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## TWO MORE CHARACTERIZATIONS OF $K$ -TRIVIALITY

NOAM GREENBERG, JOSEPH S. MILLER, BENOIT MONIN, AND DANIEL TURETSKY

ABSTRACT. We give two new characterizations of  $K$ -triviality. We show that if for all  $Y$  such that  $\Omega$  is  $Y$ -random,  $\Omega$  is  $(Y \oplus A)$ -random, then  $A$  is  $K$ -trivial. The other direction was proved by Stephan and Yu, giving us the first titular characterization of  $K$ -triviality and answering a question of Yu. We also prove that if  $A$  is  $K$ -trivial, then for all  $Y$  such that  $\Omega$  is  $Y$ -random,  $(Y \oplus A) \equiv_{\text{LR}} Y$ . This answers a question of Merkle. The other direction is immediate, so we have the second characterization of  $K$ -triviality.

The proof of the first characterization uses a new cupping result. We prove that if  $A \not\leq_{\text{LR}} B$ , then for every set  $X$  there is a  $B$ -random set  $Y$  such that  $X$  is computable from  $Y \oplus A$ .

### 1. PRELIMINARIES

We assume that the reader is familiar with basic notions from computability theory and effective randomness. For more information on these topics, we recommend either Nies [10] or Downey and Hirschfeldt [3].

The  $K$ -trivial sets have played an important role in the development of effective randomness. A set  $A \in 2^\omega$  is  $K$ -trivial if  $K(A \upharpoonright n) \leq^+ K(n)$ , where  $K$  denotes prefix-free Kolmogorov complexity. Chaitin [1] proved that such sets are always  $\Delta_2^0$ , while Solovay [15] constructed a noncomputable  $K$ -trivial set. While these results date back to the 1970s, the importance of  $K$ -triviality did not become apparent until the 2000s, when several nontrivial characterizations were discovered. In particular:

**Theorem 1.1** (Nies [9]; Hirschfeldt, Nies, and Stephan [5]). *The following are equivalent for a set  $A \in 2^\omega$ :*

- (a)  $A$  is  $K$ -trivial,
- (b)  $A$  is low for  $K$ :  $K^A(n) \geq^+ K(n)$ ,
- (c)  $A$  is low for randomness: every random set is  $A$ -random,<sup>1</sup>
- (d)  $A$  is a base for randomness: there is an  $A$ -random set  $X \geq_{\text{T}} A$ .

Nies [9] generalized (c) to LR-reducibility: we write  $A \leq_{\text{LR}} B$  to mean that every  $B$ -random set is  $A$ -random. In particular,  $A \leq_{\text{LR}} \emptyset$  means that  $A$  is low for randomness (hence  $K$ -trivial).

Much more has been proved about the  $K$ -trivial sets, including many other characterizations. But we will only need one other fact. If  $X$  is random, then we

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<sup>1</sup>Throughout this paper, we consistently use *random* to mean Martin-Löf random.

say that  $Y$  is *low for*  $X$  if  $X$  is  $Y$ -random. This notion was introduced in [5], where it is shown that a set is  $K$ -trivial if and only if it is  $\Delta_2^0$  and low for Chaitin's  $\Omega$ . However, many other sets are low for  $\Omega$ ; for example, every 2-random set. A more recent result regarding  $K$ -triviality and lowness for  $\Omega$  was used by Stephan and Yu to prove one direction of our first characterization (see the discussion before Proposition 3.2).

**Theorem 1.2** (Simpson and Stephan [14, Theorem 3.11]). *If  $S$  has PA degree and is low for  $\Omega$ , then  $S$  computes every  $K$ -trivial.*

In addition to these facts about the  $K$ -trivial sets, we will use several fairly well-known theorems from effective randomness. Van Lambalgen's theorem [16] says that  $X \oplus Y$  is random if and only if  $X$  is random and  $Y$  is  $X$ -random. Two applications allow us to show that if  $X$  is random and  $Y$  is  $X$ -random, then  $X$  is  $Y$ -random. Every set is computable from some random set. Relativizing this to  $X$ :

**Theorem 1.3** (Kučera [7]; Gács [4]). *For any sets  $X$  and  $C$ , there is an  $X$ -random set  $Y$  such that  $C \leq_T Y \oplus X$ .*

Any random set Turing below a  $Z$ -random set is also  $Z$ -random. Relativizing this to  $Y$ :

**Theorem 1.4** (Miller and Yu [8, Theorem 4.3]). *Assume that  $X \leq_T W \oplus Y$ ,  $X$  is  $Y$ -random, and  $W$  is  $Z \oplus Y$ -random. Then  $X$  is  $Z \oplus Y$ -random.*

Finally, we will use the relativized form of the “randomness preservation” basis theorem:

**Theorem 1.5** (Downey, Hirschfeldt, Miller, Nies [2]; Reimann and Slaman [13]). *If  $W$  is  $Y$ -random and  $P$  is a nonempty  $\Pi_1^0[Y]$  class, then there is a set  $S \in P$  that is low for  $W$ .*

## 2. CUPPING WITH $B$ -RANDOM SETS

As promised in the abstract, we prove the following cupping result.

**Theorem 2.1.** *Assume that  $A \not\leq_{LR} B$ . Then for any set  $X$ , there is a  $B$ -random set  $Y$  such that  $X \leq_T Y \oplus A$  (in fact, we make  $Y$  weakly 2-random relative to  $B$ ).*

Our proof uses a result of Kjos-Hanssen. We state it here in a slightly stronger form than he stated it, though without adding any essential content.

**Theorem 2.2** (Kjos-Hanssen [6]).  *$A \not\leq_{LR} B$  if and only if there is a  $\Sigma_1^0[A]$  class  $U$  of measure less than one that intersects every positive measure  $\Pi_1^0[B]$  class. Furthermore, for any  $\varepsilon > 0$ , we can ensure that  $\lambda(U) < \varepsilon$ .*

Kjos-Hanssen showed that  $A \leq_{LR} B$  if and only if each  $\Pi_1^0[A]$  class of positive measure has a  $\Pi_1^0[B]$  subclass of positive measure.<sup>2</sup> Taking the contrapositive:  $A \not\leq_{LR} B$  if and only if there is a  $\Pi_1^0[A]$  class  $T$  of positive measure that does not have a positive measure  $\Pi_1^0[B]$  subclass. So  $U = 2^\omega \setminus T$  would be a  $\Sigma_1^0[A]$  class of measure less than one that intersects every positive measure  $\Pi_1^0[B]$  class.

The fact that  $U$  can be taken to have arbitrarily small measure also follows from the work in [6]. We use this fact below, so for completeness, we sketch the argument. Assume that  $A \not\leq_{LR} B$ . So there is a  $B$ -random set  $X$  that is not  $A$ -random. Let

<sup>2</sup>This partial relativization of [6, Theorem 2.10] is stated in the proof of [6, Theorem 3.2].

$U$  be a  $\Sigma_1^0[A]$  class containing every non- $A$ -random set. We may assume, of course, that the measure of  $U$  is as small as we like. Let  $P$  be a positive measure  $\Pi_1^0[B]$  class. Relativizing a result of Kučera [7], every  $B$ -random set has a tail in  $P$ , so there is a tail  $Y$  of  $X$  in  $P$ . But  $Y$  is not  $A$ -random, so  $Y \in U$ .<sup>3</sup>

We need some basic notation for the proof of Theorem 2.1. If  $P \subseteq 2^\omega$  is measurable and  $\sigma \in 2^{<\omega}$ , let  $\lambda(P \mid \sigma)$  denote the *relative measure of  $P$  in  $[\sigma]$* , i.e.,  $\lambda(P \cap [\sigma])/\lambda([\sigma])$ . If  $\sigma \in 2^{<\omega}$  and  $W \subseteq 2^{<\omega}$ , let  $\sigma W = \{\sigma\tau : \tau \in W\}$ .

*Proof of Theorem 2.1.* Suppose that  $A \not\leq_{LR} B$ . By Theorem 2.2, there is a  $\Sigma_1^0[A]$  class  $U$  such that  $\lambda(U) < 0.1$  and  $U$  intersects every positive measure  $\Pi_1^0[B]$  class. Let  $W$  be an  $A$ -c.e. prefix-free set of strings such that  $U = [W]^\prec$ .

Let  $X$  be any set. We will construct  $Y = X(0)\sigma_0X(1)\sigma_1X(2)\sigma_2\cdots$  such that each  $\sigma_i \in W$ . In this way, it is clear that  $X \leq_T Y \oplus A$ . To ensure that  $Y$  is weakly 2-random relative to  $B$ , we build it inside a nested sequence of  $\Pi_1^0[B]$  classes of positive measure. The following claim will let us hit  $W$  and code the next bit of  $X$  while staying inside the current  $\Pi_1^0[B]$  class.

*Claim.* For any string  $\sigma \in 2^{<\omega}$  and any  $\Pi_1^0[B]$  class  $P$  such that  $\lambda(P \mid \sigma) > 0.1$ , there is a  $\tau \succeq \sigma$  such that  $\tau \in \sigma W$  and  $\lambda(P \mid \tau) \geq 0.8$ .

*Proof.* We first extend  $\sigma$  to a string  $\rho$  that has no prefix in  $\sigma W$  and such that  $\lambda(P \mid \rho) > 0.9$ . Let  $Q = 2^\omega \setminus [\sigma W]^\prec$ . As  $\lambda(Q \mid \sigma) > 0.9$  and  $\lambda(P \mid \sigma) > 0.1$ , we have  $\lambda(Q \cap P \mid \sigma) > 0$ . By the Lebesgue density theorem, there is a  $\rho \succeq \sigma$  such that  $\lambda(Q \cap P \mid \rho) > 0.9$ . In particular,  $\lambda(P \mid \rho) > 0.9$  and  $\lambda(Q \mid \rho) > 0.9$ ; the latter implies that  $\rho$  cannot have a prefix in  $\sigma W$ .

We now extend  $\rho$  to a string  $\tau$  satisfying the claim:  $\tau \in \sigma W$  and  $\lambda(P \mid \tau) \geq 0.8$ . Consider the  $\Pi_1^0(B)$  class  $\tilde{P} = \{X \in P \cap [\rho] : (\forall n \geq |\rho|) \lambda(P \mid X \upharpoonright n) \geq 0.8\}$ . In words,  $\tilde{P}$  is the subclass of  $P \cap [\rho]$  in which we remove every basic neighborhood inside  $[\rho]$  where the relative measure of  $P$  drops below 0.8. It is not hard to show that we remove at most 0.8 from the relative measure of  $P \cap [\rho]$  inside  $[\rho]$  (consider the antichain of maximal basic neighborhoods that are removed). But  $\lambda(P \mid \rho) > 0.9$ , so  $\lambda(\tilde{P} \mid \rho) > 0.1$ . In particular,  $\tilde{P}$  is a positive measure subclass of  $[\sigma]$ , so by the choice of  $U = [W]^\prec$ , it must be the case that  $[\sigma W]^\prec$  intersects  $\tilde{P}$ . Take  $\tau \in \sigma W$  such that  $\tilde{P} \cap [\tau] \neq \emptyset$ . By the definition of  $\tilde{P}$ , we have  $\lambda(P \mid \tau) \geq 0.8$ .  $\diamond$

We are ready to construct  $Y$ . We will construct it as the limit of a sequence  $\tau_0 \preceq \tau_1 \preceq \tau_2 \preceq \cdots$  of strings, while staying inside a decreasing sequence  $P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$  of  $\Pi_1^0[B]$  classes. Let  $P_0 = 2^\omega$  and let  $\tau_0$  be the empty string. We start stage  $n$  of the construction with a  $\Pi_1^0[B]$  class  $P_n$  and a string  $\tau_n = X(0)\sigma_0X(1)\cdots X(n-1)\sigma_{n-1}$  such that

$$(\star) \quad \lambda(P_n \mid \tau_n X(n)) > 0.1.$$

(Note that this is true at stage 0.) First, we want to make progress towards  $Y$  being weakly 2-random relative to  $B$ . Let  $\bigcup_{m \in \omega} R_m$  be the  $n$ th  $\Sigma_2^0[B]$  class of measure one, where  $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$  is a nested sequence of  $\Pi_1^0[B]$  classes. Pick  $m$  large enough that  $\lambda(P_n \cap R_m \mid \tau_n X(n)) > 0.1$  and let  $P_{n+1} = P_n \cap R_m$ . So as long as we ensure that  $Y \in P_{n+1}$ , we have ensured that  $Y$  is in the  $n$ th  $\Sigma_2^0[B]$  class of measure

<sup>3</sup>In fact,  $U \cap P$  has positive measure. Choose  $\sigma \in 2^{<\omega}$  such that  $Y \in [\sigma] \subseteq U$ . Then  $\tilde{P} = P \cap [\sigma] \subseteq P \cap U$  is a  $\Pi_1^0[B]$  class. Since it contains  $Y$ , which is  $B$ -random, it cannot have measure zero.

one. Now apply the claim to get  $\tau_{n+1} \succeq \tau_n X(n)$  such that  $\lambda(P_{n+1} \mid \tau_{n+1}) \geq 0.8$  and  $\tau_{n+1} \in \tau_n X(n)W$ . Let  $\sigma_n$  be the string for which  $\tau_{n+1} = \tau_n X(n)\sigma_n$ ; in particular,  $\sigma_n \in W$ . Note that  $\lambda(P_{n+1} \mid \tau_{n+1} X(n+1)) \geq 0.6 > 0.1$ , so  $(\star)$  holds at stage  $n+1$ .

Let  $Y = \bigcup_{n \in \omega} \tau_n = X(0)\sigma_0 X(1)\sigma_1 X(2)\sigma_2 \cdots$ . As promised, each  $\sigma_i$  is in  $W$ , so  $X \leq_T Y \oplus A$ . By construction,  $P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$ , and each  $\tau_n$  can be extended to an element of  $P_n$ . Therefore,  $Y \in \bigcap_{n \in \omega} P_n$ . This ensures that  $Y$  is in every  $\Sigma_2^0[B]$  class of measure one, so  $Y$  is weakly 2-random relative to  $B$ .  $\square$

### 3. LOW FOR $X$ PRESERVING

**Definition 3.1.** Let  $X$  be random. A set  $A$  is *low for  $X$  preserving* if for all  $Y$ ,  
 $Y$  is low for  $X \implies Y \oplus A$  is low for  $X$ .

This notion was recently introduced by Yu Liang, who called it *absolutely low for  $X$* . Stephan and Yu proved that every  $K$ -trivial is low for  $\Omega$  preserving (see [11, Fact 1.8]). Yu asked if the converse is true: if a set is low for  $\Omega$  preserving, is it  $K$ -trivial? We show that, in fact, this holds in general.

**Proposition 3.2.** *If  $X$  is random, then low for  $X$  preserving implies  $K$ -triviality.*

*Proof.* Assume that  $A$  is low for  $X$  preserving.

First, we claim that  $A \leq_{\text{LR}} X$ . If not, then Theorem 2.1 gives us an  $X$ -random set  $Y$  such that  $X \leq_T Y \oplus A$ . By Van Lambalgen's theorem,  $X$  is  $Y$ -random. But  $X \leq_T Y \oplus A$  implies that  $X$  is not  $(Y \oplus A)$ -random. This contradicts the assumption that  $A$  is low for  $X$  preserving. Therefore,  $A \leq_{\text{LR}} X$ .

By Theorem 1.3, there is an  $X$ -random set  $Y$  such that  $A \leq_T Y \oplus X$ . Because  $A$  is low for  $X$  preserving, we have that  $X$  is  $(Y \oplus A)$ -random. Furthermore, because  $Y$  is  $X$ -random and  $A \leq_{\text{LR}} X$ , we know that  $Y$  is  $A$ -random. Therefore, by Van Lambalgen's theorem,  $Y \oplus X$  is  $A$ -random. But  $Y \oplus X$  computes  $A$ , so  $A$  is a base for randomness. Therefore, it is  $K$ -trivial (see Theorem 1.1).  $\square$

Together with the result of Stephan and Yu, we get a new characterization of  $K$ -triviality:

**Theorem 3.3.** *A set  $A$  is  $K$ -trivial if and only if it is low for  $\Omega$  preserving.*

We actually want a slight generalization of Stephan and Yu's result.

**Lemma 3.4.** *If  $A$  is  $K$ -trivial and  $Y$  is low for  $\Omega$ , then  $Y \equiv_{\text{LR}} (Y \oplus A)$ .*

*Proof.* Let  $A$  be  $K$ -trivial and  $Y$  be low for  $\Omega$ . Let  $X$  be any  $Y$ -random. By Theorem 1.3, there is a  $Y$ -random set  $W$  such that both  $\Omega$  and  $X$  are computable from  $W \oplus Y$ . There is a nonempty  $\Pi_1^0[Y]$  class containing only members with PA degree relative to  $Y$ . So by Theorem 1.5, there is a low for  $W$  set  $S$  with PA degree relative to  $Y$ . Thus  $W$  is  $S$ -random and  $Y \leq_T S$ . By Theorem 1.4, both  $X$  and  $\Omega$  are also  $S$ -random. Since  $S$  is PA and low for  $\Omega$ , by Theorem 1.2,  $S$  computes every  $K$ -trivial. In particular,  $A \leq_T S$ . Because  $Y \oplus A \leq_T S$  and  $X$  is  $S$ -random,  $X$  is  $Y \oplus A$ -random. But  $X$  was any  $Y$ -random set, so  $Y \equiv_{\text{LR}} Y \oplus A$ .  $\square$

The property above is easily seen to imply  $K$ -triviality, giving us our second characterization of  $K$ -triviality and answering a question of Merkle (see [11]).

**Theorem 3.5.** *A set  $A$  is  $K$ -trivial if and only if for all  $Y$*

$$Y \text{ is low for } \Omega \implies Y \equiv_{\text{LR}} (Y \oplus A).$$

*Proof.* One direction is Lemma 3.4. For the other direction, assume that  $A$  has the given property. Note  $\Omega$  is  $\emptyset$ -random, so  $\emptyset \equiv_{\text{LR}} \emptyset \oplus A \equiv_{\text{LR}} A$ . In other words,  $A$  is low for randomness, hence  $K$ -trivial (see Theorem 1.1).  $\square$

It is natural to ask if low for  $X$  preserving is equivalent to  $K$ -triviality for all random  $X$ . As we shall see, this is not the case, though it is true for some  $X$ .

**Proposition 3.6.** *If  $\Omega \leq_{\text{T}} X$ , then low for  $X$  preserving is equivalent to  $K$ -triviality.*

*Proof.* One direction is given by Proposition 3.2. For the other direction, let  $A$  be  $K$ -trivial and take any  $Y$  such that  $X$  is  $Y$ -random. By (the unrelativized form of) Theorem 1.4,  $\Omega$  is also  $Y$ -random. By Lemma 3.4,  $Y \equiv_{\text{LR}} (Y \oplus A)$ . Therefore,  $X$  is  $(Y \oplus A)$ -random.  $\square$

For other  $X$ , low for  $X$  preserving is equivalent to being computable.

**Proposition 3.7.** *If  $X$  is Schnorr $[\emptyset']$  random but not 2-random, then only the computable sets are low for  $X$  preserving.*

*Proof.* We prove the contrapositive. Assume that  $A$  is not computable. If  $A$  is not  $\Delta_2^0$ , then it is not  $K$ -trivial, hence by Proposition 3.2, it is not low for  $X$  preserving. So assume that  $A$  is  $\Delta_2^0$ . By Posner–Robinson [12], there is a low set  $Y$  such that  $Y \oplus A \equiv_{\text{T}} \emptyset'$ . Because  $X$  is Schnorr $[\emptyset']$ , it is random relative to any low set,<sup>4</sup> so it is  $Y$ -random. But  $X$  is not 2-random, so it is not  $(Y \oplus A)$ -random. Therefore,  $A$  is not low for  $X$  preserving.  $\square$

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<sup>4</sup>In fact, this property characterizes Schnorr $[\emptyset']$  randomness: Yu [17] showed that  $X$  is Schnorr $[\emptyset']$  random if and only if  $X$  is  $Z$ -random for every low set  $Z$ .

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