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TWO MORE CHARACTERIZATIONS OF K -TRIVIALITY

NOAM GREENBERG, JOSEPH S. MILLER, BENOIT MONIN, AND DANIEL TURETSKY

ABSTRACT. We give two new characterizations of K -triviality. We show that if for all Y such that Ω is Y -random, Ω is $(Y \oplus A)$ -random, then A is K -trivial. The other direction was proved by Stephan and Yu, giving us the first titular characterization of K -triviality and answering a question of Yu. We also prove that if A is K -trivial, then for all Y such that Ω is Y -random, $(Y \oplus A) \equiv_{\text{LR}} Y$. This answers a question of Merkle. The other direction is immediate, so we have the second characterization of K -triviality.

The proof of the first characterization uses a new cupping result. We prove that if $A \not\leq_{\text{LR}} B$, then for every set X there is a B -random set Y such that X is computable from $Y \oplus A$.

1. PRELIMINARIES

We assume that the reader is familiar with basic notions from computability theory and effective randomness. For more information on these topics, we recommend either Nies [10] or Downey and Hirschfeldt [3].

The K -trivial sets have played an important role in the development of effective randomness. A set $A \in 2^\omega$ is K -trivial if $K(A \upharpoonright n) \leq^+ K(n)$, where K denotes prefix-free Kolmogorov complexity. Chaitin [1] proved that such sets are always Δ_2^0 , while Solovay [15] constructed a noncomputable K -trivial set. While these results date back to the 1970s, the importance of K -triviality did not become apparent until the 2000s, when several nontrivial characterizations were discovered. In particular:

Theorem 1.1 (Nies [9]; Hirschfeldt, Nies, and Stephan [5]). *The following are equivalent for a set $A \in 2^\omega$:*

- (a) A is K -trivial,
- (b) A is low for K : $K^A(n) \geq^+ K(n)$,
- (c) A is low for randomness: every random set is A -random,¹
- (d) A is a base for randomness: there is an A -random set $X \geq_{\text{T}} A$.

Nies [9] generalized (c) to LR-reducibility: we write $A \leq_{\text{LR}} B$ to mean that every B -random set is A -random. In particular, $A \leq_{\text{LR}} \emptyset$ means that A is low for randomness (hence K -trivial).

Much more has been proved about the K -trivial sets, including many other characterizations. But we will only need one other fact. If X is random, then we

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¹Throughout this paper, we consistently use *random* to mean Martin-Löf random.

say that Y is *low for* X if X is Y -random. This notion was introduced in [5], where it is shown that a set is K -trivial if and only if it is Δ_2^0 and low for Chaitin's Ω . However, many other sets are low for Ω ; for example, every 2-random set. A more recent result regarding K -triviality and lowness for Ω was used by Stephan and Yu to prove one direction of our first characterization (see the discussion before Proposition 3.2).

Theorem 1.2 (Simpson and Stephan [14, Theorem 3.11]). *If S has PA degree and is low for Ω , then S computes every K -trivial.*

In addition to these facts about the K -trivial sets, we will use several fairly well-known theorems from effective randomness. Van Lambalgen's theorem [16] says that $X \oplus Y$ is random if and only if X is random and Y is X -random. Two applications allow us to show that if X is random and Y is X -random, then X is Y -random. Every set is computable from some random set. Relativizing this to X :

Theorem 1.3 (Kučera [7]; Gács [4]). *For any sets X and C , there is an X -random set Y such that $C \leq_T Y \oplus X$.*

Any random set Turing below a Z -random set is also Z -random. Relativizing this to Y :

Theorem 1.4 (Miller and Yu [8, Theorem 4.3]). *Assume that $X \leq_T W \oplus Y$, X is Y -random, and W is $Z \oplus Y$ -random. Then X is $Z \oplus Y$ -random.*

Finally, we will use the relativized form of the “randomness preservation” basis theorem:

Theorem 1.5 (Downey, Hirschfeldt, Miller, Nies [2]; Reimann and Slaman [13]). *If W is Y -random and P is a nonempty $\Pi_1^0[Y]$ class, then there is a set $S \in P$ that is low for W .*

2. CUPPING WITH B -RANDOM SETS

As promised in the abstract, we prove the following cupping result.

Theorem 2.1. *Assume that $A \not\leq_{LR} B$. Then for any set X , there is a B -random set Y such that $X \leq_T Y \oplus A$ (in fact, we make Y weakly 2-random relative to B).*

Our proof uses a result of Kjos-Hanssen. We state it here in a slightly stronger form than he stated it, though without adding any essential content.

Theorem 2.2 (Kjos-Hanssen [6]). *$A \not\leq_{LR} B$ if and only if there is a $\Sigma_1^0[A]$ class U of measure less than one that intersects every positive measure $\Pi_1^0[B]$ class. Furthermore, for any $\varepsilon > 0$, we can ensure that $\lambda(U) < \varepsilon$.*

Kjos-Hanssen showed that $A \leq_{LR} B$ if and only if each $\Pi_1^0[A]$ class of positive measure has a $\Pi_1^0[B]$ subclass of positive measure.² Taking the contrapositive: $A \not\leq_{LR} B$ if and only if there is a $\Pi_1^0[A]$ class T of positive measure that does not have a positive measure $\Pi_1^0[B]$ subclass. So $U = 2^\omega \setminus T$ would be a $\Sigma_1^0[A]$ class of measure less than one that intersects every positive measure $\Pi_1^0[B]$ class.

The fact that U can be taken to have arbitrarily small measure also follows from the work in [6]. We use this fact below, so for completeness, we sketch the argument. Assume that $A \not\leq_{LR} B$. So there is a B -random set X that is not A -random. Let

²This partial relativization of [6, Theorem 2.10] is stated in the proof of [6, Theorem 3.2].

U be a $\Sigma_1^0[A]$ class containing every non- A -random set. We may assume, of course, that the measure of U is as small as we like. Let P be a positive measure $\Pi_1^0[B]$ class. Relativizing a result of Kučera [7], every B -random set has a tail in P , so there is a tail Y of X in P . But Y is not A -random, so $Y \in U$.³

We need some basic notation for the proof of Theorem 2.1. If $P \subseteq 2^\omega$ is measurable and $\sigma \in 2^{<\omega}$, let $\lambda(P \mid \sigma)$ denote the *relative measure of P in $[\sigma]$* , i.e., $\lambda(P \cap [\sigma])/\lambda([\sigma])$. If $\sigma \in 2^{<\omega}$ and $W \subseteq 2^{<\omega}$, let $\sigma W = \{\sigma\tau : \tau \in W\}$.

Proof of Theorem 2.1. Suppose that $A \not\leq_{\text{LR}} B$. By Theorem 2.2, there is a $\Sigma_1^0[A]$ class U such that $\lambda(U) < 0.1$ and U intersects every positive measure $\Pi_1^0[B]$ class. Let W be an A -c.e. prefix-free set of strings such that $U = [W]^\prec$.

Let X be any set. We will construct $Y = X(0)\sigma_0X(1)\sigma_1X(2)\sigma_2\cdots$ such that each $\sigma_i \in W$. In this way, it is clear that $X \leq_{\text{T}} Y \oplus A$. To ensure that Y is weakly 2-random relative to B , we build it inside a nested sequence of $\Pi_1^0[B]$ classes of positive measure. The following claim will let us hit W and code the next bit of X while staying inside the current $\Pi_1^0[B]$ class.

Claim. For any string $\sigma \in 2^{<\omega}$ and any $\Pi_1^0[B]$ class P such that $\lambda(P \mid \sigma) > 0.1$, there is a $\tau \succeq \sigma$ such that $\tau \in \sigma W$ and $\lambda(P \mid \tau) \geq 0.8$.

Proof. We first extend σ to a string ρ that has no prefix in σW and such that $\lambda(P \mid \rho) > 0.9$. Let $Q = 2^\omega \setminus [\sigma W]^\prec$. As $\lambda(Q \mid \sigma) > 0.9$ and $\lambda(P \mid \sigma) > 0.1$, we have $\lambda(Q \cap P \mid \sigma) > 0$. By the Lebesgue density theorem, there is a $\rho \succeq \sigma$ such that $\lambda(Q \cap P \mid \rho) > 0.9$. In particular, $\lambda(P \mid \rho) > 0.9$ and $\lambda(Q \mid \rho) > 0.9$; the latter implies that ρ cannot have a prefix in σW .

We now extend ρ to a string τ satisfying the claim: $\tau \in \sigma W$ and $\lambda(P \mid \tau) \geq 0.8$. Consider the $\Pi_1^0(B)$ class $\tilde{P} = \{X \in P \cap [\rho] : (\forall n \geq |\rho|) \lambda(P \mid X \upharpoonright n) \geq 0.8\}$. In words, \tilde{P} is the subclass of $P \cap [\rho]$ in which we remove every basic neighborhood inside $[\rho]$ where the relative measure of P drops below 0.8. It is not hard to show that we remove at most 0.8 from the relative measure of $P \cap [\rho]$ inside $[\rho]$ (consider the antichain of maximal basic neighborhoods that are removed). But $\lambda(P \mid \rho) > 0.9$, so $\lambda(\tilde{P} \mid \rho) > 0.1$. In particular, \tilde{P} is a positive measure subclass of $[\sigma]$, so by the choice of $U = [W]^\prec$, it must be the case that $[\sigma W]^\prec$ intersects \tilde{P} . Take $\tau \in \sigma W$ such that $\tilde{P} \cap [\tau] \neq \emptyset$. By the definition of \tilde{P} , we have $\lambda(P \mid \tau) \geq 0.8$. \diamond

We are ready to construct Y . We will construct it as the limit of a sequence $\tau_0 \preceq \tau_1 \preceq \tau_2 \preceq \cdots$ of strings, while staying inside a decreasing sequence $P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$ of $\Pi_1^0[B]$ classes. Let $P_0 = 2^\omega$ and let τ_0 be the empty string. We start stage n of the construction with a $\Pi_1^0[B]$ class P_n and a string $\tau_n = X(0)\sigma_0X(1)\cdots X(n-1)\sigma_{n-1}$ such that

$$(\star) \quad \lambda(P_n \mid \tau_n X(n)) > 0.1.$$

(Note that this is true at stage 0.) First, we want to make progress towards Y being weakly 2-random relative to B . Let $\bigcup_{m \in \omega} R_m$ be the n th $\Sigma_2^0[B]$ class of measure one, where $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ is a nested sequence of $\Pi_1^0[B]$ classes. Pick m large enough that $\lambda(P_n \cap R_m \mid \tau_n X(n)) > 0.1$ and let $P_{n+1} = P_n \cap R_m$. So as long as we ensure that $Y \in P_{n+1}$, we have ensured that Y is in the n th $\Sigma_2^0[B]$ class of measure

³In fact, $U \cap P$ has positive measure. Choose $\sigma \in 2^{<\omega}$ such that $Y \in [\sigma] \subseteq U$. Then $\tilde{P} = P \cap [\sigma] \subseteq P \cap U$ is a $\Pi_1^0[B]$ class. Since it contains Y , which is B -random, it cannot have measure zero.

one. Now apply the claim to get $\tau_{n+1} \succeq \tau_n X(n)$ such that $\lambda(P_{n+1} \mid \tau_{n+1}) \geq 0.8$ and $\tau_{n+1} \in \tau_n X(n)W$. Let σ_n be the string for which $\tau_{n+1} = \tau_n X(n)\sigma_n$; in particular, $\sigma_n \in W$. Note that $\lambda(P_{n+1} \mid \tau_{n+1} X(n+1)) \geq 0.6 > 0.1$, so (\star) holds at stage $n+1$.

Let $Y = \bigcup_{n \in \omega} \tau_n = X(0)\sigma_0 X(1)\sigma_1 X(2)\sigma_2 \cdots$. As promised, each σ_i is in W , so $X \leq_T Y \oplus A$. By construction, $P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$, and each τ_n can be extended to an element of P_n . Therefore, $Y \in \bigcap_{n \in \omega} P_n$. This ensures that Y is in every $\Sigma_2^0[B]$ class of measure one, so Y is weakly 2-random relative to B . \square

3. LOW FOR X PRESERVING

Definition 3.1. Let X be random. A set A is *low for X preserving* if for all Y ,
 Y is low for $X \implies Y \oplus A$ is low for X .

This notion was recently introduced by Yu Liang, who called it *absolutely low for X* . Stephan and Yu proved that every K -trivial is low for Ω preserving (see [11, Fact 1.8]). Yu asked if the converse is true: if a set is low for Ω preserving, is it K -trivial? We show that, in fact, this holds in general.

Proposition 3.2. *If X is random, then low for X preserving implies K -triviality.*

Proof. Assume that A is low for X preserving.

First, we claim that $A \leq_{\text{LR}} X$. If not, then Theorem 2.1 gives us an X -random set Y such that $X \leq_T Y \oplus A$. By Van Lambalgen's theorem, X is Y -random. But $X \leq_T Y \oplus A$ implies that X is not $(Y \oplus A)$ -random. This contradicts the assumption that A is low for X preserving. Therefore, $A \leq_{\text{LR}} X$.

By Theorem 1.3, there is an X -random set Y such that $A \leq_T Y \oplus X$. Because A is low for X preserving, we have that X is $(Y \oplus A)$ -random. Furthermore, because Y is X -random and $A \leq_{\text{LR}} X$, we know that Y is A -random. Therefore, by Van Lambalgen's theorem, $Y \oplus X$ is A -random. But $Y \oplus X$ computes A , so A is a base for randomness. Therefore, it is K -trivial (see Theorem 1.1). \square

Together with the result of Stephan and Yu, we get a new characterization of K -triviality:

Theorem 3.3. *A set A is K -trivial if and only if it is low for Ω preserving.*

We actually want a slight generalization of Stephan and Yu's result.

Lemma 3.4. *If A is K -trivial and Y is low for Ω , then $Y \equiv_{\text{LR}} (Y \oplus A)$.*

Proof. Let A be K -trivial and Y be low for Ω . Let X be any Y -random. By Theorem 1.3, there is a Y -random set W such that both Ω and X are computable from $W \oplus Y$. There is a nonempty $\Pi_1^0[Y]$ class containing only members with PA degree relative to Y . So by Theorem 1.5, there is a low for W set S with PA degree relative to Y . Thus W is S -random and $Y \leq_T S$. By Theorem 1.4, both X and Ω are also S -random. Since S is PA and low for Ω , by Theorem 1.2, S computes every K -trivial. In particular, $A \leq_T S$. Because $Y \oplus A \leq_T S$ and X is S -random, X is $Y \oplus A$ -random. But X was any Y -random set, so $Y \equiv_{\text{LR}} Y \oplus A$. \square

The property above is easily seen to imply K -triviality, giving us our second characterization of K -triviality and answering a question of Merkle (see [11]).

Theorem 3.5. *A set A is K -trivial if and only if for all Y*

$$Y \text{ is low for } \Omega \implies Y \equiv_{\text{LR}} (Y \oplus A).$$

Proof. One direction is Lemma 3.4. For the other direction, assume that A has the given property. Note Ω is \emptyset -random, so $\emptyset \equiv_{\text{LR}} \emptyset \oplus A \equiv_{\text{LR}} A$. In other words, A is low for randomness, hence K -trivial (see Theorem 1.1). \square

It is natural to ask if low for X preserving is equivalent to K -triviality for all random X . As we shall see, this is not the case, though it is true for some X .

Proposition 3.6. *If $\Omega \leq_{\text{T}} X$, then low for X preserving is equivalent to K -triviality.*

Proof. One direction is given by Proposition 3.2. For the other direction, let A be K -trivial and take any Y such that X is Y -random. By (the unrelativized form of) Theorem 1.4, Ω is also Y -random. By Lemma 3.4, $Y \equiv_{\text{LR}} (Y \oplus A)$. Therefore, X is $(Y \oplus A)$ -random. \square

For other X , low for X preserving is equivalent to being computable.

Proposition 3.7. *If X is Schnorr $[\emptyset']$ random but not 2-random, then only the computable sets are low for X preserving.*

Proof. We prove the contrapositive. Assume that A is not computable. If A is not Δ_2^0 , then it is not K -trivial, hence by Proposition 3.2, it is not low for X preserving. So assume that A is Δ_2^0 . By Posner–Robinson [12], there is a low set Y such that $Y \oplus A \equiv_{\text{T}} \emptyset'$. Because X is Schnorr $[\emptyset']$, it is random relative to any low set,⁴ so it is Y -random. But X is not 2-random, so it is not $(Y \oplus A)$ -random. Therefore, A is not low for X preserving. \square

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⁴In fact, this property characterizes Schnorr $[\emptyset']$ randomness: Yu [17] showed that X is Schnorr $[\emptyset']$ random if and only if X is Z -random for every low set Z .

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