see also [5, 15] for more details on this model.

Consider the financial market model defined by a Bond  $B \equiv 1$  and the risky asset price  $S$ . We recall that  $S$  is a martingale if  $\beta \leq 1$ . If  $\beta > 1$ ,  $S$  is a strict local martingale, i.e. is not a martingale. In both cases, there is no arbitrage opportunity when using predictable strategies generating portfolio processes which are bounded from below, see [4, Section 3.2]. Indeed, any self-financing portfolio process which is bounded from above is a supermartingale.

Let  $h_T$  be a European or Asian contingent claim. Suppose that  $h_T$  is bounded, which is the case for a Put option. Then, by the predictable representation theorem, the martingale  $V_t = \mathbf{E}(h_T | \mathcal{F}_t)$ ,  $t \in [0, T]$ , is a self-financing portfolio process such that  $V_T = h_T$ . We deduce that the unique price for the payoff  $h_T$  is well defined as the expectation  $V_0 = \mathbf{E}h_T$ . In the following, we illustrate our method by estimating  $V_0 = \mathbf{E}h_T$  when  $\beta > 1$  and  $\beta < 1$  respectively. The price for the Call option may be easily deduced from the Put-Call parity and the analytic expression of the expectations  $\mathbf{E}S_t$ ,  $t \in [0, T]$ , see [12].

##### 4.1. The CEV model with elasticity $\beta > 1$

Let us introduce  $\delta = (1 - 2\beta)(1 - \beta)^{-1}$ . We suppose that  $\delta > 2$  or equivalently  $\beta > 1$  so that  $S$  is a strictly positive process. The price of the Call option with strike  $K > 0$  is then given in [12] by

$$p^{Call}(S_0, K, \sigma, \delta, T) = S_0 \left( \Gamma\left(\frac{\delta}{2} - 1, \frac{S_0}{2T}\right) - \chi'^2\left(\frac{S_0}{T}, \delta - 2, \frac{\tilde{K}}{T}\right) \right) - K \chi'^2\left(\frac{\tilde{K}}{T}, \delta, \frac{S_0}{T}\right),$$

where  $\tilde{K} := \frac{K^2(1-\alpha)}{\sigma^2(1-\alpha)^2}$ ,  $\Gamma$  is the normalised incomplete gamma function

$$\Gamma(z, x) = \frac{1}{\Gamma(z)} \int_0^x t^{z-1} e^{-t} dt,$$

and  $\chi'^2(x, k, \lambda)$  is the cumulative density function of the non-central chi-squared distribution with degrees of freedom  $k$  and non-centrality parameter  $\lambda$ . The price of the Put option with strike  $K$  is deduced as

$$p^{Put}(S_0, K, \sigma, \delta, T) = p^{Call}(S_0, K, \sigma, \delta, T) - S_0 \Gamma\left(1 - \frac{\delta}{2}, \frac{S_0}{2T}\right) + K.$$

In [12], the price of the Put option is computed analytically for several values of  $\beta > 1$  when  $T = 1$ ,  $S_0 = 100$ ,  $\sigma_{LN} = 20\%$  and  $K = 90, 100, 110$ . These analytical values are compared to the ones obtained by simulating the terminal value  $X_T := \frac{S_T^{2(1-\beta)}}{\sigma^2(1-\beta)^2}$ . In the following, we compare these results to the values we obtain when using our method. Actually, we can not expect to obtain the convergence of  $S_\kappa$  towards  $S$  in the  $L^p$  sense,  $p \geq 1$ , since Condition 3.2 does not a priori hold for  $p \geq 1$ , see also Condition 3.3 (I). Indeed,  $\mathbf{E}S_\kappa(t) = S_0$  for all  $t$  since  $S_\kappa$  is a square integrable martingale. On the other hand,  $\mathbf{E}S_t < S_0$  for all  $t > 0$  as shown in [12] since  $S$  is a strict local martingale. Therefore,  $\mathbf{E}S_\kappa(t)$  does not converge to  $\mathbf{E}S_t$  as  $\kappa \rightarrow \infty$ . Nevertheless, when the payoff function  $g \geq 0$  is bounded, the dominated convergence theorem yields that  $g(S_\kappa) \rightarrow g(S)$  in the  $L^p$  sense as  $\kappa \rightarrow \infty$ ,  $p \geq 1$ , which is enough for our purpose. Precisely, since  $S = S_\kappa$  on  $\tau^\kappa \geq T$ , we get that

$$\mathbf{E}|g(S_\kappa) - g(S)|^p \leq (\max g)^p \mathbf{P}\left(\sup_{t \in [0, T]} S_t \geq \kappa\right) \rightarrow 0, \text{ as } \kappa \rightarrow \infty.$$

This implies that Theorem 3.10 still hold.

Let us compare the Picard method to the analytic one. In the following, we choose  $m = 3$  and  $\sigma_\kappa(u, x) := \sigma(u, x \wedge \kappa)$  where

$$\kappa = S_0 \left(1 + \frac{5}{1 + \beta^2}\right).$$

This choice has been empirically chosen to improve the performance of our results for the Put option prices and then implemented for the Asian options.

Recall that the volatility of the model also depends on  $\beta$ . If  $\beta = 1.5$ , we observed that the approximation error is not smaller if we increase too much  $\kappa$ . The reason is very simple: choosing  $\kappa$  very large is equivalent to suppose that  $\kappa = +\infty$  as the maximum of  $S$  over all simulated trajectories is finite since we fix a finite number of dates  $n + 1$  and a finite number of simulations  $N$ . Theoretically, if  $\kappa$  is very large, we need to increase the number of dates  $n$  as the Lipschitz constant is larger, which is more time consuming.

We choose  $N = 15000$  simulations and  $n = 1000$  so that the discretization step is  $T/n$ . We consider the case  $K = 100$ .

| $\beta$ | Picard iterations | Computation time in seconds | Analytic | Computation time in seconds | Error |
|---------|-------------------|-----------------------------|----------|-----------------------------|-------|
| 1.5     | 7.8602            | 6.38                        | 7.9696   | < 0.01                      | 1.4%  |
| 2       | 7.8995            | 6.19                        | 7.9789   | < 0.01                      | 1.01% |
| 2.5     | 7.9948            | 6.09                        | 7.9960   | < 0.01                      | 0.02% |
| 3       | 8.1332            | 6.08                        | 8.0211   | < 0.01                      | 1.4%  |
| 3.5     | 8.2779            | 6.06                        | 8.0560   | < 0.01                      | 2.76% |
| 4       | 8.4006            | 6.06                        | 8.1033   | < 0.01                      | 3.67% |
| 4.5     | 8.4886            | 6.13                        | 8.1639   | < 0.01                      | 3.98% |
| 5       | 8.5348            | 6.08                        | 8.2345   | < 0.01                      | 3.65% |
| 5.5     | 8.5344            | 5.936                       | 8.3084   | < 0.01                      | 2.72% |
| 6       | 8.5015            | 5.906                       | 8.3786   | < 0.01                      | 1.47% |
| 6.5     | 8.4499            | 6.096                       | 8.4396   | < 0.01                      | 0.13% |
| 7       | 8.3821            | 6.401                       | 8.4882   | < 0.01                      | 1.26% |

Table 1: Put option prices using Picard iterations and the analytic expression respectively for different values of  $\beta > 1$ .

Naturally, the computation time needed for the analytic method is negligible contrary to the needed time for our method which requires to simulate all the trajectories.

In the following, we compare our method to the Monte Carlo one of [12], requiring that the radius of the 95% confidence interval is less than 0.025 when applying the law of large number. The simple Monte Carlo simulation is less time consuming. Nevertheless, we shall see that our method outperforms the Monte Carlo simulations as soon as we consider an Asian Put option with several dates.

It is also natural to compare our method to a simple Doléans-Dade scheme deduced from the dynamics (4.11) where the volatility function  $\sigma(x) = \sigma x^{\beta-1}$  is replaced by  $\sigma_\kappa(x) = \sigma(x \wedge \kappa)^{\beta-1}$ . Inspired by the Doleans-Dade formula,

| $\beta$ | Picard      | Comput. Time | MC [32]     | Comput. Time | Error |
|---------|-------------|--------------|-------------|--------------|-------|
| 1.5     | 7.961482891 | 268.748      | 7.958596698 | 20.756       | 0.03% |
| 2       | 8.001945176 | 276.356      | 7.968447144 | 12.571       | 0.42% |
| 2.5     | 8.096442435 | 286.754      | 7.981469166 | 13.68        | 1.44% |
| 3       | 8.231139717 | 298.65       | 8.002688168 | 13.098       | 2.86% |
| 3.5     | 8.382673524 | 317.32       | 8.050143346 | 10.78        | 4.14% |
| 4       | 8.507583601 | 338.079      | 8.091346416 | 7.908        | 5.15% |
| 4.5     | 8.594539064 | 345.449      | 8.164748506 | 5.855        | 5.17% |
| 5       | 8.631951141 | 351.436      | 8.234170007 | 3.698        | 4.84% |

Table 2: Put option prices using Picard iterations and Monte Carlo method respectively for different values of  $\beta > 1$  so that the radius of the 95% confidence interval is less than 0.025.

we consider the Doléans-Dade scheme

$$S_{t_{i+1}}^n = S_{t_i}^n \exp \left\{ \sigma_{\kappa}(S_{t_i}^n) \Delta W_{t_{i+1}} - \frac{\sigma_{\kappa}^2(S_{t_i}^n)}{2} \Delta t_{i+1} \right\}, S_0^n = S_0, \quad i = 0, \dots, n-1.$$

In the following table, we compare the approximate prices to the analytic ones for several values of  $\beta$  by implementing the scheme until the error is smaller than the one observed in Table 1 by increasing  $N$  with the bound  $N \leq 5.10^6$ . We choose  $n = 1000$  so that the time step is  $T/n$ . The strike is  $K = 100$  and  $\kappa = 400$ . The error is still computed as the ratio between the difference of the two prices and the smallest one.

| $\beta$ | Price (truncated Doléans-Dade scheme) | Comput. time in seconds | $N$      | Error  |
|---------|---------------------------------------|-------------------------|----------|--------|
| 1.5     | 7.1412                                | 535                     | $5.10^6$ | 11.6%  |
| 2       | 7.1978                                | 555                     | $5.10^6$ | 10.85% |
| 2.5     | 7.2544                                | 553                     | $5.10^6$ | 10.22% |
| 3       | 7.3134                                | 549                     | $5.10^6$ | 9.68%  |
| 3.5     | 7.3779                                | 556                     | $5.10^6$ | 9.19%  |
| 4       | 7.4524                                | 564                     | $5.10^6$ | 8.73%  |
| 4.5     | 7.5999                                | 541                     | $5.10^6$ | 7.42%  |
| 5       | 8.0123                                | 0.014                   | 136      | 3.65%  |
| 5.5     | 8.4954                                | 0.013                   | 136      | 2.72%  |
| 6       | 10.7683                               | 523                     | $5.10^6$ | 28.5%  |
| 6.5     | 11.934                                | 518                     | $5.10^6$ | 41.4%  |
| 7       | 12.9042                               | 518                     | $5.10^6$ | 52%    |

Table 3: Put option prices with truncated Doléans-Dade scheme for different values of  $\beta$ .

We observe that the results are similar for smaller  $\kappa = 120$  as the errors are around 10% and computation times are about 520 seconds. If  $\kappa = 800$ , we get similar results. The performance is still good for  $\beta = 5$  but also  $\beta = 4$  while it is bad for  $\beta = 5.5$  contrary to the case  $\kappa = 400$ . The best results we



can get seem to be when  $\kappa = +\infty$ , i.e. without any truncation, but remains bad. With a larger  $n = 10000$  and  $N = 15000$ , errors go from 2.5% to 10.24% but computational times are larger than 14 seconds. With only  $n = 1000$ , computation times are smaller, precisely about 1.5 seconds, but the errors are similar i.e. from 2% to 20%. So, seemingly, it is more efficient to increase  $N$  rather than  $n$  if we aim to decrease the error. In any case, comparing with the results of Table 1, it is clear that our methods outperforms the simple truncated Euler scheme.

As to a direct discretization from (4.11), i.e.  $S_{t_{i+1}}^n = S_{t_i}^n + \Sigma(S_{t_i}^n)\Delta W_{t_{i+1}}$ , the Euler scheme with  $n = 1000$  and  $N = 15000$  explodes without truncation for  $\beta \geq 3$  as we may expect. For  $\beta \in \{1.5, 2, 2.5\}$ , the scheme appears to be efficient as the error is small ( $\leq 1.4\%$ ) and the computation time is also smaller ( $\simeq 1.6$  seconds). With the truncation  $\Sigma(x) = \sigma(x \wedge \kappa)^\beta$  where  $\kappa = 300$ , the errors are about 1.5% for the first values of  $\beta > 1$  while the errors go to 4.6% for  $\beta = 5$  to 769% for  $\beta = 7$ . Therefore, a truncation from the simple Euler scheme seems to be conceivable only for values of  $\beta$  smaller than 5.

At last, we shall see that our method outperforms the classical one based on Monte Carlo simulations as soon as we consider an Asian Put option of the form  $(K - (S_{T_1} - \dots - S_{T_J})/J)^+$  where  $J \geq 3$  and  $T_1 < T_2 < \dots < T_J = T$ .

Let us first consider the case  $J = 2$ . In that case, by conditioning, we rewrite  $\mathbf{E}(2K - S_{T_1} - S_{T_2})^+ = \mathbf{E}P(S_{T_1})$  where, by the Markov property,  $P(s)$  is the price of the Put option of strike  $2K - s$  in the CEV model with initial value  $s$  and horizon date  $T_2 - T_1$ . As shown in [12], it is possible to directly simulate the distribution of  $S_{T_1}$ . We then analytically deduce the price  $P(S_{T_1})$  for each simulated value of  $S_{T_1}$ . By the law of large numbers, we then deduce an approximation of  $\mathbf{E}P(S_{T_1})$ . With the same parameters as before, we obtain the following results when  $T_1 = T/2$  and  $T_2 = T = 1$ .

| $\beta$ | Picard (P) | Comput. time | MC + analytic [32] (C) | Comput. time | Error P-C | Euler (E) | Error E-C |
|---------|------------|--------------|------------------------|--------------|-----------|-----------|-----------|
| 1.5     | 6.2284     | 6.02         | 6.3147                 | 0.328        | 1.39%     | 6.3983    | 1.32%     |
| 2       | 6.2549     | 6.4          | 6.3198                 | 0.133        | 1.04%     | 6.4052    | 1.35%     |
| 2.5     | 6.3142     | 6.178        | 6.3297                 | 0.121        | 0.25%     | 6.4165    | 1.37%     |
| 3       | 6.4027     | 6.159        | 6.3438                 | 0.113        | 0.93%     | 6.4321    | 1.39%     |
| 3.5     | 6.5011     | 6.398        | 6.3625                 | 0.118        | 2.18%     | 6.4520    | 1.41%     |
| 4       | 6.5950     | 6.235        | 6.3864                 | 0.101        | 3.27%     | 6.4772    | 1.42%     |
| 4.5     | 6.6737     | 6.25         | 6.4166                 | 0.098        | 4%        | 6.5162    | 1.55%     |
| 5       | 6.7267     | 6.188        | 6.4544                 | 0.101        | 4.22%     | 6.7340    | 4.33%     |
| 5.5     | 6.7465     | 5.967        | 6.5003                 | 0.102        | 3.79%     | 7.5778    | 16.58%    |
| 6       | 6.7411     | 5.968        | 6.5530                 | 0.102        | 2.87%     | 12.7244   | 94.18%    |
| 6.5     | 6.7192     | 5.997        | 6.6102                 | 0.100        | 1.65%     | 25.4794   | 285.46%   |
| 7       | 6.6820     | 6.000        | 6.6687                 | 0.100        | 0.20%     | 51.2649   | 668.74%   |

Table 4: Asian Put option prices with two dates using Picard iterations, the Monte Carlo methods combined with the analytic expression and a truncated Euler scheme respectively. The errors are computed between the Picard method (P) and the Euler scheme (E) respectively with respect to the combined method (C).

| $\beta$ | Picard iterations | Computation time in seconds | MC+ analytic [32] | Computation time | Error |
|---------|-------------------|-----------------------------|-------------------|------------------|-------|
| 1.5     | 6.293809919       | 171.191                     | 6.307557882       | 10.375           | 0.22% |
| 2       | 6.314878756       | 173.881                     | 6.311235573       | 3.961            | 0.06% |
| 2.5     | 6.373160059       | 179.582                     | 6.320525289       | 3.429            | 0.84% |
| 3       | 6.460785842       | 186.615                     | 6.331600424       | 3.015            | 2.04% |
| 3.5     | 6.559108888       | 198.54                      | 6.347881289       | 2.524            | 3.33% |
| 4       | 6.655375191       | 205.6                       | 6.366022368       | 2.317            | 4.55% |
| 4.5     | 6.734410841       | 216.076                     | 6.390785176       | 2.13             | 5.38% |
| 5       | 6.781869529       | 219.877                     | 6.42642745        | 2.009            | 5.54% |

Table 5: Asian Put option prices with two dates using Picard iterations and the Monte Carlo methods combined with the analytic expression respectively such that the 95% confidence interval radius is smaller than 0.025.

As we may observe, it is still advantageous to combine the Monte Carlo method with the analytic one in the case of an Asian option with only two dates. Moreover, we still observe that the simple truncated Euler scheme is efficient for  $\beta \leq 5$  while it produces a large convergence error if  $\beta > 5$  contrary to the Picard iterations.

Let us now suppose that  $J = 3$ . By the Markov property, we rewrite after conditioning  $\mathbf{E}(3K - S_{T_1} - S_{T_2} - S_{T_3})^+ = \mathbf{E}P(S_{T_1})$  where  $P(s)$  is the price of the Asian Put option of strike  $3K - s$  in the CEV model with initial value  $s$ , intermediate date  $T_2 - T_1$  and horizon date  $T_3 - T_1$ . By the Monte Carlo method, we simulate  $S_{T_1}$  so that we deduce  $P(S_{T_1})$  using the pricer for two dates we have introduced before. In that case, the complexity explodes as we

need to run two successive Monte Carlo schemes. With the same parameters but only with  $\beta = 1.5$  or  $\beta = 7$  and  $T_1 = T/3$ ,  $T_2 = 2T/3$  and  $T_3 = T = 1$ , we get the following results:

| $\beta$ | Picard (P) | Comput. time | MC + analytic [32] (C) | Comput. time | Error P-C | Euler (E) | Error E-C |
|---------|------------|--------------|------------------------|--------------|-----------|-----------|-----------|
| 1.5     | 5.3788     | 61.73        | 5.2698                 | 8333         | 2.07%     | 5.8172    | 10.39%    |
| 7       | 5.6278     | 61.83        | 5.5188                 | 2306         | 1.98%     | 10.2290   | 85.35%    |

Table 6: Asian Put option price with three dates when  $\beta = 1.5$  or  $\beta = 7$  using Picard iterations, the Monte Carlo methods combined with the analytic expression and the truncated Euler scheme respectively. The errors are computed between the Picard method (P) and the Euler scheme (E) respectively with respect to the combined method (C).

| $\beta$ | Picard iterations | Computation time | CI   | MC+analytic [32] | Computation time | Error |
|---------|-------------------|------------------|------|------------------|------------------|-------|
| 1.5     | 5.345             | 1322             | 0.05 | 5.26             | 372089           | 1.62% |

Table 7: Asian Put option price with three dates using Picard iterations and the Monte Carlo methods combined with the analytic expression respectively such that the 95% confidence interval radius is smaller than 0.05.

**Conclusion:** our method appears to outperform the technique based on the Monte Carlo methods combined with the analytic expression for Asian options with more than three dates as well as the truncated Euler scheme. Notice that computational times may be shortened in the Picard scheme by replacing the time-consuming computation of the exponential function by the first terms of its Taylor expansion. In the case where  $\beta = 2$ , the simple truncated Euler scheme seems to be efficient provided that  $\beta \leq 5$  as convergence errors and computation times are small.

#### 4.2. The CEV model with elasticity $\beta \in (0, 1)$

Recall that Condition 2.3 holds when  $\beta \in (0, 1)$ . Nevertheless, the required property that there exists an absorbing point only holds when  $\beta \geq 0.5$  [12]. In this section, the parameters are  $S_0 = 100$ ,  $\sigma_{LN} = 0.5$ ,  $T = 4$ ,  $K = 100$ ,  $N = 15000$ ,  $n = 10000$  and  $m = 3$ . The parameter  $\kappa$  is empirically fixed to  $\kappa = S_0 (0.78 - 0.4 * \beta^2)$ . We also consider the Put option.

Before implementing our method, let us consider the Doleans-Dade scheme deduced from the dynamics 4.11 where the volatility function  $\sigma(x) = \sigma x^{\beta-1}$  is replaced by  $\sigma(x) = \sigma(x \vee \kappa^{-1})^{\beta-1}$ . We compare the approximate prices to the analytic ones where  $\kappa = 90$ . The error is still computed as the ratio between the difference of the two prices and the smallest one. The computation times are about 15 seconds while the errors go from 60% to 115%. For  $\kappa = +\infty$ , the computation times are similar and the errors are larger.

On the other hand, if we use the simple Euler scheme deduced from 4.11, the prices explode without truncation. With the truncation  $\kappa = 300$  and only  $n = 10000$ , we obtain the following results:

| $\beta$ | Truncated Euler scheme | Computation Time | Analytic | Computation Time | Error |
|---------|------------------------|------------------|----------|------------------|-------|
| 0       | 40.4655                | 14.58            | 39,0452  | < 0.01           | 3.63% |
| 0,1     | 39.9938                | 17.26            | 39,0089  | < 0.01           | 2.53% |
| 0,2     | 39.5656                | 16.93            | 38,9307  | < 0.01           | 1.63% |
| 0,3     | 39.2549                | 17               | 38,8210  | < 0.01           | 1.1%  |
| 0,4     | 39.0583                | 17.17            | 38,6962  | < 0.01           | 0.93% |
| 0,5     | 38.9271                | 17.10            | 38,5753  | < 0.01           | 0.91% |
| 0,6     | 38.8305                | 17.03            | 38,4724  | < 0.01           | 0.93% |
| 0,7     | 38.7365                | 16.97            | 38,3928  | < 0.01           | 0.89% |
| 0,8     | 38.6731                | 17.58            | 38,3368  | < 0.01           | 0.88% |
| 0,9     | 38,6290                | 16.83            | 38,3041  | < 0.01           | 0.85% |

Table 8: Put option prices using truncated Euler Scheme and the analytic expression respectively for different values of  $\beta < 1$ .

As we may observe, the truncated Euler scheme provides good approximations but the computation time is rather large for  $\beta \in [0, 1)$ . For  $\beta < 0$ , the errors are very large. If we only take  $n = 1000$ , and  $\beta \in [0, 1)$ , the computation time is about 1.6 seconds and the errors go from 1.25% to 3.48%.

Let us now present the approximated prices obtained by the Picard method with  $n = 10000$ .

| $\beta$ | Picard Iterations | Computation Time | Analytic | Computation Time | Error |
|---------|-------------------|------------------|----------|------------------|-------|
| 0       | 39,9628           | 5,917            | 39,0452  | < 0.01           | 2.36% |
| 0,1     | 39,6675           | 5,953            | 39,0089  | < 0.01           | 1.69% |
| 0,2     | 39,5554           | 5,961            | 38,9307  | < 0.01           | 1.61% |
| 0,3     | 39,5557           | 5,899            | 38,8210  | < 0.01           | 1.9%  |
| 0,4     | 39,6020           | 5,906            | 38,6962  | < 0.01           | 2.35% |
| 0,5     | 39,6366           | 5,988            | 38,5753  | < 0.01           | 2.76% |
| 0,6     | 39,6103           | 6,063            | 38,4724  | < 0.01           | 2.96% |
| 0,7     | 39,4664           | 6,218            | 38,3928  | < 0.01           | 2.8%  |
| 0,8     | 39,1779           | 6,310            | 38,3368  | < 0.01           | 2.2%  |
| 0,9     | 38,7374           | 6,012            | 38,3041  | < 0.01           | 1.14% |

Table 9: Put option prices using Picard iterations and the analytic expression respectively for different values of  $\beta < 1$ .

We observe that for  $\beta \in [0, 1)$ , the simple truncated Euler scheme outperforms our method when computing a European Put option as it is less time consuming. This is also the case for an Asian Put option with two dates as we may see in the following. Indeed, the truncated Euler scheme is implemented with only  $n = 1000$ ,  $\kappa = 300$ . The computation times go from 1.61 to 1.67 seconds while the errors are of same level to the ones we observe when implementing the Picard scheme. Recall that we are mainly interested in the case  $\beta \geq 0.5$ .

| $\beta$ | Picard (P) | Time  | MC + analytic [32], (C) | Time  | Error P-C | Euler (E) | Error E-C |
|---------|------------|-------|-------------------------|-------|-----------|-----------|-----------|
| 0       | 31,8518    | 5,970 | 38,0569                 | 0,082 | 19.49%    | 31.9872   | 18.98%    |
| 0.1     | 31,6319    | 5,922 | 37,3562                 | 0,093 | 18.1%     | 31.7669   | 17.59%    |
| 0.2     | 31,5461    | 5,865 | 36,5535                 | 0,099 | 15.88%    | 31.5942   | 15.69%    |
| 0.3     | 31,5399    | 5,864 | 35,2491                 | 0,102 | 10.74%    | 31.467    | 12.02%    |
| 0.4     | 31,5621    | 5,886 | 34,2660                 | 0,109 | 8.57%     | 31.3594   | 9.27%     |
| 0.5     | 31,5704    | 5,860 | 32,7486                 | 0,103 | 3.74%     | 31.2867   | 4.67%     |
| 0.6     | 31,5308    | 5,865 | 31,3049                 | 0,124 | 0.73%     | 31.2383   | 0.21%     |
| 0.7     | 31,4100    | 5,916 | 30,6201                 | 0,137 | 2.58%     | 31.1961   | 1.89%     |
| 0.8     | 31,1986    | 5,845 | 30,6708                 | 0,149 | 1.73%     | 31.1696   | 1.63%     |
| 0.9     | 30,8989    | 5,854 | 30,6531                 | 0,194 | 0.81%     | 31.1509   | 1.59%     |

Table 10: Asian Put option prices with two dates using Picard iterations (P), truncated Euler scheme (E) and the Monte Carlo methods combined with the analytic expression (C). The errors are computed between the Picard method (P) and the Euler scheme (E) respectively with respect to the combined method (C).

Clearly, the combined method derived from [12] remains the most performant for two dates. Nevertheless, with the same method, it takes several hours to estimate the price of an Asian option with three dates  $Ti/3$ ,  $i = 1, 2, 3$  and  $T = 4$ . Let us compare for three dates the asian Put option prices when

implementing either the Picard method or the truncated Euler scheme with  $\kappa = 300$  and  $\beta = 0.9$ .

| $n$    | Picard iterations | Time  | Truncated Euler scheme | Time  | Error |
|--------|-------------------|-------|------------------------|-------|-------|
| 1000   | 28.1349           | 6.282 | 28.3481                | 1.661 | 0.76% |
| 10 000 | 28.41             | 64.05 | 28.32                  | 17.25 | 0.31% |

Table 11: Asian Put option prices with three dates using Picard iterations or the truncated Euler scheme.

**Conclusion:** in the case where  $\beta \in [0, 1)$ , the simple truncated Euler scheme seems to be the best method as the computation time is small and the errors are satisfactory.

## 5. Proofs

Throughout the paper, we shall use the following convention: from a line to the next one, constants  $K, \kappa$  or  $C$  may designate different constants which are independent of any other current variables except maybe the fixed parameters of the model as the maturity date  $T$  or Lipschitz constants. In the contrary case, we shall use notations like  $C_m$  to stress the dependence on some parameter  $m$ .

*Proof of Lemma 2.6.* Let us denote  $S = S_{r,\sigma} = F(r)\mathcal{E}(\sigma)$ . Using the integration by part formula, we obtain that

$$S_t t = \int_0^t S_u du + \int_0^t u dS_u, \quad t \in [0, T].$$

On the other hand, a direct computation shows that

$$\int_0^T S_u^{(n)} du = - \int_0^T u dS_u^{(n)} + T S_T^{(n)}.$$

Since  $S_T^{(n)} = S_T$ , we deduce that

$$\int_0^T S_u du - \int_0^T S_u^{(n)} du = \int_0^T u d(S_u^{(n)} - S_u).$$

Observe that

$$\begin{aligned}
\int_0^T u dS_u^{(n)} &= \sum_{i=1}^n t_i^n \Delta S_{t_i^n}, \\
&= \sum_{i=1}^n t_i^n \int_{t_{i-1}^n}^{t_i^n} \sigma(u, S_u) S_u dW_u + \sum_{i=1}^n t_i^n \int_{t_{i-1}^n}^{t_i^n} r(u, S_u) S_u du, \\
&= \int_0^T I^{(n)}(u) \sigma(u, S_u) S_u dW_u + \int_0^T I^{(n)}(u) r(u, S_u) S_u du,
\end{aligned}$$

where  $I^{(n)}(u) = t_i$  if  $u \in [t_{i-1}, t_i]$ . It follows that

$$\int_0^T S_u du - \int_0^T S_u^{(n)} du = \int_0^T (I^{(n)}(u) - u) \sigma(u, S_u) S_u dW_u + \int_0^T (I^{(n)}(u) - u) r(u, S_u) S_u du$$

As  $|u^{(n)}(u) - u| \leq T/n$  and  $r$  and  $\sigma$  are supposed to be Lipschitz, we deduce by the Cauchy-Schwarz inequality and the Ito isometry the following sequence of inequalities:

$$\begin{aligned}
\mathbf{E}|G(S) - G(S^{(n)})| &\leq C_T \mathbf{E} \left| \int_0^T (I^{(n)}(u) - u) \sigma(u, S_u) S_u dW_u \right| \\
&\quad + C_T \mathbf{E} \left| \int_0^T (I^{(n)}(u) - u) r(u, S_u) S_u du \right|,
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}|G(S) - G(S^{(n)})| &\leq C_T \sqrt{C(\sigma) \int_0^T (I^{(n)}(u) - u)^2 \mathbf{E}(1 + S_u^4) du} \\
&\quad + C_T(r) \int_0^T |I^{(n)}(u) - u| \mathbf{E}(1 + S_u^2) du, \\
&\leq \frac{C_G \sqrt{C(\sigma)}}{n} \sqrt{\int_0^T (1 + \mathbf{E}S_u^4) du} \\
&\quad + \frac{C_T(r)}{n} \int_0^T \mathbf{E}(1 + S_u^2) du, \\
&\leq \frac{C_G \sqrt{C(\sigma)}}{n} \sqrt{\int_0^T (1 + \mathbf{E}S_u^4) du} \\
&\quad + \frac{C_T(r)}{n} \sqrt{\int_0^T \mathbf{E}(1 + S_u^2)^2 du}.
\end{aligned}$$

The conclusion follows.  $\square$

**Lemma 5.1.** *Suppose that Condition 2.3 holds. Let  $\kappa > 0$ . Then, there exists a constant  $C_\kappa$  depending on  $\kappa$  such that for all  $x, y > 0$  and  $t \in \mathbf{R}_+$ ,*

$$|\sigma_\kappa(t, x) - \sigma_\kappa(t, y)| \leq C_\kappa \sqrt{|x - y|}.$$

*Proof.* Under Conditions 2.3, we obtain the following chain of inequalities

$$\begin{aligned} |\sigma_\kappa(t, x) - \sigma_\kappa(t, y)| &= |\sigma_\kappa(t, x \vee \kappa^{-1}) - \sigma_\kappa(t, y \vee \kappa^{-1})|, \\ &\leq C_\kappa |\ln(x \vee \kappa^{-1}) - \ln(y \vee \kappa^{-1})|, \\ &\leq C_\kappa |\ln(x) \vee \ln(\kappa^{-1}) - \ln(y) \vee \ln(\kappa^{-1})|, \\ &\leq 2C_\kappa |\ln(\sqrt{x}) \vee \ln(\kappa^{-1/2}) - \ln(\sqrt{y}) \vee \ln(\kappa^{-1/2})|, \\ &\leq C_\kappa |\sqrt{x} - \sqrt{y}| \leq C_\kappa \sqrt{|x - y|}. \end{aligned}$$

The last inequalities are due to the fact that the mapping  $z \mapsto \ln(z) \vee \ln(\kappa^{-1/2})$  is Lipschitz and the square root function is Holder continuous of order 1/2.  $\square$

**Lemma 5.2.** *Suppose that Condition 2.2 holds. Let  $\kappa > 0$ . Then, there exists a constant  $C_\kappa$  depending on  $\kappa$  such that for all  $X, Y \in \mathbf{R}^2$ ,  $x, y > 0$  and  $t \in \mathbf{R}_+$ ,*

$$\begin{aligned} |\sigma_\kappa(t, e^X) - \sigma_\kappa(t, e^Y)| &\leq C_\kappa |X - Y|, \\ |\sigma_\kappa(t, x) - \sigma_\kappa(t, y)| &\leq C_\kappa \sqrt{|x - y|}. \end{aligned}$$

*Proof.*

Using the assumption and the fact that the mapping  $X \mapsto e^X \wedge \kappa$  is Lipschitz, we get that

$$\begin{aligned} |\sigma_\kappa(t, e^X) - \sigma_\kappa(t, e^Y)| &= |\sigma_\kappa(t, e^X \wedge \kappa) - \sigma_\kappa(t, e^Y \wedge \kappa)|, \\ &\leq C_\kappa |e^X \wedge \kappa - e^Y \wedge \kappa| \leq C_\kappa |X - Y|. \end{aligned}$$



On the other hand, we have

$$\begin{aligned}
|\sigma_\kappa(t, x) - \sigma_\kappa(t, y)| &= |\sigma_\kappa(t, x \wedge \kappa) - \sigma_\kappa(t, y \wedge \kappa)|, \\
&\leq C_\kappa |x \wedge \kappa - y \wedge \kappa| \leq C_\kappa \sqrt{|x \wedge \kappa - y \wedge \kappa|}, \\
&\leq C_\kappa \sqrt{|x - y|}.
\end{aligned}$$

The last inequalities hold since  $\sqrt{|x \wedge \kappa - y \wedge \kappa|}$  is bounded by  $\sqrt{2\kappa}$  and the mapping  $x \mapsto x \wedge \kappa$  is Lipschitz.  $\square$

*Proof of Lemma 3.1.* Suppose that Condition 2.2 holds. Consider the sequence of stopping times

$$\tau^\kappa := \inf\{t : S_t \geq \kappa\}. \quad (5.12)$$

Since the stopped processes  $S^{\tau^\kappa}$  and  $(S_\kappa)^{\tau^\kappa}$  satisfy the same s.d.e., we deduce that  $S_t = S_\kappa(t)$  on  $t \in [0, \tau^\kappa]$ . As  $\tau^\kappa \rightarrow \infty$ , we then conclude. Under Condition 2.3, we follow the same arguments with

$$\tau^\kappa := \inf\{t : S_t \leq \kappa^{-1}\}. \quad (5.13)$$

Note that  $\tau^\kappa \rightarrow \tau$  as  $\kappa \rightarrow \infty$ .  $\square$

**Lemma 5.3.** *Assume that Condition 3.3 (I) is satisfied. Then, there exists a constant  $C$  independent of  $\kappa$  such that  $\sup_{u \leq T} \mathbf{E}S_\kappa(u) \leq C$ .*

*Proof.* By definition, we have  $S_\kappa(u)/F_u(r_\kappa(\cdot, S_\kappa(\cdot))) \leq M_u$  where  $M_0 = 1$  and  $M$  is a local martingale which is solution to the s.d.e.

$$dM_u = M_u \sigma_\kappa(u, S_\kappa(u)) dW_u.$$

As  $r$  is supposed to be bounded, we deduce that  $S_\kappa(u) \leq CS_0 M_u$ . Let  $(\tau^n)$  be a sequence of stopping times such that  $M^{\tau^n}$  is a true martingale. Then,  $\mathbf{E}S_\kappa(u \wedge \tau^n) \leq CS_0$  and finally  $\mathbf{E}S_\kappa(u) \leq CS_0$  by virtue of by Fatou's lemma.  $\square$

**Lemma 5.4.** *Assume that Assumption 3.3 (I) is satisfied. Then, for all  $p$  such that  $p \leq 0$  or  $p \geq 1/2$ , there exists a constant  $C_p$  independent of  $\kappa$  such that  $\sup_{u \leq T} \mathbf{E}(S_\kappa(u))^p \leq C_p$ .*

*Proof.* Since  $r$  is bounded, the following inequality holds for all fixed  $q \geq 0$ :

$$(S_\kappa(u))^p \leq C_q S_0^p N_u e^{\frac{1}{2}(q-p) \int_0^u \sigma_\kappa^2(r, S_\kappa(r)) dr},$$

where

$$N_u := \exp\left\{p \int_0^u \sigma_\kappa(r, S_\kappa(r)) dW_r - \frac{1}{2}q \int_0^u \sigma_\kappa^2(r, S_\kappa(r)) dr\right\}.$$

Using the inequality  $0 \leq ab \leq (a^2 + b^2)$ , we get that

$$(S_\kappa(u))^p \leq C S_0^p \tilde{N}_u + C S_0^p e^{(q-p) \int_0^u \sigma^2(r, S_\kappa(r) \wedge \kappa) dr},$$

where  $\tilde{N} = N^2$  is a local martingale if we choose  $q = 2p^2$ . Moreover, as the function  $x \mapsto e^{(q-p)ux}$  is convex, the Jensen inequality and the hypothesis yields that

$$\begin{aligned} e^{(q-p) \int_0^u \sigma^2(r, S_\kappa(r) \wedge \kappa) dr} &\leq \frac{1}{u} \int_0^u e^{(q-p)r\sigma^2(r, S_\kappa(r) \wedge \kappa)} dr \\ &\leq C_p + \frac{C}{u} \int_0^u \left( (\log(S_\kappa(r)))^{k(q-p)T} 1_{S_\kappa(r) \wedge \kappa \geq 1} + 1_{S_\kappa(r) \wedge \kappa \leq 1} \right) dr, \end{aligned}$$

where  $k$  is a constant. Using the property

$$(\log(x \vee 1))^{k(q-p)T} \leq C_p x, \quad \forall x \geq 0,$$

and Lemma 5.3, we deduce that  $E(S_\kappa(u))^p \leq C_p$ .  $\square$

**Lemma 5.5.** *Assume that Condition 3.3 (II) is satisfied. Then, for all  $p$  such that  $p \geq 0$ , there exists a constant  $C_p$  independent of  $\kappa$  such that  $\sup_{u \leq T} \mathbf{E}(S_\kappa(u))^p \leq C_p$ .*

*Proof.* Let us consider  $N = N(p) \in \mathbf{N}$  such that  $0 \leq p \leq 2N\alpha$ . Observe that, for  $p \geq 0$ ,  $0 \leq (S_\kappa(u))^p \leq 1 + (S_\kappa(u))^{2N\alpha} 1_{S_\kappa(u) \geq 1} \leq 1 + (S_\kappa(u))^{2N\alpha}$ . Therefore, it is enough to show the lemma when  $p = 2N\alpha$ . We show by induction that the statement holds for every  $p = 2j\alpha$ ,  $0 \leq j \leq N$ . This is trivial if  $j = 0$ . Applying the Ito formula when  $q = 2(j+1)\alpha$ ,  $j \geq 0$ , we get that

$$\begin{aligned} (S_\kappa(u))^q &= (S_0)^q + \int_0^u q(S_\kappa(r))^q \sigma_\kappa(r, S_\kappa(r)) dW_r + \int_0^u q(S_\kappa(r))^q r_\kappa(r, S_\kappa(r)) dr, \\ &\quad + \frac{1}{2}q(q-1) \int_0^u (S_\kappa(r))^q \sigma_\kappa^2(r, S_\kappa(r)) dr. \end{aligned}$$

Recall that  $\sigma_\kappa$  is bounded. Using the Doleans-Dade representation of  $S_\kappa(u)$ , we deduce that  $S_\kappa(u)$  admits finite moments at any order. It follows that the Ito integral in the inequality above is a square integrable martingale. Therefore,

$$\begin{aligned}\mathbf{E}(S_\kappa(u))^q &= (S_0)^q + \int_0^u q\mathbf{E}(S_\kappa(r))^q r_\kappa(r, S_\kappa(r))dr, \\ &\quad + \frac{1}{2}q(q-1) \int_0^u \mathbf{E}(S_\kappa(r))^q \sigma_\kappa^2(r, S_\kappa(r))dr.\end{aligned}$$

We then use the inequality  $r \leq 1 + r^2$  and by Assumption 3.3 (II), we deduce that  $g^{j+1}(u) := \mathbf{E}(S_\kappa(u))^{2(j+1)\alpha}$  satisfies

$$\begin{aligned}g^{j+1}(u) &\leq \max_{j \leq N} (S_0)^{2(j+1)\alpha} + C(N^2\alpha^2 + N\alpha) \int_0^u g^j(r)dr \\ &\quad + C(N\alpha + N^2\alpha^2) \int_0^u g^{j+1}(r)dr\end{aligned}$$

Suppose that  $g^j \leq C^j$  where  $C^j$  does not depend on  $\kappa$ . Then, by Gronwall's lemma, we deduce that  $g^{j+1} \leq C^{j+1}$  where  $C^{j+1}$  does not depend on  $\kappa$ . The conclusion follows with  $j = N$  since the number of steps  $N(p)$  is finite and only depends on  $p$ .  $\square$

**Corollary 5.6.** *Assume that Condition 3.3 is satisfied. Then, for all  $p \geq 0$ , there exists a constant  $C_p$  independent of  $\kappa$  such that  $\mathbf{E} \sup_{u \leq T} (S_\kappa(u))^p \leq C_p$ .*

*Proof.* Since  $\|\cdot\|_{L^{p_1}} \leq \|\cdot\|_{L^{p_2}}$  if  $1 \leq p_1 \leq p_2$ , and  $(S_\kappa(u))^p \leq 1 + S_\kappa(u)$  when  $p \in [0, 1]$ , we may assume without loss of generality that  $p \in 2\mathbb{N}$ . Let us first suppose that Condition (I) holds. We recall that

$$S_\kappa(t) = S_0 + \int_0^t \sigma(u, S_\kappa(u) \wedge \kappa) S_\kappa(u) dW_u + \int_0^t r(u, S_\kappa(u) \wedge \kappa) S_\kappa(u) du.$$

Since  $0 \leq \sigma^2(t, x) \leq C(1+x)$  and  $r$  is bounded, we easily deduce that  $\mathbf{E} \sup_{u \leq T} (S_\kappa(u))^p \leq C_p$  by the Burkholder-Davis-Gundy and Jensen inequalities as well as Lemma 5.4.

Let us now suppose that Condition (II) holds. In this case,

$$0 \leq S_\kappa^p(u) (\sigma^p(u, S_\kappa(u) \vee \kappa^{-1}) + r^p(u, S_\kappa(u) \vee \kappa^{-1})) \leq C(S_\kappa^p(u) + S_\kappa^{(1-\alpha)p}(u)).$$

Since  $1 - \alpha \geq 0$ , we use Lemma 5.5 and conclude as previously.  $\square$

Since  $S_\kappa$  converges pointwise on  $[0, \tau]$  to  $S$ ,  $S = S_\kappa$  on  $[0, \tau^\kappa]$  where  $\tau^\kappa \rightarrow \tau$ , and  $S_t = 0$  on  $t \geq \tau$ , we apply Fatou's Lemma and deduce the following.

**Corollary 5.7.** *Assume that Condition 3.3 is satisfied. Then, for all  $p \geq 0$ , there exists a constant  $C$  such that  $E \sup_{u \leq T} (S_u)^p \leq C_p$ .*

**Remark 5.8.** *Recall that*

$$dY_t = \sigma(t, S_t)dW_t + r(t, S_t)dt - \frac{1}{2}\sigma^2(t, S_t)dt.$$

Under Condition 3.3 (I), since  $0 \leq \sigma^2(t, x) \leq C(1 + x)$ , we then deduce that  $E \sup_{u \leq T} Y_u^2 < \infty$  as the coefficient  $r$  is bounded. In particular, the process  $\int_0^\cdot \sigma(t, S_t)dW_t$  is a true martingale.

*Proof of Lemma 3.4.* If  $\sigma_\kappa = \sigma(\cdot, \cdot \wedge \kappa)$ , we consider the stopping time  $\tau^k$  defined by (5.12). We obtain that

$$\begin{aligned} \mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p &= \mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p 1_{\tau^\kappa < t} \\ &\leq \mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p 1_{\tau^\kappa < T}. \end{aligned}$$

In the case where  $\tau^k$  is defined by (5.12), we deduce that

$$\begin{aligned} \mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p &\leq \mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p 1_{\sup_{t \in [0, T]} S_t \geq \kappa} \\ &\leq \sqrt{\mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^{2p}} \sqrt{\frac{\mathbf{E} (\sup_{t \in [0, T]} S_t)^{2l}}{\kappa^{2l}}}. \end{aligned}$$

The conclusion follows. In the case where  $\sigma_\kappa = \sigma(\cdot, \cdot \vee \kappa^{-1})$ , let  $\tau^k$  be defined by (5.13). Then,

$$\begin{aligned} \mathbf{E} \sup_{t \in [0, \tau \wedge T]} |S_\kappa(t) - S_t|^p &= \mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p 1_{\tau^\kappa < t \leq \tau \wedge T} \\ &\leq \mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p 1_{\tau^\kappa < t \leq \tau} 1_{\tau \leq T} \\ &\quad + \mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p 1_{\inf_{t \in [0, T]} S_t \leq \kappa^{-1}} 1_{\tau > T}. \end{aligned}$$

Notice that the second term of the last inequality above is bounded as follows:

$$\mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p \mathbf{1}_{\inf_{t \in [0, T]} S_t \leq \kappa^{-1}} \mathbf{1}_{\tau > T} \leq \sqrt{\mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^{2p}} \gamma(\tau, \kappa),$$

where  $\gamma(\tau, \kappa) := P(\inf_{t \in [0, T]} S_t \leq \kappa^{-1}; \tau > T) \rightarrow 0$  as  $\kappa \rightarrow \infty$ . If  $t \in [\tau^\kappa, \tau]$ , where  $\tau < T$ , then

$$\begin{aligned} S_\kappa(t) &= \kappa^{-1} + \int_{\tau^\kappa}^t S_\kappa(u) \sigma_\kappa(u, S_\kappa(u)) dW_u + \int_{\tau^\kappa}^t S_\kappa(u) r_\kappa(u, S_\kappa(u)) du, \\ S_t &= \kappa^{-1} + \int_{\tau^\kappa}^t S(u) \sigma(u, S(u)) dW_u + \int_{\tau^\kappa}^t S(u) r(u, S(u)) du. \end{aligned}$$

Under Condition (II) of Condition 3.3, we have

$$\begin{aligned} |S_\kappa(u) \sigma_\kappa(u, S_\kappa(u))| + |S_\kappa(u) r_\kappa(u, S_\kappa(u))| &\leq C(|S_\kappa(u)| + |S_\kappa(u)|^{1-\alpha}), \\ |S(u) \sigma(u, S(u))| + |S(u) r(u, S(u))| &\leq C(|S(u)| + |S(u)|^{1-\alpha}). \end{aligned}$$

Using the Burkholder-Davis-Gundy inequalities and the Cauchy-Schwarz inequality, we deduce that the term  $\delta(\tau, \kappa) = \mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t) - S_t|^p \mathbf{1}_{\tau^\kappa < t < \tau} \mathbf{1}_{\tau \leq T}$  is bounded as follows:

$$\delta(\tau, \kappa) \leq C \left( \sup_{\kappa} \sqrt{\mathbf{E} \sup_{t \in [0, T]} |S_\kappa(t)|^{2p}} + \sqrt{\mathbf{E} \sup_{t \in [0, T]} |S(t)|^{2p}} \right) \sqrt{\mathbf{E} |\tau^\kappa \wedge T - \tau \wedge T|^p}.$$

By the Lebesgue theorem, since  $\tau^\kappa \rightarrow \tau$ , we deduce that  $\mathbf{E} |\tau^\kappa \wedge T - \tau \wedge T|^p \rightarrow 0$  as  $\kappa \rightarrow \infty$ . The conclusion follows.  $\square$

**Lemma 5.9.** *Let  $(g^m)_m$  be a sequence of positive functions defined on an interval  $[0, T]$ ,  $T > 0$ , such that for some  $C > 0$ , we have:*

$$g^{m+1}(t) \leq C \int_0^t g^m(u) du, \quad 0 \leq g^0 \leq C.$$

Then

$$\sup_{t \in [0, T]} g^m(t) \leq C^m \frac{t^m}{m!} \sup_{t \in [0, T]} g^0(t).$$

*Proof.* The proof relies on the following recursive reasoning. Set

$$C_g := \sup_{t \in [0, T]} g^0(t) \leq C.$$

Suppose that

$$\sup_{t \in [0, T]} g^{n-1}(t) \leq C^{n-1} \frac{t^{n-1}}{(n-1)!} C_g.$$

Then, we have

$$g^n(t) \leq C \int_0^t C^{n-1} \frac{u^{n-1}}{(n-1)!} C_g du.$$

The claim follows.  $\square$

*Proof of Lemma 3.5.* If  $\alpha \geq 2$ , recall that for all  $x, y \geq 0$ :

$$|x^{\alpha/2} - y^{\alpha/2}| \leq \frac{\alpha}{2} (\max(|x|, |y|))^{\alpha/2-1} |x - y|.$$

We deduce by Conditions 2.2 and 2.3 that

$$\left( \int_0^t \sigma_{\kappa, m+1}^2(s) - \sigma_{\kappa, m}^2(s) ds \right)^2 \leq CC_\kappa \int_0^t (\sigma_{\kappa, m+1}(s) - \sigma_{\kappa, m}(s))^2 ds,$$

by virtue of Jensen's inequality. Together with the Burkholder–Davis–Gundy inequalities, we obtain that

$$\begin{aligned} \mathbf{E} \sup_{u \leq t} |Y_{\kappa, m+1}(u) - Y_{\kappa, m}(u)|^2 &\leq CC_\kappa \mathbf{E} \int_0^t \sup_{u \leq s} |\sigma_{\kappa, m+1}(u) - \sigma_{\kappa, m}(u)|^2 ds \\ &\quad + CC_\kappa \mathbf{E} \int_0^t \sup_{u \leq s} |r_{\kappa, m+1}(u) - r_{\kappa, m}(u)| ds. \end{aligned}$$

By Conditions 2.2 or 2.3, with Lemma 5.2,

$$|\sigma_\kappa(t, e^X) - \sigma(t, e^Y)| + |r_\kappa(t, e^X) - r(t, e^Y)| \leq C_\kappa |X - Y|,$$

hence

$$|\sigma_{\kappa, m+1}(u) - \sigma_{\kappa, m}(u)| + |r_{\kappa, m+1}(u) - r_{\kappa, m}(u)| \leq C_\kappa |Y_{\kappa, m}(u) - Y_{\kappa, m-1}(u)|.$$

We deduce that

$$\mathbf{E} \sup_{u \leq t} |Y_{\kappa, m+1}(u) - Y_{\kappa, m}(u)|^2 \leq C_\kappa \int_0^t \mathbf{E} \sup_{u \leq s} |Y_{\kappa, m}(u) - Y_{\kappa, m-1}(u)|^2 ds,$$

where  $C_\kappa$  is a constant depending on  $\kappa$ . Using Lemma 5.9, we get that

$$\begin{aligned} \mathbf{E} \sup_{u \leq t} |Y_{\kappa, m+1}(u) - Y_{\kappa, m}(u)|^2 &\leq (C_\kappa)^{m+1} \mathbf{E} \sup_{u \leq t} |Y_{\kappa, 1}(u) - Y_{\kappa, 0}(u)|^2 \frac{t^m}{m!}, \\ &\leq C(C_\kappa)^{m+1} \frac{t^{m+1}}{(m+1)!} (1 + S_0^2), \end{aligned} \quad (5.14)$$

where  $C$  does not depend on  $m$  and  $\kappa$ . We deduce Inequality (3.7). Similarly, we obtain Inequality (3.8) and the claim follows.  $\square$

**Lemma 5.10.** *Let  $a_1, \dots, a_k$  be real numbers. we have the following inequality*

$$\left( \sum_{l=1}^k a_l \right)^2 \leq \sum_{l=1}^k 2^l a_l^2, \quad \left( \sum_{l=1}^k a_l \right)^4 \leq \sum_{l=1}^k 8^l a_l^4.$$

*Proof.* For any real numbers  $a, b$ , the inequality  $2ab \leq a^2 + b^2$  implies that  $(a + b)^2 \leq 2a^2 + 2b^2$ . Therefore,  $(a + b)^4 \leq (2a^2 + 2b^2)^2 \leq 8a^4 + 8b^4$ . We conclude by induction.  $\square$

*Proof of Corollary 3.6.* Using Lemma 5.10, we obtain the inequality

$$\mathbf{E} \sup_{u \leq t} |Y_{\kappa, \infty}(u) - Y_{\kappa, m}(u)|^2 \leq \sum_{j=m}^{\infty} 2^j \mathbf{E} \sup_{u \leq t} |Y_{\kappa, j+1}(u) - Y_{\kappa, j}(u)|^2.$$

We deduce the first inequality from (5.14) and, similarly, the second one. The conclusion follows.  $\square$

*Proof of Lemma 3.7.* Applying the Burkholder–Davis–Gundy and Jensen inequalities, we get that

$$\begin{aligned} \mathbf{E} \sup_{u \in [0, t]} (S_{\kappa, m+1}(u) - S_{\kappa, m}(u))^2 &\leq C \mathbf{E} \int_0^t (S_{\kappa, m+1}(s) \sigma_{\kappa, m+1}(s) - S_{\kappa, m}(s) \sigma_{\kappa, m}(s))^2 ds, \\ &\quad + 2 \mathbf{E} \int_0^t (S_{\kappa, m+1}(s) r_{\kappa, m+1}(s) - S_{\kappa, m}(s) r_{\kappa, m}(s))^2 ds, \end{aligned}$$

where we recall that  $\sigma_{\kappa, m}(s) := \sigma_{\kappa}(s, S_{\kappa, m-1}(s))$ . The first term in the r.h.s. of the inequality is bounded from above as follows:

$$\begin{aligned} &\mathbf{E} \int_0^t (S_{\kappa, m+1}(s) \sigma_{\kappa, m+1}(s) - S_{\kappa, m}(s) \sigma_{\kappa, m}(s))^2 ds \leq \\ &2 \mathbf{E} \int_0^T S_{\kappa, m+1}^2(s) [\sigma_{\kappa, m+1}(s) - \sigma_{\kappa, m}(s)]^2 ds + 2 \mathbf{E} \int_0^T (\sigma_{m, \kappa}(s) (S_{\kappa, m+1}(s) - S_{\kappa, m}(s)))^2 ds. \end{aligned}$$

The second one is similarly bounded. Recall the inequalities  $|\sigma_{\kappa}| + |r_{\kappa}| \leq C_{\kappa}$ ,  $\kappa > 0$ , and recall that, by assumption or by Lemma 5.2,

$$|\sigma_{\kappa}(t, e^X) - \sigma_{\kappa}(t, e^Y)| + |r_{\kappa}(t, e^X) - r_{\kappa}(t, e^Y)| \leq C_{\kappa} |X - Y|. \quad (5.15)$$

Moreover, using Gronwall's lemma, since  $\sigma_{\kappa}$  and  $r_{\kappa}$  are bounded by a constant depending on  $\kappa$ , we deduce the existence of  $C_{p, \kappa} > 0$  such that for all  $p$

$$\sup_m \mathbf{E} \sup_{t \in [0, T]} S_{\kappa, m}^{2p}(t) \leq C_{p, \kappa}.$$

Using the stochastic integral representation of  $S_{\kappa,m+1}$  and  $S_{\kappa,m}$  respectively, we obtain by means of Cauchy-Schwarz's inequality, Jensen's inequality and the Ito isometry that

$$\begin{aligned} \mathbf{E} \sup_{u \in [0,t]} (S_{\kappa,m+1}(u) - S_{\kappa,m}(u))^2 &\leq C_\kappa \sqrt{\mathbf{E} \sup_{u \leq T} |Y_{\kappa,m}(u) - Y_{\kappa,m-1}(u)|^4} \\ &+ C_\kappa \int_0^t \mathbf{E} \sup_{u \leq s} (S_{\kappa,m+1}(u) - S_{\kappa,m}(u))^2 ds. \end{aligned}$$

Using Gronwall's Lemma, we deduce that

$$\mathbf{E} \sup_{t \in [0,T]} (S_{\kappa,m+1}(t) - S_{\kappa,m}(t))^2 \leq C_\kappa \sqrt{\mathbf{E} \sup_{t \leq T} |Y_{\kappa,m}(t) - Y_{\kappa,m-1}(t)|^4}.$$

The conclusion follows.  $\square$

*Proof of Lemma 3.8.* By virtue of Lemma 5.10, we get the inequality

$$\begin{aligned} \mathbf{E} \sup_{t \in [0,T]} (S_{\kappa,\infty}(t) - S_{\kappa,m}(t))^2 &\leq \sum_{j=m}^{\infty} 2^j \mathbf{E} \sup_{u \in [0,t]} (S_{\kappa,j+1}(u) - S_{\kappa,j}(u))^2 \\ &\leq C_\kappa \sum_{j=m}^{\infty} \frac{(C_\kappa)^j}{\sqrt{j!}}. \end{aligned}$$

$\square$

*Proof of Lemma 3.9.* Since  $\sigma_\kappa$  and  $r_\kappa$  are bounded by a constant depending on  $\kappa$ , we deduce by the Burkholder-Davis-Gundy inequalities and the Gronwall's lemma a constant  $c_\kappa > 0$  such that for all  $m, n$ ,

$$\mathbf{E} \sup_{t \in [0,T]} |S_{\kappa,m}(t) + S_{\kappa,m,n}(t)|^4 \leq c_\kappa. \quad (5.16)$$

For a sake of simplicity, we suppose that  $r = 0$  as the proof is similar for the general case. Therefore,

$$\begin{aligned} S_{\kappa,1}(t) &= S_0 + \int_0^t S_{\kappa,1}(u) \sigma_\kappa(u, S_0) dW_u, \\ S_{\kappa,1,n}(t) &= S_0 + \int_0^t S_{\kappa,1,n}(u) \sigma_\kappa(u^{(n)}, S_0) dW_u. \end{aligned}$$

Let us introduce  $g_m(t) := \mathbf{E} \sup_{u \in [0,t]} |S_{\kappa,m}(u) - S_{\kappa,m,n}(u)|^2$ ,  $m \geq 1$ . By the



Doob's inequality,

$$\begin{aligned}
g_1(t) &\leq 4 \int_0^t (S_{\kappa,1,n}(u)\sigma_\kappa(u^{(n)}, S_0) - S_{\kappa,1}(u)\sigma_\kappa(u, S_0))^2 du, \\
g_1(t) &\leq 8 \int_0^t \sigma_\kappa^2(u^{(n)}, S_0) (S_{\kappa,1,n}(u) - S_{\kappa,1}(u))^2 du \\
&\quad + 8 \int_0^t S_{\kappa,1}^2(u) (\sigma_\kappa(u^{(n)}, S_0) - \sigma_\kappa(u, S_0))^2 du.
\end{aligned}$$

Since  $\sigma_\kappa$  is bounded by a constant  $C_\kappa$  and  $|\sigma_\kappa(u^{(n)}, S_0) - \sigma_\kappa(u, S_0)| \leq T/n$ , we deduce by (5.16) that

$$g_1(t) \leq \frac{C_\kappa}{n^2} + C_\kappa \int_0^t g(u) du, \quad t \in [0, T].$$

By the Gronwall's lemma, we deduce that  $g_1(T) \leq C_\kappa/n^2$  for some constant  $C_\kappa$  which does not depend on  $n$ . More generally, if  $m \geq 2$ , we have

$$\begin{aligned}
S_{\kappa,m}(t) &= S_0 + \int_0^t S_{\kappa,m}(u)\sigma_\kappa(u, S_{\kappa,m-1}(u))dW_u, \\
S_{\kappa,m,n}(t) &= S_0 + \int_0^t S_{\kappa,m,n}(u)\sigma_\kappa(u^{(n)}, S_{\kappa,m-1,n}(u))dW_u.
\end{aligned}$$

Therefore, by the Doob's inequality,

$$\begin{aligned}
g_m(t) &\leq 4 \int_0^t (S_{\kappa,m,n}(u)\sigma_\kappa(u^{(n)}, S_{\kappa,m-1,n}(u)) - S_{\kappa,m}(u)\sigma_\kappa(u, S_{\kappa,m-1}(u)))^2 du, \\
g_m(t) &\leq 8 \int_0^t \sigma_\kappa^2(u^{(n)}, S_{\kappa,m-1,n}(u)) (S_{\kappa,m,n}(u) - S_{\kappa,m}(u))^2 du \\
&\quad + 8 \int_0^t S_{\kappa,m}^2(u) (\sigma_\kappa(u^{(n)}, S_{\kappa,m-1,n}(u)) - \sigma_\kappa(u, S_{\kappa,m-1,n}(u)))^2 du \\
&\quad + 8 \int_0^t S_{\kappa,m}^2(u) (\sigma_\kappa(u, S_{\kappa,m-1,n}(u)) - \sigma_\kappa(u, S_{\kappa,m-1}(u)))^2 du.
\end{aligned}$$

Recall that  $\sigma_\kappa$  is bounded by a constant  $C_\kappa$  and

$$|\sigma_\kappa(u^{(n)}, S_{\kappa,m-1,n}(u)) - \sigma_\kappa(u, S_{\kappa,m-1,n}(u))| \leq C_\kappa/n.$$

Therefore, we may obtain upper bounds for the two first terms of the r.h.s. in the inequality above as we do for  $m = 1$ . At last, an upper bound for the

third term is obtained using the Cauchy-Schwarz inequality, (5.16) and the following inequality (see Lemmas 5.2 and 5.1):

$$|\sigma_\kappa(u, S_{\kappa, m-1, n}(u)) - \sigma_\kappa(u, S_{\kappa, m-1}(u))| \leq C_\kappa \sqrt{|S_{\kappa, m-1, n}(u) - S_{\kappa, m-1}(u)|}.$$

Therefore,

$$g_m(t) \leq \frac{C_\kappa}{n^2} + C_\kappa g_{m-1}(t) + C_\kappa \int_0^T g_m(u) du.$$

By the Gronwall's lemma, we then deduce a constant  $C_\kappa$  which only depends on  $\kappa$  such that

$$g_m(T) \leq \frac{C_\kappa}{n^2} + C_\kappa g_{m-1}(T).$$

By induction, we deduce that, for all  $m \geq 2$ ,

$$g_m(T) \leq n^{-2} \sum_{j=1}^m (C_\kappa)^j \leq \alpha_\kappa (C_\kappa)^m n^{-2},$$

for some constant  $\alpha_\kappa$ .  $\square$

**Lemma 5.11.** *Let  $S$  be a positive stochastic process such that  $P(\inf_{t \in [0, T]} S_t = a) = 0$  for all  $a \in (0, S_0)$ . Let  $(S^n)_{n \in \mathbb{N}}$  be a sequence of stochastic processes such that  $\mathbf{E} \sup_{t \in [0, T]} \|S_t - S_t^n\|^2 \rightarrow 0$  as  $n \rightarrow \infty$ . Let us define  $\tau^S := \inf\{t : S_t < a\}$  and  $\tau^{S^n} := \inf\{t : S_t^n < a\}$ . Then,  $\mathbf{E}|1_{\tau^S > T} - 1_{\tau^{S^n} > T}| \rightarrow 0$  as  $n \rightarrow \infty$ .*

*Proof.* Suppose that  $\mathbf{E}|1_{\tau^S > T} - 1_{\tau^{S^n} > T}| \rightarrow l > 0$  for a subsequence. Since  $|\inf_{t \in [0, T]} S_t - \inf_{t \in [0, T]} S_t^n| \leq \sup_{t \in [0, T]} \|S_t - S_t^n\|$ , we may also suppose that  $\inf_{t \in [0, T]} S_t^n$  converges to  $\inf_{t \in [0, T]} S_t$  a.s. for a common subsequence. In the case where  $\tau > T$ , i.e.  $\inf_{t \in [0, T]} S_t \geq a$ , as  $P(\inf_{t \in [0, T]} S_t = a) = 0$ , we may assume that  $\inf_{t \in [0, T]} S_t > a$  hence  $\inf_{t \in [0, T]} S_t^n > a$  for  $n$  large enough and, finally  $\tau^{S^n} > T$ . If  $\tau \leq T$ ,  $\inf_{t \in [0, T]} S_t^n < a$  and then, similarly, we obtain that  $\tau^{S^n} \leq T$  for  $n$  large enough. We deduce that  $|1_{\tau^S > T} - 1_{\tau^{S^n} > T}| \rightarrow 0$  a.s. in contradiction with  $l > 0$ .  $\square$

**Lemma 5.12.** *Let  $S$  be a positive stochastic process such that  $P(\inf_{t \in [0, T]} S_t = a) = 0$  for all  $a > 0$ . Let  $(S_n)_{n \in \mathbb{N}}$  be a sequence of stochastic processes such*

that  $\mathbf{E} \sup_{t \in [0, T]} |S_t - S_n(t)|^2 \rightarrow 0$  as  $n \rightarrow \infty$ . Let us denote by  $S_n^{(n)}$  its discrete version given in Section 2.1. Let us define  $\tau^S := \inf\{t : S_t < a\}$  and  $\tau^{S_n^{(n)}} := \inf\{t : S_n^{(n)}(t) < a\}$ . Then,  $\mathbf{E}|1_{\tau^S > T} - 1_{\tau^{S_n^{(n)}} > T}| \rightarrow 0$ .

*Proof.* Recall that  $\tau^S > T$  if and only if  $\inf_{t \in [0, T]} S_t \geq a$ . Let us first show that, for any random interval  $I \subseteq [0, T]$ ,  $\sup_{t \in I} S_n(t)$  converges in  $L^2$  to  $\sup_{t \in I} S_t$  as  $n \rightarrow \infty$ . To see it, note that for all  $t \in [0, T]$ ,  $S_t \leq S_n(t) + \sup_{t \in [0, T]} |S_t - S_n(t)|$  hence  $\sup_{t \in I} S_t \leq \sup_{t \in I} S_n(t) + \sup_{t \in [0, T]} |S_t - S_n(t)|$ . By symmetry, we get that

$$\left| \sup_{t \in I} S_n(t) - \sup_{t \in I} S_t \right| \leq \sup_{t \in [0, T]} |S_t - S_n(t)|$$

and the conclusion follows. Note that the same property holds if we replace the infimum by the supremum of  $S$  over  $I$ .

On the set  $\{\inf_{t \in [0, T]} S_t < a\}$ , as  $S$  is continuous, we may construct by a measurable selection argument a measurable interval  $I = [t_0 - \varepsilon, t_0 + \varepsilon]$ ,  $\varepsilon > 0$  a.s., such that  $\sup_{t \in I} S_t < a$ . Otherwise, we arbitrarily set  $I = [0, T]$ .

Suppose that for a subsequence  $\mathbf{E}|1_{\tau^S > T} - 1_{\tau^{S_n^{(n)}} > T}| \rightarrow l > 0$ . Consider another subsequence such that we also have  $\sup_{t \in [0, T]} |S_t - S_n(t)| \rightarrow 0$  a.s. By the first step for the infimum, if  $\inf_{t \in [0, T]} S_t > a$ , then  $\inf_{t \in [0, T]} S_n(t) > a$  if  $n$  is large enough. As  $\inf_{t \in [0, T]} S_n^{(n)}(t) \geq \inf_{t \in [0, T]} S_n(t)$ , we deduce that  $\inf_{t \in [0, T]} S_n^{(n)}(t) > a$  and, finally,  $|1_{\tau^S > T} - 1_{\tau^{S_n^{(n)}} > T}| = 0$  if  $n$  is large enough.

On the set  $\inf_{t \in [0, T]} S_t < a$ , i.e. when  $\tau^S \leq T$ , we have  $\sup_{t \in I} S_t < a$ . By the first step,  $\sup_{t \in I} S_n(t) < a$  if  $n$  is large enough. Choose  $n$  sufficiently large so that  $T/n < \varepsilon(\omega)$ . Then, the random discrete date  $t_i^n$  such that  $t_0(\omega) \in [t_i^n, t_{i+1}^n)$  satisfies  $t_i^n \in I$  hence  $S_n^{(n)}(t_0) = S_n(t_i^n) < a$ . It follows that  $\tau^{S_n^{(n)}} \leq T$ . We finally deduce by the Lebesgue theorem that  $\mathbf{E}|1_{\tau^S > T} - 1_{\tau^{S_n^{(n)}} > T}| \rightarrow 0$  in contradiction with  $l > 0$ .  $\square$

**Lemma 5.13.** *Consider the case where  $\sigma_\kappa = \sigma(\cdot, \cdot \vee \kappa^{-1})$  and  $r_\kappa = r(\cdot, \cdot \vee \kappa^{-1})$  under Condition 2.3 with  $P(\tau \leq T) > 0$ . The random variables  $Z_\kappa = \inf_{t \in [0, T]} Y_\kappa(t)$ ,  $\kappa > 0$ , and  $Z_{\kappa, m} = \inf_{t \in [0, T]} Y_{\kappa, m}(t)$ ,  $m \geq 1$ , admit a density with respect to the Lebesgue measure.*

*Proof.* The proof is based on [13][Proposition 2.1.11]. To do so, we need verify the conditions of [13][Proposition 2.1.10]. As the mapping  $h(x) = x \vee \kappa^{-1}$  is not differentiable at point  $\kappa^{-1}$ , we first approximate  $h$  by

$$\begin{aligned} h^n(x) &= \kappa^{-1}1_{(-\infty, \kappa^{-1}-n^{-1}]}(x) + x1_{[\kappa^{-1}+n^{-1}, \infty)}(x) \\ &+ \left( \kappa^{-1} + \frac{n}{4}(x - \kappa^{-1} + n^{-1})^2 \right) 1_{[\kappa^{-1}-n^{-1}, \kappa^{-1}+n^{-1}]}(x). \end{aligned}$$

We have the following properties:

$$0 \leq h^n(x) - h(x) \leq \frac{1}{4n}, \quad |(h^n)'(x) - h'(x)| \leq 1_{[\kappa^{-1}-n^{-1}, \kappa^{-1}+n^{-1}]}(x).$$

We then consider the solution  $Y_\kappa^n$  of the s.d.e. (3.5) where we replace  $\sigma_\kappa$  by

$$\sigma_\kappa^n(t, x) := \sigma(t, h^n(x)).$$

We may show that  $\sup_{t \in [0, T]} |Y_\kappa^n(t) - Y_\kappa(t)| \rightarrow 0$  in  $L^2$  as  $n \rightarrow \infty$ . Therefore, since  $Y_\kappa^n$  admits a derivative in the Malliavin sense, we also deduce that  $Y_\kappa \in \mathbf{D}^{1,2}$ . Moreover, for all  $s \leq t$ , making  $n \rightarrow \infty$ , we get that

$$\begin{aligned} \mathbf{D}_s Y_\kappa(t) &= \sigma_\kappa(s, S_0 e^{Y_\kappa(s)}) + \int_s^t \mathbf{D}_s Y_\kappa(u) h'(S_0 e^{Y_\kappa(u)}) S_0 e^{Y_\kappa(u)} \nabla_x \sigma_\kappa(t, S_0 e^{Y_\kappa(u)}) dW_u, \\ &+ \int_s^t \mathbf{D}_s Y_\kappa(u) h'(S_0 e^{Y_\kappa(u)}) S_0 e^{Y_\kappa(u)} \nabla_x \gamma_\kappa(t, S_0 e^{Y_\kappa(u)}) du, \end{aligned}$$

where  $\gamma_\kappa = r_\kappa - \sigma_\kappa^2/2$ . By the assumptions, e.g.  $X \nabla_X \sigma(t, X)$  is bounded if  $X \geq \kappa^{-1}$  and  $h'(X) = 0$  if  $X \leq \kappa^{-1}$ , and the Gronwall's lemma, we deduce that

$$\mathbf{E} \sup_{t \in [0, T]} |\mathbf{D}_s Y_\kappa(t)|^p < \infty,$$

whatever  $p \geq 2$ . By the Ito formula, we obtain an expression of  $[\mathbf{D}_s Y_\kappa(t)]^2$  for  $t \geq s$  as a stochastic integral. By the stochastic Fubini theorem, we then also deduce a representation of  $t \mapsto \int_0^t [\mathbf{D}_s Y_\kappa(t)]^2 ds$  as a stochastic integral. Using the Gronwall's lemma, we finally deduce that  $\mathbf{E} \sup_{t \leq T} \int_0^t [\mathbf{D}_s Y_\kappa(t)]^2 ds < \infty$ , i.e. the conditions of [13][Proposition 2.1.10] hold. Moreover, as  $(\mathbf{D}_s Y_\kappa(t))_{t \geq s}$  is a Doléans-Dade exponential, it is strictly positive and we finally deduce that [13][Proposition 2.1.11] applies, i.e.  $Z_\kappa = \inf_{t \in [0, T]} Y_\kappa(t)$  admits a density with respect to the Lebesgue measure. The second statement is shown similarly by induction for  $m = 0, 1, \dots$ .  $\square$

*Proof of Theorem 3.10.* Notice that we have

$$|\mathbf{E}G(S)1_{\tau>T} - \mathbf{E}G(S_{\kappa,1,n}^{(n)})1_{\tilde{\tau}^{\kappa,1,n}>T}| \leq \mathbf{E}|G(S) - G(S_{\kappa})|1_{\tau>T} \quad (5.17)$$

$$+\mathbf{E}|G(S_{\kappa}) - G(S_{\kappa,m})| \quad (5.18)$$

$$+\mathbf{E}|G(S_{\kappa,m}) - G(S_{\kappa,m,n})| \quad (5.19)$$

$$+\mathbf{E}|G(S_{\kappa,m,n}) - G(S_{\kappa,m,n}^{(n)})| \quad (5.20)$$

$$+\mathbf{E}|G(S)||1_{\tau>T} - 1_{\tilde{\tau}^{\kappa,1,n}>T}|, \quad (5.21)$$

Since  $G$  is supposed to be Lipschitz, the first term (5.17) is bounded from above by  $\varepsilon_{\kappa}$  as stated in Lemma 3.4. The second term (5.18) is bounded from above by the error  $\varepsilon_{\kappa,m}$  given by Lemma 3.8. The term (5.19) is bounded from above according to Lemma 3.9. The consistency property with respect to discretization satisfied by  $G$  allows us to estimate the fourth term (5.20). As to the last term, notice that  $G$  is of linear growth since it is Lipschitz. By Corollary 5.7, we deduce from the Cauchy-Schwarz inequality a constant  $C$  such that

$$\mathbf{E}|G(S)||1_{\tau>T} - 1_{\tilde{\tau}^{\kappa,1,n}>T}| \leq C\sqrt{\mathbf{E}|1_{\tau>T} - 1_{\tilde{\tau}^{\kappa,1,n}>T}|}.$$

It remains to show that  $\mathbf{E}|1_{\tau>T} - 1_{\tilde{\tau}^{\kappa,m,n}>T}| \rightarrow 0$  a.s. as  $\kappa, m, n \rightarrow \infty$ . Let us introduce  $\tilde{\tau}^{\kappa} := \inf\{t : S_{\kappa}(t) < \kappa^{-1}\}$ . Since  $S$  and  $S_{\kappa}$  coincides on  $[0, \tau^{\kappa}]$  and  $\tilde{\tau}^{\kappa} = \tau^{\kappa} > T$  if  $\kappa$  is large enough and  $\tau > T$ , we deduce that  $|1_{\tau>T} - 1_{\tilde{\tau}^{\kappa}>T}| \rightarrow 0$  if  $\tau > T$  as  $\kappa \rightarrow \infty$ . If  $\tau \leq T$ , we have  $\tau^{\kappa} \leq \tau$  so that  $|1_{\tau>T} - 1_{\tilde{\tau}^{\kappa}>T}| = 0$ . Therefore,  $\mathbf{E}|1_{\tau>T} - 1_{\tilde{\tau}^{\kappa}>T}| \rightarrow 0$  as  $\kappa \rightarrow +\infty$ . It remains to show that  $\mathbf{E}|1_{\tilde{\tau}^{\kappa}>T} - 1_{\tilde{\tau}^{\kappa,m,n}>T}| \rightarrow 0$  a.s. as  $\kappa, m, n \rightarrow \infty$ . Let us introduce  $\tilde{\tau}^{\kappa,m} := \inf\{t : S_{\kappa,m}(t) < \kappa^{-1}\}$ . By Proposition 3.8, Lemma 5.13 and Lemma 5.11, for fixed  $\kappa$ ,  $\mathbf{E}|1_{\tilde{\tau}^{\kappa}>T} - 1_{\tilde{\tau}^{\kappa,m}>T}| \rightarrow 0$  as  $m \rightarrow \infty$ . Therefore, it remains to show that  $\mathbf{E}|1_{\tilde{\tau}^{\kappa,m}>T} - 1_{\tilde{\tau}^{\kappa,m,n}>T}| \rightarrow 0$  a.s. as  $n \rightarrow \infty$ . By virtue of Lemma 3.9, Lemma 5.13 and Lemma 5.12,  $\mathbf{E}|1_{\tilde{\tau}^{\kappa,m}>T} - 1_{\tilde{\tau}^{\kappa,m,n}>T}| \rightarrow 0$  a.s. as  $n \rightarrow \infty$ .  $\square$

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